The Pseudoinstanton – a Lorentzian Analogue of the Instanton

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In abstract Yang–Mills theory the standard instanton construction relies on the Hodge star having real eigenvalues which makes it inapplicable in the Lorentzian case. We show that for the affine connection (i.e. connection on vectors) an instanton-type construction can be carried out in the Lorentzian setting. Namely, the Lorentzian analogue of an instanton is a metric compatible connection whose curvature is irreducible and simple ("pseudoinstanton"). We prove that a pseudoinstanton is a solution of the Yang–Mills equation for the affine connection. In fact, we prove a much stronger result: a pseudoinstanton is a stationary point of any Lorentz-invariant quadratic action with respect to the independent variation of the metric and the connection. We present examples of pseudoinstantons and discuss them within the context of non-Riemannian theories of gravity.

1 Mathematical model

We consider space-time to be a connected real 4-manifold M equipped with a Lorentzian metric g and an affine connection Γ. The 10 independent components of the (symmetric) metric tensor $g_{\mu\nu}$ and the 64 connection coefficients ${\Gamma^\lambda}_{\mu\nu}$ are the unknowns of our theory. This approach is known as metric-affine gravity. Its origins lie in the works of authors such as É. Cartan, A.S. Eddington, A. Einstein, T. Levi-Civita, E. Schrödinger and H. Weyl. A review of the more recent work in this area can be found in [1].

We define our action as

$$S := \int q(R) \tag{1}$$

where q is an $\mathrm{O}(1,3)$-invariant quadratic form on curvature R. Independent variation of the metric g and the connection Γ produces Euler–Lagrange equations which we will write symbolically as

$$\partial S/\partial g = 0, \tag{2}$$

$$\partial S/\partial \Gamma = 0. \tag{3}$$

Our objective is the study of the combined system of field equations (2), (3). This is a system of $10+64$ real nonlinear partial differential equations with $10+64$ real unknowns.

Our motivation comes from Yang–Mills theory. The Yang–Mills action for the affine connection is a special case of (1) with

$$q(R) = q_{\mathrm{YM}}(R) := {R^\kappa}_{\lambda\mu\nu}\, {R^\lambda}_\kappa{}^{\mu\nu}. \tag{4}$$

With this choice of q equation (3) is the Yang–Mills equation for the affine connection. There is a substantial bibliography devoted to the study of the system (2), (3) in the special case (4); see, for example, references in [2].

The idea of modelling gravity by means of a quadratic action goes back to H. Weyl, see end of his paper [3]. Weyl also pointed out that such an action should contain all possible invariant quadratic combinations of curvature, say, the square of Ricci curvature, the square of scalar curvature, etc. It turns out (see Appendix B.4 in [1]) that in the metric-affine setting curvature has 11 irreducible pieces. There are 16 ways of squaring these irreducible pieces to a scalar. The reason why the number of different invariant quadratic combinations is greater than the number of pieces is that some of the irreducible subspaces of curvature are isomorphic. Namely, all three 6-dimensional subspaces are isomorphic, and there are also two pairs of isomorphic 9-dimensional subspaces. The explicit formula for a general $O(1,3)$-invariant quadratic form $q(R)$ with 16 coupling constants is given in Appendix B of [4].

The paper has the following structure. In Section 3 we state and prove our main result, Theorem 1. In Sections 4 and 5 we use this theorem to construct solutions to our field equations (2), (3). Finally, in Section 6 we discuss the possible physical interpretation of our solutions.

2 Notation

Our notation follows [2, 5]. In particular, we denote local coordinates by x^μ, $\mu = 0,1,2,3$, and write $\partial_\mu := \partial/\partial x^\mu$. We define the covariant derivative of a vector function as $\nabla_\mu v^\lambda := \partial_\mu v^\lambda + \Gamma^\lambda{}_{\mu\nu} v^\nu$, torsion as $T^\lambda{}_{\mu\nu} := \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}$, curvature as $R^\kappa{}_{\lambda\mu\nu} := \partial_\mu \Gamma^\kappa{}_{\nu\lambda} - \partial_\nu \Gamma^\kappa{}_{\mu\lambda} + \Gamma^\kappa{}_{\mu\eta}\Gamma^\eta{}_{\nu\lambda} - \Gamma^\kappa{}_{\nu\eta}\Gamma^\eta{}_{\mu\lambda}$, and Ricci curvature as $Ric_{\lambda\nu} := R^\kappa{}_{\lambda\kappa\nu}$. Given a scalar function $f : M \to \mathbb{R}$ we write for brevity $\int f := \int_M f \sqrt{|\det g|}\, dx^0 dx^1 dx^2 dx^3$ where $\det g := \det(g_{\mu\nu})$. The Christoffel symbol is $\left\{{\lambda \atop \mu\nu}\right\} := \frac{1}{2} g^{\lambda\kappa}(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu})$. We define the action of the Hodge star on a rank q antisymmetric tensor as $(*Q)_{\mu_{q+1}\ldots\mu_4} := (q!)^{-1}\sqrt{|\det g|}\, Q^{\mu_1\ldots\mu_q}\varepsilon_{\mu_1\ldots\mu_4}$ where ε is the totally antisymmetric quantity.

3 Main result

The following definition plays a crucial role in our construction.

Definition 1. We call a space-time $\{M, g, \Gamma\}$ a *pseudoinstanton* if the connection is metric compatible and curvature is irreducible and simple.

Here irreducibility of curvature means that all irreducible pieces but one are identically zero. Simplicity means that the given irreducible subspace is not isomorphic to any other irreducible subspace. Metric compatibility means, as usual, that $\nabla g \equiv 0$.

It is easy to see that there are only three possible types of pseudoinstantons:

- *scalar* pseudoinstanton (all pieces of curvature apart from the scalar piece are identically zero),
- *pseudoscalar* pseudoinstanton (all pieces of curvature apart from the pseudoscalar piece are identically zero), and
- *Weyl* pseudoinstanton (all pieces of curvature apart from the Weyl piece are identically zero).

Here scalar, pseudoscalar, and Weyl curvatures are defined by the conditions

$$R_{\kappa\lambda\mu\nu} \sim g_{\kappa\mu}g_{\lambda\nu} - g_{\lambda\mu}g_{\kappa\nu}, \tag{5}$$

$$R_{\kappa\lambda\mu\nu} \sim \varepsilon_{\kappa\lambda\mu\nu}, \tag{6}$$

$$R_{\kappa\lambda\mu\nu} = R_{\mu\nu\kappa\lambda}, \qquad Ric = 0, \qquad \varepsilon^{\kappa\lambda\mu\nu}R_{\kappa\lambda\mu\nu} = 0 \tag{7}$$

respectively, with $\sim$ standing for proportionality.

Our main result is

Theorem 1. *A pseudoinstanton is a solution of the field equations* (2), (3).

Proof. Let $R_{\rm pseudo}$ be the irreducible piece of curvature corresponding to the type of our pseudoinstanton; that is, $R_{\rm pseudo}$ denotes scalar, pseudoscalar, or Weyl curvature depending on whether we are proving Theorem 1 for a scalar, pseudoscalar, or Weyl pseudoinstanton. Then for *any* curvature R we have $q(R)=q(R_{\rm pseudo})+q(R-R_{\rm pseudo})$. Here we used the fact that the piece $R_{\rm pseudo}$ is simple: if not, then we would have cross-over terms of the type $R_{\rm pseudo}\times(R-R_{\rm pseudo})$.

When we start our variation from a space-time with $R-R_{\rm pseudo}\equiv 0$ the resulting variation of $\int q(R-R_{\rm pseudo})$ is zero. Thus, the proof of Theorem 1 reduces to proving that our pseudoinstanton is a stationary point of the action $\int q(R_{\rm pseudo})$. But examination of the explicit formula for $q(R)$ given in Appendix B of [4] shows that $q(R_{\rm pseudo})=\mathrm{const}(R_{\rm pseudo},R_{\rm pseudo})_{\rm YM}$ where $(R,Q)_{\rm YM}:=R^{\kappa}{}_{\lambda\mu\nu}Q^{\lambda}{}_{\kappa}{}^{\mu\nu}$ is the Yang–Mills inner product on curvatures. Thus, the action $\int q(R_{\rm pseudo})$ is of the type studied in [2] and the result follows from Theorem 2.1 of that paper. ■

4 Riemannian pseudoinstantons

Definition 2. We call a space-time $\{M,g,\Gamma\}$ *Riemannian* if the connection is Levi-Civita (i.e. $\Gamma^{\lambda}{}_{\mu\nu}=\left\{{\lambda \atop \mu\nu}\right\}$), and *non-Riemannian* otherwise.

Note that the word "Riemannian" has a different meaning in mathematics and theoretical physics. In mathematical literature the connection is usually Levi-Civita by default and "Riemannian" indicates that the metric is definite, whereas in theoretical physics literature the metric is usually Lorentzian by default (as it is in our paper) and "Riemannian" indicates that the connection is Levi-Civita. In Definition 2 we adopt the theoretical physics terminology.

In the Riemannian case the first and third conditions (7) are automatically fulfilled, so a Riemannian space-time is a Weyl pseudoinstanton if and only if

$$Ric=0. \tag{8}$$

Therefore, according to Theorem 1, Riemannian space-times satisfying the vacuum Einstein equation (8) are solutions to our field equations (2), (3).

The above argument demonstrates both the power and the limitations of the pseudoinstanton technique. This technique allowed us to obtain an important class of solutions without having to write down explicitly the field equations. On the other hand, it did not give us all the Riemannian solutions: it is known [4] that Einstein spaces with arbitrary cosmological constant are solutions as are pp-spaces with parallel Ricci curvature, but such space-times are not necessarily pseudoinstantons.

5 Non-Riemannian pseudoinstanton

We know only one non-Riemannian solution to our field equations (2), (3). It is constructed as follows.

Let us define Minkowski space $\mathbb{M}^4$ as a real 4-manifold equipped with global coordinates (x^0,x^1,x^2,x^3) and metric $g_{\mu\nu}=\mathrm{diag}(+1,-1,-1,-1)$. Working in $\mathbb{M}^4$, let us consider a complex-valued vector field

$$A(x)=a\,\mathrm{e}^{-\mathrm{i}l\cdot x} \tag{9}$$

which is a plane wave solution of the polarized Maxwell equation

$$*dA=\pm\mathrm{i}dA. \tag{10}$$

Define torsion

$$T = \frac{1}{2}\,\mathrm{Re}(A \otimes dA) = -\frac{1}{2}\,\mathrm{Re}\,\left(\mathrm{i}\,a \otimes (l \wedge a)\,\mathrm{e}^{-2\mathrm{i}l\cdot x}\right) \tag{11}$$

and let Γ be the corresponding metric compatible connection. Straightforward calculations give the following explicit formula for curvature:

$$R = \mathrm{Re}\,(dA \otimes dA) = -\,\mathrm{Re}\left((l \wedge a) \otimes (l \wedge a)\,\mathrm{e}^{-2\mathrm{i}l\cdot x}\right). \tag{12}$$

It is easy to see that formulae (9), (10), (12) imply (7). Therefore, the space-time $\{\mathbb{M}^4, \Gamma\}$ is a Weyl pseudoinstanton, and hence, by Theorem 1, a solution of our field equations (2), (3).

For the Yang–Mills case (4) the "torsion wave" solution described above was first obtained by Singh and Griffiths: see last paragraph of Section 5 in [6] and put $k = 0$, $N = \mathrm{const}\;\mathrm{e}^{-2\mathrm{i}l\cdot x}$. Our contribution is the observation that this torsion wave remains a solution for a general quadratic action (1) and that this fact can be established without having to write down explicitly the field equations.

6 Discussion

Riemannian pseudoinstantons (see (8)) clearly model vacuum. So below we examine the non-Riemannian pseudoinstanton from Section 5 with the aim of assigning a physical meaning.

We first list some remarkable properties of our non-Riemannian pseudoinstanton.

The curvature of our non-Riemannian pseudoinstanton (see (12)) satisfies the conditions

$$R_{\kappa\lambda\mu\nu} = -R_{\lambda\kappa\mu\nu} = -R_{\kappa\lambda\nu\mu} = R_{\mu\nu\kappa\lambda}, \qquad \varepsilon^{\kappa\lambda\mu\nu}R_{\kappa\lambda\mu\nu} = 0.$$

These are the standard symmetries of curvature in Riemannian geometry. Thus, an observer of our non-Riemannian pseudoinstanton might think that they live in a Riemannian world.

The torsion of our non-Riemannian pseudoinstanton (see (11)) satisfies the conditions

$$T^{\lambda}{}_{\lambda\nu} = 0, \qquad T^{\kappa\lambda\mu}\varepsilon_{\kappa\lambda\mu\nu} = 0.$$

This means that the torsion is irreducible (purely tensor), see Appendix B in [2].

The Ricci curvature of our non-Riemannian pseudoinstanton is identically zero, which may be interpreted classically as zero mass. Also, the wave vector $2l$ is light-like, which may be interpreted quantum mechanically as zero mass.

Further on we assume that $l \neq 0$ and $a \notin \mathrm{span}\,l$, which are the necessary and sufficient conditions for non-flatness.

It is easy to check (see Section 10 in [4] for details) that our non-Riemannian pseudoinstanton has holonomy

$$B^2 := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle|\quad b \in \mathbb{C} \right\}. \tag{13}$$

Here we use the standard identification of the proper orthochroneous Lorentz group with $\mathrm{SL}(2,\mathbb{C})$, and our notation for subgroups follows that of Section 10.122 of [7]. Note that the group (13) is, up to conjugation, the unique nontrivial Abelian Lie subgroup of $\mathrm{SL}(2,\mathbb{C})$; in this statement "nontrivial" is understood as "not 1-dimensional and not a product of 1-dimensional subgroups", with dimension understood as real dimension. Note also that the (restricted) holonomy group of a (metric compatible) space-time is a subgroup of the group (13) if and only if the space-time admits a nonvanishing parallel rank 1 spinor field.

Our non-Riemannian pseudoinstanton is determined by a pair of orthogonal isotropic vectors, namely, a real vector l and a complex vector a. (We call a vector v isotropic if $v_\mu v^\mu = 0$.) By choosing different l and a we obtain a set of non-Riemannian pseudoinstantons. Any two pseudoinstantons from this set can be obtained from one another by a rescaling and a Lorentz transformation. This, however, ceases being true if we restrict ourselves to proper orthochroneous Lorentz transformations: with respect to such transformations our set of non-Riemannian pseudoinstantons has two disjoint connected components. In other words, the construction from Section 5 gives us two essentially different solutions. It is tempting to view these two essentially different solutions as a particle and antiparticle.

Assuming that our non-Riemannian pseudoinstanton is indeed a very basic model for some elementary particle, we nominate three particles as possible candidates: the photon, the graviton, and the neutrino. Our (highly speculative) arguments in favour of each of these three interpretations go as follows.

The photon interpretation is based on the observation that our non-Riemannian pseudoinstanton is constructed out of a vector field A which is a solution of Maxwell's equation $\delta dA = 0$ (this is a consequence of (10)). However, an argument against the photon interpretation is that at a quantum mechanical level the electromagnetic field should be associated with a U(1)-connection, and the presence of this structure group is not evident in our construction.

The arguments in favour of the graviton interpretation are as follows. Torsion is not an accepted physical observable but curvature is, so we base our interpretation on the analysis of the curvature of our non-Riemannian pseudoinstanton. Examination of the explicit formula (12) indicates that it is more convenient to deal with the complexified curvature $dA \otimes dA$; note also that complexification is in line with the traditions of quantum mechanics. Our complex curvature is polarized, ${}^*(dA \otimes dA) = (dA \otimes dA)^* = \pm i(dA \otimes dA)$, and purely Weyl, hence it is equivalent to a (symmetric) rank 4 spinor ζ; see subsection 1.2.3 in [8] or Appendix C in [2] for details. A rank 4 spinor corresponds to a spin 2 particle, and one naturally thinks of the graviton.

Finally, the neutrino interpretation is based on the observation that our rank 4 spinor (see previous paragraph) has additional algebraic structure: it is the 4th tensor power of a rank 1 spinor, $\zeta = \xi \otimes \xi \otimes \xi \otimes \xi$. Direct calculations [5,9] show that the rank 1 spinor field ξ satisfies Weyl's equation, which is the accepted mathematical model for the neutrino. An attractive feature of the neutrino interpretation of our non-Riemannian pseudoinstanton is that it explains [5,9] why neutrinos are always left-handed whereas antineutrinos are always right-handed.

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