Recent progress in the spectral theory of first order elliptic systems

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Typical problem in my subject area

$$-\Delta v = \lambda v \quad \text{in} \quad M \subset \mathbb{R}^3, \qquad v|_{\partial M} = 0.$$

Finding eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$ is difficult, so one introduces the counting function

$$N(\lambda) := \sum_{0<\lambda_k<\lambda} 1$$

("number of eigenvalues below a given λ") and studies the asymptotic behaviour of $N(\lambda)$ as $\lambda \to +\infty$.

Rayleigh–Jeans law (1905):

$$N(\lambda) = \frac{V}{6\pi^2}\lambda^{3/2} + o(\lambda^{3/2}) \quad \text{as} \quad \lambda \to +\infty,$$

where V is the volume of M. Rigorous proof: H.Weyl (1912).

Weyl's Conjecture (1913): one can do better and write a two-term asymptotic formula

$$N(\lambda) = \frac{V}{6\pi^2}\lambda^{3/2} - \frac{S}{16\pi}\lambda + o(\lambda) \quad \text{as} \quad \lambda \to +\infty,$$

where S is the surface area of ∂M. Proved, under certain geometric assumptions on the billiard flow, by V.Ivrii (1980).

My contribution: similar results for higher order operators.

Yu.Safarov and D.Vassiliev, *The asymptotic distribution of eigenvalues of partial differential operators*, American Mathematical Society, 1997 (hardcover), 1998 (softcover).

"In the reviewer's opinion, this book is indispensable for serious students of spectral asymptotics". Lars Hörmander for the Bulletin of the London Mathematical Society.

Will be studying the spectral problem for a system

$$Av = \lambda v$$

where A is a first order elliptic self-adjoint $m \times m$ matrix pseudo-differential operator acting on columns of m complex-valued half-densities v over a connected compact n-dimensional manifold M without boundary. The operator is not necessarily semi-bounded.

Principal symbol $A_1(x, \xi)$ is matrix-valued function on $T^*M \setminus \{0\}$.

The eigenvalues of the principal symbol are denoted $h^{(j)}(x, \xi)$. These $h^{(j)}(x, \xi)$ are assumed to be simple.

Spectral function $e(\lambda,x,x) := \sum_{0<\lambda_k<\lambda} \|v_k(x)\|^2$.

Counting function $N(\lambda) := \sum_{0<\lambda_k<\lambda} 1 = \int_M e(\lambda,x,x)\,dx$.

Want to derive, under appropriate assumptions on Hamiltonian trajectories, two-term asymptotics

$$e(\lambda,x,x) = a(x)\,\lambda^n + b(x)\,\lambda^{n-1} + o(\lambda^{n-1}),$$

$$N(\lambda) = a\lambda^n + b\lambda^{n-1} + o(\lambda^{n-1}),$$

as $\lambda \to +\infty$. Here I expect the real constants a, b and real densities $a(x)$, $b(x)$ to be related as

$$a = \int_M a(x)\,dx, \qquad b = \int_M b(x)\,dx.$$

Warning: doing microlocal analysis for systems is not easy

1 V.Ivrii, 1980, Soviet Math. Doklady.

2 V.Ivrii, 1982, Funct. Anal. Appl.

3 G.V.Rozenblyum, 1983, Journal of Mathematical Sciences.

4 V.Ivrii, 1984, Springer Lecture Notes.

5 Yu.Safarov, DSc thesis, 1989, Steklov Mathematical Institute.

6 V.Ivrii, book, 1998, Springer.

7 W.J.Nicoll, PhD thesis, 1998, University of Sussex.

8 I.Kamotski and M.Ruzhansky, 2007, Comm. PDEs.

9 O.Chervova, R.J.Downes and D.Vassiliev, 2012, preprint `http://arxiv.org/abs/1208.6015`, to appear in JST.

Correct formula for the second asymptotic coefficient

$$b(x) = -n \sum_{j=1}^{m^+} \int\limits_{h^{(j)}(x,\xi)<1} \Big([v^{(j)}]^* A_{\mathrm{sub}} v^{(j)} - \frac{i}{2}\{[v^{(j)}]^*, A_1 - h^{(j)}, v^{(j)}\}$$

$$+ \frac{i}{n-1} h^{(j)} \{[v^{(j)}]^*, v^{(j)}\} \Big)(x,\xi)\, đ\xi$$

where $v^{(j)}(x,\xi)$ are eigenvectors of principal symbol,

$$\{P,R\} := P_{x^\alpha} R_{\xi_\alpha} - P_{\xi_\alpha} R_{x^\alpha}$$

is the Poisson bracket on matrix-functions and

$$\{P,Q,R\} := P_{x^\alpha} Q R_{\xi_\alpha} - P_{\xi_\alpha} Q R_{x^\alpha}$$

is its further generalisation.

The U(1) connection

Each eigenvector $v^{(j)}(x,\xi)$ of the principal symbol is defined modulo a gauge transformation

$$v^{(j)} \mapsto e^{i\phi^{(j)}} v^{(j)}$$

where

$$\phi^{(j)} : T^*M \setminus \{0\} \to \mathbb{R}$$

is an arbitrary smooth function. Want to parallel transport eigenvector so that derivative of eigenvector along the curve is orthogonal to eigenvector itself. This gives a U(1) connection characterized by a $2n$-component real covector potential (P_α, Q^β) where

$$P_\alpha := i[v^{(j)}]^* v^{(j)}_{x^\alpha}, \qquad Q^\beta := i[v^{(j)}]^* v^{(j)}_{\xi_\beta}.$$

Covariant derivative on functions $T^*M \setminus \{0\} \to \{z \in \mathbb{C} : |z| = 1\}$ is

$$\nabla_\alpha := \partial/\partial x^\alpha - iP_\alpha, \qquad \nabla^\beta := \partial/\partial \xi_\beta - iQ^\beta.$$

The curvature of our U(1) connection is

$$R := -i \begin{pmatrix} \nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha & \nabla_\alpha \nabla^\delta - \nabla^\delta \nabla_\alpha \\ \nabla^\gamma \nabla_\beta - \nabla_\beta \nabla^\gamma & \nabla^\gamma \nabla^\delta - \nabla^\delta \nabla^\gamma \end{pmatrix}.$$

This curvature is the exterior derivative of the covector potential (with minus sign).

The scalar curvature of our U(1) connection is

$$-i(\nabla_\alpha \nabla^\alpha - \nabla^\alpha \nabla_\alpha) = -i\{[v^{(j)}]^*, v^{(j)}\}.$$

Recall: curly brackets denote Poisson bracket on matrix-functions

$$\{P, R\} := P_{x^\alpha} R_{\xi_\alpha} - P_{\xi_\alpha} R_{x^\alpha}.$$

Analysis of special case

Additional assumptions:

our manifold has dimension 3,

the number of equations in our system is 2,

our operator is differential (as opposed to pseudodifferential),

the principal symbol is trace-free.

Geometric object 1: the metric

The determinant of the principal symbol is a negative definite quadratic form

$$\det A_1(x,\xi) = -g^{\alpha\beta}\xi_\alpha\xi_\beta$$

and the coefficients $g^{\alpha\beta}(x) = g^{\beta\alpha}(x)$, $\alpha, \beta = 1, 2, 3$, can be interpreted as components of a (contravariant) Riemannian metric.

Geometric object 2: the teleparallel connection

Define an affine connection as follows. Suppose we have a covector ξ based at the point $x \in M$ and we want to construct a parallel covector $\tilde{\xi}$ based at the point $\tilde{x} \in M$. This is done by solving the linear system of equations

$$A_1(\tilde{x}, \tilde{\xi}) = A_1(x, \xi).$$

The teleparallel connection coefficients $\Gamma^{\alpha}{}_{\beta\gamma}(x)$ can be written down explicitly in terms of the principal symbol and this allows us to define yet another geometric object — the *torsion tensor*

$$T^{\alpha}{}_{\beta\gamma} := \Gamma^{\alpha}{}_{\beta\gamma} - \Gamma^{\alpha}{}_{\gamma\beta}.$$

The teleparallel connection has zero curvature and nonzero torsion. It is the opposite of the Levi-Civita connection.

Geometric object 3: the topological charge

Put

$$\mathbf{c} := -\frac{i}{2}\sqrt{\det g_{\alpha\beta}}\,\operatorname{tr}\Big((A_1)_{\xi_1}(A_1)_{\xi_2}(A_1)_{\xi_3}\Big).$$

The number $\mathbf{c}$ can take only two values, $+1$ or -1. It describes the orientation of the principal symbol $A_1(x,\xi)$ relative to the chosen orientation of local coordinates.

Simplified formula for the second asymptotic coefficient

$$b(x) = \frac{1}{8\pi^2}\Big([\,3\,\mathbf{c}\,{*T}^{\mathsf{ax}} - 2\,\mathsf{tr}\,A_{\mathsf{sub}}\,]\,\sqrt{\mathsf{det}\,g_{\alpha\beta}}\Big)(x)$$

where

$$T^{\mathsf{ax}}_{\alpha\beta\gamma} := \frac{1}{3}(T_{\alpha\beta\gamma} + T_{\gamma\alpha\beta} + T_{\beta\gamma\alpha})$$

is *axial torsion* (totally antisymmetric piece of the torsion tensor) and $*$ is the Hodge star.

Note absence of integration in ξ: it was performed explicitly.

Spectral theoretic characterization of the massless Dirac operator

Imagine a single neutrino living in a compact 3-dimensional universe. I want to find the spectrum of energy levels this neutrino can occupy. This leads to an eigenvalue problem

$$Av = \lambda v$$

where A is the massless Dirac operator, a particular 2×2 matrix first order differential operator with trace-free principal symbol.

Note: the explicit formula for the massless Dirac operator in curved space is very complicated.

Now suppose I am a spectral analyst and I want to know whether my differential operator A is a massless Dirac operator, without having to learn the differential geometry of spinors.

Theorem 1 An operator is a massless Dirac operator iff

a) subprincipal symbol is proportional to the identity matrix, and

b) second asymptotic coefficient of the spectral function is zero.

Note: the conditions of Theorem 1 are invariant under transformations of the operator $A \mapsto RAR^*$ where $R : M \to \mathrm{SU}(2)$.

Note: Theorem 1 does not feel the topological differences between different parallelizations of the manifold M.

Bottom line: the differential geometry of spinors is encoded within the microlocal analysis of PDEs.

Spectral theoretic characterization of the massless Dirac action

Working on a compact 3-manifold, consider eigenvalue problem

$$Av = \lambda w v$$

where A is a first order elliptic self-adjoint 2×2 matrix differential operator with $\operatorname{tr} A_1(x,\xi) = 0$ and $A_{\text{sub}}(x) = 0$, and $w : M \to (0, +\infty)$ is a scalar weight function.

Consider the map

$$A_1(x,\xi) \text{ and } w(x) \quad \mapsto \quad b$$

subject to the constraint "$\det A_1(x,\xi)$ is preserved".

Theorem 2 The above map is equivalent to the map

$$\text{spinor field} \quad \mapsto \quad \text{massless Dirac action.}$$

Four fundamental equations of theoretical physics

1 Maxwell's equations. Describe electromagnetism and photons.

2 Dirac equation. Describes electrons and positrons.

3 Massless Dirac equation. Describes* neutrinos and antineutrinos.

4 Linearized Einstein field equations of general relativity. Describe gravity.

All four contain the same physical constant — the speed of light.

*OK, I know that neutrinos actually have a small mass.

Accepted explanation: theory of relativity

God is a geometer. He created a 4-dimensional world parametrized by coordinates x^0, x^1, x^2, x^3 (here x^0 is time), in which distances are measured in a funny way:

$$\text{distance}^2 = -c^2(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

where c is the speed of light.

Without the term $-c^2(dx^0)^2$ this would be Pythagoras' theorem. Funny way of measuring distances is called *Minkowski metric*.

Having decided to use the Minkowski metric, God then wrote down the main equations of theoretical physics using **only geometric constructions**, i.e. using concepts such as connection, curvature etc. This way all equations have the same physical constant — the speed of light — encoded in them.

Alternative explanation

God is an analyst. He created a 3-dimensional Euclidean world, then added (absolute) time and wrote down one system of nonlinear PDEs which describes phenomena in this world. This system of PDEs has different solutions which we interpret as electromagnetism, gravity, electrons, neutrinos etc. The reason the same physical constant — the speed of light — manifests itself in all physical phenomena is because we are looking at different solutions of the **same** system of PDEs.

I cannot write down the unifying system of field equations but I can extract some geometric constructs of theoretical physics from systems of PDEs, simply by performing microlocal analysis.

Action plan: spend next 100 years studying systems of nonlinear hyperbolic PDEs, hoping to find soliton-type solutions.

Spectral asymmetry

Consider the unit torus $\mathbb{T}^3$ parameterized by cyclic coordinates x^α, $\alpha = 1, 2, 3$, of period 2π and equipped with metric

$$g_{\alpha\beta}\, dx^\alpha dx^\beta = \left[dx^1 + \epsilon(\cos x^1)dx^2 + \epsilon(\sin x^1)dx^3\right]^2 + \left[dx^2\right]^2 + \left[dx^3\right]^2.$$

Then the (double) eigenvalue of the massless Dirac operator which is closest to zero is given by the explicit formula

$$\lambda = \frac{2\sqrt{1+\epsilon^2} - 2 - \epsilon^2}{4} = -\frac{\epsilon^4}{16} + O(\epsilon^6) \qquad \text{as} \qquad \epsilon \to 0.$$