Is God a geometer or an analyst?

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Basic example of a problem in my subject area: acoustic resonator. Suppose we are studying the vibrations of air

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x_1^2} - \frac{\partial^2\varphi}{\partial x_2^2} - \frac{\partial^2\varphi}{\partial x_3^2} = 0$$

in a bounded domain $\Omega \subset \mathbb{R}^3$ subject to boundary conditions

$$\left.\frac{\partial\varphi}{\partial n}\right|_{\partial\Omega} = 0.$$

Here φ is the velocity potential and c is the speed of sound.

Seek solutions in the form $\varphi(t, x_1, x_2, x_3) = e^{-i\omega t}\psi(x_1, x_2, x_3)$ where ω is the unknown natural frequency.

This leads to an eigenvalue problem:

$$-\Delta\psi = \lambda\psi \quad \text{in} \quad \Omega, \qquad \partial\psi/\partial n|_{\partial\Omega} = 0,$$

where Δ is the Laplacian and $\lambda := \frac{\omega^2}{c^2}$ is the spectral parameter.

Finding eigenvalues $0 = \lambda_1 < \lambda_2 \le \lambda_3 \le \ldots$ is difficult, so one introduces the counting function

$$N(\lambda) := \sum_{0 \le \lambda_k < \lambda} 1$$

("number of eigenvalues below a given λ") and studies the asymptotic behaviour of $N(\lambda)$ as $\lambda \to +\infty$.

Rayleigh–Jeans law (1905):

$$N(\lambda) = \frac{V}{6\pi^2}\lambda^{3/2} + o(\lambda^{3/2}) \quad \text{as} \quad \lambda \to +\infty$$

where V is the volume of the resonator.

Rayleigh's "proof" of the Rayleigh–Jeans law

Suppose Ω is a cube with side length a. Then the eigenvalues and eigenfunctions can be calculated explicitly:

$$\psi_{\mathbf{k}} = \cos\left(\frac{\pi k_1 x_1}{a}\right)\cos\left(\frac{\pi k_2 x_2}{a}\right)\cos\left(\frac{\pi k_3 x_3}{a}\right),$$

$$\lambda_{\mathbf{k}} = \frac{\pi^2}{a^2}\|\mathbf{k}\|^2 = \frac{\pi^2}{a^2}(k_1^2 + k_2^2 + k_3^2),$$

where $\mathbf{k} = (k_1, k_2, k_3)$ and k_1, k_2, k_3 are nonnegative integers.

$N(\lambda)$ is the number of integer lattice points in the nonnegative octant of a ball of radius $\frac{a}{\pi}\sqrt{\lambda}$, so

$$N(\lambda) \approx \frac{1}{8}\left(\frac{4}{3}\pi\left(\frac{a}{\pi}\sqrt{\lambda}\right)^3\right) = \frac{a^3}{6\pi^2}\lambda^{3/2} = \frac{V}{6\pi^2}\lambda^{3/2}.$$

Jeans' contribution to the Rayleigh–Jeans law:

"It seems to me that Lord Rayleigh has introduced an unnecessary factor 8 by counting negative as well as positive values of his integers".

1910: Lorentz visits Göttingen at Hilbert's invitation and delivers a series of lectures "Old and new problems in physics". Lorentz states the Rayleigh–Jeans law as a mathematical conjecture. Hermann Weyl is in the audience.

1912: Weyl publishes a rigorous proof of Rayleigh–Jeans law. Almost incomprehensible.

Comprehensible proof: in R.Courant and D.Hilbert, *Methods of Mathematical Physics* (1924).

Courant's method. Approximate domain Ω by a collection of small cubes, setting Dirichlet or Neumann boundary conditions on boundaries of cubes. Setting extra Dirichlet conditions raises the eigenvalues whereas setting extra Neumann conditions lowers the eigenvalues. Remains only to

- choose size of cubes correctly (in relation to λ) and

- estimate contribution of bits of domain near the boundary (we throw them out).

General statement of the problem. Let M be a compact n-dimensional manifold with boundary ∂M. Consider the spectral problem for an elliptic self-adjoint semi-bounded from below differential operator of even order $2m$:

$$Au = \lambda u \quad \text{on} \quad M, \qquad (B^{(j)}u)\Big|_{\partial M} = 0, \quad j = 1, \ldots, m.$$

Has been proven (by many authors over many years) that

$$N(\lambda) = a\lambda^{n/(2m)} + o(\lambda^{n/(2m)}) \quad \text{as} \quad \lambda \to +\infty$$

where the constant a is written down explicitly.

Weyl's Conjecture (1913): one can do better and prove two-term asymptotic formulae for the counting function. Say, for the case of the Laplacian in 3D with Neumann boundary conditions Weyl's Conjecture reads

$$N(\lambda) = \frac{V}{6\pi^2}\lambda^{3/2} - \frac{S}{16\pi}\lambda + o(\lambda) \quad \text{as} \quad \lambda \to +\infty,$$

where S is the surface area of ∂M. For a general partial differential operator of order $2m$ Weyl's Conjecture reads

$$N(\lambda) = a\lambda^{n/(2m)} + b\lambda^{(n-1)/(2m)} + o(\lambda^{(n-1)/(2m)}) \quad \text{as} \quad \lambda \to +\infty$$

where the constant b can also be written down explicitly.

For the case of a second order operator Weyl's Conjecture was proved by V.Ivrii in 1980. I proved it for operators of arbitrary order in 1984.

My main research publication:

Yu.Safarov and D.Vassiliev, *The asymptotic distribution of eigenvalues of partial differential operators*, American Mathematical Society, 1997 (hardcover), 1998 (softcover).

"In the reviewer's opinion, this book is indispensable for serious students of spectral asymptotics". Lars Hörmander for the Bulletin of the London Mathematical Society.

Idea of proof

Key word: *microlocal analysis*. L.Hörmander (Fields Medal 1962).

Introduce time t and study the "hyperbolic" equation

$$Au = \left(i\frac{\partial}{\partial t}\right)^{2m} u.$$

Construct the operator $U(t) := e^{-itA^{1/(2m)}}$. This operator is called the *propagator* and it provides the "solution" to the Cauchy problem (initial value problem) for our "hyperbolic" equation. The propagator can be constructed explicitly, modulo an integral operator with smooth kernel, in the form of a *Fourier integral operator*. This is a way of doing the Fourier transform for operators with variable coefficients.

A Fourier integral operator is an oscillatory integral. Similar to Feynman diagrams, the variability of coefficients playing role of perturbation. Unlike Feynman diagrams, 100% rigorous.

Having constructed the propagator, recover information about the spectrum using *Fourier Tauberian theorems*. These allow us to perform the inverse Fourier transform from variable t (time) to variable λ (spectral parameter) using incomplete information, with control of error terms.

Similar to Tauberian theorems used in analytic number theory.

Example: vibrations of a plate

$$\Delta^2 u = \lambda u \quad \text{in} \quad \Omega \subset \mathbb{R}^2, \qquad u|_{\partial\Omega} = \partial u/\partial n|_{\partial\Omega} = 0.$$

Then

$$N(\lambda) = \frac{S}{4\pi}\lambda^{1/2} + \frac{\beta L}{4\pi}\lambda^{1/4} + o(\lambda^{1/4}) \quad \text{as} \quad \lambda \to +\infty$$

where S is the area of the plate, L is the length of the boundary and

$$\beta = -1 - \frac{\Gamma(3/4)}{\sqrt{\pi}\,\Gamma(5/4)} \approx -1.763.$$

The first asymptotic term was derived by Courant (1922).

Inverting the formula and switching to frequencies $\lambda_N^{1/2}$, we get

$$\lambda_N^{1/2} = \frac{4\pi}{S}N - \frac{2\sqrt{\pi}\,\beta L}{S^{3/2}}\sqrt{N} + o(\sqrt{N}\,) \quad \text{as} \quad N \to +\infty.$$

What I have been doing during the last year

Looking at systems

$$Av = \lambda v$$

where A is a first order elliptic self-adjoint first order $m \times m$ matrix differential operator acting on complex-valued m-columns v over an n-dimensional compact manifold M without boundary. The operator is not necessarily semi-bounded.

Example: massless Dirac operator on the torus $\mathbb{T}^3$

$$-i \begin{pmatrix} \frac{\partial}{\partial x^3} & \frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2} \\ \frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2} & -\frac{\partial}{\partial x^3} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Warning: doing microlocal analysis for systems is not easy

1 V.Ivrii, 1980, Soviet Math. Doklady.

2 V.Ivrii, 1982, Funct. Anal. Appl.

3 G.V.Rozenblyum, 1983, Journal of Mathematical Sciences.

4 V.Ivrii, 1984, Springer Lecture Notes.

5 Yu.Safarov, DSc thesis, 1989, Steklov Mathematical Institute.

6 V.Ivrii, book, 1998, Springer.

7 W.J.Nicoll, PhD thesis, 1998, University of Sussex.

8 I.Kamotski and M.Ruzhansky, 2007, Comm. PDEs.

9 O.Chervova, R.J.Downes and D.Vassiliev, 2012, preprint `http://arxiv.org/abs/1208.6015`, to appear in JST.

The circle group $\mathsf{U}(1)$

$$\mathsf{U}(1) = \{z \in \mathbb{C} : \; |z| = 1\}.$$

Here the group operation is multiplication.

Why the circle group U(1) is relevant

Look at matrix differential operator A, keep only leading (first order) derivative and replace each $\partial/\partial x^\alpha$ by $i\xi_\alpha$, $\alpha = 1, \ldots, n$, to get a matrix-function $A_1(x,\xi)$. A physicist would call ξ *momentum* and write p instead of ξ.

The matrix-function $A_1(x,\xi)$ is called *principal symbol*.

Let $v(x,\xi)$ be an eigenvector of the principal symbol.

Problem: $v(x,\xi)$ is not defined uniquely. It is is defined modulo a gauge transformation $v \mapsto e^{i\phi}v$ where $\phi : T^*M \setminus \{0\} \to \mathbb{R}$ is an arbitrary smooth function. This gives rise to a U(1) connection which, in turn, generates curvature.

This curvature term was missed by all previous authors.

Physical meaning of the U(1) connection

In theoretical physics a U(1) connection is always associated with electromagnetism. The corresponding curvature is tensor is the electromagnetic (Faraday) tensor.

I have shown that inside *any* system of partial differential equations with variable coefficients there is an intrinsic electromagnetic field. This is an abstract mathematical fact which remained unnoticed until recently.

One has to take account of this intrinsic electromagnetic field in order to get correct formulae.

What I am doing now

Analysing special case:

our manifold has dimension 3,

the number of equations in our system is 2,

the principal symbol is trace-free.

Turns out, I get a lot of geometry coming out of my spectral analysis.

Geometric object 1: the metric

The determinant of the principal symbol is a negative definite quadratic form

$$\det A_1(x,\xi) = -g^{\alpha\beta}\xi_\alpha\xi_\beta$$

and the coefficients $g^{\alpha\beta}(x) = g^{\beta\alpha}(x)$, $\alpha, \beta = 1, 2, 3$, can be interpreted as components of a (contravariant) Riemannian metric.

Geometric object 2: the teleparallel connection

Define an affine connection as follows. Suppose we have a covector ξ based at the point $x \in M$ and we want to construct a parallel covector $\tilde{\xi}$ based at the point $\tilde{x} \in M$. This is done by solving the linear system of equations

$$A_1(\tilde{x}, \tilde{\xi}) = A_1(x, \xi).$$

The teleparallel connection coefficients ${\Gamma^\alpha}_{\beta\gamma}(x)$ can be written down explicitly in terms of the principal symbol and this allows us to define yet another geometric object — the *torsion tensor*

$${T^\alpha}_{\beta\gamma} := {\Gamma^\alpha}_{\beta\gamma} - {\Gamma^\alpha}_{\gamma\beta}.$$

The teleparallel connection has zero curvature and nonzero torsion. It is the opposite of the Levi-Civita connection.

Geometric object 3: spinor field

Spinor = quaternion $a + bi + cj + dk$, where $a, b, c, d \in \mathbb{R}$ and $i^2 = j^2 = k^2 = ijk = -1$.

Where does spinor come from? 2×2 trace-free Hermitian matrix-function $A_1(x, \xi)$ is not fully defined by its determinant (metric). Remaining degrees of freedom are described by a spinor.

Geometric object 4: massless Dirac action

Action = variational functional for corresponding diff. operator.

Turns out, the second asymptotic coefficient of the counting function has the geometric meaning of the massless Dirac action.

Bottom line: the differential geometry of spinors is encoded within the microlocal analysis of PDEs.

Four fundamental equations of theoretical physics

1 Maxwell's equations. Describe electromagnetism and photons.

2 Dirac equation. Describes electrons and positrons.

3 Massless Dirac equation. Describes* neutrinos and antineutrinos.

4 Linearized Einstein field equations of general relativity. Describe gravity.

All four contain the same physical constant — the speed of light.

*OK, I know that neutrinos actually have a small mass.

Accepted explanation: theory of relativity

God is a geometer. He created a 4-dimensional world parametrized by coordinates x^0, x^1, x^2, x^3 (here x^0 is time), in which distances are measured in a funny way:

$$\text{distance}^2 = -c^2(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

where c is the speed of light.

Without the term $-c^2(dx^0)^2$ this would be Pythagoras' theorem. Funny way of measuring distances is called *Minkowski metric*.

Having decided to use the Minkowski metric, God then wrote down the main equations of theoretical physics using **only geometric constructions**, i.e. using concepts such as connection, curvature etc. This way all equations have the same physical constant — the speed of light — encoded in them.

Alternative explanation

God is an analyst. He created a 3-dimensional Euclidean world, then added (absolute) time and wrote down one system of nonlinear PDEs which describes phenomena in this world. This system of PDEs has different solutions which we interpret as electromagnetism, gravity, electrons, neutrinos etc. The reason the same physical constant — the speed of light — manifests itself in all physical phenomena is because we are looking at different solutions of the **same** system of PDEs.

I cannot write down the unifying system of field equations but I can extract many geometric constructs of theoretical physics from systems of PDEs, simply by performing microlocal analysis.

Action plan: spend next 100 years studying systems of nonlinear hyperbolic PDEs, hoping to find soliton-type solutions.

My PhD students who helped me a lot in recent years

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