Teleparallel reformulation of the Weyl equation

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10 July 2007

18th International Conference on General Relativity and Gravitation

Sydney Australia

Will study Weyl's equation (massless Dirac).

Formulating Weyls's equation requires:
(a) spinors,
(b) Pauli matrices,
(c) covariant differentiation.

My reformulation of Weyl's equation requires:
(a) differential forms,
(b) wedge product,
(c) exterior differentiation.

Traditional formulation of Weyl's equation

Work on a 4-manifold with prescribed Lorentzian metric g.

Dynamical variable is a 2-component spinor ξ.

Weyl's equation

$$i\sigma^{\alpha}{}_{a\dot{b}}\nabla_{\alpha}\xi^{a} = 0.$$

Weyl's Lagrangian

$$L_{\mathrm{Weyl}}(\xi) := \frac{i}{2}(\bar{\xi}^{\dot{b}}\sigma^{\alpha}{}_{a\dot{b}}\nabla_{\alpha}\xi^{a} - \xi^{a}\sigma^{\alpha}{}_{a\dot{b}}\nabla_{\alpha}\bar{\xi}^{\dot{b}}) * 1.$$

Teleparallel formulation of Weyl's equation

Work on a 4-manifold with prescribed Lorentzian metric g.

Dynamical variable is coframe ϑ^j, $j = 0, 1, 2, 3$, i.e. quartet of real covector fields satisfying metric constraint

$$g = o_{jk}\,\vartheta^j \otimes \vartheta^k$$

where $o_{jk} = o^{jk} := \mathrm{diag}(1, -1, -1, -1)$.

Notion of parallelism: each covector field ϑ^j, $j = 0, 1, 2, 3$, is parallel by definition.

Parallelism $\Longrightarrow$ connection. Curvature $R = 0$.

Field strength: torsion $T = o_{jk}\,\vartheta^j \otimes d\vartheta^k$.

Irreducible piece of field strength: axial (totally antisymmetric) torsion

$$T = \frac{1}{3} o_{jk}\,\vartheta^j \wedge d\vartheta^k.$$

Put $l = \vartheta^0 + \vartheta^3$ and define Lagrangian

$$\boxed{L = l \wedge T^{\text{axial}}}$$

Theorem 1 *Above Lagrangian is, up to a non-linear change of variable, Weyl's Lagrangian.*

Proof of Theorem 1 Perform transformation

$$\begin{pmatrix}\vartheta^0\\ \vartheta^1\\ \vartheta^2\\ \vartheta^3\end{pmatrix} \mapsto \begin{pmatrix} 1+\frac{1}{2}|f|^2 & \operatorname{Re} f & \operatorname{Im} f & \frac{1}{2}|f|^2 \\ \operatorname{Re} f & 1 & 0 & \operatorname{Re} f \\ \operatorname{Im} f & 0 & 1 & \operatorname{Im} f \\ -\frac{1}{2}|f|^2 & -\operatorname{Re} f & -\operatorname{Im} f & 1-\frac{1}{2}|f|^2 \end{pmatrix} \begin{pmatrix}\vartheta^0\\ \vartheta^1\\ \vartheta^2\\ \vartheta^3\end{pmatrix}$$

where $f : M \to \mathbb{C}$ is an arbitrary scalar function.

Metric and Lagrangian are invariant! Hence, solutions come in equivalence classes. Geometric meaning of these equivalence classes?

We are looking at an Abelian subgroup of the Lorentz group. Geometric fact: cosets of this subgroup can be identified with spinors.

Remains to perform very long calculation ... $\square$

Big worry

In my construction I took

$$l = \vartheta^0 + \vartheta^3$$

but I could have as well taken

$$l = l_j \vartheta^j$$

where the l_j are real constants satisfying

$$o^{jk} l_j l_k = 0.$$

Would have still gotten Weyl's equation!

This extra degree of freedom is worrying.

Constants l_j should have a physical meaning.

Origin of Lagrangian $L=l\wedge T^{\text{axial}}$

Consider Lagrangian

$$L=\|T^{\text{axial}}\|^2 * 1.$$

Let metric be Minkowski. Look for explicit solutions of the Euler–Lagrange equation.

Explicit solution:

$$\begin{pmatrix}\vartheta^0\\ \vartheta^1\\ \vartheta^2\\ \vartheta^3\end{pmatrix}=\begin{pmatrix}1&0&0&0\\ 0&\cos(x^0+x^3)&\pm\sin(x^0+x^3)&0\\ 0&\mp\sin(x^0+x^3)&\cos(x^0+x^3)&0\\ 0&0&0&1\end{pmatrix}\begin{pmatrix}\boldsymbol{\vartheta}^0\\ \boldsymbol{\vartheta}^1\\ \boldsymbol{\vartheta}^2\\ \boldsymbol{\vartheta}^3\end{pmatrix}$$

where $\boldsymbol{\vartheta}^j$ is a constant coframe.

Call this *plane wave solution* with *momentum*

$$l=\boldsymbol{\vartheta}^0+\boldsymbol{\vartheta}^3.$$

Can similarly write down plane wave solution with momentum $l=l_j\boldsymbol{\vartheta}^j$ where $o^{jk}l_jl_k=0$.

Now look for solutions that are not necessarily plane wave. Formal perturbation argument: the ϑ^j are no longer constant but slowly varying. Write linearized field equation dropping second derivatives. Choose convenient basis.

Theorem 2 *Formal perturbation argument gives the following equation in $\mathbb{C}^3$:*

$$\begin{pmatrix} \sigma^\alpha (i\nabla \pm \frac{1}{2}\, l)_\alpha & 0 \\ & 0 \\ 0 \quad 0 & 0 \end{pmatrix} \begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \end{pmatrix} = 0.$$

Proof of Theorem 2 Being written down ...□

Bottom line: I think that the “true” Lagrangian for the neutrino is

$$L = \|T^{\mathrm{axial}}\|^2 * 1.$$

Comparison with Maxwell's equation

	Maxwell's equation	My equation
Dynamical variable	Covector field A	Quartet of covector fields ϑ^j
Field strength	2-form dA	3-form T^{axial}
Lagrangian	$\|dA\|^2 * 1$	$\|T^{\text{axial}}\|^2 * 1$

Dirac's equation

Klein's interpretation of mass: prescribed oscillation along 5th coordinate. Separation of variables gives mass term.

Conjecture: my Lagrangian $L = \|T^{\text{axial}}\|^2 * 1$ in dimension 5 generates the Dirac equation. "Separation of variables" = one rotation of coframe as we move along the 5th coordinate.

Dirac's equation with electromagnetic field

Electromagnetism in Dirac's equation:

$$\nabla \to \nabla + iA.$$

Kaluza's interpretation of electromagnetism: perturbation (shear) of the extended metric

$$\begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & -1 \end{pmatrix} \to \begin{pmatrix} g_{\alpha\beta} - A_\alpha A_\beta & A_\alpha \\ A_\beta & -1 \end{pmatrix}.$$

Conjecture: Kaluza's construction works in my model to give Dirac's equation with electromagnetic field.

End goal of project

To derive all Quantum Electrodynamics from the Lagrangian

$$L = \|T^{\mathrm{axial}}\|^2 * 1.$$

Bibliography

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