Rotational elasticity

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Group at UCL involved in rotational elasticity:

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4) Rob Downes,
5) Yuri Obukhov,
6) Dmitri Vassiliev.

We are part of the *UCL Institute of Origins*.

Describing a 3-dimensional elastic medium

(a) Classical elasticity: displacements only.

(b) Cosserat elasticity: displacements and rotations. See E.Cosserat and F.Cosserat, *Théorie des Corps Déformables*, 1909. Reprinted by Cornell and now available from Amazon.

(c) Rotational elasticity: rotations only.

Classical and rotational elasticity are two limit cases of Cosserat.

Motivation for rotational elasticity.

(a) Curiosity.

(b) MacCullagh, 1839. Tried modelling world aether in terms of rotational elasticity. Inadequate mathematical apparatus.

(c) A. Einstein and É. Cartan, 1920s. Teleparallelism = absolute parallelism = fernparallelismus. Formal geometric definition of teleparallelism: curvature is zero but torsion is nonzero. Compare with general relativity: torsion is zero but curvature is nonzero.

Note: Cartan knew the Cosserat book. He wrote that he drew inspiration from the 'beautiful' work of the Cosserat brothers.

(d) Ericksen fluid. See J. L. Ericksen, Twist waves in liquid crystals, *Q. Jl Mech. Appl. Math.* **21** (1968) 463-465.

Elastic medium occupies $\mathbb{R}^3$. To describe rotations of material points I attach to each geometric point of $\mathbb{R}^3$ a *coframe*.

A coframe ϑ is a triple ϑ^j, $j = 1,2,3$, of orthonormal covector fields. Each ϑ^j has hidden tensor index: $\vartheta^j = \vartheta^j{}_\alpha$, $\alpha = 1,2,3$.

Same in plain English: a coframe is a field of orthonormal bases.

Can think of the coframe as a field of orthogonal matrices $\vartheta^j{}_\alpha$.

NB. Coframe lives separately from Cartesian coordinates. It is not aligned with coordinate lines.

The coframe ϑ is a dynamical variable (unknown quantity).

The other dynamical variable is a density ρ (unlike in Böhmer's talk where $\rho = 1$).

Measuring rotational deformations

The natural measure of rotational deformations is *torsion*

$$T := \vartheta^1 \otimes d\vartheta^1 + \vartheta^2 \otimes d\vartheta^2 + \vartheta^3 \otimes d\vartheta^3.$$

Torsion is a rank 3 tensor antisymmetric in the last pair of indices.

Convenient to switch to tensor $\overset{*}{T}_{\alpha\beta} := \frac{1}{2} T_\alpha{}^{\gamma\delta} \varepsilon_{\gamma\delta\beta}$.

$\overset{*}{T}$ is a rank 2 tensor without symmetries and with arbitrary trace. Sometimes called *dislocation density tensor*. Explicit formulae:

$$\overset{*}{T} = \vartheta^1 \otimes \operatorname{curl} \vartheta^1 + \vartheta^2 \otimes \operatorname{curl} \vartheta^2 + \vartheta^3 \otimes \operatorname{curl} \vartheta^3,$$

$$\overset{*}{T}_{\alpha\beta} = \sum_{j=1}^{3} \begin{pmatrix} \vartheta^j{}_1 \partial_2 \vartheta^j{}_3 - \vartheta^j{}_1 \partial_3 \vartheta^j{}_2 & \vartheta^j{}_1 \partial_3 \vartheta^j{}_1 - \vartheta^j{}_1 \partial_1 \vartheta^j{}_3 & \vartheta^j{}_1 \partial_1 \vartheta^j{}_2 - \vartheta^j{}_1 \partial_2 \vartheta^j{}_1 \\ \vartheta^j{}_2 \partial_2 \vartheta^j{}_3 - \vartheta^j{}_2 \partial_3 \vartheta^j{}_2 & \vartheta^j{}_2 \partial_3 \vartheta^j{}_1 - \vartheta^j{}_2 \partial_1 \vartheta^j{}_3 & \vartheta^j{}_2 \partial_1 \vartheta^j{}_2 - \vartheta^j{}_2 \partial_2 \vartheta^j{}_1 \\ \vartheta^j{}_3 \partial_2 \vartheta^j{}_3 - \vartheta^j{}_3 \partial_3 \vartheta^j{}_2 & \vartheta^j{}_3 \partial_3 \vartheta^j{}_1 - \vartheta^j{}_3 \partial_1 \vartheta^j{}_3 & \vartheta^j{}_3 \partial_1 \vartheta^j{}_2 - \vartheta^j{}_3 \partial_2 \vartheta^j{}_1 \end{pmatrix}.$$

Irreducible decomposition of rotational deformations

$$\overset{*}{T} = \overset{*}{T}{}^{\text{ax}} + \overset{*}{T}{}^{\text{vec}} + \overset{*}{T}{}^{\text{ten}}$$

where

$$\begin{aligned}
\overset{*}{T}{}^{\text{ax}}_{\alpha\beta} &:= \frac{\overset{*}{T}{}^{\gamma}{}_{\gamma}}{3}\, g_{\alpha\beta}\,, \\
\overset{*}{T}{}^{\text{vec}}_{\alpha\beta} &:= \frac{\overset{*}{T}_{\alpha\beta} - \overset{*}{T}_{\beta\alpha}}{2}\,, \\
\overset{*}{T}{}^{\text{ten}}_{\alpha\beta} &:= \frac{\overset{*}{T}_{\alpha\beta} + \overset{*}{T}_{\beta\alpha}}{2} - \overset{*}{T}{}^{\text{ax}}_{\alpha\beta}\,.
\end{aligned}$$

Adjectives *axial*, *vector* and *tensor*

Potential energy

$$P(x^0) = \int_{\mathbb{R}^3} \Big(c^{\mathrm{ax}} \|\overset{*}{T}{}^{\mathrm{ax}}\|^2 + c^{\mathrm{vec}} \|\overset{*}{T}{}^{\mathrm{vec}}\|^2 + c^{\mathrm{ten}} \|\overset{*}{T}{}^{\mathrm{ten}}\|^2 \Big) \rho \, dx^1 dx^2 dx^3.$$

Here x^1, x^2, x^3 are Cartesian coordinates and x^0 is time.

Kinetic energy

$$K(x^0) = c^{\mathrm{kin}} \int_{\mathbb{R}^3} \|\omega\|^2 \rho \, dx^1 dx^2 dx^3$$

where $\omega = \frac{1}{2} *(\vartheta^1 \wedge \partial_0 \vartheta^1 + \vartheta^2 \wedge \partial_0 \vartheta^2 + \vartheta^3 \wedge \partial_0 \vartheta^3)$ is the (pseudo)vector of angular velocity. Explicit formula for angular velocity:

$$\omega_\alpha = \frac{1}{2} \sum_{j=1}^{3} \begin{pmatrix} \vartheta^j{}_2 \partial_0 \vartheta^j{}_3 - \vartheta^j{}_3 \partial_0 \vartheta^j{}_2 \\ \vartheta^j{}_3 \partial_0 \vartheta^j{}_1 - \vartheta^j{}_1 \partial_0 \vartheta^j{}_3 \\ \vartheta^j{}_1 \partial_0 \vartheta^j{}_2 - \vartheta^j{}_2 \partial_0 \vartheta^j{}_1 \end{pmatrix}.$$

Action (variational functional) of rotational elasticity

$$S(\vartheta,\rho)=\int_{\mathbb{R}}(P(x^0)-K(x^0))dx^0=\int_{\mathbb{R}\times\mathbb{R}^3}L(\vartheta,\rho)\,dx^0dx^1dx^2dx^3$$

where

$$L(\vartheta,\rho)=\Big(c^{\mathrm{ax}}\|\overset{*}{T}{}^{\mathrm{ax}}\|^2+c^{\mathrm{vec}}\|\overset{*}{T}{}^{\mathrm{vec}}\|^2+c^{\mathrm{ten}}\|\overset{*}{T}{}^{\mathrm{ten}}\|^2-c^{\mathrm{kin}}\|\omega\|^2\Big)\rho$$

is the Lagrangian density.

Euler–Lagrange equations: vary coframe ϑ and density ρ.

Model is physically linear but geometrically nonlinear.

Solving Euler–Lagrange equations

Varying coframe is difficult because of kinematic constraint: covectors ϑ^j, $j = 1, 2, 3$, have to remain orthonormal. Could use Euler angles (yaw, pitch, and roll) but this is inconvenient.

Most convenient description of rotations in $\mathbb{R}^3$: switch to spinors

$$\boxed{\text{coframe } \vartheta \text{ and density } \rho > 0}$$

$$\Updownarrow$$

$$\boxed{\text{nonvanishing 2-component complex spinor field } \xi \text{ modulo sign}}$$

My Lagrangian density $L(\xi)$ is a rational function of ξ, $\bar{\xi}$ and partial derivatives of ξ, $\bar{\xi}$.

Expressing density ρ and coframe ϑ via spinor $\xi^a = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}$:

$$\rho = \bar{\xi}^{\dot{1}}\xi^1 + \bar{\xi}^{\dot{2}}\xi^2,$$

$$\vartheta^1{}_\alpha = \rho^{-1}\,\mathrm{Re}\begin{pmatrix} (\xi^1)^2 - (\xi^2)^2 \\ i(\xi^1)^2 + i(\xi^2)^2 \\ -2\xi^1\xi^2 \end{pmatrix},$$

$$\vartheta^2{}_\alpha = \rho^{-1}\,\mathrm{Im}\begin{pmatrix} (\xi^1)^2 - (\xi^2)^2 \\ i(\xi^1)^2 + i(\xi^2)^2 \\ -2\xi^1\xi^2 \end{pmatrix},$$

$$\vartheta^3{}_\alpha = \rho^{-1}\begin{pmatrix} \bar{\xi}^{\dot{2}}\xi^1 + \bar{\xi}^{\dot{1}}\xi^2 \\ i\bar{\xi}^{\dot{2}}\xi^1 - i\bar{\xi}^{\dot{1}}\xi^2 \\ \bar{\xi}^{\dot{1}}\xi^1 - \bar{\xi}^{\dot{2}}\xi^2 \end{pmatrix}.$$

Plane wave solutions

Look first for *plane wave* solutions

$$\xi(x^0,x^1,x^2,x^3)=e^{-i(p_0x^0+p_1x^1+p_2x^2+p_3x^3)}\zeta\,.$$

Theorem 1 *My Euler–Lagrange equation admits plane wave solutions with velocities* $\sqrt{\dfrac{4c^{\mathrm{ax}}+2c^{\mathrm{ten}}}{3c^{\mathrm{kin}}}}$ *and* $\sqrt{\dfrac{c^{\mathrm{vec}}+c^{\mathrm{ten}}}{2c^{\mathrm{kin}}}}$ *.*

Purely axial material

$$c^{\mathsf{ax}} \neq 0, \qquad c^{\mathsf{vec}} = c^{\mathsf{ten}} = 0.$$

Potential energy feels only the axial deformation, i.e. only the trace of the dislocation density tensor $\overset{*}{T}$.

For convenience normalise the kinetic term by setting $c^{\mathsf{kin}} = \frac{4}{3}c^{\mathsf{ax}}$.

Look for *stationary* solutions

$$\xi(x^0, x^1, x^2, x^3) = e^{-ip_0x^0}\eta(x^1, x^2, x^3).$$

Theorem 2 *In the stationary setting my Euler–Lagrange equation is equivalent to a pair of massless Dirac equations*

$$i\begin{pmatrix} \mp\partial_0 + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \mp\partial_0 - \partial_3 \end{pmatrix}\begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} = 0.$$

Nonlinear second order PDEs

which reduce to pairs of linear first order PDEs

Consider a pair of formally self-adjoint first order linear differential operators $A_\pm$ acting on smooth vector functions $u : \mathbb{R}^n \to \mathbb{C}^m$.

Corresponding Lagrangian densities are $L_\pm(u) := \mathrm{Re}(u^* A_\pm u)$.

Define new Lagrangian density

$$L(u) := \frac{L_+(u)\, L_-(u)}{L_+(u) - L_-(u)} \; . \tag{1}$$

Work with vector functions such that $L_+(u) \neq L_-(u)$.

Lemma 1 *A vector function u is a solution of the Euler–Lagrange equation for the Lagrangian density (1) if and only if it is a solution of $A_+u = 0$ or $A_-u = 0$.*

Example illustrating the use of Lemma 1

The pair of linear first order ordinary differential equations

$$iu' \pm u = 0$$

is equivalent to a single nonlinear second order equation

$$\left(\frac{\bar{u}u' - u\bar{u}'}{2|u|^2}\,u\right)' + \frac{(\bar{u}u')^2 - (u\bar{u}')^2}{4|u|^4}\,u + u = 0\,.$$

Same trick works for systems of PDEs with variable coefficients.

Can we incorporate mass and electromagnetic field into rotational elasticity?

Yes. Start with Minkowski space and perform perform a Kaluza–Klein extension. The extra space coordinate takes over the role previously played by time.

This way we derive the massive Dirac equation in dimension 1+2.

In dimension 1+3 we get an equation which is slightly different from Dirac. Is it better or worse than the Dirac equation?

Do I really believe that world aether is made up of rotating points?

No. I believe that world aether is described by a system of nonlinear hyperbolic PDEs. The way to analyse this system is to start from the linearised system, write down a solution explicitly in the form of a "wave packet" and then treat the nonlinearity as a perturbation. This will hopefully give, in the end, a soliton with the properties of an elementary particle.

How is this related to rotational elasticity? Fact: *the principal symbol of a traceless linear elliptic first order differential operator in 3D acting on a pair of complex half-densities is equivalent to a frame.* Varying the frame means that we are making our system of PDEs nonlinear.

Is there a good mathematical technique for the analysis of linear hyperbolic PDEs?

Yes. It is called *microlocal analysis*. Sources:

1) book by L.Hörmander (in 4 volumes),
2) book by Yu.Safarov and D.Vassiliev,
3) papers by Yu.Safarov (1980s) and PhD Thesis of Wilfred Nicoll (Sussex, 1999).

Item 3) is important for dealing with systems.

My conjecture: all the stuff I described today (rotational elasticity, teleparallelism, torsion, Dirac equation etc) is already present in microlocal analysis, only nobody has noticed it.

Summary

- Rotational elasticity is a fun subject.
- Nobody has previously studied rotational elasticity.
- Rotational elasticity may lead to an alternative description of fermions (neutrinos, electrons) in quantum mechanics.

Papers and preprints can be found on my web page

http://www.homepages.ucl.ac.uk/~ucahdva/

My talks (including this one) are also on my web page.