Teleparallelism: difficult word but simple way of reinterpreting the Dirac equation

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Dirac's equation is a system of 4 homogeneous linear PDEs for 4 complex unknowns in dimension 1+3.

Formulating Dirac's equation requires:
(a) spinors,
(b) Pauli matrices,
(c) covariant derivative.

My reformulation of Dirac's equation requires:
(a) differential forms,
(b) wedge product,
(c) exterior derivative.

Describing a deformable continuous medium

(a) Classical elasticity: displacements only.

(b) Cosserat elasticity (multipolar elasticity): displacements and rotations. See, for example, Truesdell's *First course in rational continuum mechanics*.

(c) Teleparallelism (absolute parallelism): rotations only.

Teleparallelism in Euclidean 3-space

Cartesian coordinates x^α, $\alpha = 1, 2, 3$.

Euclidean metric $g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Euclidean distance squared $= g_{\alpha\beta} dx^\alpha dx^\beta$.

Coframe $\{\vartheta^1, \vartheta^2, \vartheta^3\}$: triad of covector fields satisfying metric constraint

$$g = \vartheta^1 \otimes \vartheta^1 + \vartheta^2 \otimes \vartheta^2 + \vartheta^3 \otimes \vartheta^3.$$

NB. Coframe lives separately from Cartesian coordinates (not aligned with coordinate lines).

Notion of parallelism: each covector field ϑ^k, $k = 1, 2, 3$, is parallel by definition.

Parallelism $\Longrightarrow$ connection. Curvature $R = 0$.

Terminology: if $R = 0$ spacetime is called *flat* or *teleparallel* or *Weitzenböck*.

Field strength: torsion

$$T = \vartheta^1 \otimes d\vartheta^1 + \vartheta^2 \otimes d\vartheta^2 + \vartheta^3 \otimes d\vartheta^3.$$

Analogue of strain tensor.

Irreducible piece of field strength: axial (totally antisymmetric) torsion

$$T^{\mathrm{axial}} = \frac{1}{3}(\vartheta^1 \wedge d\vartheta^1 + \vartheta^2 \wedge d\vartheta^2 + \vartheta^3 \wedge d\vartheta^3).$$

Analogue of shear.

Possible Lagrangians

$$L = T^{\mathrm{axial}}, \tag{1}$$

$$L = \|T^{\mathrm{axial}}\|^2 * 1. \tag{2}$$

Action (variational functional) $\int L\,.$

Vary action with respect to coframe subject to metric constraint to get Euler–Lagrange equation, a nonlinear PDE for unknown coframe.

Lagrangian (1) gives first order equation, Lagrangian (2) gives second order equation.

Weyl's equation (massless Dirac)

Dimension is now $1+3$.

Coframe $\{\vartheta^0, \vartheta^1, \vartheta^2, \vartheta^3\}$.

$$g = \vartheta^0 \otimes \vartheta^0 - \vartheta^1 \otimes \vartheta^1 - \vartheta^2 \otimes \vartheta^2 - \vartheta^3 \otimes \vartheta^3.$$

$$T = \vartheta^0 \otimes d\vartheta^0 - \vartheta^1 \otimes d\vartheta^1 - \vartheta^2 \otimes d\vartheta^2 - \vartheta^3 \otimes d\vartheta^3.$$

$$T^{\mathrm{axial}} = \frac{1}{3}(\vartheta^0 \wedge d\vartheta^0 - \vartheta^1 \wedge d\vartheta^1 - \vartheta^2 \wedge d\vartheta^2 - \vartheta^3 \wedge d\vartheta^3).$$

Put $l = \vartheta^0 + \vartheta^3$ and define Lagrangian

$$\boxed{L = l \wedge T^{\mathrm{axial}}}$$

Theorem 1 *The corresponding Euler–Lagrange eq-n is, up to change of variable, Weyl's eq-n.*

Proof of Theorem 1 Perform transformation

$$\begin{pmatrix}\vartheta^0\\ \vartheta^1\\ \vartheta^2\\ \vartheta^3\end{pmatrix}\mapsto\begin{pmatrix}1+\frac{1}{2}|f|^2 & \mathrm{Re}\,f & \mathrm{Im}\,f & \frac{1}{2}|f|^2\\ \mathrm{Re}\,f & 1 & 0 & \mathrm{Re}\,f\\ \mathrm{Im}\,f & 0 & 1 & \mathrm{Im}\,f\\ -\frac{1}{2}|f|^2 & -\mathrm{Re}\,f & -\mathrm{Im}\,f & 1-\frac{1}{2}|f|^2\end{pmatrix}\begin{pmatrix}\vartheta^0\\ \vartheta^1\\ \vartheta^2\\ \vartheta^3\end{pmatrix}$$

where $f : M \to \mathbb{C}$ is an arbitrary scalar function.

Metric and Lagrangian are invariant! Hence, solutions come in equivalence classes. Geometric meaning of these equivalence classes?

We are looking at an Abelian subgroup of the Lorentz group. Geometric fact: cosets of this subgroup can be identified with spinors.

Details in *Phys. Rev.* **D75**, 025006 (2007).

Origin of Lagrangian $L = l \wedge T^{\mathrm{axial}}$

Consider Lagrangian

$$L = \|T^{\mathrm{axial}}\|^2 * 1 \tag{3}$$

(special case of Cosserat elasticity).

Let metric be Minkowski and let $\{\boldsymbol{\vartheta}^0, \boldsymbol{\vartheta}^1, \boldsymbol{\vartheta}^2, \boldsymbol{\vartheta}^3\}$ be a constant coframe. Plane wave solution:

$$\begin{pmatrix}\vartheta^0\\ \vartheta^1\\ \vartheta^2\\ \vartheta^3\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \cos(x^0+x^3) & \pm\sin(x^0+x^3) & 0\\ 0 & \mp\sin(x^0+x^3) & \cos(x^0+x^3) & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \begin{pmatrix}\boldsymbol{\vartheta}^0\\ \boldsymbol{\vartheta}^1\\ \boldsymbol{\vartheta}^2\\ \boldsymbol{\vartheta}^3\end{pmatrix}$$

Look for solutions which are not necessarily plane wave, with metric not necessarily Minkowski.

Formal perturbation argument: linearization of Lagr-n (3) about plane wave gives Lagr-n

$$L = l \wedge T^{\mathrm{axial}} \tag{4}$$

where $l = \vartheta^0 + \vartheta^3$.

Comparison with Maxwell's equation

	Maxwell's equation	My equation
Dynamical variable	Covector field A	Quartet of covector fields $\{\vartheta^0, \vartheta^1, \vartheta^2, \vartheta^3\}$
Field strength	2-form dA	3-form T^{axial}
Lagrangian	$\|dA\|^2 * 1$	$\|T^{\mathrm{axial}}\|^2 * 1$

Dirac's equation

What is the geometric meaning of mass m?

Klein's interpretation of mass, as illustrated by Klein–Gordon equation in Minkowski space

$$(\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2)\psi + m^2\psi = 0.$$

Introduce 5th coordinate: $(x^0, x^1, x^2, x^3, \underline{x^4})$.

Consider wave equation in extended space

$$(\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2 - \partial_4^2)\psi = 0.$$

Separate out the variable x^4: $\psi \sim e^{-imx^4}$.

Conjecture: Klein's construction works in my model to give Dirac's equation with mass. "Separation of variables" means one full rotation of coframe as we move along the 5th coordinate.

Dirac's equation with electromagnetic field

Electromagnetism in Dirac's equation:

$$\nabla \to \nabla + iA.$$

Kaluza's interpretation of electromagnetism: perturbation (shear) of the extended metric

$$\begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & -1 \end{pmatrix} \;\to\; \begin{pmatrix} g_{\alpha\beta} - A_\alpha A_\beta & A_\alpha \\ A_\beta & -1 \end{pmatrix}.$$

NB. Kaluza did not devise above substitution for use in quantum mechanics. In fact, at the time (1921) quantum mechanics didn't exist.

Conjecture: Kaluza's construction works in my model to give Dirac's equation with electromagnetic field.

Summary

Result 1. New representation for the Weyl Lagrangian (massless Dirac Lagrangian):

$$L = l \wedge T^{\mathrm{axial}} \tag{5}$$

where $l = \vartheta^0 + \vartheta^3$.

Result 2. The Lagrangian (5) is the result of formal linearization of the Lagrangian

$$L = \|T^{\mathrm{axial}}\|^2 * 1 \tag{6}$$

about a plane wave.

Conjecture. The Dirac Lagrangian can also be derived from the Lagrangian (6) with a Kaluza–Klein extension.