Teleparallelism: difficult word but simple way of reinterpreting the Dirac equation

Dmitri Vassiliev

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Will study Weyl's equation (massless Dirac).

Weyl's equation is a system of 2 homogeneous linear partial differential equations for 2 complex unknowns in dimension 1+3.

Formulating Weyls's equation requires:
(a) spinors,
(b) Pauli matrices,
(c) covariant differentiation.

My reformulation of Weyl's equation requires:
(a) differential forms,
(b) wedge product,
(c) exterior differentiation.

Traditional formulation of Weyl's equation

Work on 4-manifold with Lorentzian metric $g_{\alpha\beta}$

Deal with 2-component spinors ξ_a or $\eta_{\dot{b}}$.

Raise and lower spinor indices using "metric spinor" $\epsilon_{ab} = \epsilon_{\dot{a}\dot{b}} = \epsilon^{ab} = \epsilon^{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Pauli matrices $\sigma^{\alpha}{}_{a\dot{b}}$ defined by condition

$$\sigma^{\alpha}{}_{a\dot{b}}\sigma^{\beta c\dot{b}} + \sigma^{\beta}{}_{a\dot{b}}\sigma^{\alpha c\dot{b}} = 2g^{\alpha\beta}\delta_a{}^c.$$

Covariant derivative of a spinor field

$$\nabla_\mu \xi^a = \partial_\mu \xi^a + \Gamma^a{}_{\mu b}\xi^b$$

where

$$\Gamma^a{}_{\mu b} = \frac{1}{4}\sigma_\alpha{}^{a\dot{c}}\left(\partial_\mu \sigma^\alpha{}_{b\dot{c}} + \left\{\begin{matrix} \alpha \\ \mu\beta \end{matrix}\right\}\sigma^\beta{}_{b\dot{c}}\right),$$

$$\left\{\begin{matrix} \alpha \\ \mu\beta \end{matrix}\right\} = \frac{1}{2}g^{\alpha\kappa}(\partial_\mu g_{\beta\kappa} + \partial_\beta g_{\mu\kappa} - \partial_\kappa g_{\mu\beta}).$$

Weyl's equation

$$i\sigma^{\alpha}{}_{a\dot{b}}\nabla_{\alpha}\xi^{a}=0.$$

Weyl's Lagrangian

$$L_{\mathsf{Weyl}}(\xi):=\frac{i}{2}(\bar{\xi}^{\dot{b}}\sigma^{\alpha}{}_{a\dot{b}}\nabla_{\alpha}\xi^{a}-\xi^{a}\sigma^{\alpha}{}_{a\dot{b}}\nabla_{\alpha}\bar{\xi}^{\dot{b}})*1.$$

Describing a deformable continuous medium

(a) Classical elasticity: displacements only.

(b) Cosserat elasticity (multipolar elasticity): displacements and rotations. See, for example, Truesdell's *First course in rational continuum mechanics*.

(c) Teleparallelism (absolute parallelism): rotations only.

Teleparallelism in Euclidean 3-space

Cartesian coordinates x^α, $\alpha = 1, 2, 3$.

Euclidean metric $g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Euclidean distance squared $= g_{\alpha\beta} dx^\alpha dx^\beta$.

Coframe $\{\vartheta^1, \vartheta^2, \vartheta^3\}$: triad of covector fields satisfying metric constraint

$$g = \vartheta^1 \otimes \vartheta^1 + \vartheta^2 \otimes \vartheta^2 + \vartheta^3 \otimes \vartheta^3.$$

NB. Coframe lives separately from Cartesian coordinates (not aligned with coordinate lines).

Notion of parallelism: each covector field ϑ^k, $k = 1, 2, 3$, is parallel by definition.

Parallelism $\Longrightarrow$ connection. Curvature $R = 0$.

Terminology: if $R = 0$ spacetime is called *flat* or *teleparallel* or *Weitzenböck*.

Field strength: torsion

$$T = \vartheta^1 \otimes d\vartheta^1 + \vartheta^2 \otimes d\vartheta^2 + \vartheta^3 \otimes d\vartheta^3.$$

Analogue of strain tensor.

Irreducible piece of field strength: axial (totally antisymmetric) torsion

$$T^{\text{axial}} = \frac{1}{3}(\vartheta^1 \wedge d\vartheta^1 + \vartheta^2 \wedge d\vartheta^2 + \vartheta^3 \wedge d\vartheta^3).$$

Analogue of shear.

Possible Lagrangians

$$L = T^{\text{axial}}, \tag{1}$$
$$L = \|T^{\text{axial}}\|^2 * 1. \tag{2}$$

Action (variational functional) $\int L\,.$

Vary action with respect to coframe subject to metric constraint to get Euler–Lagrange equation, a nonlinear PDE for unknown coframe.

Lagrangian (1) gives first order equation, Lagrangian (2) gives second order equation.

Teleparallel formulation of Weyl's equation

Dimension is now $1+3$.

Coframe $\{\vartheta^0, \vartheta^1, \vartheta^2, \vartheta^3\}$.

$$g = \vartheta^0 \otimes \vartheta^0 - \vartheta^1 \otimes \vartheta^1 - \vartheta^2 \otimes \vartheta^2 - \vartheta^3 \otimes \vartheta^3.$$

$$T = \vartheta^0 \otimes d\vartheta^0 - \vartheta^1 \otimes d\vartheta^1 - \vartheta^2 \otimes d\vartheta^2 - \vartheta^3 \otimes d\vartheta^3.$$

$$T^{\text{axial}} = \frac{1}{3}(\vartheta^0 \wedge d\vartheta^0 - \vartheta^1 \wedge d\vartheta^1 - \vartheta^2 \wedge d\vartheta^2 - \vartheta^3 \wedge d\vartheta^3).$$

Put $l = \vartheta^0 + \vartheta^3$ and define Lagrangian

$$\boxed{L = l \wedge T^{\text{axial}}}$$

Theorem 1 *The above Lagrangian is, up to change of variable, Weyl's Lagrangian.*

Proof of Theorem 1 Perform transformation

$$\begin{pmatrix}\vartheta^0\\ \vartheta^1\\ \vartheta^2\\ \vartheta^3\end{pmatrix}\mapsto\begin{pmatrix}1+\frac{1}{2}|f|^2 & \mathrm{Re}\,f & \mathrm{Im}\,f & \frac{1}{2}|f|^2\\ \mathrm{Re}\,f & 1 & 0 & \mathrm{Re}\,f\\ \mathrm{Im}\,f & 0 & 1 & \mathrm{Im}\,f\\ -\frac{1}{2}|f|^2 & -\mathrm{Re}\,f & -\mathrm{Im}\,f & 1-\frac{1}{2}|f|^2\end{pmatrix}\begin{pmatrix}\vartheta^0\\ \vartheta^1\\ \vartheta^2\\ \vartheta^3\end{pmatrix}$$

where $f: M \to \mathbb{C}$ is an arbitrary scalar function.

Metric and Lagrangian are invariant! Hence, solutions come in equivalence classes. Geometric meaning of these equivalence classes?

We are looking at an Abelian subgroup of the Lorentz group. Geometric fact: cosets of this subgroup can be identified with spinors.

Remains to perform very long calculation ... □

Big worry

In my construction I took

$$l = \vartheta^0 + \vartheta^3$$

but I could have as well taken

$$l = l_0\vartheta^0 + l_1\vartheta^1 + l_2\vartheta^2 + l_3\vartheta^3$$

where the l_j are real constants satisfying

$$(l_0)^2 - (l_1)^2 - (l_2)^2 - (l_3)^2 = 0.$$

Would have still gotten Weyl's equation!

This extra degree of freedom is worrying.

Constants l_j should have a physical meaning.

Origin of Lagrangian $L = l \wedge T^{\text{axial}}$

Consider Lagrangian

$$L = \|T^{\text{axial}}\|^2 * 1$$

(special case of Cosserat elasticity).

Let metric be Minkowski. Look for explicit solutions of the Euler–Lagrange equation.

Explicit solution:

$$\begin{pmatrix}\vartheta^0\\ \vartheta^1\\ \vartheta^2\\ \vartheta^3\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \cos(x^0+x^3) & \pm\sin(x^0+x^3) & 0\\ 0 & \mp\sin(x^0+x^3) & \cos(x^0+x^3) & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\boldsymbol{\vartheta}^0\\ \boldsymbol{\vartheta}^1\\ \boldsymbol{\vartheta}^2\\ \boldsymbol{\vartheta}^3\end{pmatrix}$$

where $\{\boldsymbol{\vartheta}^0, \boldsymbol{\vartheta}^1, \boldsymbol{\vartheta}^2, \boldsymbol{\vartheta}^3\}$ is a constant coframe.

Call this *plane wave solution* with *momentum*

$$\boldsymbol{\vartheta}^0 + \boldsymbol{\vartheta}^3.$$

Can similarly write down plane wave solution with momentum $l_0\boldsymbol{\vartheta}^0 + l_1\boldsymbol{\vartheta}^1 + l_2\boldsymbol{\vartheta}^2 + l_3\boldsymbol{\vartheta}^3$ where $(l_0)^2 - (l_1)^2 - (l_2)^2 - (l_3)^2 = 0$.

Look for solutions that are not necessarily plane wave, with metric not necessarily Minkowski.

Formal perturbation argument.

Observe that for a plane wave solution

$$T^{\text{axial}} = \pm\frac{2}{3} * l \tag{3}$$

where $l = l_0\vartheta^0 + l_1\vartheta^1 + l_2\vartheta^2 + l_3\vartheta^3$.

Now linearize the quadratic Lagrangian

$$L = \|T^{\text{axial}}\|^2 * 1 \tag{4}$$

at the point (3).

Theorem 2 *My original Lagrangian*

$$L = l \wedge T^{\text{axial}}$$

is the linearization of (4).

Should really be looking at the Lagrangain (4)!

Comparison with Maxwell's equation

	Maxwell's equation	My equation
Dynamical variable	Covector field A	Quartet of covector fields $\{\vartheta^0, \vartheta^1, \vartheta^2, \vartheta^3\}$
Field strength	2-form dA	3-form T^{axial}
Lagrangian	$\|dA\|^2 * 1$	$\|T^{\text{axial}}\|^2 * 1$

Summary

Result 1. New representation for the Weyl Lagrangian (massless Dirac Lagrangian):

$$L = l \wedge T^{\text{axial}}$$

where $l = \vartheta^0 + \vartheta^3$.

Result 2. The above Lagrangian is the formal linearization of the Lagrangian

$$L = \|T^{\text{axial}}\|^2 * 1$$

about a plane wave solution.

Bibliography

[1] *Phys. Rev.* **D75**, 025006 (2007).

[2] Preprint gr-qc/0702020.