# Rotational elasticity

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## What we do at UCL
## (Böhmer, Burnett, Chervova, Downes, Obukhov, Vassiliev)

Describing a 3-dimensional elastic medium

(a) Classical elasticity: displacements only.

(b) Cosserat elasticity: displacements and rotations. See E.Cosserat and F.Cosserat, *Théorie des Corps Déformables*, 1909.

(c) Rotational elasticity: rotations only.

Classical and rotational elasticity are two limit cases of Cosserat.

Classical elasticity is respectable, rotational elasticity is crazy.

Motivation for rotational elasticity.

(a) Curiosity.

(b) MacCullagh, 1839. Tried modelling world aether in terms of rotational elasticity. Inadequate mathematical apparatus.

(c) A. Einstein and É. Cartan, 1920s. Teleparallelism = absolute parallelism = fernparallelismus. Cartan knew the Cosserat book. Drew inspiration from 'beautiful' work of the Cosserat brothers.

Elastic medium occupies $\mathbb{R}^3$. To describe rotations of material points I attach to each geometric point of $\mathbb{R}^3$ a *coframe*.

A coframe $\vartheta$ is a triple $\vartheta^j$, $j = 1, 2, 3$, of orthonormal covector fields. Each $\vartheta^j$ has hidden tensor index: $\vartheta^j = \vartheta^j{}_\alpha$, $\alpha = 1, 2, 3$.

Same in plain English: a coframe is a field of orthonormal bases.

Can think of the coframe as a field of orthogonal matrices $\vartheta^j{}_\alpha$.

**NB.** Coframe lives separately from Cartesian coordinates. It is not aligned with coordinate lines.

The coframe $\vartheta$ is an unknown quantity (dynamical variable).

The other dynamical variable is a density $\rho$.

## Measuring rotational deformations

The natural measure of rotational deformations is *torsion*

$$T := \vartheta^1 \otimes d\vartheta^1 + \vartheta^2 \otimes d\vartheta^2 + \vartheta^3 \otimes d\vartheta^3.$$

Torsion is a rank 3 tensor antisymmetric in the last pair of indices.

Convenient to switch to tensor $\overset{*}{T}_{\alpha\beta} := \frac{1}{2} T_\alpha{}^{\gamma\delta} \varepsilon_{\gamma\delta\beta}$ .

Tensor $\overset{*}{T}$ is a rank 2 tensor without any symmetries and arbitrary trace. It is called *dislocation density tensor*. Explicit formulae:

$$\overset{*}{T} = \vartheta^1 \otimes \operatorname{curl} \vartheta^1 + \vartheta^2 \otimes \operatorname{curl} \vartheta^2 + \vartheta^3 \otimes \operatorname{curl} \vartheta^3,$$

$$\overset{*}{T}_{\alpha\beta} = \sum_{j=1}^{3} \begin{pmatrix} \vartheta^j{}_1 \partial_2 \vartheta^j{}_3 - \vartheta^j{}_1 \partial_3 \vartheta^j{}_2 & \vartheta^j{}_1 \partial_3 \vartheta^j{}_1 - \vartheta^j{}_1 \partial_1 \vartheta^j{}_3 & \vartheta^j{}_1 \partial_1 \vartheta^j{}_2 - \vartheta^j{}_1 \partial_2 \vartheta^j{}_1 \\ \vartheta^j{}_2 \partial_2 \vartheta^j{}_3 - \vartheta^j{}_2 \partial_3 \vartheta^j{}_2 & \vartheta^j{}_2 \partial_3 \vartheta^j{}_1 - \vartheta^j{}_2 \partial_1 \vartheta^j{}_3 & \vartheta^j{}_2 \partial_1 \vartheta^j{}_2 - \vartheta^j{}_2 \partial_2 \vartheta^j{}_1 \\ \vartheta^j{}_3 \partial_2 \vartheta^j{}_3 - \vartheta^j{}_3 \partial_3 \vartheta^j{}_2 & \vartheta^j{}_3 \partial_3 \vartheta^j{}_1 - \vartheta^j{}_3 \partial_1 \vartheta^j{}_3 & \vartheta^j{}_3 \partial_1 \vartheta^j{}_2 - \vartheta^j{}_3 \partial_2 \vartheta^j{}_1 \end{pmatrix}.$$

## Irreducible decomposition of rotational deformations

$$\overset{*}{T} = \overset{*}{T}{}^{\mathsf{ax}} + \overset{*}{T}{}^{\mathsf{vec}} + \overset{*}{T}{}^{\mathsf{ten}}$$

where

$$\begin{aligned}
\overset{*}{T}{}^{\mathsf{ax}}_{\alpha\beta} &:= \frac{\overset{*}{T}{}^{\gamma}{}_{\gamma}}{3}\, g_{\alpha\beta}\,, \\
\overset{*}{T}{}^{\mathsf{vec}}_{\alpha\beta} &:= \frac{\overset{*}{T}_{\alpha\beta} - \overset{*}{T}_{\beta\alpha}}{2}\,, \\
\overset{*}{T}{}^{\mathsf{ten}}_{\alpha\beta} &:= \frac{\overset{*}{T}_{\alpha\beta} + \overset{*}{T}_{\beta\alpha}}{2} - \overset{*}{T}{}^{\mathsf{ax}}_{\alpha\beta}\,.
\end{aligned}$$

Adjectives *axial*, *vector* and *tensor*

## Potential energy

$$P(x^0) = \int_{\mathbb{R}^3} \Big( c^{\mathrm{ax}} \|\overset{*}{T}{}^{\mathrm{ax}}\|^2 + c^{\mathrm{vec}} \|\overset{*}{T}{}^{\mathrm{vec}}\|^2 + c^{\mathrm{ten}} \|\overset{*}{T}{}^{\mathrm{ten}}\|^2 \Big) \rho \, dx^1 dx^2 dx^3.$$

Here $x^1$, $x^2$, $x^3$ are Cartesian coordinates and $x^0$ is time.

## Kinetic energy

$$K(x^0) = c^{\mathrm{kin}} \int_{\mathbb{R}^3} \|\omega\|^2 \rho \, dx^1 dx^2 dx^3$$

where $\omega = \frac{1}{2} *(\vartheta^1 \wedge \partial_0 \vartheta^1 + \vartheta^2 \wedge \partial_0 \vartheta^2 + \vartheta^3 \wedge \partial_0 \vartheta^3)$ is the (pseudo)vector of angular velocity. Explicit formula for angular velocity:

$$\omega_\alpha = \frac{1}{2} \sum_{j=1}^{3} \begin{pmatrix} {\vartheta^j}_2 \partial_0 {\vartheta^j}_3 - {\vartheta^j}_3 \partial_0 {\vartheta^j}_2 \\ {\vartheta^j}_3 \partial_0 {\vartheta^j}_1 - {\vartheta^j}_1 \partial_0 {\vartheta^j}_3 \\ {\vartheta^j}_1 \partial_0 {\vartheta^j}_2 - {\vartheta^j}_2 \partial_0 {\vartheta^j}_1 \end{pmatrix}.$$

## Action (variational functional) of rotational elasticity

$$S(\vartheta,\rho)=\int_{\mathbb{R}}(P(x^0)-K(x^0))dx^0=\int_{\mathbb{R}\times\mathbb{R}^3}L(\vartheta,\rho)\,dx^0dx^1dx^2dx^3$$

where

$$L(\vartheta,\rho)=\Big(c^{\mathrm{ax}}\|\overset{*}{T}{}^{\mathrm{ax}}\|^2+c^{\mathrm{vec}}\|\overset{*}{T}{}^{\mathrm{vec}}\|^2+c^{\mathrm{ten}}\|\overset{*}{T}{}^{\mathrm{ten}}\|^2-c^{\mathrm{kin}}\|\omega\|^2\Big)\rho$$

is the Lagrangian density.

Euler–Lagrange equations: vary coframe $\vartheta$ and density $\rho$.

Model is physically linear but geometrically nonlinear.

## Solving Euler–Lagrange equations

Varying coframe is difficult because of kinematic constraint: covectors $\vartheta^j$, $j = 1, 2, 3$, have to remain orthonormal. Could use Euler angles (yaw, pitch, and roll) but this is inconvenient.

Most convenient description of rotations in $\mathbb{R}^3$: switch to spinors

$$\boxed{\text{coframe } \vartheta \text{ and density } \rho > 0}$$

$$\Updownarrow$$

$$\boxed{\text{nonvanishing 2-component complex spinor field } \xi \text{ modulo sign}}$$

My Lagrangian density $L(\xi)$ is a rational function of $\xi$, $\bar{\xi}$ and partial derivatives of $\xi$, $\bar{\xi}$.

Expressing density $\rho$ and coframe $\vartheta$ via spinor $\xi^a = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}$:

$$\rho = \bar{\xi}^{\dot{1}}\xi^1 + \bar{\xi}^{\dot{2}}\xi^2,$$

$$\vartheta^1{}_\alpha = \rho^{-1}\,\mathrm{Re}\begin{pmatrix} (\xi^1)^2 - (\xi^2)^2 \\ i(\xi^1)^2 + i(\xi^2)^2 \\ -2\xi^1\xi^2 \end{pmatrix},$$

$$\vartheta^2{}_\alpha = \rho^{-1}\,\mathrm{Im}\begin{pmatrix} (\xi^1)^2 - (\xi^2)^2 \\ i(\xi^1)^2 + i(\xi^2)^2 \\ -2\xi^1\xi^2 \end{pmatrix},$$

$$\vartheta^3{}_\alpha = \rho^{-1}\begin{pmatrix} \bar{\xi}^{\dot{2}}\xi^1 + \bar{\xi}^{\dot{1}}\xi^2 \\ i\bar{\xi}^{\dot{2}}\xi^1 - i\bar{\xi}^{\dot{1}}\xi^2 \\ \bar{\xi}^{\dot{1}}\xi^1 - \bar{\xi}^{\dot{2}}\xi^2 \end{pmatrix}.$$

## Plane wave solutions

Look first for plane wave solutions

$$\xi(x^0, x^1, x^2, x^3) = e^{-i(p_0 x^0 + p_1 x^1 + p_2 x^2 + p_3 x^3)}\zeta\,.$$

**Theorem 1** *My Euler–Lagrange equation admits plane wave solutions with velocities* $\sqrt{\frac{4c^{\mathrm{ax}} + 2c^{\mathrm{ten}}}{3c^{\mathrm{kin}}}}$ *and* $\sqrt{\frac{c^{\mathrm{vec}} + c^{\mathrm{ten}}}{2c^{\mathrm{kin}}}}$ *.*

## Purely axial material

$$c^{\mathsf{ax}} \neq 0, \qquad c^{\mathsf{vec}} = c^{\mathsf{ten}} = 0.$$

Potential energy feels only the axial deformation, i.e. only the trace of the dislocation density tensor $\overset{*}{T}$.

For convenience normalise the kinetic term by setting $c^{\mathsf{kin}} = \frac{4}{3} c^{\mathsf{ax}}$.

Look for stationary solutions

$$\xi(x^0, x^1, x^2, x^3) = e^{-ip_0 x^0} \eta(x^1, x^2, x^3).$$

**Theorem 2** *In the stationary setting my Euler–Lagrange equation is equivalent to a pair of massless Dirac equations*

$$\begin{pmatrix} \mp p_0 + i\partial_3 & i\partial_1 + \partial_2 \\ i\partial_1 - \partial_2 & \mp p_0 - i\partial_3 \end{pmatrix} \begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix} = 0.$$

## Nonlinear second order PDEs

## which reduce to pairs of linear first order PDEs

Consider a pair of formally self-adjoint first order linear differential operators $A_\pm$ acting on smooth vector functions $v:\mathbb{R}^n\to\mathbb{C}^m$.

Define corresponding Lagrangians $L_\pm(v):=\mathrm{Re}(v^*A_\pm v)$.

Define new Lagrangian

$$L(v)=\frac{L_+(v)\,L_-(v)}{L_+(v)-L_-(v)}\;. \tag{1}$$

Work with vector functions such that $L_+(v)\ne L_-(v)$.

**Lemma 1** *A vector function $v$ is a solution of the Euler–Lagrange equation for the Lagrangian density (1) if and only if it is a solution of $A_+v=0$ or $A_-v=0$.*

## Example illustrating the use of Lemma 1

The pair of linear first order ordinary differential equations

$$iv' \pm v = 0$$

is equivalent to a single nonlinear second order equation

$$\frac{d}{dx}\left(\frac{\bar{v}v' - v\bar{v}'}{2|v|^2}\,v\right) + \frac{(\bar{v}v')^2 - (v\bar{v}')^2}{4|v|^4}\,v + v = 0\,.$$

Same trick works for systems of PDEs with variable coefficients.

## Summary

- Rotational elasticity is an interesting subject.
- Nobody has studied rotational elasticity.
- Rotational elasticity may provide an alternative description of fermions (neutrinos, electrons) in quantum mechanics.

Papers and preprints can be found on my web page

`http://www.homepages.ucl.ac.uk/~ucahdva/`

My talks (including this one) are also on my web page.