Spectral theory of differential operators: what it is all about and what is its use

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Inaugural lecture

Free oscillations of a pendulum

$u(t)$ is horizontal displacement at time t.

Differential equation:

$$\frac{d^2u}{dt^2} = -\frac{g}{L}\,u\,,$$

where $g \approx 9.81\,\frac{\mathrm{m}}{\mathrm{sec}^2}\,, \quad L = 1\,\mathrm{m}\,.$

Solution to differential equation:

$$u(t) = A\sin(2\pi f t)$$

where $f = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \approx 0.5\,\frac{\text{cycles}}{\text{sec}}$ is the frequency.

We call f the **natural frequency** or **eigen-frequency** or **resonance frequency**.

Where does the "resonance" come from? If the pendulum is excited by a periodic external force $\sin(2\pi f_{\mathrm{e}}\,t)$ then $u(t) = \dfrac{\sin(2\pi f_{\mathrm{e}}\,t)}{4\pi^2(f^2 - f_{\mathrm{e}}^2)}$.

Vibration of a string

$u(x,t)$ is deflection at point x and time t

Differential equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

subject to boundary conditions

$$u|_{x=0} = u|_{x=1} = 0.$$

Look for solutions which oscillate in time:

$$u(x,t) = v(x)\sin(2\pi f t)\,.$$

The natural frequencies are

$$f_n = \frac{n}{2}\,, \qquad n = 1, 2, \ldots$$

f_1 is called **fundamental tone**.

$f_2, f_3, \ldots$ are called **overtones**.

More complicated spectral problems

1. Vibration of a membrane.

2. Acoustic or electromagnetic resonator.

3. Vibration of an elastic body.

4. Vibration of a thin elastic shell.

5. Vibration of a shell contacting fluid.

6. "Trapped modes" in waveguides.

7. "Fractal drums".

8. Energy levels of electrons in an atom.

9. Energy levels of electrons in a molecule.

10. Energy levels of electrons in a crystal.

Etc etc.

Vibration of air in a resonator

$u(x, y, z, t)$ is pressure at point (x, y, z) and time t.

Differential equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial t^2}$$

subject to boundary condition $u = 0$ on walls.

Spectrum of natural frequencies $f_1, f_2, \ldots$

Rayleigh–Jeans formula (1905): for large n

$$f_n \approx \left(\frac{3n}{4\pi V}\right)^{1/3}$$

where V is the volume of the resonator.

Weyl's conjecture (1913)

It may be possible to get better asymptotic formulae for natural frequencies, i.e., formulae which take account of boundary phenomena.

Example: for an acoustic resonator

$$f_n \approx \left(\frac{3n}{4\pi V}\right)^{1/3} + \frac{S}{16V}\,.$$

Here S is the surface area of the walls.

Weyl's conjecture was proved for second order equations by V.Ivrii and R.Melrose (1980).

My contribution: proof of Weyl's conjecture for higher order equations.

The asymptotic distribution of eigenvalues of partial differential operators, AMS, 1997 (hardcover), 1998 (softcover). Jointly with Yu.Safarov.

Vibration of a plate

$u(x, y, t)$ is deflection at point (x, y) and time t.

Differential equation:

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^2} = \frac{\partial^2 u}{\partial t^2}$$

subject to boundary conditions $u = \frac{\partial u}{\partial n} = 0$ on the edge.

My formula (1987):

$$f_n \approx \frac{2n}{S} + 2.813\, L \sqrt{\frac{n}{S^3}}\,.$$

Here S is the surface area of the plate and L is the length of the edge.

First term found by R.Courant (1922).

1st experience of applying for grant in UK

Letter from SERC dated 19 November 1992:

Perhaps the applicant could have spelled out in greater detail the actual application of the proposed analysis to the problems (or, rather, their solutions) alluded to in the proposal which are of practical significance.

Recent years: obsession with Dirac's eq-n

Dirac's equation describes electron in electromagnetic field. Could not make sense of it.

Part of bigger problem.

Two theories forming the basis of modern theoretical physics:

- General theory of relativity (what happens on an astronomical scale).

- Quantum mechanics (what happens on a microscopic scale).

This . . . reminds one of a palace which has two wings; the left wing is built of imperishable marble, the right wing of inferior wood.

A.Einstein

Teleparallelism

Basic assumption: every point of the space-time continuum can rotate and rotations of different points are independent.

Rotation of a point is described by attaching to it an orthonormal basis of covectors

$$\vartheta^0,\ \vartheta^1,\ \vartheta^2,\ \vartheta^3.$$

Field strength is a 3-form called *axial torsion*

$$T^{\mathrm{axial}} = \frac{1}{3}(\vartheta^0\wedge d\vartheta^0 - \vartheta^1\wedge d\vartheta^1 - \vartheta^2\wedge d\vartheta^2 - \vartheta^3\wedge d\vartheta^3).$$

Lagrangian density is

$$L = \|T^{\mathrm{axial}}\|^2 \sqrt{|\det g|}\ .$$

End goal of project: to derive all Quantum Electrodynamics from the above Lagrangian.