# Dirac equation as a special case of Cosserat elasticity

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Aim of talk: to **understand** Dirac's equation.

Will look ar Dirac's equation: a system of 4 homogeneous linear partial differential equations for 4 complex unknowns in dimension 1+3.

Formulating Dirac's equation requires:
(a) spinors,
(b) Pauli matrices,
(c) covariant differentiation.

My reinterpretation of Dirac's eq-n will require:
(a) differential forms,
(b) wedge product,
(c) exterior differentiation.

Price I will pay: my model will be nonlinear.

## Formulation of Dirac's equation

Work on 4-manifold with Lorentzian metric $g_{\alpha\beta}$

Unknown quantity is a pair of 2-component spinors $\xi_a$ and $\eta_{\dot{a}}$ .

Raise and lower spinor indices using "metric spinor" $\epsilon_{ab} = \epsilon_{\dot{a}\dot{b}} = \epsilon^{ab} = \epsilon^{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Pauli matrices ${\sigma^\alpha}_{a\dot{b}}$ defined by condition

$$ {\sigma^\alpha}_{a\dot{b}}\sigma^{\beta c\dot{b}} + {\sigma^\beta}_{a\dot{b}}\sigma^{\alpha c\dot{b}} = 2g^{\alpha\beta}{\delta_a}^c. $$

Covariant derivative of a spinor field

$$ \nabla_\mu \xi^a = \partial_\mu \xi^a + {\Gamma^a}_{\mu b}\xi^b, $$

$$ \nabla_\mu \eta^{\dot{a}} = \partial_\mu \eta^{\dot{a}} + {\bar{\Gamma}^{\dot{a}}}_{\mu\dot{b}}\eta^{\dot{b}} $$

where

$$\Gamma^{a}{}_{\mu b} = \frac{1}{4}\sigma_{\alpha}{}^{a\dot{c}}\left(\partial_\mu \sigma^{\alpha}{}_{b\dot{c}} + \begin{Bmatrix} \alpha \\ \mu\beta \end{Bmatrix} \sigma^{\beta}{}_{b\dot{c}}\right),$$

$$\begin{Bmatrix} \alpha \\ \mu\beta \end{Bmatrix} = \frac{1}{2} g^{\alpha\kappa}(\partial_\mu g_{\beta\kappa} + \partial_\beta g_{\mu\kappa} - \partial_\kappa g_{\mu\beta}),$$

$$\bar{\Gamma}^{\dot{a}}{}_{\mu\dot{b}} = \overline{\Gamma^{a}{}_{\mu b}}.$$

Dirac's equation:

$$\begin{aligned} i\sigma^{\alpha a\dot{b}} \nabla_\alpha \eta_{\dot{b}} &= m\xi^{a}, \\ i\sigma^{\alpha}{}_{a\dot{b}} \nabla_\alpha \xi^{a} &= m\eta_{\dot{b}} \end{aligned}$$

where $m$ is mass.

Will deal first with massless Dirac ($m = 0$). Then Dirac's equation reduces to Weyl's eq-n

$$i\sigma^{\alpha}{}_{a\dot{b}}\nabla_{\alpha}\xi^{a} = 0.$$

More precisely, massless Dirac reduces to a *pair* of Weyl's equations.

Weyl's equation is simpler: system of 2 homogeneous linear PDEs for 2 complex unknowns.

Why is massless Dirac interesting?

- Retains main features of massive Dirac.
- Describes neutrino.
- Separation of variables $\xi \sim e^{-imx^3}$ in massless Dirac equation in dimension 1+3 gives massive Dirac equation in dimension 1+2.

**Describing a deformable continuous medium**

(a) Classical elasticity: displacements only.

(b) Cosserat elasticity: displacements and rotations. See

E. Cosserat and F. Cosserat, *Théorie des Corps Déformables*, A. Hermann et Fils, Paris, 1909.

(c) Teleparallelism (absolute parallelism, fernparallelismus): rotations only.

## Teleparallelism in Euclidean 3-space

Work in $\mathbb{R}^3$ equipped with standard metric $g$ and Cartesian coordinates $x^\alpha$, $\alpha = 1, 2, 3$.

A *coframe* $\{\vartheta^1, \vartheta^2, \vartheta^3\}$ is a triad of covector fields satisfying metric constraint

$$g = \vartheta^1 \otimes \vartheta^1 + \vartheta^2 \otimes \vartheta^2 + \vartheta^3 \otimes \vartheta^3.$$

Same in plain English: a coframe is a field of orthonormal bases.

**NB.** Coframe lives separately from Cartesian coordinates (not aligned with coordinate lines).

Coframe will play the role of unknown quantity (dynamical variable).

Natural measure of deformation: torsion

$$T=\vartheta^1\otimes d\vartheta^1+\vartheta^2\otimes d\vartheta^2+\vartheta^3\otimes d\vartheta^3.$$

Analogue of strain tensor.

Irreducible pieces of torsion $T^{(1)}$, $T^{(2)}$, $T^{(3)}$.

Lagrangian density

$$L=\left(c_1\left\|T^{(1)}\right\|^2+c_2\left\|T^{(2)}\right\|^2+c_3\left\|T^{(3)}\right\|^2\right)\rho$$

where $\rho$ is a density. Either take $\rho=\sqrt{\det g}$ or view $\rho$ as a dynamical variable.

Action (variational functional) $\int L\,dx^1dx^2dx^3$ .

Vary action to get Euler–Lagrange equations.

**NB.** I keep the metric fixed.

## My model

Need only one irreducible piece of torsion: axial (totally antisymmetric) torsion

$$T^{(1)} = \frac{1}{3}(\vartheta^1 \wedge d\vartheta^1 + \vartheta^2 \wedge d\vartheta^2 + \vartheta^3 \wedge d\vartheta^3).$$

Analogue of trace free part of strain tensor.

I like my own notation

$$d\vartheta := 3T^{(1)} = \vartheta^1 \wedge d\vartheta^1 + \vartheta^2 \wedge d\vartheta^2 + \vartheta^3 \wedge d\vartheta^3.$$

"Exterior derivative of coframe".

My Lagrangian density

$$\boxed{L = \|d\vartheta\|^2 \rho}$$

Dynamical variables: coframe $\vartheta$ and density $\rho$.

My model is conformally invariant!

## Introducing time into my model

$$\dot\vartheta := \vartheta^1 \wedge \dot\vartheta^1 + \vartheta^2 \wedge \dot\vartheta^2 + \vartheta^3 \wedge \dot\vartheta^3.$$

$*\dot\vartheta$ is the vector of *angular velocity*.

$$\boxed{L = (\|\dot\vartheta\|^2 - \|d\vartheta\|^2)\rho}$$

Model remains conformally invariant!

## Solving Euler–Lagrange equations

Switch to spinors:

$$\boxed{\text{coframe } \vartheta \text{ and density } \rho}$$

$$\Updownarrow$$

$$\boxed{\text{nonvanishing spinor field } \xi \text{ modulo sign}}$$

Lagrangian density $L(\xi)$ is a rational function of $\xi$, $\bar\xi$ and partial derivatives of $\xi$, $\bar\xi$.

Look first for plane wave solutions

$$\xi \sim e^{-i(\omega t + k\cdot x)}, \qquad \omega \neq 0.$$

**Theorem 1** *Plane wave solutions in my model are the same as for a pair of Weyl equations*

$$\left[\sigma^0 \partial_t \pm \left(\sigma^1 \partial_1 + \sigma^2 \partial_2 + \sigma^3 \partial_3\right)\right]_{a\dot{b}} \xi^a = 0.$$

## Visualising plane wave solutions

Up to a rigid orthogonal transformation

$$\vartheta^1_\alpha = \begin{pmatrix} \cos\varphi \\ \pm\sin\varphi \\ 0 \end{pmatrix}, \quad \vartheta^2_\alpha = \begin{pmatrix} \mp\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}, \quad \vartheta^3_\alpha = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\rho = \text{const},$$

where

$$\varphi := 2|\omega|(t + x^3).$$

Looking at a travelling wave of rotations.

## Perturbations of plane waves

Idea: seek spinor field in the form

slowly varying amplitude $\times e^{-i(\omega t+k\cdot x)}$.

Deriving equation for perturbed plane wave:

- substitution $\xi \mapsto \xi e^{-i(\omega t+k\cdot x)}$ in Lagrangian;
- write down Euler–Lagrange equation;
- linearize Euler–Lagrange equation;
- drop second derivatives;
- substitution $\xi \mapsto \xi e^{i(\omega t+k\cdot x)}$ in equation.

**Theorem 2** *Perturbations of plane wave solutions are described by a pair of Weyl equations*

$$\left[\sigma^0\partial_t \pm \left(\sigma^1\partial_1 + \sigma^2\partial_2 + \sigma^3\partial_3\right)\right]_{a\dot{b}}\xi^a = 0.$$

## Relativistic version of my model

Work in Minkowski space.

Coordinates $x^\alpha$, $\alpha = 0, 1, 2, 3$.

Metric $g_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1)$.

Coframe $\quad \{\vartheta^0, \vartheta^1, \vartheta^2, \vartheta^3\}$.

$$g = \vartheta^0 \otimes \vartheta^0 - \vartheta^1 \otimes \vartheta^1 - \vartheta^2 \otimes \vartheta^2 - \vartheta^3 \otimes \vartheta^3.$$

$$T = \vartheta^0 \otimes d\vartheta^0 - \vartheta^1 \otimes d\vartheta^1 - \vartheta^2 \otimes d\vartheta^2 - \vartheta^3 \otimes d\vartheta^3.$$

$$d\vartheta = \vartheta^0 \wedge d\vartheta^0 - \vartheta^1 \wedge d\vartheta^1 - \vartheta^2 \wedge d\vartheta^2 - \vartheta^3 \wedge d\vartheta^3.$$

Lagrangian density $\quad L = \|d\vartheta\|^2 \rho$.

## Comparing the relativistic and nonrelativistic models

- Relativistic model has 3 extra degrees of freedom (Lorentz boosts in 3 directions).

- Relativistic model admits gauge transformations: transformations of coframe which preserve $d\vartheta$ up to scaling. This appears to gauge out 3 degrees of freedom.

*Conjecture:* the two models are equivalent.

**Theorem 3** *At the asymptotic level (plane waves and their formal perturbations) solutions of the relativistic and nonrelativistic models differ only by gauge transformations.*

## Comparison with Maxwell's equation

| | Maxwell's equation | My equation |
|---|---|---|
| Dynamical variable | Covector field $A$ | Quartet of orthonormal covector fields $\{\vartheta^0, \vartheta^1, \vartheta^2, \vartheta^3\}$ and density $\rho$ |
| Field strength | 2-form $dA$ | 3-form $d\vartheta$ |
| Lagrangian density | $\|dA\|^2\sqrt{\lvert\det g\rvert}$ | $\|d\vartheta\|^2\rho$ |
| Gauge degrees of freedom | 1 | 3 |
| Conformal invariance | Yes | Yes |

## Massive Dirac equation

What is the geometric meaning of mass $m$?

Klein's interpretation of mass, as illustrated by Klein–Gordon equation in Minkowski space

$$(\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2)\psi + m^2\psi = 0.$$

Introduce 5th coordinate: $(x^0, x^1, x^2, x^3, \underline{x^4})$.

Consider wave equation in extended space

$$(\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2 - \partial_4^2)\psi = 0.$$

Separate out the variable $x^4$: $\psi \sim e^{-imx^4}$.

*Conjecture:* Klein's construction works in my model to give massive Dirac equation. "Separation of variables" means one full rotation of coframe as we move along the 5th coordinate.

## Dirac's equation with electromagnetic field

Electromagnetism in Dirac's equation:

$$\nabla \to \nabla + iA.$$

Kaluza's interpretation of electromagnetism: perturbation (shear) of the extended metric

$$\begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & -1 \end{pmatrix} \to \begin{pmatrix} g_{\alpha\beta} - A_\alpha A_\beta & A_\alpha \\ A_\beta & -1 \end{pmatrix}.$$

**NB.** Kaluza did not devise above substitution for use in quantum mechanics. In fact, at the time (1921) quantum mechanics didn't exist.

*Conjecture:* Kaluza's construction works in my model to give massive Dirac equation with electromagnetic field.

## Summary

I am suggesting a new equation which is, in effect, a generalisation of Maxwell's equation.

- Dynamical variables: coframe and density.
- Lagrangian: quadratic in torsion and conformally invariant.

## End goal of project

To derive all (or as much as possible) of Quantum Electrodynamics from above equation.

## Spin-off

There is a class of beautiful nonlinear PDEs arising in Cosserat elasticity which has never been studied by analysts.

## Bibliography

Precursor paper

*Phys. Rev.* **D75**, 025006 (2007).

Different Lagrangian (linear in torsion) but ideologically similar to today's talk:

- metric is fixed;
- coframe is dynamical variable;
- only axial piece of torsion is used.