Dirac and Weyl equations from Cosserat elasticity

James Burnett, Olga Chervova and Dmitri Vassiliev

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Mathematical Foundations meeting

Describing a deformable continuous medium

(a) Classical elasticity: displacements only.

(b) Cosserat elasticity: displacements and rotations. See

E. Cosserat and F. Cosserat, *Théorie des Corps Déformables*, 1909. Available from Amazon.

(c) Teleparallelism (absolute parallelism, fernparallelismus): rotations only.

Teleparallelism in Euclidean 3-space

Work in \mathbb{R}^3 equipped with standard metric and Cartesian coordinates x^{α} , $\alpha = 1, 2, 3$.

A frame ϑ is a triplet ϑ^j , j = 1, 2, 3, of orthonormal vector fields. Each vector field ϑ^j has a hidden tensor index: $\vartheta^j = \vartheta^j_{\alpha}$, $\alpha = 1, 2, 3$.

Same in plain English: a frame is a field of orthonormal bases.

NB. Frame lives separately from Cartesian coordinates (not aligned with coordinate lines).

My model

The frame ϑ is an unknown quantity (dynamical variable).

The other dynamical variable is a density ρ .

My Lagrangian density $L(\vartheta, \rho)$ is chosen from the condition of conformal invariance.

Action (variational functional) $\int L(\vartheta, \rho) dx^1 dx^2 dx^3$.

Vary action with respect to frame ϑ and density ρ to get Euler–Lagrange equations.

Difference with existing models

1. I assume metric to be fixed (prescribed).

2. My Lagrangian has never been considered.

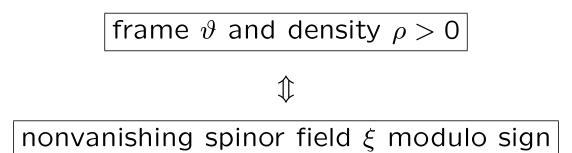
Introducing time into my model

Standard Newtonian construction: write down angular velocity, write down kinetic energy etc.

My model remains conformally invariant, only now in the Lorentzian sense.

Solving Euler–Lagrange equations

Switch to spinors:



My Lagrangian density $L(\xi)$ is a rational function of ξ , $\overline{\xi}$ and partial derivatives of ξ , $\overline{\xi}$.

Quasi-stationary solutions

Look first for quasi-stationary solutions

 $\xi(t, x^1, x^2, x^3) = e^{-i\omega t} \eta(x^1, x^2, x^3), \qquad \omega \neq 0.$

Theorem 1 In the quasi-stationary case my Euler–Lagrange equation is equivalent to a pair of Weyl equations.

"Weyl equation" = "massless Dirac equation".

Proof Amazing fact: my Lagrangian factorises into a product of two Weyl Lagrangians! Result follows from factorisation. □

Plane waves

Special case of quasi-stationary solution

$$\xi(t, x^1, x^2, x^3) = e^{-i(\omega t + k \cdot x)}\eta$$

where η is constant spinor. This is *plane wave*.

Corollary 1 Plane wave solutions in my model are the same as for a pair of Weyl equations.

Perturbations of plane waves

Idea: seek spinor field in the form

slowly varying amplitude $\times e^{-i(\omega t + k \cdot x)}$.

Theorem 2 Perturbations of plane wave solutions in my model are described by a pair of Weyl equations.

Relativistic version of my model

Work in Minkowski 4-space instead of Euclidean 3-space. Write x^0 instead of t. Frame now has 4 elements, each of these has 4 components.

Comparing the relativistic

and nonrelativistic models

Relativistic model has 3 extra degrees of freedom (Lorentz boosts in 3 directions) and, consequently, 3 extra field equations.

Theorem 3 At the perturbative level the 3 extra field equations are automatically satisfied.

Conclusion: my nonrelativistic model possesses relativistic invariance at the perturbative level.

Incorporating mass m and electromagnetic (co)vector potential A

Introduce 5th coordinate: $(x^0, x^1, x^2, x^3, \underline{x^4})$.

O.Klein (1926): prescribe oscillation $\sim e^{-imx^4}$ along extra coordinate, then separate variables.

T.Kaluza (1921): perturb extended metric

$$\begin{pmatrix} g_{\alpha\beta} & 0\\ 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} g_{\alpha\beta} - \frac{1}{m^2} A_{\alpha} A_{\beta} & \frac{1}{m} A_{\alpha} \\ \\ \frac{1}{m} A_{\beta} & -1 \end{pmatrix}$$

This is *shear*.

NB. Kaluza–Klein extension destroys conformal invariance!

Theorem 4 In special case with no dependence on x^3 my model is equivalent to the massive Dirac equation with electromagnetic field.

Summary

New mathematical model for fermions.

- Spacetime viewed as Cosserat continuum.
- Lagrangian chosen from the condition of conformal invariance.
- Mass and electromagnetic field incorporated via Kaluza–Klein extension.