Is God a geometer or an analyst?

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My background

Analysis of partial differential eq-s (PDEs) and spectral theory.

Example of a problem in this subject area: acoustic resonator. Suppose we are studying the vibrations of air

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\varphi}{\partial z^2} = 0$$

in a domain $\Omega \subset \mathbb{R}^3$ subject to boundary conditions

$$\left.\frac{\partial\varphi}{\partial n}\right|_{\partial\Omega} = 0.$$

Here φ is potential of displacements and c is the speed of sound.

Seek solutions in the form $\varphi(t,x,y,z) = e^{-i\omega t}\psi(x,y,z)$ where ω is the unknown natural frequency.

This leads to an eigenvalue problem:

$$-\Delta\psi = \lambda\psi \quad \text{in} \quad \Omega,$$

$$\partial\psi/\partial n|_{\partial\Omega} = 0,$$

where $\lambda := \omega^2/c^2$ is the spectral parameter.

Finding eigenvalues $0 = \lambda_1 < \lambda_2 \le \lambda_3 \le \ldots$ is difficult, so one introduces the counting function

$$N(\lambda) := \sum_{0 \le \lambda_k < \lambda} 1$$

("number of eigenvalues below a given λ").

Rayleigh–Jeans law:

$$N(\lambda) = \frac{V}{6\pi^2}\lambda^{3/2} + o(\lambda^{3/2}) \quad \text{as} \quad \lambda \to +\infty$$

where V is the volume of the resonator.

People like Rayleigh, J.H.Jeans, P.J.W.Debye and L.Onsager performed their analysis non-rigorously, but this motivated the emergence of an important subject area within pure mathematics — the spectral theory of PDEs. Mathematical techniques currently used in this subject area are very advanced, resembling number theoretic techniques.

Four fundamental equations of theoretical physics

1 Maxwell's equations. Describe electromagnetism and photons.

2 Dirac equation. Describes electrons and positrons.

3 Massless Dirac equation. Describes* neutrinos and antineutrinos.

4 Linearized Einstein field equations of general relativity. Describe gravity.

All four contain the same physical constant — the speed of light.

*OK, I know that neutrinos actually have a small mass.

Accepted explanation

God is a geometer. He created a 4-dimensional world parametrized by coordinates x^0, x^1, x^2, x^3 (here x^0 is time), in which distances are measured in a funny way:

$$\text{distance}^2 = -c^2(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

where c is the speed of light.

Without the term $-c^2(dx^0)^2$ this would be Pythagoras' theorem. We do not notice the term $-c^2(dx^0)^2$ in everyday life and think that we live in a 3-dimensional Euclidean world in which Pythagoras' theorem is true*. Funny way of measuring distances is called *Minkowski metric*.

*OK, maybe in a slightly modified version if the world is curved.

Having decided to use the Minkowski metric, God then wrote down the main equations of theoretical physics using **only geometric constructions**. This way all equations have the same physical constant — the speed of light — encoded in them.

Geometric concepts used in modern theoretical physics: metric, connection, spinor, Dirac equation, Dirac Lagrangian etc. Theoretical physics became part of differential geometry.

Alternative explanation

God is an analyst. He created a 3-dimensional Euclidean world, then added (absolute) time and wrote down one system of non-linear PDEs which describes phenomena in this world. This system of PDEs has different solutions which we interpret as electromagnetism, gravity, electrons, neutrinos etc. The reason the same physical constant — the speed of light — manifests itself in all physical phenomena is because we are looking at different solutions of the **same** system of PDEs.

Only problem: nobody knows how to write down the unifying system of field equations.

What I am currently doing

Analysis of a general linear first order hyperbolic system of PDEs for a pair of unknown complex scalar fields depending on time and 3 spatial variables. The coefficients of this system are not assumed to be constant and, moreover, can later be used as dynamical variables (poor man's way of dealing with nonlinearity).

Example of such a hyperbolic system:

$$\begin{pmatrix} \frac{1}{c}p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & \frac{1}{c}p_0 - p_3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

where $p_k = -i\partial/\partial x^k$, $k = 0, 1, 2, 3$.

I am constructing solutions in the form of a "wave packet", i.e. many waves combined into one packet.

Rigorous mathematical way of defining a "wave packet":

1 Concept of *wave group* (or *unitary group*) This is the operator providing the solution to the Cauchy problem (initial value problem) for our hyperbolic system.

2 Concept of *Fourier integral operator*. Allows to construct the wave group explicitly in the form of an oscillatory integral. Somewhat similar to Feynman diagrams, the variability of coefficients playing role of perturbation. Unlike Feynman diagrams, 100% rigorous.

Key word: *microlocal analysis*. L.Hörmander (Fields Medal 1962).

What I have done to date

I discovered that the microlocal analysis of a system of hyperbolic equations leads to the emergence of geometric objects such as

1 metric,

2 U(1) connection,

3 spinor, spin connection and massless Dirac Lagrangian.

Paper *The spectral function of a first order system*, jointly with O.Chervova and R.J.Downes (in preparation, approx 45 pages).

When I state the problem (system of hyperbolic PDEs) there is no geometry involved. There is no metric, no connections, no spinors etc. I state the problem as an analyst: let us just look at a system of hyperbolic PDEs.

The point I am making is that geometric objects such as metric, U(1) connection, spinor, spin connection, massless Dirac Lagrangian etc **inevitably arise** in the process of microlocal analysis. In other words, these geometric objects inevitably arise when you construct a "wave packet".

Warning: microlocal analysis is not easy

Especially difficult for systems. Main contributors for systems:

1 V.Ivrii, 1980, Soviet Math. Doklady.

2 V.Ivrii, 1982, Funct. Anal. Appl.

3 V.Ivrii, book, 1984, Springer Lecture Notes.

4 Yu.Safarov, DSc thesis, 1989, Steklov Mathematical Institute.

5 V.Ivrii, book, 1998, approx 800 pages, Springer.

6 V.Ivrii, future book. 2012? Approx 3000 pages?

Microlocal interpretation of metric

Consider a hyperbolic system, say,

$$\begin{pmatrix} \frac{1}{c}p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & \frac{1}{c}p_0 - p_3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

and view the p's not as differential operators but as components of a covector (momentum). Then the 2×2 matrix in above formula is called *principal symbol*. Taking the determinant of the principal symbol we get the Minkowski metric:

$$-\begin{vmatrix} \frac{1}{c}p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & \frac{1}{c}p_0 - p_3 \end{vmatrix} = -\frac{1}{c^2}p_0^2 + p_1^2 + p_2^2 + p_3^2\,.$$

Can invert metric and rewrite it in terms of position variables.

Microlocal interpretation of U(1) connection

New result: **any** system of hyperbolic PDEs contains inside it a U(1) connection.

"Microlocal analysis" means, in plain English, that locally (in position space and momentum space) one approximates the solution by means of a plane wave. A plane wave is defined modulo multiplication by $e^{i\phi}$, $\phi \in \mathbf{R}$. As the coefficients of our system are variable, one needs to connect plane waves from different points. Hence, a U(1) connection.

I give an explicit formula for the corresponding vector potential and curvature. This curvature term has been missing in **all** previous publications on the microlocal analysis of systems.

Differences with traditional electromagnetism

At the moment my U(1) connection is slightly different from the U(1) connection of traditional electromagnetism:

1 My “electromagnetic field” is intrinsic, as opposed to the extrinsic electromagnetic field appearing in massive Dirac equation.

2 My “vector potential” has 8 components as opposed to 4 components in traditional electromagnetism.

3 My “electromagnetic tensor” has 64 components as opposed to 16 components in traditional electromagnetism.

Microlocal interpretation of spinors, take 1

Imagine a single neutrino living in a compact 3-dimensional universe. I want to find the spectrum of energy levels this neutrino can occupy. This leads to an eigenvalue problem

$$Av = \lambda v$$

where A is the massless Dirac operator (a 2×2 matrix first order partial differential operator) and λ is the spectral parameter (energy).

Note: the explicit formula for the massless Dirac operator in curved space is very complicated.

Now suppose I am a spectral analyst and I want to know whether my differential operator A is a massless Dirac operator.

New result: an operator is a massless Dirac operator iff

a) its principal symbol is trace-free,

b) subprincipal symbol is proportional to the identity matrix, and

c) second asymptotic coefficient of the spectral function is zero.

Here "spectral function" is $e(\lambda, x, x) := \sum_{0<\lambda_k<\lambda} \|v_k(x)\|^2$.

Bottom line: the geometry of spinors is encoded within the microlocal analysis of PDEs. Microlocal analysis is used to write down the second asymptotic coefficient of the spectral function.

Microlocal interpretation of spinors, take 2 (more radical)

Consider an eigenvalue problem

$$Av = \lambda s v$$

where A is a 2×2 matrix first order partial differential operator with trace-free principal symbol and zero subprincipal symbol, and $s = s(x) > 0$ is a scalar weight function. View coefficients as dynamical variables and consider the map

principal symbol of operator and scalar weight $\mapsto$
second asymptotic coefficient of counting function.

New result: the above map is equivalent to the map

metric and spinor field $\mapsto$ **static massless Dirac action**.

Summary

I can extract the main geometric constructs of theoretical physics from systems of PDEs, simply by performing microlocal analysis of "wave packets".

Action plan

Spend the next 100 years meticulously studying systems of non-linear hyperbolic PDEs, hoping to find soliton-type solutions.