# Dirac equation as a special case of Cosserat elasticity

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Aim of talk: to **understand** Dirac's equation.

Will look ar Dirac's equation: a system of 4 homogeneous linear partial differential equations for 4 complex unknowns in dimension 1+3.

Formulating Dirac's equation requires:
(a) spinors,
(b) Pauli matrices,
(c) covariant differentiation.

My reinterpretation of Dirac's eq-n will require:
(a) differential forms,
(b) wedge product,
(c) exterior differentiation.

Price I will pay: my model will be nonlinear.

## Formulation of Dirac's equation

Work on 4-manifold with Lorentzian metric $g_{\alpha\beta}$

Unknown quantity is a pair of 2-component spinors $\xi_a$ and $\eta_{\dot{a}}$ . Such a pair is called "bispinor" .

Raise and lower spinor indices using "metric spinor" $\epsilon_{ab} = \epsilon_{\dot{a}\dot{b}} = \epsilon^{ab} = \epsilon^{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Pauli matrices $\sigma^{\alpha}{}_{a\dot{b}}$ defined by condition

$$\sigma^{\alpha}{}_{a\dot{b}}\sigma^{\beta c\dot{b}} + \sigma^{\beta}{}_{a\dot{b}}\sigma^{\alpha c\dot{b}} = 2g^{\alpha\beta}\delta_a{}^c.$$

Covariant derivative of a spinor field

$$\nabla_\mu \xi^a = \partial_\mu \xi^a + \Gamma^a{}_{\mu b}\xi^b,$$

$$\nabla_\mu \eta^{\dot{a}} = \partial_\mu \eta^{\dot{a}} + \bar{\Gamma}^{\dot{a}}{}_{\mu\dot{b}}\eta^{\dot{b}}$$

where

$$\Gamma^{a}{}_{\mu b} = \frac{1}{4}\sigma_{\alpha}{}^{a\dot{c}}\left(\partial_{\mu}\sigma^{\alpha}{}_{b\dot{c}} + \left\{ \begin{matrix} \alpha \\ \mu\beta \end{matrix} \right\} \sigma^{\beta}{}_{b\dot{c}}\right),$$

$$\left\{ \begin{matrix} \alpha \\ \mu\beta \end{matrix} \right\} = \frac{1}{2} g^{\alpha\kappa}(\partial_{\mu} g_{\beta\kappa} + \partial_{\beta} g_{\mu\kappa} - \partial_{\kappa} g_{\mu\beta}),$$

$$\bar{\Gamma}^{\dot{a}}{}_{\mu\dot{b}} = \overline{\Gamma^{a}{}_{\mu b}}.$$

Dirac's equation:

$$\begin{aligned} \sigma^{\alpha a\dot{b}}(i\nabla + A)_{\alpha}\eta_{\dot{b}} &= m\xi^{a}, \\ \sigma^{\alpha}{}_{a\dot{b}}(i\nabla + A)_{\alpha}\xi^{a} &= m\eta_{\dot{b}} \end{aligned}$$

where $m$ is mass and $A$ is the electromagnetic (co)vector potential.

**Describing a deformable continuous medium**

(a) Classical elasticity: displacements only.

(b) Cosserat elasticity: displacements and rotations. See

E. Cosserat and F. Cosserat, *Théorie des Corps Déformables*, A. Hermann et Fils, Paris, 1909.

(c) Teleparallelism (absolute parallelism, fernparallelismus): rotations only.

## My model

Initially work on 3-manifold $M$ equipped with prescribed positive metric $g$.

A *coframe* $\{\vartheta^1, \vartheta^2, \vartheta^3\}$ is a triad of covector fields satisfying metric constraint

$$g = \vartheta^1 \otimes \vartheta^1 + \vartheta^2 \otimes \vartheta^2 + \vartheta^3 \otimes \vartheta^3.$$

Same in plain English: a coframe is a field of orthonormal bases.

**NB.** Coframe lives separately from local coordinates (not aligned with coordinate lines).

Coframe will play the role of unknown quantity (dynamical variable).

Measure of deformation: the 3-form

$$T^{\mathsf{ax}} := \frac{1}{3}(\vartheta^1 \wedge d\vartheta^1 + \vartheta^2 \wedge d\vartheta^2 + \vartheta^3 \wedge d\vartheta^3).$$

Called "axial torsion of teleparallel connection".

The 3-form $T^{\mathsf{ax}}$ is conformally covariant. Let

$$\vartheta^j \mapsto e^h \vartheta^j$$

where $h : M \to \mathbb{R}$ is an arbitrary scalar function. Then

$$g \mapsto e^{2h} g,$$

$$T^{\mathsf{ax}} \mapsto e^{2h} T^{\mathsf{ax}}$$

without the derivatives of $h$ appearing.

My Lagrangian density

$$\boxed{L = \|T^{\mathrm{ax}}\|^2 \rho}$$

where $\rho$ is an additional dynamical variable.

My Lagrangian is conformally invariant!

Action (variational functional) $\int L \, dx^1 dx^2 dx^3$ .

Vary action with respect to coframe $\vartheta$ and density $\rho$ to get Euler–Lagrange equations.

**Difference with existing models**

1. I assume metric to be fixed (prescribed).

2. My Lagrangian has never been considered.

## Introducing time into my model

Switch to 4-manifold with Lorentzian metric $g$.

$$\text{Coframe} \quad \{\vartheta^0, \vartheta^1, \vartheta^2, \vartheta^3\}.$$

$$g = \vartheta^0 \otimes \vartheta^0 - \vartheta^1 \otimes \vartheta^1 - \vartheta^2 \otimes \vartheta^2 - \vartheta^3 \otimes \vartheta^3.$$

$$T^{\mathrm{ax}} = \frac{1}{3}(\vartheta^0 \wedge d\vartheta^0 - \vartheta^1 \wedge d\vartheta^1 - \vartheta^2 \wedge d\vartheta^2 - \vartheta^3 \wedge d\vartheta^3).$$

$$\text{Lagrangian density} \quad L = \|T^{\mathrm{ax}}\|^2 \rho.$$

The resulting system of equations is not yet the Dirac system. Need to incorporate mass $m$ and electromagnetic (co)vector potential $A$.

## Kaluza-Klein extension

Introduce 5th coordinate: $(x^0, x^1, x^2, x^3, \underline{x^4})$.

O.Klein (1926): prescribe oscillation $\sim e^{-imx^4}$ along extra coordinate, then separate variables.

Will use **bold** for extended quantities.

Extended coframe $\{\boldsymbol{\vartheta}^0, \boldsymbol{\vartheta}^1, \boldsymbol{\vartheta}^2, \boldsymbol{\vartheta}^3, \boldsymbol{\vartheta}^4\}$:

$$\boldsymbol{\vartheta}^0_{\boldsymbol{\alpha}} = \begin{pmatrix} \vartheta^0_\alpha \\ 0 \end{pmatrix}, \qquad \boldsymbol{\vartheta}^3_{\boldsymbol{\alpha}} = \begin{pmatrix} \vartheta^3_\alpha \\ 0 \end{pmatrix}, \qquad \boldsymbol{\vartheta}^4_{\boldsymbol{\alpha}} = \begin{pmatrix} 0_\alpha \\ 1 \end{pmatrix},$$

$$(\boldsymbol{\vartheta}^1 + i\boldsymbol{\vartheta}^2)_{\boldsymbol{\alpha}} = \begin{pmatrix} (\vartheta^1 + i\vartheta^2)_\alpha \\ 0 \end{pmatrix} e^{-2imx^4}.$$

Coordinate $x^4$ parametrises circle of radius $\dfrac{1}{2m}$.
Coframe makes full turn in the $\boldsymbol{\vartheta}^1, \boldsymbol{\vartheta}^2$-plane as we move along the circle.

Axial torsion of extended coframe

$$\mathbf{T}^{\text{ax}} = \frac{1}{3}(\boldsymbol{\vartheta}^0 \wedge d\boldsymbol{\vartheta}^0 - \boldsymbol{\vartheta}^1 \wedge d\boldsymbol{\vartheta}^1 - \boldsymbol{\vartheta}^2 \wedge d\boldsymbol{\vartheta}^2 - \boldsymbol{\vartheta}^3 \wedge d\boldsymbol{\vartheta}^3 - \underbrace{\boldsymbol{\vartheta}^4 \wedge d\boldsymbol{\vartheta}^4}_{=0}).$$

T.Kaluza (1921): electromagnetism is a perturbation (shear) of the extended metric

$$\begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & -1 \end{pmatrix} \to \begin{pmatrix} g_{\alpha\beta} - \frac{1}{m^2}A_\alpha A_\beta & \frac{1}{m}A_\alpha \\ & \\ \frac{1}{m}A_\beta & -1 \end{pmatrix} =: \mathbf{g}_{\boldsymbol{\alpha\beta}}\,.$$

Note: with electromagnetism, extended coframe and extended metric no longer agree

$$\mathbf{g} \neq \boldsymbol{\vartheta}^0\otimes\boldsymbol{\vartheta}^0-\boldsymbol{\vartheta}^1\otimes\boldsymbol{\vartheta}^1-\boldsymbol{\vartheta}^2\otimes\boldsymbol{\vartheta}^2-\boldsymbol{\vartheta}^3\otimes\boldsymbol{\vartheta}^3-\boldsymbol{\vartheta}^4\otimes\boldsymbol{\vartheta}^4.$$

I don't care (in this talk).

Consider Lagrangian density

$$\boxed{L = \|\mathbf{T}^{\mathsf{ax}}\|^2\rho}$$

Dynamical variables: original (unextended) coframe $\vartheta$ and density $\rho$. Note: conformal invariance is destroyed by Kaluza–Klein extension.

*Question:* do I get the Dirac equation?

*Answer:* almost.

**Comparing my model with Dirac equation**

Where are the spinors in my model?

Geometric fact: in dimension 1+3

$$\boxed{\text{coframe } \vartheta \text{ and density } \rho \neq 0}$$

$$\Updownarrow$$

$$\boxed{\text{bispinor } \xi_a,\ \eta_{\dot a} \text{ subject to constraint } \operatorname{Im} \xi^a \bar\eta_a = 0}$$

My nonlinear field equations can be rewritten in the same language as the Dirac equation.

Remains to compare the two.

## Special case: no dependence on $x^3$

Suppose that $g$ and $A$ do not depend on $x^3$.

Suppose also that

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & 0 \\ g_{10} & g_{11} & g_{12} & 0 \\ g_{20} & g_{21} & g_{22} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad A_\alpha = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ 0 \end{pmatrix}.$$

Seek solutions which do not depend on $x^3$.

Problem simplifies: no need for a bispinor and no need for constraint $\operatorname{Im} \xi^a \bar{\eta}_a = 0$. Single spinor $\xi^a$ plays the role of dynamical variable.

Main result of this talk:

**Theorem 1** *In the special case when there is no dependence on* $x^3$ *my nonlinear field equations are equivalent to the Dirac equation.*

**Proof** My Lagrangian density $L$ factorises as

$$L(\xi) = \frac{L_+(\xi)L_-(\xi)}{L_+(\xi) - L_-(\xi)}$$

where

$$L_\pm(\xi) := \left[\frac{1}{2}(\bar{\xi}^{\dot{b}}\sigma^\alpha{}_{a\dot{b}}(i\nabla+A)_\alpha\xi^a - \xi^a\sigma^\alpha{}_{a\dot{b}}(i\nabla-A)_\alpha\bar{\xi}^{\dot{b}}) \pm m\sigma^3{}_{a\dot{b}}\xi^a\bar{\xi}^{\dot{b}}\right]\sqrt{|\det g|}\,.$$

Use also scaling covariance of Dirac Lagrangian:

$$L_\pm(e^h\xi) = e^{2h}L_\pm(\xi)$$

where $h$ is an arbitrary real scalar function. $\square$

## Summary

New mathematical model for the electron.

- Spacetime viewed as Cosserat continuum.
- Lagrangian chosen from the condition of conformal invariance.
- Mass and electromagnetic field incorporated via Kaluza–Klein extension.

## What is to be done?

Develop a proper spectral theory for my model.

## Spin-off

There is a class of beautiful nonlinear PDEs arising in Cosserat elasticity which has never been studied by analysts.