Dirac equation for dummies or theory of elasticity for the seriously advanced

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Dirac's equation is a model for

(a) electron and positron (massive case),

(b) neutrino and antineutrino (massless case).

Formulating Dirac's equation requires:

(i) spinors,

(ii) Pauli matrices,

(iii) covariant differentiation.

Also, logical issues with Dirac's equation:

(iv) difficult to distinguish particle from antiparticle, both in massive and massless cases,

(v) electromagnetism doesn't admit a sensible geometric interpretation ($\partial \mapsto \partial + iA$).

My model requires:

(i) differential forms,

(ii) wedge product,

(iii) exterior differentiation.

Also, logical issues are resolved:

(iv) easy to distinguish particle from antiparticle, both in massive and massless cases,

(v) electromagnetism admits a sensible geometric interpretation.

Price I will pay: my model will be nonlinear.

Formulation of Dirac's equation

Start with massless Dirac equation.

Also assume metric to be flat (no gravity).

Unknown quantity is a 2-component spinor

$$\xi = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}.$$

Pair of "scalar" complex-valued functions of time t and Euclidean coordinates x^1, x^2, x^3.

Massless Dirac equation:

$$\pm\partial_t \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} = \begin{pmatrix} \partial_3 & \partial_1 + i\partial_2 \\ \partial_1 - i\partial_2 & -\partial_3 \end{pmatrix} \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}.$$

Equation transforms in complicated way under rotations of coordinate system. Spinor is "square root" of a complex isotropic vector.

Describing a deformable continuous medium

(a) Classical elasticity: displacements only.

(b) Cosserat elasticity: displacements and rotations. See

E. Cosserat and F. Cosserat, *Théorie des Corps Déformables*, 1909. Available from Amazon.

(c) Teleparallelism (absolute parallelism, fernparallelismus): rotations only.

Teleparallelism in Euclidean 3-space

Work in $\mathbb{R}^3$ equipped with standard metric

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and Cartesian coordinates x^α, $\alpha = 1, 2, 3$.

A *coframe* $\{\vartheta^1, \vartheta^2, \vartheta^3\}$ is a triplet of covector fields satisfying metric constraint

$$g = \vartheta^1 \otimes \vartheta^1 + \vartheta^2 \otimes \vartheta^2 + \vartheta^3 \otimes \vartheta^3.$$

Same in plain English: a coframe is a field of orthonormal bases.

Coframe lives separately from Cartesian coordinates (not aligned with coordinate lines).

Coframe will play the role of unknown quantity (dynamical variable).

Measure of deformation: the 3-form

$$T^{\mathsf{ax}} := \frac{1}{3}(\vartheta^1 \wedge d\vartheta^1 + \vartheta^2 \wedge d\vartheta^2 + \vartheta^3 \wedge d\vartheta^3).$$

Called "axial torsion of teleparallel connection".

The 3-form T^{ax} is conformally covariant. Let

$$\vartheta^j \mapsto e^h \vartheta^j, \qquad j = 1, 2, 3,$$

where $h : \mathbb{R}^3 \to \mathbb{R}$ is arbitrary scalar function. Then

$$g \mapsto e^{2h} g,$$

$$T^{\mathsf{ax}} \mapsto e^{2h} T^{\mathsf{ax}}$$

without the derivatives of h appearing.

My Lagrangian density

$$\boxed{L = \|T^{\text{ax}}\|^2 \rho}$$

where ρ is an additional dynamical variable.

My Lagrangian is conformally invariant!

Action (variational functional) $\int L\, dx^1 dx^2 dx^3$.

Vary action with respect to coframe ϑ and density ρ to get Euler–Lagrange equations.

Difference with existing models

1. I assume metric to be fixed (prescribed).

2. My Lagrangian has never been considered.

Introducing time into my model

Define 2-form

$$\dot{\vartheta} := \vartheta^1 \wedge \dot{\vartheta}^1 + \vartheta^2 \wedge \dot{\vartheta}^2 + \vartheta^3 \wedge \dot{\vartheta}^3.$$

Note: $*\dot{\vartheta}$ is the vector of *angular velocity*.

$$\boxed{L = (\|\dot{\vartheta}\|^2 - \|T^{\mathrm{ax}}\|^2)\rho}$$

Model remains conformally invariant, only now in the Lorentzian sense.

Solving Euler–Lagrange equations

Switch to spinors:

$$\boxed{\text{coframe } \vartheta \text{ and density } \rho > 0}$$

$$\Updownarrow$$

$$\boxed{\text{nonvanishing spinor field } \xi \text{ modulo sign}}$$

Lagrangian density $L(\xi)$ is a rational function of ξ, $\bar{\xi}$ and partial derivatives of ξ, $\bar{\xi}$.

Look first for quasi-stationary solutions

$$\xi(t,x^1,x^2,x^3) = e^{-i\omega t}\eta(x^1,x^2,x^3), \qquad \omega \neq 0.$$

Theorem 1 *In the quasi-stationary case my Euler–Lagrange equation is equivalent to a pair of massless Dirac equations.*

Proof My Lagrangian density L factorises as

$$L(\eta) = \frac{L_+(\eta)L_-(\eta)}{L_+(\eta) - L_-(\eta)}$$

where $L_\pm(\eta)$ are the Dirac Lagrangians. Use also scaling covariance of Dirac Lagrangians:

$$L_\pm(e^h\eta) = e^{2h}L_\pm(\eta)$$

where $h : \mathbb{R}^3 \to \mathbb{R}$ is arbitrary scalar function. □

Special case of quasi-stationary solution

$$\xi(t,x^1,x^2,x^3)=e^{-i(\omega t+k\cdot x)}\eta$$

where η is constant spinor. This is *plane wave*.

Corollary 1 *Plane wave sol-s in my model are same as for pair of massless Dirac equations.*

Visualising plane wave solutions

Up to a rigid orthogonal transformation

$$\vartheta^1_\alpha=\begin{pmatrix}\cos\varphi\\ \pm\sin\varphi\\ 0\end{pmatrix},\quad \vartheta^2_\alpha=\begin{pmatrix}\mp\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix},\quad \vartheta^3_\alpha=\begin{pmatrix}0\\ 0\\ 1\end{pmatrix},$$

$$\rho=1,$$

where

$$\varphi:=2|\omega|(t+x^3).$$

Looking at a travelling wave of rotations.

Perturbations of plane waves

Idea: seek spinor field in the form

slowly varying amplitude $\times e^{-i(\omega t + k\cdot x)}$.

Deriving equation for perturbed plane wave:

- substitution $\xi \mapsto \xi e^{-i(\omega t + k\cdot x)}$ in Lagrangian;
- write down Euler–Lagrange equation;
- linearize Euler–Lagrange equation;
- drop second derivatives;
- substitution $\xi \mapsto \xi e^{i(\omega t + k\cdot x)}$ in equation.

Theorem 2 *Perturbations of plane wave solutions in my model are described by a pair of massless Dirac equations.*

Analytic challenges

- My model is nonlinear.
- My model does not appear to fit into standard scheme of hyperbolic systems of PDEs.
- Don't know how to handle situation $\rho = 0$.
- Cannot set problem rigorously in terms of function spaces.
- Cannot justify formal asymptotic analysis.

Relativistic version of my model

Work in Minkowski space.

Coordinates x^α, $\alpha = 0, 1, 2, 3$.

$$\text{Metric} \quad g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Coframe $\{\vartheta^0, \vartheta^1, \vartheta^2, \vartheta^3\}$.

$$g = \vartheta^0 \otimes \vartheta^0 - \vartheta^1 \otimes \vartheta^1 - \vartheta^2 \otimes \vartheta^2 - \vartheta^3 \otimes \vartheta^3.$$

$$T^{\mathrm{ax}} = \frac{1}{3}(\vartheta^0 \wedge d\vartheta^0 - \vartheta^1 \wedge d\vartheta^1 - \vartheta^2 \wedge d\vartheta^2 - \vartheta^3 \wedge d\vartheta^3).$$

Lagrangian density $\quad L = \|T^{\mathrm{ax}}\|^2 \rho$.

Comparing the relativistic and nonrelativistic models

Relativistic model has 3 extra degrees of freedom (Lorentz boosts in 3 directions) and, consequently, 3 extra field equations.

Theorem 3 *At the asymptotic level (plane waves and their formal perturbations) the 3 extra field equations are automatically satisfied.*

Conclusion: my nonrelativistic model possesses relativistic invariance at the asymptotic level.

Kaluza-Klein extension

Introduce 5th coordinate: $(x^0, x^1, x^2, x^3, \underline{x^4})$.

O.Klein (1926): prescribe oscillation $\sim e^{-imx^4}$ along extra coordinate, then separate variables.

Will use **bold** for extended quantities.

Extended coframe $\{\boldsymbol{\vartheta}^0, \boldsymbol{\vartheta}^1, \boldsymbol{\vartheta}^2, \boldsymbol{\vartheta}^3, \boldsymbol{\vartheta}^4\}$:

$$\boldsymbol{\vartheta}^0_{\boldsymbol{\alpha}} = \begin{pmatrix} \vartheta^0_\alpha \\ 0 \end{pmatrix}, \qquad \boldsymbol{\vartheta}^3_{\boldsymbol{\alpha}} = \begin{pmatrix} \vartheta^3_\alpha \\ 0 \end{pmatrix}, \qquad \boldsymbol{\vartheta}^4_{\boldsymbol{\alpha}} = \begin{pmatrix} 0_\alpha \\ 1 \end{pmatrix},$$

$$(\boldsymbol{\vartheta}^1 + i\boldsymbol{\vartheta}^2)_{\boldsymbol{\alpha}} = \begin{pmatrix} (\vartheta^1 + i\vartheta^2)_\alpha \\ 0 \end{pmatrix} e^{-2imx^4}.$$

Coordinate x^4 parametrises circle of radius $\frac{1}{2m}$. Coframe makes full turn in the ϑ^1, ϑ^2-plane as we move along the circle.

Axial torsion of extended coframe

$$\mathbf{T}^{\text{ax}} = \frac{1}{3}(\boldsymbol{\vartheta}^0 \wedge d\boldsymbol{\vartheta}^0 - \boldsymbol{\vartheta}^1 \wedge d\boldsymbol{\vartheta}^1 - \boldsymbol{\vartheta}^2 \wedge d\boldsymbol{\vartheta}^2 - \boldsymbol{\vartheta}^3 \wedge d\boldsymbol{\vartheta}^3 - \underbrace{\boldsymbol{\vartheta}^4 \wedge d\boldsymbol{\vartheta}^4}_{=0}).$$

T.Kaluza (1921): electromagnetism is a perturbation (shear) of the extended metric

$$\begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} g_{\alpha\beta} - \frac{1}{m^2}A_\alpha A_\beta & \frac{1}{m}A_\alpha \\ & \\ \frac{1}{m}A_\beta & -1 \end{pmatrix} =: \mathbf{g}_{\boldsymbol{\alpha\beta}}\,.$$

Note: with electromagnetism, extended coframe and extended metric no longer agree

$$\mathbf{g} \ne \boldsymbol{\vartheta}^0 \otimes \boldsymbol{\vartheta}^0 - \boldsymbol{\vartheta}^1 \otimes \boldsymbol{\vartheta}^1 - \boldsymbol{\vartheta}^2 \otimes \boldsymbol{\vartheta}^2 - \boldsymbol{\vartheta}^3 \otimes \boldsymbol{\vartheta}^3 - \boldsymbol{\vartheta}^4 \otimes \boldsymbol{\vartheta}^4.$$

I don't care (in this talk).

Consider Lagrangian density

$$\boxed{L = \|\mathbf{T}^{\text{ax}}\|^2 \rho}$$

Dynamical variables: original (unextended) coframe ϑ and density ρ. Note: conformal invariance is destroyed by Kaluza–Klein extension.

Theorem 4 *In special case with no dependence on x^3 my model is equivalent to the massive Dirac equation with electromagnetic field.*

Summary

New mathematical model for fermions.

- Spacetime viewed as Cosserat continuum.

- Lagrangian chosen from the condition of conformal invariance.

- Mass and electromagnetic field incorporated via Kaluza–Klein extension.

What is to be done?

Perform mathematical analysis of my model.

Spin-off

There is a class of beautiful nonlinear PDEs arising in Cosserat elasticity which has never been studied by analysts.