Massless Dirac equation as a special case of Cosserat elasticity

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Dirac's equation is a model for the electron and its antiparticle, positron.

Aim of talk: to **understand** Dirac's equation.

Formulating Dirac's equation requires:
(a) spinors,
(b) Pauli matrices,
(c) covariant differentiation.
Also, difficult to distinguish electron from positron.

My reinterpretation of Dirac's eq-n will require:
(a) differential forms,
(b) wedge product,
(c) exterior differentiation.
Also, easy to distinguish electron from positron.

Price I will pay: my model will be nonlinear.

Formulation of Dirac's equation

For simplicity will deal with *massless* Dirac equation. Model for the neutrino and antineutrino.

Will also assume metric to be flat (no gravity).

Unknown quantity is a 2-component spinor

$$\xi = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}.$$

Pair of "scalar" complex-valued functions of time t and Euclidean coordinates x^1, x^2, x^3.

Massless Dirac equation:

$$\partial_t \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} = \pm \begin{pmatrix} \partial_3 & \partial_1 + i\partial_2 \\ \partial_1 - i\partial_2 & -\partial_3 \end{pmatrix} \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}.$$

Equation transforms in a complicated way under rotations of coordinate system. Spinor is "square root" of a complex isotropic vector.

Describing a deformable continuous medium

(a) Classical elasticity: displacements only.

(b) Cosserat elasticity: displacements and rotations. See

E. Cosserat and F. Cosserat, *Théorie des Corps Déformables*, A. Hermann et Fils, Paris, 1909.

(c) Teleparallelism (absolute parallelism, fernparallelismus): rotations only.

Teleparallelism in Euclidean 3-space

Work in $\mathbb{R}^3$ equipped with standard metric

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and Cartesian coordinates x^α, $\alpha = 1, 2, 3$.

A *coframe* $\{\vartheta^1, \vartheta^2, \vartheta^3\}$ is a triad of covector fields satisfying metric constraint

$$g = \vartheta^1 \otimes \vartheta^1 + \vartheta^2 \otimes \vartheta^2 + \vartheta^3 \otimes \vartheta^3.$$

Same in plain English: a coframe is a field of orthonormal bases.

NB. Coframe lives separately from Cartesian coordinates (not aligned with coordinate lines).

Coframe will play the role of unknown quantity (dynamical variable).

Measure of deformation: the 3-form

$$T^{\mathsf{ax}} := \frac{1}{3}(\vartheta^1 \wedge d\vartheta^1 + \vartheta^2 \wedge d\vartheta^2 + \vartheta^3 \wedge d\vartheta^3).$$

Called "axial torsion of teleparallel connection".

The 3-form T^{ax} is conformally covariant. Let

$$\vartheta^j \mapsto e^h \vartheta^j, \qquad j = 1, 2, 3,$$

where h is arbitrary real scalar function. Then

$$g \mapsto e^{2h} g,$$

$$T^{\mathsf{ax}} \mapsto e^{2h} T^{\mathsf{ax}}$$

without the derivatives of h appearing.

My Lagrangian density

$$\boxed{L = \|T^{\mathrm{ax}}\|^2 \rho}$$

where ρ is an additional dynamical variable.

My Lagrangian is conformally invariant!

Action (variational functional) $\int L \, dx^1 dx^2 dx^3$.

Vary action with respect to coframe ϑ and density ρ to get Euler–Lagrange equations.

Difference with existing models

1. I assume metric to be fixed (prescribed).

2. My Lagrangian has never been considered.

Introducing time into my model

Define 2-form

$$\dot\vartheta := \vartheta^1 \wedge \dot\vartheta^1 + \vartheta^2 \wedge \dot\vartheta^2 + \vartheta^3 \wedge \dot\vartheta^3.$$

Note: $*\dot\vartheta$ is the vector of *angular velocity*.

$$\boxed{L = (\|\dot\vartheta\|^2 - \|T^{\mathrm{ax}}\|^2)\rho}$$

Model remains conformally invariant, only now in the Lorentzian sense.

Solving Euler–Lagrange equations

Switch to spinors:

$$\boxed{\text{coframe } \vartheta \text{ and density } \rho \neq 0}$$

$$\Updownarrow$$

$$\boxed{\text{nonvanishing spinor field } \xi \text{ modulo sign}}$$

Lagrangian density $L(\xi)$ is a rational function of ξ, $\bar\xi$ and partial derivatives of ξ, $\bar\xi$.

Look first for quasi-stationary solutions

$$\xi(t,x^1,x^2,x^3)=e^{-i\omega t}\eta(x^1,x^2,x^3),\quad \omega\neq 0. \quad (1)$$

Theorem 1 *Let ξ be of the form (1). Then my Euler–Lagrange equation is equivalent to a pair of massless Dirac equations.*

Proof My Lagrangian density L factorises as

$$L(\xi)=\frac{L_+(\xi)L_-(\xi)}{L_+(\xi)-L_-(\xi)}$$

where $L_\pm(\xi)$ are the Dirac Lagrangians. Use also scaling covariance of Dirac Lagrangians:

$$L_\pm(e^h\xi)=e^{2h}L_\pm(\xi)$$

where h is an arbitrary real scalar function. □

Special case of quasi-stationary solution

$$\xi(t,x^1,x^2,x^3)=e^{-i(\omega t+k\cdot x)}, \qquad \omega\neq 0.$$

This is a *plane wave*.

Corollary 1 *Plane wave solutions in my model are the same as for a pair of massless Dirac equations.*

Visualising plane wave solutions

Up to a rigid orthogonal transformation

$$\vartheta^1_\alpha=\begin{pmatrix}\cos\varphi\\ \pm\sin\varphi\\ 0\end{pmatrix}, \quad \vartheta^2_\alpha=\begin{pmatrix}\mp\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix}, \quad \vartheta^3_\alpha=\begin{pmatrix}0\\ 0\\ 1\end{pmatrix},$$

$$\rho=1,$$

where

$$\varphi:=2|\omega|(t+x^3).$$

Looking at a travelling wave of rotations.

Perturbations of plane waves

Idea: seek spinor field in the form

slowly varying amplitude $\times e^{-i(\omega t + k\cdot x)}$.

Deriving equation for perturbed plane wave:

- substitution $\xi \mapsto \xi e^{-i(\omega t + k\cdot x)}$ in Lagrangian;
- write down Euler–Lagrange equation;
- linearize Euler–Lagrange equation;
- drop second derivatives;
- substitution $\xi \mapsto \xi e^{i(\omega t + k\cdot x)}$ in equation.

Theorem 2 *Perturbations of plane wave solutions in my model are described by a pair of massless Dirac equations.*

Relativistic version of my model

Work in Minkowski space.

Coordinates x^α, $\alpha = 0, 1, 2, 3$.

Metric $g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

Coframe $\{\vartheta^0, \vartheta^1, \vartheta^2, \vartheta^3\}$.

$$g = \vartheta^0 \otimes \vartheta^0 - \vartheta^1 \otimes \vartheta^1 - \vartheta^2 \otimes \vartheta^2 - \vartheta^3 \otimes \vartheta^3.$$

$$T^{\mathrm{ax}} = \frac{1}{3}(\vartheta^0 \wedge d\vartheta^0 - \vartheta^1 \wedge d\vartheta^1 - \vartheta^2 \wedge d\vartheta^2 - \vartheta^3 \wedge d\vartheta^3).$$

Lagrangian density $L = \|T^{\mathrm{ax}}\|^2 \rho$.

Comparing the relativistic and nonrelativistic models

Relativistic model has 3 extra degrees of freedom (Lorentz boosts in 3 directions) and, consequently, 3 extra field equations.

Theorem 3 *At the asymptotic level (plane waves and their formal perturbations) the 3 extra field equations are automatically satisfied.*

My nonrelativistic model possesses relativistic invariance at the asymptotic level.

Incorporating mass m and electromagnetic (co)vector potential A

Kaluza–Klein extension.

Introduce 5th coordinate: $(x^0, x^1, x^2, x^3, \underline{x^4})$.

O.Klein (1926): mass m is prescribed oscillation $\xi \sim e^{-imx^4}$ along extra coordinate. Mass appears when we separate out the variable x^4.

T.Kaluza (1921): electromagnetism is a perturbation (shear) of the extended metric

$$\begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} g_{\alpha\beta} - \frac{1}{m^2} A_\alpha A_\beta & \frac{1}{m} A_\alpha \\ \\ \frac{1}{m} A_\beta & -1 \end{pmatrix}.$$

Mathematical difficulties

- Symbol of linearized system is degenerate (has determinant 0).

- Don't know how to handle situation $\rho = 0$.

- Cannot set problem rigorously in terms of function spaces.

- Cannot justify formal asymptotic analysis.

Summary

New mathematical model for fermions.

- Spacetime viewed as Cosserat continuum.
- Lagrangian chosen from the condition of conformal invariance.
- Mass and electromagnetic field incorporated via Kaluza–Klein extension.

What is to be done?

Perform mathematical analysis of my model.

Spin-off

There is a class of beautiful nonlinear PDEs arising in Cosserat elasticity which has never been studied by analysts.