

Dirichlet L -series at $s = 0$ and the scarcity of Euler systems

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We study Euler systems for \mathbb{G}_m over a number field k . Motivated by a distribution-theoretic idea of Coleman, we formulate a conjecture regarding the existence of such systems that is elementary to state and yet strictly finer than Kato’s equivariant Tamagawa number conjecture for Dirichlet L -series at $s = 0$. To investigate the conjecture, we develop an abstract theory of ‘Euler limits’ and, in particular, prove the existence of canonical ‘restriction’ and ‘localisation’ sequences in this theory. By using this approach we obtain a variety of new results, ranging from a proof, modulo standard μ -vanishing hypotheses, of our central conjecture in the case k is \mathbb{Q} or imaginary quadratic to a proof of the ‘minus part’ of Kato’s conjecture in the case k is totally real. In proving these results, we also show that higher-rank Euler systems for a wide class of p -adic representations control the structure of Iwasawa-theoretic Selmer groups in the manner predicted by ‘main conjectures’.

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1. Introduction

The mysterious link between L -series and arithmetic, manifestations of which include the analytic class number formula and the Conjecture of Birch and Swinnerton-Dyer, is a key theme in modern arithmetic geometry. The theory of Euler systems has, since the 1980s, been crucial to investigations of this link, though finding concrete examples of such systems has proved to be a difficult problem. In the present article, we consider in detail the apparent scarcity of Euler systems in the setting of the multiplicative group \mathbb{G}_m over number fields, and find that this scarcity itself, when made precise, has important consequences regarding the formulation and study of special value conjectures.

To help motivate our approach we recall that, in 1989, Coleman conjectured a striking, and intrinsically global, distribution-theoretic analogue of the fact that norm-compatible families of units in towers of local cyclotomic fields arise by evaluating a power series at roots of unity, as had been proved in [Col79]. Hitherto, however, a resolution of this conjecture has seemed out of reach, with comparatively little supporting evidence and no proof strategy apparent (see [Seo01] or [BS21b] for a discussion of the history).

To generalise Coleman’s idea, we note that, after suitable reinterpretation, his conjecture implies that every Euler system (of rank one) for \mathbb{G}_m over \mathbb{Q} should, modulo certain torsion considerations, arise as a product of Galois-conjugates of the system of cyclotomic units. We then further note that, for any number field k , the collection of Euler systems of any given rank for \mathbb{G}_m over k is a module over the algebra $\mathcal{R}_k := \varprojlim_E \mathbb{Z}[\mathrm{Gal}(E/k)]$, where E runs over all finite abelian extensions of k (and the transition maps in the inverse limit are the natural projection maps). Then, roughly speaking, our central conjecture will assert that every Euler system over k that satisfies certain natural, and explicit, auxiliary conditions should be an \mathcal{R}_k -multiple of the ‘Rubin–Stark system’ c_k^{RS} that is defined in [Rub96]. This straightforward prediction (which, in the sequel, we refer to as the ‘Scarcity Conjecture’) is stated precisely as Conjecture (2.24) and lies at the centre of our approach.

At this point, it is important to note \mathcal{R}_k is neither Noetherian nor compact and hence that various standard algebraic techniques cannot be applied to the study of these questions. To overcome such difficulties, we develop an abstract theory of ‘Euler limits’ that simultaneously incorporates, amongst other things, the theories of inverse limits, Euler systems and Perrin-Riou functionals. This general theory is then the main theoretical advance that we make in this article and can be expected to have applications beyond those that we discuss here. In particular, the theory concerns systems that are defined integrally (that is, over \mathbb{Z}) rather than either p -adically (for a fixed prime p) or adically, and hence crucially incorporates techniques from both classical, and (what one might call) ‘horizontal’, Iwasawa theory. For instance, these differing techniques can be combined with a detailed analysis of the classical embedding problem for number fields to prove the existence of canonical p -adic ‘restriction’ and global ‘localisation’ exact sequences for modules arising from Euler limits, and also to carefully analyse completion functors in this setting.

These general results can then be used to reduce the classification of Euler systems to a family of p -primary problems (for all p) and thereby to prove, under natural hypotheses, that the Scarcity Conjecture is implied by the sort of divisibilities in ‘Iwasawa Main Conjecture’-type statements that Euler systems are already expected to satisfy (and can often be verified using existing techniques). This approach therefore provides both a conceptual underpinning of the Scarcity Conjecture and also an effective means of obtaining strong supporting evidence, such as the following result concerning Euler systems of rank one.

Theorem A. *After inverting 2 and all primes that ramify in k , and assuming standard μ -vanishing hypotheses, the Scarcity Conjecture is valid if k is either \mathbb{Q} or imaginary quadratic.*

This result is later stated precisely as Theorem (6.12), whilst concrete evidence in support of the Scarcity Conjecture for Euler systems of rank greater than one can be found, for example,

in Theorems (5.9) and (6.16).

With some additional effort, the methods developed here will allow one to prove the result of Theorem A without inverting any rational primes. We also stress that, whilst its proof relies on key properties of cyclotomic and elliptic units, for the reasons given above the content of Theorem A goes far beyond what has previously been shown in these classical settings. For example, the argument used to prove it also leads to a proof of Coleman’s original distribution-theoretic conjecture and, beyond that, to an explicit characterisation of all of the distributions that are considered by Coleman (many of which are not cyclotomic in nature). For brevity, however, the treatment of these important issues concerning Theorem A is deferred to the supplementary article [Bul+23].

In the present article, we instead choose to focus on the link between our study of Euler systems and the ‘equivariant’ strengthening of the Tamagawa Number Conjecture of Bloch and Kato [BK90] that is formulated by Kato in [Kat93a; Kat93b] (where the conjecture is referred to as the ‘generalised Iwasawa Main Conjecture’). More precisely, our approach is related to Kato’s conjecture in the case of leading terms at zero of Dirichlet L -series and, for ease, we refer to this as ‘eTNC(\mathbb{G}_m)’. We recall, in particular, that eTNC(\mathbb{G}_m) is known to imply a wide range of previously formulated refinements of Stark’s Conjecture (cf. Remark (3.3)).

To explain this link, we recall that, aside from the analytically-defined system c_k^{RS} , there also exists (unconditionally) a family of Galois-cohomological Euler systems c_k^{coh} for \mathbb{G}_m over k (as constructed by Sano and the second author in [BS21a]). Then, roughly speaking, a special case of the Scarcity Conjecture implies that $c_k^{\text{coh}} = \lambda \cdot c_k^{\text{RS}}$ for an element λ of \mathcal{R}_k , the analytic class number formula implies that any such element λ belongs to \mathcal{R}_k^\times , and the resulting relation between c_k^{RS} and c_k^{coh} implies eTNC(\mathbb{G}_m) over all abelian extensions of k . In this way, then, eTNC(\mathbb{G}_m) is seen to be a direct consequence of the analytic class number formula and the scarcity of Euler systems, thereby providing a straightforward philosophy to underpin eTNC(\mathbb{G}_m). To the best of the authors’ knowledge, excluding the basic analogy to Deligne’s proof of the Weil Conjectures for varieties over function fields, no heuristic of any sort has previously been available for Kato’s conjecture.

In fact, it turns out that the Scarcity Conjecture is strictly finer than eTNC(\mathbb{G}_m) and also encodes precise information about the structure of the Selmer module of \mathbb{G}_m over the abelian closure of k (for details see [Bul+23]). Fortunately, however, the theory of Euler limits is flexible enough to provide an effective strategy for proving eTNC(\mathbb{G}_m) without requiring one to first prove the Scarcity Conjecture. The resulting approach then has a significant advantage over previous strategies in this context since it avoids delicate issues relating to Iwasawa-theoretic descent that have been key obstacles to progress on eTNC(\mathbb{G}_m).

The point here is that, for any given prime p , the methods of p -adic Iwasawa theory involve passing to the limit over \mathbb{Z}_p -power extensions of k and therefore primarily concern extensions that ramify at p -adic places. Hence, when considering extensions of k that are unramified at any such places, this approach can introduce undesired Euler factors that are not invertible (in the presence of ‘trivial zeroes’) and so cannot easily be removed. Whilst previous strategies to deal with this issue have relied on the deep conjectures of Gross–Kuz’min, of Leopoldt and of Mazur–Rubin and Sano, the validity of which remain restricted to a small number of well-known cases, our theory completely avoids any reliance on the first two of these conjectures and only depends on the third conjecture in the (much easier) setting of tamely ramified cyclic extensions since the Euler systems we study are essentially characterised by their values on extensions that ramify at all p -adic places.

In this way we can therefore directly leverage existing techniques to obtain concrete new evidence for Kato’s conjecture such as in the following result. Before stating this result, we recall that eTNC(\mathbb{G}_m) predicts, for each finite abelian extension of number fields K/k , an equality of graded invertible $\mathbb{Z}[\text{Gal}(K/k)]$ -modules (cf. Remark (3.3) (a)).

Theorem B. *Fix a finite abelian extension of number fields K/k and set $G := \text{Gal}(K/k)$. Then $e\text{TNC}(\mathbb{G}_m)$ for K/k is valid in both of the following cases:*

- (a) *k is totally real, K is CM and one extends scalars from $\mathbb{Z}[G]$ to $\mathbb{Z}[1/2][G](1 - \tau)$, where τ denotes the element of G corresponding with complex conjugation;*
- (b) *k is imaginary quadratic and such that for every finite abelian extension (and every prime) a certain Iwasawa μ -invariant vanishes, and one inverts all prime divisors of the number of roots of unity in k .*

Claim (a) of this result verifies what is often referred to as the ‘minus part’ $e\text{TNC}(\mathbb{G}_m)^-$ of $e\text{TNC}(\mathbb{G}_m)$, and our approach reduces its proof to a statement in Iwasawa theory that is easily seen to follow from the ‘Strong Brumer–Stark Conjecture’ recently proved (outside 2) by Dasgupta and Kakde in [DK23]. In particular, in this setting the horizontal Iwasawa theory of Euler limits allows us, via Theorem (4.10), to avoid technical hypotheses used in other attempts to derive $e\text{TNC}(\mathbb{G}_m)^-$ from the seminal results of [DK23], either directly as in Nickel [Nic21] (though the deduction in loc. cit. of the p -part of $e\text{TNC}(\mathbb{G}_m)^-$ for extensions that are tamely ramified at p does play a role in our argument) or by strengthening relevant aspects of the arguments in [DK23] as in Atsuta and Kataoka [AK21b], whilst at the same time avoiding difficult hypotheses related to the Gross–Kuz’min Conjecture that arose in earlier attempts to use Iwasawa theory in this context. The result of claim (a) also itself has interesting consequences: for example, it implies the validity, after inverting 2, of the ‘integral Gross–Stark Conjecture’ from [Gro88] (cf. Remark (3.3) (b)) and this fact has recently been used by Honnor to remove, up to 2-power torsion, the ‘root of unity ambiguity’ in the p -adic analytic formula for Brumer–Stark units that is proved by Dasgupta and Kakde in [DK21, Th. 1.6] (for details see [Hon22]). In addition, claim (a) also combines with work of Atsuta and Kataoka [AK21a] to imply an explicit description for the Fitting ideals of the minus parts of class groups in the relevant cases.

Claim (b) of Theorem B will be made precise in §6.3.1 and strongly improves upon previous results towards $e\text{TNC}(\mathbb{G}_m)$ over imaginary quadratic fields. In addition, in recent work of Hofer and the first author [BH21], it is shown that the hypothesis on μ -invariants that occurs in this result, and which Iwasawa has conjectured to always hold, can sometimes be avoided. However, this observation relies on techniques that seem to be restricted to the setting of imaginary quadratic fields, whilst the approach used here appears, in principle, to be completely general and thereby applicable to many different contexts.

The arguments proving Theorem B also lead us, at the same time, to a new proof of $e\text{TNC}(\mathbb{G}_m)$ for K/k in the case that $k = \mathbb{Q}$ and one extends scalars from $\mathbb{Z}[G]$ to $\mathbb{Z}[1/2][G]$. This result was first proved as the main result of Greither and the second author in [BG03], but the proof obtained here is much simpler since it avoids the extensive, and delicate, descent calculations in Iwasawa theory that are key to the argument of loc. cit.

Going beyond special cases, our approach provides an effective strategy for deriving the validity of the Scarcity Conjecture over an arbitrary number field k , and hence also of $e\text{TNC}(\mathbb{G}_m)$ and the numerous conjectures it implies, from a single, explicit, ‘integrality’ prediction of Rubin [Rub96] concerning the system c_k^{RS} (that is usually referred to as the ‘Rubin–Stark Conjecture’). In addition, at this stage, the theory of higher-rank Euler, Kolyvagin and Stark systems developed by Sakamoto, Sano and the second author in [BSS19a] already allows us to partially achieve this goal, and thereby to obtain strong evidence for the Scarcity Conjecture over any k . In particular, as a key aspect of these arguments we prove (in an appendix) that higher-rank Euler systems for a wide class of p -adic representations control the structure of Selmer groups in precisely the manner predicted by ‘Main Conjectures’ in this setting. In fact, to complete the deduction of the Scarcity Conjecture from the Rubin–Stark Conjecture, it would suffice to extend the latter result to an ‘equivariant’ setting in just the same degree of generality as has already been established for Euler systems of rank one in [Rub00], and we will discuss this problem elsewhere. We note, however, that the Rubin–Stark Conjecture itself is of a differ-

ent nature and that its verification may well require both an explicit construction of c_k^{RS} by geometric means and the proof of an appropriate explicit reciprocity law, as has already been achieved in the case $k = \mathbb{Q}$ by Urban in [Urb20].

In brief, the main contents of this article are as follows. In § 2 we define, for each non-negative rank, several special families of Euler systems over k , prove some important preliminary results and then formulate the Scarcity Conjecture. In § 3 we establish the precise link between the Scarcity Conjecture and $\text{eTNC}(\mathbb{G}_m)$. In § 4 we introduce an abstract theory of ‘Euler limits’ and prove key results in this theory. In § 5 we prove a ‘Uniformisation Theorem’ for Euler systems over k that are valued in a field, combine this with general results concerning Euler limits to derive explicit criteria for the validity of the Scarcity Conjecture and of $\text{eTNC}(\mathbb{G}_m)$ and finally use higher-rank Kolyvagin systems to provide evidence for these criteria. In § 6 we combine criteria established in § 5 with several existing results to obtain further evidence in support of the Scarcity Conjecture and also prove precise versions of Theorems A and B. Finally, in an appendix we prove new results concerning the ‘Iwasawa Main Conjecture’ for p -adic representations over arbitrary number fields.

For the reader’s convenience, we end this section by specifying some general notation that will be used throughout the article. Given a commutative ring R , an R -module M and an integer $r \geq 0$, we write $\bigwedge_R^r M$ for the r^{th} -exterior power over R of M . Given a homomorphism $f: M \rightarrow N$ of R -modules we write $f^{(r)}$ for the induced map $\bigwedge_R^r M \rightarrow \bigwedge_R^r N$. Except in cases of ambiguity we will often abuse notation and simply refer to this map as f .

We write $M^* := \text{Hom}_R(M, R)$ for the R -linear dual of M and, if \mathfrak{p} denotes a prime ideal of R , then we let $M_{\mathfrak{p}}$ be the localisation of M at \mathfrak{p} .

For an abelian group A we denote by A_{tor} its torsion-subgroup and by $A_{\text{tf}} = A/A_{\text{tor}}$ its torsion-free quotient. The Pontryagin dual of A will be denoted by $A^{\vee} = \text{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$. If there is no confusion possible, we often shorten the functor $(-) \otimes_{\mathbb{Z}} A$ to just $(-) \cdot A$ (or even $(-)A$) and, if A is also a \mathbb{Z}_p -module, similarly for the functor $(-) \otimes_{\mathbb{Z}_p} A$. If A is finite, we denote by $\hat{A} = \text{Hom}_{\mathbb{Z}}(A, \mathbb{C}^{\times})$ its character group and write $\mathbf{1}_A$ for the trivial character, and for any $\chi \in \hat{A}$ we write e_{χ} for the primitive idempotent $|A|^{-1} \sum_{\sigma \in A} \chi(\sigma) \sigma^{-1}$ of $\mathbb{C}[A]$. Furthermore, $N_A = \sum_{\sigma \in A} \sigma \in \mathbb{Z}[A]$ denotes the norm element of A .

The duals M^* and M^{\vee} of a $\mathbb{Z}[A]$ -module M will in general be endowed with the contragredient A -action.

Finally, for any finite set \mathcal{S} of prime numbers, we often use the subring of \mathbb{Q} defined by

$$\mathbb{Z}_{\mathcal{S}} := \mathbb{Z}[1/p \mid p \in \mathcal{S}]$$

(so that $\mathbb{Z}_{\emptyset} = \mathbb{Z}$).

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Finally we would like to point out that this article incorporates various ideas from an earlier unpublished version of an article of Hofer and the first author [BH21] and we are very grateful to Martin Hofer for his permission to include these.

2. Euler systems for \mathbb{G}_m

For any number field E we write $S_\infty(E)$ and $S_{\text{fin}}(E)$ for the sets of archimedean and finite (non-archimedean) places of E respectively, and $S_p(E)$ for the subset of $S_{\text{fin}}(E)$ comprising all places that lie above a rational prime p . Given an extension F/E we write $S_{\text{ram}}(F/E)$ for the places of E that ramify in F .

If S is a set of places of E , we denote by S_F the set of places of F that lie above those contained in S . We will however omit the explicit reference to the field in case it is clear from the context. For example, $\mathcal{O}_{F,S}$ will denote the ring of $(S_F \cap S_{\text{fin}}(F))$ -integers of F . We also define $Y_{F,S}$ to be the free abelian group on S_F and write $X_{F,S}$ for the subgroup thereof comprising elements whose coefficients sum to zero.

For any finite place v of E we write $\text{ord}_v: E^\times \rightarrow \mathbb{Z}$ for the normalised valuation at v and Nv for its absolute norm $|\mathcal{O}_E/\mathfrak{p}_v|$, where \mathfrak{p}_v is the ideal of \mathcal{O}_E corresponding to v . We also write $E(\mathfrak{m})$ for the (narrow) ray class field of a given modulus \mathfrak{m} of E .

If F/E is a Galois extension and v is a finite place of E that is unramified in F , then we write Frob_v for the arithmetic Frobenius of v in $\text{Gal}(F/E)$. We recall that an archimedean place of E is said to ‘ramify’ in F if its decomposition group is non-trivial (and hence has order two); for each archimedean place v of E that is unramified (that is, does not ramify) in F , we will therefore write Frob_v for the trivial element of $\text{Gal}(F/E)$.

2.1. Multi-rank Euler systems

2.1.1. Abelian extensions

We fix an algebraic closure $\overline{\mathbb{Q}}$ of \mathbb{Q} and refer to finite extensions of \mathbb{Q} in $\overline{\mathbb{Q}}$ as ‘number fields’. We then fix a number field k and for any Galois extension K of k in $\overline{\mathbb{Q}}$ set

$$\mathcal{G}_K := \text{Gal}(K/k) \quad \text{and} \quad S(K) := S_{\text{ram}}(K/k).$$

We also write \mathcal{K} for the maximal abelian extension of k in $\overline{\mathbb{Q}}$ and

$$\Omega = \Omega(k)$$

for the set of finite extensions of k in \mathcal{K} that are ramified at at least one finite place (so that $S(K) \not\subseteq S_\infty(k)$).

A ‘modulus’ of k is a formal product \mathfrak{m} of places of k and gives rise to an associated ray class field extension $k(\mathfrak{m})$ of k of conductor \mathfrak{m} . The extension $k(\mathfrak{m})/k$ is finite, abelian and such that $S(k(\mathfrak{m}))$ is contained in the set of places that divide \mathfrak{m} . In particular, the ray classfield $k(1)$ of conductor equal to the empty product of places is the Hilbert classfield of k .

For a number field E , we write μ_E for the \mathbb{Z} -torsion subgroup of E^\times . For a natural number M we write

$$\mu_M := \{x \in \overline{\mathbb{Q}}^\times : x^M = 1\}$$

for the group of roots of unity in $\overline{\mathbb{Q}}$ of order dividing M .

The following result records an observation of Rubin concerning ray classfields that will be useful in the sequel.

(2.1) Lemma. *Fix a natural number M and a finite place \mathfrak{q} of k that does not divide M . Then the ray class field $k(\mathfrak{q})$ has the following properties.*

- (i) $S(k(\mathfrak{q})) \subseteq \{\mathfrak{q}\}$ and so $S(k(\mathfrak{q})) \cap S_\infty(k) = \emptyset$.
- (ii) *The inertia degree of \mathfrak{q} in $k(\mathfrak{q})$ is equal to $[k(\mathfrak{q}) : k(1)]$ and is divisible by M if and only if \mathfrak{q} splits completely in $k(\mu_M, (\mathcal{O}_k^\times)^{1/M})$.*

Proof. Claim (i) and the first assertion in claim (ii) are clear. The second assertion in claim (ii) is proved by Rubin in [Rub00, Lem. 4.1.2] but, for completeness, we give the argument.

Class field theory identifies the inertia subgroup $\text{Gal}(k(\mathfrak{q})/k(1))$ of \mathfrak{q} in $\mathcal{G}_{k(\mathfrak{q})}$ with the cokernel of the natural map $\theta_{\mathfrak{q}} : \mathcal{O}_k^{\times} \rightarrow (\mathcal{O}_k/\mathfrak{q})^{\times}$. Hence, if M divides $[k(\mathfrak{q}) : k(1)]$, then M divides $|(\mathcal{O}_k^{\times}/\mathfrak{q})^{\times}| = N_{\mathfrak{q}} - 1$ and so \mathfrak{q} splits completely in $k(\mu_M)$.

In addition, any such \mathfrak{q} splits completely in $k(\mu_M, (\mathcal{O}_k^{\times})^{1/M})$ if and only if every element of \mathcal{O}_k^{\times} is an M -th power in the completion $k_{\mathfrak{q}}$ of k at \mathfrak{q} . Then, since \mathfrak{q} does not divide M , Hensel's Lemma implies that this last condition is satisfied if and only if every element of $\text{im}(\theta_{\mathfrak{q}})$ is an M -th power in the cyclic group $(\mathcal{O}_k/\mathfrak{q})^{\times}$, or equivalently the order of $\text{cok}(\theta_{\mathfrak{q}})$ is divisible by M . This proves claim (ii). \square

2.1.2. Euler systems

For each field K in Ω we write V_K for the set $S_{\infty}(k) \setminus S(K)$ of archimedean places of k that split completely in K and set

$$r_K := \begin{cases} |V_K|, & \text{if } K \neq k, \\ |S_{\infty}(k)| - 1, & \text{if } K = k. \end{cases}$$

We write $\mathcal{P}(S_{\infty}(k))$ for the power set of $S_{\infty}(k)$ and \mathbb{N}_0 for the set of non-negative integers.

(2.2) Definition. A *rank function* (or simply a *rank*) for k is a function $\mathbf{r} : \Omega \rightarrow \mathbb{N}_0$ that factors through the function $\Omega \rightarrow \mathcal{P}(S_{\infty}(k))$ sending each K to V_K (so that $\mathbf{r}(K) = \mathbf{r}(K')$ if $V_K = V_{K'}$). The *maximal rank function* \mathbf{r}_{\max} for k is the function $K \mapsto r_K$.

The motivation for the terminology ‘maximal’ used above will become apparent in Lemma (2.6) (b) below. In addition, for any given field k , we will usually identify a non-negative integer r with the (constant) rank function that sends each K in Ω to r .

Finally, for any pair of fields $E, F \in \Omega$ with $E \subseteq F$, any non-negative integer s , and any field \mathcal{Q} , we write

$$N_{F/E}^s : \mathcal{Q} \cdot \bigwedge_{\mathbb{Z}[\mathcal{G}_F]}^s \mathcal{O}_{F,S(F)}^{\times} \rightarrow \mathcal{Q} \cdot \bigwedge_{\mathbb{Z}[\mathcal{G}_E]}^s \mathcal{O}_{E,S(E)}^{\times}$$

for the homomorphism of $\mathcal{Q}[\mathcal{G}_F]$ -modules induced by the field-theoretic norm $N_{F/E} : F^{\times} \rightarrow E^{\times}$.

(2.3) Definition. Let \mathbf{r} be a rank function for k and \mathcal{Q} a field. A \mathcal{Q} -rational *Euler system* for k of rank \mathbf{r} is a collection of elements

$$(c_E)_E \in \prod_{E \in \Omega} \mathcal{Q} \cdot \bigwedge_{\mathbb{Z}[\mathcal{G}_E]}^{\mathbf{r}(E)} \mathcal{O}_{E,S(E)}^{\times}$$

that satisfy the following ‘Euler system distribution relations’: for every pair of fields E and F in Ω with $E \subseteq F$, there is, in the graded module $\bigoplus_{i=0}^{\infty} (\mathcal{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_E]}^i \mathcal{O}_{E,S(E)}^{\times})$, an equality

$$N_{F/E}^{\mathbf{r}(F)}(c_F) = \left(\prod_{v \in S(F) \setminus S(E)} (1 - \text{Frob}_v^{-1}) \right) c_E. \quad (1)$$

We write $\text{ES}_k^{\mathbf{r}}(\mathcal{Q})$ for the $\mathcal{Q}[[\mathcal{G}_K]]$ -module of \mathcal{Q} -rational Euler systems for k of rank \mathbf{r} . If \mathbf{r} is the maximal rank function \mathbf{r}_{\max} , then we abbreviate $\text{ES}_k^{\mathbf{r}}(\mathcal{Q})$ to $\text{ES}_k(\mathcal{Q})$.

(2.4) Remark. If any place v in $S(F) \setminus S(E)$ is archimedean, then v splits completely in E and so the corresponding Euler factor $1 - \text{Frob}_v^{-1}$ in the equality (1) is 0. In this case, therefore, one has $N_{F/E}^{\mathbf{r}(F)}(c_F) = 0$.

In the sequel it will often be useful to consider certain ‘projections’ of an Euler system to suitable subsets of Ω . More precisely, for any subset $\mathcal{X} \subseteq \Omega$, we write $\text{ES}_k^{\mathbf{r}, \mathcal{X}}(\mathcal{Q})$ for the image of $\text{ES}_k^{\mathbf{r}}(\mathcal{Q})$ under the natural projection map

$$\mathcal{Q}^{\mathcal{X}} : \prod_{E \in \Omega} \mathcal{Q} \cdot \bigwedge_{\mathbb{Z}[\mathcal{G}_E]}^{\mathbf{r}(E)} \mathcal{O}_{E,S(E)}^{\times} \rightarrow \prod_{E \in \mathcal{X}} \mathcal{Q} \cdot \bigwedge_{\mathbb{Z}[\mathcal{G}_E]}^{\mathbf{r}(E)} \mathcal{O}_{E,S(E)}^{\times}.$$

(2.5) Remark. For a subset \mathcal{V} of $\mathcal{P}(S_\infty(k))$, we write $\Omega^\mathcal{V}(k)$ for the subset of $\Omega(k)$ comprising all fields K such that $V_K \in \mathcal{V}$. If $\mathcal{V} = \{V\}$ for some subset $V \subseteq S_\infty(k)$, then we abbreviate $\Omega^\mathcal{V}(k)$ to $\Omega^V(k)$ and, for the set $\mathcal{X} = \Omega^V(k)$, we shorten all adornments \mathcal{X} (as, for example, for the map $\varrho^\mathcal{X}$ defined above) to V .

To end this subsection, we record an important first consequence of the Euler system relations. For any field $K \in \Omega(k)$, we abbreviate the trivial character $\mathbf{1}_{\mathcal{G}_K}$ of \mathcal{G}_K to $\mathbf{1}_K$. For any finite set of places Π of k we set

$$r_\Pi(\chi) := \dim_{\mathbb{C}}(e_\chi \mathbb{C}\mathcal{O}_{K,\Pi}^\times) = \begin{cases} |\{v \in (\Pi \cup S_\infty(k)) \mid \chi(\mathcal{G}_{K,v}) = 1\}| & \text{if } \chi \neq \mathbf{1}_K, \\ |\Pi \cup S_\infty(k)| - 1 & \text{if } \chi = \mathbf{1}_K, \end{cases} \quad (2)$$

where $\mathcal{G}_{K,v} \subseteq \mathcal{G}_K$ denotes the decomposition group at the place v . We then define Υ_K to be the subset of $\widehat{\mathcal{G}_K}$ comprising all characters χ for which one has $r_{S(K)}(\chi) = r_K$. In particular, since Υ_K is a union of orbits of the natural action of $\text{Aut}(\mathbb{C})$ on $\widehat{\mathcal{G}_K}$, we obtain an idempotent of $\mathbb{Q}[\mathcal{G}_K]$ by setting

$$e_K := \sum_{\chi \in \Upsilon_K} e_\chi, \quad (3)$$

where, for each χ , we write e_χ for the primitive idempotent $|\mathcal{G}_K|^{-1} \sum_{g \in \mathcal{G}_K} \chi(g^{-1})g$ of $\mathbb{C}[\mathcal{G}_K]$.

(2.6) Lemma. *If $c = (c_E)_E \in \text{ES}_k^r(\mathcal{Q})$ is an Euler system for k with coefficients in a subfield \mathcal{Q} of \mathbb{C} , then for all fields $E \in \Omega$ the following claims are valid.*

- (a) $(1 - e_E) \cdot c_E = 0$.
- (b) If $r(E) > r_E$, then $c_E = 0$.

Proof. Let $\chi \in \widehat{\mathcal{G}_E}$ and write E_χ for the subfield of E cut out by χ . Consider the injective map

$$\nu_{E/E_\chi} : \mathcal{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_{E_\chi}]}^{r_E} \mathcal{O}_{E_\chi, S(E)}^\times \rightarrow \mathcal{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_E]}^{r_E} \mathcal{O}_{E, S(E)}^\times, \quad a \mapsto \begin{cases} [E : E_\chi]^{1-r_E} a & \text{if } r_E \geq 1, \\ N_{\text{Gal}(E/E_\chi)} a & \text{if } r_E = 0. \end{cases}$$

We also remark that N_{E/E_χ}^0 agrees with the natural restriction map $\mathcal{Q}[\mathcal{G}_E] \rightarrow \mathcal{Q}[\mathcal{G}_{E_\chi}]$. We then have

$$[E : E_\chi] \cdot e_\chi \cdot c_E = e_\chi \cdot N_{\text{Gal}(E/E_\chi)} \cdot c_E = e_\chi (\nu_{E/E_\chi} \circ N_{E/E_\chi}^{r(E)})(c_E)$$

and so it suffices to prove that $e_\chi N_{E/E_\chi}^{r(E)}(c_E) = 0$ whenever $\chi \notin \Upsilon_E$.

To do this, we first consider the case that $\chi \neq \mathbf{1}_E$. Let l be a large-enough odd prime number such that $E \cap k(\mu_l) = k$. By Chebotarev's Density Theorem we may then choose a $\mathfrak{p} \in S_{\text{fin}}(k) \setminus S(E)$ that has full decomposition group in E_χ and splits completely in $k(\mu_l, (\mathcal{O}_k)^{1/l})$. By Lemma (2.1) the ray class field $F := k(\mathfrak{p})$ is then a non-trivial extension of the Hilbert class field of k . It follows that F , and hence also $E_\chi F$, is a ramified extension of k , which implies that the value $c_{E_\chi F}$ is well-defined. We also note that $r(E) = r(EF)$. The Euler systems distribution relations now imply that

$$(1 - \text{Frob}_{\mathfrak{p}}^{-1}) \cdot N_{E/E_\chi}^{r(E)}(c_E) = N_{EF/E_\chi}^{r(E)}(c_{EF}) = \left(\prod_{v \in S(E) \setminus S(E_\chi)} (1 - \text{Frob}_v^{-1}) \right) \cdot N_{E_\chi F/E_\chi}^{r(E)}(c_{E_\chi F}).$$

If $\chi \notin \Upsilon_E$, then there exists a place $v \in S(E) \setminus S(E_\chi)$ such that $\chi(\text{Frob}_v^{-1}) = 1$, hence e_χ annihilates the above Euler product. On the other hand, $1 - \chi(\text{Frob}_{\mathfrak{p}}^{-1}) \neq 0$ because $\chi \neq \mathbf{1}_E$ and $\text{Frob}_{\mathfrak{p}}$ generates \mathcal{G}_{E_χ} , so we conclude that $e_\chi N_{E/E_\chi}^{r(E)}(c_E) = 0$, as required to prove claim (a) for non-trivial characters.

To deal with the trivial character, we note that $\mathbf{1}_E \in \Upsilon_E$ if and only if $|S(E) \cup S_\infty(k)| = r_E + 1$ or, equivalently (since $r_E \leq |S_\infty(k)|$), $|S(E)| = 1$. Assume this condition is not satisfied. Then we may factor the conductor of E as \mathfrak{pm} , where \mathfrak{m} is a non-trivial modulus and \mathfrak{p} is a (finite or archimedean) place. It is enough to prove $N_{k(\mathfrak{mp})/k}^{r(E)}(c_{k(\mathfrak{mp})}) = 0$ and this follows from

$$N_{k(\mathfrak{mp})/k}^{r(E)}(c_{k(\mathfrak{mp})}) = N_{k(\mathfrak{m})/k}^{r(E)}((1 - \text{Frob}_{\mathfrak{p}}^{-1})c_{k(\mathfrak{m})}) = 0.$$

Turning now to the proof of claim (b), we observe that by (2) we have

$$e_\chi c_E \in \bigwedge_{e_\chi \mathbb{C}[\mathcal{G}_E]}^{r(E)} e_\chi \mathbb{C} \mathcal{O}_{E,S(E)}^\times = 0$$

whenever χ is contained in Υ_E and $r(E) > r_E$. Since $e_E c_E = c_E$ by claim (a), this shows that $c_E = 0$, as desired. \square

2.2. Special families

For the purpose of arithmetic applications an Euler system must be ‘integral’ in a suitable sense. However, defining a precise notion of integrality for higher-rank systems is a delicate task. In this subsection we use exterior biduals and T -modification, pioneered by Rubin [Rub96] and Gross [Gro88], respectively, to address this question. We then introduce certain special families of such systems that will play an important role in our theory.

If R is a commutative Noetherian ring, then for any R -module M we write $M^* = \text{Hom}_R(M, R)$ for its dual. We recall that, for each non-negative integer s , the s -th *exterior power bidual* of M is defined to be the module

$$\bigcap_R^s M := \left(\bigwedge_R^s M^* \right)^*.$$

We note that if $R = \mathbb{Z}[A]$ for a finite abelian group A , then $\bigcap_R^s M$ coincides with the lattice first introduced by Rubin in [Rub96, § 2] (cf. [BS21a, Rk. A.9]).

2.2.1. Rubin lattices

For a finite abelian extension K of k we write $\mathcal{P}_K^{\text{ad}}$ for the collection of non-empty finite sets of places of k that are disjoint from $S(K) \cup S_\infty(k)$ and contain no place that divides $|\mu_K|$. We refer to elements of $\mathcal{P}_K^{\text{ad}}$ as ‘admissible sets for K ’. For any such T in $\mathcal{P}_K^{\text{ad}}$ it is easy to check that the group

$$\mathcal{O}_{K,S(K),T}^\times = \{u \in \mathcal{O}_{K,S(K)}^\times \mid u \equiv 1 \pmod{T}\}$$

of ‘ T -modified $S(K)$ -units’ of K is a finite index subgroup of $\mathcal{O}_{K,S(K)}^\times$ that is \mathbb{Z} -torsion free. For T in $\mathcal{P}_K^{\text{ad}}$ we set

$$\delta_T = \delta_{T,K} := \prod_{v \in T} (1 - Nv \cdot \text{Frob}_v^{-1}) \in \mathbb{Z}[\mathcal{G}_K],$$

and then, for any integer $s \geq 0$, define a \mathcal{G}_K -submodule of $\mathbb{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_K]}^s \mathcal{O}_{K,S(K)}^\times$ by setting

$$\mathfrak{L}_K^s := \left\{ a \in \mathbb{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_K]}^s \mathcal{O}_{K,S(K)}^\times \mid \delta_{T,K}(a) \in \bigcap_{\mathbb{Z}[\mathcal{G}_K]}^s \mathcal{O}_{K,S(K),T}^\times \text{ for all } T \in \mathcal{P}_K^{\text{ad}} \right\}.$$

If $s = r_K$, then we will suppress the superscript r_K in the notation.

The argument of [Tat84, Ch. IV, Lem. 1.1] implies that $\text{Ann}_{\mathbb{Z}[\mathcal{G}_K]}(\mu_K)$ is generated over \mathbb{Z} by the set $\{\delta_{T,K} \mid T \in \mathcal{P}_K^{\text{ad}}\}$. By combining this fact together with the defining conditions of \mathfrak{L}_K^s , one deduces that

$$\text{Ann}_{\mathbb{Z}[\mathcal{G}_K]}(\mu_K) \cdot \mathfrak{L}_K^s \subseteq \bigcap_{\mathbb{Z}[\mathcal{G}_K]}^s \mathcal{O}_{K,S(K)}^\times. \quad (4)$$

In particular, since $\bigcap_{\mathbb{Z}[\mathcal{G}_K]}^s \mathcal{O}_{K,S(K)}^\times$ is finitely generated and $\text{Ann}_{\mathbb{Z}[\mathcal{G}_K]}(\mu_K)$ contains $|\mu_K|$, it follows that \mathfrak{L}_K^s is finitely generated as a $\mathbb{Z}[\mathcal{G}_K]$ -module.

(2.7) Remark. If $s \leq 1$, then the result of [Tat84, Ch. IV, Prop. 1.2] implies that the inclusion (4) uniquely characterises \mathfrak{L}_K^s . In other words, in this case, an element a of $\mathbb{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_K]}^s \mathcal{O}_{K,S(K)}^\times$ belongs to \mathfrak{L}_K^s if and only if $\text{Ann}_{\mathbb{Z}[\mathcal{G}_K]}(\mu_K) \cdot a \subseteq \bigcap_{\mathbb{Z}[\mathcal{G}_K]}^s \mathcal{O}_{K,S(K)}^\times$. Similarly, if p is a prime such that $\mu_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is \mathcal{G}_K -cohomologically trivial, then the arguments of Popescu in [Pop02, Thm. 5.5.1 (3)] show that, for any $s \geq 0$, an element a of $\mathbb{Q}_p \bigwedge_{\mathbb{Z}[\mathcal{G}_K]}^s \mathcal{O}_{K,S(K)}^\times$ belongs to $\mathbb{Z}_p \mathfrak{L}_K^s$ if and only if $\text{Ann}_{\mathbb{Z}_p[\mathcal{G}_K]}(\mu_K \otimes_{\mathbb{Z}} \mathbb{Z}_p) \cdot a \subseteq \bigcap_{\mathbb{Z}_p[\mathcal{G}_K]}^s (\mathbb{Z}_p \mathcal{O}_{K,S(K)}^\times)$.

In general, the inclusion (4) allows one to explicitly relate \mathfrak{L}_K^s to the lattices defined by Popescu in [Pop02, Def. 2.1.1]. To be more precise, setting

$$U_{K,S(K)}^{\text{ab}} := \{u \in \mathcal{O}_{K,S(K)}^\times \mid K(u^{1/|\mu_K|})/k \text{ is abelian}\},$$

one finds that an element a of $\mathbb{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_K]}^s \mathcal{O}_{K,S(K)}^\times$ satisfies $\text{Ann}_{\mathbb{Z}[\mathcal{G}_K]}(\mu_K) \cdot a \subseteq \bigcap_{\mathbb{Z}[\mathcal{G}_K]}^s \mathcal{O}_{K,S(K)}^\times$ if and only if one has $|\mu_K| \cdot f(a) \in (U_{K,S(K)}^{\text{ab}})^{**}$ for all $f \in \bigwedge_{\mathbb{Z}[\mathcal{G}_K]}^{s-1} (\mathcal{O}_{K,S(K)}^\times)^*$.

(2.8) Remark. The explicit description of the modules $\text{Ann}_{\mathbb{Z}[\mathcal{G}_K]}(\mu_K)$ that is given above has the following useful consequence: for any finite abelian extensions K and K' of k with $K \subseteq K'$, the homomorphism $\text{Ann}_{\mathbb{Z}[\mathcal{G}_{K'}]}(\mu_{K'}) \rightarrow \text{Ann}_{\mathbb{Z}[\mathcal{G}_K]}(\mu_K)$ that is induced by the restriction map $\mathbb{Z}[\mathcal{G}_{K'}] \rightarrow \mathbb{Z}[\mathcal{G}_K]$ is surjective (cf. [Bur+23, Lem. 3.9]).

The following result establishes several useful properties of the lattices \mathfrak{L}_K^s .

(2.9) Lemma. *Fix a finite abelian extension E of k with $K \subseteq E$ and consider the injection*

$$\nu_{E/K}: \mathbb{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_K]}^s \mathcal{O}_{K,S(E)}^\times \rightarrow \mathbb{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_E]}^s \mathcal{O}_{E,S(E)}^\times, \quad a \mapsto \begin{cases} [E:K]^{1-s} a & \text{if } s > 0, \\ N_{\text{Gal}(E/K)} a & \text{if } s = 0. \end{cases}$$

Then the following claims are valid.

- (a) $\nu_{E/K}$ restricts to an injection of \mathcal{G}_K -modules from \mathfrak{L}_K^s to $(\mathfrak{L}_E^s)^{\text{Gal}(E/K)}$,
- (b) the composite $\nu_{E/K} \circ N_{E/K}^s$ coincides with multiplication by $N_{\text{Gal}(E/K)}$,
- (c) for every non-zero element $a \in \mathfrak{L}_K^s$ there exists a finite set \mathcal{N} of integers, depending only on a , such that $\nu_{E/K}(a)$ is not divisible in \mathfrak{L}_E^s by any integer outside \mathcal{N} ,
- (d) $\nu_{E/k}$ induces an isomorphism $|\mu_k|^{-1} \bigwedge_{\mathbb{Z}}^s (\mathcal{O}_{k,S(E)}^\times)^{**} \cong (\mathfrak{L}_E^s)^{\mathcal{G}_E}$.

Proof. To prove claim (a) we must show that for each a in \mathfrak{L}_K^s and each T in $\mathcal{P}_E^{\text{ad}}$ one has $\delta_{T\nu_{E/K}}(a) \in \bigcap_{\mathbb{Z}[\mathcal{G}_E]}^s \mathcal{O}_{E,S(E),T}^\times$. Since $S(K) \subseteq S(E)$ and $\mu_K \subseteq \mu_E$, one has $\mathcal{P}_E^{\text{ad}} \subseteq \mathcal{P}_K^{\text{ad}}$. In addition, for each T in $\mathcal{P}_E^{\text{ad}}$ one has that $\mathcal{O}_{E,S(E),T}^\times$ is \mathbb{Z} -torsion free and hence reflexive. By Lemma (2.10) below we therefore have that

$$\nu_{E/K}(\delta_{K,T} a) = \delta_{E,T\nu_{E/K}}(a) \in \bigcap_{\mathbb{Z}[\mathcal{G}_E]}^s \mathcal{O}_{E,S(E),T}^\times,$$

as required to prove claim (a).

Claim (b) is straightforward to check from the definitions (see also [BKS16, Rk. 4.14]).

To prove claim (c), let $a \in \mathfrak{L}_K^s$ be an element with the property that $\nu_{E/K}(a) = N \cdot x$ for some $x \in \mathfrak{L}_E^s$. Then $b := \nu_{E/K}^{-1}(x)$ is an element of $\mathbb{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_K]}^s \mathcal{O}_{K,S(K)}^\times$ and we now claim that δ_{Tb} belongs to $\bigcap_{\mathbb{Z}[\mathcal{G}_K]}^s \mathcal{O}_{K,S(K),T}^\times$ for every set T in $\mathcal{P}_E^{\text{ad}}$.

If T belongs to $\mathcal{P}_E^{\text{ad}}$, then $\delta_{T,E} x \in (\bigcap_{\mathbb{Z}[\mathcal{G}_E]}^s \mathcal{O}_{E,S(E),T}^\times)^{\text{Gal}(E/K)}$ and so Lemma (2.10) implies that $\delta_{T,K} b \in \bigcap_{\mathbb{Z}[\mathcal{G}_K]}^s \mathcal{O}_{K,S(E),T}^\times$. Moreover, since the cokernel of the inclusion $\mathcal{O}_{K,S(K),T}^\times \rightarrow \mathcal{O}_{K,S(E),T}^\times$ is torsion-free, the natural restriction map $(\mathcal{O}_{K,S(E),T}^\times)^* \rightarrow (\mathcal{O}_{K,S(K),T}^\times)^*$ is surjective and so one has

$$\bigcap_{\mathbb{Z}[\mathcal{G}_K]}^s \mathcal{O}_{K,S(K),T}^\times = \left(\mathbb{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_K]}^s \mathcal{O}_{K,S(K)}^\times \right) \cap \left(\bigcap_{\mathbb{Z}[\mathcal{G}_K]}^s \mathcal{O}_{K,S(E),T}^\times \right)$$

by the argument of [BKS16, Lem. 4.7 (ii)]. Thus, b is contained in $\bigcap_{\mathbb{Z}[\mathcal{G}_K]}^s \mathcal{O}_{K,S(K),T}^\times$ and hence satisfies the defining condition of \mathfrak{L}_K^s at all sets T in $\mathcal{P}_E^{\text{ad}}$. Since $\text{Ann}_{\mathbb{Z}[\mathcal{G}_K]}(\mu_K)$ is generated by $\delta_{T,K}$ with T ranging over $\mathcal{P}_E^{\text{ad}}$, we deduce that b belongs to the lattice

$$\tilde{\mathfrak{L}}_K^s := \{c \in \mathbb{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_K]}^s \mathcal{O}_{K,S(K)}^\times \mid \text{Ann}_{\mathbb{Z}[\mathcal{G}_K]}(\mu_K) \cdot c \in \bigcap_{\mathbb{Z}[\mathcal{G}_K]}^s \mathcal{O}_{K,S(K)}^\times\}.$$

The equation $a = N \cdot b$ therefore shows that a is divisible by N in $\tilde{\mathfrak{L}}_K^s$. Now, $\tilde{\mathfrak{L}}_K^s$ is a finitely generated \mathbb{Z} -module and so the claim follows by taking \mathcal{N} to be the finite set of integers that

a is divisible by in $\tilde{\mathfrak{L}}_K^s$.

To prove claim (d) we take $a \in (\mathfrak{L}_E^s)^{\mathcal{G}_E}$ and set $b := \nu_{E/k}^{-1}(a)$, which is a well-defined element of $\mathbb{Q} \wedge_{\mathbb{Z}}^s \mathcal{O}_{k,S(E)}^\times$ because $\nu_{E/k}$ defines an isomorphism $\mathbb{Q} \wedge_{\mathbb{Z}}^s \mathcal{O}_{k,S(E)}^\times \cong (\mathbb{Q} \wedge_{\mathbb{Z}[\mathcal{G}_E]}^s \mathcal{O}_{E,S(E)}^\times)^{\mathcal{G}_E}$. Given any $T \in \mathcal{P}_E^{\text{ad}}$, one has that $\delta_{T,E}a$ belongs to $(\bigcap_{\mathbb{Z}[\mathcal{G}_E]}^s \mathcal{O}_{E,S(E),T}^\times)^{\mathcal{G}_E}$ and hence, by Lemma (2.10), the element $\delta_{T,k}b$ is contained in $\bigwedge_{\mathbb{Z}}^s \mathcal{O}_{k,S(E),T}^\times$. In particular, $|\mu_k|a \in \bigwedge_{\mathbb{Z}}^s (\mathcal{O}_{k,S(E)}^\times)^{**}$. Conversely, if $|\mu_k|a \in \bigwedge_{\mathbb{Z}}^s (\mathcal{O}_{k,S(E)}^\times)^{**}$, then the argument of [Pop02, Thm. 5.5.1 (3)] shows that $\delta_T a \in \bigwedge_{\mathbb{Z}}^s \mathcal{O}_{k,S(E),T}^\times$ for all $T \in \mathcal{P}_k^{\text{ad}}$. As in the proof of (a), we then conclude that $\nu_{E/k}(a)$ belongs to $(\mathfrak{L}_E^s)^{\mathcal{G}_E}$, as required. \square

The following algebraic result is an analogue of [BS21a, Prop. A.4].

(2.10) Lemma. *Let G be a finite abelian group and M a finitely generated $\mathbb{Z}[G]$ -module. Then, for each subgroup $H \subseteq G$, there exists an isomorphism of $\mathbb{Z}[G/H]$ -modules*

$$\bigcap_{\mathbb{Z}[G/H]}^s (M^{**})^H \xrightarrow{\cong} \left(\bigcap_{\mathbb{Z}[G]}^s M \right)^H$$

that is induced by the assignment $N_H^s a \mapsto N_{Ha}$ on $\mathbb{Q} \wedge_{\mathbb{Z}[G/H]}^s M^H \rightarrow \mathbb{Q} \wedge_{\mathbb{Z}[G]}^s M$.

Proof. We may assume that $s > 0$. Choose a free presentation $F_1 \rightarrow F_0 \rightarrow M^* \rightarrow 0$ of the linear dual M^* of M , where F_0 and F_1 are finitely generated free $\mathbb{Z}[G]$ -modules. Dualising this presentation then gives an exact sequence

$$0 \longrightarrow M^{**} \longrightarrow F_0^* \longrightarrow F_1^*.$$

Upon applying the general result of Sakamoto [Sak20, Lem. B.12] to both this sequence and its H -invariants, we therefore obtain a composite isomorphism of the required form

$$\begin{aligned} \bigcap_{\mathbb{Z}[G/H]}^s (M^{**})^H &\cong \ker \left\{ \bigwedge_{\mathbb{Z}[G/H]}^s (F_0^*)^H \rightarrow (F_0^*)^H \otimes_{\mathbb{Z}[G/H]} \bigwedge_{\mathbb{Z}[G/H]}^{s-1} (F_1^*)^H \right\} \\ &\cong \ker \left\{ \left(\bigwedge_{\mathbb{Z}[G]}^s F_0^* \right)^H \rightarrow \left(F_0^* \otimes_{\mathbb{Z}[G]} \bigwedge_{\mathbb{Z}[G]}^{s-1} F_1^* \right)^H \right\} \\ &\cong \left(\bigcap_{\mathbb{Z}[G]}^s M \right)^H, \end{aligned}$$

where the second isomorphism is induced by the isomorphism $\bigwedge_{\mathbb{Z}[G/H]}^s (F_0^*)^H \cong (\bigwedge_{\mathbb{Z}[G]}^s F_0^*)^H$ that sends each $N_H^s a$ to N_{Ha} . \square

(2.11) Remark. Let \mathcal{S} be a finite set of prime numbers. Then the proof of Lemma (2.9) shows that all of the stated claims remain valid for the $\mathbb{Z}_{\mathcal{S}}[\mathcal{G}_K]$ -lattices $\mathbb{Z}_{\mathcal{S}}\mathfrak{L}_K^s$ if in claim (c) one restricts to integers outside \mathcal{N} that are coprime to \mathcal{S} .

2.2.2. Integrality restrictions

Let \mathcal{R} be a subring of \mathbb{R} and \mathcal{X} a subset of $\Omega(k)$. We now introduce several $\mathcal{R}[\mathcal{G}_K]$ -submodules of $\text{ES}_k^{r,\mathcal{X}}(\mathbb{R})$ that play an important role in our theory (and are in part motivated by the properties of the ‘Rubin-Stark Euler system’ discussed in the next section).

We start by specifying the notion of ‘integrality’ that is central to our approach.

(2.12) Definition. *A system c in $\text{ES}_k^{r,\mathcal{X}}(\mathbb{R})$ is said to be \mathcal{R} -integral if one has*

$$c_E \in \mathcal{R} \cdot \mathfrak{L}_E^{r(E)} \quad \text{for all } E \in \mathcal{X}. \quad (5)$$

We write $\text{ES}_k^{r,\mathcal{X}}(\mathcal{R})$ for the $\mathcal{R}[\mathcal{G}_K]$ -submodule of $\text{ES}_k^{r,\mathcal{X}}(\mathbb{Q})$ comprising \mathcal{R} -integral systems. (If $\mathcal{X} = \Omega(k)$, then we suppress explicit reference to \mathcal{X} in the notation.)

In most cases, the module $\text{ES}_k^r(\mathcal{R})$ contains a non-zero submodule that is elementary in nature. To describe the submodule, we let \mathfrak{p} be a place of k of residue characteristic p and E a field in $\Omega(k)$ with $S(E) = \{\mathfrak{p}\}$. In this setting, Lemma (2.9) (d) implies that the map $\nu_{E/k}$ induces an isomorphism

$$|\mu_k|^{-1} \cdot \bigwedge_{\mathbb{Z}}^{r(E)} (\mathcal{O}_{k,\{\mathfrak{p}\}}^\times)^{**} \xrightarrow{\simeq} (\mathcal{L}_E^{r(E)})_{\mathcal{G}_E}.$$

Set $k(\mathfrak{p}^\infty) := \bigcup_{n \in \mathbb{N}} k(\mathfrak{p}^n)$ and write $t_{\mathfrak{p}}$ for the cardinality of $(\mathcal{G}_{k(\mathfrak{p}^\infty)})_{\text{tor}}$. For each $E \in \Omega(k)$ and each element a of the lattice $|\mu_k|^{-1} \cdot \bigwedge_{\mathcal{R}}^{r(E)} (\mathcal{R}\mathcal{O}_{k,\{\mathfrak{p}\}}^\times)^{**}$, we set

$$c_{\mathfrak{p}}(a)_E := \begin{cases} [E : k]^{-1} \cdot t_{\mathfrak{p}} \cdot \nu_{E/k}(a) & \text{if } E \subseteq k(\mathfrak{p}^\infty), \\ 0 & \text{otherwise.} \end{cases}$$

Then, if either p is a unit in \mathcal{R} or the extension $k(\mathfrak{p}^\infty)/k$ is finite, the family

$$c_{\mathfrak{p}}(a) := (c_{\mathfrak{p}}(a)_E)_{E \in \Omega(k)}$$

belongs to $\text{ES}_k^r(\mathcal{R})$. In this way one obtains a canonical homomorphism of $\mathcal{R}[[\mathcal{G}_{\mathcal{K}}]]$ -modules

$$\prod_{\mathfrak{p}} |\mu_k|^{-1} \bigwedge_{\mathcal{R}}^{r(k(\mathfrak{p}))} (\mathcal{R}\mathcal{O}_{k,\{\mathfrak{p}\}}^\times)^{**} \rightarrow \text{ES}_k^r(\mathcal{R}), \quad (a_{\mathfrak{p}})_{\mathfrak{p}} \mapsto \left(\prod_{\mathfrak{p}} c_{\mathfrak{p}}(a_{\mathfrak{p}}) \right), \quad (6)$$

where the product is over all \mathfrak{p} in $S_{\text{fin}}(k)$ that divide a prime $p \in \mathcal{R}^\times$ or are such that $k(\mathfrak{p}^\infty)/k$ is finite. For any system in the image of this map, the components are trivial except possibly on fields of prime-power conductor and there are no non-trivial distribution relations between components at fields of coprime conductors. This fact motivates the following definition.

(2.13) Definition. *The image $\text{ES}_k^r(\mathcal{R})^{\text{iso}}$ of the map (6) is the $\mathcal{R}[[\mathcal{G}_{\mathcal{K}}]]$ -module of **isolated** (\mathcal{R} -integral) Euler systems.*

A conceptual description of the module $\text{ES}_k^r(\mathcal{R})^{\text{iso}}$ will be given in Lemma (5.4) below. For the moment, we note only that $\text{ES}_k^r(\mathcal{R})^{\text{iso}}$ is non-trivial if $r(E) \leq |S_\infty(k)|$ for all $E \in \Omega^{S_\infty(k)}(k)$ and there exists a prime number $p \in \mathcal{R}^\times$ or a place \mathfrak{p} in $S_{\text{fin}}(k)$ for which $k(\mathfrak{p}^\infty)/k(1)$ is finite and non-trivial.

(2.14) Remark. The subset of $S_{\text{fin}}(k)$ comprising those \mathfrak{p} for which $k(\mathfrak{p}^\infty)/k(1)$ is a finite non-trivial extension is empty if and only if k is either \mathbb{Q} or imaginary quadratic. To explain this, we fix a prime number p that splits completely in k and let \mathfrak{p} denote a place of k above p . Global class field theory then identifies $\text{Gal}(k(\mathfrak{p}^\infty)/k(1))$ with the quotient of \mathbb{Z}_p^\times by the image of the global units of k^\times . If k is not \mathbb{Q} or an imaginary quadratic field, then the global units are infinite and so said quotient must be finite. This combines with Lemma (2.1) (b) to imply that $k(\mathfrak{p}^\infty)/k(1)$ is finite and non-trivial if \mathfrak{p} lies above a prime number p that splits completely in $k(\mu_\ell, (\mathcal{O}_k^\times)^{1/\ell})$ for some prime number $\ell \neq p$. On the other hand, if k is either \mathbb{Q} or imaginary quadratic, then every extension $k(\mathfrak{p}^\infty)/k$ is infinite.

The following submodule of $\text{ES}_k(\mathcal{R}) = \text{ES}_k^{r_{\text{max}}}(\mathcal{R})$ will also play an important role in our theory.

(2.15) Definition. *An (\mathcal{R} -integral) symmetric Euler system is a family c in $\text{ES}_k(\mathcal{R})$ that has the following property: there exists an element $c_k \in \mathcal{R} \bigwedge_{\mathbb{Z}}^{r_k} \mathcal{O}_k^\times$ (the ‘initial value’ of c) such that for all $E \in \Omega(k)$ and $\mathfrak{p} \in S_{\text{fin}}(k) \cap S(E)$ there is, in $\bigoplus_{i=0}^{\infty} (\mathcal{R} \bigwedge_{\mathbb{Z}}^i \mathcal{O}_{k,S(E)}^\times)$, an equality*

$$N_{E/k}^{r_E}(c_E) = \left(\prod_{v \in S(E) \setminus \{\mathfrak{p}\}} (1 - \text{Frob}_v^{-1}) \right) \cdot \text{Ord}_{\mathfrak{p}}^{-1}(c_k),$$

where

$$\text{Ord}_{\mathfrak{p}} : \mathbb{R} \bigwedge_{\mathbb{Z}}^{r_k+1} \mathcal{O}_{k,\{\mathfrak{p}\}}^\times \rightarrow \mathbb{R} \bigwedge_{\mathbb{Z}}^{r_k} \mathcal{O}_k^\times$$

is the isomorphism induced by the normalised valuation $\text{ord}_{\mathfrak{p}} : \mathcal{O}_{k,\{\mathfrak{p}\}}^\times \rightarrow \mathbb{Z}$ at \mathfrak{p} . The collection $\text{ES}_k(\mathcal{R})^{\text{sym}}$ of such systems is an $\mathcal{R}[[\mathcal{G}_{\mathcal{K}}]]$ -submodule of $\text{ES}_k(\mathcal{R})$.

(2.16) Remark. Fix c in $\text{ES}_k(\mathcal{R})$, a finite place \mathfrak{p} of k and a field $E \in \Omega$ with $E \subseteq k(\mathfrak{p}^\infty)$. Then the element $N_{E/k}^{r_E}(c_E)$ of $\mathbb{R} \wedge_{\mathbb{Z}}^{r_k+1} \mathcal{O}_{k,\{\mathfrak{p}\}}^\times$ is independent of the choice of E , and c is symmetric (in the above sense) if and only if $\text{Ord}_{\mathfrak{p}}(N_{E/k}^{r_E}(c_E))$ is independent of \mathfrak{p} .

The following consequence of the Chebotarev's Density Theorem for symmetric systems is important in the sequel.

(2.17) Lemma. *If \mathcal{R} denotes either $\mathbb{Z}_{\mathcal{S}}$ for a finite set of prime numbers \mathcal{S} or \mathbb{Z}_p for a prime number p , then $\text{ES}_k(\mathcal{R})^{\text{iso}} \cap \text{ES}_k(\mathcal{R})^{\text{sym}} = \{0\}$.*

Proof. It is enough to show that for every system c in $\text{ES}_k(\mathcal{R})^{\text{iso}} \cap \text{ES}_k(\mathcal{R})^{\text{sym}}$ the initial value c_k vanishes. Indeed, if \mathfrak{p} is any place of k and $E \subseteq k(\mathfrak{p}^\infty)$ is a finite ramified extension of k , then the assumption $c_k = 0$ implies $N_{E/k}^{r_E}(c_E) = 0$ and hence that $c_E = 0$ because, by construction, c_E is fixed by \mathcal{G}_E .

Now, if we suppose there exists a finite place \mathfrak{p} of k that neither divides a prime in \mathcal{S} nor is such that $k(\mathfrak{p}^\infty)/k$ is finite, then by definition $c_{k(\mathfrak{p})} = 0$ and so the initial value condition implies that $c_k = 0$. We may therefore assume that all but finitely many places of k are such that $k(\mathfrak{p}^\infty)/k$ is a finite extension.

Then, since c is isolated, the element $c_{k(\mathfrak{p})}$ is fixed by $\mathcal{G}_{k(\mathfrak{p})}$ for every such \mathfrak{p} and so the element

$$(\nu_{k(\mathfrak{p})/k} \circ \text{Ord}_{\mathfrak{p}}^{-1})(c_k) = (-1)^{r_k} \cdot (\nu_{k(\mathfrak{p})/k} \circ N_{k(\mathfrak{p})/k}^{r_k+1})(c_{k(\mathfrak{p})}) = (-1)^{r_k} \cdot [k(\mathfrak{p}) : k] c_{k(\mathfrak{p})}$$

is divisible by $[k(\mathfrak{p}) : k]$ in $\mathfrak{L}_{k(\mathfrak{p})}$. It now follows from Lemma (2.9) (d) that $\text{Ord}_{\mathfrak{p}}^{-1}(c_k)$ is divisible by $[k(\mathfrak{p}) : k]$ in $|\mu_k|^{-1} \wedge_{\mathbb{Z}_{\mathcal{S}}}^{r_k+1} \mathbb{Z}_{\mathcal{S}} \mathcal{O}_{k,\{\mathfrak{p}\}}^\times$, and hence that c_k is divisible by $[k(\mathfrak{p}) : k]$ in $|\mu_k|^{-1} \wedge_{\mathbb{Z}_{\mathcal{S}}}^{r_k} \mathbb{Z}_{\mathcal{S}} \mathcal{O}_k^\times$. Let p be a prime number that is not invertible in \mathcal{R} . It then suffices to show the degrees $[k(\mathfrak{p}) : k]$ are divisible by an unbounded power of p (as \mathfrak{p} varies).

To do this, we fix a natural number n and consider the finite Galois extension L of k obtained by adjoining p^n -th roots of all elements in \mathcal{O}_k^\times . Then it is enough to note that, by Chebotarev's Density Theorem, there exist infinitely many places \mathfrak{p} of k , not lying above p , that are completely split in L and, for any such place, the degree of $k(\mathfrak{p})/k(1)$ is divisible by p^n (by Lemma (2.1)). This concludes the proof of the Lemma. \square

To describe another distinguished family of Euler systems, we fix finite extensions L and K of k in \mathcal{K} with $K \subseteq L$ and $V_L = V_K$, and set $H := \text{Gal}(L/K) \subseteq \mathcal{G}_L$. We also fix a place v in $S(L) \setminus S_\infty(k)$ that splits completely in K and is *tamely ramified* in L and a place $w = w_K$ of K above v . We write $\text{rec}_v: K^\times \rightarrow H$ for the composite of the canonical embedding of K into its completion K_w at w and the local reciprocity map $K_w^\times \rightarrow H_w \subseteq H$, where H_w denotes the decomposition group of w inside H . Writing $I(H)$ for the kernel of the projection map $\mathbb{Z}[\mathcal{G}_L] \rightarrow \mathbb{Z}[\mathcal{G}_K]$, we thereby obtain a canonical homomorphism of $\mathbb{Z}[\mathcal{G}_K]$ -modules

$$\text{Rec}'_v: \mathcal{O}_{K,S(K)}^\times \rightarrow I(H)/I(H)^2, \quad a \mapsto \sum_{\sigma \in \mathcal{G}_K} (\text{rec}_v(\sigma a) - 1) \sigma^{-1} \quad (7)$$

and hence also an induced composite homomorphism

$$\begin{aligned} \text{Rec}_v: \bigcap_{\mathbb{Z}[\mathcal{G}_K]}^{r_K+1} \mathcal{O}_{K,S(L),T}^\times &\xrightarrow{\text{Rec}'_v} \left(\bigcap_{\mathbb{Z}[\mathcal{G}_K]}^{r_K} \mathcal{O}_{K,S(L),T}^\times \otimes_{\mathbb{Z}[\mathcal{G}_K]} (I(H)/I(H)^2) \right) \\ &\xrightarrow{\nu_{L/K}} \left(\bigcap_{\mathbb{Z}[\mathcal{G}_L]}^{r_K} \mathcal{O}_{L,S(L),T}^\times \otimes_{\mathbb{Z}[\mathcal{G}_L]} (\mathbb{Z}[\mathcal{G}_L]/I(H)^2) \right) \end{aligned}$$

in which $\nu_{L/K}$ is induced by the isomorphism $\bigcap_{\mathbb{Z}[\mathcal{G}_K]}^{r_K} \mathcal{O}_{K,S(L),T}^\times \xrightarrow{\cong} \left(\bigcap_{\mathbb{Z}[\mathcal{G}_L]}^{r_K} \mathcal{O}_{L,S(L),T}^\times \right)^H$ from Lemma (2.10) and the inclusion $I(H) \subset \mathbb{Z}[\mathcal{G}_L]$.

We further recall from [Rub96, Lem. 5.1 (iii)] that the map

$$\text{Ord}_v: K^\times \rightarrow \mathbb{Z}[\mathcal{G}_K], \quad a \mapsto \sum_{\sigma \in \mathcal{G}_K} \text{ord}_w(\sigma a) \sigma^{-1}$$

induces an isomorphism (which we denote by the same symbol)

$$\mathrm{Ord}_v : e_K \mathbb{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_K]}^{r_K+1} \mathcal{O}_{K,S(L)}^\times \xrightarrow{\cong} e_K \mathbb{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_K]}^{r_K} \mathcal{O}_{K,S(K)}^\times.$$

We finally write $P_{L/K,\{v\}}$ for the Euler factor $\prod_{v' \in S(L) \setminus (S(K) \cup \{v\})} (1 - \mathrm{Frob}_{v'}^{-1})$.

(2.18) Definition. Fix a finite set of prime numbers \mathcal{S} . Then a **($\mathbb{Z}_{\mathcal{S}}$ -integral) congruence Euler system** is a family c in $\mathrm{ES}_k(\mathbb{Z}_{\mathcal{S}})$ that has the following property: for all data $L/K, T$ and v as above and every prime $p \notin \mathcal{S}$ that divides $2d_k$ (with d_k the absolute discriminant of k), the element $\mathrm{Ord}_v^{-1}(\delta_{K,Tc_K})$ belongs to $\bigcap_{\mathbb{Z}_p[\mathcal{G}_K]}^{r_K+1} \mathbb{Z}_p \mathcal{O}_{K,S(L),T}^\times$ and is such that

$$\begin{aligned} \delta_{L,T} \sum_{\sigma \in H} \sigma c_L \otimes \sigma^{-1} &= (-1)^{r_K} \cdot (\mathrm{Rec}_v \circ \mathrm{Ord}_v^{-1})(P_{L/K,\{v\}} \cdot \delta_{K,Tc_K}) \\ &\text{in } \left(\bigcap_{\mathbb{Z}[\mathcal{G}_L]}^{r_K} \mathcal{O}_{L,S(L),T}^\times \otimes_{\mathbb{Z}[\mathcal{G}_L]} (\mathbb{Z}[\mathcal{G}_L]/I(H)^2) \otimes_{\mathbb{Z}} \mathbb{Z}_p \right). \end{aligned}$$

The collection $\mathrm{ES}_k(\mathbb{Z}_{\mathcal{S}})^{\mathrm{con}}$ of all such systems is a $\mathbb{Z}_{\mathcal{S}}[\mathcal{G}_K]$ -submodule of $\mathrm{ES}_k(\mathbb{Z}_{\mathcal{S}})$.

(2.19) Remark. Whilst congruence relations play an important role in early articles concerning (rank one) Euler systems, such as Thaine [Tha88], it was subsequently shown by Rubin [Rub00, Ch. IV, § 8] that a weaker Iwasawa-theoretic form of these congruences can be directly deduced from Euler distribution relations. However, distribution relations on their own are not sufficient for our theory (see, for example, Remark (2.25) (d)) and the congruences described above provide an appropriate replacement in higher rank for the congruences used by Thaine. The precise form of these congruences is motivated by conjectures of Mazur–Rubin and Sano (see Lemma (2.22) (iii) and Remark (2.23)) and the role that they play in our approach is described in Proposition (5.5).

2.3. The Scarcity Conjecture

2.3.1. The Rubin–Stark Euler system

Let \mathcal{S} denote the set $S_\infty(k) \cup S_{\mathrm{fin}}(k)$ of all places of k . Then, by fixing a bijection $\mathcal{S} \cong \mathbb{N}$ we may regard \mathcal{S} as a totally ordered set $(\mathcal{S}, \preceq) = \{v_i\}_{i \in \mathbb{N}}$ so that $v_i \preceq v_j$ if and only if $i \leq j$. We choose this bijection in such a way that the first $|S_\infty(k)|$ places in \mathcal{S} are the places in $S_\infty(k)$. Throughout this article all exterior powers $\bigwedge_{v \in \Sigma}$ indexed over a given finite set of places Σ of k will be arranged with respect to the ordering \preceq . We also fix, for each place $v \in \mathcal{S}$, an extension \bar{v} of v to our fixed choice of algebraic closure $\bar{\mathbb{Q}}$ of \mathbb{Q} . For any subfield $K \subseteq \bar{k}$ we write v_K for the restriction of \bar{v} to K , and also set

$$S^*(K) := S(K) \cup S_\infty(k).$$

Let K be a finite abelian extension of k and assume to be given disjoint finite sets of places Σ and T of k such that Σ contains $S^*(K)$. Then, for any character $\chi \in \widehat{\mathcal{G}_K}$, the **T -modified Σ -imprimitive Dirichlet L -series** for K/k and χ is defined by setting

$$L_{k,\Sigma,T}(\chi, s) = \prod_{v \in T} (1 - \chi(\mathrm{Frob}_v) N v^{1-s}) \cdot \prod_{v \notin \Sigma} (1 - \chi(\mathrm{Frob}_v) N v^{-s})^{-1},$$

where s is a complex variable of real part $\mathrm{Re}(s) > 1$. It is well-known that $L_{k,\Sigma,T}(\chi, s)$ admits a meromorphic continuation to \mathbb{C} . We recall from [Tat84, Ch. I, Prop. 3.4] that the order of vanishing of $L_{k,\Sigma,T}(\chi, s)$ at $s = 0$ is given by the number $r_\Sigma(\chi)$ defined in (2).

We moreover define the leading term of the T -modified Σ -imprimitive equivariant L -series of K/k to be

$$\theta_{K/k,\Sigma,T}^*(0) = \sum_{\chi \in \widehat{\mathcal{G}_K}} \left(\lim_{s \rightarrow 0} s^{-r_\Sigma(\chi)} L_{k,\Sigma,T}(\chi, s) \right) e_{\chi^{-1}}$$

and note $\theta_{K/k,\Sigma,T}^*(0)$ belongs to $\mathbb{R}[\mathcal{G}_K]^\times$. Finally, we recall that the Dirichlet regulator map

$$\lambda_{K,\Sigma} : \mathcal{O}_{K,\Sigma}^\times \rightarrow \mathbb{R} \otimes_{\mathbb{Z}} X_{K,\Sigma}, \quad a \mapsto - \sum_{w \in \Sigma_K} \log |a|_w \cdot w$$

induces an isomorphism of $\mathbb{R}[\mathcal{G}_K]$ -modules

$$\lambda_{K,\Sigma}^{(r)} : \mathbb{R} \otimes_{\mathbb{Z}} \bigwedge_{\mathbb{Z}[\mathcal{G}_K]}^r \mathcal{O}_{K,\Sigma}^\times \xrightarrow{\cong} \mathbb{R} \otimes_{\mathbb{Z}} \bigwedge_{\mathbb{Z}[\mathcal{G}_K]}^r X_{K,\Sigma}, \quad (8)$$

that will often also be abbreviated to $\lambda_{K,\Sigma}$ (if no confusion is possible).

(2.20) Definition.

- (a) Let K be any finite abelian extension of k , $V \subsetneq \Sigma$ a finite set of places that split completely in K/k , set $r := |V|$ and fix a place $\mathfrak{p} \in \Sigma \setminus V$. The r -th order **Rubin–Stark element** for the data $K/k, \Sigma, T$ and V is the unique element $\varepsilon_{K/k,\Sigma,T}^V$ of $\mathbb{R} \otimes_{\mathbb{Z}} \bigwedge_{\mathbb{Z}[\mathcal{G}_K]}^r \mathcal{O}_{K,\Sigma}^\times$ such that

$$\lambda_{K,\Sigma}(\varepsilon_{K/k,\Sigma,T}^V) = e_{K,\Sigma,r} \theta_{K/k,\Sigma,T}^*(0) \cdot \bigwedge_{v \in V} (v_K - \mathfrak{p}_K),$$

where $e_{K,\Sigma,r}$ denotes the sum of e_χ over all characters $\chi \in \widehat{\mathcal{G}_K}$ with $r_\Sigma(\chi) = r$. (This element $\varepsilon_{K/k,\Sigma,T}^V$ does not depend on the choice of place $\mathfrak{p} \in \Sigma \setminus V$ – see [San15, Prop. 3.3].)

- (b) The **Rubin–Stark system** for k is the family

$$\varepsilon_k = (\varepsilon_{K/k})_K \in \prod_{K \in \Omega(k)} \mathbb{R} \otimes_{\mathbb{Z}} \bigwedge_{\mathbb{Z}[\mathcal{G}_K]}^{r_K} \mathcal{O}_{K,S(K)}^\times,$$

where we set $\varepsilon_{K/k} := \varepsilon_{K/k,S^*(K),\emptyset}^V$.

(2.21) Examples. In several cases, the above definition can be made more explicit for $r := r_K$.

- (a) (*Cyclotomic units*) Take k to be \mathbb{Q} , K to be a finite real abelian extension of \mathbb{Q} , and V to be $S_\infty(\mathbb{Q}) = \{v_0\}$. Then one has

$$\varepsilon_{K/\mathbb{Q}} = \frac{1}{2} \otimes N_{\mathbb{Q}(\xi_m)/K}(1 - \xi_m) \in \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O}_{K,S(K)}^\times,$$

Here $m = m_K$ is the conductor of K and $\xi_m = \iota^{-1}(e^{2\pi i/m})$ where $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ is the embedding corresponding to the choice of place \bar{v}_1 fixed at the beginning of the section. (See [Tat84, Ch. III, § 5] for a proof of this fact.)

- (b) (*Stickelberger elements*) Let k be a totally real field, K a finite abelian CM extension of k , and $V = \emptyset$. In this setting the Rubin–Stark element is given by

$$\varepsilon_{K/k} = \theta_{K/k,S^*(K),\emptyset}(0) := \sum_{\chi \in \widehat{\mathcal{G}_K}} L_{k,\Sigma,T}(\chi, 0) e_\chi^{-1}.$$

- (c) (*Elliptic units*) Let k be an imaginary quadratic field and $K \in \Omega(k)$. Fix a place $\mathfrak{p} \in S(K)$ and write $\mathfrak{f} = \mathfrak{p}^n$ for a power of \mathfrak{p} large enough so that the natural map $\mathcal{O}_k^\times \rightarrow (\mathcal{O}_k/\mathfrak{f})^\times$ is injective. Write \mathfrak{m} for the conductor of K and let $\mathfrak{a} \subsetneq \mathcal{O}_k$ be an auxiliary prime ideal coprime to $6\mathfrak{f}\mathfrak{m}$. Then the Rubin–Stark Conjecture holds for E (see, for example, [Tat84, Ch. IV, Prop. 3.9]) with the elliptic unit

$$\varepsilon_{K/k} = (\text{Frob}_{\mathfrak{a}} - N_{\mathfrak{a}})^{-1} \cdot N_{k(\mathfrak{f}\mathfrak{m})/K}(\psi(1; \mathfrak{f}\mathfrak{m}, \mathfrak{a})) \in \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O}_{K,S(K)}^\times.$$

(This follows from Kronecker’s second limit formula; see, for instance, [Fla09, Lem. 2.2 e]).

In the next result we record several key properties of the Rubin–Stark system. In claim (ii) of this result we refer to the central conjecture formulated by Rubin in [Rub96] and in claim (iii) to the ‘refined class number formula’ that was conjectured independently by Mazur and Rubin in [MR16] and by Sano in [San14].

(2.22) Lemma. (i) The system ε_k is symmetric and its initial value $\varepsilon_{k,k} \in \mathbb{R} \bigwedge_{\mathbb{Z}}^{r_k} \mathcal{O}_k^\times$ is such that

$$\lambda_{k,S_\infty(k)}(\varepsilon_{k,k}) = -\zeta_k^*(0) \cdot \bigwedge_{v \in S_\infty(k) \setminus \{v_1\}} (v - v_1) \in \mathbb{R} \bigwedge_{\mathbb{Z}}^{r_k} X_{k,S_\infty(k)},$$

with $\zeta_k^*(0) = \theta_{k/k,S_\infty(k)}^*(0)$ the leading term of the Dedekind ζ -function of k .

(ii) The system ε_k is \mathbb{Z} -integral if the Rubin–Stark Conjecture [Rub96, Conj. B'] for L/k is valid for all L in Ω .

(iii) The system ε_k is a congruence system if the Mazur–Rubin–Sano Conjecture [BKS16, Conj. 5.4] for L/k is valid for all L in Ω .

Proof. The fact that ε_k validates the distribution relations in Definition (2.3), and hence belongs to $\text{ES}_k(\mathbb{R})$, is proved by Rubin in [Rub96, Prop. 6.1] (see also [San14, Prop. 3.5]). To prove claim (i), it therefore suffices to show that, for every field $E \in \Omega(k)$ and any choice of place $\mathfrak{p} \in S_{\text{fin}}(k) \cap S(E)$, one has

$$N_{E/k}^{r_E}(\varepsilon_{E/k}) = \left(\prod_{v \in S(E) \setminus \{\mathfrak{p}\}} (1 - \text{Frob}_v^{-1}) \right) \cdot \text{Ord}_{\mathfrak{p}}^{-1}(\varepsilon_{k,k})$$

in $\bigoplus_{i=0}^{\infty} (\mathbb{R} \bigwedge_{\mathbb{Z}}^i \mathcal{O}_{k,S(E)}^\times)$. We also note that this condition is satisfied trivially except in the case that the conductor of E is a power of a prime \mathfrak{p} and $V_E = S_\infty(k)$.

It is therefore enough to show that the specified element $\varepsilon_{k,k}$ has the required property in the case that the only finite place contained in $S^*(E)$ is \mathfrak{p} and $V_E = S_\infty(k)$. If we write $v^*: \mathbb{R} Y_{k,S^*(E)} \rightarrow \mathbb{R}$ for the dual map of a place $v \in S^*(E)$ (considered as an element of $Y_{k,S^*(E)}$), then the map $\bigwedge_{v \in S_\infty(k) \setminus \{v_1\}} v^*$ defines an isomorphism $\mathbb{R} \bigwedge_{\mathbb{Z}}^{r_k} X_{k,S_\infty(k)} \cong \mathbb{R}$ with the property that

$$\left(\bigwedge_{v \in S_\infty(k) \setminus \{v_1\}} v^* \right) (\zeta_k^*(0) \cdot \bigwedge_{v \in S_\infty(k) \setminus \{v_1\}} (v - v_1)) = \zeta_k^*(0).$$

It therefore suffices to show that

$$\left(\left(\bigwedge_{v \in S_\infty(k) \setminus \{v_1\}} v^* \right) \circ \lambda_{k,S_\infty(k)}^{(r_k)} \circ \text{Ord}_{\mathfrak{p}} \circ N_{E/k}^{r_E} \right) (\varepsilon_{E/k}) = -\zeta_k^*(0)$$

To do this, we first note that, by [Rub96, Prop. 6.1], one has $N_{E/k}^{r_E}(\varepsilon_{E/k,S^*(E),\emptyset}^{V_E}) = \varepsilon_{k/k,S^*(E),\emptyset}^{V_E}$, which is the unique element of $\mathbb{R} \bigwedge_{\mathbb{Z}}^{r_E} \mathcal{O}_{k,S^*(E)}^\times$ such that

$$\begin{aligned} \lambda_{k,S^*(E)}^{(r_E)}(\varepsilon_{k/k,S^*(E),\emptyset}^{V_E}) &= \zeta_{k,S^*(E)}^*(0) \cdot \bigwedge_{v \in S_\infty(k)} (v - \mathfrak{p}) \\ &= -\zeta_{k,S^*(E)}^*(0) \cdot (\mathfrak{p} - v_1) \wedge (v_2 - v_1) \wedge \cdots \wedge (v_{r_k+1} - v_1). \end{aligned}$$

Moreover, it is easy to see that $(\log N_{\mathfrak{p}}) \cdot \text{ord}_{\mathfrak{p}} = \mathfrak{p}^* \circ \lambda_{k,S^*(E)}$. In addition, for any subset $M \subseteq S^*(E) \setminus \{v_1\}$ one has $(\bigwedge_{v \in M} v^*) \circ \lambda_{k,M \cup \{v_1\}}^{(|M|)} = \bigwedge_{v \in M} (v^* \circ \lambda_{k,M \cup \{v_1\}})$ and, for any M' containing M , the restriction of $\lambda_{k,M'}$ to $\mathbb{R} \mathcal{O}_{k,M}^\times$ is equal to $\lambda_{k,M \cup \{v_1\}}$. It follows that

$$\begin{aligned} \left(\bigwedge_{v \in S_\infty(k) \setminus \{v_1\}} v^* \right) \circ \lambda_{k,S_\infty(k)}^{(r_k)} \circ \text{Ord}_{\mathfrak{p}} &= \text{ord}_{\mathfrak{p}} \wedge \left(\bigwedge_{v \in S_\infty(k) \setminus \{v_1\}} (v^* \circ \lambda_{k,S^*(E)}) \right) \\ &= (\log N_{\mathfrak{p}})^{-1} \cdot (\mathfrak{p}^* \circ \lambda_{k,S^*(E)}) \wedge \left(\bigwedge_{v \in S_\infty(k) \setminus \{v_1\}} (v^* \circ \lambda_{k,S^*(E)}) \right) \\ &= (-1)^{r_k} \cdot (\log N_{\mathfrak{p}})^{-1} \cdot \bigwedge_{v \in S^*(E) \setminus \{v_1\}} (v^* \circ \lambda_{k,S^*(E)}) \\ &= (-1)^{r_k} \cdot (\log N_{\mathfrak{p}})^{-1} \cdot \left(\bigwedge_{v \in S^*(E) \setminus \{v_1\}} v^* \right) \circ \lambda_{k,S^*(E)}^{(r_E)}. \end{aligned}$$

We may thus calculate that

$$\begin{aligned}
& \left(\left(\bigwedge_{v \in S_\infty(k) \setminus \{v_1\}} v^* \right) \circ \lambda_{k, S_\infty(k)}^{(r_k)} \circ \text{Ord}_{\mathfrak{p}} \circ N_{E/k}^{r_E} \right) (\varepsilon_{E/k}) \\
&= (-1)^{r_k} \cdot (\log \mathbf{N}\mathfrak{p})^{-1} \cdot \left(\bigwedge_{v \in S^*(E) \setminus \{v_1\}} v^* \right) \left(-\zeta_{k, S^*(E)}^*(0) \cdot (\mathfrak{p} - v_1) \wedge (v_2 - v_1) \wedge \cdots \wedge (v_{r_k+1} - v_1) \right) \\
&= (-1)^{r_k} \cdot (-1)^{r_k} \cdot (-1) \cdot (\log \mathbf{N}\mathfrak{p})^{-1} \cdot \zeta_{k, S^*(E)}^*(0) \\
&= -\zeta_k^*(0),
\end{aligned}$$

as required to prove claim (i).

Claims (ii) and (iii) follow directly from the statements of the respective conjectures. \square

(2.23) Remark. The strong restrictions imposed on v and p in Definition (2.18), and the fact that only a single place in $S(L) \setminus S_\infty(k)$ is considered, means that the conditions required to ensure ε_k belongs to $\text{ES}_k(\mathbb{Z})^{\text{con}}$ are much weaker than are the general properties predicted by Mazur–Rubin and Sano. This fact plays a key role in later sections.

2.3.2. Statement of the Scarcity Conjecture

For any subset \mathcal{X} of $\Omega(k)$ and any finite set \mathcal{S} of prime numbers, we consider the $\mathbb{Z}_{\mathcal{S}}[[\mathcal{G}_{\mathcal{K}}]]$ -submodules of $\text{ES}_k^{\mathcal{X}}(\mathbb{Z}_{\mathcal{S}})$ that are obtained by setting

$$\text{ES}_k^{\mathcal{X}}(\mathbb{Z}_{\mathcal{S}})^{\text{sym}} := \varrho^{\mathcal{X}}(\text{ES}_k(\mathbb{Z}_{\mathcal{S}})^{\text{sym}}) \quad \text{and} \quad \text{ES}_k^{\mathcal{X}}(\mathbb{Z}_{\mathcal{S}})^{\text{con}} := \varrho^{\mathcal{X}}(\text{ES}_k(\mathbb{Z}_{\mathcal{S}})^{\text{con}}).$$

In particular, we note that Lemma (2.22) implies, modulo conjectures of Rubin and of Mazur–Rubin–Sano, that the ‘ \mathcal{X} -restricted Rubin–Stark system’

$$\varepsilon_k^{\mathcal{X}} := \varrho^{\mathcal{X}}(\varepsilon_k) = (\varepsilon_{E/k})_{E \in \mathcal{X}}$$

belongs to both $\text{ES}_k^{\mathcal{X}}(\mathbb{Z}_{\mathcal{S}})^{\text{sym}}$ and $\text{ES}_k^{\mathcal{X}}(\mathbb{Z}_{\mathcal{S}})^{\text{con}}$.

We now state the central conjecture of this article. This conjecture simultaneously continues ideas of Coleman, of Rubin and of Sano and three of the current authors (for more details see Remarks (3.3) (b) and (3.11) below).

(2.24) Conjecture (The ‘Scarcity Conjecture’). *For each finite set of prime numbers \mathcal{S} , and each subset \mathcal{X} of $\Omega(k)$ that is disjoint from $\Omega^{\mathcal{Q}}(k)$, one has*

$$\text{ES}_k^{\mathcal{X}}(\mathbb{Z}_{\mathcal{S}})^{\text{sym}} \cap \text{ES}_k^{\mathcal{X}}(\mathbb{Z}_{\mathcal{S}})^{\text{con}} = \mathbb{Z}_{\mathcal{S}}[[\mathcal{G}_{\mathcal{K}}]] \cdot \varepsilon_k^{\mathcal{X}}.$$

(2.25) Remark.

- (a) The requirement that \mathcal{X} is disjoint from $\Omega^{\mathcal{Q}}(k)$ (and hence that \mathcal{X} contains no field K with $r_K = 0$) is forced by the fact Euler systems of rank zero and of positive rank are seemingly of an essentially different nature. In particular, the recent work of Sakamoto in [Sak22] suggests many Euler systems of rank zero do not belong to the submodule generated by the system of Stickelberger elements discussed in Example (2.21)(b).
- (b) If $\mathcal{X} \cap \Omega^{S_\infty(k)}(k) = \emptyset$, then (as a direct consequence of the definition of symmetric systems) one has $\text{ES}_k^{\mathcal{X}}(\mathbb{Z}_{\mathcal{S}})^{\text{sym}} = \text{ES}_k^{\mathcal{X}}(\mathbb{Z}_{\mathcal{S}})$ and so Conjecture (2.24) predicts that the $\mathbb{Z}_{\mathcal{S}}[[\mathcal{G}_{\mathcal{K}}]]$ -module $\text{ES}_k^{\mathcal{X}}(\mathbb{Z}_{\mathcal{S}})^{\text{con}}$ is generated by $\varepsilon_k^{\mathcal{X}}$.
- (c) If \mathcal{S} contains all prime numbers that divide $2d_k$, then $\text{ES}_k(\mathbb{Z}_{\mathcal{S}})^{\text{con}} = \text{ES}_k(\mathbb{Z}_{\mathcal{S}})$ and so Conjecture (2.24) predicts that the $\mathbb{Z}_{\mathcal{S}}[[\mathcal{G}_{\mathcal{K}}]]$ -module $\text{ES}_k^{\mathcal{X}}(\mathbb{Z}_{\mathcal{S}})^{\text{sym}}$ is generated by $\varepsilon_k^{\mathcal{X}}$.
- (d) If $k = \mathbb{Q}$, $\mathcal{S} = \emptyset$ and $\mathcal{X} = S_\infty(\mathbb{Q}) = \{\infty\}$, then (as a special case of (c)) Conjecture (2.24) predicts $\text{ES}_{\mathbb{Q}}^{\{\infty\}}(\mathbb{Z})^{\text{sym}}$ is generated over $\mathbb{Z}[[\mathcal{G}_{\mathcal{K}}]]$ by the cyclotomic system $\varepsilon_{\mathbb{Q}}^{\{\infty\}}$ described in Example (2.21)(a). In Theorem (6.12) this prediction is proved after replacing \mathbb{Z} by $\mathbb{Z}[1/2]$. In addition, in the supplementary article [Bul+23], we resolve the remaining 2-primary difficulties in order to fully verify the prediction, show that a system in $\text{ES}_{\mathbb{Q}}^{\{\infty\}}(\mathbb{Z})$

belongs to $\mathbb{Z}[[\mathcal{G}_K]] \cdot \varepsilon_{\mathbb{Q}}^{\{\infty\}}$ if and only if it verifies the classical congruences considered by Thaine in [Tha88] and prove that the set $\text{ES}_{\mathbb{Q}}^{\{\infty\}}(\mathbb{Z}) \setminus \mathbb{Z}[[\mathcal{G}_K]] \cdot \varepsilon_{\mathbb{Q}}^{\{\infty\}}$ is non-empty.

- (e) The theory developed below will in fact suggest the possibility of formulating a stronger version of Conjecture (2.24) (see Remark (5.3)).

The interest of Conjecture (2.24) will be explained, at least partly, by the main result of the next section. Then, in the remainder of the article, we shall develop, and apply, a general strategy for the proof of the conjecture (and hence of the results discussed in the Introduction).

3. Scarcity and Tamagawa numbers

3.1. Statement of the main result

For a finite group Δ we write e_{Δ} for the idempotent $|\Delta|^{-1} \sum_{\delta \in \Delta} \delta$ of the group ring $\mathbb{Q}[\Delta]$. For each subset V of $S_{\infty}(k)$, and any finite abelian extension K of k , we then define an idempotent of $\mathbb{Q}[\mathcal{G}_K]$ by setting

$$\epsilon_{K,V} := \prod_{v \in (S_{\infty} \setminus V)} (1 - e_{\mathcal{G}_{K,v}}) \cdot \prod_{v \in V} e_{\mathcal{G}_{K,v}}. \quad (9)$$

For any subset \mathcal{V} of the power set $\mathcal{P}(S_{\infty}(k))$ of $S_{\infty}(k)$ we then define

$$\epsilon_{K,\mathcal{V}} := \sum_{V \in \mathcal{V}} \epsilon_{K,V}.$$

For convenience, in the case $\mathcal{V} = \{V_K\}$ we also use the abbreviation

$$\epsilon_K := \epsilon_{K,\{V_K\}} = \prod_{\substack{v \in S_{\infty} \\ |\mathcal{G}_{K,v}| \neq 1}} (1 - e_{\mathcal{G}_{K,v}}) \quad (10)$$

(and we note that, in general, this element differs from the idempotent e_K defined in (3)).

(3.1) Remark. We record several useful properties of the above elements.

- (a) For each V one can check that $\epsilon_{K,V} = \sum_{\chi} e_{\chi}$, where χ runs over all characters in $\widehat{\mathcal{G}}_K$ with the property that, for each $v \in S_{\infty}$, one has $\chi(\mathcal{G}_{K,v}) = 1$ if and only if $v \in V$. This description implies that $\epsilon_{K,V}$ is orthogonal to $\epsilon_{K,V'}$ for any other subset V' of $S_{\infty}(k)$ and hence that, for each set \mathcal{V} , the element $\epsilon_{K,\mathcal{V}}$ is an idempotent.
- (b) If $V_K = S_{\infty}$ (as is the case, for example, if k is totally imaginary or K is totally real), then $\epsilon_K = 1$. If k is totally real and K is CM, then $V_K = \emptyset$ and $\epsilon_K = (1 - \tau)/2$, where τ denotes the element of \mathcal{G}_K induced by complex conjugation. In general, it is clear that ϵ_K belongs to $2^{-|S_{\infty}| + r_K} \cdot \mathbb{Z}[\mathcal{G}_K]$.
- (c) If K' is any finite abelian extension of k that contains K , then, for each set \mathcal{V} , the projection map $\mathbb{Q}[\mathcal{G}_{K'}] \rightarrow \mathbb{Q}[\mathcal{G}_K]$ sends $\epsilon_{K',\mathcal{V}}$ to $\epsilon_{K,\mathcal{V}}$.

In the sequel we write $\text{TNC}(h^0(\text{Spec } K), \epsilon_{K,\mathcal{V}} \mathbb{Z}_{\mathcal{S}}[\mathcal{G}_K])$ to refer to Kato's 'equivariant Tamagawa Number Conjecture' for the pair $(h^0(\text{Spec } K), \epsilon_{K,\mathcal{V}} \mathbb{Z}_{\mathcal{S}}[\mathcal{G}_K])$.

In this section we shall prove the following result.

(3.2) Theorem. *Let K be a finite abelian extension of k and fix a subset \mathcal{V} of $\mathcal{P}(S_{\infty}(k))$ that does not contain the empty set. Then Conjecture (2.24) for the pair $(\Omega^{\mathcal{V}}(k), \mathcal{S})$ implies $\text{TNC}(h^0(\text{Spec } K), \epsilon_{K,\mathcal{V}} \mathbb{Z}_{\mathcal{S}}[\mathcal{G}_K])$.*

(3.3) Remark. (a) The conjecture $\text{TNC}(h^0(\text{Spec } K), \epsilon_{K,\mathcal{V}} \mathbb{Z}_{\mathcal{S}}[\mathcal{G}_K])$ was formulated (up to an ambiguity of signs) by Kato in [Kat93a; Kat93b] and is stated precisely in, for example, [BKS16, Conj. 3.1].

- (b) The phrase ‘eTNC(G_m) for K/k ’ that is used in the Introduction refers to the conjecture $\text{TNC}(h^0(\text{Spec } K), \mathbb{Z}[\mathcal{G}_K])$. The validity of this conjecture has been shown to imply a wide range of more explicit conjectures including the Rubin–Stark Conjecture, the ‘refined class number formula’ conjectured by Mazur and Rubin and Sano, the ‘integral Gross–Stark conjecture’ (from [Gro88]), the ‘Lifted Root Number Conjecture’ of Gruenberg, Ritter and Weiss [GRW99], and the central conjecture of Chinburg in [Chi83]. Details of these, and other similar, deductions can be found, for example, in [BF98] and [BKS16].

3.2. The analytic class number formula

As a first step in the proof of Theorem (3.2), we will establish, in Proposition (3.6) below, a useful reinterpretation of the analytic class number formula in our setting. Before doing so, however, it is convenient to review the properties of an important family of complexes.

For any finite abelian extension E/k , finite set of places $\Sigma \supseteq S^*(E)$, and finite set T of places of k that is disjoint from Σ , we use the T -modified ‘Weil-étale cohomology’ complex $\text{R}\Gamma_{c,T}((\mathcal{O}_{E,\Sigma})_{\mathcal{W}}, \mathbb{Z})$ of the constant sheaf \mathbb{Z} that is constructed in [BKS16, Prop. 2.4], and consider its linear dual

$$C_{E,\Sigma,T}^\bullet = \text{R Hom}_{\mathbb{Z}}(\text{R}\Gamma_{c,T}((\mathcal{O}_{E,\Sigma})_{\mathcal{W}}, \mathbb{Z}), \mathbb{Z})[-2].$$

Whenever $T = \emptyset$ we will suppress the respective subscript in the notation. Lemma (3.4) below is taken from [Bur+23, Prop. 3.1] and sets out the essential properties of the complex $C_{E,\Sigma,T}^\bullet$. For a commutative noetherian ring R we write $D(R)$ for the derived category of R -modules and $D^{\text{perf}}(R)$ for the full triangulated subcategory of $D(R)$ comprising complexes that are ‘perfect’ (that is, isomorphic in $D(R)$ to a bounded complex of finitely generated projective R -modules).

(3.4) Lemma. *For any data E/k , Σ and T as above, the complex $C_{E,\Sigma,T}^\bullet$ belongs to $D^{\text{perf}}(\mathbb{Z}[\mathcal{G}_E])$ and has all of the following properties.*

- (a) *The complex $C_{E,\Sigma,T}^\bullet$ is acyclic outside degrees zero and one, and there are canonical identifications of $\mathbb{Z}[\mathcal{G}_E]$ -modules*

$$H^0(C_{E,\Sigma,T}^\bullet) = \mathcal{O}_{E,\Sigma,T}^\times, \quad H^1(C_{E,\Sigma,T}^\bullet)_{\text{tor}} = \text{Cl}_{E,\Sigma,T} \quad \text{and} \quad H^1(C_{E,\Sigma,T}^\bullet)_{\text{tf}} = X_{E,\Sigma}.$$

Here we write $\text{Cl}_{E,\Sigma,T}$ for the quotient of the group of fractional ideals of $\mathcal{O}_{E,\Sigma}$ that are prime to T_E by the subgroup of principal ideals with a generator congruent to 1 modulo all places in T_E (so $\text{Cl}_{E,\Sigma,T}$ is the ‘ Σ_E -ray class group mod T_E ’ of E).

- (b) *Let T' be a finite set of places of k that contains T and is disjoint from Σ , and write $\mathbb{F}_{E,T'\setminus T}^\times$ for the \mathcal{G}_E -module $\bigoplus_{w \in (T' \setminus T)_E} (\mathcal{O}_E/\mathfrak{p}_w)^\times$, where \mathfrak{p}_w is the prime ideal of \mathcal{O}_E corresponding with the place w . Then there exists a canonical exact triangle in $D^{\text{perf}}(\mathbb{Z}[\mathcal{G}_E])$*

$$C_{E,\Sigma,T'}^\bullet \longrightarrow C_{E,\Sigma,T}^\bullet \longrightarrow \mathbb{F}_{E,T'\setminus T}^\times[0] \longrightarrow \quad .$$

- (c) *Let Σ' be a finite set of places of k that contains Σ and is disjoint from T . Then there exists a canonical exact triangle in $D^{\text{perf}}(\mathbb{Z}[\mathcal{G}_E])$*

$$C_{E,\Sigma,T}^\bullet \longrightarrow C_{E,\Sigma',T}^\bullet \longrightarrow \bigoplus_{v \in \Sigma' \setminus \Sigma} \left[\mathbb{Z}[\mathcal{G}_E] \xrightarrow{1 - \text{Frob}_v^{-1}} \mathbb{Z}[\mathcal{G}_E] \right] \longrightarrow \quad , \quad (11)$$

where each complex that occurs in the direct sum is concentrated in degrees zero and one.

- (d) *If F is a finite abelian extension of k with $E \subseteq F$ and $S(F) \subseteq \Sigma$, then there exists a natural isomorphism $C_{F,\Sigma,T}^\bullet \otimes_{\mathbb{Z}[\mathcal{G}_F]}^{\mathbb{L}} \mathbb{Z}[\mathcal{G}_E] \cong C_{E,\Sigma,T}^\bullet$ in $D^{\text{perf}}(\mathbb{Z}[\mathcal{G}_E])$.*

(3.5) Remark. For any set $T \in \mathcal{P}_E^{\text{ad}}$, the $\mathbb{Z}[\mathcal{G}_E]$ -module $\mathbb{F}_{E,T}^\times$ that occurs in Lemma (3.4) (b) lies in an exact sequence of $\mathbb{Z}[\mathcal{G}_E]$ -modules of the form

$$0 \longrightarrow \bigoplus_{v \in T} \mathbb{Z}[\mathcal{G}_E] \xrightarrow{(x \mapsto x \cdot \delta_{\{v\}})_v} \bigoplus_{v \in T} \mathbb{Z}[\mathcal{G}_E] \longrightarrow \mathbb{F}_{E,T}^\times \longrightarrow 0,$$

(cf. [Chi85, (4.16)]). This sequence implies that $\text{Fitt}_{\mathbb{Z}[\mathcal{G}_E]}^0(\mathbb{F}_{E,T}^\times) = \mathbb{Z}[\mathcal{G}_E] \cdot \delta_{T,E}$.

Fix an integer $r \geq 0$. As before, we write $e_{E,\Sigma,r}$ for the idempotent of $\mathbb{Q}[\mathcal{G}_E]$ which is defined as the sum of all primitive orthogonal idempotents e_χ associated to characters $\chi \in \mathcal{G}_E$ such that $r_\Sigma(\chi) = r$. For any primitive summand e_χ of $e_{E,\Sigma,r}$ we then have that

$$e_\chi \mathbb{C}X_{E,\Sigma} \cong e_\chi \mathbb{C}Y_{E,V_\chi} \cong e_\chi \mathbb{C}[\mathcal{G}_E]^r,$$

where (with \mathfrak{p} a fixed finite place as in Definition (2.20))

$$V_\chi = \begin{cases} \{v \in \Sigma \mid \chi(v) = 1\} & \text{if } \chi \neq \mathbf{1}_E, \\ \Sigma \setminus \{\mathfrak{p}\} & \text{if } \chi = \mathbf{1}_E \text{ and } S_\infty(k) \cup \{\mathfrak{p}\} \subseteq \Sigma, \\ \Sigma \setminus \{v_1\} & \text{if } \chi = \mathbf{1}_E \text{ and } S_\infty(k) = \Sigma. \end{cases}$$

In particular, $e_{E,\Sigma,r} \mathbb{Q}X_{E,\Sigma}$ is a free $e_{E,\Sigma,r} \mathbb{Q}[\mathcal{G}_E]$ -module of rank r (for which we have fixed a basis by virtue of our fixed choice of extension to E for every place of k at the beginning of §2.3.1 and, in the case of the trivial character, also of the places v_0 and v_1). Given this, we consider the following composite ‘projection’ map

$$\begin{aligned} \Theta_{E/k,\Sigma,T}^r: \text{Det}_{\mathbb{Z}[\mathcal{G}_E]}(C_{E,\Sigma,T}^\bullet) &\hookrightarrow \text{Det}_{\mathbb{Q}[\mathcal{G}_E]}(\mathbb{Q}C_{E,\Sigma,T}^\bullet) \\ &\xrightarrow{\sim} \text{Det}_{\mathbb{Q}[\mathcal{G}_E]}(\mathbb{Q}H^0(C_{E,\Sigma,T}^\bullet)) \otimes_{\mathbb{Q}[\mathcal{G}_E]} \text{Det}_{\mathbb{Q}[\mathcal{G}_E]}^{-1}(\mathbb{Q}H^1(C_{E,\Sigma,T}^\bullet)) \\ &\xrightarrow{\cdot e_{E,\Sigma,r}} \left(e_{E,\Sigma,r} \mathbb{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_E]}^r \mathcal{O}_{E,\Sigma}^\times \right) \otimes_{\mathbb{Q}[\mathcal{G}_E]} \left(e_{E,\Sigma,r} \mathbb{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_E]}^r X_{E,\Sigma}^* \right) \\ &\xrightarrow{\sim} e_{E,\Sigma,r} \mathbb{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_E]}^r \mathcal{O}_{E,\Sigma}^\times, \end{aligned} \quad (12)$$

where the second map is the passage-to-cohomology map, the third arrow is multiplication by the idempotent $e_{E,\Sigma,r}$, and the last map is induced by our fixed choice of basis for the $e_{E,\Sigma,r} \mathbb{Q}[\mathcal{G}_E]$ -free module $e_{E,\Sigma,r} \mathbb{Q}X_{E,\Sigma}$.

In the sequel we fix, for each prime p , an algebraic closure $\overline{\mathbb{Q}_p}$ of \mathbb{Q}_p and write \mathbb{C}_p for its completion.

(3.6) Proposition. *Fix a finite abelian extension K of k and a subset $V \subsetneq \Sigma$ of cardinality r that comprises places splitting completely in K . Then, for each prime p , the following assertions are equivalent:*

- (a) $\mathbb{Z}_p \cdot \text{im}(\Theta_{K/k,\Sigma,T}^r) = \mathbb{Z}_p[\mathcal{G}_K] \cdot \varepsilon_{K/k,\Sigma,T}^V$,
- (b) $\mathbb{Z}_p \cdot \text{im}(\Theta_{K/k,\Sigma,T}^r) \subseteq \mathbb{Z}_p[\mathcal{G}_K] \cdot \varepsilon_{K/k,\Sigma,T}^V$,
- (c) $\mathbb{Z}_p \cdot \text{im}(\Theta_{K/k,\Sigma,T}^r) \supseteq \mathbb{Z}_p[\mathcal{G}_K] \cdot \varepsilon_{K/k,\Sigma,T}^V$.

Proof. We shall show that condition (b) implies (a). The converse is clear, and the proof that (c) implies (a) is analogous.

We first note that if T' is a finite set of places which is both disjoint from Σ and such that $T'' := T \cup T'$ is admissible, then $\delta_{T'} = \delta_{T',K}$ is a non-zero divisor. Since $\varepsilon_{K/k,\Sigma,T''}^V = \delta_{T'} \varepsilon_{K/k,\Sigma,T}^V$ and $\text{im}(\Theta_{K/k,\Sigma,T''}^r) = \delta_{T'} \cdot \text{im}(\Theta_{K/k,\Sigma,T}^r)$ (the latter as a consequence of the exact triangle in Lemma (3.4) (b) and Remark (3.5)) we may therefore assume that T is admissible. In this case, it is known that the complex $\mathbb{Z}_p \otimes_{\mathbb{Z}} C_{K,\Sigma,T}^\bullet$ admits a representative of the form $[P \rightarrow P]$ with P a free $\mathbb{Z}_p[\mathcal{G}_K]$ -module of finite rank (cf. [BKS16, §5.4]). Moreover, $\mathbb{Z}_p \cdot \text{Det}_{\mathbb{Z}[\mathcal{G}_K]}(C_{K,\Sigma,T}^\bullet)$ is a free $\mathbb{Z}_p[\mathcal{G}_K]$ -module of rank one. We fix a basis \mathfrak{z}_K of the latter and set $z_K^b := \Theta_{K/k,\Sigma,T}^r(\mathfrak{z}_K)$.

By assumption, we may write $z_K^b = q_K \cdot \varepsilon_{K/k,\Sigma,T}^V$ for some $q_K \in \mathbb{Z}_p[\mathcal{G}_K]$. By construction of the map $\Theta_{K/k,\Sigma,T}^r$, the element z_K^b is annihilated by the idempotent $1 - e_{K,\Sigma,r}$. Since the same is true for the Rubin–Stark element $\varepsilon_{K/k,\Sigma,T}^V$, it suffices to prove that $q_K e_{K,\Sigma,r}$ is a unit in $\mathbb{Z}_p[\mathcal{G}_K] e_{K,\Sigma,r}$.

As before we write $\lambda_{K,\Sigma}$ for the Σ -Dirichlet regulator map of K . Let $\varpi = \varpi_{K/k,\Sigma,T}$ be the

unique element of $\mathbb{C}_p \text{Det}_{\mathbb{Z}[\mathcal{G}_K]}(C_{K,\Sigma,T}^\bullet)$ that corresponds with the leading term $\theta_{K/k,\Sigma,T}^*(0)$ under the isomorphism $\vartheta_{\mathbb{C}_p[\mathcal{G}_K],\lambda_{K,\Sigma}}$ defined in Lemma (3.7) below. In particular, one has an equality $(\mathbb{C}_p \otimes_{\mathbb{Z}} \Theta_{K/k,\Sigma,T}^r)(\varpi) = \varepsilon_{K/k,\Sigma,T}^V$.

Let $x \in \mathbb{C}_p[\mathcal{G}_K]$ be the unique element such that $\varpi = x\mathfrak{z}_K$. By construction, $(q_K - x)\mathfrak{z}_K$ is contained in the kernel of $\Theta_{K/k,\Sigma,T}^r$ and so the injectivity of $\Theta_{K/k,\Sigma,T}^r$ on the $e_{K,\Sigma,r}$ -part implies that $q_K e_{K,\Sigma,r} = x e_{K,\Sigma,r}$. It therefore suffices to show that $x e_{K,\Sigma,r}$ is a unit in $\mathbb{Z}_p[\mathcal{G}_K]e_{K,\Sigma,r}$.

To do this, we shall show that the norm $N_{\mathbb{Z}_p[\mathcal{G}_K]e_{K,\Sigma,r}/\mathbb{Z}_p}(x e_{K,\Sigma,r})$ is a unit in \mathbb{Z}_p . Given this, Lemma (3.7) (a) then implies the claim.

Note that the set $\Upsilon_{K,\Sigma,r}$ of all characters $\chi \in \widehat{\mathcal{G}_K}$ with the property that $r_\Sigma(\chi) = r$ is stable under the action of the absolute Galois group $G_{\mathbb{Q}}$, whence $\sum_{\chi \in \Upsilon_{K,\Sigma,r}} \chi$ is a rational valued character. By the Artin induction theorem (see [Tat84, Ch. II, Thm. 1.2]) there thus exists a natural number m and for each subgroup H of \mathcal{G}_E an integer m_H such that

$$m \cdot \sum_{\chi \in \Upsilon_{K,\Sigma,r}} \chi = \sum_H m_H \cdot \text{Ind}_H^{\mathcal{G}_K}(\mathbf{1}_H) = \sum_H m_H \cdot \left(\sum_{\chi(H)=1} \chi \right).$$

Writing $\pi_{K/K^H}: \mathbb{Z}_p[\mathcal{G}_K] \rightarrow \mathbb{Z}_p[\mathcal{G}_{K^H}]$ for the natural restriction map, we then deduce that

$$N_{\mathbb{Z}_p[\mathcal{G}_K]e_{K,\Sigma,r}/\mathbb{Z}_p}(x e_{K,\Sigma,r})^m = \left(\prod_{\chi \in \Upsilon_{K,\Sigma,r}} \chi(x) \right)^m = \prod_H N_{\mathbb{Z}_p[\mathcal{G}_{K^H}]/\mathbb{Z}_p}(\pi_{K/K^H}(x))^{m_H}$$

and it therefore suffices to show that each factor $N_{\mathbb{Z}_p[\mathcal{G}_{K^H}]/\mathbb{Z}_p}(\pi_{K/K^H}(x))$ is a unit in \mathbb{Z}_p .

By construction, $\pi_{K^H/K}(x)\mathfrak{z}_{K^H}$ is the unique element of $\mathbb{C}_p \text{Det}_{\mathbb{Z}[\mathcal{G}_{K^H}]}(C_{K^H,\Sigma,T}^\bullet)$ that corresponds with the leading term $\theta_{K^H/k,\Sigma,T}^*$ via the map $\vartheta_{\mathbb{C}_p[\mathcal{G}_{K^H}],\lambda_{K^H,\Sigma}}$ defined in Lemma (3.7) below. It then follows from Lemma (3.7) that

$$\mathcal{F}_{\mathbb{C}_p[\mathcal{G}_{K^H}]/\mathbb{C}_p}(\pi_{K^H/K}(x)\mathfrak{z}_{K^H,T}) = N_{\mathbb{C}_p[\mathcal{G}_{K^H}]/\mathbb{C}_p}(\pi_{K^H/K}(x)) \cdot \mathcal{F}_{\mathbb{Z}_p[\mathcal{G}_{K^H}]/\mathbb{Z}_p}(\mathfrak{z}_{K^H,T}) \quad (13)$$

corresponds, by [Tat84, Ch. IV, § 1, Prop. 1.8], under $\vartheta_{\mathbb{C}_p[\mathcal{G}_{K^H}],\lambda_{K^H,\Sigma}}$ with

$$N_{\mathbb{C}_p[\mathcal{G}_{K^H}]/\mathbb{C}_p}(\theta_{K^H/k,\Sigma,T}^*(0)) = \theta_{K^H/K^H,\Sigma,T}^*(0) = \zeta_{K^H,\Sigma,T}^*(0),$$

where $\zeta_{K^H,\Sigma,T}^*(0)$ denotes the leading term of the Σ -truncated and T -modified Dedekind ζ -function of K^H at $s = 0$. The analytic class number formula therefore implies that the element in (13) is a basis of $\mathbb{Z}_p \text{Det}_{\mathbb{Z}[\mathcal{G}_{K^H}]}(C_{K^H,\Sigma,T}^\bullet)$. On the other hand, since $\mathfrak{z}_{K^H,T}$ is a basis of $\mathbb{Z}_p \text{Det}_{\mathbb{Z}[\mathcal{G}_{K^H}]}(C_{K^H,\Sigma,T}^\bullet)$, the element $\mathcal{F}_{\mathbb{Z}_p[\mathcal{G}_{K^H}]/\mathbb{Z}_p}(\mathfrak{z}_{K^H,T})$ is a \mathbb{Z}_p -basis as well (by Lemma (3.7) (b)). We therefore deduce that $N_{\mathbb{Z}_p[\mathcal{G}_{K^H}]/\mathbb{Z}_p}(\pi_{K^H/K}(x))$ is a unit in \mathbb{Z}_p , as claimed. \square

In the following general algebraic result we fix a commutative Noetherian ring R and a finitely generated free R -algebra A . We write $N_{A/R}: A \rightarrow R$ for the ‘norm’ map that sends each a in A to the determinant of the R -linear endomorphism $x \mapsto ax$ of A .

(3.7) Lemma. *Suppose to be given a perfect complex C^\bullet of A -modules of the form $P_0 \rightarrow P_1$, where P_0 and P_1 are finitely generated free A -modules, and P_0 is placed in degree 0. Then the following claims are valid.*

- (a) *An element $a \in A$ is a unit if and only if $N_{A/R}(a)$ is a unit in R .*
- (b) *There is an $N_{A/R}$ -semilinear map $\mathcal{F}_{A/R}: \text{Det}_A(C^\bullet) \rightarrow \text{Det}_R(\tilde{C}^\bullet)$, where in the last term \tilde{C}^\bullet indicates C^\bullet , regarded as a complex of R -modules. The map $\mathcal{F}_{A/R}$ sends each A -basis of $\text{Det}_A(C^\bullet)$ to an R -basis of $\text{Det}_R(\tilde{C}^\bullet)$.*
- (c) *Assume that R and A are semi-simple algebras and that $\lambda: H^0(C^\bullet) \cong H^1(C^\bullet)$ is an isomorphism of A -modules. Define the isomorphism*

$$\vartheta_{C^\bullet,\lambda}: \text{Det}_A(C^\bullet) \cong \text{Det}_A(H^0(C^\bullet)) \otimes_A \text{Det}_A(H^1(C^\bullet))^{-1} \cong A,$$

where the first map is the canonical ‘passage-to-cohomology’ isomorphism and the second the composite of $\text{Det}_A(\lambda) \otimes_A 1$ and the evaluation map on $\text{Det}_A(H^1(C^\bullet))$. Then there exists a commutative diagram

$$\begin{array}{ccc} \text{Det}_A(C^\bullet) & \xrightarrow{\vartheta_{C^\bullet, \lambda}} & A \\ \mathcal{F}_{A/R} \downarrow & & \downarrow N_{A/R} \\ \text{Det}_R(\tilde{C}^\bullet) & \xrightarrow{\vartheta_{\tilde{C}^\bullet, \lambda}} & R. \end{array}$$

Proof. The first claim is a restatement of [Bou74, Ch. III, § 9.6, Prop. 3]. The second and third claims are both derived via an explicit computation from the general result of [Bou74, Ch. III, § 9.6, Prop. 6]. \square

3.3. The proof of Theorem (3.2)

3.3.1. Commutative diagrams in Galois cohomology

We now suppose to be given finite sets of places $U \subseteq U'$ of k that each contain the set $S^*(E) = S_\infty(k) \cup S(E/k)$. We then define an isomorphism of $\mathbb{Z}[\mathcal{G}_E]$ -modules

$$\begin{aligned} \Delta_{E, U', U} : \text{Det}_{\mathbb{Z}[\mathcal{G}_E]}(C_{E, U', T}^\bullet) &\xrightarrow{\sim} \text{Det}_{\mathbb{Z}[\mathcal{G}_E]}(C_{E, U, T}^\bullet) \otimes_{\mathbb{Z}[\mathcal{G}_E]} \bigotimes_{v \in U' \setminus U} \text{Det}_{\mathbb{Z}[\mathcal{G}_E]} \left[\mathbb{Z}[\mathcal{G}_E] \xrightarrow{1 - \text{Frob}_v^{-1}} \mathbb{Z}[\mathcal{G}_E] \right] \\ &\xrightarrow{\sim} \text{Det}_{\mathbb{Z}[\mathcal{G}_E]}(C_{E, U, T}^\bullet), \end{aligned}$$

where the first map is induced by the exact triangle in Proposition (3.4) (c) and the second by trivialising the latter determinant with respect to the identity.

Given a further finite abelian extension F of k with $E \subseteq F$, we then write

$$i_{F/E} : \text{Det}_{\mathbb{Z}[\mathcal{G}_F]}(C_{F, S^*(F), T}^\bullet) \rightarrow \text{Det}_{\mathbb{Z}[\mathcal{G}_E]}(C_{E, S^*(E), T}^\bullet) \quad (14)$$

for the composite homomorphism of $\mathbb{Z}[\mathcal{G}_F]$ -modules

$$\text{Det}_{\mathbb{Z}[\mathcal{G}_F]}(C_{F, S^*(F), T}^\bullet) \longrightarrow \text{Det}_{\mathbb{Z}[\mathcal{G}_F]}(C_{F, S^*(F), T}^\bullet) \otimes_{\mathbb{Z}[\mathcal{G}_F]} \mathbb{Z}[\mathcal{G}_E] \quad (15)$$

$$\xrightarrow{\sim} \text{Det}_{\mathbb{Z}[\mathcal{G}_E]}(C_{F, S^*(F), T}^\bullet \otimes_{\mathbb{Z}[\mathcal{G}_F]}^{\mathbb{L}} \mathbb{Z}[\mathcal{G}_E]) \quad (16)$$

$$\xrightarrow{\sim} \text{Det}_{\mathbb{Z}[\mathcal{G}_E]}(C_{E, S^*(F), T}^\bullet) \quad (17)$$

$$\xrightarrow{\sim} \text{Det}_{\mathbb{Z}[\mathcal{G}_E]}(C_{E, S^*(E), T}^\bullet).$$

Here the first map is the canonical projection, the second is induced by the base change property of the determinant functor, the third is the isomorphism in Proposition (3.4) (d) and the final map is $\Delta_{E, S^*(F), S^*(E)}$.

In the sequel a key role will be played by the functoriality of the projections $\Theta_{E/k, S^*(E), T}^{rE}$ with respect to the various maps that occur in the descent homomorphisms $i_{F/E}$. This behaviour is explicitly described in the following result.

(3.8) Lemma. *Let E and F be finite abelian extensions of k such that $E \subseteq F$. Let $V' \subseteq S^*(F)$ be the set of places which split completely in E/k and write $r' = |V'|$ for its cardinality. For simplicity, we also put $V = V_E$, $r = r_E$, and $W = V' \setminus V$.*

(a) *The following diagram commutes:*

$$\begin{array}{ccc} \text{Det}_{\mathbb{Z}[\mathcal{G}_F]}(C_{F, S^*(F), T}^\bullet) & \xrightarrow{\Theta_{F, S^*(F), T}^{r'}} & \mathbb{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_F]}^{r'} \mathcal{O}_{F, S^*(F), T}^\times \\ \downarrow & & \downarrow N_{F/E}^{r'} \\ \text{Det}_{\mathbb{Z}[\mathcal{G}_E]}(C_{E, S^*(F), T}^\bullet) & \xrightarrow{\Theta_{E, S^*(F), T}^{r'}} & \mathbb{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_E]}^{r'} \mathcal{O}_{E, S^*(F), T}^\times, \end{array}$$

where the left-hand vertical map is the composition of (15), (16), and (17).

- (b) Assume $F \in \Omega(k)$ and $E \in \Omega(k) \cup \{k\}$, and put $\omega_{F/E}^* = (-1)^{r|W|}$ if $E \in \Omega(k)$ and $\omega_{F/k}^* = -1$. The following diagram commutes:

$$\begin{array}{ccc} \mathrm{Det}_{\mathbb{Z}[\mathcal{G}_E]}(C_{E,S^*(F),T}^\bullet) & \xrightarrow{\Theta_{E,S^*(F),T}^{r'}} & \mathbb{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_E]}^{r'} \mathcal{O}_{E,S^*(F),T}^\times \\ \wr \downarrow \Delta_{E,S^*(F),W'} & & \downarrow \omega_{F/E}^* \cdot \mathrm{Ord}_{E,W} \\ \mathrm{Det}_{\mathbb{Z}[\mathcal{G}_E]}(C_{E,W',T}^\bullet) & \xrightarrow{\Theta_{E,W',T}^r} & \mathbb{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_E]}^r \mathcal{O}_{E,S^*(F) \setminus W,T}^\times, \end{array}$$

where $W' := S^*(F) \setminus W$ and $\mathrm{Ord}_{E,W}$ denotes the exterior product over v in W of the maps

$$\mathrm{Ord}_{E,v}: \mathcal{O}_{E,S^*(F),T}^\times \rightarrow \mathbb{Z}[\mathcal{G}_E], \quad a \mapsto \sum_{\sigma \in \mathcal{G}_E} \mathrm{ord}_{vE}(\sigma a) \sigma^{-1}.$$

- (c) Suppose that $E \neq k$ and that $U \subseteq U'$ are finite sets of places of k containing $S^*(E)$. Then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Det}_{\mathbb{Z}[\mathcal{G}_E]}(C_{E,U',T}^\bullet) & \xrightarrow{\Theta_{E/k,U',T}^r} & \mathbb{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_E]}^r \mathcal{O}_{E,U',T}^\times \\ \wr \downarrow \Delta_{E,U',U} & & \uparrow \prod (1 - \mathrm{Frob}_v^{-1}) \\ \mathrm{Det}_{\mathbb{Z}[\mathcal{G}_E]}(C_{E,U,T}^\bullet) & \xrightarrow{\Theta_{E/k,U,T}^r} & \mathbb{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_E]}^r \mathcal{O}_{E,U,T}^\times, \end{array}$$

where the product on the right hand side ranges over all places in $U' \setminus U$.

- (d) Suppose that $U \subseteq U'$ are finite sets of places of k containing $S^*(F)$. Then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Det}_{\mathbb{Z}[\mathcal{G}_F]}(C_{F,U',T}^\bullet) & \xrightarrow{\Delta_{F,U',U}} & \mathrm{Det}_{\mathbb{Z}[\mathcal{G}_F]}(C_{F,U,T}^\bullet) \\ \downarrow & & \downarrow \\ \mathrm{Det}_{\mathbb{Z}[\mathcal{G}_E]}(C_{E,U',T}^\bullet) & \xrightarrow{\Delta_{E,U',U}} & \mathrm{Det}_{\mathbb{Z}[\mathcal{G}_E]}(C_{E,U,T}^\bullet), \end{array}$$

where the vertical arrows are the composition of the relevant instances of (15), (16), and (17).

Proof. For parts (a) and (c) see the proof of [Bur+23, Thm. 3.8].

To prove part (b), we recall that the left hand vertical map in the given square is induced by the exact triangle (11) in Proposition (3.4) (c) applied with the sets $\Sigma' = S^*(F)$ and $\Sigma = S^*(F) \setminus W$. To prove commutativity, we may first base change to $\mathbb{Q}[\mathcal{G}_E]$. Then, by the definition of the projection maps $\Theta_{E/k,S^*(F),T}^{r'}$ and $\Theta_{E/k,S^*(F) \setminus W,T}^r$ (which involves passing to cohomology and trivialising the top degree cohomology), it suffices to show that the composite homomorphism

$$\begin{aligned} e_E \mathbb{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_E]}^{r'} \mathcal{O}_{E,S^*(F),T}^\times &= e_E \mathrm{Det}_{\mathbb{Q}[\mathcal{G}_E]}(\mathbb{Q}H^0(C_{E,S^*(F),T}^\bullet)) \\ &\longrightarrow e_E \mathrm{Det}_{\mathbb{Q}[\mathcal{G}_E]}(\mathbb{Q}H^0(C_{E,S^*(F) \setminus W,T}^\bullet)) = e_E \mathbb{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_E]}^r \mathcal{O}_{E,S^*(F) \setminus W,T}^\times, \end{aligned}$$

that is induced by the exact triangle (11) along with the trivialisation

$$\mathrm{Det}_{\mathbb{Z}[\mathcal{G}_E]} \left(\left[\mathbb{Z}[\mathcal{G}_E] \xrightarrow{0} \mathbb{Z}[\mathcal{G}_E] \right] \right) \cong \mathbb{Z}[\mathcal{G}_E]$$

for all $v \in W$, coincides with the map $\omega_{F/E}^* \cdot \mathrm{Ord}_W$ that is defined in the statement of part (b). To do this, we note that the Artin–Verdier duality theorem identifies $C_{E,S}^\bullet$ with the complex that is denoted by Ψ_S in [BF98, §3.2]. In particular, the argument of [BF98, Prop. 3.2] implies the long exact sequence in cohomology of the triangle (11) splits into the two short exact

sequences

$$\begin{aligned} 0 &\longrightarrow \mathbb{Q}\mathcal{O}_{E,S^*(F)\setminus W,T}^\times \longrightarrow \mathbb{Q}\mathcal{O}_{E,S^*(F),T}^\times \xrightarrow{f} \mathbb{Q}Y_{E,W} \longrightarrow 0 \\ 0 &\longrightarrow \mathbb{Q}X_{E,S^*(F)\setminus W} \longrightarrow \mathbb{Q}X_{E,S^*(F)} \longrightarrow \mathbb{Q}Y_{E,W} \longrightarrow 0, \end{aligned}$$

in which the map f is given by

$$f: \mathcal{O}_{E,S^*(F),T}^\times \rightarrow Y_{E,W} \cong \bigoplus_{v \in W} \mathbb{Z}[\mathcal{G}_E], \quad a \mapsto \sum_{v \in W} \text{ord}_v(\sigma a) \sigma^{-1}.$$

Given this description, the required claim follows by a straightforward, and explicit, calculation (just as in the argument of [BKS16, Lem. 4.2, 4.3]).

The commutativity of the diagram in (d) follows from a standard functoriality property of the determinant functor and the fact that the triangles (11) and isomorphisms in Proposition (3.4) (d) combine into a canonical commutative diagram

$$\begin{array}{ccccccc} C_{F,U,T}^\bullet & \longrightarrow & C_{F,U',T}^\bullet & \longrightarrow & \bigoplus_{v \in U' \setminus U} \left[\mathbb{Z}[\mathcal{G}_F] \xrightarrow{1 - \text{Frob}_v^{-1}} \mathbb{Z}[\mathcal{G}_F] \right] & \longrightarrow & \\ \downarrow & & \downarrow & & \downarrow & & \\ C_{E,U,T}^\bullet & \longrightarrow & C_{E,U',T}^\bullet & \longrightarrow & \bigoplus_{v \in U' \setminus U} \left[\mathbb{Z}[\mathcal{G}_E] \xrightarrow{1 - \text{Frob}_v^{-1}} \mathbb{Z}[\mathcal{G}_E] \right] & \longrightarrow & . \end{array}$$

This proves the claimed result. \square

3.3.2. Completion of the proof

The link between the constructions in § 3.3.1 and the theory of integral Euler systems is provided by the following result.

(3.9) Proposition. *The collection of morphisms $(\Theta_{E/k,S^*(E)}^{r_E})_{E \in \Omega(k)}$ induces a homomorphism of $\mathbb{Z}[[\mathcal{G}_K]]$ -modules*

$$\Theta_k: \varprojlim_{E \in \Omega(k)} \text{Det}_{\mathbb{Z}[\mathcal{G}_E]}(C_{E,S^*(E)}^\bullet) \rightarrow \text{ES}_k(\mathbb{Z})^{\text{sym}} \cap \text{ES}_k(\mathbb{Z})^{\text{con}},$$

where the transition morphisms in the limit are the maps $i_{F/E}$ defined in (14) for $E \subseteq F$.

Proof. Existing results in the literature imply $\text{im}(\Theta_k)$ is contained in $\text{ES}_k(\mathbb{Z})^{\text{con}}$. More precisely, the argument of [Bur+23, Thm. 3.8 (i)] shows that every c in $\text{im}(\Theta_k)$ satisfies the Euler system distribution relations; any such c also satisfies the integrality condition (5) with $\mathcal{R} = \mathbb{Z}$ because the proof of Theorem 3.8 (ii) of *loc. cit.* implies $\text{im}(\Theta_{E/k,S^*(E)}^{r_E}) \subseteq \mathfrak{L}_E$ for every $E \in \Omega(k)$; finally, the argument of [BKS16, Thm. 5.16] shows that any such c validates all of the congruences that occur in the Mazur–Rubin–Sano Conjecture.

It therefore suffices for us to fix a system c in $\text{im}(\Theta_k)$ and show that it is symmetric. To do this, we fix a field $E \in \Omega(k)$. We note first that if either $V_E \neq S_\infty(k)$ or $|S(E)| > 1$, then $N_{E/k}^{r_E}(c_E) = 0$ by Lemma (2.6) (a) and so the explicit condition in Definition (2.15) is satisfied trivially in this case. We may therefore assume in the sequel that both $V_E = S_\infty(k)$ and $S(E) = \{\mathfrak{p}\}$ for some finite place \mathfrak{p} .

By definition, there exists a system $\mathfrak{z} = (\mathfrak{z}_E)_E \in \varprojlim_{E \in \Omega(k)} \text{Det}_{\mathbb{Z}[\mathcal{G}_E]}(C_{E,S^*(E)}^\bullet)$ such that one has $\Theta_{E/k,S^*(E)}^{r_E}(\mathfrak{z}_E) = c_E$. Observe that \mathfrak{z} also uniquely specifies an element \mathfrak{z}_k of $\text{Det}_{\mathbb{Z}}(C_{k,S_\infty(k)}^\bullet)$.

Now, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Det}_{\mathbb{Z}[\mathcal{G}_E]}(C_{E,S^*(E)}^\bullet) & \xrightarrow{\Theta_{E/k,S^*(E)}^{r_E}} & \bigcap_{\mathbb{Z}[\mathcal{G}_E]}^{r_E} \mathcal{O}_{E,S(E)}^\times \\
\downarrow & & \downarrow N_{E/k}^{r_E} \\
\mathrm{Det}_{\mathbb{Z}}(C_{k,S^*(E)}^\bullet) & \xrightarrow{\Theta_{k/k,S^*(E)}^{r_E}} & \bigwedge_{\mathbb{Z}}^{r_E} \mathcal{O}_{k,\{\mathfrak{p}\}}^\times \\
\Delta_{k,S^*(E),S_\infty(k)} \downarrow & & \downarrow -\mathrm{Ord}_{\mathfrak{p}} \\
\mathrm{Det}_{\mathbb{Z}}(C_{k,S_\infty(k)}^\bullet) & \xrightarrow{\Theta_{k/k,S_\infty(k)}^{r_E}} & \bigwedge_{\mathbb{Z}}^{r_E-1} \mathcal{O}_k^\times
\end{array}$$

in which the upper and lower squares are the relevant instances of the diagrams in Lemma (3.8) (a) and Lemma (3.8) (b). From the commutativity of this diagram, it then follows that c satisfies the condition in Definition (2.15) with its initial value c_k equal to $\Theta_{k/k,S_\infty(k)}^{r_k}(\mathfrak{z}k)$. \square

If we now assume Conjecture (2.24) to be valid for the pair $(\mathcal{X}, \mathcal{S})$, then every system c in $\varrho^\mathcal{X}(\mathrm{im}(\Theta_k))$ can be written in the form $c = q \cdot \varepsilon_k^\mathcal{X}$ for some $q \in \mathbb{Z}_{\mathcal{S}}[[\mathcal{G}_{\mathcal{K}}]]$. This shows that Conjecture (2.24) gives rise to an inclusion $\varrho^\mathcal{X}(\mathrm{im}(\Theta_k)) \subseteq \mathbb{Z}_{\mathcal{S}}[[\mathcal{G}_{\mathcal{K}}]]\varepsilon_k^\mathcal{X}$. Theorem (3.2) therefore now follows immediately upon applying the following result with \mathcal{X} taken to be the set $\Omega^\mathcal{V}(k)$ (which is easily seen to satisfy the hypotheses required by the result) for every prime number p outside \mathcal{S} .

(3.10) Proposition. *Let K/k be a finite abelian extension, p a prime number and \mathcal{V} a subset of $\mathcal{P}(S_\infty(k))$. Let \mathcal{X} be a subset of $\Omega(k)$ that satisfies the following condition: for each intermediate field L of K/k such that $V_L \in \mathcal{V}$, one has*

- $L \in \mathcal{X}$ if L/k is ramified,
- $k(\mathfrak{p})L \in \mathcal{X}$ for almost all $\mathfrak{p} \in S_{\mathrm{fin}}(k)$ if L/k is unramified.

Then, if $\varrho^\mathcal{X}(\mathrm{im}(\Theta_k)) \subseteq \mathbb{Z}_p[[\mathcal{G}_{\mathcal{K}}]]\varepsilon_k^\mathcal{X}$, the conjecture $\mathrm{TNC}(h^0(\mathrm{Spec} K), \epsilon_{K,\mathcal{V}}\mathbb{Z}_p[[\mathcal{G}_K]])$ is valid.

Proof. For each χ in $\widehat{\mathcal{G}}_K$, we denote the subfield $K^{\ker(\chi)}$ of K by $K(\chi)$. We also write $\Upsilon_{K,\mathcal{V}}^\diamond$ for the subset of $\widehat{\mathcal{G}}_K$ comprising characters χ with $\chi(\epsilon_{K,\mathcal{V}}) \neq 0$. Then for each $\chi \in \Upsilon_{K,\mathcal{V}}^\diamond$, the properties in Remark (3.1) (a) and (c) combine to imply $V_{K(\chi)}$ belongs to \mathcal{V} . In particular, if $K(\chi)/k$ is unramified, then we can fix a (finite) prime \mathfrak{p}_χ of k that is inert in the (cyclic) extension $K(\chi)$ and is such that the compositum of $K(\chi)$ and $k(\mathfrak{p}_\chi)$ belongs to \mathcal{X} . Then, for each χ in $\Upsilon_{K,\mathcal{V}}^\diamond$, the extension of k that is defined by setting

$$K_\chi := \begin{cases} K(\chi) & \text{if } \chi \text{ is ramified,} \\ K(\chi)k(\mathfrak{p}_\chi) & \text{otherwise,} \end{cases}$$

belongs to \mathcal{X} .

We now write N for the compositum of K and the fields K_χ for each χ in $\Upsilon_{K,\mathcal{V}}^\diamond$, set $\Gamma := \mathcal{G}_N$ and regard $\epsilon_{K,\mathcal{V}}$ as an idempotent of $\mathbb{Q}[\Gamma]$ in the obvious way (this idempotent does not usually identify with $\epsilon_{N,\mathcal{V}}$ but this fact is not important in the sequel).

Then, by the known functoriality properties of the equivariant Tamagawa Number Conjecture (cf. [BF01, Prop. 4.1]), it is sufficient to show that the inclusion $\varrho^\mathcal{X}(\mathrm{im}(\Theta_k)) \subseteq \mathbb{Z}_p[[\mathcal{G}_{\mathcal{K}}]]\varepsilon_k^\mathcal{X}$ implies the validity of $\mathrm{TNC}(h^0(\mathrm{Spec} N), \epsilon_{K,\mathcal{V}}\mathbb{Z}_p[[\Gamma]])$.

Then, since the $\mathbb{Z}[\Gamma]$ -module $\mathrm{Det}_{\mathbb{Z}[\Gamma]}(C_{N,S^*(N)}^\bullet)$ is locally free, Roiter's Lemma [CR81, (31.6)] allows us to choose an element \mathfrak{z}_N of $\mathrm{Det}_{\mathbb{Z}[\Gamma]}(C_{N,S^*(N)}^\bullet)$ that generates a (free, rank one) $\mathbb{Z}[\Gamma]$ -submodule of finite prime-to- p index. In particular, if we set $D_{N,S^*(N)}^\bullet := \mathbb{Z}_p \otimes_{\mathbb{Z}} C_{N,S^*(N)}^\bullet$, then the image $1 \otimes \mathfrak{z}_N$ of \mathfrak{z}_N in the free $\mathbb{Z}_p[\Gamma]$ -module $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{Det}_{\mathbb{Z}[\Gamma]}(C_{N,S^*(N)}^\bullet) \cong \mathrm{Det}_{\mathbb{Z}_p[\Gamma]}(D_{N,S^*(N)}^\bullet)$ is a basis.

Now, since the transition maps $i_{F/E}$ are surjective (by construction), we can lift \mathfrak{z}_N to an element a of the corresponding inverse limit $\varprojlim_{E \in \Omega(k)} \mathrm{Det}_{\mathbb{Z}[\mathcal{G}_E]}(C_{E,S^*(E)}^\bullet)$. This element a then

gives rise, by Proposition (3.9), to an Euler system $z^b := \Theta_k(a)$. In particular, if we assume that $\varrho^{\mathcal{X}}(\text{im}(\Theta_k))$ is contained in $\mathbb{Z}_p[[\mathcal{G}_K]]\varepsilon_k^{\mathcal{X}}$, then there exists an element $q = (q_E)_E$ of $\mathbb{Z}_p[[\text{Gal}(\mathcal{K}/k)]]$ such that $\varrho^{\mathcal{X}}(z^b) = q \cdot \varepsilon_k^{\mathcal{X}}$. More concretely, therefore, this element q would satisfy

$$z_E^b = q_E \cdot \varepsilon_{E/k} \quad \text{for every } E \in \mathcal{X}. \quad (18)$$

We claim that these equalities imply that the element $\varepsilon_{K,\mathcal{V}} \cdot q_N$ is a unit of $\varepsilon_{K,\mathcal{V}}\mathbb{Z}_p[\Gamma]$. To show this, we recall that for each χ in $\Upsilon_{K,\mathcal{V}}^\circ$ the field K_χ is a subfield of N .

This implies that the element $1 \otimes z_{K_\chi}^b$ is equal to the image of the $\mathbb{Z}_p[\Gamma]$ -basis $1 \otimes \mathfrak{z}_N$ of $\text{Det}_{\mathbb{Z}_p[\Gamma]}(D_{N,S^*(N)}^\bullet)$ under the composite homomorphism

$$\text{Det}_{\mathbb{Z}_p[\Gamma]}(D_{N,S^*(N)}^\bullet) \xrightarrow{i_{N/K_\chi}} \text{Det}_{\mathbb{Z}_p[\mathcal{G}_{K_\chi}]}(D_{K_\chi,S^*(K_\chi)}^\bullet) \xrightarrow{\Theta_{K_\chi/k,S^*(K_\chi)}^{r_{K_\chi}}} \mathbb{Z}_p\mathfrak{L}_{K_\chi}.$$

It therefore follows from the equality (18) (with $E = K_\chi$) that one has

$$(\Theta_{K_\chi/k,S^*(K_\chi)}^{r_{K_\chi}} \circ i_{N/K_\chi})(\mathfrak{z}_N) = z_{K_\chi}^b = q_{K_\chi} \cdot \varepsilon_{K_\chi/k}. \quad (19)$$

This combines with the implication (b) \Rightarrow (a) of Proposition (3.6) (with K replaced by K_χ and V by V_{K_χ}) to imply that the elements $z_{K_\chi}^b$ and $\varepsilon_{K_\chi/k}$ generate the same $\mathbb{Z}_p[\mathcal{G}_{K_\chi}]$ -submodule of $\mathbb{Z}_p\mathfrak{L}_{K_\chi}$. In particular, since this submodule is free as an $e_{K_\chi}\mathbb{Z}_p[\mathcal{G}_{K_\chi}]$ -module, we deduce that the element $e_{K_\chi} \cdot q_{K_\chi}$ is a unit in $e_{K_\chi}\mathbb{Z}_p[\mathcal{G}_{K_\chi}]$. In addition, since $r_{S^*(K_\chi)}(\chi) = r_{K_\chi}$, the orthogonality relations of characters imply that $\chi(e_{K_\chi}) = 1$ and so the element $\chi(q_N) = \chi(q_{K_\chi})$ is a unit in $\mathbb{Z}_p[\text{im } \chi]$.

We have, by now, shown that $\chi(q_N)$ is a unit in $\mathbb{Z}_p[\text{im}(\chi)]$ for every χ in $\Upsilon_{K,\mathcal{V}}^\circ$. Since the set $\Upsilon_{K,\mathcal{V}}^\circ$ is stable under the action of G_Q on $\widehat{\Gamma}$, we can therefore deduce that the element

$$N_{\varepsilon_{K,\mathcal{V}}\mathbb{Z}_p[\Gamma]/\mathbb{Z}_p}(\varepsilon_{K,\mathcal{V}} \cdot q_N) = \prod_{\chi \in \Upsilon_{K,\mathcal{V}}^\circ} \chi(q_N) = \prod_{\chi \in \Upsilon_{K,\mathcal{V}}^\circ/\sim} N_{\mathbb{Z}_p[\text{im}(\chi)]/\mathbb{Z}_p}(\chi(q_N))$$

is a unit in \mathbb{Z}_p . By Lemma (3.7) (a) this in turn proves that $\varepsilon_{K,\mathcal{V}} \cdot q_N$ is a unit in $\varepsilon_{K,\mathcal{V}}\mathbb{Z}_p[\Gamma]$ as claimed above.

This fact implies that $\varepsilon_{K,\mathcal{V}}q_N^{-1}\mathfrak{z}_N$ is an $\varepsilon_{K,\mathcal{V}}\mathbb{Z}_p[\Gamma]$ -basis of $\varepsilon_{K,\mathcal{V}}\text{Det}_{\mathbb{Z}_p[\Gamma]}(D_{N,S^*(N)}^\bullet)$. To prove the Proposition it therefore suffices, by [BKS17, Prop. 2.5], to verify for every character $\chi \in \Upsilon_{K,\mathcal{V}}^\circ$ that the composite homomorphism

$$\begin{aligned} e_\chi \mathbb{C}_p \text{Det}_{\mathbb{Z}_p[\Gamma]}(D_{N,S^*(N)}^\bullet) &\xrightarrow{f_{N/K(\chi)}} e_\chi \mathbb{C}_p \text{Det}_{\mathbb{Z}_p[\mathcal{G}_{K(\chi)}]}(D_{K(\chi),S^*(N)}^\bullet) \\ &\xrightarrow{\Theta_{K(\chi)/k,S^*(N)}^{r_\chi}} e_\chi \mathbb{C}_p \bigwedge_{\mathbb{Z}_p[\mathcal{G}_{K(\chi)}]}^{r_\chi} U_{K(\chi),S^*(N)} \end{aligned}$$

sends $e_\chi q_N^{-1}\mathfrak{z}_N$ to $e_\chi \varepsilon_{K(\chi)/k,S^*(N)}^{V_\chi}$. Here V_χ is any choice of subset of $S^*(N)$ of cardinality $r_\chi := r_{S^*(N)}(\chi)$ that only contains places that split completely in $K(\chi)$, and the map $f_{N/K(\chi)}$ appearing above is induced by the composite of (15), (16), and (17).

To verify this we set $W := V_\chi \setminus V_{K_\chi}$ and $m := |W|$, and claim that it suffices to demonstrate the equality

$$(\text{Ord}_{K(\chi),W} \circ \Theta_{K(\chi)/k,S^*(N)}^{r_\chi} \circ f_{N/K(\chi)})(e_\chi q_N^{-1}\mathfrak{z}_N) = (-1)^{mr_\chi} \cdot e_\chi \varepsilon_{K(\chi)/k,S^*(N)\setminus W,\emptyset}^{V_{K_\chi}}. \quad (20)$$

Indeed, by the result of [San14, Prop. 3.6] one knows that

$$\text{Ord}_{K(\chi),W}(\varepsilon_{K(\chi)/k,S^*(N),\emptyset}^{V_\chi}) = (-1)^{mr_\chi} \cdot \varepsilon_{K(\chi)/k,S^*(N)\setminus W,\emptyset}^{V_{K_\chi}}.$$

In particular, the injectivity of the map $\text{Ord}_{K(\chi),W}$ on the $e_{K(\chi)/k,S^*(N),\emptyset}$ -isotypic component of $\bigwedge_{\mathbb{Z}_p[\mathcal{G}_{K(\chi)}]}^{r_\chi} U_{K(\chi),S^*(N)}$ (as is proved in [Rub96, Lem. 5.1 (i)]) then combines with the previous two equalities to imply the claim.

Turning now to the verification of (20), we first note (19) implies that one has

$$(\Theta_{K_\chi/k,S^*(K_\chi)}^{r_{K_\chi}} \circ i_{N/K_\chi})(q_N^{-1}\mathfrak{z}_N) = \varepsilon_{K_\chi/k}. \quad (21)$$

Using Lemma (3.8), we may therefore calculate that

$$\begin{aligned}
& (\Theta_{K(\chi), \Pi(K(\chi))}^{r_{K_\chi}} \circ \Delta_{K(\chi), S^*(N), S^*(K_\chi)} \circ f_{N/K(\chi)})(e_\chi q_N^{-1} \mathfrak{z}_N) \\
&= (\Theta_{K(\chi), S^*(K_\chi)}^{r_{K_\chi}} \circ \Delta_{K(\chi), S^*(N), S^*(K_\chi)} \circ f_{K_\chi/K(\chi)} \circ f_{N/K_\chi})(e_\chi q_N^{-1} \mathfrak{z}_N) \\
&= (\Theta_{K(\chi), S^*(K_\chi)}^{r_{K_\chi}} \circ f_{K_\chi/K(\chi)} \circ \Delta_{K_\chi, S^*(N), S^*(K_\chi)} \circ f_{N/K_\chi})(e_\chi q_N^{-1} \mathfrak{z}_N) \\
&= (\mathbb{N}_{K_\chi/K(\chi)}^{r_{K_\chi}} \circ \Theta_{K_\chi, S^*(K_\chi)}^{r_\chi} \circ i_{N/K_\chi})(e_\chi q_N^{-1} \mathfrak{z}_N) \\
&= \mathbb{N}_{K_\chi/K(\chi)}^{r_{K_\chi}}(e_\chi \varepsilon_{K_\chi/k}) \\
&= e_\chi \varepsilon_{K(\chi)/k, S^*(K_\chi), \emptyset}^{V_{K_\chi}}. \tag{22}
\end{aligned}$$

Here the first equality follows from the fact that $f_{N/K(\chi)} = f_{K_\chi/K(\chi)} \circ f_{N/K_\chi}$, the second from Lemma (3.8) (d), the third from Lemma (3.8) (a) and the definition of the map i_{N/K_χ} , the fourth from (21), and the last from the properties of Rubin–Stark elements.

To proceed, we set $W' = S^*(N) \setminus W$ and calculate

$$\begin{aligned}
& (-1)^{mr_\chi} \cdot (\text{Ord}_{K(\chi), W} \circ \Theta_{K(\chi)/k, S^*(N)}^{r_\chi} \circ f_{N/K(\chi)})(e_\chi q_N^{-1} \mathfrak{z}_N) \\
&= (\Theta_{K(\chi), W'}^{r_{K_\chi}} \circ \Delta_{K(\chi), S^*(N), W'} \circ f_{N/K(\chi)})(e_\chi q_N^{-1} \mathfrak{z}_N) \\
&= (\Theta_{K(\chi), W'}^{r_{K_\chi}} \circ \Delta_{K(\chi), S^*(K_\chi), W'} \circ \Delta_{K(\chi), S^*(N), S^*(K_\chi)} \circ f_{N/K(\chi)})(e_\chi q_N^{-1} \mathfrak{z}_N) \\
&= \left(\prod_{v \in W' \setminus S^*(K_\chi)} (1 - \chi(\text{Frob}_v)^{-1}) \right) \cdot (\Theta_{K(\chi), S^*(K_\chi)}^{r_{K_\chi}} \circ \Delta_{K(\chi), S^*(N), S^*(K_\chi)} \circ f_{N/K(\chi)})(e_\chi q_N^{-1} \mathfrak{z}_N) \\
&= \left(\prod_{v \in W' \setminus S^*(K_\chi)} (1 - \chi(\text{Frob}_v)^{-1}) \right) \cdot e_\chi \varepsilon_{K(\chi)/k, S^*(K_\chi), \emptyset}^{V_{K_\chi}} \\
&= e_\chi \varepsilon_{K(\chi)/k, W', \emptyset}^{V_{K_\chi}}.
\end{aligned}$$

Here the first equality follows via an application of Lemma (3.8) (b), the second by the definition of the respective Δ homomorphisms, the third by appealing to Lemma (3.8) (c), the fourth by (22), and the last by properties of Rubin–Stark elements.

This concludes the proof of (20), and hence of Proposition (3.10). \square

(3.11) Remark. The results of this section show that the conjecture formulated by Sano and three of the present authors in [Bur+23] cannot be valid. To discuss this we write k^s for the maximal abelian extension of k in which all places in $S_\infty = S_\infty(k)$ split completely, Ω for the collection $\Omega^{S_\infty}(k)$ of finite ramified extensions of k in k^s , \mathcal{R}^s for the algebra $\mathbb{Z}[\text{Gal}(k^s/k)]$ and \mathcal{A}^s for the ideal $\varprojlim_{E \in \Omega} \text{Ann}_{\mathbb{Z}[\mathcal{G}_E]}(\mu_E)$ of \mathcal{R}^s , where the limit is taken with respect to the natural projection maps $\mathbb{Z}[\mathcal{G}_{E'}] \rightarrow \mathbb{Z}[\mathcal{G}_E]$ for fields $E \subseteq E'$ in Ω . Then the central conjecture (Conjecture 2.5) of loc. cit. predicts the \mathcal{R}^s -module ES_k of Euler systems defined in [Bur+23, Def. 2.3] is such that $\mathcal{A}^s \cdot \text{ES}_k \subseteq \mathcal{A}^s \cdot \varrho^{S_\infty}(\text{im}(\Theta_k))$. However, the validity of this prediction is not consistent with the result of Proposition (3.9) for two reasons: firstly, the intersection $\mathcal{A}^s \cdot (\text{ES}_k \cap \text{ES}_k^{S_\infty}(\mathbb{Z})^{\text{iso}})$ is in general non-zero (and, in this regard, recall Lemma (2.17)) and, secondly, the final observation in Remark (2.25)(d) implies that, in general, there are systems in $\mathcal{A}^s \cdot \text{ES}_k$ that are not symmetric.

4. The theory of Euler limits

In this section we develop a general theory of ‘Euler limits’ that is vital for our approach to the analysis of Euler systems.

4.1. Definitions and examples

Throughout this section, we fix a number field k and, for convenience, write Ω instead of $\Omega(k)$ for the set of finite abelian extensions of k that are ramified at (at least) one finite place. Similarly, we abbreviate $S_\infty(k)$ and $S_p(k)$ to S_∞ and S_p respectively. We also fix a subring \mathcal{R} of \mathbb{C} with field of fractions \mathcal{Q} .

(4.1) Definition. Given a subset \mathcal{X} of Ω , we define a **triangular system** on \mathcal{X} to consist of the following data for every pair of fields E and F in \mathcal{X} with $E \subseteq F$:

- an $\mathcal{R}[\mathcal{G}_F]$ -module $M_{F,E}$;
- maps of $\mathcal{R}[\mathcal{G}_F]$ -modules $\rho_{F/E}: M_{F,F} \rightarrow M_{F,E}$ and $j_{F/E}: M_{E,E} \rightarrow M_{F,E}$.

In the case $E = F$, we usually abbreviate $M_{F,E}$ to M_E and refer to it as a **diagonal term** of the system.

(4.2) Examples. Triangular systems arise naturally in several ways.

- (a) (*Projective systems*) Each projective system $(\widetilde{M}_F, \widetilde{\rho}_{F/E})_{F \in \mathcal{X}}$ of $\mathcal{R}[\mathcal{G}_F]$ -modules gives rise to a triangular system by setting $M_{F,E} := \widetilde{M}_E$ for each $E \subseteq F$ and taking $\rho_{F/E}$ to be $\widetilde{\rho}_{F/E}$ and $j_{F/E}$ to be the identity map on \widetilde{M}_E .
- (b) (*Biduals of units*) For any rank function \mathbf{r} one obtains a triangular system on \mathcal{X} by setting, for each E and F in \mathcal{X} with $E \subseteq F$,

$$M_{F,E} := \begin{cases} \mathcal{R}\mathcal{L}_E^{\mathbf{r}(E)} & \text{if } E = F, \\ \bigoplus_{i=0}^{\infty} \mathcal{Q} \wedge_{\mathbb{Z}[\mathcal{G}_E]}^i \mathcal{O}_{E,S(F)}^\times & \text{if } E \neq F, \end{cases}$$

and by taking $\rho_{F/E}$ to be the map induced by the norm $\mathcal{O}_{F,S(F)}^\times \rightarrow \mathcal{O}_{E,S(F)}^\times$ and $j_{F/E}$ to be the (injective) map induced by the inclusion $\mathcal{O}_{E,S(E)}^\times \subseteq \mathcal{O}_{E,S(F)}^\times$.

- (c) (*Exterior powers of duals*) For any non-negative integer t , and any pair of fields $E, F \in \mathcal{X}$ with $E \subseteq F$, the composite homomorphism

$$(\mathcal{O}_{F,S(F)}^\times)^* \rightarrow \text{Hom}_{\mathbb{Z}[\mathcal{G}_F]}(\mathcal{O}_{E,S(E)}^\times, \mathbb{Z}[\mathcal{G}_F]^{\text{Gal}(F/E)}) \rightarrow (\mathcal{O}_{E,S(E)}^\times)^*,$$

where the first map is restriction and the second is induced by sending $N_{\text{Gal}(F/E)}$ to 1, induces a homomorphism of $\mathbb{Z}[\mathcal{G}_F]$ -modules $\Phi_{F/E}^t: \bigwedge_{\mathbb{Z}[\mathcal{G}_F]}^t (\mathcal{O}_{F,S(F)}^\times)^* \rightarrow \bigwedge_{\mathbb{Z}[\mathcal{G}_E]}^t (\mathcal{O}_{E,S(E)}^\times)^*$. Given a rank function \mathbf{t} , we therefore obtain a triangular system by setting

$$M_{F,E} := \begin{cases} \bigwedge_{\mathcal{R}[\mathcal{G}_E]}^{\mathbf{t}(E)} (\mathcal{R}\mathcal{O}_{E,S(E)}^\times)^* & \text{if } E = F, \\ \bigoplus_{i=0}^{\infty} \bigwedge_{\mathcal{R}[\mathcal{G}_E]}^i (\mathcal{R}\mathcal{O}_{E,S(E)}^\times)^* & \text{if } E \neq F, \end{cases}$$

and by taking $\rho_{F/E}$ to be $\Phi_{F/E}^{\mathbf{t}(E)}$ and $j_{F/E}$ to be the natural inclusion map.

We next associate to each triangular system a natural notion of limit.

(4.3) Definition. Fix a finite set Π of places of k and, for each field E in Ω , set

$$\Pi(E) := \Pi \cup S(E) = \Pi \cup S_{\text{ram}}(E/k).$$

(Note, in particular, that $\Pi(E) = S(E)$ if $\Pi = \emptyset$ and $\Pi(E) = S^*(E)$ if $\Pi = S_\infty(k)$.)

- (a) The Π -relative **Euler factor** for an extension F/E of fields in Ω is the element

$$P_{F/E,\Pi} := \prod_{v \in S(F) \setminus \Pi(E)} (1 - \text{Frob}_v^{-1}) \in \mathbb{Z}[\mathcal{G}_E].$$

(In particular, if $\Pi = \emptyset$, then $P_{F/E,\Pi}$ is the standard factor $\prod_{v \in S(F) \setminus S(E)} (1 - \text{Frob}_v^{-1})$ that also occurs in the definition of an Euler system.)

- (b) Let $\mathcal{T} := \{(M_{F,E}, \rho_{F/E}, j_{F/E})\}_{E,F \in \mathcal{X}}$ be a triangular system on a subset \mathcal{X} of Ω . Then the Π -relative **Euler limit** of \mathcal{T} is the $\mathcal{R}[[\mathcal{G}_{\mathcal{K}}]]$ -submodule

$$\left\{ (a_F)_F \in \prod_{F \in \mathcal{X}} M_F \mid \rho_{F/E}(a_F) = P_{F/E, \Pi} \cdot j_{F/E}(a_E) \text{ for all } E, F \in \mathcal{X} \text{ with } E \subseteq F \right\}$$

of the direct product $\prod_{F \in \mathcal{X}} M_F$. We denote this submodule by $\varprojlim^{\Pi} \mathcal{T}$ or, more simply (when data defining the system \mathcal{T} is clear from context), by $\varprojlim_{F \in \mathcal{X}}^{\Pi} M_F$.

In the rest of this section we fix a set of places Π as above. In addition, for a (not necessarily finite) extension of fields $E \subseteq F$ that are abelian over k , we write

$$\pi_{F/E}: \mathbb{C}[[\mathcal{G}_F]] \rightarrow \mathbb{C}[[\mathcal{G}_E]]$$

for the natural restriction map (and we use the same notation to denote the restriction of $\pi_{F/E}$ to $R[[\mathcal{G}_F]]$ for any subring R of \mathbb{C}).

(4.4) Examples. The concept of Euler limit incorporates several natural constructions.

- (a) (*Euler systems*) If we take $\mathcal{X} = \Omega$ and $\Pi = \emptyset$, then elements of the Euler limit associated to the triangular system on \mathcal{X} described in Example (4.2) (b) are Euler systems of rank r that are \mathcal{R} -integral in the sense of Definition (2.12).
- (b) (*Perrin-Riou functionals*) Take $\mathcal{X} = \Omega$ and $\Pi = \emptyset$. If we define the module of *Perrin-Riou functionals* $\text{PR}_k^t(\mathcal{R})$ to be the Euler limit of the triangular system defined in Example (4.2) (c) with transition maps $\rho_{F/E} := P_{F/E, \emptyset} \cdot \Phi_{F/E}^{t(E)}$, then, for any pair of rank functions \mathbf{t} and \mathbf{s} with $\mathbf{s}(E) \geq \mathbf{t}(E)$ for all $E \in \Omega(k)$, and any subring \mathcal{R} of \mathbb{C} , the argument of Perrin-Riou in [PR98, Lem. 1.2.3] (see also [Rub96, §6]) shows that the assignment $((f_E)_E, (c_E)_E) \mapsto (f_E(c_E))_E$ induces a homomorphism of $\mathcal{R}[[\mathcal{G}_{\mathcal{K}}]]$ -modules

$$\text{PR}_k^t(\mathcal{R}) \otimes_{\mathcal{R}[[\mathcal{G}_{\mathcal{K}}]]} \text{ES}_k^s(\mathcal{R}) \rightarrow \text{ES}_k^{s-t}(\mathcal{R}).$$

This method of ‘rank-reduction’ has been used in the literature to obtain useful classical (rank-one) Euler systems from higher-rank Euler systems.

- (c) (*Inverse limits*) Each projective system $(M_F, \rho_{F/E})_{F \in \mathcal{X}}$ of $\mathcal{R}[[\mathcal{G}_F]]$ -modules gives rise to an associated projective system $(M'_F, \rho'_{F/E})_{F \in \mathcal{X}}$ in which one has $M'_F := M_F$ and $\rho'_{F/E} := P_{F/E, \Pi} \cdot \rho_{F/E}$ for all $E \subseteq F$. The identity map on $\prod_{F \in \mathcal{X}} M_F$ restricts to induce an embedding of the inverse limit $\varprojlim_{F \in \mathcal{X}} M_F$ into the Euler limit $\varprojlim_{F \in \mathcal{X}}^{\Pi} M'_F$ of the triangular system associated to $(M'_F, \rho'_{F/E})_{F \in \mathcal{X}}$ by Example (4.2) (a).
- (d) (*Idempotent projections*) Let $(M_F, \rho_{F/E})_{F \in \mathcal{X}}$ be a projective system of \mathbb{Z} -torsion-free $\mathcal{R}[[\mathcal{G}_F]]$ -modules and recall the idempotents

$$e_{F, \Pi} := e_{F, S_{\infty}(k) \cup \Pi(F), r_F}$$

from Definition (2.20). (We note, in particular, that if $\Pi = \emptyset$, then $e_{F, \Pi}$ is equal to the idempotent e_F defined in (3).) We define a triangular system $(M'_{F,E}, \rho'_{F/E}, j_{F/E})_{F \in \mathcal{X}}$ by setting

$$M'_{F,E} := \begin{cases} e_{F, \Pi} M_F & \text{if } F = E, \\ \mathcal{Q} \otimes_{\mathcal{R}} M_E & \text{if } F \neq E, \end{cases}$$

and $\rho'_{F/E} := P_{F/E, \Pi} \cdot \rho_{F/E}$, and taking $j_{F/E}$ to be the natural inclusion of $e_{F, \Pi} M_F$ into $\mathcal{Q} \otimes_{\mathcal{R}} M_F$. We then write $\varprojlim_{F \in \mathcal{X}}^{\Pi} e_{F, S} M_F$ for the Euler limit of the triangular system $(M'_{F,E}, \rho'_{F/E}, j_{F/E})_{F \in \mathcal{X}}$. A straightforward calculation shows that for fields E and F in Ω with $E \subseteq F$ one has $P_{F/E, \Pi} \cdot (\pi_{F/E}(e_{F, \Pi}) - e_{E, \Pi}) = 0$. This fact implies that the assignment

$$\varprojlim_{F \in \mathcal{X}} M_F \rightarrow \varprojlim_{F \in \mathcal{X}}^{\Pi} e_{F, \Pi} M_F, \quad (m_F)_F \mapsto (e_{F, \Pi} m_F)_F$$

is a well-defined map of $\mathcal{R}[[\mathcal{G}_{\mathcal{K}}]]$ -modules.

In the next two sections we shall prove the existence of natural exact sequences relating respectively to ‘restriction’ and ‘localisation’ functors that are associated to the Euler limits of triangular systems.

4.2. The p -adic restriction exact sequence

4.2.1. Statement of the p -adic restriction sequence

In the sequel we will consider subsets \mathcal{X} of Ω that are ‘large’ in the sense that they satisfy the following hypothesis.

(4.5) Hypothesis (Closure Hypothesis). Fix a subset \mathcal{V} of $\mathcal{P}(S_\infty(k))$ and let \mathcal{X} be a subset of $\Omega^\mathcal{V}(k)$ that has the following two closure properties:

- (i) (*Composita*) If $K \in \mathcal{X}$ and $E \in \Omega^{S_\infty(k)}(k)$, then $KE \in \mathcal{X}$.
- (ii) (*Subfields*) If $K \in \mathcal{X}$, then for every intermediate field L of K/k with $V_L \in \mathcal{V}$, one has
 - $L \in \mathcal{X}$ if L/k is ramified,
 - $k(\mathfrak{p})L \in \mathcal{X}$ for almost all $\mathfrak{p} \in S_{\text{fin}}(k)$ if L/k is unramified.

(4.6) Examples. The following two families of extensions satisfy Hypothesis (4.5) (for suitable sets \mathcal{V}) and will play an important role in the sequel.

- (a) Let \mathcal{V} be any subset of $\mathcal{P}(S_\infty(k))$. Then the set $\mathcal{X} = \Omega^\mathcal{V}(k)$ satisfies Hypothesis (4.5) with respect to \mathcal{V} .
- (b) Let k be a totally real field. Then the set \mathcal{X} of all finite abelian CM extensions of k that are ramified at some finite prime satisfies Hypothesis (4.5) with respect to $\mathcal{V} = \{\emptyset\}$.

For each finite set of places Σ of k , we write \mathcal{X}_Σ for the subset of \mathcal{X} comprising fields that are ramified at every place in Σ . We then have a natural ‘restriction’ map on Euler limits

$$\text{res}_\Sigma^\Pi: \varprojlim_{F \in \mathcal{X}}^\Pi M_F \rightarrow \varprojlim_{F \in \mathcal{X}_\Sigma}^\Pi M_F. \quad (23)$$

(4.7) Remark. For $E \in \mathcal{X}$ and $F \in \mathcal{X}_\Sigma$ with $E \subseteq F$ (if such F exists), the defining relation of the Euler limit relative to $\Pi \cup \Sigma$ has no Euler factor for the extension E/F . In many cases, therefore, the limit $\varprojlim_{F \in \mathcal{X}_\Sigma}^\Pi M_F$ can be regarded as a submodule of $\varprojlim_{F \in \mathcal{X}}^{\Pi \cup \Sigma} M_F$.

In the next result we shall establish an important property of the maps res_Σ^Π for triangular systems that satisfy the following natural hypothesis.

(4.8) Hypothesis. Let p be a prime number. The data $\{(M_{F,E}, \rho_{F/E}, j_{F/E})\}_{E,F \in \mathcal{X}}$ constitutes a triangular system in which the transition maps $j_{F/E}$ are injective (and will be suppressed in the notation) and, for all pairs of fields $E \subseteq F$ contained in Ω , there exists an injective homomorphism of \mathcal{G}_E -modules $\iota_{F/E}: M_E \rightarrow M_F^{\text{Gal}(F/E)}$, that has both of the following properties:

- (i) the restriction of the composite $\iota_{F/E} \circ \rho_{F/E}$ to the full pre-image M'_E in M_F of M_E under $\rho_{F/E}$ factors through multiplication by $N_{\text{Gal}(F/E)}$,
- (ii) there exists an integer N , depending only on M_E , such that for every m in M_E and every integer t , one has $m \in p^t \cdot M_E$ if $\iota_{F/E}(m) \in p^{t+N} \cdot M_F$.

(4.9) Examples. Hypothesis (4.8) is satisfied in several natural cases with \mathcal{R} a Dedekind domain and p a prime number that is not a unit in \mathcal{R} . For example, one may take \mathcal{R} to be \mathbb{Z}_p or \mathbb{Z}_S with a finite set of prime numbers S that does not contain p .

- (a) Hypothesis (4.8) (ii) is satisfied in this case if, for each E , the module M_E is a \mathbb{Z} -torsion-free and there exists a natural number N , depending only on E , such that, for every F , the p -part of the order of $\text{coker}(\iota_{F/E})_{\text{tor}}$ divides p^N . To justify this, set $\iota := \iota_{F/E}$ and note that if $\iota(m) = p^t \cdot y$ for some integer $t \geq 0$ and $y \in M_F$, then the image of y in $\text{coker}(\iota)$ is annihilated by p^t , and hence also, by assumption, by cp^N for some natural number c that is coprime with p . If z is the (unique) element of M_E with $\iota(z) = cp^N \cdot y$, then $cp^N \cdot m = p^t \cdot z \in p^t M_E$. Thus, because M_E is \mathbb{Z} -torsion-free and c is a unit modulo p^{t-N} , we conclude that m belongs to $p^{t-N} M_E$, as required.
- (b) The triangular system associated (via Example (4.2) (a)) to the canonical projective system $(\mathcal{R}[\mathcal{G}_E], \pi_{F/E})$ satisfies Hypothesis (4.8) with $\iota_{F/E}$ induced by the map $\mathcal{R}[\mathcal{G}_E] \rightarrow \mathcal{R}[\mathcal{G}_F]$ sending each a to $N_{\text{Gal}(F/E)} \cdot \tilde{a}$ for any (and therefore every) $\tilde{a} \in \mathcal{R}[\mathcal{G}_F]$ with $\pi_{F/E}(\tilde{a}) = a$. For this choice, the validity of (i) is clear and the validity of (ii) follows from Example (4.9) (a) and the fact that $\text{coker}(\iota_{F/E})$ vanishes.
- (c) Lemma (2.9) (c) (see also Remark (2.11)) shows that the triangular system of biduals in Example (4.2) (b) satisfies Hypothesis (4.8) with $\iota_{F/E}$ taken to be the map $\nu_{F/E}$ from Lemma (2.9).

In the sequel we shall use the field

$$k\langle p \rangle := k(\mu_{p^{s+1}}, (\mathcal{O}_k^\times)^{1/p^{\min\{s,1\}}}). \quad (24)$$

The following is the main result of this subsection.

(4.10) Theorem (The p -adic restriction sequence). *We assume to be given the following data:*

- an odd prime number p ,
- a subset \mathcal{X} of Ω that satisfies Hypothesis (4.5),
- a triangular system $\{(M_{F,E}, \rho_{F/E}, j_{F/E})\}_{E,F \in \mathcal{X}}$ that satisfies Hypothesis (4.8) with respect to p and is such that each diagonal term M_E is a $\mathbb{Z}_p[\mathcal{G}_E]$ -lattice,
- finite subsets Π and Σ of $S_{\text{fin}}(k)$.

For each field F in \mathcal{X} write $k_\Pi(F)$ for the composite of all extensions of k in $k\langle p \rangle$ in which at least one place of $S_p(k) \setminus \Pi(F)$ splits completely. Then the sequence of $\mathbb{Z}_p[\mathcal{G}_\mathcal{X}]$ -modules

$$0 \longrightarrow \varprojlim_{F \in \mathcal{X}}^\Pi M_F^{\text{Gal}(F/F \cap k_\Pi(F))} \xrightarrow{\subseteq} \varprojlim_{F \in \mathcal{X}}^\Pi M_F \xrightarrow{\overline{\text{res}}_\Sigma^\Pi} \varprojlim_{F \in \mathcal{X}_\Sigma}^\Pi (M_F/M_F^{\text{Gal}(F/F \cap k_\Pi(F))}),$$

in which $\overline{\text{res}}_\Sigma^\Pi$ denotes the map induced by res_Σ^Π , is exact.

(4.11) Remark. The result of Theorem (4.10) is equivalent to asserting that the kernel of $\overline{\text{res}}_\Sigma^\Pi$ is independent of the choice of Σ . In the stated generality, however, it seems more difficult to obtain concrete information about the cokernel of $\overline{\text{res}}_\Sigma^\Pi$.

(4.12) Remark. Our methods also prove a variant of Theorem (4.10) for $p = 2$, see Proposition (4.21) and Remark (4.24) for more details.

The proof of Theorem (4.10) will be given in § 4.2.5 after we first establish several necessary preliminary results. We end this section by recording an important consequence of Theorem (4.10) that will be derived in § 4.2.6.

(4.13) Corollary (The global restriction sequence). *We assume to be given the following data:*

- a finite set of prime numbers \mathcal{S} ,
- a subset \mathcal{X} of Ω that satisfies Hypothesis (4.5),

- a triangular system $\{(M_{F,E}, \rho_{F/E}, j_{F/E})\}_{E,F \in \mathcal{X}}$ for which each diagonal term M_E is a $\mathbb{Z}_S[\mathcal{G}_E]$ -lattice and Hypothesis (4.8) is satisfied for all but finitely many p ,
- finite subsets Π and Σ of $S_{\text{fin}}(k)$.

Then the sequence of $\mathbb{Z}_S[\mathcal{G}_K]$ -modules

$$0 \longrightarrow \varprojlim_{F \in \mathcal{X}}^{\Pi} M_F^{\mathcal{G}_F} \xrightarrow{\subseteq} \varprojlim_{F \in \mathcal{X}} M_F \xrightarrow{\overline{\text{res}}_{\Sigma}^{\Pi}} \varprojlim_{F \in \mathcal{X}_{\Sigma}}^{\Pi} (M_F/M_F^{\mathcal{G}_F}),$$

in which $\overline{\text{res}}_{\Sigma}^{\Pi}$ denotes the map induced by $\text{res}_{\Sigma}^{\Pi}$, is exact.

4.2.2. Constructing extensions with prescribed local behaviour

A key step in the proof of Theorem (4.10) is provided by the following technical result concerning the existence of cyclic extensions of number fields with certain prescribed local behaviour.

(4.14) Proposition. *Assume to be given data of the following form:*

- a prime number p ,
- a finite abelian extension K of k ,
- a finite set \mathcal{T} of places of k that contains $S_p(k)$ and is such that the \mathcal{T} -classgroup of k is trivial.

Write s for the greatest integer such that k contains a primitive p^s -th root of unity, fix a natural number n with $n > s$, define a finite Galois extension of k by

$$k(p, \mathcal{T}, n) := \begin{cases} k(\mu_{p^{s+n}}, (\mathcal{O}_{k, \mathcal{T}}^{\times})^{1/p^s}) & \text{if } p \text{ is odd,} \\ k(\mu_{p^{s+n+1}}, (\mathcal{O}_{k, \mathcal{T}}^{\times})^{1/p^{s+1}}) & \text{if } p = 2, \end{cases}$$

and set

$$L = L(p, \mathcal{T}, n) := K \cap k(p, \mathcal{T}, n).$$

Then, for any element σ of \mathcal{G}_K whose restriction to L is trivial, there exists a Galois extension $E = E(p, n, \mathcal{T}, \sigma)$ of k that has all of the following properties:

- (i) \mathcal{G}_E is a cyclic p -group,
- (ii) every place $v \in \mathcal{T}$ is unramified in E and the order of Frob_v on E is at least p^n ,
- (iii) $S(E)$ contains no infinite place and at most two finite places,
- (iv) for each $v \in S(E)$ the following claims are valid:
 - (a) the inertia subgroup of v in \mathcal{G}_E has order at least p^n ,
 - (b) v is unramified in K and the restriction of Frob_v to K is equal to σ .

To prove this result, we shall use several important results in the theory of embedding problems that are obtained by Neukirch in [Neu73, § 7 and § 8]. For the convenience of the reader, we shall therefore begin by fixing a natural number n , setting

$$q := p^n$$

and reviewing Neukirch's approach to the problem of determining whether there exists a cyclic Galois extension of k of degree q that realises a given family of local extensions.

We write G_k for the absolute Galois group of k and G_{k_v} for the decomposition subgroup in G_k of each place v of k . Then, for a given collection of places v and a morphism $\varphi_v : G_{k_v} \rightarrow \mathbb{Z}/q\mathbb{Z}$ for each such v , we ask whether there exists a surjective morphism $G_k \rightarrow \mathbb{Z}/q\mathbb{Z}$ whose restriction to G_{k_v} agrees with φ_v for all v in the given set.

To be more precise, we fix a set of places \mathcal{M} of k that contains both $S_\infty(k)$ and $S_p(k)$ and a finite subset \mathcal{T} of \mathcal{M} . We then define

$$\Lambda(\mathcal{M}, \mathcal{T}) := \prod_{v \in \mathcal{T}} \{0\} \times \prod'_{v \in \mathcal{M} \setminus \mathcal{T}} \text{Hom}(G_{k_v}, \mathbb{Z}/q\mathbb{Z}) \times \prod_{v \notin \mathcal{M}} \text{Hom}_{\text{nr}}(G_{k_v}, \mathbb{Z}/q\mathbb{Z})$$

where $\text{Hom}_{\text{nr}}(G_{k_v}, \mathbb{Z}/q\mathbb{Z})$ is the subset of $\text{Hom}(G_{k_v}, \mathbb{Z}/q\mathbb{Z})$ comprising all morphisms that are ‘non-ramified’ at v (that is, vanish on the inertia subgroup) and \prod' is the restricted product (that is, the subset of the cartesian product comprising all elements $(\varphi_v)_v$ with the property that φ_v is non-ramified for all but finitely many v). We then consider

$$\Delta(k, \mathbb{Z}/q\mathbb{Z})_{\mathcal{T}}^{\mathcal{M}} := \text{coker} \left(\text{Hom}(G_k, \mathbb{Z}/q\mathbb{Z}) \rightarrow \left(\prod'_v \text{Hom}(G_{k_v}, \mathbb{Z}/q\mathbb{Z}) \right) / \Lambda(\mathcal{M}, \mathcal{T}) \right)$$

where in the restricted product v runs over all places of k .

We now assume to be given a morphism $\phi_0: G_k \rightarrow \mathbb{Z}/q\mathbb{Z}$ that is unramified outside \mathcal{M} and, for each $v \in \mathcal{T}$, a morphism $\varphi_v: G_{k_v} \rightarrow \mathbb{Z}/q\mathbb{Z}$. We then write $\phi_{0,v}$ for the restriction of ϕ_0 to G_{k_v} and define η to be the class in $\Delta(k, \mathbb{Z}/q\mathbb{Z})_{\mathcal{T}}^{\mathcal{M}}$ of the collection $(\eta_v)_v$ that is defined by setting

$$\eta_v = \begin{cases} \phi_{0,v} - \varphi_v & \text{if } v \in \mathcal{T}, \\ \phi_{0,v} & \text{if } v \notin \mathcal{T}. \end{cases} \quad (25)$$

(4.15) Remark. The class η is a natural ‘obstruction’ to solving the problem at hand since it vanishes if and only if there exists a morphism $\phi: G_k \rightarrow \mathbb{Z}/q\mathbb{Z}$ that is unramified outside \mathcal{M} and, for all $v \in \mathcal{T}$, coincides with φ_v when restricted to G_{k_v} .

To investigate the vanishing of η , we introduce the following group of units:

$$U_{\mathcal{T}}^{\mathcal{M}}(q) := \{a \in k^\times \mid \forall v \notin \mathcal{M} : \text{ord}_v(a) \equiv 0 \pmod{q}, \forall v \in \mathcal{M} \setminus \mathcal{T} : a \in (k_v^\times)^q\}.$$

(4.16) Lemma ([Neu73, Satz (7.1)]). *There is a canonical isomorphism*

$$\Delta(k, \mathbb{Z}/q\mathbb{Z})_{\mathcal{T}}^{\mathcal{M}} \xrightarrow{\cong} \text{Hom}(U_{\mathcal{T}}^{\mathcal{M}}(q)/(k^\times)^q, \mathbb{Z}/q\mathbb{Z})$$

that can be explicitly described as follows. Let f be an element of $\Delta(k, \mathbb{Z}/q\mathbb{Z})_{\mathcal{T}}^{\mathcal{M}}$ represented by a family $(f_v)_v$ of morphisms $f_v: G_{k_v} \rightarrow \mathbb{Z}/q\mathbb{Z}$. Then the above isomorphism sends f to

$$U_{\mathcal{T}}^{\mathcal{M}}(q)/(k^\times)^q \rightarrow \mathbb{Z}/q\mathbb{Z}, \quad a \mapsto \sum_v f_v(\text{rec}_v(a)),$$

where $\text{rec}_v: k_v^\times \rightarrow G_{k_v}^{\text{ab}}$ is the local reciprocity map and $G_{k_v}^{\text{ab}}$ is the abelianisation of G_{k_v} .

We next use Kummer theory to give a convenient Galois-theoretic description of $\Delta(k, \mathbb{Z}/q\mathbb{Z})_{\mathcal{T}}^{\mathcal{M}}$. To do this, we set

$$k_1 := k(\mu_q) \quad \text{and} \quad k_2 = k_{2,\mathcal{T}}^{\mathcal{M}} := k_1(\sqrt[q]{a} \mid a \in U_{\mathcal{T}}^{\mathcal{M}}(q)).$$

(4.17) Remark. The Galois extension k_2/k_1 is referred to as the ‘obstruction extension’ by Neukirch and has the following functorial behaviour. Let \mathcal{M}' be a set of places of k that is disjoint from $\mathcal{M} \cup \mathcal{T}$, and write $H_{\mathcal{M}'}$ for the subgroup of $\text{Gal}(k_{2,\mathcal{T}}^{\mathcal{M}}/k_1)$ generated by the collection $\{\text{Frob}_v \mid v \in \mathcal{M}'_{k_1}\}$. Then, since every place in \mathcal{M}'_{k_1} splits completely in $k_{2,\mathcal{T}}^{\mathcal{M} \cup \mathcal{M}'}$, one has $k_{2,\mathcal{T}}^{\mathcal{M} \cup \mathcal{M}'} \subseteq (k_{2,\mathcal{T}}^{\mathcal{M}})^{H_{\mathcal{M}'}}$ and $\text{Gal}(k_{2,\mathcal{T}}^{\mathcal{M} \cup \mathcal{M}'}/k_1)$ is a quotient of $\text{Gal}(k_{2,\mathcal{T}}^{\mathcal{M}}/k_1)/H_{\mathcal{M}'}$. (In fact, one has equality in both instances if p is odd, cf. [Neu73, Satz (7.3)].)

The following result is a straightforward extension of [Neu73, Thm. 7.4] that incorporates the case $p = 2$.

(4.18) Lemma. *The canonical map*

$$\text{Gal}(k_2/k_1) \rightarrow \text{Hom}_{\mathbb{Z}}(U_{\mathcal{T}}^{\mathcal{M}}(q)/(k^\times)^q, \mu_q), \quad \sigma \mapsto \{a \mapsto \sigma(\sqrt[q]{a})/\sqrt[q]{a}\}$$

is injective. Its cokernel is of order dividing two and vanishes if p is odd.

Proof. We write Q for the quotient module $(U_{\mathcal{T}}^{\mathcal{M}}(a) \cap (k_1^{\times})^q)/(k^{\times})^q$, and consider the tautological exact sequence

$$0 \longrightarrow Q \longrightarrow U_{\mathcal{T}}^{\mathcal{M}}(q)/(k^{\times})^q \longrightarrow U_{\mathcal{T}}^{\mathcal{M}}(q)/(U_{\mathcal{T}}^{\mathcal{M}}(q) \cap (k_1^{\times})^q) \longrightarrow 0.$$

Then, since Kummer theory identifies $\text{Gal}(k_2/k_1)$ with $\text{Hom}_{\mathbb{Z}}(U_{\mathcal{T}}^{\mathcal{M}}(q)/(U_{\mathcal{T}}^{\mathcal{M}}(a) \cap (k_1^{\times})^q), \mu_q)$, we can take Kummer duals of the above sequence to obtain an exact sequence

$$0 \longrightarrow \text{Gal}(k_2/k_1) \longrightarrow \text{Hom}_{\mathbb{Z}}(U_{\mathcal{T}}^{\mathcal{M}}(q)/(k^{\times})^q, \mathbb{Z}/q\mathbb{Z}) \longrightarrow \text{Hom}_{\mathbb{Z}}(Q, \mathbb{Z}/q\mathbb{Z}) \longrightarrow 0.$$

This sequence directly proves the first claim and also shows that the order of the cokernel of the given map is equal to the order of Q .

To bound this order, we observe that Q identifies with a subgroup of the kernel

$$\ker(k^{\times}/(k^{\times})^q \rightarrow k_1^{\times}/(k_1^{\times})^q) \cong H^1(\text{Gal}(k_1/k), \mu_q),$$

and that, by [Neu73, Satz (4.8)], the latter cohomology group vanishes if either q is odd or if $k \cap \mathbb{Q}(\mu_{2^n})$ is complex.

It therefore only remains to consider the case that $q = 2^n$ and $\sqrt{-1} \notin k$. In this case, the inflation-restriction sequence combines with the aforementioned vanishing result to give an isomorphism

$$H^1(\text{Gal}(k(\sqrt{-1})/k), \mu_q) \cong H^1(\text{Gal}(k_1/k), \mu_q).$$

In addition, since $\text{Gal}(k(\sqrt{-1})/k)$ is cyclic, a Herbrand quotient argument implies that

$$|H^1(\text{Gal}(k(\sqrt{-1})/k), \mu_q)| = |\widehat{H}^0(\text{Gal}(k(\sqrt{-1})/k), \mu_q)| = |\{\pm 1\}| = 2,$$

as required to conclude the proof of the second claim. \square

The following observation regarding the field k_2 defined above will also be useful later on.

(4.19) Lemma. *Let $q := p^n$ with n a natural number as before, and set*

$$k_3 = k_{3, \mathcal{T}} := k(\mu_q, (\mathcal{O}_{k, \mathcal{T}}^{\times})^{1/q}).$$

Then the following claims are valid.

- (a) *One has $k_2 \subseteq k_3$, with equality if every place in $\mathcal{M} \setminus \mathcal{T}$ splits completely in k_3 .*
- (b) *Write s for the greatest integer such that k contains a primitive p^s -th root of unity and assume $n > s$. Then the maximal abelian extension of k in k_3 is equal to $k(\mu_{p^{n+s}}, (\mathcal{O}_{k, \mathcal{T}}^{\times})^{1/p^s})$ if p is odd or $\sqrt{-1} \in k$, and is otherwise contained in $k(\mu_{2^{n+s+1}}, (\mathcal{O}_{k, \mathcal{T}}^{\times})^{1/2^{s+1}})$.*

Proof. To prove the first part of claim (a) it is sufficient, by Kummer theory, to show that the class of any element of $U_{\mathcal{T}}^{\mathcal{M}}(q)$ in $k^{\times}/(k^{\times})^q$ is represented by an element of $\mathcal{O}_{k, \mathcal{T}}^{\times}$. To do this, we fix a in $U_{\mathcal{T}}^{\mathcal{M}}(q)$ and write $Z(a)$ for the (finite) set of places v of k that are outside \mathcal{T} and satisfy $\text{ord}_v(a) \neq 0$. Then, since the \mathcal{T} -classgroup of k is assumed to be trivial, for each $v \in Z(a)$, one has that the ideal v is equal to an ideal that is only supported at \mathcal{T} times an element π_v of k^{\times} . It follows that π_v belongs to $\mathcal{O}_{k, \mathcal{T} \cup \{v\}}^{\times}$ and satisfies $\text{ord}_v(\pi_v) = 1$. By definition of $U_{\mathcal{T}}^{\mathcal{M}}(q)$, one has $\text{ord}_v(a) \equiv 0 \pmod{q}$ for all $v \in Z(a)$ and so $\pi_a := \prod_{v \in Z(a)} \pi_v^{-\text{ord}_v(a)}$ belongs to $(k^{\times})^q$. The element $a \cdot \pi_a$ can then be checked to belong to $\mathcal{O}_{k, \mathcal{T}}^{\times}$ and have the same class as a in $k^{\times}/(k^{\times})^q$. This proves that k_2 is contained in k_3 , as claimed in (a).

If, in addition, every place in $\mathcal{M} \setminus \mathcal{T}$ splits completely in k_3 , then the definition of k_3 implies that every element of $\mathcal{O}_{k, \mathcal{T}}^{\times}$ must belong to $(k_v^{\times})^q$ for every such place $v \in \mathcal{M} \setminus \mathcal{T}$. We deduce that $\mathcal{O}_{k, \mathcal{T}}^{\times}$ is contained in $U_{\mathcal{T}}^{\mathcal{M}}(q)$ in this case, and hence that k_3 is contained in k_2 . This proves the second part of claim (a).

To prove claim (b), we write L' for the maximal abelian extension of k in k_3 . Then L' contains $k_1 = k(\mu_q)$ and so, by Kummer theory, there exists a subgroup U of $(\mathcal{O}_{k, \mathcal{T}}^{\times} \cdot (k_1^{\times})^q)/(k_1^{\times})^q$ such that $\text{Gal}(L'/k_1) \cong \text{Hom}(U, \mu_q)$.

Since L' is abelian over k , the conjugation action of \mathcal{G}_{k_1} on $\text{Gal}(L'/k_1)$ is trivial. To make this condition explicit, we fix elements $\tau \in \text{Gal}(L'/k_1)$ and $\rho \in \mathcal{G}_{k_1}$, a lift $\tilde{\rho}$ of ρ to $\text{Gal}(L'/k)$ and an element a of $\mathcal{O}_{k,\mathcal{T}}^\times$ that represents a class in U . Then, for a suitable integer $n(a, \tilde{\rho})$, one has

$$\tilde{\rho}(\sqrt[q]{a}) = \zeta_q^{n(a, \tilde{\rho})} \sqrt[q]{\rho(a)} = \zeta_q^{n(a, \tilde{\rho})} \sqrt[q]{a}.$$

In particular, since $\tilde{\rho} \circ \tau = \tau \circ \tilde{\rho}$ in $\text{Gal}(L'/k_1)$, one therefore has

$$\tilde{\rho}(\tau(\sqrt[q]{a})) = \tau(\tilde{\rho}(\sqrt[q]{a})) = \tau(\zeta_q^{n(a, \tilde{\rho})} \sqrt[q]{a}) = \zeta_q^{n(a, \tilde{\rho})} (\tau(\sqrt[q]{a})/\sqrt[q]{a}) \sqrt[q]{a} = (\tau(\sqrt[q]{a})/\sqrt[q]{a}) \tilde{\rho}(\sqrt[q]{a}),$$

and so the element $\tau(\sqrt[q]{a})/\sqrt[q]{a}$ of $\mu_q \subset k_1$ is fixed by $\tilde{\rho}$. Since this is true for every element ρ of \mathcal{G}_{k_1} , it follows that $\tau(\sqrt[q]{a})/\sqrt[q]{a}$ belongs to k^\times and hence must be a p^s -th root of unity.

We next write $\langle a \rangle$ for the subgroup of $k_1^\times/(k_1^\times)^q$ that is generated by a , and note that Kummer theory gives a natural isomorphism

$$\text{Gal}(k_1(\sqrt[q]{a})/k_1) \xrightarrow{\cong} \text{Hom}(\langle a \rangle, \mu_q), \quad \tau \mapsto \tau(\sqrt[q]{a})/\sqrt[q]{a}.$$

In particular, since the above argument shows that the order of every element in the image of this isomorphism is at most p^s , the order of $\langle a \rangle$ is at most p^s , and so the element a^{p^s} belongs to the kernel of the natural map

$$\theta : k^\times/(k^\times)^q \rightarrow k_1^\times/(k_1^\times)^q.$$

In addition, θ is injective if either p is odd or $\sqrt{-1} \in k$, and in all other cases the order of $\ker(\theta)$ divides two (cf. the proof of Lemma (4.18)).

For simplicity, we now assume that we are in the first of these cases (and merely note that the second case can be dealt with in the same fashion).

In this case, the above argument shows that a^{p^s} belongs to $(k^\times)^q$ and hence that

$$a^{p^s} = b^q$$

for some element b of $\mathcal{O}_{k,\mathcal{T}}^\times$. One therefore has $\sqrt[q]{a} = \zeta \cdot \sqrt[p^s]{b}$ for a root of unity ζ of order dividing $qp^s = p^{s+n}$ (so that $\zeta \in k(\mu_{p^{s+n}})$), and hence that

$$k_1(\sqrt[q]{a}) \subset k(\mu_{p^{s+n}}, (\mathcal{O}_{k,\mathcal{T}}^\times)^{1/p^s}).$$

Since the field L' is generated by all such fields $k_1(\sqrt[q]{a})$, it must therefore be contained in $k_3 \cap k(\mu_{p^{s+n}}, (\mathcal{O}_{k,\mathcal{T}}^\times)^{1/p^s}) = k(\mu_{p^{s+n}}, (\mathcal{O}_{k,\mathcal{T}}^\times)^{1/p^s})$, as required to complete the proof. \square

Proof of Proposition (4.14): We are now ready to prove Proposition (4.14), the notation and hypotheses of which we henceforth assume. In addition, in this argument we take

$$q := \begin{cases} p^{s+n} & \text{if } p \text{ is odd,} \\ p^{s+n+1} & \text{if } p = 2, \end{cases}$$

and use the fields

$$k_1 := k(\mu_q), \quad k_3 := k_1((\mathcal{O}_{k,\mathcal{T}}^\times)^{1/q}) \quad \text{and} \quad F := K \cdot k_3.$$

It is clear that $L \subseteq k_3$, and Lemma (4.19) (b) implies $K \cap k_3 \subseteq L$. Thus, since the restriction to L of the element σ fixed in the statement of Proposition (4.14) is trivial, we can use the canonical isomorphism

$$\text{Gal}(F/L) \cong \text{Gal}(K/L) \times \text{Gal}(k_3/L)$$

to choose an element $\bar{\sigma}$ of $\text{Gal}(F/L)$ whose restrictions to K and k_3 are respectively equal to σ and the identity automorphism. By Chebotarev's Density Theorem, applied to the Galois extension F/k , we can then choose a place \mathfrak{q} of k with the property that $\bar{\sigma}$ is equal to Frob_Ω for some place Ω of F lying above \mathfrak{q} . This condition ensures that \mathfrak{q} splits completely in k_3 and that the restriction of $\text{Frob}_\mathfrak{q}$ to K is equal to σ . In particular, by Lemma (2.1) (and the definition of k_3), it follows that the degree of $k(\mathfrak{q})/k(1)$ is divisible by q . Now, since $\text{Gal}(k(\mathfrak{q})/k(1))$ is isomorphic to a quotient of the unit group $(\mathcal{O}_k/\mathfrak{q})^\times$, it is cyclic and so this argument implies

$\mathcal{G}_{k(\mathfrak{q})}$ contains an element of order q . This fact in turn implies $k(\mathfrak{q})$ contains a subfield E_0 that is a cyclic degree q extension of k and so gives rise to a surjective homomorphism

$$\phi_0: G_k \rightarrow \text{Gal}(E_0/k) \cong \mathbb{Z}/q\mathbb{Z}.$$

For each place v we denote the restriction of ϕ_0 to G_{k_v} by $\phi_{0,v}$.

We next write \mathcal{T}' for the subset of \mathcal{T} comprising all places v whose decomposition group $D_v(E_0/k)$ in \mathcal{G}_{E_0} has order less than p^n (and so is identified, under the isomorphism $\text{Gal}(E_0/k) \cong \mathbb{Z}/q\mathbb{Z}$ fixed above, with a subgroup of $(qp^{-n+1}\mathbb{Z})/(q\mathbb{Z})$). We note, in particular, that for each $v \in \mathcal{T}'$ the restricted morphism $\phi_{0,v}$ is a composite of the form

$$\phi_{0,v}: G_{k_v} \rightarrow D_v(E_0/k) \hookrightarrow (qp^{-n+1}\mathbb{Z})/q\mathbb{Z} \subseteq \mathbb{Z}/q\mathbb{Z}.$$

For each place v in \mathcal{T} we now define a morphism φ_v in the following way:

- if $v \in \mathcal{T} \setminus \mathcal{T}'$, then we set $\varphi_v := \phi_{0,v}$;
- if $v \in \mathcal{T}'$, then we take φ_v to be the composite

$$\mathcal{G}_{k_v} \rightarrow \text{Gal}(k_v^{\text{nr},n}/k_v) \cong (qp^{-n}\mathbb{Z})/q\mathbb{Z} \subseteq \mathbb{Z}/q\mathbb{Z},$$

where $k_v^{\text{nr},n}$ denotes the unique non-ramified extension of k_v of degree p^n and the first arrow is the canonical projection.

This data gives rise to an obstruction class $\eta = (\eta_v)_v$ in the sense of (25) for which one has

$$\eta_v = \begin{cases} \phi_{0,v} - \varphi_v, & \text{if } v \in \mathcal{T}' \\ 0, & \text{if } v \in \mathcal{T} \setminus \mathcal{T}' \\ \phi_{0,v}, & \text{if } v \notin \mathcal{T}. \end{cases} \quad (26)$$

To analyse this class we set $\mathcal{M} := \mathcal{T} \cup \{\mathfrak{q}\}$. We also note that, for the fixed choice of \mathfrak{q} , one has that the field $k_2 = k_{2,\mathcal{T}}^{\mathcal{M}} := k_1(U_{\mathcal{T}}^{\mathcal{M}}(q)^{1/q})$ defined earlier coincides with k_3 by Lemma (4.19)(a). Now, since the explicit definition of each of the morphisms $\phi_{0,v}$ and φ_v implies η is divisible by 2 in $\Delta(k, \mathbb{Z}/q\mathbb{Z})_{\mathcal{T}}^{\mathcal{M}}$, it belongs to the image of the injective map

$$\text{Gal}(k_3/k_1) = \text{Gal}(k_2/k_1) \rightarrow \Delta(k, \mathbb{Z}/q\mathbb{Z})_{\mathcal{T}}^{\mathcal{M}}$$

defined in Lemma (4.18). Writing τ for the pre-image of η under this map, we claim next that τ acts as the identity on L .

To verify this, we first show that every element of $\text{Gal}(k_3/k_1)$ of order dividing p^n acts as the identity on L . To do this, we fix $\sigma \in \text{Gal}(k_3/k_1)$ with σ^{p^n} trivial and $a \in \mathcal{O}_{k,\mathcal{T}}^\times$. Then, since L is contained in the field $k(p, \mathcal{T}, n)$ that is generated over k_1 by elements of the form $\sqrt[q]{a}$ with $t := qp^{-n}$, it suffices to show $\sigma(\sqrt[q]{a}) = \sqrt[q]{a}$. Let ξ_a be the unique q -th root of unity with the property that $\sigma(\sqrt[q]{a}) = \xi_a \sqrt[q]{a}$. Then, since $\xi_a \in k_1$, one has $\sigma(\xi_a) = \xi_a$ and so $\sigma^m(\sqrt[q]{a}) = \xi_a^m \sqrt[q]{a}$ for all integers m . In particular, since σ^{p^n} is trivial, the order of ξ_a divides p^n and so we obtain the required equality via the computation

$$\sigma(\sqrt[q]{a}) = (\sigma(\sqrt[q]{a}))^{p^n} = (\xi_a \sqrt[q]{a})^{p^n} = \sqrt[q]{a}.$$

To prove τ acts as the identity on L , it is therefore enough to show that τ^{p^n} is trivial. Note also that, if p is odd, this will be true if η is divisible by p^s in $\Delta(k, \mathbb{Z}/q\mathbb{Z})_{\mathcal{T}}^{\mathcal{M}}$, whilst if $p = 2$ then the cokernel of $\text{Gal}(k''/k') \rightarrow \Delta(k, \mathbb{Z}/q\mathbb{Z})_{\mathcal{T}}^{\mathcal{M}}$ has order at most 2 and so it will be true if η is divisible by 2^{s+1} . However, since for each $v \in \mathcal{T}'$ the morphisms $\phi_{0,v}$ and φ_v are, by their construction, divisible by p^s (resp. p^{s+1} if $p = 2$), it is clear that this condition is satisfied.

Next, we observe that Lemma (4.19)(b) implies $K \cap k_3 = L$, and so the canonical isomorphism

$$\text{Gal}(Kk_3/L) \cong \text{Gal}(k_3/L) \times \text{Gal}(K/L)$$

implies the existence of a unique element g of $\text{Gal}(Kk_3/L)$ that restricts to k_3 and K to respectively give τ and the element σ fixed in Proposition (4.14).

Now, since g belongs to the abelian subgroup $\text{Gal}(Kk_3/Lk_1)$ of $\text{Gal}(Kk_3/L)$, we can use Chebotarev's Density Theorem to fix a place \mathfrak{A} of Lk_1 with all of the following properties:

- \mathfrak{A} has absolute degree one;
- the rational prime lying beneath \mathfrak{A} is unramified in Kk_3 ;
- the restriction of $\text{Frob}_{\mathfrak{A}}$ to Kk_3 is equal to g .

The place \mathfrak{a} of k lying beneath such a place \mathfrak{A} is then unramified in Kk_3 and such that the restrictions to K and k_3 of $\text{Frob}_{\mathfrak{a}}$ are respectively equal to σ and τ .

Thus, if we set $\mathcal{M}' = \mathcal{T} \cup \{\mathfrak{q}, \mathfrak{a}\}$, then it follows from Remark (4.17) that the obstruction class η vanishes in $\Delta(k, \mathbb{Z}/q\mathbb{Z})_{\mathcal{T}'}^{\mathcal{M}'}$. As a consequence of Remark (4.15) and the explicit specification (26) of η , we can therefore deduce the existence of a field E_1 that has all of the properties (i), (ii), (iii) and (iv)(b) that are stated in Proposition (4.14). To complete the proof, it therefore suffices to show there exists such a field E_1 that also has the property (iv)(a).

To do this, we set $h := \text{ord}_p(|\text{cl}_k|)$ and repeat the above argument with n replaced by $n' := h + 2n$ in order to obtain a cyclic extension E_1 of k that has properties (i), (ii), (iii) and (iv)(b) with respect to n' (rather than n). In particular, for this field there exists a place v_1 in $S(E_1)$ such that the inertia subgroup of v_1 in \mathcal{G}_{E_1} has order at least p^{2n} . If $S(E_1) = \{v_1\}$, then this field E_1 already has all of the required properties.

We can therefore assume that $S(E_1) = \{v_1, v_2\}$ for some place $v_2 \neq v_1$. If the inertia subgroup $I(v_2)$ of v_2 in \mathcal{G}_{E_1} has order at least p^n , then the field E_1 again has all required properties. On the other hand, if the order of $I(v_2)$ is less than p^n , then we need only consider its fixed field $E := E_1^{I(v_2)}$ in order to obtain a cyclic extension of k that is unramified outside v_1 , in which v_1 has inertia subgroup of order at least p^n , and each place in \mathcal{T} has decomposition group of order at least p^n . In this case, therefore, the field E has all of the required properties (i), (ii), (iii) and (iv).

This concludes the proof of Proposition (4.14). \square

4.2.3. Consequences of the Chebotarev Density Theorem

Our proof of Theorem (4.10) will also rely on the technical consequence of Chebotarev's Density Theorem that is described in the following result. In this result we use the fields $k(p, \mathcal{T}, n)$ that are defined in the statement of Proposition (4.14).

(4.20) Lemma. *Fix a prime p and a finite place \mathfrak{p} of k . Set $\mathcal{T} := S_p(k) \cup \{\mathfrak{p}\}$ and, for any extension L of k define the field*

$$L(p) := L \cap k(p, \mathcal{T}) \quad \text{with} \quad k(p, \mathcal{T}) := \bigcup_{n \in \mathbb{N}} k(p, \mathcal{T}, n).$$

Let \mathcal{X} be a subset of Ω that satisfies Hypothesis (4.5) and $\{(M_{F,E}, \rho_{F/E}, j_{F/E})\}_{E,F \in \mathcal{X}}$ a triangular system that satisfies Hypothesis (4.8) and is such that each diagonal term M_E is a $\mathbb{Z}_p[\mathcal{G}_E]$ -lattice. Fix an element $(a_L)_L$ of $\varprojlim_{L \in \mathcal{X}}^{\Pi} M_L$ and assume F is a field in \mathcal{X} that has the following property:

- for any field $L \in \mathcal{X}$ with $F \subseteq L$ the element a_L is fixed by $\text{Gal}(L/L(p))$.

Then, for any field $K \in \mathcal{X}$, the element a_K is fixed by $\text{Gal}(K/K(p))$ provided that all of the following conditions are satisfied:

- (i) $k \subseteq K \subseteq F$.
- (ii) $S(F) \setminus \Pi(K) = \{\mathfrak{p}\}$.
- (iii) $V_K = V_F$.

Proof. Fix a field K as above and an element σ of $\text{Gal}(K/K(p))$ and write $P(\sigma)$ for the subset of $S_{\text{fin}}(k) \setminus (\Pi(K) \cup S_p(k))$ comprising places v for which the restriction of Frob_v^{-1} to K agrees with σ . Then, for any natural number t , Proposition (4.14) implies the existence of a finite cyclic p -extension E of k that belongs to $\Omega^{S_\infty(k)}(k)$, is unramified outside $P(\sigma)$ and is such that the order of the decomposition subgroup of \mathfrak{p} in \mathcal{G}_E is at least p^t .

We note that, since all archimedean places split in E/k , Hypothesis (4.5) (i) guarantees that both of the composite fields

$$F' := EF \quad \text{and} \quad K' := EK$$

belong to \mathcal{X} .

Now, by assumption, one has $a_{F'} \in M_{F'}^{\text{Gal}(F'/F'(p))}$ and so the Euler limit relations combine with the stated conditions (ii) and (iii) to imply that the element

$$(1 - \text{Frob}_{\mathfrak{p}}^{-1}) \cdot a_{K'} = P_{F'/K', \Pi} \cdot a_{K'} = \rho_{F'/K'}(a_{F'})$$

is fixed by the image $\text{Gal}(K'/K'(p))$ of $\text{Gal}(F'/F'(p))$ in $\mathcal{G}_{K'}$.

Let us now write H for the intersection of $\text{Gal}(K'/K'(p))$ and the subgroup of $\mathcal{G}_{K'}$ generated by $\text{Frob}_{\mathfrak{p}}$. Then any element τ of H is a power of $\text{Frob}_{\mathfrak{p}}$ and so $(1 - \tau) \cdot a_{K'}$ is fixed by H . However, the element $(1 - \tau) \cdot a_{K'}$ is also annihilated by the norm N_H and hence, since $M_{F'}$ is \mathbb{Z} -torsion free, must vanish. It follows that $a_{K'}$ is fixed by H .

Write $M'_{K'}$ for the full pre-image in $M_{K'}$ of M_K under $\rho_{K'/K}$. Then, since $(a_L)_L$ belongs to $\varprojlim_{L \in \mathcal{X}}^{\Pi} M_L$, one has $a_{K'} \in M'_{K'}$ and so Hypothesis (4.8) (i) implies that

$$(\iota_{K'/K} \circ \rho_{K'/K})(a_{K'}) = N_{\text{Gal}(K'/K)} \cdot f(a_{K'})$$

for a suitable endomorphism f of $M'_{K'}$. Setting

$$n := |S(K') \setminus \Pi(K)|,$$

we can therefore deduce that the element

$$\begin{aligned} \iota_{K'/K}((1 - \sigma)^n \cdot a_K) &= \iota_{K'/K}(P_{K'/K, \Pi} \cdot a_K) \\ &= (\iota_{K'/K} \circ \rho_{K'/K})(a_{K'}) \\ &= N_{\text{Gal}(K'/K)} \cdot f(a_{K'}) \\ &= N_{\text{Gal}(K'/K)/H} \cdot N_H \cdot f(a_{K'}) \\ &= N_{\text{Gal}(K'/K)/H} \cdot |H| \cdot f(a_{K'}) \end{aligned}$$

is divisible by $|H|$ in $M_{K'}$. Note that restriction to E induces an isomorphism

$$\begin{aligned} \text{Gal}(K'/K'(p)K) &= \text{Gal}(K'/K) \cap \text{Gal}(K'/K'(p)) \cong \text{Gal}(E/E \cap K) \cap \text{Gal}(E/E(p)) \\ &= \text{Gal}(E/E(p)(E \cap K)). \end{aligned}$$

Now, as E is unramified at p , the field $E(p)$ is contained in the maximal unramified extension of k in $k(p, \mathcal{T})$. That is, we can find an integer s that only depends on K , k , p and \mathfrak{p} and is such that, for any value of $t \geq s$ the group $\text{Gal}(E/E(p)(E \cap K))$ must be contained in the unique subgroup of the (cyclic) group \mathcal{G}_E that has index p^s . However, by construction, the decomposition group of \mathfrak{p} in \mathcal{G}_E has order at least p^t and so the order of its intersection with $\text{Gal}(E/E(p)(E \cap K))$ must be at least p^{t-s} . This implies, in particular, that $|H| \geq p^{t-s}$.

We now write e_1 for the idempotent in $\mathbb{Q}[\langle \sigma \rangle]$ that is associated to the trivial character of $\langle \sigma \rangle$. Then $1 - e_1$ is contained in the augmentation ideal of $\mathbb{Q}[\langle \sigma \rangle]$. Since the latter ideal is generated by $1 - \sigma$, it follows that $1 - e_1 = x(1 - \sigma)$ for some $x \in \mathbb{Q}[\langle \sigma \rangle]$, and hence also

$$\begin{aligned} x^{n-1}(1 - \sigma)^n &= (1 - \sigma)(x(1 - \sigma))^{n-1} = (1 - \sigma)(1 - e_1)^{n-1} \\ &= (1 - \sigma)(1 - e_1) = (1 - \sigma). \end{aligned}$$

Thus, if we now fix a natural number z such that $z \cdot x \in \mathbb{Z}[\langle \sigma \rangle]$, then this computation combines with the previous discussion to imply that the element

$$\iota_{K'/K}(z^{n-1} \cdot (1 - \sigma) \cdot a_K) = z^{n-1} \cdot x^{n-1} \cdot \iota_{K'/K}((1 - \sigma)^n \cdot a_K)$$

is divisible by p^{t-s} in $M_{K'}$.

Now, since $n \leq |S(K') \setminus S(K)| \leq |S(E)|$, property (iii) in Proposition (4.14) implies that $n \leq 2$. By taking t large, Hypothesis (4.8) (ii) therefore implies that $z \cdot (1 - \sigma) \cdot a_K = 0$ and hence, as M_K is torsion free, that $(1 - \sigma) \cdot a_K = 0$. Since σ was an arbitrary element of $\text{Gal}(K/K(p))$, we have therefore shown that a_K is fixed by $\text{Gal}(K/K(p))$, as required. \square

4.2.4. Reduction steps

The following result provides a key reduction step in the proof of Theorem (4.10).

(4.21) Proposition. *Fix a prime number p and a subset \mathcal{X} of Ω that satisfies Hypothesis (4.5). Let $\{(M_{F,E}, \rho_{F/E}, j_{F/E})\}_{E, F \in \mathcal{X}}$ be a triangular system that satisfies Hypothesis (4.8) and is such that each diagonal term M_E is both \mathbb{Z} -torsion-free and satisfies $\bigcap_{i \in \mathbb{N}} (p^i M_E) = \{0\}$. Let Π and Σ be finite subsets of $S_{\text{fin}}(k)$ and let \mathcal{T} be a finite set of places of k that contains $\Sigma \cup S_p(k)$ and is such that the \mathcal{T} -class group of k vanishes. For each extension L of k set*

$$L(p) := L \cap k(p, \mathcal{T}),$$

where the field $k(p, \mathcal{T})$ is as defined in Lemma (4.20). Then the following sequence is exact

$$0 \longrightarrow \left(\varprojlim_{F \in \mathcal{X}} M_F \right)^{\text{Gal}(K/k(p, \mathcal{T}))} \xrightarrow{\subseteq} \varprojlim_{F \in \mathcal{X}} M_F \longrightarrow \varprojlim_{F \in \mathcal{X}_\Sigma} (M_F / M_F^{\text{Gal}(F/F(p))}),$$

where the third arrow is induced by the restriction map res_Σ^Π .

Proof. We need to show that if $a = (a_K)_K$ is any element of $\varprojlim_{K \in \mathcal{X}}^\Pi M_K$ for which $a_{K'}$ belongs to $M_{K'}^{\text{Gal}(K'/K'(p))}$ for every K' in \mathcal{X}_Σ , then one has $a_K \in M_K^{\text{Gal}(K/K(p))}$ for every K in \mathcal{X} . To do this, we fix such an element a and argue by induction on the quantity

$$n(K) := |\Sigma \setminus S(K)|.$$

If $n(K) = 0$, then K/k is ramified at every place in Σ and so belongs to \mathcal{X}_Σ . In this case, therefore, the given hypothesis on a directly implies that a_K belongs to $M_K^{\text{Gal}(K/K(p))}$, as required.

We therefore now assume $n(K) > 0$ and that the assertion holds true for all fields $L \in \mathcal{X}$ for which one has $n(L) < n(K)$. Then, since $n(K) > 0$, we may fix a (finite) place \mathfrak{p} in $\Sigma \setminus S(K)$. We now apply the result of Lemma (4.22) below, with $\Sigma' = \{\mathfrak{p}\}$ to obtain a chain of fields in Ω

$$k = L_0 \subset L_1 \subset \cdots \subset L_t \subset L$$

with $|S(L_i) \setminus S(L_{i-1})| = 1$ for $1 \leq i \leq t$, $S(L) \setminus S(L_t) = \{\mathfrak{p}\}$ and $S(L) \cap S_\infty(k) = \emptyset$. We set $F := LK$ and $F_i := L_i K$ for each i with $1 \leq i \leq t$ and thereby obtain a chain of fields in \mathcal{X}

$$K = F_0 \subseteq L_1 \subseteq \cdots \subseteq F_t \subseteq F$$

with $|S(F_i) \setminus S(F_{i-1})| \leq 1$ for $1 \leq i \leq t$, $S(F) \setminus S(F_t) = \{\mathfrak{p}\}$ and $V_K = V_F$.

We shall now argue by a downwards induction on i (for $0 \leq i \leq t$) that for any field E in \mathcal{X} containing F_i the element a_E belongs to $M_E^{\text{Gal}(E/E(p))}$. We note in particular that, since $F_0 = K$, this result for $i = 0$ implies the claimed result.

For the inductive argument we first take a field E_t in \mathcal{X} that contains F_t and write E for the field $E_t F = E_t L$. Then E belongs to \mathcal{X} and $n(E) < n(K)$ since $\mathfrak{p} \in S(F) \subseteq S(E)$, and so the inductive hypothesis implies a_E belongs to $M_E^{\text{Gal}(E/E(p))}$.

In addition, if either $S(E_t) = S(E)$ or $\mathfrak{p} \in \Pi$, then the Euler factor $P_{\Pi, E/E_t}$ is trivial and so the Euler limit relations directly imply that $a_{E_t} = \rho_{E/E_t}(a_E)$ is contained in $M_{E_t}^{\text{Gal}(E_t/E_t(p))}$. On the other hand, if $S(E) \setminus \Pi(E_t) = \{\mathfrak{p}\}$ then, as $V_{E_t} = V_E$, we may apply Lemma (4.20) (with F and K taken to be E and E_t) to deduce that a_{E_t} is again fixed by $\text{Gal}(E_t/E_t(p))$, as required. This verifies the inductive base and then the inductive step is established by the same argument with the roles of F , F_t and \mathfrak{p} played by F_i , F_{i-1} and the unique place in $S(F_i) \setminus S(F_{i-1})$ (if such a place exists). This proves the claimed result. \square

(4.22) Lemma. *Let Σ' be a finite subset of $S_{\text{fin}}(k)$. Then there exists a non-negative integer t and a chain of fields in Ω*

$$k = L_0 \subset L_1 \subset \cdots \subset L_t \subset L$$

such that $|S(L_i) \setminus S(L_{i-1})| = 1$ for $1 \leq i \leq t$, $S(L) \setminus S(L_t) = \Sigma'$ and $S(L) \cap S_\infty(k) = \emptyset$.

Proof. For each finite place v of k and natural number n , write $U_v^{(n)}$ for the group of units in the ring of integers \mathcal{O}_v of the completion k_v that are congruent to 1 modulo $(v\mathcal{O}_v)^n$. Then Hensel's Lemma implies that for any integer ℓ there exists a natural number n_v with the property that $(U_v^{(n_v)})^\ell = U_v^{(n_v + \text{ord}_v(\ell))}$ (for details see [Neu11, Ch. II, Lem. 3.5]).

We define a conductor $\mathfrak{m} := \prod_{v'} v' \prod_v v^{n_v + \text{ord}_v(2)}$ of k , where v' runs over all real archimedean places of k and v over places in $S_2(k)$ that are not in Σ' , and write $k(\mathfrak{m})$ for the ray class field of k of conductor \mathfrak{m} . We fix an odd prime ℓ that does not ramify in k , set $k[\ell] := k(\mu_\ell, (\mathcal{O}_k^\times)^{1/\ell})$ and consider the diagram

$$\begin{array}{ccc} & k(\mathfrak{m}) \cdot k[\ell] & \\ & \swarrow \quad \searrow & \\ k(\mathfrak{m}) & & k[\ell] \\ & \nwarrow \quad \nearrow & \\ & k & \end{array}$$

Then ℓ is prime to $|\mu_k|$ so that the maximal abelian extension of k in $k[\ell]$ is $k(\mu_\ell)$ (see the argument of Lemma (4.19)(b)) and so is totally ramified at all places in $S_\ell(k)$. One therefore has $k(\mathfrak{m}) \cap k[\ell] = k$. In particular, for each element σ of $\mathcal{G}_{k(\mathfrak{m})}$ we can use Chebotarev's Density Theorem to fix a place \mathfrak{p}_σ of k that is outside Σ' , splits completely in $k[\ell]$ and has Frobenius on $k(\mathfrak{m})$ equal to σ . We note that the degree of $k(\mathfrak{p}_\sigma)/k(1)$ is divisible by ℓ (by Lemma (2.1)) and hence that $k(\mathfrak{p}_\sigma)/k$ is ramified at \mathfrak{p}_σ (but not at any archimedean place).

Write $k_{\mathfrak{m},1}$ for the subgroup of k^\times comprising elements x with $|x|_v > 0$ for all real places of k and $x \equiv 1 \pmod{v^{n_v + \text{ord}_v(2)}\mathcal{O}_v}$ for all v in $S_2(k)$, and recall that the \mathfrak{m} -ray class group $\text{cl}_{k,\emptyset,\mathfrak{m}}$ of k is defined as the group of fractional ideals that are coprime to \mathfrak{m} modulo its subgroup of principal ideals with a generator in $k_{\mathfrak{m},1}$. Since the set $\Upsilon := \{\mathfrak{p}_\sigma \mid \sigma \in \mathcal{G}_{k(\mathfrak{m})}\}$ generates $\text{cl}_{k,\emptyset,\mathfrak{m}}$, we find that the ideal $\prod_{v \in \Sigma'} \mathfrak{p}$ is equal to an ideal only supported on places in Υ times an element x of $k_{\mathfrak{m},1}$. This element x therefore is integral outside $\Sigma' \cup \Upsilon$ and, moreover, has all of the following properties:

- $\text{ord}_{\mathfrak{p}}(x) = 1$ if $\mathfrak{p} \in \Sigma'$,
- $x \equiv 1 \pmod{v^{n_v + \text{ord}_v(2)}} if v is 2-adic and $v \notin \Upsilon$,$
- $|x|_v > 0$ if v is a real archimedean place of k .

For any such x the quadratic extension $L' := k(\sqrt{x})/k$ is unramified outside $S_2(k) \cup \Sigma' \cup \Upsilon$. Further, since (by construction and the earlier observation concerning principal units) x is a square in k_v^\times for all 2-adic places v outside Σ' , any such v splits completely in L' and so L'/k is unramified outside $\Sigma' \cup \Upsilon$. Finally we note that L'/k is ramified at each place \mathfrak{p} in Σ' because x is a uniformiser in $k_{\mathfrak{p}}$ for all such \mathfrak{p} . The field L' is therefore such that $\Sigma' \subseteq S(L) \subseteq \Sigma' \cup \Upsilon$. We now set $t := |\mathcal{G}_{k(\mathfrak{m})}|$ and fix an ordering $\{\mathfrak{p}_j : 1 \leq j \leq t\}$ of the places $\{\mathfrak{p}_\sigma : \sigma \in \mathcal{G}_{k(\mathfrak{m})}\}$. We define L to be the compositum of L' and the fields $\{k(\mathfrak{p}_j) : 1 \leq j \leq t\}$ and for each integer i with $1 \leq i \leq t$ we write L_i for the maximal extension of k in L that is unramified outside $\{\mathfrak{p}_j : 1 \leq j \leq i\}$. This gives a chain of fields of the required sort since $S(L_i) = \{\mathfrak{p}_j : 1 \leq j \leq i\}$ for $1 \leq i \leq t$ (since each extension $k(\mathfrak{p}_j)/k$ is ramified at precisely \mathfrak{p}_j) and hence both $S(L) = \Upsilon \cup \Sigma'$ and $S(L) \setminus S(L_t) = \Sigma'$. \square

In the remainder of the argument we use the field $k\langle p \rangle$ defined in (24).

(4.23) Lemma. *Assume to be given data of the following sort:*

- an odd prime number p ,

- a subset \mathcal{X} of Ω that satisfies Hypothesis (4.5),
- a triangular system $\{(M_{F,E}, \rho_{F/E}, j_{F/E})\}_{E, F \in \mathcal{X}}$ that satisfies Hypothesis (4.8) with respect to p and is such that each diagonal term M_E is a $\mathbb{Z}_p[\mathcal{G}_E]$ -lattice,
- finite subsets Π and Σ of $S_{\text{fin}}(k)$.

Then the sequence of $\mathbb{Z}_p[[\mathcal{G}_{\mathcal{K}}]]$ -modules

$$0 \longrightarrow \varprojlim_{F \in \mathcal{X}}^{\Pi} M_F^{\text{Gal}(F/F \cap k\langle p \rangle)} \xrightarrow{\subseteq} \varprojlim_{F \in \mathcal{X}}^{\Pi} M_F \xrightarrow{\text{res}_{\Sigma}^{\Pi}} \varprojlim_{F \in \mathcal{X}_{\Sigma}}^{\Pi} (M_F/M_F^{\text{Gal}(F/F(p))})$$

is exact, where the fields $F(p)$ are as defined in Proposition (4.21).

Proof. Fix a family $(a_{F'})_{F'}$ in $\ker(\text{res}_{\Sigma}^{\Pi})$ and a field F in \mathcal{X} . Then, by Proposition (4.21), a_F is fixed by $\text{Gal}(F/F(p))$ where, we recall, $F(p)$ denotes $F \cap k(\mu_{p^\infty}, (\mathcal{O}_{k, \mathcal{T}}^\times)^{1/p^s})$ for a suitable finite set of places \mathcal{T} of k that contains $\Sigma \cup S_p(k)$, and so it suffices to prove a_F is fixed by the group $\text{Gal}(F(p)/F \cap k\langle p \rangle)$. To do this, we note that Lemma (4.19) (b) implies that

$$F(p) \cap k(\mu_p, (\mathcal{O}_k^\times)^{1/p}) = F \cap k(\mu_{p^\infty}, (\mathcal{O}_{k, \mathcal{T}}^\times)^{1/p^s}) \cap k(\mu_p, (\mathcal{O}_k^\times)^{1/p}) = F \cap k\langle p \rangle$$

and hence that there is an isomorphism of Galois groups

$$\text{Gal}(F(p)(\mu_p, (\mathcal{O}_k^\times)^{1/p})/(F \cap k\langle p \rangle)) \cong \text{Gal}(F(p)/F \cap k\langle p \rangle) \times \text{Gal}(k(\mu_p, (\mathcal{O}_k^\times)^{1/p})/(F \cap k\langle p \rangle)).$$

For any σ in $\text{Gal}(F(p)/F \cap k\langle p \rangle)$ and natural number d , we can therefore use Chebotarev's Density Theorem to fix a set $\Sigma' = \Sigma'(d, \sigma) = \{\mathfrak{q}_i\}_{1 \leq i \leq d}$ of d places of k that do not belong to $\Pi \cup \mathcal{T}$ and are such that every \mathfrak{q}_i has the following properties: \mathfrak{q}_i is unramified in F and such that the restriction of $\text{Frob}_{\mathfrak{q}_i}$ to $F(p)$ is equal to σ^{-1} , \mathfrak{q}_i is totally split in $k(\mu_p, (\mathcal{O}_k^\times)^{1/p})$ and hence, by Lemma (2.1), such that the ramification degree e_i of \mathfrak{q}_i in $k(\mathfrak{q}_i)$ is divisible by p .

Write F'_d for the compositum of F and the fields $k(\mathfrak{q}_i)$ for $1 \leq i \leq d$, and I for the subgroup of $\text{Gal}(F'_d/F)$ generated by the inertia subgroups in $\mathcal{G}_{F'_d}$ of each of the places \mathfrak{q}_i . Then, since $F'_d(p)/k$ is unramified at every place in Σ' , I is contained in $\text{Gal}(F'_d/F'_d(p))$ and so $a_{F'_d}$ is fixed by I . In particular, since $S(F'_d) \setminus \Pi(F) = \Sigma'$, the Euler distribution relations combine with Hypothesis (4.8) (a) to imply that the element

$$\begin{aligned} \iota_{F'_d/F}((1 - \sigma)^d \cdot a_F) &= \iota_{F'_d/F} \left(\left(\prod_{v \in \Sigma'} (1 - \text{Frob}_v^{-1}) \right) \cdot a_F \right) = \iota_{F'_d/F}(P_{F'_d/F, \Pi} \cdot a_F) \\ &= (\iota_{F'_d/F} \circ \rho_{F'_d/F, \Pi})(a_{F'_d}) \\ &= p^d \cdot (\iota_{F'_d/F} \circ \rho_{(F'_d)^I/F, \Pi})(a_{F'_d}) \end{aligned}$$

is divisible by p^d in $M_{F'_d}$. It then follows from Hypothesis (4.8) (ii) that $(1 - \sigma)^d \cdot a_F$ belongs to $p^{d-N} \cdot M_F$ for a natural number N that is independent of d .

Now, since M_F is \mathbb{Z} -torsion-free, the morphism

$$M_F \cong \mathbb{Z}[\langle \sigma \rangle] \otimes_{\mathbb{Z}[\langle \sigma \rangle]} M_F \longrightarrow \bigoplus_{\chi} (\mathbb{Z}_p[\chi] \otimes_{\mathbb{Z}[\langle \sigma \rangle]} M_F)_{\text{tf}}, \quad \sigma \otimes m \mapsto (\chi(\sigma) \otimes m)_{\chi}$$

is injective, where the sum ranges over all characters of $\langle \sigma \rangle$, we set $\mathbb{Z}_p[\chi] := \mathbb{Z}_p[\text{im}(\chi)]$ and write N_{tf} for the quotient of a finitely generated $\mathbb{Z}_p[\chi]$ -module by its torsion subgroup. It is therefore enough for us to prove that $(1 - \chi(\sigma)) \otimes a_F$ vanishes in $(\mathbb{Z}_p[\chi] \otimes_{\mathbb{Z}[\langle \sigma \rangle]} M_F)_{\text{tf}}$ for all such non-trivial characters χ . To do this, we note that, because χ is a p -power order character, the field $\mathbb{Q}_p(\chi) := \mathbb{Q}_p(\text{im}(\chi))$ is a totally ramified extension of \mathbb{Q}_p with valuation ring $\mathbb{Z}_p[\chi]$ and uniformiser $\pi_{\chi} := 1 - \chi(\sigma)$. In particular, one has $p = u \cdot \pi_{\chi}^{[\mathbb{Q}_p(\chi) : \mathbb{Q}_p]}$ for some unit u of $\mathbb{Z}_p[\chi]$. It then follows from $(1 - \sigma)^d a_F \in p^{d-N} \cdot M_F$ and $[\mathbb{Q}_p(\chi) : \mathbb{Q}_p] \geq p - 1$ that

$$\begin{aligned} (\chi(\sigma) - 1) \otimes a_F &\in \pi_{\chi}^{(d-N)[\mathbb{Q}_p(\chi) : \mathbb{Q}_p] - (d-1)} (\mathbb{Z}_p[\chi] \otimes_{\mathbb{Z}[\langle \sigma \rangle]} M_F)_{\text{tf}} \\ &\subseteq \pi_{\chi}^{d(p-2) - N(p-1) + 1} (\mathbb{Z}_p[\chi] \otimes_{\mathbb{Z}[\langle \sigma \rangle]} M_F)_{\text{tf}}. \end{aligned}$$

Since, if p is odd, the exponent $d(p-2) - N(p-1) + 1$ is unbounded as d increases, this shows that $(\chi(\sigma) - 1) \otimes a_F$ is divisible by an arbitrarily large power of π_χ in the $\mathbb{Z}_p[\chi]$ -lattice $(\mathbb{Z}_p[\chi] \otimes_{\mathbb{Z}[\langle \sigma \rangle]} M_F)_{\text{tf}}$. It follows that the element $(\chi(\sigma) - 1) \otimes a_F$ of $(\mathbb{Z}_p[\chi] \otimes_{\mathbb{Z}[\langle \sigma \rangle]} M_F)_{\text{tf}}$ vanishes, as required. \square

(4.24) Remark. If $p = 2$, then the above argument proves the following variant of Lemma (4.23): for every element $a = (a_F)_{F \in \mathcal{X}}$ of $\ker(\text{res}_{\Sigma}^{\Pi})$ and every field F in Ω one has $a_F = \sigma^2(a_F)$ for all σ in $\text{Gal}(F/F \cap k(\mu_{2^{s+1}}, (\mathcal{O}_k^\times)^{1/2}))$.

4.2.5. The proof of Theorem (4.10)

To complete the proof of Theorem (4.10) we will need the following two lemmas.

(4.25) Lemma. *Let \mathcal{X} be a subset of Ω that satisfies Hypothesis (4.5) and fix a triangular system $\{(M_{F,E}, \rho_{F/E}, j_{F/E})\}_{E, F \in \mathcal{X}}$ that satisfies Hypothesis (4.8) with respect to the prime number p . Suppose we have $\bigcap_{i \in \mathbb{N}} p^i M_E = \{0\}$ for every $E \in \mathcal{X}$. Fix finite subsets Π and Σ of $S_{\text{fin}}(k)$. Let H be an open subgroup of $\mathcal{G}_{\mathcal{X}}$.*

If $a = (a_E)_E$ is an element of $\varprojlim_{E \in \mathcal{X}}^{\Pi} M_E$ with the property that a_E is fixed by H whenever E belongs to \mathcal{X}_{Σ} , then a_F vanishes whenever $\Pi(F)$ contains $S_p(k)$.

Proof. Fix a F in \mathcal{X} with $S_p(k) \subseteq \Pi(F)$ and for every natural number n write F_n for the n -th layer of the cyclotomic \mathbb{Z}_p -extension F_{∞} of F .

Then, by Lemma (4.22) (with $\Sigma' = \Sigma$), there exists an extension L of k such that $\Sigma \subseteq S(L)$ and $S(L) \cap S_{\infty}(k) = \emptyset$. In particular, for every $n \in \mathbb{N}$ the field $L_n := L \cdot F_n$ belongs to \mathcal{X}_{Σ} and so, by assumption, the element a_{L_n} is fixed by H . Since $S(L_n) \setminus \Pi(F_n) = S(L) \setminus \Pi(F)$, the Euler limit relations then show that, for every n , the element

$$\left(\prod_{v \in S(L) \setminus \Pi(F)} (1 - \text{Frob}_v) \right) \cdot a_{F_n} = \rho_{L_n/F_n}(a_{L_n}) \in M_{F_n}$$

is fixed by H .

Now, since F_{n+1}/F_n is unramified outside $S_p(k)$, one has $P_{F_{n+1}/F_n, \Pi} = 1$ and so the Euler limit relations imply the family $a_{F_{\infty}} := (a_{F_n})_{n \in \mathbb{N}}$ defines an element of $\varprojlim_{n \in \mathbb{N}} (\mathbb{Z}_p \otimes_{\mathbb{Z}} M_{F_n})$, where the limit is taken with respect to the maps ρ_{F_{n+1}/F_n} .

In addition, if we fix any non-trivial element γ of the open subgroup $\text{Gal}(F_{\infty}/F) \cap H$ of $\text{Gal}(F_{\infty}/k)$, then the last displayed equation implies that $a_{F_{\infty}}$ is annihilated by the element $(\gamma - 1) \cdot \prod_{v \in S(L) \setminus \Pi(F)} (1 - \text{Frob}_v)$ of $\mathbb{Z}_p[[\mathcal{G}_{F_{\infty}}]]$. In particular, since each of the elements γ and Frob_v for $v \in S(L) \setminus \Pi(F)$ generates an open subgroup of $\mathcal{G}_{F_{\infty}}$, we may apply Lemma (4.26) below to deduce that $(a_{F_n})_{n \in \mathbb{N}}$ vanishes, and hence that a_F vanishes, as claimed. \square

(4.26) Lemma. *Fix a field $K \in \mathcal{X}$ and set $K_{\infty} := Kk_{\infty}$, where k_{∞} is any \mathbb{Z}_p -extension of k in which no finite place splits completely. Let $\{(M_{K_m}, \varphi_{K_m/K_n})\}_{n \geq 0}$ be a projective system of $\mathbb{Z}_p[\mathcal{G}_{K_n}]$ -modules (where K_n denotes the n -th layer of K_{∞}/K) that satisfies Hypothesis (4.8) with respect to p and $\bigcap_{i \in \mathbb{N}} p^i M_{K_n} = \{0\}$ for every $n \in \mathbb{N}$. Then, for any generator γ of an open subgroup of $\mathcal{G}_{K_{\infty}}$, the element $\gamma - 1$ acts injectively on $\varprojlim_n M_{K_n}$.*

Proof. Fix an element $a = (a_n)_n$ of this limit such that $(\gamma - 1)a = 0$. Then each a_n is fixed by the restriction of γ to K_n .

Now fix an integer $N \geq 0$. By assumption, $L := K_{\infty}^{(\gamma)}$ is a finite extension of K and so $K_N L$ is a finite extension of k as well, hence $K_N L = K_n$ for some n . In addition, the discussion above implies that, for any $m \geq n$, the element a_m is invariant under the action of γ , hence contained in $M_{K_m}^{\text{Gal}(K_m/K_n)}$. Hypothesis (4.8) (i) therefore implies that

$$\iota_{K_m/K_n}(a_n) = (\iota_{K_m/K_n} \circ \varphi_{K_m/K_n})(a_m) = N_{\text{Gal}(K_m/K_n)} \cdot f(a_m) = [K_m : K_n] \cdot f(a_m)$$

for a suitable endomorphism f of M_{K_m} . This shows $\iota_{K_m/K_n}(a_n)$ is divisible by $[K_m : K_n] = p^{m-n}$ in $M_{K_m}^{\text{Gal}(K_m/K_n)}$. Since p^{m-n} is unbounded as m tends to infinity, this fact combines with Hypothesis (4.8) (ii) to imply a_n is divisible by an unbounded power of p in the $\mathbb{Z}_p[\mathcal{G}_{K_n}]$ -lattice M_{K_n} , and hence that a_n must vanish. In particular, $a_N = 0$ and, since N was chosen arbitrarily, it follows that $a = 0$, as claimed. \square

Proof of Theorem (4.10): In the setting of Theorem (4.10), we fix a family $a = (a_{F'})_{F'}$ that belongs to $\ker(\text{res}_{\Sigma}^{\Pi})$. It then follows from Lemma (4.23) that a is fixed by $\text{Gal}(\mathcal{K}/k\langle p \rangle)$. Since the latter is an open subgroup of $\mathcal{G}_{\mathcal{K}}$, Lemma (4.25) implies that a_F vanishes for every F in Ω for which $\Pi(F)$ contains $S_p(k)$.

To proceed, we now fix a field F in Ω and write

$$k^s := k_{\Pi}^s(F) \quad \text{and} \quad k^{\text{nr}} := k_{\Pi}^{\text{nr}}(F)$$

for the composites of all extensions of k in $k\langle p \rangle$ in which at least one place of $S_p(k) \setminus \Pi(F)$ splits completely, respectively is non-ramified. We note, in particular, that

$$k(F) := k_{\Pi}(F) \subseteq k^s \subseteq k^{\text{nr}}.$$

We have already observed that a_F vanishes if F is ramified at every place in $S_p(k) \setminus \Pi$, so for the remainder of this argument we may assume that at least one place in $S_p(k) \setminus \Pi$ is unramified in F . It follows that $F \cap k\langle p \rangle$ is contained in $k^{\text{nr}}(F)$ and hence that a_F is fixed by $\text{Gal}(F/F \cap k^{\text{nr}})$. To prove that a_F is fixed by $\text{Gal}(F/F \cap k^s)$ it is enough to choose an arbitrary element σ of $\text{Gal}(F/F \cap k^s)$ and show that $(\sigma - 1)a_F$ vanishes, or equivalently (by the above observation) that $e_{\chi}(\sigma - 1)a_F$ vanishes for every character χ of \mathcal{G}_F that factors through $\mathcal{G}_{F \cap k^{\text{nr}}}$ but not through $\mathcal{G}_{F \cap k^s}$.

We fix such a χ and claim that there can exist no place $v \in S_p(k) \setminus \Pi$ of k that is both unramified in F and such that $\chi(\text{Frob}_v) = 1$. Indeed, since any such place v splits completely in the fixed field F_{χ} of $\ker(\chi)$ in F , one would have $F_{\chi} \cap k^{\text{nr}}(F) \subset F \cap k^s$ which contradicts the fact that χ does not factor through $\mathcal{G}_{F \cap k^s}$. We may therefore assume that $\chi(\text{Frob}_v) \neq 1$ for every v in $S_p(k) \setminus \Pi(F)$.

Fix an integer n that is large enough to ensure the n -th layer F_n in the cyclotomic \mathbb{Z}_p -extension of F is such that $S(F_n) = S_p(k)$. Then the Π -relative Euler system relations imply that

$$\chi(P_{F_n/F, \Pi}) \cdot e_{\chi} \cdot a_F = e_{\chi} \cdot P_{F_n/F, \Pi} \cdot a_F = e_{\chi} \cdot \rho_{F_n/F}(a_{F_n}) = 0.$$

In particular, since the element

$$\chi(P_{F_n/F, \Pi}) = \prod_{v \in S_p(k) \setminus \Pi(F)} (1 - \chi(\text{Frob}_v))^{-1}$$

is non-zero, the above equality implies that the element $e_{\chi} a_F$ vanishes, as required.

This completes the proof of Theorem (4.10). \square

4.2.6. The proof of Corollary (4.13)

In the setting of Corollary (4.13), we suppose to be given an element $a = (a_F)_F$ of $\varprojlim_{F \in \mathcal{X}}^{\Pi} M_F$ with the property that a_F is fixed by \mathcal{G}_F whenever F belongs to \mathcal{X}_{Σ} .

Then, by assumption, we may choose a large enough odd prime number p such that all of the following hold: p does not belong to \mathcal{S} , p does not ramify in k and Hypothesis (4.8) holds for the triangular system $\{(M_{F,E}, \rho_{F/E}, j_{F/E})\}_{E, F \in \mathcal{X}}$. In particular, $\{(\mathbb{Z}_p \otimes_{\mathbb{Z}} M_{F,E}, \rho_{F/E}, j_{F/E})\}_{E, F \in \mathcal{X}}$ is then a triangular system that satisfies Hypothesis (4.8) and is such that each diagonal term M_E is a $\mathbb{Z}_p[\mathcal{G}_E]$ -lattice.

In addition, Lemma (4.25) implies that the element a_F is fixed by \mathcal{G}_F whenever F belongs to $\mathcal{X}_{S_p(k)}$ and so a belongs to the kernel of the map $\text{res}_{S_p(k)}^{\Pi}$ that occurs in Theorem (4.10) for the

triangular system $\{(\mathbb{Z}_p \otimes_{\mathbb{Z}} M_{F,E}, \rho_{F/E}, j_{F/E})\}_{E,F \in \mathcal{X}}$. We may therefore apply the latter result to deduce that each element a_F is fixed by the group $\text{Gal}(F/F \cap k_{\Pi}(F))$.

Now, the assumption that p does not ramify in k implies that the extension $k(\mu_p)/k$ is totally ramified at all p -adic places of k and hence that $k_{\Pi}(F) = k$ whenever $S_p(k) \setminus \Pi(F)$ is non-empty. For any such F , the element a_F is therefore fixed by \mathcal{G}_F . On the other hand, if $S_p(k) \subseteq \Pi(F)$, then we have already seen above that a_F is fixed by \mathcal{G}_F .

We have therefore proved that the element a_F is fixed by \mathcal{G}_F for every F in \mathcal{X} , as required to prove Corollary (4.13). \square

4.3. The localisation exact sequence

In this section we consider the case that the subset Π of $S_{\text{fin}}(k)$ that is fixed above is empty. We assume to be given both a projective system $\{(M_F, \varphi_{F/E})\}_{F \in \mathcal{X}}$ and a triangular system $\{(N_{F,E}, \rho_{F/E}, j_{F/E})\}_{E,F \in \mathcal{X}}$ of $\mathcal{R}[\mathcal{G}_F]$ -modules. With e_F the idempotent of $\mathbb{Q}[\mathcal{G}_F]$ defined in (3), we also assume to be given a family of injective $\mathcal{R}[\mathcal{G}_F]$ -module homomorphisms $\beta_F: e_F M_F \rightarrow N_F$ that gives rise to an (injective) homomorphism of $\mathcal{R}[\mathcal{G}_{\mathcal{K}}]$ -modules of the form

$$\beta = (\beta_F)_F: \varprojlim_{F \in \mathcal{X}} e_F M_F \rightarrow \varprojlim_{F \in \mathcal{X}} N_F, \quad (27)$$

where the left hand Euler limit is as defined in Example (4.4) (d).

Our aim is then to investigate the composite homomorphism

$$\beta_*: \varprojlim_{F \in \mathcal{X}} e_F M_F \rightarrow \varprojlim_{F \in \mathcal{X}} e_F M_F \xrightarrow{\beta} \varprojlim_{F \in \mathcal{X}} N_F, \quad (m_F)_F \mapsto (e_F m_F)_F \mapsto \beta((e_F m_F)_F) \quad (28)$$

under the assumption that $(M_F, \varphi_{F/E})_{F \in \mathcal{X}}$ satisfies the following technical hypothesis.

(4.27) Hypothesis. For each $E \subseteq F$, there exist injective maps $\iota_{F/E}: M_E \rightarrow M_F$ such that $(M_F, \iota_{F/E})_{F \in \mathcal{X}}$ is an inductive system of $\mathcal{R}[\mathcal{G}_{\mathcal{K}}]$ -modules for which both of the following conditions are satisfied: There exists a natural number s (independent of F/E) such that one has both

$$\iota_{F/E} \circ \varphi_{F/E} = N_{\text{Gal}(F/E)}^s \quad \text{and} \quad \varphi_{F/E} \circ \iota_{F/E} = [F : E]^s.$$

(4.28) Example. For any Dedekind domain \mathcal{R} , the discussion of Example (4.9) (b) shows that the canonical projective system $(\mathcal{R}[\mathcal{G}_E], \pi_{F/E})$ satisfies Hypothesis (4.27) with $s = 1$.

We recall the idempotents e_K defined in (10) and write Υ_K^{\diamond} for the set of all characters χ in $\widehat{\mathcal{G}_K}$ for which $\chi(e_K) \neq 0$. In other words, Υ_K^{\diamond} is the set of all characters χ such that for every place $v \in S_{\infty}(k)$ one has $\chi(v) = 1$ if and only if $v \in V_K$. We note, in particular, that the set Υ_K defined in §2.1 is contained in Υ_K^{\diamond} , and hence that the idempotent e_K is such that $e_K e_K = e_K$. We can now state the main result of this section.

(4.29) Theorem. *We assume to be given data of the following sort:*

- a finite set of prime numbers \mathcal{S} ,
- a subset V of $S_{\infty}(k)$,
- a subset \mathcal{X} of Ω that satisfies Hypothesis (4.5) with $\mathcal{V} = \{V\}$,
- a projective system of $\mathbb{Z}_{\mathcal{S}}[\mathcal{G}_F]$ -lattices $\{(M_F, \varphi_{F/E})\}_{F \in \mathcal{X}}$ that satisfies Hypothesis (4.27),
- a triangular system $\{(N_{F,E}, \rho_{F/E}, j_{F/E})\}_{E,F \in \mathcal{X}}$ such that each diagonal term is a $\mathbb{Z}_{\mathcal{S}}[\mathcal{G}_E]$ -lattice,
- a homomorphism β of $\mathcal{R}[\mathcal{G}_{\mathcal{K}}]$ -modules of the form (27).

Then, for any such collection of data, the following sequence is exact:

$$0 \longrightarrow \varprojlim_{F \in \mathcal{X}} \epsilon_F M_F \xrightarrow{\beta_*} \varprojlim_{F \in \mathcal{X}}^{\emptyset} N'_F \xrightarrow{\Delta} \prod_{p \notin \mathcal{S}} \left(\frac{\varprojlim_{F \in \mathcal{X}}^{\emptyset} (\mathbb{Z}_p \otimes_{\mathbb{Z}} N_F)}{\beta_*^{(p)}(\ker \theta_p)} \right).$$

Here, by abuse of notation, we write β_* for the homomorphism that is induced by the map in (28) (and the equalities $e_F \epsilon_F = e_F$) and we set $N'_F := N_F \cap (\mathbb{Q} \otimes_{\mathbb{Z}} \beta_{F,*}(\epsilon_F M_F))$. We also write Δ for the map induced by the diagonal map

$$\prod_{F \in \mathcal{X}} N_F \longrightarrow \prod_{F \in \mathcal{X}} \left(\prod_{p \notin \mathcal{S}} (\mathbb{Z}_p \otimes_{\mathbb{Z}} N_F) \right)$$

(where the internal direct product is over all rational primes p outside \mathcal{S}), θ_p for the canonical boundary map

$$\varprojlim_{F \in \mathcal{X}} (\mathbb{Z}_p \otimes_{\mathbb{Z}} \epsilon_F M_F) \rightarrow \varprojlim_{n \in \mathbb{N}} (\varprojlim_F^1 (\epsilon_F M_F)) [p^n],$$

and $\beta_*^{(p)}$ for the map

$$\varprojlim_{F \in \mathcal{X}} (\mathbb{Z}_p \otimes_{\mathbb{Z}} \epsilon_F M_F) \rightarrow \varprojlim_{F \in \mathcal{X}}^{\emptyset} (\mathbb{Z}_p \otimes_{\mathbb{Z}} N_F)$$

that is defined by \mathbb{Z}_p -linearly extending each map $\beta_{*,F}: \epsilon_F M_F \rightarrow N_F$ induced by β_F .

An application of this result will play a key role in the proof of Theorem A from the introduction. Its proof will be completed in the last paragraph of this section after we have first established several necessary auxiliary results.

4.3.1. Generic Euler limits

In this paragraph we investigate Euler limits that arise from systems of $\mathcal{Q}[\mathcal{G}_F]$ -modules.

(4.30) Proposition. *Let $\mathcal{X} \subseteq \Omega$ be a subset that satisfies Hypothesis (4.5) with $\mathcal{V} = \{V\}$ for some $V \subseteq S_{\infty}(k)$. Suppose to be given a projective system $\{(M_F, \varphi_{F/E})\}_{F \in \mathcal{X}}$ of $\mathcal{Q}[\mathcal{G}_F]$ -modules that satisfies Hypothesis (4.27), and take $\Pi = \emptyset$. Then the following claims are valid.*

- (a) Write Δ for the map $\varprojlim_{F \in \mathcal{X}} \epsilon_F M_F \rightarrow \prod_{F \in \mathcal{X}} e_1 \epsilon_F M_F$ sending $(m_F)_F$ to $(e_1 \epsilon_F m_F)_F$. Then the following sequence of $\mathcal{R}[[\mathcal{G}_{\mathcal{K}}]]$ -modules

$$0 \longrightarrow \varprojlim_{F \in \mathcal{X}} \epsilon_F M_F \xrightarrow{(a_F)_{F \mapsto (e_F a_F)_F}} \varprojlim_{F \in \mathcal{X}}^{\emptyset} e_F M_F \xrightarrow{(b_F)_{F \mapsto (e_1 b_F)_F}} \left(\prod_{F \in \mathcal{X}} e_1 \epsilon_F M_F \right) / \text{im}(\Delta),$$

is exact, where the Euler limit is as defined in Example (4.4) (d).

- (b) Let $\mathcal{R} \rightarrow \mathcal{R}'$ be an injective morphism of subrings of \mathbb{C} and set $M'_F := \mathcal{R}' \otimes_{\mathcal{R}} M_F$ for each $F \in \mathcal{X}$. Then the following sequence is exact

$$0 \longrightarrow \left(\prod_{F \in \mathcal{X}} \epsilon_F M_F \right) \cap \left(\varprojlim_{F \in \mathcal{X}} \epsilon_F M'_F \right) \xrightarrow{\subseteq} \varprojlim_{F \in \mathcal{X}} \epsilon_F M'_F \xrightarrow{(a_F)_{F \mapsto (e_F a_F)_F}} \varprojlim_{F \in \mathcal{X}}^{\emptyset} ((e_F M'_F) / (e_F M_F)),$$

where the Euler limit is as defined in Example (4.4) (d).

Proof. To prove part (a) we recall (from Example (4.4) (d)) that the second arrow is indeed a well-defined homomorphism of $\mathcal{R}[[\mathcal{G}_{\mathcal{K}}]]$ -modules. It is also clear that the image of the second map is contained in the kernel of the third map. To prove part (a) it is therefore enough to show that the canonical map

$$\varprojlim_{F \in \mathcal{X}} \epsilon_F (1 - e_1) M_F \rightarrow \varprojlim_{F \in \mathcal{X}}^{\emptyset} (1 - e_1) e_F M_F, \quad (a_F)_F \mapsto (e_F a_F)_F \quad (29)$$

is an isomorphism. To do this, we introduce the following notation: let H be the Hilbert class field of k and write $\widehat{\mathcal{G}}_H / \sim$ for the set of conjugacy classes of the natural action of $G_{\mathcal{Q}}$ on

$\widehat{\mathcal{G}}_H$. For each non-trivial χ in $\widehat{\mathcal{G}}_H / \sim$, we fix a place \mathfrak{p}_χ in $S_{\text{fin}}(k)$ such that $\chi(\mathfrak{p}_\chi) \neq 1$ and $[k(\mathfrak{p}_\chi) : H] > 1$. If now $F \in \mathcal{X}$ is a field and $\chi \in \widehat{\mathcal{G}}_F$ an unramified character, then we set $F_\chi := k(\mathfrak{p}_\chi)F^{\ker(\chi)}$. For any ramified character χ of \mathcal{G}_F we let $F_\chi := F^{\ker(\chi)}$. Then, in all cases, the field F_χ belongs to \mathcal{X} as a consequence of Hypothesis (4.5) (ii).

We shall now prove that the assignment

$$(a_F)_F \mapsto (\widetilde{a}_F)_F := \left(a_F + \sum_{\chi \in \Upsilon_F^\circ \setminus \{\mathbf{1}_F\}} [FF_\chi : F_\chi]^{-s} e_\chi(\varphi_{FF_\chi/F} \circ \iota_{FF_\chi/F_\chi})(a_{F_\chi}) \right)_F,$$

defines an inverse to the map (29). To do this, we first claim that

$$(1 - e_1)a_F = \sum_{\chi \in \Upsilon_F \setminus \{\mathbf{1}_F\}} [FF_\chi : F_\chi]^{-s} e_\chi(\varphi_{FF_\chi/F} \circ \iota_{FF_\chi/F_\chi})(a_{F_\chi}) \quad (30)$$

for every field $F \in \mathcal{X}$. Let $\chi \in \Upsilon_F$ be a non-trivial character and note that

$$e_\chi a_F = e_\chi \varphi_{FF_\chi/F}(a_{FF_\chi})$$

because the Euler factor $P_{FF_\chi/F, \emptyset}$ is either 1 or $1 - \text{Frob}_{\mathfrak{p}_\chi}^{-1}$. Similarly, since $\chi \in \Upsilon_F$, we also have $\chi(P_{FF_\chi/F_\chi, \emptyset}) \neq 0$ and this implies that

$$e_\chi a_{F_\chi} = e_\chi \varphi_{FF_\chi/F_\chi}(a_{FF_\chi}).$$

We may thus calculate

$$\begin{aligned} e_\chi a_F &= e_\chi \varphi_{FF_\chi/F}(a_{FF_\chi}) \\ &= [FF_\chi : F_\chi]^{-s} e_\chi \varphi_{FF_\chi/F}(\mathbf{N}_{\text{Gal}(FF_\chi/F_\chi)}^s a_{FF_\chi}) \\ &= [FF_\chi : F_\chi]^{-s} e_\chi(\varphi_{FF_\chi/F} \circ \iota_{FF_\chi/F_\chi} \circ \varphi_{FF_\chi/F_\chi})(a_{FF_\chi}) \\ &= [FF_\chi : F_\chi]^{-s} e_\chi(\varphi_{FF_\chi/F} \circ \iota_{FF_\chi/F_\chi})(a_{F_\chi}), \end{aligned} \quad (31)$$

where the third equality is a consequence of Hypothesis (4.27) (i). We have hence shown that \widetilde{a}_F is equal to the sum $\sum_{\chi \in \Upsilon_F^\circ \setminus \{\mathbf{1}_F\}} [FF_\chi : F_\chi]^{-s} e_\chi(\varphi_{FF_\chi/F} \circ \iota_{FF_\chi/F_\chi})(a_{F_\chi})$.

It is clear by construction that $e_F \widetilde{a}_F = a_F$ and also, given any element b of $\varprojlim_{F \in \mathcal{X}} (1 - e_1) \epsilon_F M_F$, that one has $((e_F b_F)_F)_F = b_F$. It therefore remains to show that $(\widetilde{a}_F)_F$ defines an element in $\varprojlim_{F \in \mathcal{X}} (1 - e_1) \epsilon_F M_F$, and to do this we suppose to be given a pair of fields $E, F \in \mathcal{X}$ such that $E \subseteq F$. One then has

$$\begin{aligned} \varphi_{F/E}(\widetilde{a}_F) &= \varphi_{F/E} \left(\sum_{\chi \in \Upsilon_F^\circ \setminus \{\mathbf{1}_F\}} [FF_\chi : F_\chi]^{-s} e_\chi(\varphi_{FF_\chi/F} \circ \iota_{FF_\chi/F_\chi})(a_{F_\chi}) \right) \\ &= \sum_{\chi \in \Upsilon_F^\circ \setminus \{\mathbf{1}_F\}} [FF_\chi : F_\chi]^{-s} \pi_{F/E}(e_\chi)(\varphi_{FF_\chi/E} \circ \iota_{FF_\chi/F_\chi})(a_{F_\chi}) \\ &= \sum_{\chi \in \Upsilon_F^\circ \setminus \{\mathbf{1}_F\}} [FF_\chi : F_\chi]^{-s} [FF_\chi : EF_\chi]^s \pi_{F/E}(e_\chi)(\varphi_{EF_\chi/E} \circ \iota_{EF_\chi/F_\chi})(a_{F_\chi}) \\ &= \sum_{\chi \in \Upsilon_E^\circ \setminus \{\mathbf{1}_E\}} [EE_\chi : E_\chi]^{-s} e_\chi(\varphi_{EE_\chi/E} \circ \iota_{EE_\chi/E_\chi})(a_{E_\chi}), \end{aligned}$$

where the third equality is a consequence of Hypothesis (4.27) (i) and the last line uses the fact that $\pi_{F/E}(\epsilon_F) = \epsilon_E$ since E and F are both contained in \mathcal{X} .

Turning to part (b), it is enough to prove that

$$\left(\prod_{F \in \mathcal{X}} (\epsilon_F M_F + (1 - e_F) \epsilon_F M'_F) \right) \cap \varprojlim_{F \in \mathcal{X}} \epsilon_F M'_F \subseteq \prod_{F \in \mathcal{X}} \epsilon_F M_F. \quad (32)$$

To verify this, we fix a field $F \in \mathcal{X}$ and introduce the following notation. For any field $E \in \Omega$ we let $\Xi(E) \subseteq \widehat{\mathcal{G}}_F$ be the (possibly empty) subset comprising all characters χ such that $F_\chi = E$. Note that by construction each field F_χ depends only on the class of χ in Υ_F° / \sim and hence that $\Xi(E)$ is stable under the action of G_Q . This implies that the associated idempotent

$e_{\Xi(E)} = \sum_{\chi \in \Xi(E)} e_\chi$ belongs to $\mathbb{Q}[\mathcal{G}_F]$.

To investigate the F -component m_F of an element $m = (m_E)_E$ of the left hand side of (32) we use the decomposition

$$\begin{aligned}
m_F &= 1 \cdot m_F = \left(\sum_{\chi \in \widehat{\mathcal{G}}_F} e_\chi \right) \cdot m_F = \sum_{\chi \in \widehat{\mathcal{G}}_F} e_\chi m_F \\
&= \sum_{\chi \in \Upsilon_F^\circ} [FF_\chi : F_\chi]^{-s} e_\chi (\varphi_{FF_\chi/F} \circ \iota_{FF_\chi/F_\chi})(m_{F_\chi}) \\
&= \sum_{E \in \mathcal{X}} \left(\sum_{\chi \in \Xi(E)} [FE : E]^{-s} e_\chi (\varphi_{FE/F} \circ \iota_{FE/E})(m_E) \right) \\
&= \sum_{E \in \mathcal{X}} [FE : E]^{-s} e_{\Xi(E)} (\varphi_{FE/F} \circ \iota_{FE/E})(m_E), \tag{33}
\end{aligned}$$

where the fourth equality follows from (31) and the fifth from the fact that each field F_χ belongs to \mathcal{X} .

We shall now use an induction on the number $n(F)$ of (non-archimedean) prime divisors of the conductor of F to show that $m_F \in \epsilon_F M_F$ for all F .

Let us first assume that F has prime-power conductor. From (33) we see that it is enough to show that $e_{\Xi(E)} m_E \in \epsilon_E M_E$ for all fields E which are of the form $F = E_\chi$ for some character $\chi \in \Upsilon_F^\circ$. Fix such a field E . Now, $\chi \in \Upsilon_E$ by the construction of $E = F_\chi$ (which in the case that χ is unramified involves the choice of a prime ideal with full decomposition group in the kernel field of χ) and so $e_\chi(1 - e_E)\epsilon_E = 0$. By assumption m_E belongs to $\epsilon_E M_E + (1 - e_E)\epsilon_E M'_E$ and so we obtain $e_{\Xi(E)} m_E \in \epsilon_E M_E$, as claimed.

Now assume to be given a natural number n and suppose that for every field E in \mathcal{X} such that $n(E) \leq n$ one has that $m_E \in \epsilon_E M_E$. Fix a field F in \mathcal{X} such that $n(F) = n + 1$. Let E be of the form F_χ for some $\chi \in \Upsilon_F^\circ$. If χ is ramified, then F_χ is the kernel field of χ and, in particular, a subfield of F . Clearly, we therefore have $n(E) \leq n(F)$. If χ is unramified, on the other hand, then F_χ is defined to be a prime-power conductor field and so $n(E) = 1 < n(F)$. In both cases therefore $n(E) \leq n(F)$.

If $n(E) < n(F)$, then, by the induction hypothesis, one has that $m_E \in \epsilon_E M_E$. On the other hand, if $n(E) = n(F)$, and $y_F \in \epsilon_F M_F$ and $i_F \in (1 - e_F)\epsilon_F M'_F$ are such that $m_F = y_F + i_F$, then, by reversing the calculation in (31), one has

$$\begin{aligned}
[FE : E]^{-s} e_{\Xi(E)} (\varphi_{FE/F} \circ \iota_{FE/E})(m_E) &= \sum_{\chi \in \Xi(E)} e_\chi \cdot m_F = \sum_{\chi \in \Xi(E)} e_\chi (y_F + i_F) \\
&= \sum_{\chi \in \Xi(E)} e_\chi y_F = e_{\Xi(E)} y_F,
\end{aligned}$$

where the third equality is valid since, under the present hypothesis, each χ in $\Xi(E)$ is not trivial on the decomposition group of any prime divisor of the conductor of F so that one has $e_\chi = e_\chi e_F$ and hence also $e_\chi(i_F) = 0$.

These observations imply that the element $e_{\Xi(E)} (\varphi_{FE/F} \circ \iota_{FE/E})(m_E)$ belongs to $\epsilon_E M_E$ for every subfield E of L that is of the form F_χ for some $\chi \in \Upsilon_F$ and hence, via the decomposition (33), that m_F belongs to $\epsilon_F M_F$, as required to complete the proof of part (b). \square

4.3.2. Profinite completions

In the setting of Theorem (4.29), we now assume to be given an element a of the Euler limit $\varprojlim_{F \in \mathcal{X}}^\circ N'_F$ and consider the modules

$$M := \varprojlim_{F \in \mathcal{X}} \epsilon_F M_F, \quad M(a) := \varprojlim_{F \in \mathcal{X}}^\circ (\beta_{*,F}(\epsilon_F M_F) + \mathbb{Z}_S[\mathcal{G}_F]a_F), \quad X = X(a) := M(a)/\beta_*(M).$$

Then, to prove Theorem (4.29), we must show that X vanishes if, for all primes $p \notin \mathcal{S}$ and all fields $F \in \mathcal{X}$, the element a_F belongs to a suitable submodule of $\mathbb{Z}_p \otimes_{\mathbb{Z}} N_F$.

In this paragraph we use completion functors to study the vanishing of X .

To be more precise, for any \mathbb{Z} -algebra H and H -module A , we shall use the modules

$$\widehat{A} := \varprojlim_{n \in \mathbb{N}} A/(nA) \quad \text{and} \quad \widehat{A}^p := \varprojlim_{n \in \mathbb{N}} A/(p^n A).$$

Observe that the assignment $A \mapsto \widehat{A}$ (resp. $A \mapsto \widehat{A}^p$) defines a functor from the category of H -modules to the category of \widehat{H} -modules (resp. \widehat{H}^p -modules). For the reader's convenience, we recall some useful properties of these functors in the following result.

(4.31) Lemma. *For an H -module A and prime p the following claims are valid.*

- (a) *If A is finitely generated as an abelian group, then the natural maps $A \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \rightarrow \widehat{A}$ and $\mathbb{Z}_p \otimes_{\mathbb{Z}} A \rightarrow \widehat{A}^p$ are bijective.*
- (b) *$(-)^p$ is an idempotent functor.*
- (c) *If A is \mathbb{Z} -torsion-free, then every short exact sequence $0 \rightarrow A_1 \rightarrow A_2 \rightarrow A \rightarrow 0$ of H -modules induces an exact sequence $0 \rightarrow \widehat{A}_1 \rightarrow \widehat{A}_2 \rightarrow \widehat{A} \rightarrow 0$ of \widehat{H} -modules and similarly for the functor $(-)^p$,*
- (d) *If A is \mathbb{Z} -torsion-free, then so too are the groups \widehat{A} and \widehat{A}^p .*

Proof. Claim (a) is well-known. In addition, claim (b) is both straightforward to prove directly and also follows immediately from the general result [Mat78, Th. 15] of Matlis (since \widehat{A}^p is equal to the completion of the \mathbb{Z} -module A at the ideal generated by p).

For both claims (c) and (d), it is enough to consider the functor $A \rightarrow \widehat{A}$. To prove claim (c) in this case we note first that, since A is torsion-free, for each natural number n the Snake Lemma applies to the following exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \xrightarrow{\psi} & A_2 & \xrightarrow{\phi} & A & \longrightarrow & 0 \\ & & \downarrow n & & \downarrow n & & \downarrow n & & \\ 0 & \longrightarrow & A_1 & \xrightarrow{\psi} & A_2 & \xrightarrow{\phi} & A & \longrightarrow & 0 \end{array}$$

to give an exact sequence $0 \rightarrow A_1/nA_1 \xrightarrow{\psi/n} A_2/nA_2 \xrightarrow{\phi/n} A/nA \rightarrow 0$. It is then enough to note that the latter sequences are compatible (with respect to the natural projection maps) as n varies and that, by the Mittag-Leffler criterion, exactness of the sequences is preserved when one passes to the inverse limit over n since, for each multiple m of n , the projection map $A_1/mA_1 \rightarrow A_1/nA_1$ is surjective.

Finally, to prove claim (d) we must show that if $x = (x_n)_n$ is an element of \widehat{A} with the property that $px = 0$ for some prime p , then $x = 0$. But, since A is torsion-free, for each n the element x_{np} is the image in $A/(npA)$ of an element \hat{x}_{np} of nA . Since x_n is equal to the image of \hat{x}_{np} in $A/(nA)$ one therefore has $x_n = 0$, as required. \square

We can now state the main result of this paragraph.

(4.32) Proposition. *The natural composite homomorphism of R -modules*

$$X \rightarrow \widehat{X} \rightarrow \prod_{p \notin \mathcal{S}} \widehat{X}^p,$$

where the product runs over all rational primes, is injective.

Before proving this result, we establish a preliminary result.

(4.33) Lemma. *If, for all primes $p \notin \mathcal{S}$, one has $a \in \beta_*(\ker \theta_p)$, then the module X is \mathbb{Z} -torsion-free.*

Proof. We fix a prime $p \notin \mathcal{S}$ and shall then demonstrate that X has no element of order p . By assumption, there exists an element $x^{(p)} = (x_E^{(p)})_{E \in \mathcal{X}}$ of $\ker \theta_p \subseteq \varprojlim_{E \in \mathcal{X}} (\mathbb{Z}_p \otimes_{\mathbb{Z}} \epsilon_E M_E)$ such that $\beta_*^{(p)}(x^{(p)})$ agrees with a inside $\varprojlim_{E \in \mathcal{X}}^{\emptyset} (\mathbb{Z}_p \otimes_{\mathbb{Z}} N_E)$. In particular, one has that $\beta_{*,E}(\epsilon_E M_E) + \mathbb{Z}_{\mathcal{S}}[\mathcal{G}_E]a_E$ is a submodule of $\mathbb{Z}_p \otimes_{\mathbb{Z}} \beta_{*,E}(\epsilon_E M_E) = \beta_{*,E}^{(p)}(\mathbb{Z}_p \otimes_{\mathbb{Z}} \epsilon_E M_E)$. One therefore has an injective map

$$X = M(a)/\beta_*(M) \hookrightarrow \left(\varprojlim_{E \in \mathcal{X}} (\mathbb{Z}_p \otimes_{\mathbb{Z}} \beta_{*,E}(\epsilon_E M_E)) \right) / \beta_*(M)$$

and it is now to prove that the quotient on the right hand side is p -torsion-free. To do this, we write $M^{(p)}$ for the limit $\varprojlim_{E \in \mathcal{X}} (\mathbb{Z}_p \otimes_{\mathbb{Z}} \epsilon_E M_E)$ and first claim that one has $\beta_*^{(p)}(M^{(p)}) = \varprojlim_{E \in \mathcal{X}} (\mathbb{Z}_p \otimes_{\mathbb{Z}} \beta_{*,E}(\epsilon_E M_E))$. Indeed, since the kernel of $\beta_{*,E}^{(p)}$ is a finitely generated \mathbb{Z}_p -module and hence a compact Hausdorff space, one has that $\varprojlim_{E \in \mathcal{X}}^1 (\ker \beta_{*,E}^{(p)})$ vanishes. Passing to the limit (over $E \in \mathcal{X}$) of the exact sequences

$$0 \longrightarrow \ker \beta_{*,E}^{(p)} \longrightarrow \mathbb{Z}_p \otimes_{\mathbb{Z}} \epsilon_E M_E \xrightarrow{\beta_{*,E}^{(p)}} \beta_{*,E}^{(p)}(\mathbb{Z}_p \otimes_{\mathbb{Z}} \epsilon_E M_E) \longrightarrow 0$$

then gives an exact sequence

$$0 \longrightarrow \varprojlim_{E \in \mathcal{X}} (\ker \beta_{*,E}^{(p)}) \longrightarrow M^{(p)} \xrightarrow{\beta_*^{(p)}} \varprojlim_{E \in \mathcal{X}} (\mathbb{Z}_p \otimes_{\mathbb{Z}} \beta_{*,E}(\epsilon_E M_E)) \longrightarrow 0 \quad (34)$$

and hence the claimed identification. To prove that the quotient of $\beta_*^{(p)}(M^{(p)})$ by $\beta_*(M)$ is \mathbb{Z} -torsion-free, it is enough to prove that the outer terms in the exact sequence

$$0 \rightarrow \left(\varprojlim_{F \in \mathcal{X}} \beta_{*,F}(\epsilon_F M_F) \right) / \beta_*(M) \rightarrow \left(\beta_*^{(p)}(M^{(p)}) \right) / \beta_*(M) \rightarrow \left(\beta_*^{(p)}(M^{(p)}) \right) / \left(\varprojlim_{F \in \mathcal{X}} \beta_{*,F}(\epsilon_F M_F) \right) \rightarrow 0 \quad (35)$$

are each p -torsion-free. As for the module on the left, we first note that passing to the limit (over F in \mathcal{X}) of the exact sequences

$$0 \longrightarrow \ker \beta_{*,F} \longrightarrow \epsilon_F M_F \xrightarrow{\beta_{*,F}} \beta_{*,F}(\epsilon_F M_F) \longrightarrow 0$$

combines with the injectivity of β_* to imply the exactness of the sequence

$$0 \longrightarrow M \xrightarrow{\beta_*} \varprojlim_{F \in \mathcal{X}} \beta_{*,F}(\epsilon_F M_F) \longrightarrow \varprojlim_{F \in \mathcal{X}}^1 (\ker \beta_{*,F}).$$

It is therefore enough to prove that $\varprojlim_{F \in \mathcal{X}}^1 (\ker \beta_{*,F})$ is p -torsion-free. To this end, we recall that each $\beta_F: \epsilon_F M_F \rightarrow N_F$ is assumed to be injective, which implies that also the induced map $\beta_F^{(p)}: (\mathbb{Z}_p \otimes_{\mathbb{Z}} \epsilon_F M_F) \rightarrow (\mathbb{Z}_p \otimes_{\mathbb{Z}} N_F)$ is injective. The injectivity of the map $\beta^{(p)}: \varprojlim_{F \in \mathcal{X}} (\mathbb{Z}_p \otimes_{\mathbb{Z}} \epsilon_F M_F) \rightarrow \varprojlim_{F \in \mathcal{X}}^{\emptyset} (\mathbb{Z}_p \otimes_{\mathbb{Z}} N_F)$ then combines with Proposition (4.30) (a) to imply that $\beta_*^{(p)}: \varprojlim_{F \in \mathcal{X}} (\mathbb{Z}_p \otimes_{\mathbb{Z}} \epsilon_F M_F) \rightarrow \varprojlim_{F \in \mathcal{X}}^{\emptyset} (\mathbb{Z}_p \otimes_{\mathbb{Z}} N_F)$ is injective. This proves that $\varprojlim_{F \in \mathcal{X}} (\ker \beta_{*,F}^{(p)})$, being the kernel of $\beta_*^{(p)}$ by the exact sequence (34), vanishes. Upon noting that $\ker \beta_{*,F}^{(p)}$ identifies with $\mathbb{Z}_p \otimes_{\mathbb{Z}} \ker \beta_{*,F}$ (as \mathbb{Z}_p is a flat \mathbb{Z} -module) and that, as already observed before, $\varprojlim_{F \in \mathcal{X}}^1 (\ker \beta_{*,F}^{(p)})$ vanishes, passing to the limit (over F in \mathcal{X}) of the exact sequences

$$0 \longrightarrow \ker \beta_{*,F} \longrightarrow \mathbb{Z}_p \otimes_{\mathbb{Z}} \ker \beta_{*,F} \longrightarrow (\mathbb{Z}_p/\mathbb{Z}) \otimes_{\mathbb{Z}} \ker \beta_{*,F} \longrightarrow 0$$

therefore gives an isomorphism

$$\varprojlim_{F \in \mathcal{X}} \left((\mathbb{Z}_p/\mathbb{Z}) \otimes_{\mathbb{Z}} \ker \beta_{*,F} \right) \xrightarrow{\cong} \varprojlim_{F \in \mathcal{X}}^1 (\ker \beta_{*,F}).$$

Since $(\mathbb{Z}_p/\mathbb{Z}) \otimes_{\mathbb{Z}} \ker \beta_{*,F}$ is p -torsion-free for every F , this shows that $\varprojlim_{F \in \mathcal{X}}^1 \ker \beta_{*,F}$ is p -torsion-free, as required to conclude that the module on the left in (35) is p -torsion-free. Turning now to the module on the right in (35), we pass to the limit (over F in \mathcal{X}) of the exact sequences

$$0 \longrightarrow \beta_{*,F}(\epsilon_F M_F) \longrightarrow \mathbb{Z}_p \otimes_{\mathbb{Z}} \beta_{*,F}(\epsilon_F M_F) \longrightarrow (\mathbb{Z}_p/\mathbb{Z}) \otimes_{\mathbb{Z}} \beta_{*,F}(\epsilon_F M_F) \longrightarrow 0$$

to deduce that the quotient $(\beta_*^{(p)}(M^{(p)}))/(\varprojlim_{F \in \mathcal{X}} \beta_{*,F}(\epsilon_F M_F))$ identifies with a submodule of the projective limit $\varprojlim_F ((\mathbb{Z}_p/\mathbb{Z}) \otimes_{\mathbb{Z}} \beta_{*,F}(\epsilon_F M_F))$. Since each group $(\mathbb{Z}_p/\mathbb{Z}) \otimes_{\mathbb{Z}} \beta_{*,F}(\epsilon_F M_F)$ is p -torsion-free, said quotient is therefore p -torsion-free as well, as required.

This concludes the proof of the lemma. \square

We now prove Proposition (4.32). At the outset we note that the natural map $\widehat{X} \rightarrow \prod_{p \notin \mathcal{S}} \widehat{X}^p$ is injective by virtue of the Chinese Remainder Theorem, and so it suffices to show that the same is true of the homomorphism $i: X \rightarrow \widehat{X}$.

Given that X is \mathbb{Z} -torsion-free (by Lemma (4.33)), we may appeal to Lemma (4.31) to obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{\beta_*} & M(a) & \longrightarrow & X \longrightarrow 0 \\ & & \downarrow i_2 & & \downarrow i_1 & & \downarrow i \\ 0 & \longrightarrow & \widehat{M} & \xrightarrow{\widehat{\beta}_*} & M(a)^\wedge & \longrightarrow & \widehat{X} \longrightarrow 0. \end{array}$$

To proceed, we first note that the map i_1 , and therefore also i_2 , is injective. Indeed, the kernel of i_1 is equal to the intersection $\bigcap_{n \in \mathbb{N}} (n \cdot M(a))$ and so for any element $x \in \ker(i_1)$ one has that, for each field $E \in \Omega^V$, the value x_E is divisible by every natural number. Since x_E is contained in N_E , which is a finitely generated $\mathbb{Z}_{\mathcal{S}}$ -module by assumption, it then follows that $x_E = 0$.

By using the maps i_1 and i_2 we may, and will, identify M and $M(a)$ with their images inside \widehat{M} and $M(a)^\wedge$, respectively. These identifications then combine with the Snake Lemma to induce an isomorphism $\ker(i) \cong (M(a) \cap \widehat{\beta}_*(\widehat{M}))/\beta_*(M)$, with the intersection taking place in $M(a)^\wedge$. We are therefore reduced to verifying the equality $\widehat{\beta}_*(\widehat{M}) \cap M(a) = \beta_*(M)$.

For this purpose let $m = (m_n)_n$ be an element of $\widehat{M} = \varprojlim_n M/(nM)$ such that $\widehat{\beta}_*(m) = (\beta_*(m_n))_n$ belongs to $M(a)$. If we set $M_E(a) = \mathbb{Z}_{\mathcal{S}}[\mathcal{G}_E]a_E + \beta_{E,*}(\epsilon_E M_E)$, then by assumption the image $\widehat{\beta}_*(m)_E = (\beta_E(e_E m_{n,E}))_n$ of $\widehat{\beta}_*(m)$ under the natural map $M(a)^\wedge \rightarrow M_E(a)^\wedge$ belongs to $M_E(a) \subseteq \mathbb{Q} \otimes_{\mathbb{Z}} \beta_{E,*}(\epsilon_E M_E)$. We may therefore find a natural number that annihilates the image of $\widehat{\beta}_*(m)_E$ inside the quotient

$$\beta_{E,*}(\epsilon_E M_E)^\wedge / \beta_{E,*}(\epsilon_E M_E) \cong \beta_{E,*}(\epsilon_E M_E) \otimes_{\mathbb{Z}} (\widehat{\mathbb{Z}}/\mathbb{Z}).$$

The latter module is however \mathbb{Z} -torsion-free and so we deduce that $\widehat{\beta}_*(m)_E$ is contained in $\beta_{E,*}(\epsilon_E M_E)$. By the injectivity of β , we have therefore proved that $e_E m_E$ is contained in $\epsilon_E M_E$, which is to say that $m_E \in \epsilon_E M_E + (1 - e_E)\epsilon_E \widehat{M}_E$. It now remains to prove that

$$\widehat{M} \cap \left(\prod_E (\epsilon_E M_E + (1 - e_E)\epsilon_E \widehat{M}_E) \right) = M.$$

To do this, it is enough to show that any element m of the above intersection belongs to M . For any such m we write $(m_E)_E$ for its image under the natural map $\widehat{M} \rightarrow \varprojlim_E (\epsilon_E M_E)^\wedge$. Then, by applying Proposition (4.30) (b) to the ring extension $\mathbb{Z} \rightarrow \widehat{\mathbb{Z}}$ and projective system $(\mathbb{Q} \otimes_{\mathbb{Z}} \epsilon_E M_E, \varphi_{F/E})$, we deduce that the element $(m_E)_E$ of the limit $\varprojlim_E (\mathbb{Q} \otimes_{\mathbb{Z}} \epsilon_E \widehat{M}_E)$ belongs to $\varprojlim_E (\mathbb{Q} \otimes_{\mathbb{Z}} \epsilon_E M_E)$. By Lemma (4.34) below, this then implies that m belongs to M , as required. \square

(4.34) Lemma. Let $\{(Q_E, \rho_{F/E})\}_{E \in \Omega^V}$ be a projective system of finitely generated $\mathbb{Z}_S[\mathcal{G}_E]$ -lattices and set $Q := \varprojlim_E Q_E$. Then the following sequence is exact:

$$0 \longrightarrow Q \longrightarrow \widehat{Q} \longrightarrow \prod_E \frac{\mathbb{Q} \otimes_{\mathbb{Z}} \widehat{Q}_E}{\mathbb{Q} \otimes_{\mathbb{Z}} Q_E}.$$

Here the rightmost arrow is induced by the natural map $\widehat{Q} \rightarrow \varprojlim_E \widehat{Q}_E$.

Proof. The map $Q \rightarrow \widehat{Q}$ is injective because each Q_E is a finitely generated abelian group. To prove exactness of the sequence we therefore fix an element $m = (m_n)_n \in \widehat{Q} = \varprojlim_n Q/(nQ)$ with the property that for each E the image m_E of m under the natural map $\widehat{Q} \rightarrow \widehat{Q}_E$ is contained in $\mathbb{Q} \otimes_{\mathbb{Z}} Q_E$ and then show that m is in fact contained in Q .

To do this we note that $\widehat{Q}_E \cap (\mathbb{Q} \otimes_{\mathbb{Z}} Q_E) = Q_E$ (because $\widehat{\mathbb{Z}} \cap \mathbb{Q} = \mathbb{Z}$) and so the assumption implies that, for every field E , there exists an element $x_E \in Q_E$ such that the sequence m_E agrees with x_E in \widehat{Q}_E . This is to say that, if m_E is given by the family $(m_{E,n})_n$, then we have $m_{E,n} - x_E \in nQ_E$ for all natural numbers n . Since Q_E is \mathbb{Z} -torsion-free, we can therefore write $m_{E,n} - x_E = nz_{E,n}$ for some unique element $z_{E,n} \in Q_E$.

We first claim that the elements $(x_E)_E$ define an element of Q . For each $n \in \mathbb{N}$ we have that

$$\begin{aligned} \rho_{F/E}(x_F) - x_E &= \rho_{F/E}(m_{F,n} - nz_{F,n}) - x_E \\ &= m_{E,n} - x_E - n\rho_{F/E}(z_{F,n}) \\ &= n(\rho_{F/E}(z_{F,n}) - z_{E,n}) \in nQ_E. \end{aligned}$$

However, Q_E , being a finitely generated \mathbb{Z}_S -module, has no non-zero divisible elements and so we must have that $\rho_{F/E}(x_F) = x_E$, as claimed. This shows that $x := (x_E)_E$ defines an element of Q , and uniqueness of the elements $z_{n,E}$ implies that the same is true for $(z_{n,E})_E$. We have therefore proved that $x - m_n \in nQ$ for all n , which is to say that $x = m$ in \widehat{Q} , as desired. This completes the proof of the claimed result. \square

4.3.3. The proof of Theorem (4.29)

Before turning to the proof of Theorem (4.29), we provide the following auxilliary result regarding the compatibility of Euler limits with p -adic completions.

(4.35) Lemma. Let \mathcal{X} be a subset of Ω and $\{(C_{E,F}, \rho_{F/E}, j_{F/E})\}_{F,E \in \mathcal{X}}$ a triangular system with the property that each map $j_{F/E}$ is injective and each diagonal term C_E is p -torsion-free. Then, for any finite subset Π of $S_{\text{fin}}(k)$, the natural map

$$\left(\varprojlim_{E \in \mathcal{X}}^{\Pi} C_E\right)^{\wedge, p} \rightarrow \varprojlim_{E \in \mathcal{X}}^{\Pi} \widehat{C}_E^p, \quad ((m_{E,n})_{E \in \mathcal{X}})_{n \in \mathbb{N}} \mapsto ((m_{E,n})_{n \in \mathbb{N}})_{E \in \mathcal{X}}$$

is both well-defined and injective.

Proof. Since the maps $j_{F/E}$ are injective, as shall for brevity omit explicit reference to them. We set $C := \varprojlim_{E \in \mathcal{X}}^{\Pi} C_E$ and, for every $E \in \mathcal{X}$, we take the limit (over $n \in \mathbb{N}$) of the natural maps $C/p^n \rightarrow C_E/p^n$ to obtain a map

$$C^{\wedge, p} \rightarrow \widehat{C}_E^p. \quad (36)$$

By taking the product of these maps over $E \in \mathcal{X}$ we then obtain the further homomorphism

$$C^{\wedge, p} \rightarrow \prod_E \widehat{C}_E^p,$$

which we claim has image inside $\varprojlim_{E \in \mathcal{X}}^{\Pi} \widehat{C}_E^p$. To show this we suppose to be given an element $m = ([m_n])_n$ of $C^{\wedge, p}$ and, for every $E \in \mathcal{X}$, write m_E for the image of m under the map (36). For every n and $F \in \mathcal{X}$ containing E , we have that

$$[\rho_{F/E}(m_F)] = [\rho_{F/E}(m_{n,F})] = [P_{F/E,\Pi} \cdot m_{n,E}] = [P_{F/E,\Pi} \cdot m_E] \quad \text{in } C_E/p^n$$

(here $m_{n,F}$ denotes the image of $m_n \in C$ in C_E). Since the above equality holds for all n , this shows that $\rho_{F/E}(m_F) = P_{F/E,\Pi} \cdot m_E$, as required.

To prove injectivity, let us now assume that the system $(m_E)_E$ is trivial in $\varprojlim_{E \in \mathcal{X}}^\Pi \widehat{C}_E^p$. That is to say that, for every $E \in \mathcal{X}$, one has that $m_E = 0$, and hence that for each n the element $m_{n,E}$ is divisible by p^n in C_E . We can therefore find $z_{n,E}$ such that $m_{n,E} = p^n z_{n,E}$. We now fix a natural number n and claim that the collection $z_n := (z_{n,E})_E$ is an element of C . Indeed, for every $F \in \mathcal{X}$ containing E , one has that

$$p^n \cdot \rho_{F/E}(z_{n,F}) = \rho_{F/E}(p^n z_{n,F}) = \rho_{F/E}(m_{n,F}) = P_{F/E,\Pi} \cdot m_{n,E} = p^n \cdot (P_{F/E,\Pi} \cdot z_{n,E}).$$

Since C_E is assumed to be p -torsion-free, we deduce that $\rho_{F/E}(z_{n,F}) = z_{n,E}$, as required.

We have therefore proved that one has $m_n = p^n z_n$ in C , and this shows that each m is trivial. \square

We are finally in a position to carry out the proof of Theorem (4.29).

Proof (of Theorem (4.29)): At the outset we note that the map β_* is injective as a consequence of Proposition (4.30) (a), and that the image of β_* is clearly contained in the kernel of Δ . In order to establish Theorem (4.29), we therefore need to prove that any element a of the kernel of Δ belongs to the image of β_* or, equivalently, that the class of such an element a in the quotient X vanishes. By Proposition (4.32) it is enough for this purpose to verify that the class of a in \widehat{X}^p vanishes for every $p \notin \mathcal{S}$. To do this, we first clarify the nature of the map θ_p that appears in the statement of Theorem (4.29).

Fix a prime number $p \notin \mathcal{S}$ and let n be a natural number. Then, by passing to the limit over $E \in \mathcal{X}$ of the tautological short exact sequences

$$0 \rightarrow \epsilon_E M_E \xrightarrow{\cdot p^n} \epsilon_E M_E \rightarrow (\epsilon_E M_E)/p^n \rightarrow 0$$

one obtains a canonical short exact sequence

$$0 \rightarrow \epsilon_E M/p^n \rightarrow \varprojlim_{E \in \mathcal{X}} (\epsilon_E M_E/p^n) \xrightarrow{\theta_{p,n}} (\varprojlim_{E \in \mathcal{X}}^1 (\epsilon_E M_E))[p^n] \rightarrow 0. \quad (37)$$

We define the map θ_p in Theorem (4.29) (a) to be the composite homomorphism

$$\begin{aligned} \varprojlim_{E \in \mathcal{X}} (\mathbb{Z}_p \otimes_{\mathbb{Z}} \epsilon_E M_E) &\cong \varprojlim_{E \in \mathcal{X}} (\varprojlim_n ((\epsilon_E M_E)/p^n)) \\ &\cong \varprojlim_n (\varprojlim_{E \in \mathcal{X}} ((\epsilon_E M_E)/p^n)) \\ &\longrightarrow \varprojlim_n (\varprojlim_{E \in \mathcal{X}}^1 (\epsilon_E M_E))[p^n] \end{aligned}$$

in which the two isomorphisms are the canonical identifications and the unlabelled arrow is the limit (over n) of the maps $\theta_{p,n}$ in (37).

Having defined θ_p , we are now ready to complete the proof of Theorem (4.29). As a is assumed to belong to the kernel of Δ , there exists an element $u^{(p)} = (u_E^{(p)})_{E \in \mathcal{X}}$ of $\ker \theta_p$ such that a is equal to $\beta_*^{(p)}(u^{(p)})$ in $\varprojlim_{F \in \mathcal{X}}^\varnothing (\mathbb{Z}_p \otimes_{\mathbb{Z}} N_F)$. In particular, one has that a_E belongs to the image of $\beta_{*,E}^{(p)}$ for all $E \in \mathcal{X}$, and hence that $\beta_{*,E}^{(p)}(\mathbb{Z}_p \otimes_{\mathbb{Z}} \epsilon_E M_E) + \mathbb{Z}_p[\mathcal{G}_E]a_E$ coincides with $\beta_{*,E}^{(p)}(\mathbb{Z}_p \otimes_{\mathbb{Z}} \epsilon_E M_E)$. Lemma (4.35) therefore gives an injective map

$$i_2: M(a)^{\wedge,p} \hookrightarrow \varprojlim_{E \in \mathcal{X}}^\varnothing (\beta_{*,E}^{(p)}(\mathbb{Z}_p \otimes_{\mathbb{Z}} \epsilon_E M_E)).$$

We then obtain a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widehat{M}^p & \xrightarrow{\widehat{\beta}_*} & M(a)^{\wedge,p} & \longrightarrow & \widehat{X}^p \longrightarrow 0 \\ & & \downarrow i_1 & & \downarrow i_2 & & \\ 0 & \longrightarrow & \varprojlim_{E \in \mathcal{X}} (\mathbb{Z}_p \otimes_{\mathbb{Z}} \epsilon_E M_E) & \xrightarrow{\beta_*^{(p)}} & \varprojlim_{E \in \mathcal{X}}^\varnothing (\beta_{*,E}^{(p)}(\mathbb{Z}_p \otimes_{\mathbb{Z}} \epsilon_E M_E)), & & \end{array} \quad (38)$$

where the top line is exact as a consequence of Lemma (4.31) (c) (this uses that X is \mathbb{Z} -torsion-free by Lemma (4.33)). To prove that the class of a in \widehat{X}^p vanishes it is now enough, by the exactness of the top line in (38), that a coincides, in $M(a)^{\wedge, p}$, with an element in the image of $\widehat{\beta}_*^p$.

The explicit definition of θ_p and the exactness of the sequence (37) combine to imply the existence of a pre-image $x^{(p)} = ((x_E^{(p)})_{E \in \mathcal{X}})_{n \in \mathbb{N}}$ of $u^{(p)}$ under the natural map $i_1: \widehat{M}^p \rightarrow \varprojlim_{E \in \mathcal{X}_{S_p}} (\mathbb{Z}_p \otimes_{\mathbb{Z}} \epsilon_E M_E)$ that sends $x^{(p)}$ to $((x_E^{(p)})_{n \in \mathbb{N}})_{E \in \mathcal{X}}$. By the commutativity of the square in (38) we then have that

$$(i_2 \circ \widehat{\beta}_*^p)(x^{(p)}) = (\beta_*^{(p)} \circ i_1)(x^{(p)}) = \beta_*^{(p)}(u^{(p)}) = i_2(a).$$

The injectivity of i_2 then implies that a is equal to $\widehat{\beta}_*^p(x^{(p)})$, as required.

This concludes the proof of Theorem (4.29). \square

4.3.4. p -adic Euler limits

In this final paragraph we provide a useful result on p -adic Euler limits. To state this result we fix a prime p and an isomorphism $\mathbb{C} \cong \mathbb{C}_p$, which we use to regard \mathbb{Q}_p as a subfield of \mathbb{C} . We write S_p for the set of p -adic places of k .

(4.36) Proposition. *Let \mathcal{X} be a subset of Ω that satisfies Hypothesis (4.5) with $\mathcal{V} = \{V\}$ for some $V \subseteq S_\infty(k)$, and let $\{(M_F, \varphi_{F/E})\}_{F \in \mathcal{X}_{S_p}}$ be a projective system of $\mathbb{Z}_p[\mathcal{G}_F]$ -lattices that satisfies Hypothesis (4.27). If, for each element $m \in M_E \setminus \{0\}$, there exists a natural number d (that depends only on m and p) such that the element $\iota_{F/E}(m)$ cannot be divisible in M_F by any power p^t for $t > d$, then the following map is bijective:*

$$\alpha: \varprojlim_{F \in \mathcal{X}_{S_p}} \epsilon_F M_F \rightarrow \varprojlim_{F \in \mathcal{X}_{S_p}}^\varnothing \epsilon_F M_F, \quad (m_F)_F \mapsto (e_F m_F)_F.$$

Proof. Assume to be given a \mathbb{Z}_p -extension k_∞ of k in which no finite place splits completely (for example, one may take k_∞ to be the cyclotomic \mathbb{Z}_p -extension of k). Since all infinite places split completely in k_∞ , Hypothesis (4.5) (i) then implies that, for any field $K \in \mathcal{X}$, every finite extension of k contained in the composite $K_\infty = K k_\infty$ is also contained in \mathcal{X} . We write K_n for the n -th layer of K_∞/K and let $\Lambda_K = \mathbb{Z}_p[[\text{Gal}(K_\infty/k)]]$ be the relevant (equivariant) Iwasawa algebra.

Fix a field $E \in \mathcal{X}_{S_p}$ and let $(a_F)_F$ be an element of $\varprojlim_{F \in \mathcal{X}_{S_p}}^\varnothing \epsilon_F M_F$. Since $S_p \subseteq S(E)$ and E_∞/E is unramified outside p , we have $S(E) = S(E_n)$ for all $n \geq 0$. The defining property of the Euler limit therefore simplifies to $a_{E_n} = \varphi_{E_n/E_n}(a_{E_n})$ for all $m \geq n$. It follows that the family $(a_{E_n})_n$ defines an element of $\varprojlim_n \epsilon_{E_n} M_{E_n}$. Now, each term in the exact sequence

$$0 \longrightarrow (\epsilon_{E_n} M_{E_n})[e_{E_n}] \longrightarrow \epsilon_{E_n} M_{E_n} \xrightarrow{\cdot e_{E_n}} e_{E_n} M_{E_n} \longrightarrow 0$$

is a finitely generated \mathbb{Z}_p -module and hence endowed with the structure of a compact Hausdorff topological group. Consequently, we obtain an exact sequence

$$0 \longrightarrow \varprojlim_n ((\epsilon_{E_n} M_{E_n})[e_{E_n}]) \longrightarrow \varprojlim_n (\epsilon_{E_n} M_{E_n}) \longrightarrow \varprojlim_n (e_{E_n} M_{E_n}) \longrightarrow 0$$

and now claim that the term on the left hand side vanishes.

Since no finite prime splits completely in E_∞/E , we can find an integer $N \geq 0$ such that, for all $n \geq N$, each character $\chi \in \widehat{\mathcal{G}_{E_n}} \setminus \Upsilon_{E_n}$ factors through \mathcal{G}_{E_n} . For any $\sigma \in \text{Gal}(E_n/E_n)$ and $x \in (\epsilon_{E_n} M_{E_n})[e_{E_n}]$ we therefore have $\sigma \cdot x = \sigma(1 - e_{E_n})x = (1 - e_{E_n})x = x$.

This shows that $\varprojlim_n ((\epsilon_{E_n} M_{E_n})[e_{E_n}])$ is contained in the submodule of $\varprojlim_n (\epsilon_{E_n} M_{E_n})$ comprising all elements invariant under the action of $\text{Gal}(E_\infty/E_n)$. By Lemma (4.26), we have that $(\varprojlim_n \epsilon_{E_n} M_{E_n})^{\text{Gal}(E_\infty/E_n)}$ vanishes. It follows that $\varprojlim_n ((\epsilon_{E_n} M_{E_n})[e_{E_n}])$ vanishes as well,

as claimed.

The discussion above shows the family $(a_{E_n})_n$ converges to a unique element $b_{E_\infty} = (b_{E_n})_n$ in

$$\varprojlim_n (\epsilon_{E_n} M_{E_n}) \cong \varprojlim_n (e_{E_n} M_{E_n}) \quad (39)$$

with the property that $e_{E_n} b_{E_n} = e_{E_n} a_{E_n}$ for all n . We next claim that the family $b = (b_E)_{E \in \mathcal{X}_{S_p}}$ defines an element of $\varprojlim_E (\epsilon_E M_E)$. To do this, we suppose to be given a field $F \in \mathcal{X}_{S_p}$ which contains E and first note that it is enough to prove that $\varphi_{F_\infty/E_\infty}(b_{F_\infty}) = b_{E_\infty}$, where $\varphi_{F_\infty/E_\infty}$ is the map $\varprojlim_n (\epsilon_{F_n} M_{F_n}) \rightarrow \varprojlim_n (\epsilon_{E_n} M_{E_n})$ induced by the maps φ_{F_n/E_n} . Now, for each $m \in \mathbb{N}$ we have

$$\begin{aligned} \pi_{F_m/E_m}(e_{F_m}) \cdot P_{F_m/E_m, \emptyset} \cdot \varphi_{F_m/E_m}(b_{F_m}) &= P_{F_m/E_m, \emptyset} \cdot \varphi_{F_m/E_m}(e_{F_m} b_{F_m}) \\ &= P_{F_m/E_m, \emptyset} \cdot \varphi_{F_m/E_m}(e_{F_m} a_{F_m}) \\ &= \pi_{F_m/E_m}(e_{F_m}) \cdot P_{F_m/E_m, \emptyset} \cdot a_{F_m} \\ &= \pi_{F_m/E_m}(e_{F_m}) \cdot P_{F_m/E_m, \emptyset} \cdot b_{E_m}, \end{aligned}$$

where the last equality uses that $\pi_{F_m/E_m}(e_{F_m}) \cdot e_{E_m} = \pi_{F_m/E_m}(e_{F_m})$. Taking the limit as before, we see that

$$P_{F/E, \emptyset} \cdot \varphi_{F_\infty/E_\infty}(b_{F_\infty}) = P_{F/E, \emptyset} \cdot b_{E_\infty}. \quad (40)$$

Since no finite place splits completely in E_∞/E , for each place $v \in S(F) \setminus S(E)$ the associated Frobenius automorphism Frob_v generates an open subgroup of $\text{Gal}(E_\infty/k)$. Given this, Lemma (4.26) asserts that $1 - \text{Frob}_v$, and hence also $P_{F/E, \emptyset}$, acts injectively on $\varprojlim_n (\epsilon_{E_n} M_{E_n})$. We conclude that $\varphi_{F/E}(b_F) = b_E$, as claimed.

To establish the proposition, it now remains to prove that the assignment $a \mapsto b$ defines an inverse ψ to the natural map α . Suppose to be given a family $m = (m_F) \in \varprojlim_F (\epsilon_F M_F)$, then our construction yields $e_{E_n} m_{E_n} = e_{E_n} (\psi \circ \alpha)(m)_{E_n}$ for all n . The isomorphism (39) hence shows that $m_{E_n} = (\psi \circ \alpha)(a)_{E_n}$ for all n . In particular, $m_E = (\psi \circ \alpha)(m)_E$ and so $m = (\psi \circ \alpha)(m)$. Conversely, $\alpha \circ \psi = \text{id}$ holds by construction. \square

5. Euler systems and Euler limits

In this section we derive concrete arithmetic consequences of the results on Euler limits that were established above.

5.1. The Uniformisation Theorem

If \mathbf{r} and \mathbf{r}' are rank functions for k (as in Definition (2.2)), then we say that ‘ \mathbf{r} is greater than \mathbf{r}' ’, respectively ‘ \mathbf{r} is at most \mathbf{r}' ’ (and write ‘ $\mathbf{r} > \mathbf{r}'$ ’, respectively ‘ $\mathbf{r} \leq \mathbf{r}'$ ’) if one has $\mathbf{r}(E) > \mathbf{r}'(E)$, respectively $\mathbf{r}(E) \leq \mathbf{r}'(E)$, for all $E \in \Omega(k)$.

Before stating the next result, we also recall that if \mathbf{r} is the maximal rank function \mathbf{r}_{\max} for k , then the notation $\text{ES}_k^{\mathbf{r}}(\mathcal{Q})$ is abbreviated to $\text{ES}_k(\mathcal{Q})$.

(5.1) Theorem. *Fix a subfield \mathcal{Q} of \mathbb{C} such that $\text{ES}_k(\mathcal{Q})$ contains the Rubin–Stark system ε_k . Then the following claims are valid.*

- (a) *For any rank \mathbf{r} with $\mathbf{r} > \mathbf{r}_{\max}$, the module $\text{ES}_k^{\mathbf{r}}(\mathcal{Q})$ vanishes.*
- (b) *For any rank \mathbf{r} with $\mathbf{r} \leq \mathbf{r}_{\max}$, there exists an isomorphism of $\mathcal{Q}[[\mathcal{G}_K]]$ -modules*

$$\text{PR}_k^{\mathbf{r}_{\max} - \mathbf{r}}(\mathcal{Q}) \xrightarrow{\simeq} \text{ES}_k^{\mathbf{r}}(\mathcal{Q}), \quad (f_E)_E \mapsto (f_E(\varepsilon_{E/k}))_E,$$

where we use the module of Perrin–Riou functionals defined in Example (4.4) (b).

- (c) *In maximal rank, one has $\text{ES}_k(\mathcal{Q}) = \mathcal{Q}[[\mathcal{G}_K]] \cdot \varepsilon_k + \text{ES}_k(\mathcal{Q})^{\mathcal{G}_K}$.*
- (d) *The $\mathcal{Q}[[\mathcal{G}_K]]$ -module generated by ε_k is free of rank one and equal to $\text{ES}_k(\mathcal{Q})^{\text{sym}}$.*

Proof. Claim (a) follows directly from Lemma (2.6) (b). To prove claim (b), we first note that the Rubin–Stark system ε_k is an Euler system by Lemma (2.22), and that the image of the explicit map in claim (b) is contained in $\text{ES}_k^r(\mathcal{Q})$ (cf. Example (4.4) (b)). It therefore suffices to show that every system $c = (c_E)_E \in \text{ES}_k^r(\mathcal{Q})$ arises uniquely in this way.

Fix a field E in $\Omega(k)$. Then the $\mathbb{R}[\mathcal{G}_E]$ -module isomorphism $\mathbb{R}\mathcal{O}_{E,S(E)}^\times \cong \mathbb{R}X_{E,S^*(E)}$ induced by the Dirichlet regulator implies that there exists an (in general, non-canonical) isomorphism of $\mathcal{Q}[\mathcal{G}_E]$ -modules $\mathcal{Q}\mathcal{O}_{E,S(E)}^\times \cong \mathcal{Q}X_{E,S^*(E)}$, and hence that the $e_E\mathcal{Q}[\mathcal{G}_E]$ -module $e_E\mathcal{Q}\mathcal{O}_{E,S(E)}^\times \cong e_E\mathcal{Q}X_{E,S^*(E)} = e_E\mathcal{Q}Y_{E,V_E}$ is free of rank r_E . It follows that the $e_E\mathcal{Q}[\mathcal{G}_E]$ -module

$$e_E\mathcal{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_E]}^{r_E} \mathcal{O}_{E,S(E)}^\times = \bigwedge_{e_E\mathcal{Q}[\mathcal{G}_E]}^{r_E} e_E\mathcal{Q}\mathcal{O}_{E,S(E)}^\times$$

is free of rank one. We now first claim that the Rubin–Stark element $\varepsilon_{E/k}$ is a basis of this free module. To do this, we write $\varepsilon_{E/k} = q \cdot a$ for some $q \in e_E\mathcal{Q}[\mathcal{G}_E]$ and $e_E\mathcal{Q}[\mathcal{G}_E]$ -basis a . We then have $e_\chi \cdot q \neq 0$ for all $\chi \in \Upsilon_E$ by definition of Rubin–Stark elements. Since Υ_E is exactly the set of characters on which e_E is supported, this shows that q is a unit of the ring $e_E\mathcal{Q}[\mathcal{G}_E]$ whence the claim.

Next we note that, by Lemma (2.6), the system c is such that $c_E \in e_E\mathcal{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_E]}^{r(E)} \mathcal{O}_{E,S(E)}^\times$, and hence that the assignment $\varepsilon_{E/k} \mapsto c_E$ extends linearly to give a (well-defined) map of $\mathcal{Q}[\mathcal{G}_E]$ -modules

$$e_E\mathcal{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_E]}^{r_E} \mathcal{O}_{E,S(E)}^\times \rightarrow e_E\mathcal{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_E]}^{r(E)} \mathcal{O}_{E,S(E)}^\times.$$

By Lemma (5.2) below, this map can be regarded as an element f_E of $e_E\mathcal{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_E]}^{r_E-r(E)} (\mathcal{O}_{E,S(E)}^\times)^*$ and, to complete the proof of claim (b), it suffices to show that these elements combine to define an element

$$f = (f_E)_E \in \varprojlim_{E \in \Omega(k)}^\emptyset e_E\mathcal{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_E]}^{r_E-r(E)} (\mathcal{O}_{E,S(E)}^\times)^*,$$

where the Euler limit is taken with respect to the transition maps $\Phi_{E/E'}$ defined in Example (4.2)(c). To see this we note that, for every extension F/E , one has

$$\begin{aligned} P_{F/E,\emptyset} \cdot (\Phi_{F/E}(f_F))(\varepsilon_{E/k}) &= (\Phi_{F/E}(f_F))(N_{F/E}^{r(F)}(\varepsilon_{F/k})) = N_{F/E}^{r(F)}(f_F(\varepsilon_{F/k})) \\ &= N_{F/E}^{r(F)}(c_F) = P_{F/E,\emptyset} \cdot c_E \\ &= P_{F/E,\emptyset} \cdot f_E(\varepsilon_{E/k}). \end{aligned}$$

In particular, since $\varepsilon_{E/k}$ is an $e_E\mathcal{Q}[\mathcal{G}_E]$ -basis of $e_E\mathcal{Q} \bigwedge_{\mathbb{Z}[\mathcal{G}_E]}^{r_E} \mathcal{O}_{E,S(E)}^\times$, this equality implies the required equality of maps $P_{F/E,\emptyset} \cdot (\Phi_{F/E}(f_F)) = P_{F/E,\emptyset} \cdot f_E$.

To prove claim (c) it is enough to show that every Euler system c in $\text{ES}_k(\mathcal{Q})$ belongs to $\mathcal{Q}[[\mathcal{G}_K]] \cdot \varepsilon_k + \text{ES}_k(\mathcal{Q})^{\mathcal{G}_K}$. To do this we fix such a c and note that claim (b) implies the existence of a unique element $q = (q_E)_E$ of $\varprojlim_{E \in \Omega(k)}^\emptyset e_E\mathcal{Q}[\mathcal{G}_E]$ with the property $c = q \cdot \varepsilon_k$. We write q' for the element $(e_1 q_E)_E$ of $(\varprojlim_{E \in \Omega(k)}^\emptyset e_1\mathcal{Q}[\mathcal{G}_E])^{\mathcal{G}_K}$. Then Proposition (4.30) (a) implies that there exists an element $l = (l_E)_E$ of $\mathcal{Q}[[\mathcal{G}_K]]$ such that $e_E l_E = q_E - q'_E$. This in turn implies

$$c = (c_E)_E = (q_E \varepsilon_{E/k})_E = l \cdot \varepsilon_k + q' \cdot \varepsilon_k \in \mathcal{Q}[[\mathcal{G}_K]] \cdot \varepsilon_k + \text{ES}_k(\mathcal{Q})^{\mathcal{G}_K},$$

as required to prove claim (c).

To prove the first part of claim (d), we suppose to be given an element $q = (q_E)_E$ of $\mathbb{R}[[\text{Gal}(\mathcal{K}/k)]]$ that annihilates ε_k , so that one has

$$q_E \cdot \varepsilon_{E/k} = 0 \quad \text{for all } E \in \Omega(k). \quad (41)$$

Let $E \in \Omega(k)$ be such a field. For any character $\chi \in \widehat{\mathcal{G}_E}$ we then define a field E_χ as follows. If χ is ramified, then we take E_χ to be the field $E^{\ker(\chi)}$ cut out by χ . If χ is unramified, then we choose an auxiliary prime ideal \mathfrak{p} that has full decomposition group in $\text{Gal}(E^{\ker(\chi)}/k)$ and set $E_\chi := E^{\ker(\chi)} \cdot k(\mathfrak{p})$. Then, in order to show that $q_E = 0$ we may assume, without loss of

generality, that E contains E_χ for all $\chi \in \widehat{\mathcal{G}}_E$.

Given any $\chi \in \widehat{\mathcal{G}}_E$ we have $\chi(q_E) = \chi(q_{E_\chi})$ in \mathbb{C} , where, by a slight abuse of notation, we have written χ for the \mathbb{C} -linear extensions $\mathbb{C}[\mathcal{G}_E] \rightarrow \mathbb{C}$ and $\mathbb{C}[\mathcal{G}_{E_\chi}] \rightarrow \mathbb{C}$ of χ . Now, $r_{S(E_\chi)}(\chi) = r_{E_\chi}$ and so $e_\chi \varepsilon_{E_\chi/k} \neq 0$ by the definition of Rubin–Stark elements. The equality (41) for the field E_χ therefore implies that $\chi(q_{E_\chi}) = 0$. This shows that $\chi(q_E) = 0$ for all $\chi \in \widehat{\mathcal{G}}_E$, as is required to conclude that $q_E = 0$.

Since the system ε_k is symmetric (by Lemma (2.22)(a)), it therefore remains to prove that every system c in $\text{ES}_k(\mathcal{Q})^{\text{sym}}$ can be written as $c = q \cdot \varepsilon_k$ for some $q \in \mathcal{Q}[[\mathcal{G}_K]]$. By (a) we have that $c = q \cdot \varepsilon_k$ for some $q = (q_E)_E \in \varprojlim_{E \in \Omega(k)}^{\varnothing} e_E \mathcal{Q}[\mathcal{G}_E]$ and by Proposition (4.30) (a) it is enough to prove that the family $(e_1 e_E q_E)_E$ belongs to the image of the map

$$\varprojlim_E \mathcal{Q}[\mathcal{G}_E] \rightarrow \prod_E \mathcal{Q}[\mathcal{G}_E], \quad (a_E)_E \mapsto (e_1 e_E a_E)_E. \quad (42)$$

To verify the latter condition, it is enough to consider fields E in $\Omega(k)$ with the property that $V_E = S_\infty(k)$ and $|S(E)| = r_E + 1$ (since, otherwise, one has $e_1 e_E = 0$). We therefore fix such a field E and write \mathfrak{p} for the unique place in $S(E) \cap S_{\text{fin}}(k)$. Then, since both c and ε_k are symmetric, one has

$$\begin{aligned} \pi_{E/k}(q_E) \cdot \varepsilon_{k,k} &= \pi_{E/k}(q_E) \cdot (\text{Ord}_{\mathfrak{p}} \circ N_{E/k}^{r_E})(\varepsilon_{E/k}) \\ &= (\text{Ord}_{\mathfrak{p}} \circ N_{E/k}^{r_E})(e_1 q_E \varepsilon_{E/k}) \\ &= (\text{Ord}_{\mathfrak{p}} \circ N_{E/k}^{r_E})(e_1 c_E) \\ &= c_k. \end{aligned}$$

It follows that, if F is any other such field, then $\pi_{E/k}(q_E) \cdot \varepsilon_{k,k} = \pi_{F/k}(q_F) \cdot \varepsilon_{k,k}$, and hence $\pi_{E/k}(q_E) = \pi_{F/k}(q_F)$ since the explicit characterisation of $\varepsilon_{k,k}$ in Lemma (2.22) (a) shows that it does not vanish. In particular, if we set $q_k := \pi_{E/k}(q_E)$ (which does not depend on the choice of E by the above discussion), then the element $(e_1 q_k)_E$ of $\varprojlim_E \mathcal{Q}[\mathcal{G}_E]$ is a preimage of the element $(e_1 e_E q_E)_E$ under the map (42). This therefore concludes the proof of claim (d). \square

(5.2) Lemma. *Let R be a ring and F a finitely generated free R -module of rank d . For any integer $0 \leq s \leq d$ we have an isomorphism*

$$\begin{aligned} \bigwedge_R^s F^* &\rightarrow \text{Hom}_R \left(\bigwedge_R^d F, \bigwedge_R^{d-s} F \right), \\ f_1 \wedge \cdots \wedge f_s &\mapsto \left\{ m_1 \wedge \cdots \wedge m_d \mapsto \sum_{\sigma} \text{sgn}(\sigma) \det(f_i(m_{\sigma(j)}))_{1 \leq i, j \leq s} m_{\sigma(s+1)} \wedge \cdots \wedge m_{\sigma(d)} \right\}, \end{aligned}$$

where the sum runs over all permutations σ of the set $\{1, 2, \dots, d\}$ with the property that both $\sigma(1) < \cdots < \sigma(s)$ and $\sigma(s+1) < \cdots < \sigma(d)$.

Proof. This is a straightforward exercise that we leave to the reader. \square

(5.3) Remark. In view of Theorem (5.1) (b) it is perhaps tempting to strengthen Conjecture (2.24) by predicting that any Euler system of rank at most r_{max} that satisfies an appropriate analogue of the requirement to be a symmetric congruence system should arise from the Rubin–Stark system via rank reduction (as per Example (4.4) (b)). We do not discuss this possibility any further here, except to mention that conjectures connecting particular examples of Euler systems to a higher-rank Euler system have previously appeared in the literature (see, for example, [B+19, Conj. 3.5.1]) and have also been verified in special cases (see [BL19]).

5.2. Consequences for integral Euler systems

In this section we show that the results of Theorems (4.10), (4.29) and (5.1) combine to reduce the proof of Conjecture (2.24) to consideration of the individual components of Euler systems.

This result provides us with an effective means of providing evidence in support of Conjecture (2.24) and is proved as Theorem (5.7) in §5.2.2. However, before discussing it we must first prove two auxiliary results concerning the modules of isolated and congruence systems (from Definitions (2.13) and (2.18) respectively).

5.2.1. Isolated and Congruence systems

The following result gives a useful interpretation of the module of isolated Euler systems.

(5.4) Lemma. *Let \mathcal{R} denote either $\mathbb{Z}_{\mathcal{S}}$ for a finite set of prime numbers \mathcal{S} or \mathbb{Z}_p for a prime p . Then, for each choice of \mathcal{X} , one has $\text{ES}_k^{\mathcal{X}}(\mathcal{R})^{\text{iso}} = \text{ES}_k^{\mathcal{X}}(\mathcal{R})^{\mathcal{G}_{\mathcal{K}}}$.*

Proof. The explicit construction of isolated systems implies directly that $\text{ES}_k^{\mathcal{X}}(\mathcal{R})^{\text{iso}}$ is contained in $\text{ES}_k^{\mathcal{X}}(\mathcal{R})^{\mathcal{G}_{\mathcal{K}}}$. It therefore suffices to show that if a system c in $\text{ES}_k^{\mathcal{X}}(\mathcal{R})$ is $\mathcal{G}_{\mathcal{K}}$ -invariant, then it is isolated. In addition, Lemma (2.6) implies that the component c_F of any such c at a field F in \mathcal{X} vanishes if either $S(F)$ contains an archimedean place or at least two finite places. Since isolated Euler systems have the same property, we can therefore assume in the sequel that $\mathcal{X} = \Omega^{S_{\infty}(k)}(k)$.

Next we note that, if F belongs to \mathcal{X} and E is an intermediate field of F/k , then the $\mathcal{G}_{\mathcal{K}}$ -invariance of c implies that

$$c_F = [F : E]^{-1} \cdot (\nu_{F/E} \circ \text{N}_{F/E}^{r(F)})(c_F).$$

We now fix a place \mathfrak{p} in $S_{\text{fin}}(k)$ and suppose that $F \subseteq k(\mathfrak{p}^{\infty})$ is a ramified finite extension of k . Then the element $a'_{\mathfrak{p}} := \text{N}_{F/k}^{r(F)}(c_F)$ is independent of F and the above displayed equality implies that

$$c_F = [F : k]^{-1} \nu_{F/k}(a'_{\mathfrak{p}}).$$

Moreover, by Lemma (2.9) (d), the element $a'_{\mathfrak{p}}$ belongs to $|\mu_k|^{-1} \bigwedge_{\mathcal{R}}^{r(k(\mathfrak{p}))} (\mathcal{RO}_{k, \{\mathfrak{p}\}}^{\times})^{**}$. In particular, since c is assumed to be fixed by $\mathcal{G}_{\mathcal{K}}$, Lemma (2.9) (d) implies that c_F is in the image of $\nu_{F/k}$ and so one also has the equation

$$a'_{\mathfrak{p}} = [F : k] \cdot \nu_{F/k}^{-1}(c_F).$$

This shows that $a'_{\mathfrak{p}}$ is divisible by $[F : k]$ in $|\mu_k|^{-1} \bigwedge_{\mathcal{R}}^{r(k(\mathfrak{p}))} (\mathcal{RO}_{k, \{\mathfrak{p}\}}^{\times})^{**}$ for every such F . Writing $t_{\mathfrak{p}}$ for the cardinality of the finite group $(\mathcal{G}_{k(\mathfrak{p}^{\infty})})_{\text{tor}}$, we thus conclude that $c_F = [F : k]^{-1} t_{\mathfrak{p}} \nu_{F/k}(a_{\mathfrak{p}})$ for some element $a_{\mathfrak{p}}$ of $|\mu_k|^{-1} \bigwedge_{\mathcal{R}}^{r(k(\mathfrak{p}))} (\mathcal{RO}_{k, \{\mathfrak{p}\}}^{\times})^{**}$.

To complete the proof that c is isolated, it is therefore enough to show that $c_F = 0$ whenever $S(F) = \{\mathfrak{p}\}$ for some prime \mathfrak{p} in $S_{\text{fin}}(k)$ the residue characteristic of which is not a unit in \mathcal{R} (so \mathfrak{p} divides no prime in \mathcal{S} if $\mathcal{R} = \mathbb{Z}_{\mathcal{S}}$ or \mathfrak{p} is a p -adic prime if $\mathcal{R} = \mathbb{Z}_p$) and is such that the extension $k(\mathfrak{p}^{\infty})/k$ is infinite.

To do this we write p for the residue characteristic of such a place \mathfrak{p} and fix a \mathbb{Z}_p -extension F_{∞} of F that is contained in $k(\mathfrak{p}^{\infty})$. Then, writing F_n for the n -th layer of F_{∞}/F , one has

$$\nu_{F_n/F}(c_F) = (\nu_{F_n/F} \circ \text{N}_{F_n/F}^{r(F)})(c_{F_n}) = \text{N}_{\text{Gal}(F_n/F)} \cdot c_{F_n} = [F_n : F] \cdot c_{F_n}$$

and so $\nu_{F_n/F}(c_F)$ is divisible by $[F_n : F] = p^n$ in the lattice $(\mathcal{RL}_{F_n}^{r(F)})^{\mathcal{G}_{F_n}}$. Taking account of Lemma (2.9) (c), this then implies that c_F vanishes, as required. \square

The following result explains the significance of congruence systems to our approach and its proof occupies the rest of this section.

(5.5) Proposition. *Fix a prime number p and a system c in $\text{ES}_k(\mathbb{Z}_p)^{\text{con}}$ that is fixed by an open subgroup of $\mathcal{G}_{\mathcal{K}}$. Then, for every E in $\Omega(k)$ with $r_E > 0$, the component c_E is fixed by \mathcal{G}_E .*

Proof: At the outset we fix an open subgroup H of \mathcal{G}_K that fixes c and write $N := \mathcal{K}^H$ for its fixed field. We also fix a field K in $\Omega(k)$ with $r_K > 0$ and write \mathcal{Z}_n for the set of non-archimedean places of k that split completely in $N \cdot K(\mu_{p^n}, (\mathcal{O}_k^\times)^{1/p^n})$. In addition, we fix a finite set T of places of k that is admissible for K and disjoint from $\mathcal{Z}_1 \cup S_{\text{ram}}(K/\mathbb{Q})$. Note that T is then also admissible for all fields $K \cdot k(\mathfrak{q})$ with $\mathfrak{q} \in \mathcal{Z}_1$.

Set $c_{K,T} := \delta_{T,K} \cdot c_K$. Then, since $\delta_{T,K}$ is a non-zero divisor in $\mathbb{Z}_p[\mathcal{G}_K]$, it is enough to show that $(\sigma - 1)c_{K,T}$ vanishes for every $\sigma \in \mathcal{G}_K$. For this purpose it suffices to prove that $z \cdot (\sigma - 1)c_{K,T}$ vanishes for some non-zero integer z because $\mathbb{Z}_p \mathfrak{L}_{K,T}^{r_K}$ is \mathbb{Z} -torsion free.

To do this, it is enough to find a non-zero z such that, for every $f \in \bigwedge_{\mathbb{Z}_p[\mathcal{G}_K]}^{r_K-1} U_{K,S(K),T}^*$ (with $U_{K,S(K),T} := \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_{K,S(K),T}^\times$) and $\sigma \in \mathcal{G}_K$, the element $z(\sigma - 1)f(c_{K,T})$ of $U_{K,S(K),T}^{**} \cong U_{K,S(K),T}$ vanishes. Indeed, $U_{K,S(K),T}$ embeds into a free $\mathbb{Z}_p[\mathcal{G}_K]$ -module P of finite rank with \mathbb{Z} -torsion-free cokernel (cf. [BKS16, Rk. 5.11]). Every such f can thus be lifted to an element of $\bigwedge_{\mathbb{Z}_p[\mathcal{G}_K]}^{r_K-1} P^*$. The claim then implies $z(\sigma - 1)g(c_{K,T})$ vanishes for every g in $\bigwedge_{\mathbb{Z}_p[\mathcal{G}_K]}^{r_K} P^*$, and hence that $z(\sigma - 1)c_{K,T}$ vanishes as an element of $\bigwedge_{\mathbb{Z}_p[\mathcal{G}_K]}^{r_K} P$. Since the natural map $\bigcap_{\mathbb{Z}_p[\mathcal{G}_K]}^{r_K} U_{K,S(K),T} \rightarrow \bigwedge_{\mathbb{Z}_p[\mathcal{G}_K]}^{r_K} P$ is injective by [Sak20, Lem. C.1], we can therefore deduce that $z(\sigma - 1)c_{K,T}$ vanishes, as required.

We now fix $f \in \bigwedge_{\mathbb{Z}_p[\mathcal{G}_K]}^{r_K-1} U_{K,S(K),T}^* = \varprojlim_{n \in \mathbb{N}} (\bigwedge_{\mathbb{Z}[\mathcal{G}_K]}^{r_K-1} (\mathcal{O}_{K,S(K),T}^\times)^*)/p^n$ and write f as a family $f = (f_n)_{n \in \mathbb{N}}$ of classes represented by elements f_n of $\bigwedge_{\mathbb{Z}[\mathcal{G}_K]}^{r_K-1} (\mathcal{O}_{K,S(K),T}^\times)^*$. Note that since \mathbb{Z}_p is \mathbb{Z} -flat, one can use [Sak20, Lem. B.12] to obtain a similar isomorphism

$$\bigcap_{\mathbb{Z}_p[\mathcal{G}_K]}^{r_K} U_{K,S(K),T} \cong \varprojlim_{n \in \mathbb{N}} \left(\bigcap_{\mathbb{Z}[\mathcal{G}_K]}^{r_K} \mathcal{O}_{K,S(K),T}^\times \right) / p^n$$

that allows us to regard $c_{K,T}$ as a family $c_{K,T} = (c_{K,T}^{(n)})_{n \in \mathbb{N}}$ of classes represented by elements $c_{K,T}^{(n)}$ of $\bigcap_{\mathbb{Z}[\mathcal{G}_K]}^{r_K} \mathcal{O}_{K,S(K),T}^\times$. One then has $z(\sigma - 1)f(c_{K,T}) = ((\sigma - 1)f_n(c_{K,T}^{(n)}))_{n \in \mathbb{N}}$ and we need to show that $z(\sigma - 1)f_n(c_{K,T}^{(n)})$, as an element of K^\times , is divisible by p^n for each n in order to verify that $z(\sigma - 1)f(c_{K,T})$ vanishes.

Let us therefore now fix $n \in \mathbb{N}$ and show that $z(\sigma - 1)f_n(c_{K,T}^{(n)})$ is divisible by p^n for a natural number z that does not depend on n . To do this we take $z = 2[N : k]$ and apply the criterion in the following result.

(5.6) Lemma. *Fix a in $\mathcal{O}_{K,S(K)}^\times$ with the property that, for every $\mathfrak{q} \in \mathcal{Z}_n$ and every \mathfrak{q} -adic place \mathfrak{Q} of K , a is congruent to a p^n -th power modulo \mathfrak{Q} when viewed as an element of the valuation ring $\mathcal{O}_{K_\mathfrak{Q}}$ of the completion $K_\mathfrak{Q}$ of K at \mathfrak{Q} under the canonical embedding $\iota_\mathfrak{Q} : K \hookrightarrow K_\mathfrak{Q}$. Then, for every $\sigma \in \mathcal{G}_K$ the element $2[N : k](\sigma - 1)a$ belongs to $(K^\times)^{p^n}$.*

Proof. Let \mathfrak{Q} be a place of the stated kind and write \bar{a} for the class of $\iota_\mathfrak{Q}(a)$ in the residue field $\mathbb{F}_\mathfrak{Q} := \mathcal{O}_{K_\mathfrak{Q}}/\mathfrak{m}_\mathfrak{Q}$. Then, by assumption, the polynomial $X^{p^n} - \bar{a} \in \mathbb{F}_\mathfrak{Q}[X]$ has a root in $\mathbb{F}_\mathfrak{Q}$. Since the characteristic of $\mathbb{F}_\mathfrak{Q}$ and p are coprime, Hensel's Lemma implies $X^{p^n} - a$ has a root in $\mathcal{O}_{K_\mathfrak{Q}}$ and hence that a is a p^n -th power in $K_\mathfrak{Q}$. Since \mathfrak{Q} was arbitrary, it follows that a belongs to the kernel of the diagonal map

$$K^\times / (K^\times)^{p^n} \rightarrow \prod_{\mathfrak{Q} \in (\mathcal{Z}_n)_K} K_\mathfrak{Q}^\times / (K_\mathfrak{Q}^\times)^{p^n}.$$

We now claim that the kernel of this map only contains classes represented by elements of K^\times that are p^n -th powers in $F_n := NK(\mu_{p^n}, (\mathcal{O}_k^\times)^{1/p^n})$. To justify this claim, we let b denote an element of K^\times that represents such a class in the kernel of Δ . Then every place of F_n lying above a place in \mathcal{Z}_n , and hence every place of k that splits completely in F_n , splits completely in $F_n(\sqrt[p^n]{b})$. Since every Galois extension is uniquely determined by the set of places that split completely in the extension, it follows that $F_n(\sqrt[p^n]{b})$ is contained in, and hence equal to, F_n . In particular, b is a p^n -th power in F_n .

At this stage, we have proved that the class of a belongs to the kernel of the natural map

$\lambda: K^\times/(K^\times)^{p^n} \rightarrow F_n^\times/(F_n^\times)^{p^n}$. To prove the claimed result, it therefore suffices to show that elements in $\ker(\lambda)$ are annihilated by $2[N:k](\sigma-1)$ for all $\sigma \in \mathcal{G}_K$. To do this, we note the inflation-restriction sequence identifies $\ker(\lambda)$ with $H^1(\text{Gal}(F_n/K), \mu_{p^n})$. In addition, setting $K_n := K(\mu_{p^n})$, [Neu73, Satz (4.8)] implies that

$$H^1(\text{Gal}(NK_n/NK), \mu_{p^n}) = \begin{cases} 0 & \text{if } p \text{ is odd,} \\ H^1(\text{Gal}(NK_2/NK), \mu_4) & \text{if } p = 2, \end{cases}$$

is annihilated by 2. It therefore follows from the exact sequence

$$0 \rightarrow H^1(\text{Gal}(NK/K), \mu_{p^n} \cap NK) \rightarrow H^1(\text{Gal}(NK_n/K), \mu_{p^n}) \rightarrow H^1(\text{Gal}(NK_n/NK), \mu_{p^n})$$

that $H^1(\text{Gal}(NK_n/K), \mu_{p^n})$ is annihilated by $2[NK:K]$ (and hence also by $2[N:k]$). Now, we also have the exact sequence

$$0 \rightarrow H^1(\text{Gal}(NK_n/K), \mu_{p^n}) \rightarrow H^1(\text{Gal}(F_n/K), \mu_{p^n}) \rightarrow H^1(\text{Gal}(F_n/NK_n), \mu_{p^n})$$

and so it suffices to show that $H^1(\text{Gal}(F_n/NK_n), \mu_{p^n})$ is fixed by \mathcal{G}_K in order to prove the lemma. To this end, we note that the perfectness of the Kummer pairing gives a \mathcal{G}_k -equivariant isomorphism

$$\begin{aligned} H^1(\text{Gal}(F_n/NK_n), \mu_{p^n}) &= \text{Hom}(\text{Gal}(F_n/NK_n), \mu_{p^n}) \\ &\cong (\mathcal{O}_k^\times \cdot (NK_n^\times)^{p^n} / (NK_n^\times)^{p^n}) \\ &\cong \mathcal{O}_k^\times / (\mathcal{O}_k^\times \cap (NK_n^\times)^{p^n}). \end{aligned}$$

Since elements in this quotient module are clearly fixed by \mathcal{G}_K , the claimed result follows. \square

Returning now to the proof of Proposition (5.5), we are reduced to showing that, if we fix a place $\mathfrak{q} \in \mathcal{Z}_n$ and set $L = L(\mathfrak{q}) := Kk(\mathfrak{q})$, then $f_n(c_{K,T}^{(n)})$ is a p^n -th power modulo any place of K lying above \mathfrak{q} . Since $(f_m(c_{K,T}^{(m)}))_{m \in \mathbb{N}}$ is a compatible family, it is therefore enough to show that $f_m(c_{K,T}^{(m)})$ is a p^n -th power for any sufficiently large $m \geq n$.

Let χ be a character of $\mathcal{G}_{L(\mathfrak{q})}$ with $e_\chi \cdot e_{L(\mathfrak{q})} \neq 0$. By construction, \mathfrak{q} splits completely in N and so χ cannot factor through $N \cap L(\mathfrak{q})$. On the other hand, by assumption $c_{L(\mathfrak{q})}$ is fixed by $\text{Gal}(L(\mathfrak{q})/L(\mathfrak{q}) \cap N)$. We therefore must have that $e_\chi \cdot c_{L(\mathfrak{q})}$ vanishes. This shows that $e_{L(\mathfrak{q})}c_{L(\mathfrak{q})}$, and hence also $c_{L(\mathfrak{q})}$ by Lemma (2.6), vanishes.

In particular, $c_{L,T}$ identifies with a family $(c_{L,T}^{(m)})_{m \in \mathbb{N}}$ in which each $c_{L,T}^{(m)}$ is divisible by p^m . We can thus fix an m that is large enough to ensure $\mathcal{N}_H(c_{L,T}^{(m)}) := \sum_{\sigma \in H} \sigma c_{L,T}^{(m)} \otimes \sigma^{-1}$ is trivial when viewed in the finite p -group $(\prod_{\mathbb{Z}[\mathcal{G}_L]}^{r_K} \mathcal{O}_{L,S(L),T}^\times \otimes_{\mathbb{Z}[\mathcal{G}_L]} (I(H)/I(H)^2))$, where $H := \text{Gal}(L/K)$ and $I(H)$ the kernel of $\mathbb{Z}[\mathcal{G}_L] \rightarrow \mathbb{Z}[\mathcal{G}_K]$. The assumption that c is a congruence system (and so satisfies the congruences in Definition (2.18) with $v = \mathfrak{q}$ so $P_{L/K, \{v\}} = 1$) then combines with the injectivity of the map

$$\left(\prod_{\mathbb{Z}[\mathcal{G}_K]}^{r_K} \mathcal{O}_{K,S(K),T}^\times \otimes_{\mathbb{Z}[\mathcal{G}_K]} (I(H)/I(H)^2) \right) \rightarrow \left(\prod_{\mathbb{Z}[\mathcal{G}_L]}^{r_K} \mathcal{O}_{L,S(L),T}^\times \otimes_{\mathbb{Z}[\mathcal{G}_K]} (I(H)/I(H)^2) \right)$$

induced by $\nu_{L/K}$ (cf. [San14, Lem. 2.11]) to imply that the element

$$(\text{Rec}_\mathfrak{q} \circ \text{Ord}_\mathfrak{q}^{-1})(c_{K,T}^{(m)}) \in \left(\prod_{\mathbb{Z}[\mathcal{G}_K]}^{r_K} \mathcal{O}_{K,S(K),T}^\times \otimes_{\mathbb{Z}[\mathcal{G}_K]} (I(H)/I(H)^2) \right)$$

vanishes. In the quotient group $I(H)/I(H)^2$ one therefore has

$$\begin{aligned} \text{Rec}'_\mathfrak{q}(f_n(c_{K,T}^{(m)})) &= (\text{Rec}'_\mathfrak{q} \circ f_n \circ \text{Ord}_\mathfrak{q} \circ \text{Ord}_\mathfrak{q}^{-1})(c_{K,T}^{(m)}) \\ &= \pm (\text{Ord}_\mathfrak{q} \wedge f_n)((\text{Rec}_\mathfrak{q} \circ \text{Ord}_\mathfrak{q}^{-1})(c_{K,T}^{(m)})) = 0, \end{aligned} \tag{43}$$

where the map $\text{Rec}'_\mathfrak{q}$ is as defined in (7).

To proceed, we write I_H for the augmentation ideal of $\mathbb{Z}[H]$. Then the composite isomorphism

$$I(H)/I(H)^2 \cong I_H/I_H^2 \otimes_{\mathbb{Z}} \mathbb{Z}[G/H] \cong H \otimes_{\mathbb{Z}} \mathbb{Z}[G/H]$$

(in which the first isomorphism is described in [San14, (3)] and the second is induced by sending, for each $h \in H$, the element $h - 1$ of I_H to h), combines with the equality (43) and the explicit definition of $\text{Rec}'_{\mathfrak{q}}$ to imply that, for some fixed place Ω_1 of K above \mathfrak{q} and all $\sigma \in \mathcal{G}_K$, the element $\text{rec}_{\Omega_1}(\sigma f_m(c_{K,T}^{(m)}))$ of $H \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is trivial. It follows that, for every place Ω of K above \mathfrak{q} , the element $\text{rec}_{\Omega}(f_m(c_{K,T}^{(m)}))$ of $H \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is trivial.

Now, the local reciprocity map $\text{rec}_{\Omega}: K_{\Omega}^{\times} \rightarrow \text{Gal}(L_{\Omega'}/K_{\Omega})$ maps $\mathcal{O}_{K_{\Omega}}^{\times}$ onto the inertia subgroup \mathcal{I} of $\text{Gal}(L_{\Omega'}/K_{\Omega})$ and so induces an isomorphism

$$\mathcal{O}_{K_{\Omega}}^{\times}/N_{L_{\Omega'}/K_{\Omega}}(\mathcal{O}_{L_{\Omega'}}^{\times}) \xrightarrow{\cong} \mathcal{I}.$$

In addition, class field theory implies $N_{L_{\Omega'}/K_{\Omega}}(\mathcal{O}_{L_{\Omega'}}^{\times})$ contains $1 + \Omega^l$ for any natural number l for which the l -th upper ramification subgroup of $\text{Gal}(L_{\Omega'}/K_{\Omega})$ vanishes. In particular, because the quotient of $1 + \Omega$ by $1 + \Omega^m$ is a q -group, where q is the residue characteristic of Ω and hence prime to p , the above isomorphism implies $\mathcal{I} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is isomorphic to a quotient of the group $\mathbb{F}_{\Omega}^{\times} \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong (\mathcal{O}_{K_{\Omega}}^{\times}/(1 + \Omega)) \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Since $\mathbb{F}_{\Omega}^{\times}$ is a cyclic group, it therefore follows that rec_{Ω} induces an isomorphism

$$\mathbb{F}_{\Omega}^{\times} \otimes_{\mathbb{Z}} (\mathbb{Z}_p/\mathcal{I}\mathbb{Z}_p) \xrightarrow{\cong} \mathcal{I} \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

and hence that the image of $f_m(c_{K,T}^{(m)})$ in $\mathbb{F}_{\Omega}^{\times} \otimes_{\mathbb{Z}} (\mathbb{Z}_p/\mathcal{I}\mathbb{Z}_p)$ must vanish.

Now, by assumption, \mathfrak{q} belongs to \mathcal{Z}_n and so Lemma (2.1) (b) implies p^n divides $[k(\mathfrak{q}) : k(1)]$ and hence also $|\mathcal{I}|$. As a consequence, the above observation implies that $f_m(c_{K,T}^{(m)})$ is a p^n -th power in $\mathbb{F}_{\Omega}^{\times}$. Since this is true for all \mathfrak{q} in \mathcal{Z}_n and all places Ω of K above \mathfrak{q} , we can therefore conclude the proof of Proposition (5.5) by applying Lemma (5.6). \square

5.2.2. A reduction to the Scarcity Conjecture

In the next result we investigate the extent to which global properties of an Euler system can be determined by analysis of its individual components. In particular, in claim (a) we show that Euler systems are uniquely determined, up to multiplication by isolated systems, by their components at certain sparse families of fields. Then in claim (b) we reduce the proof of Conjecture (2.24) to the verification that the components of Euler systems have certain explicit properties. Subsequently, in §5.3 and §6, we provide concrete evidence in support of the containments that respectively occur in (b) (ii) and (iii) of this result.

(5.7) Theorem. *Fix a finite set \mathcal{S} of prime numbers and a subset \mathcal{X} of $\Omega(k)$ that satisfies Hypothesis (4.5). Then the following claims are valid.*

- (a) *Fix a rank function \mathbf{r} for k and a finite set of places \mathcal{M} of k , and write $\mathcal{X}_{\mathcal{M}}(k)$ for the subset of \mathcal{X} comprising fields E which are ramified at all places in \mathcal{M} . Then, up to multiplication by isolated systems, each system c in $\text{ES}_k^{\mathbf{r}, \mathcal{X}}(\mathbb{Z}_{\mathcal{S}})$ is uniquely determined by its values c_E for fields E in $\mathcal{X}_{\mathcal{M}}(k)$.*
- (b) *Assume \mathcal{X} is contained in $\Omega^V(k)$ for some non-empty subset V of $S_{\infty}(k)$ and that $\varepsilon_k^{\mathcal{X}}$ belongs to $\text{ES}_k^{\mathcal{X}}(\mathbb{Z}_{\mathcal{S}})^{\text{con}}$. Then, for every c in $\text{ES}_k^{\mathcal{X}}(\mathbb{Z}_{\mathcal{S}})^{\text{sym}} \cap \text{ES}_k^{\mathcal{X}}(\mathbb{Z}_{\mathcal{S}})^{\text{con}}$, the following assertions are equivalent:*
 - (i) *The system c belongs to $\mathbb{Z}_{\mathcal{S}}[[\mathcal{G}_K]]\varepsilon_k^{\mathcal{X}}$.*
 - (ii) *For every field K in \mathcal{X} one has $c_K \in \mathbb{Z}_{\mathcal{S}}[\mathcal{G}_K] \cdot \varepsilon_{K/k}$.*
 - (iii) *c_K belongs to $\mathbb{Z}_p[\mathcal{G}_K] \cdot \varepsilon_{K/k}$ for every prime number $p \notin \mathcal{S}$ and every field K in $\mathcal{X}_{\mathcal{S}_p}$.*

Proof. To prove claim (a) we consider the triangular system $\{(N_{F,E}, \rho_{F/E})\}_{E,F \in \mathcal{X}}$ of Example (4.2) (b) with $\mathcal{R} = \mathbb{Z}_{\mathcal{S}}$ (so that $N_{F,E} = \mathbb{Z}_{\mathcal{S}}\mathcal{L}_E^{\mathbf{r}(E)}$) and $\rho_{F/E}$ is the relevant norm map for each

extension F/E). We note that each module $\mathbb{Z}_S \mathfrak{L}_E^{r(E)}$ is a $\mathbb{Z}_S[\mathcal{G}_E]$ -lattice and that this triangular system validates Hypothesis (4.27) for every $p \notin \mathcal{S}$ by Example (4.9) (c). Given these facts, Theorem (5.7) (a) follows directly upon applying Corollary (4.13) to this triangular system.

Next we observe that the implications “(i) \Rightarrow (ii)” and “(ii) \Rightarrow (iii)” in claim (b) are clearly valid, and hence that it suffices to prove that condition (iii) implies (i).

To do this we assume c satisfies condition (iii) and claim first that, for each prime $p \notin \mathcal{S}$, this implies the existence of an element q of $\mathbb{Z}_p[[\mathcal{G}_K]]$ such that

$$\varrho^{\mathcal{X}_{S_p}}(c) = \varrho^{\mathcal{X}_{S_p}}(q \cdot \varepsilon_k). \quad (44)$$

To justify this we fix, for every $K \in \mathcal{X}_{S_p}$ an element $q_K^{(p)}$ of $\mathbb{Z}_p[\mathcal{G}_K]$ with $c_K = q_K^{(p)} \cdot \varepsilon_{K/k}$, and show that the family $\tilde{q} := (q_K^{(p)} e_K)_{K \in \mathcal{X}_{S_p}}$ defines an element of $\varprojlim_{K \in \mathcal{X}_{S_p}}^{\emptyset} \mathbb{Z}_p[\mathcal{G}_K] e_K$, where the Euler limit is defined as in Example (4.4) (d). For this purpose we take fields K and L in \mathcal{X}_{S_p} with $K \subseteq L$ and note that, since $\varepsilon_{L/k} = e_L \cdot \varepsilon_{L/k}$, one has

$$\begin{aligned} P_{L/K, \emptyset} \cdot \pi_{L/K}(q_L^{(p)} e_L) \cdot \varepsilon_{K/k} &= \pi_{L/K}(q_L^{(p)} e_L) \cdot N_{L/K}^{r_L}(\varepsilon_{L/k}) \\ &= N_{L/K}^{r_L}(c_L) \\ &= P_{L/K, \emptyset} \cdot c_K \\ &= P_{L/K, \emptyset} \cdot q_K^{(p)} e_K \cdot \varepsilon_{K/k}. \end{aligned}$$

Since $\varepsilon_{K/k}$ generates a free $\mathbb{Z}_p[\mathcal{G}_K] e_K$ -module of rank one, this calculation implies the element $P_{L/K, \emptyset} \cdot (\pi_{L/K}(q_L^{(p)} e_L) - q_K^{(p)} e_K)$ vanishes and hence proves \tilde{q} belongs to $\varprojlim_{K \in \mathcal{X}_{S_p}}^{\emptyset} \mathbb{Z}_p[\mathcal{G}_K] e_K$, as required. We can now apply Proposition (4.36) to deduce \tilde{q} lifts to $\varprojlim_{K \in \mathcal{X}_{S_p}} \mathbb{Z}_p[\mathcal{G}_K]$ and hence to $\mathbb{Z}_p[[\mathcal{G}_K]]$, and any lift q of \tilde{q} to $\mathbb{Z}_p[[\mathcal{G}_K]]$ satisfies the claimed equality (44).

If we now take $\{(N_{F,E}, \rho_{F/E})\}_{E, F \in \mathcal{X}}$ to be the triangular system with $N_{F,E} = \mathbb{Z}_p \mathfrak{L}_E^{r(E)}$ and $\rho_{F/E}$ the relevant norm map for each extension F/E , then we can reinterpret the equality (44) as asserting that $c - q\varepsilon_k^{\mathcal{X}}$ belongs to the kernel of the restriction map

$$\text{res}_{S_p} : \varprojlim_{E \in \mathcal{X}}^{\emptyset} N_E \rightarrow \varprojlim_{E \in \mathcal{X}_{S_p}}^{\emptyset} N_E.$$

In particular, since the triangular system under consideration satisfies Hypothesis (4.8) for p , we can apply Theorem (4.10) (with $\Pi = \emptyset$ and $\Sigma = S_p(k)$) to deduce that, if p is odd, then $c - q\varepsilon_k^{\mathcal{X}}$ is fixed by $\text{Gal}(\mathcal{K}/k_p)$, where k_p denotes the composite of all subextensions of the field $k\langle p \rangle$ (from (24)) in which at least one p -adic place splits completely.

Now if p does not divide $2d_k$, then $k\langle p \rangle = k(\mu_p)$ is totally ramified at all places in $S_p(k)$ so $k_p = k$, and hence $c - q\varepsilon_k^{\mathcal{X}}$ is fixed by \mathcal{G}_K . On the other hand, if p divides $2d_k$, then Theorem (4.10) (resp., if $p = 2$, Proposition (4.21) combined with the observation that any subextension of $k\langle p, \mathcal{T} \rangle$ that is unramified at a p -adic place is finite over k) implies $c - q\varepsilon_k^{\mathcal{X}}$ is fixed by an open subgroup of \mathcal{G}_K . Thus, since both c and ε_k , and hence also $c - q\varepsilon_k^{\mathcal{X}}$, are assumed to be congruence systems, we can apply Proposition (5.5) to deduce that $c - q\varepsilon_k^{\mathcal{X}}$ is also fixed by \mathcal{G}_K in this case.

Having proved that $c - q\varepsilon_k^{\mathcal{X}}$ is fixed by \mathcal{G}_K in all cases, we next observe that $c - q\varepsilon_k^{\mathcal{X}}$ is a symmetric system (this uses that ε_k is symmetric by Lemma (2.22) (i)). It therefore follows from Lemma (2.17) and Lemma (5.4) that $c = q\varepsilon_k^{\mathcal{X}}$, and hence that

$$c \in \mathbb{Z}_p[[\mathcal{G}_K]] \cdot \varepsilon_k^{\mathcal{X}}. \quad (45)$$

To proceed, we now consider the projective system $\{(M_F, \varphi_{F/E})\}_{F \in \mathcal{X}}$ defined by taking $M_F = \mathbb{Z}_S[\mathcal{G}_F]$ and $\varphi_{F/E} = \pi_{F/E}$ (which automatically satisfies Hypothesis (4.27), see Example (4.28)). In this case, we then obtain a map β of the sort that occurs in Theorem (4.29) by means of the assignment

$$\varprojlim_{E \in \mathcal{X}}^{\emptyset} e_E \mathbb{Z}_S[\mathcal{G}_E] \rightarrow \text{ES}_k^{\mathcal{X}}(\mathbb{Z}_S) \subseteq \varprojlim_{E \in \mathcal{X}}^{\emptyset} \mathbb{Z}_S \mathfrak{L}_E, \quad (q_E)_E \mapsto (q_E \varepsilon_{E/k})_E.$$

We now note that Theorem (5.1) (b) implies both that this map is injective and further that for each system c in $\text{ES}_k^{\mathcal{X}}(\mathbb{Z}_{\mathcal{S}})$ there exists a unique element $a = (a_E)_E$ of $\varprojlim_{E \in \mathcal{X}} e_E \mathcal{Q}[\mathcal{G}_E]$ such that $c_E = a_E \cdot \varepsilon_{E/k}$ for every E in \mathcal{X} .

Given this fact, the condition in (b)(i) of Theorem (5.7) follows directly upon combining the containments (45) with the result of Theorem (4.29) (b) and noting that, in this case, one has $\beta_*(\mathbb{Z}_{\mathcal{S}}[[\mathcal{G}_{\mathcal{K}}]]) = \mathbb{Z}_{\mathcal{S}}[[\mathcal{G}_{\mathcal{K}}]] \cdot \varepsilon_k^{\mathcal{X}}$ and $\beta_*^{(p)}(\ker \theta_p) = \mathbb{Z}_p[[\mathcal{G}_{\mathcal{K}}]] \cdot \varepsilon_k^{\mathcal{X}}$. This therefore completes the proof that (b)(iii) implies (b)(i). \square

(5.8) Remark. Fix a prime number p , a subset \mathcal{X} of $\Omega(k)$ that satisfies Hypothesis (4.5) and an Euler system c in $\text{ES}_k(\mathbb{Z}_p)$. Then the argument used to prove the implication ‘(iii) \Rightarrow (i)’ in Theorem (5.7) (a) also shows that $\varrho^{\mathcal{X}_{S_p}}(c)$ belongs to $\mathbb{Z}_p[[\mathcal{G}_{\mathcal{K}}]] \cdot \varepsilon_k^{\mathcal{X}_{S_p}}$ if and only if c_K belongs to $\mathbb{Z}_p[\mathcal{G}_K] \cdot \varepsilon_{K/k}$ for every field K in \mathcal{X}_{S_p} .

5.3. Integral Euler systems and Kolyvagin systems

We now explain how the observations made in § 3.2 can be combined with the theory of higher-rank Kolyvagin systems recently developed by Sakamoto, Sano and the second author to provide concrete evidence in support of the containment in Theorem (5.7) (b) (ii).

To do this, we fix a field E in $\Omega^{\{S_{\infty}(k)\}}(k)$ and an odd prime number p . We write F and L for the maximal extensions of k in E of degree a power of p and coprime to p , respectively. We set $G := \mathcal{G}_E$, $P := \mathcal{G}_F$ and $H := \mathcal{G}_L$ and we regard P and H as subgroups of G in the obvious way. In particular, for each character $\chi: H \rightarrow \overline{\mathbb{Q}_p}^{\times}$, the idempotent $e_{\chi} := |H|^{-1} \sum_{h \in H} \chi(h) h^{-1}$ can be viewed as an element of $\mathbb{Z}_p[\text{im}(\chi)][G]$ and hence acts on the image $1 \otimes m$ in $\mathbb{Z}_p[\text{im}(\chi)] \otimes_{\mathbb{Z}} M$ of an element m of a $\mathbb{Z}[G]$ -module M .

Finally, we write ω_p for the p -adic Teichmüller character of k .

(5.9) Theorem. *Fix an abelian extension E/k as above and a homomorphism $\chi: H \rightarrow \overline{\mathbb{Q}_p}^{\times}$. Assume the Rubin–Stark Conjecture holds for all abelian extensions of k and, in addition, that all of the following conditions are satisfied.*

- (a) E contains the Hilbert p -class field of k ;
- (b) $\chi \neq \omega_p$ and, if $p = 3$, also $\chi^2 \neq \omega_p$;
- (c) χ is not trivial on the decomposition subgroup of any place in $S_{\text{ram}}(F/k)$;
- (d) χ is not trivial on the inertia subgroup of at least one place in $S_{\text{ram}}(E/k)$;

Then, for every Euler system c in $\text{ES}_k(\mathbb{Z})$ one has $e_{\chi}(1 \otimes c_E) \in \mathbb{Z}_p[\text{im} \chi][\mathcal{G}_E] \cdot (1 \otimes \varepsilon_{E/k})$.

Proof. Write L_{χ} for the fixed field of $\ker(\chi)$ in L . Then, after taking into account the Euler system distribution relations (for both c and ε_k), it is enough to prove the stated claim after replacing L by L_{χ} (and hence E by the compositum $L_{\chi}F$). In the sequel we will therefore assume that χ is a faithful character of H .

Then, in this case, the stated conditions imply that all of the hypotheses of [Bur+23, Thm. 4.1] are satisfied. Hence, if we set $r := |S_{\infty}(k)|$, then the latter result (which relies on the theory of higher-rank Kolyvagin systems) implies that, for each system c in $\text{ES}_k(\mathbb{Z})$, there is a containment

$$e_{\chi}(1 \otimes c_E) \in e_{\chi}(\mathbb{Z}_p[\text{im}(\chi)] \otimes_{\mathbb{Z}} \text{im}(\Theta_{E/k, S(E)}^r)).$$

In addition, by combining this containment in the case $c = \varepsilon_k$ together with the argument of Proposition (3.6) one finds that

$$e_{\chi}(\mathbb{Z}_p[\text{im}(\chi)] \otimes_{\mathbb{Z}} \text{im}(\Theta_{E/k, S(E)}^r)) = e_{\chi}(\mathbb{Z}_p[\text{im} \chi][\mathcal{G}_E] \cdot (1 \otimes \varepsilon_{E/k})) \subseteq \mathbb{Z}_p[\text{im} \chi][\mathcal{G}_E] \cdot (1 \otimes \varepsilon_{E/k}).$$

The claimed result now follows directly upon combining this inclusion with the previous containment. \square

6. Iwasawa theory

The key feature of condition (b) (iii) in Theorem (5.7) is that it only concerns abelian extensions K of k that are ramified at all p -adic places. For this reason, in this section we are able to use techniques of ‘equivariant’ Iwasawa theory to explicitly reinterpret the condition and thereby, in several important cases, deduce its validity from existing results and conjectures. In this way we shall, in particular, complete the proofs of all of the results that are stated in the Introduction.

6.1. An Iwasawa-theoretic reduction

6.1.1. Statement of the main results

For each prime p we fix a subextension $k_\infty = k_\infty^p$ of \mathcal{K}/k for which the set of places of k that split completely in k_∞ is equal to $S_\infty(k)$ and, in addition, the group $\text{Gal}(k_\infty/k)$ is topologically isomorphic to \mathbb{Z}_p^d for an integer $d > 0$. (For example, one can take k_∞ to be the cyclotomic \mathbb{Z}_p -extension of k .)

For any finite abelian extension E of k , we set

$$E_\infty = E_\infty^p := Ek_\infty^p \quad \text{and} \quad \Lambda_{p,E} := \mathbb{Z}_p \llbracket \text{Gal}(E_\infty^p/k) \rrbracket.$$

For any such E , any finite set Σ of places of k and any finite subset T of $S_{\text{fin}}(k)$ that is disjoint from Σ , we also write $\text{Cl}_{E,\Sigma,T}$ for the (Σ_E, T_E) -ray class group of E (as discussed in Lemma (3.4) (a)). For any abelian extension L of k we then define

$$\begin{aligned} U_{L,\Sigma,T} &:= \varprojlim_E (\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_{E,\Sigma,T}^\times), & \text{Cl}_{L,\Sigma,T}^p &:= \varprojlim_E (\mathbb{Z}_p \otimes_{\mathbb{Z}} \text{Cl}_{E,\Sigma,T}), \\ X_{L,\Sigma}^p &:= \varprojlim_E (\mathbb{Z}_p \otimes_{\mathbb{Z}} X_{E,\Sigma}), & Y_{L,\Sigma}^p &:= \varprojlim_E (\mathbb{Z}_p \otimes_{\mathbb{Z}} Y_{E,\Sigma}). \end{aligned}$$

Here each limit is taken over all finite extensions E of k in L and the respective transition morphisms are induced in the first two cases by the norm maps $N_{E'/E}$ and the last two cases by the restriction of places maps.

We note that if T is both disjoint from the set $S_p = S_p(k)$ of p -adic places of k and also ‘admissible’ for E (in the sense of § 2.2.1), then any Euler system c in $\text{ES}_k(\mathbb{Z}_S)$ gives rise to a norm-coherent sequence

$$c_{E_\infty^p, T} := (\delta_T \cdot c_F)_F \in \varprojlim_F \bigcap_{\mathbb{Z}_p[\mathcal{G}_F]}^{r_E} U_{F, S(E_\infty^p), T} \cong \bigcap_{\Lambda_{p,E}}^{r_E} U_{E_\infty^p, S(E_\infty^p), T},$$

where the limit is taken over all fields F that belong to $\Omega(k)$ and are contained in E_∞^p and is with respect to the norm maps $N_{F'/F}^{r_E}$ (here the isomorphism is a consequence of the general observation [Sak20, Lem. B.15] of Sakamoto). In this context we further recall that any element η of $\bigcap_{\Lambda_{p,E}}^{r_E} U_{E_\infty^p, S(E_\infty^p), T}$ is by definition a map $\bigwedge_{\Lambda_{p,E}}^{r_E} U_{E_\infty^p, S(E_\infty^p), T}^* \rightarrow \Lambda_{p,E}$ and so gives rise to an ideal $\text{im}(\eta)$ of $\Lambda_{p,E}$.

The following result is the main observation that we shall make in § 6 and lists a variety of explicit conditions that are sufficient to ensure either the validity of the Tamagawa Number Conjecture or that an Euler system validates the Scarcity Conjecture. In this result we refer to the field $k\langle p \rangle$ defined in (24).

(6.1) Theorem. *We assume to be given data of the following sort.*

- A subset V of $S_\infty(k)$ and a subset \mathcal{X} of $\Omega(k)$ that satisfies Hypothesis (4.5) with $\mathcal{V} = \{V\}$.
- A finite set \mathcal{S} of prime numbers that contains 2 if $V \neq S_\infty(k)$.
- For every prime $p \notin \mathcal{S}$, an extension k_∞^p of k as specified at the beginning of this section.

For every field K in \mathcal{X} and every prime $p \notin \mathcal{S}$, the idempotent ϵ_K identifies with an element of the algebra $\Lambda_{p,K}$, and we assume that the above data satisfies the following two hypotheses:

- (i) The Rubin–Stark system $\varepsilon_k^{\mathcal{X}}$ is integral outside \mathcal{S} in the sense of Definition (2.12).
- (ii) For every prime $p \notin \mathcal{S}$, every field K in \mathcal{X}_{S_p} and every height-one prime \mathfrak{p} of $\Lambda_{p,K}$ in the support of $\varepsilon_K \Lambda_{p,K}$, the $(\Lambda_{p,K})_{\mathfrak{p}}$ -module $\mathrm{Cl}^p(K_{\infty}^p)_{\mathfrak{p}}$ has finite projective dimension.

Then the following claims are valid.

- (a) $\mathrm{TNC}(h^0(\mathrm{Spec} E), \mathbb{Z}_p[\mathcal{G}_E] \varepsilon_{E,V})$ is valid for every field E in \mathcal{X} if, for every K and p as in hypothesis (ii) above, there exists a set T in $\mathcal{P}_K^{\mathrm{ad}}$ such that

$$\mathrm{im}(\varepsilon_{K_{\infty}^p, T})^{**} \subseteq \mathrm{Fitt}_{\Lambda_{p,K}}^0(\mathrm{Cl}_{K_{\infty}^p, S(K), T}^p)^{**} \cdot \mathrm{Fitt}_{\Lambda_{p,K}}^0(X_{K_{\infty}^p, S(K)}^p)^{**},$$

and, in addition, at least one of the following conditions holds:

- (i) p does not divide $2d_k$.
- (ii) $V \neq \emptyset$ and ε_k is a congruence system in the sense of Definition (2.18).
- (iii) p is odd and $\mathrm{TNC}(h^0(\mathrm{Spec} F), \mathbb{Z}_p[\mathcal{G}_F] \varepsilon_{F,V})$ is valid for every field F that belongs to the subset \mathcal{Y} of \mathcal{X} that is defined in either of the following ways:
 - \mathcal{Y} comprises all fields $k\langle p \rangle L$ with L a tamely ramified cyclic p -extension of k that belongs to $\Omega^{S_{\infty}}(k)$.
 - \mathcal{Y} comprises all finite extensions of $k\langle p \rangle$ in $k\langle p \rangle k_{\infty}^p$ that belong to \mathcal{X} .
- (b) Assume $V \neq \emptyset$ and $\varepsilon_k \in \mathrm{ES}_k^{\mathcal{X}}(\mathbb{Z}_S)^{\mathrm{con}}$. Then a system c in $\mathrm{ES}_k^{\mathcal{X}}(\mathbb{Z}_S)^{\mathrm{con}} \cap \mathrm{ES}_k^{\mathcal{X}}(\mathbb{Z}_S)^{\mathrm{sym}}$ belongs to $\mathbb{Z}_S[[\mathcal{G}_K]] \cdot \varepsilon_k^{\mathcal{X}}$ provided that, for every K and p as in hypothesis (ii) above, there exists a set T in $\mathcal{P}_K^{\mathrm{ad}}$ for which one has

$$\mathrm{im}(c_{K_{\infty}^p, T})^{**} \subseteq \mathrm{Fitt}_{\Lambda_{p,K}}^0(\mathrm{Cl}_{K_{\infty}^p, S(K), T}^p)^{**} \cdot \mathrm{Fitt}_{\Lambda_{p,K}}^0(X_{K_{\infty}^p, S(K)}^p)^{**}. \quad (46)$$

This result will be proved in §6.1.4, where we shall also (in Lemma (6.6)) record some useful facts about the ideals that occur in the inclusions displayed in claims (a) and (b).

6.1.2. Preliminary observations

Before proving Theorem (6.1), it is convenient to take a slightly more general point of view and, for this, we let E be any finite abelian extension of k .

Then, for a given prime p , the ring $\Lambda_{p,E}$ need not be regular. To take account of the difficulties that this causes, we refer to a height-one prime \mathfrak{p} of $\Lambda_{p,E}$ as ‘regular’ if the localisation of $\Lambda_{p,E}$ at \mathfrak{p} is a regular local ring and we label any height-one prime that is not regular as ‘singular’ (note that this terminology differs slightly from that in [BKS17, §3C1] if \mathfrak{p} contains p). For convenience, we record several general properties of localisation in such rings that will be useful in later arguments.

(6.2) Lemma. *For each prime p , the following claims are valid.*

- (a) If I and J are ideals of $\Lambda_{p,E}$, then one has $I^{**} \subseteq J^{**}$ if and only if $I_{\mathfrak{p}} \subseteq J_{\mathfrak{p}}$ for all prime ideals \mathfrak{p} of $\Lambda_{p,E}$ of height at most one.
- (b) Write $\Lambda_{p,E}$ for the subring $\mathbb{Z}_p[[\mathrm{Gal}(E_{\infty}^p/E)]]$ of $\Lambda_{p,E}$. Then each singular prime of $\Lambda_{p,E}$ contains p and is outside the support of any finitely generated $\Lambda_{p,E}$ -module that is both torsion, and has vanishing μ -invariant, as a $\Lambda_{p,E}$ -module.

Proof. Claim (a) follows, for example, from a general result of Sakamoto in [Sak20, Lem. C.13]. The first assertion of claim (b) follows from the argument of [BG03, Lem. 6.2 (ii)] and the second from an application of Nakayama’s Lemma (as in [BG03, Lem. 6.3]). \square

(6.3) Remark. In connection with Lemma (6.2) (a) we remark that the ‘reflexive hull’ I^{**} of an ideal I of $\Lambda_{p,E}$ can naturally be interpreted as the ideal of $\Lambda_{p,E}$ that is obtained as the image of $I^{**} \subseteq (\Lambda_{p,E})^{**}$ under the evaluation map $(\Lambda_{p,E})^{**} \cong \Lambda_{p,E}$.

We now fix a prime p and, for simplicity, suppress many explicit references to it in the notation of this section (thereby writing E_∞ in place of E_∞^p etc.)

We write $S(E_\infty)$ for the set of places of k that ramify in E_∞ and note that, by assumption, $S(k_\infty)$ only contains finite places.

We also fix a set T in $\mathcal{P}_E^{\text{ad}}$ that is disjoint from $S(k_\infty)$ and consider the p -completion

$$D_{E,S(E_\infty),T}^\bullet := \mathbb{Z}_p \otimes_{\mathbb{Z}}^{\mathbb{L}} C_{E,S(E_\infty),T}^\bullet$$

of the complex $C_{E,S(E_\infty),T}^\bullet$ introduced in §3.2, as well as the complex

$$D_{E_\infty,T}^\bullet := \text{R}\lim_{\substack{\longleftarrow \\ F}} D_{F,S(E_\infty),T}^\bullet,$$

where in the limit F ranges over all number fields contained in E_∞/k and the transition maps are induced by the relevant instances of the isomorphism in Lemma (3.4) (d).

Then, since each complex $D_{F,S(E_\infty),T}^\bullet$ can be represented by a bounded complex of finitely generated \mathbb{Z}_p -modules, and hence of compact Hausdorff spaces, the result of Lemma (3.4) (a) induces (upon passing to the limit over all intermediate fields of E_∞/E) both an identification $H^0(D_{E_\infty,T}^\bullet) = U_{E_\infty,S(E_\infty),T}$ and a natural exact sequence

$$0 \longrightarrow \text{Cl}_{E_\infty,S(E_\infty),T}^p \longrightarrow H^1(D_{E_\infty,T}^\bullet) \longrightarrow X_{E_\infty,S^*(E_\infty)}^p \longrightarrow 0. \quad (47)$$

In addition, since T belongs to $\mathcal{P}_E^{\text{ad}}$, for every finite subextension E' of E_∞/E , the group $U_{E',S(E'),T}$ is \mathbb{Z}_p -torsion free (cf. the general result of [NSW08, Prop. (1.6.12)]). By a well-known argument in homological algebra (see, for example, [BH21, Prop. 3.2]), it then follows that $D_{E_\infty,T}^\bullet$ admits a ‘quadratic standard representative’ (in the sense of [BS21a, §A.2]) of the form

$$P_{E_\infty} \longrightarrow P_{E_\infty}, \quad (48)$$

where P_{E_∞} is a free Λ_E -module of finite rank and the first term occurs in degree zero.

In the next result we shall also use the homomorphism of Λ_E -modules $\Theta_{E_\infty,T}$ that is defined by means of the following composite (where, for simplicity, we abbreviate Λ_E to Λ)

$$\begin{aligned} \epsilon_E \text{Det}_\Lambda(D_{E_\infty,T}^\bullet) &\hookrightarrow \epsilon_E Q(\Lambda) \otimes_\Lambda \text{Det}_\Lambda(D_{E_\infty,T}^\bullet) \\ &\cong (\epsilon_E Q(\Lambda) \otimes_\Lambda \text{Det}_\Lambda(H^0(D_{E_\infty,T}^\bullet))) \otimes_{Q(\Lambda)} (\epsilon_E Q(\Lambda) \otimes_\Lambda \text{Det}_\Lambda(H^1(D_{E_\infty,T}^\bullet))) \\ &\cong \left(\epsilon_E Q(\Lambda) \otimes_\Lambda \bigwedge_{\Lambda}^{r_E} U_{K_\infty,S(E_\infty),T} \right) \otimes \left(\epsilon_E Q(\Lambda) \otimes_\Lambda \bigwedge_{\Lambda}^{r_E} Y_{E_\infty,V_E}^p \right)^* \\ &\cong \epsilon_E Q(\Lambda) \otimes_\Lambda \bigwedge_{\Lambda}^{r_E} U_{E_\infty,S(E_\infty),T}. \end{aligned}$$

Here the second isomorphism is the natural ‘passage-to-cohomology’ map and the last isomorphism is due to our fixed choice of extensions of places in V_E to E_∞ (and hence of isomorphism $Y_{E_\infty,V_E}^p \cong \Lambda_K^{r_E}$). We recall that this map can be explicitly described in terms of certain ‘rank reduction maps’ (see [BS21a, Lem. A.7 (i)]) and that, by using this explicit description, it can be shown that the image of $\Theta_{E_\infty,T}$ is contained in $\bigcap_{\Lambda_E}^{r_E} U_{E_\infty,S(E_\infty),T}$, and that $\Theta_{E_\infty,T}$ agrees with the limit (over all finite layers F/k of E_∞/k) of the maps $\epsilon_E(\mathbb{Z}_p \otimes_{\mathbb{Z}} \Theta_{F,S(E_\infty),T}^{r_E})$ that are defined in §3.3.1 (see [BD21, Lem. 3.12 and Lem. 3.19] for details).

The following result will play a key role in the proof of Theorem (6.1).

(6.4) Proposition. *Fix a prime p and a finite abelian extension E of k with the property that $V_E = S_\infty(k)$ if $p = 2$.*

Then, for every set T in $\mathcal{P}_E^{\text{ad}}$ that is disjoint from $S(k_\infty^p)$, the following claims are valid.

(a) *Let $\mathfrak{z}_{E_\infty^p}$ be an $\epsilon_E \Lambda_{p,E}$ -basis of $\epsilon_E \text{Det}_{\Lambda_{p,E}}(D_{E_\infty^p,T}^\bullet)$ and set $z_{E_\infty^p}^b := \Theta_{E_\infty^p,T}(\mathfrak{z}_{E_\infty^p})$. Then one has*

$$\text{im}(z_{E_\infty^p}^b)^{**} = \text{Fitt}_{\Lambda_{p,E}}^0(\text{Cl}_{E_\infty^p,S(E_\infty^p),T}^p)^{**} \cdot \text{Fitt}_{\Lambda_{p,E}}^0(X_{E_\infty^p,S(E_\infty^p)}^p)^{**}. \quad (49)$$

(b) *Assume that, for every height-one prime \mathfrak{p} of $\Lambda_{p,E}$ that is contained in the support of $\epsilon_E \Lambda_{p,E}$, the $(\Lambda_{p,E})_{\mathfrak{p}}$ -module $(\text{Cl}_{E_\infty^p}^p)_{\mathfrak{p}}$ has finite projective dimension.*

(i) An element η of $(\bigcap_{\Lambda_{p,E}}^{r_E} U_{E_\infty, S(E_\infty), T})[1 - \epsilon_E]$ belongs to $\text{im}(\Theta_{E_\infty, T})$ if and only if

$$\text{im}(\eta)^{**} \subseteq \text{Fitt}_{\Lambda_{p,E}}^0(\text{Cl}_{E_\infty, S(E_\infty), T}^p)^{**} \cdot \text{Fitt}_{\Lambda_{p,E}}^0(X_{E_\infty, S(E_\infty)}^p)^{**}. \quad (50)$$

(ii) If the family of Rubin–Stark elements $\varepsilon_{E_\infty, T} := (\varepsilon_{F/k, S(E_\infty), T}^{V_F})_F$ satisfies (50), then there exists an $\epsilon_E \Lambda_{p,E}$ -basis $\mathcal{L}_{E_\infty/k, T}$ of $\epsilon_E \text{Det}_{\Lambda_{p,E}}(D_{E_\infty, T}^\bullet)$ such that

$$\Theta_{E_\infty, T}(\mathcal{L}_{E_\infty/k, T}) = \varepsilon_{E_\infty, T}.$$

Proof. As p is fixed during this proof, we drop all adornments p to lighten notation and we further abbreviate $\Lambda_{p,E}$ to Λ .

We then first note that Y_{E_∞, V_E} and, due to our assumption that $V_E = S_\infty(k)$ if $p = 2$, also $Y_{E_\infty, S_\infty(k) \setminus V_E}$ are both projective Λ -modules. We can therefore find a Λ -module Z such that the exact sequence (47) induces an isomorphism

$$H^1(D_{E_\infty, T}^\bullet) \cong Z \oplus Y_{E_\infty, S_\infty(k) \setminus V_E}^p \oplus Y_{E_\infty, V_E}^p \quad (51)$$

and, setting $S(E_\infty)_{\text{fin}} := S(E_\infty) \cap S_{\text{fin}}(k)$, an exact sequence

$$0 \longrightarrow \text{Cl}_{S(E_\infty), T}^p(E_\infty) \longrightarrow Z \longrightarrow X_{E_\infty, S(E_\infty)_{\text{fin}}}^p \longrightarrow 0. \quad (52)$$

Moreover, by choosing a section to the (surjective) composite map

$$P_{E_\infty} \rightarrow H^1(D_{E_\infty, T}^\bullet) \rightarrow Y_{E_\infty, V_E}^p$$

in which the first map is induced by the representative (48) of the complex $D_{E_\infty, T}^\bullet$ and the second by the isomorphism (51), we may identify Y_{E_∞, V_E}^p with a free direct summand of P_{E_∞} . In this way, we deduce the existence of a projective, and hence free (since Λ is semilocal), Λ -submodule P'_{E_∞} of P_{E_∞} for which there is an isomorphism of Λ -modules

$$P_{E_\infty} \cong P'_{E_\infty} \oplus Y_{E_\infty, V_E}^p, \quad (53)$$

and also an exact sequence of Λ -modules

$$0 \longrightarrow U_{E_\infty, S(E_\infty), T} \longrightarrow P_{E_\infty} \longrightarrow P'_{E_\infty} \longrightarrow Z \oplus Y_{E_\infty, S_\infty(k) \setminus V_E}^p \longrightarrow 0,$$

in which the third arrow is the composite of the differential of (48) and the projection $P_\infty \rightarrow P'_{E_\infty}$ induced by (53) and the fourth the restriction to P'_{E_∞} of the map $P_{E_\infty} \rightarrow Z \oplus Y_{E_\infty, S_\infty(k) \setminus V_E}^p$ induced by (48) and the decomposition (51).

By now applying [BD21, Lem. 2.7 (c)] to this sequence and recalling that $\Theta_{E_\infty, T}$ can be explicitly described as a rank reduction map (see [BS21a, Lem. A.7 (i)]), we derive an equality

$$\begin{aligned} \text{im}(z_{E_\infty}^b)^{**} &= \text{Fitt}_\Lambda^0(Z \oplus Y_{E_\infty, S_\infty(k) \setminus V_E}^p)^{**} \\ &= \text{Fitt}_\Lambda^0(Z)^{**} \cdot \text{Fitt}_\Lambda^0(Y_{E_\infty, S_\infty(k) \setminus V_E}^p)^{**}. \end{aligned} \quad (54)$$

Next we note that if \mathfrak{p} is a regular height-one prime of Λ , then $\Lambda_{\mathfrak{p}}$ is a discrete valuation domain. In particular, since over such a ring initial Fitting ideals are multiplicative on short exact sequences, we may deduce from the above equality and the exact sequence (52) that

$$\begin{aligned} \text{im}(z_{E_\infty}^b)_{\mathfrak{p}} &= (\text{im}(z_{E_\infty}^b)^{**})_{\mathfrak{p}} \\ &= \text{Fitt}_{\Lambda_{\mathfrak{p}}}^0(\text{Cl}_{E_\infty, S(E_\infty), T}^p)_{\mathfrak{p}} \cdot \text{Fitt}_{\Lambda_{\mathfrak{p}}}^0(X_{E_\infty, S(E_\infty)_{\text{fin}}}^p)_{\mathfrak{p}} \cdot \text{Fitt}_{\Lambda_{\mathfrak{p}}}^0(Y_{E_\infty, S_\infty(k) \setminus V_E}^p)_{\mathfrak{p}} \\ &= \text{Fitt}_{\Lambda_{\mathfrak{p}}}^0(\text{Cl}_{E_\infty, S(E_\infty), T}^p)_{\mathfrak{p}} \cdot \text{Fitt}_{\Lambda_{\mathfrak{p}}}^0(X_{E_\infty, S(E_\infty)}^p)_{\mathfrak{p}}, \end{aligned}$$

where the last equality is true because $S_\infty(k) \setminus V_E = S_\infty(k) \cap S(E_\infty)$ and so there exists a natural exact sequence

$$0 \longrightarrow X_{E_\infty, S(E_\infty)_{\text{fin}}}^p \longrightarrow X_{E_\infty, S(E_\infty)}^p \longrightarrow Y_{E_\infty, S_\infty(k) \setminus V_E}^p \longrightarrow 0. \quad (55)$$

Since $Q(\Lambda)$ is a semi-simple ring, similar arguments also show that the above description of $\text{im}(z_{E_\infty}^b)_{\mathfrak{p}}$ is valid for any prime \mathfrak{p} of Λ of height zero.

To consider singular height-one primes of Λ , we note that the Λ -module $Y_{E_\infty, S(E_\infty)_{\text{fin}}}^p$ is isomorphic to the direct sum $\bigoplus_v \mathbb{Z}_p[\mathcal{G}_{E_\infty}/\mathcal{G}_{E_\infty, v}]$, where v runs over $S(E_\infty)_{\text{fin}}$ and $\mathcal{G}_{E_\infty, v}$ is the decomposition group of v in \mathcal{G}_{E_∞} . In particular, since no finite place splits completely in k_∞/k , it follows that $Y_{E_\infty, S(E_\infty)_{\text{fin}}}^p$ is isomorphic to a direct sum of modules that are finitely generated over a power series ring (over \mathbb{Z}_p) in at most $d-1$ variables and therefore, by Lemma (6.2) (b), that the localisation of $Y_{E_\infty, S(E_\infty)_{\text{fin}}}^p$ at any height-one singular prime \mathfrak{p} of Λ vanishes.

This in turn implies that, for any such \mathfrak{p} , the localisation $(X_{E_\infty, S(E_\infty)_{\text{fin}}}^p)_{\mathfrak{p}}$ vanishes and hence that the exact sequence (52) induces a natural isomorphism $(\text{Cl}_{E_\infty, S(E_\infty), T}^p)_{\mathfrak{p}} \cong \mathbb{Z}_p$. This isomorphism then combines with the equality (54) to imply that

$$\begin{aligned} \text{im}(z_{E_\infty}^b)_{\mathfrak{p}} &= \text{Fitt}_{\Lambda}^0(\text{Cl}_{E_\infty, S(E_\infty), T}^p)_{\mathfrak{p}} \cdot \text{Fitt}_{\Lambda}^0(Y_{E_\infty, S_\infty(k) \setminus V_E}^p)_{\mathfrak{p}} \\ &= \text{Fitt}_{\Lambda}^0(\text{Cl}_{E_\infty, S(E_\infty), T}^p)_{\mathfrak{p}} \cdot \text{Fitt}_{\Lambda}^0(X_{E_\infty, S(E_\infty)}^p)_{\mathfrak{p}}, \end{aligned}$$

where the second equality is true since $(X_{E_\infty, S(E_\infty)_{\text{fin}}}^p)_{\mathfrak{p}}$ vanishes and (55) is exact.

At this stage we have established the last displayed equality for all primes of Λ of height at most one and so it follows from Lemma (6.2) (a) that the claimed equality (49) holds.

To proceed, we first note that the ‘only if’ part of claim (i) in (b) follows from (a). To prove the ‘if’ part, we observe that the definition of $\Theta_{E_\infty, T}$ ensures that $z_{E_\infty}^b$ is a generator of the $\epsilon_E Q(\Lambda)$ -module spanned by $(\bigcap_{\Lambda}^{r_E} U_{E_\infty, S(E_\infty), T})[1-\epsilon_E]$. Given an element η of $(\bigcap_{\Lambda}^{r_E} U_{E_\infty, S(E_\infty), T})[1-\epsilon_E]$, we can therefore fix an element q of $\epsilon_E Q(\Lambda)$ with the property that

$$\eta = q \cdot z_{E_\infty}^b. \quad (56)$$

This equality then combines with (49) and the assumed inclusion (50) to imply an inclusion

$$q \cdot \text{im}(z_{E_\infty}^b)^{**} = \text{im}(\eta)^{**} \subseteq \text{im}(z_{E_\infty}^b)^{**}. \quad (57)$$

We next claim that, for every height-one prime ideal \mathfrak{p} of Λ that is contained in the support of $\epsilon_E \Lambda$, the ideal $\text{Fitt}_{\Lambda}^0(Z)_{\mathfrak{p}}$ is principal and generated by a non-zero divisor. Indeed, if \mathfrak{p} is regular, this is automatically satisfied because $\Lambda_{\mathfrak{p}}$ is a discrete valuation ring. If \mathfrak{p} is singular, on the other hand, then the exact sequence (52) implies that $Z_{\mathfrak{p}}$ is isomorphic to $(\text{Cl}_{E_\infty, S(E_\infty), T}^p)_{\mathfrak{p}}$. In addition, since we assume that no finite place splits completely in k_∞/k , Lemma (6.2) (b) implies that the natural maps $(\text{Cl}_{E_\infty, \emptyset, T}^p)_{\mathfrak{p}} \rightarrow (\text{Cl}_{E_\infty, S(E_\infty), T}^p)_{\mathfrak{p}}$ and $(\text{Cl}_{E_\infty, \emptyset, T}^p)_{\mathfrak{p}} \rightarrow (\text{Cl}_{E_\infty}^p)_{\mathfrak{p}}$ are both bijective (cf. the argument of [BD21, Lem. 4.11]). Thus, the claim follows in this case from our assumption that the projective dimension of the $\Lambda_{\mathfrak{p}}$ -module $\text{Cl}^p(E_\infty)_{\mathfrak{p}}$ is finite and hence at most one (as a consequence of the Auslander–Buchsbaum formula since Λ is Gorenstein and \mathfrak{p} has height one).

In addition, one has $\text{Fitt}_{\Lambda}^0(Y_{E, S_\infty(k) \setminus V_E}^p) = \Lambda \epsilon_E$, and so it follows from (54) that, if \mathfrak{p} is in the support of $\epsilon_E \Lambda$, then the ideal $\text{im}(z_{E_\infty}^b)_{\mathfrak{p}}$ is generated by a non-zero divisor in $\Lambda_{\mathfrak{p}} = (\epsilon_E \Lambda)_{\mathfrak{p}}$. We therefore deduce from (57), by cancellation, that q belongs to $\Lambda_{\mathfrak{p}}$. Since, by construction, q belongs to $\epsilon_E Q(\Lambda)$, the last assertion is also clear both for height-one primes \mathfrak{p} that are not in the support of $\epsilon_E \Lambda$ and for primes of height zero, and so Lemma (6.2) (a) implies that q belongs to $(q\Lambda)^{**} \subseteq (\epsilon_E \Lambda)^{**} = \epsilon_E \Lambda$, as required to prove claim (b) (i).

To prove claim (b) (ii) we note that, for each finite extension E' of E in E_∞ one has $V_{E'} = V_E$ and hence $r_{E'} = r_E$. Using this fact, we write $z_{E'}^b$ for the image of $z_{E_\infty}^b$ under the natural projection map

$$\bigcap_{\Lambda}^{r_E} U_{E_\infty, S(E_\infty), T} \rightarrow \bigcap_{\mathbb{Z}_p[\mathcal{G}_{E'}]}^{r_{E'}} U_{E', S(E_\infty), T}.$$

Then, by claim (b) (i), we know that $\varepsilon_{E'/k, S(E_\infty), T}^{V_{E'}}$ belongs to $\mathbb{Z}_p[\mathcal{G}_{E'}] \cdot z_{E'}^b = \mathbb{Z}_p \cdot \text{im}(\Theta_{E', S(E_\infty), T}^{r_{E'}})$ and hence, by Proposition (3.6), that $\mathbb{Z}_p \cdot \text{im}(\Theta_{E', S(E_\infty), T}^{r_{E'}}) = \mathbb{Z}_p[\mathcal{G}_{E'}] \cdot \varepsilon_{E'/k, S(E_\infty), T}^{V_{E'}}$.

By passing to the limit over all such fields E' and recalling that $\Theta_{E_\infty, T}$ agrees with the limit of the maps $\mathbb{Z}_p \otimes_{\mathbb{Z}} \Theta_{E', S(E_\infty), T}^{r_{E'}}$, these equalities combine to imply that the image of $\Theta_{E_\infty, T}$ is

generated as a \mathbb{A} -module by $\varepsilon_{E_\infty, T}$. As a consequence, the element q that verifies (56) with η taken to be $\varepsilon_{E_\infty, T}$ must be a unit of $\epsilon_E \mathbb{A}$.

Given this observation, it is then easily checked that the element $\mathcal{L}_{E_\infty/k, T} := q^{-1} \cdot \mathfrak{z}_{E_\infty}$ has the properties that are required to validate claim (b) (ii). \square

6.1.3. Criteria for the validity of the equivariant Tamagawa Number Conjecture

We next establish a concrete link between Theorems (5.7) and (6.1) (a).

To do this we fix a subset \mathcal{X} of $\Omega(k)$ that satisfies Hypothesis (4.5) with respect to a (singleton) subset $\mathcal{V} = \{V\}$ of $\mathcal{P}(S_\infty(k))$. We also fix a prime p and, as in Proposition (3.9), define a homomorphism of $\mathbb{Z}_p[[\mathcal{G}_K]]$ -modules

$$\Theta_k^p: \varprojlim_{E \in \Omega(k)} \text{Det}_{\mathbb{Z}_p[[\mathcal{G}_E]]}(\mathbb{Z}_p \otimes_{\mathbb{Z}} C_{E, S^*(E)}^\bullet) \rightarrow \text{ES}_k(\mathbb{Z}_p),$$

where the limit is taken with respect to the maps $i_{F/E}$ defined in (14). We also use the idempotents $\epsilon_{E, \mathcal{V}}$ of $\mathbb{Q}[[\mathcal{G}_E]]$ defined in (9) and the field $k\langle p \rangle$ defined in (24).

(6.5) Lemma. *Let p be odd and write k_p for the composite of all subextensions of $k\langle p \rangle$ in which at least one p -adic place of k splits completely. Then $\text{TNC}(h^0(\text{Spec } K), \mathbb{Z}_p[[\mathcal{G}_K]]\epsilon_{K, \mathcal{V}})$ is valid for all $K \in \mathcal{X}$ provided that the following two conditions are satisfied:*

- (i) *The image of Θ_k^p is contained in $\mathbb{Z}_p[[\mathcal{G}_K]] \cdot \varepsilon_k^{\mathcal{X}} + \text{ES}_k^{\mathcal{X}}(\mathbb{Z}_p)^{\text{Gal}(K/k_p)}$.*
- (ii) *At least one of the following holds:*

- *$\text{TNC}(h^0(\text{Spec } F), \epsilon_{K, \mathcal{V}} \mathbb{Z}_p[[\mathcal{G}_F]])$ holds for all extensions of the form $F = k_p \cdot E$ with E a tamely ramified cyclic p -extension of k that belongs to $\Omega^{S_\infty(k)}(k)$.*
- *$\text{TNC}(h^0(\text{Spec } F), \epsilon_{F, \mathcal{V}} \mathbb{Z}_p[[\mathcal{G}_F]])$ holds for all extensions of the form $F = k_p \cdot k_n^p$ with k_n^p the n -th layer of a \mathbb{Z}_p -extension k_∞^p of k in which no finite place splits completely.*

Proof. Since $\text{im}(\Theta_k) \subseteq \text{im}(\Theta_k^p)$, Proposition (3.10) implies that $\text{TNC}(h^0(\text{Spec } K), \mathbb{Z}_p[[\mathcal{G}_K]]\epsilon_{K, \mathcal{V}})$ is valid for all $K \in \mathcal{X}$ provided that $\varrho^{\mathcal{X}}(\text{im}(\Theta_k^p))$ is contained in $\mathbb{Z}_p[[\mathcal{G}_K]] \cdot \varepsilon_k^{\mathcal{X}}$.

To verify the latter inclusion we note that, for every field $E \in \Omega(k)$, the $\mathbb{Z}_p[[\mathcal{G}_E]]$ -module $\text{Det}_{\mathbb{Z}_p[[\mathcal{G}_E]]}(\mathbb{Z}_p \otimes_{\mathbb{Z}} C_{E, S^*(E)}^\bullet)$ is free of rank one. Since all of the transition maps $i_{F/E}$ are surjective, it follows that the $\mathbb{Z}_p[[\mathcal{G}_K]]$ -module $\varprojlim_{E \in \Omega(k)} \text{Det}_{\mathbb{Z}_p[[\mathcal{G}_E]]}(\mathbb{Z}_p \otimes_{\mathbb{Z}} C_{E, S^*(E)}^\bullet)$ is free of rank one (see, for example, the argument of [Bur+23, Prop. 3.7]). We may therefore choose a basis $\mathfrak{z} = (\mathfrak{z}_E)_{E \in \Omega(k)}$ of the latter module and write z^b for the associated Euler system $\Theta_k^p(\mathfrak{z})$. Then, by condition (i), there exists an element

$$r_p = (r_{p, E})_{E \in \Omega(k)} \in \varprojlim_{E \in \Omega(k)} (\mathbb{Z}_p[[\mathcal{G}_E]]\epsilon_{E, \mathcal{V}})$$

such that $r_{p, E} \cdot \varepsilon_{E/k} - z_E^b$ is fixed by $\text{Gal}(E/E \cap k_p)$ for all $E \in \mathcal{X}$. It therefore suffices to show that either of the conditions stated in (ii) implies that $r_{p, E} \cdot \varepsilon_{E/k} - z_E^b$ vanishes for all $E \in \mathcal{X}$, or equivalently that

$$e_\chi \cdot (r_{p, E} \cdot \varepsilon_{E/k} - z_E^b) = 0 \tag{58}$$

for all $E \in \mathcal{X}$ and all characters χ of \mathcal{G}_E .

Now, since $r_{p, E} \cdot \varepsilon_{E/k} - z_E^b$ is fixed by $\text{Gal}(E/E \cap k_p)$, the equality (58) is clear unless χ factors through \mathcal{G}_{k_p} and so we may assume that χ factors through \mathcal{G}_{k_p} . Then, since $e_\chi \cdot (r_{p, E} \cdot \varepsilon_{E/k} - z_E^b)$ vanishes if $r_{p, E_\chi} \cdot \varepsilon_{E_\chi/k} - z_{E_\chi}^b$ vanishes (cf. the argument of Lemma (2.6)), we can further assume that E is a subfield of k_p and hence that every finite place in $S(E)$ is p -adic.

To proceed, we write M for the subset of \mathcal{X} comprising all extensions F of E for which $\text{TNC}(h^0(\text{Spec } F), \mathbb{Z}_p[[\mathcal{G}_F]]\epsilon_{F, \mathcal{V}})$ is valid and for which one has that $\epsilon_{F, \mathcal{V}} \cdot (r_{p, F} \cdot \varepsilon_{F/k} - z_F^b)$ vanishes. Then, for any such field F in M , the preimage $\mathcal{L}_{F/k}^V$ of $\epsilon_{F, \mathcal{V}} \theta_{F/k, S(F)}^*$ under the isomorphism

$$\mathbb{C}_p \otimes_{\mathbb{Z}} \text{Det}_{\mathbb{Z}[[\mathcal{G}_F]]}(C_{F, S(F)}^\bullet) \cong \mathbb{C}_p[[\mathcal{G}_F]]$$

that is induced by the Dirichlet regulator $\lambda_{F,S(F)}$ (as in Lemma (3.7)(c)) is a $\mathbb{Z}_p[\mathcal{G}_F]\epsilon_{F,V}$ -basis of $\epsilon_{F,V} \text{Det}_{\mathbb{Z}_p[\mathcal{G}_F]}(\mathbb{Z}_p \otimes_{\mathbb{Z}} C_{F,S^*(F)}^\bullet)$. We can therefore fix a unit $q_{p,F}$ in $(\mathbb{Z}_p[\mathcal{G}_F]\epsilon_{F,V})^\times$ such that $q_{p,F} \cdot \mathfrak{z}_{F/k} = \mathcal{L}_{F/k}^V$ and, for this unit, one has both

$$q_{p,F} \cdot z_F^b = \epsilon_{F,V} \epsilon_{F/k} \quad \text{and} \quad \pi_{F/E}(q_{p,F}) \cdot z_E^b = \epsilon_{E,V} \epsilon_{E/k}, \quad (59)$$

(cf. [BKS16, Thm. 5.14]). The first of these equations combines with the assumed vanishing of $\epsilon_{F,V} \cdot (r_{p,F} \epsilon_{F/k} - z_F^b)$ to imply that we have

$$\epsilon_{F,V} \cdot z_F^b = \epsilon_{F,V} \cdot r_{p,F} \epsilon_{F/k} = r_{p,F} q_{p,F} \cdot z_E^b.$$

Since z_E^b generates a free $\epsilon_F \mathbb{Z}_p[\mathcal{G}_F]$ -module of rank one, we deduce that the difference $q_{p,F}^{-1} - r_{p,F}$ is annihilated by $\epsilon_{F,V} \epsilon_F$.

Now, if χ is a character of \mathcal{G}_F with $e_\chi \epsilon_{F,V} (1 - e_F) \neq 0$, then χ must vanish on the decomposition group of at least one finite place in $S(F)$. The element $q_{p,F}^{-1} - r_{p,F}$ of $\mathbb{Z}_p[\mathcal{G}_F]\epsilon_{F,V}$ is therefore fixed by every element of the subgroup $H_F := \bigcap_{v \in S(F) \setminus S_\infty(k)} \mathcal{G}_{F,v}$ of \mathcal{G}_F , where $\mathcal{G}_{F,v}$ denotes the decomposition subgroup in \mathcal{G}_F of each place v . As a consequence, the element

$$\pi_{F/E}(q_{p,F}^{-1}) - r_{p,E} = \pi_{F/E}(q_{p,F}^{-1} - r_{p,F})$$

is divisible by the order of $H_{F,1} := H_F \cap \text{Gal}(F/E)$ and so the second equality in (59) implies

$$z_E^b - r_{p,E} \cdot \epsilon_{E,V} \epsilon_{E/k} = z_E^b - r_{p,E} \pi_{F/E}(q_{p,F}) \cdot z_E^b = \pi_{F/E}(q_{p,F}) (\pi_{F/E}(q_{p,F}^{-1}) - r_{p,E}) \cdot z_E^b$$

is divisible by $|H_{F,1}|$ in the lattice \mathfrak{L}_K . In particular, if we can show that, under either of the conditions stated in (ii), the p -part of $|H_{F,1}|$ is unbounded as F ranges over M , then we could deduce the required vanishing of $r_{p,E} \cdot \epsilon_{E,V} \epsilon_{E/k} - z_E^b$.

To do this, we let n be any natural number. Then the result of Proposition (4.14) (with $\mathcal{T} = S_p(k)$, $K = k_p$ and $\sigma = 1$) provides a cyclic Galois extension L_n of k in which all p -adic places have decomposition group of order at least p^n and all places in $S(L_n)$ are non-archimedean, totally split in k_p and have inertia subgroup of order at least p^n .

In particular, if we assume the first condition in (ii), then $\text{TNC}(h^0(\text{Spec } k_p L_n), \mathbb{Z}_p[\mathcal{G}_{k_p L_n}] \epsilon_{k_p L_n, V})$, and hence also $\text{TNC}(h^0(\text{Spec } F), \mathbb{Z}_p[\mathcal{G}_F] \epsilon_{F,V})$ is valid for the compositum $F_n := L_n \cdot E$. In addition, F_n belongs to \mathcal{X} by Hypothesis (4.5) (i). To prove that F_n belongs to the set M , we need to justify the vanishing of $\epsilon_{F_n, V} \cdot (r_{p,F_n} \epsilon_{F_n/k} - z_{F_n}^b)$ in this case. To do this, it suffices to prove that $e_\chi \cdot (r_{p,F_n} \epsilon_{F_n/k} - z_{F_n}^b)$ for any character χ of \mathcal{G}_F with $e_\chi \epsilon_{F, \mathcal{X}} e_F$. Since $(r_{p,F_n} \epsilon_{F_n/k} - z_{F_n}^b)$ is fixed by $\text{Gal}(F_n/F_n \cap k_p)$, the required vanishing is valid for any character of \mathcal{G}_{F_n} that does not factor through k_p . If it a character χ of \mathcal{G}_{F_n} factors through k_p , on the other hand, then by construction it vanishes on the decomposition group of a place in $S(L_n) \subseteq S(F_n)$ and so one has $e_\chi e_{F_n} = 0$. This proves that F_n belongs to M , as claimed.

Moreover, in this case $H_{F_n,1}$ contains the unique cyclic subgroup of $\text{Gal}(F_n/K)$ that is of order p^n (note that K/k is unramified at any place in $S(E_n)$ and hence that said subgroup of order p^n , being contained in the inertia subgroup of such a place, must be contained in $\text{Gal}(F_n/K)$) and so the order of $H_{F_n,1}$ is divisible by an arbitrarily large power of p as n varies, as required. To consider the second condition in (ii), we recall k_∞^p is assumed to be a \mathbb{Z}_p -extension in which no finite place splits completely, and hence that there exists an integer m with the property that every p -adic place has full decomposition group in $k_p^\infty k_p / k_m^p k_p$. Thus, if we assume the second condition in (ii), then similar arguments show that $F_n := k_{n+m}^p k_p$ is a field in M with the property that $|H_{F_n,1}|$ is divisible by p^n , as required.

This concludes the proof of the stated result. \square

6.1.4. The proof of Theorem (6.1)

In this section, we prove Theorem (6.1) and also establish some useful facts about the ideals that occur in Theorem (6.1) (a) and (b).

We start by proving Theorem (6.1) and so fix data as in the statement of that result.

It is convenient to prove claim (b) first and to do this we fix an Euler system c in $\text{ES}_k^{\mathcal{X}}(\mathbb{Z}_{\mathcal{S}})$. We then also fix a prime $p \notin \mathcal{S}$, a field K in $\mathcal{X}_{S_p(k)}$ and an admissible set T in $\mathcal{P}_K^{\text{ad}}$ with respect to which c satisfies the condition (46). (For brevity, we shall often shorten the notations used below by omitting adornments K or T where, we feel, no confusion is possible.)

Then, by Proposition (6.4) (a), we know that $c_{K_{\infty}^p, T}$ belongs to $\text{im}(\Theta_{K_{\infty}^p, T})$. In addition, Proposition (6.4) (b) combines with the given assumption that the Rubin–Stark system satisfies (46) to imply that $\text{im}(\Theta_{K_{\infty}^p, T})$ is generated over $\Lambda_{p, K}$ by $\varepsilon_{K_{\infty}, T}$, and hence that there exists an element $q_{p, K}$ of $\varepsilon_K \Lambda_{p, K}$ such that $c_{K_{\infty}^p, T} = q_{p, K} \cdot \varepsilon_{K_{\infty}^p, T}$.

Taking images under the projection map $\bigcap_{\Lambda_{p, K}}^{r_K} U_{K_{\infty}^p, S(K), T} \rightarrow \bigcap_{\mathbb{Z}_p[\mathcal{G}_K]}^{r_K} U_{K, S(K), T}$, it follows that $\delta_{T, K} \cdot c_K = \delta_{T, K} \cdot q_{p, K} \cdot \varepsilon_{K/k}$ and hence, since $\delta_{T, K}$ is a non-zero divisor in $\mathbb{Z}_p[\mathcal{G}_K]$, that

$$c_K = q_{p, K} \cdot \varepsilon_{K/k} \in \mathbb{Z}_p[\mathcal{G}_K] \cdot \varepsilon_{K/k}.$$

In particular, since this containment is valid for all primes p outside \mathcal{S} , we find that condition (b) (iii) in Theorem (5.7) is satisfied in this case. The implication ‘(iii) \Rightarrow (i)’ in Theorem (5.7) now gives that c belongs to $\mathbb{Z}_{\mathcal{S}}[[\mathcal{G}_{\mathcal{K}}]]\varepsilon_k^{\mathcal{X}}$, as claimed in Theorem (6.1) (b).

Turning now to the proof of claim (a) of Theorem (6.1), we recall that, by Proposition (3.10), it is enough to show that if c is any Euler system that belongs to the image of the map Θ_k defined in Proposition (3.9), then $\varrho^{\mathcal{X}}(c)$ belongs to $\mathbb{Z}_p[[\mathcal{G}_{\mathcal{K}}]] \cdot \varepsilon_k^{\mathcal{X}}$. In particular, it is sufficient to verify the latter inclusion for every system c in the image of Θ_k^p .

As a first step in this direction we note the exact triangle in Lemma (3.4) (b) combines with Remark (3.5) to imply that, for any finite subextension F of K_{∞}^p/k , one has $\text{im}(\Theta_{F, S(K), T}) = \delta_{T, F} \cdot \text{im}(\Theta_{F, S(K), \emptyset})$. In particular, the p -adic completion of this module contains $c_{F, T}$ for every such F and, by taking the limit over such F , we deduce that the element $c_{K_{\infty}^p, T}$ belongs to $\text{im}(\Theta_{K_{\infty}^p, T})$.

Next we recall that, as observed earlier, the assumed validity of (46) for the Rubin–Stark system implies that $\text{im}(\Theta_{K_{\infty}^p, T})$ is equal to $\Lambda_{p, K} \cdot \varepsilon_{K_{\infty}, T}$, and hence, by an argument similar to above, that c_K belongs to $\mathbb{Z}_p[\mathcal{G}_K] \cdot \varepsilon_{K/k}$.

By Remark (5.8), one therefore has that c belongs to $\mathbb{Z}_p[[\mathcal{G}_{\mathcal{K}}]]\varepsilon_k^{\mathcal{X}} + \text{ES}_k^{\mathcal{X}}(\mathbb{Z}_p)^{\text{Gal}(K/k\langle p \rangle)}$.

In the remainder of this argument we now explain how this containment combines with either of the conditions stated in Theorem (6.1) (a) to imply the inclusion $c \in \mathbb{Z}_p[[\mathcal{G}_{\mathcal{K}}]]\varepsilon_k^{\mathcal{X}}$ that is required to complete the proof of the claim.

If $p \nmid 2d_k$, as assumed in condition (a) (i) of Theorem (6.1), then $k\langle p \rangle = k$ and so c belongs to $\mathbb{Z}_p[[\mathcal{G}_{\mathcal{K}}]]\varepsilon_k^{\mathcal{X}} + \text{ES}_k^{\mathcal{X}}(\mathbb{Z}_p)^{\mathcal{G}_{\mathcal{K}}}$. Since c is symmetric by Proposition (3.9), we deduce from Lemma (2.17) that we must in fact have that c belongs to $\mathbb{Z}_p[[\mathcal{G}_{\mathcal{K}}]]\varepsilon_k^{\mathcal{X}}$, as required.

Let us next suppose that condition (a) (ii) is valid. That is, $V \neq \emptyset$ and ε_k is a congruence system. In this case we may use Proposition (5.5) to deduce that also in this case c belongs to $\mathbb{Z}_p[[\mathcal{G}_{\mathcal{K}}]]\varepsilon_k^{\mathcal{X}} + \text{ES}_k^{\mathcal{X}}(\mathbb{Z}_p)^{\mathcal{G}_{\mathcal{K}}}$. Since c is assumed to be symmetric, the same argument as above then shows that c belongs to $\mathbb{Z}_p[[\mathcal{G}_{\mathcal{K}}]]\varepsilon_k^{\mathcal{X}}$. This proves the claim in the cases of conditions (i) and (ii).

Lastly, if we assume condition (iii), then the containment $\text{im} \Theta_k^p \subseteq \mathbb{Z}_p[[\mathcal{G}_{\mathcal{K}}]]\varepsilon_k^{\mathcal{X}} + \text{ES}_k^{\mathcal{X}}(\mathbb{Z}_p)^{\text{Gal}(K/k\langle p \rangle)}$ directly combines with Lemma (6.5) to also imply the claim in this case, thereby concluding the proof of Theorem (6.1). \square

The next result establishes some useful facts about the ideals in Theorem (6.1) (a) and (b).

(6.6) Lemma. *Let E be a finite abelian extension of k , $p \notin \mathcal{S}$ a prime number, $T \in \mathcal{P}_E^{\text{ad}}$ an admissible set disjoint from $S(k_{\infty}^p)$, and c an Euler system in $\text{ES}_k(\mathbb{Z}_{\mathcal{S}})$. Then the following claims are valid.*

- (a) $\text{Fitt}_{\Lambda_{p, E}}^0(X_{E_{\infty}^p, S(k_{\infty}^p)}^p)^{**} \cdot \text{im}(c_{E_{\infty}^p, T})^{**} \subseteq \text{Fitt}_{\Lambda_{p, E}}^0(X_{E_{\infty}^p, S(E_{\infty}^p)}^p)^{**}$.
- (b) If $|S(k_{\infty}^p)| = 1$, then $\text{im}(c_{E_{\infty}^p, T})^{**} \subseteq \text{Fitt}_{\Lambda_{p, E}}^0(X_{E_{\infty}^p, S(E_{\infty}^p)}^p)^{**}$.

(c) If the \mathbb{Z}_p -rank of $\mathcal{G}_{E_\infty^p}$ is at least two, and every place in $S(k_\infty^p) \cap S_{\text{fin}}(k)$ is finitely decomposed in k_∞^p/k , then $\text{Fitt}_{\Lambda_{p,E}}^0(X_{E_\infty^p, S(k_\infty^p)}^{p,*}) = \Lambda_{p,E}$.

(d) If the \mathbb{Z}_p -rank of $\mathcal{G}_{E_\infty^p}$ is equal to one, and the module $(\text{Cl}_{E_\infty^p, S(k_\infty^p)}^p)^{\text{Gal}(E_\infty^p/E_n)}$ is finite for every natural number n (here E_n denotes the n -th layer of E_∞^p/E), then

$$\begin{aligned} & \text{Fitt}_{\Lambda_{p,E}}^0(\text{Cl}_{E_\infty^p, S(E_\infty^p), T}^p)^{**} \cdot \text{Fitt}_{\Lambda_{p,E}}^0(X_{E_\infty^p, S(E_\infty^p)}^{p,*})^{**} \\ &= \text{Fitt}_{\Lambda_{p,E}}^0(\text{Cl}_{E_\infty^p, S(E_\infty^p), T}^p)^{**} \cap \text{Fitt}_{\Lambda_{p,E}}^0(X_{E_\infty^p, S(E_\infty^p)}^{p,*})^{**}. \end{aligned}$$

Proof. As p is fixed in this proof, we often suppress explicit reference to it in the notation, and we also abbreviate $\Lambda_{p,E}$ to simply Λ .

The exact sequence

$$0 \longrightarrow X_{E_\infty, S(k_\infty)}^p \longrightarrow X_{E_\infty, S(E_\infty)}^p \longrightarrow Y_{E_\infty, S(E_\infty) \setminus S(k_\infty)}^p \longrightarrow 0$$

reduces part (a) to the claim that, for each height one prime \mathfrak{p} of Λ , one has

$$\text{im}(c_{E_\infty, T})_{\mathfrak{p}} \subseteq \text{Fitt}_{\Lambda}^0(Y_{E_\infty, S(E_\infty) \setminus S(k_\infty)}^p)_{\mathfrak{p}}.$$

To verify this, we write Δ for the (finite) torsion subgroup of \mathcal{G}_{E_∞} and fix a splitting of groups $\mathcal{G}_{E_\infty} \cong \Delta \times \Gamma$. We note that if \mathfrak{p} belongs to the support of $Y_{E_\infty, S(E_\infty) \setminus S(k_\infty)}^p$, then $p \notin \mathfrak{p}$ and there exists a character χ in $\widehat{\Delta}$ and a height-one prime \wp_χ of the ring $\Lambda_\chi := \mathbb{Z}_p[\text{im } \chi][[\Gamma]]$ such that $\Lambda_{\mathfrak{p}} = \Lambda_{\chi, \wp_\chi}$ (cf. [BKS17, §3C1]).

Setting $E_\chi := E^{\ker(\chi)}$ and $L := E_\infty^\Gamma$, the elements $N_{\text{Gal}(L/E_\chi)}$ and e_χ are units in $\Lambda_{\mathfrak{p}}$ and so

$$c_{E_\infty, T} \in \Lambda_{\mathfrak{p}} \cdot N_{\text{Gal}(L/E_\chi)} c_{E_\infty, T} = \Lambda_{\chi, \wp_\chi} \cdot P_{L/E_\chi, \wp} \cdot c_{E_{\chi, \infty}, T},$$

where the Euler factor $P_{L/E_\chi, \wp}$ is as defined in Definition (4.3) (a). This required inclusion is therefore true since

$$\Lambda_{\chi, \wp_\chi} \cdot P_{L/E_\chi, \wp} = e_\chi \text{Fitt}_{\Lambda_\chi}^0(Y_{E_{\chi, \infty}, S(E_\infty) \setminus S(k_\infty)}^p)_{\wp_\chi} = \text{Fitt}_{\Lambda}^0(Y_{E_\infty, S(E_\infty) \setminus S(k_\infty)}^p)_{\mathfrak{p}}.$$

By the above discussion, it is sufficient to prove, for each $\chi \in \widehat{\Delta}$, that the ideal $\text{im}(c_{E_{\chi, \infty}, T})_{\wp_\chi}$ is contained in $e_\chi \text{Fitt}_{\Lambda_\chi}^0(X_{E_{\chi, \infty}, S(k_\infty)}^p)_{\wp_\chi}$ in order to establish claim (b). Note that the module $e_\chi X_{E_{\chi, \infty}, S(k_\infty) \setminus S_\infty}^p$ vanishes if the unique p -adic place $\wp \in S(k_\infty) \setminus S_\infty$ satisfies $\chi(\wp) \neq 1$ and so we may therefore assume that \wp is completely split in E_χ/k .

We now write n for the unique integer with the property that $E_{\chi, n}$ is the decomposition field of \wp in $E_{\chi, \infty}/k$. The ideal $\text{Fitt}_{\Lambda_\chi}^0(X_{E_{\chi, \infty}, S(k_\infty)}^p)_{\wp_\chi}$ is then generated by $\gamma^{p^n} - 1$, where γ is any choice of topological generator of $\Gamma_{E_\chi} := \text{Gal}(E_{\chi, \infty}/E_\chi)$. It therefore suffices to prove that $c_{E_{\chi, \infty}, T}$ is divisible by $\gamma^{p^n} - 1$ in $\bigcap_{\Lambda_\chi}^{r_{E_\chi}} U_{E_{\chi, \infty}, S(E_{\chi, \infty})}$. To do this, we observe that

$$N_{E_{\chi, m}/E_{\chi, n}}(c_{E_{\chi, m}, T}) = (1 - \text{Frob}_\wp^{-1}) \cdot c_{E_{\chi, n}} = 0$$

for all $m > n$ since \wp is assumed to split completely in $E_{\chi, n}/k$. This shows that $c_{E_{\chi, \infty}, T}$ is in the kernel of the natural corestriction map

$$\bigcap_{\Lambda_\chi}^{r_{E_\chi}} U_{E_{\chi, \infty}, S(E_{\chi, \infty}), T} \rightarrow \bigcap_{\mathbb{Z}_p[\mathcal{G}_{E_\chi, n}]}^{r_{E_\chi}} U_{E_{\chi, n}, S(E_{\chi, \infty}), T}$$

which, by the argument of [BD21, Thm. 3.8 (b)], is equal to $(\gamma^{p^n} - 1) \bigcap_{\Lambda_\chi}^{r_{E_\chi}} U_{E_{\chi, \infty}, S(K_{\chi, p}), T}$. This completes the proof of (b).

To prove (c), it is enough to note that the given assumption ensures that $X_{E_\infty, S(k_\infty)}^p$ is a finitely generated \mathbb{Z}_p -module and hence pseudo-null as a Λ -module if $\text{rk}_{\mathbb{Z}_p}(\mathcal{G}_{E_\infty}) > 1$.

Regarding claim (d), we first observe that, by Lemma (6.2) (a), it suffices to verify the stated equality after localisation at each height-one prime \mathfrak{p} of Λ .

To do this we note that, since no finite place splits completely in k_∞/k , the Λ -module $X_{E_\infty, S(E_\infty)}^p$ is annihilated by an element of the form $\gamma^{p^n} - 1$, where γ denotes a choice of topological generator of Γ . From this, it follows that any prime \mathfrak{p} in the support of $X_{E_\infty, S(E_\infty)}^p$ must contain an element of the form $\gamma^{p^n} - 1$.

We fix such a prime \mathfrak{p} . Then, since the stated assumption implies that the kernel and cokernel of multiplication by $\gamma^{p^n} - 1$ on $\mathrm{Cl}_{E_\infty, S(k_\infty)}^p$, and hence also on $\mathrm{Cl}_{E_\infty, S(E_\infty), T}^p$ (cf. [BH21, Lem. 4.5]), are finite, it follows that $\gamma^{p^n} - 1$ acts bijectively on $(\mathrm{Cl}_{E_\infty, S(E_\infty), T}^p)_{\mathfrak{p}}$. Hence, since $\gamma^{p^n} - 1$ belongs to \mathfrak{p} , the tensor product $(\mathrm{Cl}_{E_\infty, S(E_\infty), T}^p)_{\mathfrak{p}} \otimes_{\Lambda_{\mathfrak{p}}} (\Lambda_{\mathfrak{p}}/(\mathfrak{p}\Lambda_{\mathfrak{p}}))$ vanishes. By appealing to Nakayama's Lemma, this in turn implies that $(\mathrm{Cl}_{E_\infty, S(E_\infty), T}^p)_{\mathfrak{p}}$ vanishes.

This argument shows that any height-one prime \mathfrak{p} of Λ that belongs to the support of $X_{E_\infty, S(E_\infty)}^p$ cannot also belong to the support of $\mathrm{Cl}_{E_\infty, S(E_\infty), T}^p$. The displayed equality in claim (d) follows easily from this fact. \square

(6.7) Remark. The module $(\mathrm{Cl}_{E_\infty, S(k_\infty)}^p)^{\mathrm{Gal}(E_\infty^p/E)}$ that occurs in Lemma (6.6) (d) is known to be finite if either E is an abelian extension of \mathbb{Q} or if a single place of E ramifies in E_∞^p , and also in several other situations including cases in which k is imaginary quadratic (cf. [BH21, Rks. 4.4 and 4.13] for an overview of results). Further, if k_∞^p is the cyclotomic \mathbb{Z}_p -extension of k , then the Gross–Kuz'min Conjecture predicts that this module is always finite.

6.2. Results in rank zero and the minus part of Kato's Conjecture

In this section we focus on the case of CM extensions of totally real fields and, in particular, prove claim (a) of Theorem B in the Introduction.

To do this we fix a (finite, abelian) CM extension K of a totally real field k . We write K^+ for the maximal totally real subfield of K and τ for the (unique) non-trivial element of $\mathrm{Gal}(K/K^+)$. We then write e^- for the idempotent $(1 - \tau)/2$ and define the *minus part* of a $\mathbb{Z}[\mathcal{G}_K]$ -module M by setting

$$M^- := (\mathbb{Z}[1/2][\mathcal{G}_K]e^-) \otimes_{\mathbb{Z}[\mathcal{G}_K]} M.$$

We note that the assignment $M \mapsto M^-$ gives an exact functor from the category of $\mathbb{Z}[\mathcal{G}_K]$ -modules to the category of modules over the ring $\mathbb{Z}[\mathcal{G}_K]^- = \mathbb{Z}[1/2][\mathcal{G}_K]e^-$.

In this section, for each prime p we always take the field k_∞^p (as fixed in §6.1.1) to be the cyclotomic \mathbb{Z}_p -extension of k . For each abelian extension E of k and each non-negative integer n we write E_n for the unique intermediate field of E_∞^p/E of degree p^n .

We shall prove claim (a) of Theorem B by combining Theorem (6.1) (a) with the following seminal result of Dasgupta and Kakde.

(6.8) Theorem (Dasgupta–Kakde). *Fix an extension K/k as above, an odd prime p and a set T in $\mathcal{P}_K^{\mathrm{ad}}$ that is disjoint from $S_p(K)$. Then one has*

$$\theta_{K_\infty^p/k, S^*(K_\infty^p), T}(0) \in \mathrm{Fitt}_{\Lambda_{K_\infty^p}}^0 \left(\varprojlim_n (\mathrm{Cl}_{K_n^p, \emptyset, T}^p)^{\vee, -} \right)^\#, \quad (60)$$

where the superscript $\#$ indicates that $\mathcal{G}_{K_\infty^p}$ acts via the involution $\mathcal{G}_{K_\infty^p} \rightarrow \mathcal{G}_{K_\infty^p}$ that sends $\sigma \mapsto \sigma^{-1}$.

Proof. The verification of the Strong Brumer–Stark Conjecture given in [DK23, Cor. 3.8] implies, for every natural number n , a containment

$$\theta_{K_n^p/k, S^*(K_n^p), T}(0) \in \mathrm{Fitt}_{\mathbb{Z}_p[\mathcal{G}_{K_n^p}]^-}^0 \left((\mathrm{Cl}_{K_n^p, \emptyset, T}^p)^{\vee, -} \right)^\#. \quad (61)$$

In addition, the natural maps $\mathrm{Cl}_{K_n^p, \emptyset}^{p, -} \rightarrow \mathrm{Cl}_{K_{n+1}^p, \emptyset}^{p, -}$ are injective (by [Was97, Prop. 13.26]) and so the Pontryagin-dual maps $(\mathrm{Cl}_{K_{n+1}^p, \emptyset, T}^p)^{\vee, -} \rightarrow (\mathrm{Cl}_{K_n^p, \emptyset, T}^p)^{\vee, -}$ are surjective. Given this fact, the claimed containment is obtained by simply passing to the limit over n of (61) and then taking account of the general result of Greither and Kurihara in [GK08, Thm. 2.1]. \square

A key step in the deduction of Theorem B (a) will be provided by the following technical result. This result reinterprets the containment in Theorem (6.8) in a way that is better suited to our purposes and also, at the same time, verifies condition (ii) of Theorem (6.1) in the relevant case.

(6.9) Proposition. *Fix a finite abelian CM extension K of a totally real field k and an odd prime p . Then the following claims are valid.*

- (a) *For every height-one prime \mathfrak{p} of $\Lambda_{p,K}^-$, the $(\Lambda_{p,K}^-)_{\mathfrak{p}}$ -module $(\text{Cl}_{K_{\infty}^p}^{p,-})_{\mathfrak{p}}$ has projective dimension one.*
- (b) *For each set T in $\mathcal{P}_K^{\text{ad}}$ that is disjoint from $S_p(K)$ one has*

$$\theta_{K_{\infty}^p/k, S^*(K_{\infty}^p), T}(0) \in \text{Fitt}_{\Lambda_{p,K}^-}^0(\text{Cl}_{K_{\infty}^p, S(K_{\infty}^p), T}^{p,-})^{**} \cdot \text{Fitt}_{\Lambda_{p,K}^-}^0(X_{K_{\infty}^p, S(K_{\infty}^p)}^{p,-})^{**}. \quad (62)$$

Proof. Regarding p as clear from context, we abbreviate $\Lambda_{p,K}$, K_{∞}^p and k_{∞}^p to Λ_K , K_{∞} and k_{∞} respectively.

To prove claim (a), it is enough to consider a singular prime \mathfrak{p} of Λ_K . To deal with this case we fix a set T in $\mathcal{P}_K^{\text{ad}}$ that is disjoint from $S_p(k)$.

We then note first that, since no finite place splits completely in k_{∞}/k , Lemma (6.2) (b) implies the natural projection maps

$$(\text{Cl}_{K_{\infty}}^p)_{\mathfrak{p}} \leftarrow (\text{Cl}_{K_{\infty}, \emptyset, T}^p)_{\mathfrak{p}} \quad \text{and} \quad (\text{Cl}_{K_{\infty}, \emptyset, T}^p)_{\mathfrak{p}} \rightarrow (\text{Cl}_{K_{\infty}, S(K_{\infty}), T}^p)_{\mathfrak{p}}$$

are both bijective. For the same reason, one finds that the \mathfrak{p} -localisation of the Λ_K^- -module $X_{K_{\infty}, S^*(K_{\infty})}^{p,-} = X_{K_{\infty}, S(K_{\infty}) \setminus S_{\infty}(k)}^{p,-}$ vanishes. Upon taking minus parts of the exact sequence (47), it follows that the $\Lambda_{K, \mathfrak{p}}^-$ -module $(\text{Cl}_{K_{\infty}, S(K_{\infty}), T}^{p,-})_{\mathfrak{p}}$, and hence also $(\text{Cl}_{K_{\infty}}^{p,-})_{\mathfrak{p}}$, is isomorphic to $H^1(D_{K_{\infty}, T}^{\bullet})_{\mathfrak{p}}^-$.

Next we note that $H^1(D_{K_{\infty}, T}^{\bullet})^-$ is a Λ_K^- -torsion module and so, by analysing the exact sequence

$$0 \longrightarrow U_{K_{\infty}, S(K_{\infty}), T}^- \longrightarrow P_{K_{\infty}}^- \longrightarrow P_{K_{\infty}}^- \longrightarrow H^1(D_{K_{\infty}, T}^{\bullet})^- \longrightarrow 0$$

obtained by taking minus parts of the representative (48) of the complex $D_{K_{\infty}, T}^{\bullet}$, we deduce that $U_{K_{\infty}, S(K_{\infty}), T}^-$ is Λ_K^- -torsion as well. On the other hand, $U_{K_{\infty}, S(K_{\infty}), T}^-$ embeds into the Λ_K^- -torsion free module $P_{K_{\infty}}^-$ and so must in fact vanish.

One therefore obtains a short exact sequence of Λ_K^- -modules

$$0 \longrightarrow P_{K_{\infty}}^- \longrightarrow P_{K_{\infty}}^- \longrightarrow H^1(D_{K_{\infty}, T}^{\bullet})^- \longrightarrow 0. \quad (63)$$

This sequence directly implies that the torsion $\Lambda_{K, \mathfrak{p}}^-$ -module $(\text{Cl}_{K_{\infty}}^{p,-})_{\mathfrak{p}} \cong H^1(D_{K_{\infty}, T}^{\bullet})_{\mathfrak{p}}^-$ is of projective dimension one, as required to prove claim (a).

To prepare for the proof of claim (b), we first consider an arbitrary finite abelian CM extension E of k , and a set T in $\mathcal{P}_E^{\text{ad}}$ that is disjoint from $S_p(k)$, and show that, for every height-one prime \mathfrak{p} of Λ_E^- one has

$$\theta_{E_{\infty}/k, S^*(E_{\infty}), T}(0) \in \text{Fitt}_{\Lambda_E^-}^0(\text{Cl}_{E_{\infty}, S(E_{\infty}), T}^{p,-})_{\mathfrak{p}} \cdot \text{Fitt}_{\Lambda_E^-}^0(Y_{E_{\infty}, S_p(k)}^{p,-})_{\mathfrak{p}}. \quad (64)$$

To show this we observe that, for each \mathfrak{p} , there exists a natural number n , an $(n \times n)$ -matrix A with coefficients in $\Lambda_{E, \mathfrak{p}}^-$, and an exact sequence of $\Lambda_{E, \mathfrak{p}}^-$ -modules of the form

$$0 \longrightarrow (\Lambda_{E, \mathfrak{p}}^-)^n \xrightarrow{v \mapsto A \cdot v} (\Lambda_{E, \mathfrak{p}}^-)^n \longrightarrow (\text{Cl}_{E_{\infty}, \emptyset, T}^{p,-})_{\mathfrak{p}} \longrightarrow 0.$$

Indeed, if \mathfrak{p} is regular, then the existence of such a sequence is clear since $(\text{Cl}_{E_{\infty}, \emptyset, T}^{p,-})_{\mathfrak{p}}$ is a finitely generated torsion module over the discrete valuation domain $\Lambda_{E, \mathfrak{p}}^-$ and, if \mathfrak{p} is singular, then it follows by localising the exact sequence (63) (with K replaced by E) at \mathfrak{p} .

Upon taking $\Lambda_{E,\mathfrak{p}}$ -linear duals of the above exact sequence one derives an isomorphism of $\Lambda_{E,\mathfrak{p}}$ -modules

$$\alpha(\mathrm{Cl}_{E_\infty,\emptyset,T}^{p,-})_{\mathfrak{p}} \cong (\Lambda_{E,\mathfrak{p}}^-)^{n,*} / (A^\# \cdot (\Lambda_{E,\mathfrak{p}}^-)^{n,*}) \cong \left((\Lambda_{E,\mathfrak{p}}^-)^n / (A \cdot (\Lambda_{E,\mathfrak{p}}^-)^n) \right)^\# \cong (\mathrm{Cl}_{E_\infty,\emptyset,T}^{p,-})_{\mathfrak{p}}^\#,$$

where we write $\alpha(-) := \mathrm{Ext}_{\Lambda_E}^1(-, \Lambda_E)$ for the Iwasawa adjoint of a Λ_E -module and $A^\#$ for the matrix obtained from A by applying the involution $\#$ to each of its entries. We can then consider the composite isomorphism of $\Lambda_{E,\mathfrak{p}}$ -modules

$$(\mathrm{Cl}_{E_\infty,\emptyset,T}^{p,-})_{\mathfrak{p}} \cong \alpha(\mathrm{Cl}_{E_\infty,\emptyset,T}^{p,-})_{\mathfrak{p}}^\# \cong \varinjlim_n \mathrm{Cl}_{E_n,\emptyset,T}^{p,-} \vee \cong \varprojlim_n (\mathrm{Cl}_{E_n,\emptyset,T}^{p,-})^\vee_{\mathfrak{p}},$$

where the second isomorphism follows from [Was97, Prop. 15.34] and the third is an easy consequence of the fact that taking Pontryagin duals is an exact functor. In particular, this isomorphism combines with Theorem (6.8) to imply that, for every height-one prime \mathfrak{p} of Λ_E^- , there is a containment

$$\theta_{E_\infty/k, S^*(E_\infty), T}(0) \in \mathrm{Fitt}_{\Lambda_E^-}^0(\mathrm{Cl}_{E_\infty,\emptyset,T}^{p,-})_{\mathfrak{p}}. \quad (65)$$

To derive (64) from here, we note that, since $S_p(k) \subseteq S(E_\infty)$, there exists a canonical exact sequence of Λ_E -modules

$$U_{E_\infty, S(E_\infty), T} \longrightarrow Y_{E_\infty, S_p(k)}^p \longrightarrow \mathrm{Cl}_{E_\infty, \emptyset, T}^p \longrightarrow \mathrm{Cl}_{E_\infty, S(E_\infty), T}^p \longrightarrow 0$$

that induces, upon taking minus parts, a short exact sequence of Λ_E^- -modules

$$0 \longrightarrow Y_{E_\infty, S_p(k)}^{p,-} \longrightarrow \mathrm{Cl}_{E_\infty, \emptyset, T}^{p,-} \longrightarrow \mathrm{Cl}_{E_\infty, S(E_\infty), T}^{p,-} \longrightarrow 0.$$

Now, since the torsion $\Lambda_{E,\mathfrak{p}}^-$ -module $(\mathrm{Cl}_{E_\infty, S(E_\infty), T}^{p,-})_{\mathfrak{p}}$ has projective dimension one (by the above argument), its zeroth Fitting ideal contains a non-zero divisor. Given these observations, a general property of Fitting ideals (cf. [Gre04, Prop. 2.2.3]) combines with the above short exact sequence to imply that

$$\mathrm{Fitt}_{\Lambda_E^-}^0(\mathrm{Cl}_{E_\infty, \emptyset, T}^{p,-})_{\mathfrak{p}} = \mathrm{Fitt}_{\Lambda_E^-}^0(\mathrm{Cl}_{E_\infty, S(E_\infty), T}^{p,-})_{\mathfrak{p}} \cdot \mathrm{Fitt}_{\Lambda_E^-}^0(Y_{E_\infty, S_p(k)}^{p,-})_{\mathfrak{p}}.$$

Given this equality, the claimed containment (64) follows directly from (65).

Turning now to the proof of claim (b), we observe that, by Lemma (6.2) (a), the containment (62) can be verified after localisation at height-one primes of Λ_K^- . In addition, if \mathfrak{p} is a singular height-one prime, then the argument in claim (a) implies that the localisation of $X_{K_\infty, S(K_\infty)}^{p,-}$ at \mathfrak{p} vanishes, and hence that its zeroth Fitting ideal over $(\Lambda_K^-)_{\mathfrak{p}}$ is equal to $(\Lambda_K^-)_{\mathfrak{p}}$. In this case, therefore, the localisation at \mathfrak{p} of the containment (62) is a direct consequence of (64) with E taken to be K .

In the remainder of this argument we may therefore assume that \mathfrak{p} is a regular height-one prime of Λ_K^- . To investigate this case, we fix a group isomorphism $\mathcal{G}_{K_\infty} \cong \Delta \times \Gamma$ where Δ is finite and Γ isomorphic to \mathbb{Z}_p , and we set $L := K_\infty^\Gamma$. For each totally odd character χ of Δ , we write L_χ for the kernel field of χ in L and Λ_χ for the ring Λ_{L_χ} . We also note that there exists such a character χ with the property that $\Lambda_{K,\mathfrak{p}}^-$ identifies with the localisation of Λ_χ^- at a suitable height-one prime ideal \wp_χ (cf. [BKS17, §3C1]).

Writing $P_{K_\infty/L_\chi, \infty}$ for the Euler factor $P_{K_\infty/L_\chi, \infty, \emptyset}$ in Λ_χ that is defined as in Definition (4.3) (a), we claim first that

$$\begin{aligned} (\Lambda_K^-)_{\mathfrak{p}} \cdot \theta_{K_\infty/k, S^*(K_\infty), T} &= (\Lambda_\chi^-)_{\wp_\chi} \cdot P_{K_\infty/L_\chi, \infty} \cdot \theta_{L_\chi, \infty/k, S^*(L_\chi, \infty), T} \\ &\in P_{K_\infty/L_\chi, \infty} \cdot \mathrm{Fitt}_{\Lambda_\chi^-}^0(\mathrm{Cl}_{L_\chi, \infty, S(L_\chi, \infty), T}^{p,-})_{\wp_\chi} \cdot \mathrm{Fitt}_{\Lambda_\chi^-}^0(Y_{L_\chi, \infty, S_p(k)}^{p,-})_{\wp_\chi} \\ &= \mathrm{Fitt}_{\Lambda_\chi^-}^0(\mathrm{Cl}_{L_\chi, \infty, S(K_\infty), T}^{p,-})_{\wp_\chi} \cdot \mathrm{Fitt}_{\Lambda_\chi^-}^0(X_{L_\chi, \infty, S(K_\infty)}^{p,-})_{\wp_\chi}. \end{aligned}$$

Here the first equality follows directly from the functorial properties of Dirichlet L -series and the containment from (64) with $E = L_\chi$. The second equality is valid because explicit computation

(as in [Fla04, Lem. 5.5]) shows that

$$P_{K_\infty/L_{\chi,\infty}} \cdot \text{Fitt}_{\Lambda_\chi^-}^0(Y_{L_{\chi,\infty},S_p(k)}^{p,-})_{\wp_\chi} = \text{Fitt}_{\Lambda_\chi^-}^0(Y_{L_{\chi,\infty},S(K_\infty)}^{p,-})_{\wp_\chi} = \text{Fitt}_{\Lambda_\chi^-}^0(X_{L_{\chi,\infty},S(K_\infty)}^{p,-})_{\wp_\chi}$$

and, in addition, one has $\text{Fitt}_{\Lambda_\chi^-}^0(\text{Cl}_{L_{\chi,\infty},S(L_{\chi,\infty}),T}^{p,-}) \subseteq \text{Fitt}_{\Lambda_\chi^-}^0(\text{Cl}_{L_{\chi,\infty},S(K_\infty),T}^{p,-})$ since the natural map $\text{Cl}_{L_{\chi,\infty},S(L_{\chi,\infty}),T}^p \rightarrow \text{Cl}_{L_{\chi,\infty},S(K_\infty),T}^p$ is surjective.

To derive the \mathfrak{p} -localisation of (62) from the above containment, it is then enough to note that there are equalities of ideals

$$\begin{aligned} & \text{Fitt}_{\Lambda_\chi^-}^0(\text{Cl}_{L_{\chi,\infty},S(K_\infty),T}^{p,-})_{\wp_\chi} \cdot \text{Fitt}_{\Lambda_\chi^-}^0(X_{L_{\chi,\infty},S(K_\infty)}^{p,-})_{\wp_\chi} \\ &= \text{Fitt}_{\Lambda_\chi^-}^0(H^1(D_{L_{\chi,\infty},S(K_\infty),T}^\bullet)^-)_{\wp_\chi} \\ &= \text{Fitt}_{\Lambda_K^-}^0(H^1(D_{K_\infty,S(K_\infty),T}^\bullet)^-)_{\mathfrak{p}} \\ &= \text{Fitt}_{\Lambda_K^-}^0(\text{Cl}_{K_\infty,S(K_\infty),T}^{p,-})_{\mathfrak{p}} \cdot \text{Fitt}_{\Lambda_K^-}^0(X_{K_\infty,S(K_\infty)}^{p,-})_{\mathfrak{p}}. \end{aligned}$$

Here the first and third equalities follow from the relevant cases of the exact sequence (47) (and the fact that the primes \mathfrak{p} and \wp_χ are regular). To derive the second equality we note that the result of Lemma (3.4) (d) induces (upon applying $\mathbb{Z}_p \otimes_{\mathbb{Z}} -$ and then passing to the limit over intermediate fields E of K_∞/k) an isomorphism $D_{K_\infty,S(K_\infty),T}^\bullet \otimes_{\Lambda_K}^{\mathbb{L}} \Lambda_\chi \cong D_{L_{\chi,\infty},S(K_\infty),T}^\bullet$ in $D^{\text{perf}}(\Lambda_\chi)$. This isomorphism in turn induces an isomorphism of Λ_χ -modules

$$H^1(D_{K_\infty,S(K_\infty),T}^\bullet) \otimes_{\Lambda_K} \Lambda_{L_\chi} \cong H^1(D_{L_{\chi,\infty},S(K_\infty),T}^\bullet),$$

which directly implies the second equality in the above display.

This concludes the proof of claim (b). □

By combining Proposition (6.9) with a particular case of the criterion of Theorem (6.1) (a), we can now finally derive the main result of this section (which, we observe, verifies Theorem B(a) in the Introduction).

(6.10) Theorem. *Assume k is a totally real field. Then, for any finite abelian CM extension E of k , the conjecture $\text{TNC}(h^0(\text{Spec } E), \mathbb{Z}[\mathcal{G}_E]^-)$ is valid.*

Proof. At the outset, we note that the Rubin–Stark system ε_k^\wp is \mathbb{Z} -integral in the sense of Definition (2.12). Indeed, given the explicit description of ε_k^\wp that follows, in this setting, from Example (2.21) (b), this fact is a well-known consequence of work of Deligne and Ribet in [DR80] (cf. [Gro88, Prop. 3.7]).

Next we fix an odd prime p and write \mathcal{Y}_p for the set of all fields of the form $k\langle p \rangle L$, where $k\langle p \rangle$ is as defined in (24) and L is a totally real, tamely ramified cyclic p -extension of k .

Then, since p is odd and k is totally real, the field $k\langle p \rangle$ is by its definition contained in $k(\mu_p)$. In particular, every field F in \mathcal{Y}_p is a tamely ramified abelian extension of k for which one has $\epsilon_{F,\emptyset} = e^-$ and so the validity of $\text{TNC}(h^0(\text{Spec } F), \mathbb{Z}_p[\mathcal{G}_F]\epsilon_{F,\emptyset})$ is proved by Nickel in [Nic21, Thm. 2].

Given this fact, the validity of $\text{TNC}(h^0(\text{Spec } E), \mathbb{Z}_p[\mathcal{G}_E]\epsilon_{E,\emptyset})$ follows directly upon combining Proposition (6.9) with the criterion of Theorem (6.1) (a) (iii) with $V = \emptyset$, \mathcal{X} the set of all finite abelian CM extensions of k (cf. Example (4.6) (b)), $\mathcal{S} = \{2\}$, k_∞^p the cyclotomic \mathbb{Z}_p -extension of k and $\mathcal{Y} = \mathcal{Y}_p$.

Then, since $\text{TNC}(h^0(\text{Spec } E), \mathbb{Z}_p[\mathcal{G}_E]\epsilon_{E,\emptyset})$ is valid for every odd prime p , to deduce the validity of $\text{TNC}(h^0(\text{Spec } E), \mathbb{Z}[\mathcal{G}_E]^-)$ it is now enough to simply note that $\epsilon_{E,\emptyset} = e^-$. □

(6.11) Remark. In [JN20] Johnston and Nickel use arguments similar to those in the proof of Proposition (6.9) to deduce the ‘equivariant Iwasawa Main Conjecture’ of Ritter and Weiss from the known validity of the Strong Brumer–Stark Conjecture.

6.3. Results in rank one

In this section we use the classical theory of (rank-one) Euler systems to investigate the inclusion (46) in cases where both $|S_\infty(k)| = 1$ (so that k is either \mathbb{Q} or an imaginary quadratic field) and $\mathcal{V} = \{S_\infty(k)\}$. In this way we shall, in particular, complete the proofs of Theorems A and B, as stated in the Introduction.

6.3.1. Statement of the main results

For a p -adic place \mathfrak{p} of k we write $k_\infty^{\mathfrak{p}}$ for the maximal \mathbb{Z}_p -power extension of k that is unramified outside \mathfrak{p} .

Then Theorem (6.1) enables us to prove the following unconditional result.

(6.12) Theorem. *Let \mathcal{S} be a finite set of prime numbers that contains all divisors of $|\mu_k|$. Then Conjecture (2.24) for the pair $(\Omega^{S_\infty(k)}(k), \mathcal{S})$ is valid in the following cases:*

- (a) $k = \mathbb{Q}$,
- (b) k is an imaginary quadratic field and, for all primes $p \notin \mathcal{S}$, all p -adic places \mathfrak{p} of k and all fields K in $\Omega_{S_p(k)}(k)$ either $\text{Gal}(Kk_\infty^{\mathfrak{p}}/k)$ is p -torsion free or the $\mathbb{Z}_p[[\text{Gal}(Kk_\infty^{\mathfrak{p}}/K)]]$ -module $\text{Cl}^p(Kk_\infty^{\mathfrak{p}})$ has vanishing Iwasawa μ -invariant.

(6.13) Remark. The hypothesis on μ -invariants in Theorem (6.12) (b) is valid for (K, p) if p splits in k . This was proven by Gillard [Gil85] if $p > 3$ and by Oukhaba–Viguié [OV16] if $p \in \{2, 3\}$. Moreover, the Theorem of Ferrero–Washington [FW79] combines with a well-known result of Iwasawa to imply that the relevant μ -invariant (at some prime number p) vanishes if the degree $[K : k]$ is a power of p (cf. [BH21, Prop. 5.6]).

(6.14) Remark. As noted in the Introduction, one can also combine Theorem (6.12) (a) with Theorem (3.2) to obtain a simpler proof of the known validity of $\text{TNC}(h^0(\text{Spec } K), R_K[\mathcal{G}_K])$ for any finite abelian extension K of $k = \mathbb{Q}$, with $R_K = \mathbb{Z}$ if K is real and $R_K = \mathbb{Z}[1/2]$ otherwise.

6.3.2. The proof of Theorem (6.12)

We begin by establishing a useful technical result.

(6.15) Lemma. *Fix a number field k , an Euler system c in $\text{ES}_k(\mathbb{Z}_S)$, a prime number p outside S and a field K in $\Omega_{S_p(k)}(k)$. Set $K_\infty := Kk_\infty^{\mathfrak{p}}$, and write Ω' for the set of finite extensions of K in K_∞ . Then the following claims are valid.*

- (a) *For each E in Ω' , write \mathcal{C}_E for the $\mathbb{Z}_p[\mathcal{G}_E]$ -submodule of $U_{E, S(E)}$ generated by the set $\{\delta_{T, E} \cdot c_E : T \in \mathcal{P}_E^{\text{ad}}\}$. Then, for any field E' in Ω' that contains E , the field theoretic norm $N_{E'/E}^{r_{E'}}$ maps $\mathcal{C}_{E'}$ to \mathcal{C}_E .*
- (b) *Set $\mathcal{C}_{K_\infty} := \varprojlim_{E \in \Omega'} \mathcal{C}_E$ and $\mu_{K_\infty} := \varprojlim_{E \in \Omega'} (\mathbb{Z}_p \otimes_{\mathbb{Z}} \mu_E)$, where the transition maps are induced by field-theoretic norms (and, in the first case, the result of claim (a)). Then, for any regular height-one prime \mathfrak{p} of \mathbb{A} that belongs to the support of $\mathbb{A}_{p, K} \mathcal{C}_{K_\infty, T}$ one has*

$$\text{Fitt}_{\mathbb{A}_{p, K}}^0(\mathcal{C}_{K_\infty}/(\mathbb{A}_{p, K} \mathcal{C}_{K_\infty, T}))_{\mathfrak{p}} = \delta_{T, K_\infty} \cdot \text{Ann}_{\mathbb{A}_{p, K}}(\mu_{K_\infty})_{\mathfrak{p}}^{-1},$$

where δ_{T, K_∞} is the element of $\mathbb{A}_{p, K}$ given by the family $(\delta_{T, E})_{E \in \Omega'}$.

Proof. We abbreviate $\mathbb{A}_{p, K}$ to \mathbb{A} . For each pair of fields E and E' in Ω' with $E \subseteq E'$, one has $S(E') = S(E)$ and $r_{E'} = r_E$. This implies that $N_{E'/E}^{r_{E'}}(\mathcal{C}_{E'}) = \mathcal{C}_E$. Since $\mathcal{P}_{E'}^{\text{ad}}$ is a subset of $\mathcal{P}_E^{\text{ad}}$, it is therefore clear that $N_{E'/E}^{r_{E'}}$ maps $\mathcal{C}_{E'}$ to \mathcal{C}_E , as required to prove claim (a).

To prove the equality claimed in (b) we recall that, for E in Ω' , the module $\text{Ann}_{\mathbb{Z}_p[\mathcal{G}_E]}(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mu_E)$

is generated over \mathbb{Z}_p by the set $\{\delta_{T,E} \mid T \in \mathcal{P}_E^{\text{ad}}\}$ (cf. § 2.2.1). This fact gives rise to an exact commutative diagram of $\mathbb{Z}_p[\mathcal{G}_E]$ -modules in which all vertical maps are surjective:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}_p[\mathcal{G}_E]\delta_{T,E} & \longrightarrow & \text{Ann}_{\mathbb{Z}_p[\mathcal{G}_E]}(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mu_E) & \longrightarrow & \text{Ann}_{\mathbb{Z}_p[\mathcal{G}_E]}(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mu_E)/(\mathbb{Z}_p[\mathcal{G}_E]\delta_{T,E}) & \longrightarrow & 0 \\ & & \downarrow x \mapsto x \cdot c_E & & \downarrow x \mapsto x \cdot c_E & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}_p[\mathcal{G}_E]\delta_{T,E}c_E & \longrightarrow & \mathcal{C}_E & \longrightarrow & \mathcal{C}_E/(\mathbb{Z}_p[\mathcal{G}_E]\delta_{T,E}c_E) & \longrightarrow & 0. \end{array}$$

By passing to the limit over E of these diagrams (which preserves exactness since all occurring modules are compact) and then localising at \mathfrak{p} , we deduce the existence of an exact commutative diagram of $\Lambda_{\mathfrak{p}}$ -modules in which all vertical maps are surjective:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Lambda_{\mathfrak{p}}\delta_{T,K_{\infty}} & \longrightarrow & \text{Ann}_{\Lambda}(\mu_{K_{\infty}})_{\mathfrak{p}} & \longrightarrow & \text{Ann}_{\Lambda}(\mu_{K_{\infty}})_{\mathfrak{p}}/(\Lambda\delta_{T,K_{\infty}})_{\mathfrak{p}} & \longrightarrow & 0 \\ & & \downarrow \lambda\delta_{T \mapsto \lambda \cdot c_{K_{\infty},T}} & & \downarrow x \mapsto x \cdot c_{K_{\infty},T} & & \downarrow & & \\ 0 & \longrightarrow & \Lambda_{\mathfrak{p}}c_{K_{\infty},T} & \longrightarrow & \mathcal{C}_{K_{\infty},\mathfrak{p}} & \longrightarrow & \mathcal{C}_{K_{\infty},\mathfrak{p}}/(\Lambda_{\mathfrak{p}}c_{K_{\infty},T}) & \longrightarrow & 0. \end{array}$$

Now, since $\Lambda_{\mathfrak{p}}$ is a discrete valuation domain, if the module $(\Lambda c_{K_{\infty},T})_{\mathfrak{p}}$ does not vanish, then it is free of rank one and so the first vertical arrow in this diagram is bijective. Similarly, the non-zero submodule $\text{Ann}_{\Lambda}(\mu_{K_{\infty}})_{\mathfrak{p}}$ of $\Lambda_{\mathfrak{p}}$ is free of rank one and so the second vertical arrow is also bijective. Applying the Snake Lemma to the diagram, we deduce that the third vertical arrow is bijective, and hence that, for any prime \mathfrak{p} as in the statement one has

$$\begin{aligned} \text{Fitt}_{\Lambda}^0(\mathcal{C}_{K_{\infty}}/(\Lambda c_{K_{\infty},T}))_{\mathfrak{p}} &= \text{Fitt}_{\Lambda_{\mathfrak{p}}}^0(\mathcal{C}_{K_{\infty},\mathfrak{p}}/(\Lambda c_{K_{\infty},T})_{\mathfrak{p}}) \\ &= \text{Fitt}_{\Lambda_{\mathfrak{p}}}^0\left(\text{Ann}_{\Lambda}(\mu_{K_{\infty}})_{\mathfrak{p}}/(\Lambda\delta_{T,K_{\infty}})_{\mathfrak{p}}\right) = \delta_{T,K_{\infty}} \cdot \text{Ann}_{\Lambda}(\mu_{K_{\infty}})_{\mathfrak{p}}^{-1}, \end{aligned}$$

as claimed. \square

We shall now prove Theorem (6.12). To do this we fix a field k that is either \mathbb{Q} or imaginary quadratic.

At the outset we recall that for any such k the module of isolated systems $\text{ES}_k^{\circ,\mathcal{X}}(\mathbb{Z}_{\mathcal{S}})$ vanishes (cf. Remark (2.14)) and hence that Theorem (6.12) will follow if we can verify that the conditions of Theorem (6.1) are satisfied in this case.

We further recall that the integrality of the Rubin–Stark system ε_k for such a field k follows from the explicit descriptions recalled in Example (2.21).

Next, we fix a rational prime $p \notin \mathcal{S}$ and, if k is imaginary quadratic, an element \mathfrak{p} of the set $S_p(k)$ of p -adic places of k (this involves a choice if p splits in k). As p is fixed, we also occasionally suppress dependency on p in the notation. We take k_{∞} to be the cyclotomic \mathbb{Z}_p -extension if $k = \mathbb{Q}$ and to be $k_{\infty}^{\mathfrak{p}}$ if k is imaginary quadratic. We write ∞ for the unique archimedean place of k and abbreviate the set of fields $\Omega_{S_p(k)}^{\{\infty\}}(k)$ to $\Omega_p(k)$.

If $k = \mathbb{Q}$, then condition (ii) in Theorem (6.1) holds by the Theorem of Ferrero–Washington [FW79]. If k is imaginary quadratic, then condition (ii) is satisfied due to the explicit assumption on $k_{\infty}^{\mathfrak{p}}$ in (b). Moreover, condition (i) in (a) is satisfied by assumption on \mathcal{S} . In this way we are reduced to verifying the containment (46).

To do this, we fix a field K in $\Omega_p(k)$ and set $\Lambda := \Lambda_K$. Note that if either k is \mathbb{Q} , or both k is imaginary quadratic and p splits in k , then Remark (6.7) implies that the modules $\text{Cl}_{S(k_{\infty})}^p(K_{\infty})^{\text{Gal}(K_{\infty}/K_n)}$ are finite for all $n > 0$. In these cases, therefore, an application of claims (a) and (d) of Lemma (6.6) reduces us to verifying, for all Euler systems $c \in \text{ES}_k(\mathbb{Z}_{\mathcal{S}})$, all fields $K \in \Omega_p(k)$ and (for each K) a suitably chosen set T in $\mathcal{P}_K^{\text{ad}}$, that one has

$$\text{im}(c_{K_{\infty},T})^{**} \subseteq \text{Fitt}_{\Lambda}^0(\text{Cl}_{S(K),T}^p(K_{\infty}))^{**}. \quad (66)$$

On the other hand, if k is imaginary quadratic and p does not split in k , then the \mathbb{Z}_p -rank of $\mathcal{G}_{K_{\infty}}$ is two and, by claims (a) and (c) of Lemma (6.6), we are again reduced to verifying the

inclusion (66).

To verify (66) it is in turn enough, by Lemma (6.2) (a), to argue locally at height-one primes of Λ . At the outset we remark that if Λ contains a singular height-one prime, then $\text{Gal}(Kk_\infty^{\mathfrak{p}}/k)$ has a non-trivial element of order p . In this case, consequently, the given assumption asserts the vanishing of the μ -invariant of $\text{Cl}^p(Kk_\infty^{\mathfrak{p}})$, and hence also of that of $\text{Cl}_{S(K),T}^p(Kk_\infty^{\mathfrak{p}})$. Hence, for any singular height-one prime \mathfrak{p} of Λ , Lemma (6.2) (b) implies that the localisation at \mathfrak{p} of the right hand side of (66) is equal to $(\Lambda)_{\mathfrak{p}}$ and so the claimed inclusion is clear in this case.

We can therefore assume in the rest of the argument that \mathfrak{p} is a regular height-one prime. To proceed in this case, we set $\mathbb{F}_{K_\infty, T}^{\times, p} := \varprojlim_E (\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{F}_{E, T}^{\times})$ with the limit ranging over all finite subextensions of K_∞/k and the transition maps being induced by the relevant norm maps. Then, by applying $\mathbb{Z}_p \otimes_{\mathbb{Z}} -$ to the exact sequences in Remark (3.5) and then passing to the limit over E , one computes that

$$\text{Fitt}_{\Lambda}^0(\mathbb{F}_{K_\infty, T}^{\times, p}) = \Lambda \cdot \delta_{T, K_\infty}, \quad (67)$$

where δ_{T, K_∞} denotes the element $(\delta_{T, E})_E$ of Λ . Further, by applying $\mathbb{Z}_p \otimes_{\mathbb{Z}} -$ to the long exact cohomology sequence of the exact triangle in Lemma (3.4) (b) and then passing to the limit over all such E of the resulting exact sequences, one obtains an exact sequence of Λ -modules

$$0 \rightarrow U_{K_\infty, S(K), T} \rightarrow U_{K_\infty, S(K)} \rightarrow \mathbb{F}_{K_\infty, T}^{\times, p} \rightarrow \text{Cl}_{S(K), T}^p(K_\infty) \rightarrow \text{Cl}_{S(K)}^p(K_\infty) \rightarrow 0.$$

Since the general result of Lemma (A.10) implies that $\text{Fitt}_{\Lambda}^0(U_{K_\infty, S(K), T}/(\Lambda c_{K_\infty, T}))_{\mathfrak{p}}$ is equal to $\text{im}(c_{K_\infty, T})_{\mathfrak{p}}$, this sequence combines with the equality (67) to reduce the proof of (the \mathfrak{p} -localisation of) (66) to the proof of an inclusion

$$\text{Fitt}_{\Lambda}^0(U_{K_\infty, S(K)}/(\Lambda c_{K_\infty, T}))_{\mathfrak{p}} \subseteq \delta_{T, K_\infty} \cdot \text{Fitt}_{\Lambda}^0(\text{Cl}_{S(K)}^p(K_\infty))_{\mathfrak{p}}.$$

Before applying Lemma (6.15) in this setting, we note that the Λ -module μ_{K_∞} is pseudo-null. Indeed, μ_{K_∞} is finite if either k is \mathbb{Q} (since K_∞ is then totally real) or if k is imaginary quadratic and p is split in k (since k_∞/k is unramified outside the single place \mathfrak{p}), whilst if k is imaginary quadratic and p does not split in k , then μ_{K_∞} is contained in the finitely generated \mathbb{Z}_p -module $\mathbb{Z}_p(1)$ and so is pseudo-null since, in this case, the \mathbb{Z}_p -rank of $\text{Gal}(K_\infty/k)$ is two.

In particular, since the quotient of Λ by $\text{Ann}_{\Lambda}(\mu_{K_\infty})$ is isomorphic to μ_{K_∞} , the pseudo-nullity of the latter module implies that $\text{Ann}_{\Lambda}(\mu_{K_\infty})_{\mathfrak{p}} = \Lambda_{\mathfrak{p}}$. In view of this equality, the result of Lemma (6.15) (b) implies that the above inclusion is valid provided that the Λ -module \mathcal{C}_{K_∞} that occurs in the latter result is such that

$$\text{Fitt}_{\Lambda}^0(U_{K_\infty, S(K)}/\mathcal{C}_{K_\infty})_{\mathfrak{p}} \subseteq \text{Fitt}_{\Lambda}^0(\text{Cl}_{S(K)}^p(K_\infty))_{\mathfrak{p}}.$$

To verify this inclusion, we can assume \mathfrak{p} is in the support of \mathcal{C}_{K_∞} (otherwise $(U_{K_\infty, S(K)}/\mathcal{C}_{K_\infty})_{\mathfrak{p}} = (U_{K_\infty, S(K)})_{\mathfrak{p}}$ is a free $\Lambda_{\mathfrak{p}}$ -module and so its Fitting ideal vanishes), and hence that the $\Lambda_{\mathfrak{p}}$ -module $\mathcal{C}_{K_\infty, \mathfrak{p}}$ is free of rank one. Given this, the general result of Lemma (A.10) implies

$$\text{Fitt}_{\Lambda}^0(U_{K_\infty, S(K)}/\mathcal{C}_{K_\infty})_{\mathfrak{p}} = \Lambda_{K, \mathfrak{p}} \cdot \{f(x) \mid x \in \mathcal{C}_{K_\infty}, f \in U_{K_\infty, S(K)}^*\}. \quad (68)$$

To analyse this equality we note the argument of Lemma (6.15) shows that $x = \lambda \cdot c_{K_\infty}$ for a suitable element λ of $\text{Ann}_{\Lambda}(\mu_{K_\infty})$ and then, following Remark (2.8), we fix a pre-image λ' of λ under the projection map $\varprojlim_{E \in \Omega(k)} \text{Ann}_{\mathbb{Z}_p[\mathcal{G}_E]}(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mu_E) \rightarrow \text{Ann}_{\Lambda}(\mu_{K_\infty})$.

We write \mathcal{L} for the maximal abelian pro- p extension of k in which all archimedean places split completely and, for each field E in $\Omega(k)$ that is contained in \mathcal{L} , we set

$$c'_E := \lambda'_{EK} \left(\prod_{v \in \Pi(E) \setminus S(EK)} (1 - \text{Frob}_v^{-1}) \right) \cdot c_{EK},$$

with $\Pi := S_\infty(k)$ (so that $\Pi(E) = S^*(E) = S_\infty(k) \cup S(E)$).

Then the family $c' := (c'_E)_E$ can be seen to belong to the module $\text{ES}_{\Pi}^1(\text{Ind}_{G_k}^{G_K}(\mathbb{Z}_p)(1), \mathcal{L})$ of p -adic Euler systems introduced in Definition (A.2) and is also such that $c'_{k_\infty} = x$. In view of

(68), the required inclusion will therefore be true if we can show that for any p -adic system c in $\text{ES}_{\Pi}^1(\text{Ind}_{G_k}^{G_K}(\mathbb{Z}_p)(1), \mathcal{L})$ and any homomorphism f in $U_{K_\infty, S(K)}^*$ one has

$$f(c_{k_\infty}) \in \text{Fitt}_{\Lambda}^0(\text{Cl}_{K_\infty, S(K)}^p)_{\mathfrak{p}}. \quad (69)$$

In the rest of the argument we shall explain how this containment follows from results of Rubin. To do this, we write Δ for the torsion subgroup $\text{Gal}(K_\infty/k_\infty)$ of $\text{Gal}(K_\infty/k)$ and note that there exists a character χ of $\widehat{\Delta}$ and a height-one prime \wp_χ of the ring $\Lambda_\chi := \Lambda \otimes_{\mathbb{Z}_p[\Delta]} \mathbb{Z}_p[\text{im } \chi]$ such that $\Lambda_{\mathfrak{p}} = (\Lambda_\chi)_{\wp_\chi}$ (cf. [BKS17, §3C1]).

We also recall that, by the argument of Rubin in [Rub00, Prop. 6.2.1 and Cor. 6.2.2], the canonical Selmer structure gives rise to a natural isomorphism of dual Selmer modules

$$(H_{\mathcal{F}_{\text{can}}}^1(\mathcal{O}_{k, S(K)}, \Lambda^\vee(1))^\vee) \otimes_{\Lambda} \Lambda_\chi \cong H_{\mathcal{F}_{\text{can}}}^1(\mathcal{O}_{k, S(K)}, \Lambda_\chi^\vee(1)(\chi^{-1}))^\vee \cong H_{\mathcal{F}_{\text{can}}}^1(\mathcal{O}_{k, S(K)}, \mathcal{T}_{\chi, k_\infty}^\vee)^\vee,$$

where $\mathcal{T}_{\chi, k_\infty}$ denotes the induction from G_{k_∞} to G_k of the G_k -representation $\mathbb{Z}_p[\text{im } \chi](1)(\chi^{-1})$. In particular, if one combines this isomorphism with the canonical isomorphism

$$\text{Cl}_{K_\infty, S(K)}^p \cong H_{\mathcal{F}_{\text{can}}}^1(\mathcal{O}_{k, S(K)}, \Lambda^\vee(1))^\vee \quad (70)$$

(coming, for example, from [Rub00, Prop. 1.6.1]), then one obtains an isomorphism of $\Lambda_{\mathfrak{p}}$ -modules of the form $(\text{Cl}_{K_\infty, S(K)}^p)_{\mathfrak{p}} \cong H_{\mathcal{F}_{\text{can}}}^1(\mathcal{O}_{k, S(K)}, \mathcal{T}_{\chi, k_\infty}^\vee)_{\wp_\chi}^\vee$. This isomorphism in turn implies an equality of ideals

$$\begin{aligned} \text{Fitt}_{\Lambda}^0(\text{Cl}_{K_\infty, S(K)}^p)_{\mathfrak{p}} &= \text{Fitt}_{\Lambda_\chi}^0(H_{\mathcal{F}_{\text{can}}}^1(\mathcal{O}_{k, S(K)}, \mathcal{T}_{\chi, k_\infty}^\vee)_{\wp_\chi}^\vee) \\ &= \text{char}_{\Lambda_\chi}(H_{\mathcal{F}_{\text{can}}}^1(\mathcal{O}_{k, S(K)}, \mathcal{T}_{\chi, k_\infty}^\vee)_{\wp_\chi}^\vee), \end{aligned}$$

where the second equality follows from the fact that, by the structure theorem for finitely generated modules over a discrete valuation ring R with maximal ideal \mathfrak{m} , one has $\text{Fitt}_R^0(M) = \mathfrak{m}^{\text{length}_R(M)}$ for any finitely generated torsion R -module M .

This last displayed equality reduces the proof of (69) to the verification of an inclusion

$$\text{im}(c^\chi) \subseteq \text{char}_{\Lambda_\chi}(H_{\mathcal{F}_{\text{can}}}^1(\mathcal{O}_{k, S(K)}, \mathcal{T}_{\chi, k_\infty}^\vee)_{\wp_\chi}^\vee),$$

where c^χ denotes the p -adic Euler system in $\text{ES}_{\Pi}^1(\mathcal{T}_\chi, \mathcal{L})$ that is obtained from c via twisting by χ (as in [Rub00, Prop. 2.4.2]). It is therefore enough to note that this inclusion follows directly upon taking account of the observation of Rubin in [Rub00, Rem. 2.1.3] and then applying the general result of [Rub00, Thm. 2.3.3].

This completes the proof of Theorem (6.12). \square

6.4. Results in higher rank

The main result of this section provides further evidence for the inclusion (46) in the case that p is odd and $|V_K| > 1$. This result complements the evidence for Conjecture (2.24) that is provided by Theorem (5.9) and depends on proving a higher-rank analogue of a well-known result in the Iwasawa theory of (rank-one) Euler systems. However, since the latter result is best understood in terms of more general p -adic representations we have, for clarity, deferred its treatment to Appendix A.

Throughout the section we fix an odd prime p and a finite abelian extension K/k of prime-to- p degree for which $V_K \neq \emptyset$. We take the field $k_\infty = k_\infty^p$ to be the cyclotomic \mathbb{Z}_p -extension of k , set $\Gamma_K := \text{Gal}(K_\infty/K)$ and fix a splitting of groups $\mathcal{G}_{K_\infty} \cong \mathcal{G}_K \times \Gamma_K$. Via this splitting, we regard $\mathbb{Z}_p[\mathcal{G}_K]$ as a subring of $\Lambda_K = \Lambda_{p, K}$. In this way, for each Λ_K -module M , and each character χ in $\widehat{\mathcal{G}_K}$, we obtain a $\mathbb{Z}_p[\text{im}(\chi)] \otimes_{\mathbb{Z}_p} \Lambda_K$ -module by setting

$$M^\chi := M \otimes_{\mathbb{Z}_p[\mathcal{G}_K]} \mathbb{Z}_p[\text{im}(\chi)].$$

We write ω_p for the p -adic Teichmüller character of k , regarded as a \mathbb{C} -valued character via some fixed isomorphism $\mathbb{C}_p \cong \mathbb{C}$. We then write Ψ_K for the subset of $\widehat{\mathcal{G}_K}$ comprising all characters χ that satisfy the following conditions:

- χ is not totally odd;
- $U_{K,S(K_\infty)}^\chi$ is \mathbb{Z}_p -torsion-free (which is automatic unless $\chi = \omega_p$);
- if both $p = 3$ and $\chi^2 = \omega_p$, then $\text{Cl}_{K_\infty,S(K_\infty)}^{(p),\chi}$ is finite.

We note that Ψ_K is closed under the action of the absolute Galois group of \mathbb{Q}_p on $\text{Hom}(\mathcal{G}_K, \overline{\mathbb{Q}_p})$, and hence that the sum

$$e_{\Psi_K} := \sum_{\chi \in \Psi_K} e_\chi$$

defines an idempotent of $\mathbb{Z}_p[\mathcal{G}_K] \subset \Lambda_K$. In addition, the defining assumptions on characters in Ψ_K imply that, for each n , the \mathbb{Z}_p -module $e_{\Psi_K} U_{K_n, S(K_\infty)}$ is torsion-free and hence, just as at the beginning of §6.1.1, there is a natural isomorphism

$$\varprojlim_n (e_{\Psi_K} \bigcap_{\mathbb{Z}_p[\mathcal{G}_{K_n}]^{r_K}} U_{K_n, S(K_\infty)}) \cong e_{\Psi_K} \bigcap_{\Lambda_K}^{r_K} U_{K_\infty, S(K_\infty)}.$$

Via this isomorphism, any Euler system c in $\text{ES}_k(\mathbb{Z}_S)$ gives rise to an element $e_{\Psi_K} c_{K_\infty}$ of $e_{\Psi_K} \bigcap_{\Lambda_K}^{r_K} U_{K_\infty, S(K_\infty)}$, and hence also to an ideal $\text{im}(e_{\Psi_K} c_{K_\infty})$ of $e_{\Psi_K} \Lambda_K$.

(6.16) Theorem. *Fix a system c in $\text{ES}_k(\mathbb{Z}_S)$. Then, for each prime p outside S , one has*

$$\text{im}(e_{\Psi_K} c_{K_\infty})^{**} \subseteq e_{\Psi_K} \cdot \text{Fitt}_{\Lambda_K}^0(\text{Cl}_{K_\infty, S(K_\infty)}^p)^{**}.$$

In particular, if $p > 3$, $\widehat{\mathcal{G}}_K$ contains no totally odd characters, $U_{K, S(K_\infty)}$ is \mathbb{Z}_p -torsion free, and $\text{Cl}_{K_\infty, S(K_\infty)}^{p,\chi}$ is finite for both $\chi = \mathbf{1}_K$ and $\chi = \omega_{K,p}$, then one has

$$\text{im}(c_{K_\infty})^{**} \subseteq \text{Fitt}_{\Lambda_K}^0(\text{Cl}_{K_\infty, S(K_\infty)}^p)^{**}.$$

Proof. For each character χ in $\widehat{\mathcal{G}}_K$ we set $K_\chi := K^{\ker(\chi)}$, $\mathcal{G}_\chi := \mathcal{G}_{K_\chi}$, $\mathcal{R}_\chi := \mathbb{Z}_p[\text{im}(\chi)]$ and $\Lambda_\chi := \Lambda_K^\chi$. We also define a p -adic representation of G_k by setting $\mathcal{T}_\chi := \mathcal{R}_\chi(1)(\chi^{-1})$.

At the outset, we note that the second assertion of the theorem follows immediately from the first assertion and the fact the stated hypotheses imply $\Psi_K = \widehat{\mathcal{G}}_K$ so that $e_{\Psi_K} = 1$.

Concerning the first assertion, it is enough to prove, for each χ in Ψ_K , that the displayed inclusion is valid after replacing e_{Ψ_K} and $\text{Fitt}_{\Lambda_K}^0(\text{Cl}_{K_\infty, S(K_\infty)}^p)$ by e_χ and $\text{Fitt}_{\Lambda_\chi}^0(\text{Cl}_{K_\infty, S(K_\infty)}^{p,\chi})$ respectively. Indeed, this is true because we may verify the claimed inclusion after passing from Λ_K to its faithfully flat ring extension $\mathcal{O} \otimes_{\mathbb{Z}_p} \Lambda_K$ with \mathcal{O} the \mathbb{Z}_p -algebra generated by the values of all characters $\chi \in \Psi_K$.

In addition, if, for any such χ , the module $\text{Cl}_{K_\infty, S(K_\infty)}^{p,\chi}$ is finite, then the required inclusion is obvious since then $\text{Cl}_{K_\infty, S(K_\infty)}^{p,\chi}$ vanishes after localising at every height one prime of Λ_χ . Hence, for the rest of the argument we can, and will, fix a character χ such that $\chi \neq \omega_{K,p}$ and, if $p = 3$, also $\chi^2 \neq \omega_p$.

We write \mathcal{L} for the maximal abelian pro- p extension of k and note that all archimedean places split completely in \mathcal{L} (as p is odd) and that for any finite extension E of k in \mathcal{L} the integer $r_{\mathcal{T}_\chi}$ defined in Hypothesis (A.3) (i) below is equal to $r := r_{K_\chi} = r_{EK_\chi}$.

With $\text{ES}^r(\mathcal{T}_\chi, \mathcal{L})$ denoting the module of p -adic Euler systems specified in Definition (A.2) (with $\Pi = S_\infty(k) \cup S_p(k)$), we next define a non-zero map of $\mathbb{Z}[[\mathcal{G}_K]]$ -modules

$$(-)^\chi: \text{ES}_k(\mathbb{Z}_S) \rightarrow \text{ES}^r(\mathcal{T}_\chi, \mathcal{L}), \quad c \mapsto c^\chi := (\text{Tw}_{E,\chi}^r(c_{EK_\chi}))_{E \in \Omega(\mathcal{L}/k)} \quad (71)$$

in the following way. For each E in $\Omega(\mathcal{L}/k)$, the ring homomorphism $\mathbb{Z}_p[\mathcal{G}_\chi] \rightarrow \mathcal{R}_\chi$ induced by χ gives rise to a map of $\mathbb{Z}_p[\mathcal{G}_E]$ -modules

$$\begin{aligned} \text{Tw}_{E,\chi}^r: \bigcap_{\mathbb{Z}_p[\mathcal{G}_{EK_\chi}]^r} U_{EK_\chi, \Pi(EK_\chi)} &\rightarrow \left(\bigcap_{\mathbb{Z}_p[\mathcal{G}_{EK_\chi}]^r} U_{EK_\chi, \Pi(EK_\chi)} \right) \otimes_{\mathbb{Z}_p[\mathcal{G}_\chi]} \mathcal{R}_\chi \\ &\xrightarrow{\cong} \bigcap_{\mathcal{R}_\chi[\mathcal{G}_E]^r} H^1(\mathcal{O}_{EK_\chi, \Pi(EK_\chi)}, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p[\mathcal{G}_\chi]} \mathcal{R}_\chi \\ &\xrightarrow{\cong} \bigcap_{\mathcal{R}_\chi[\mathcal{G}_E]^r} H^1(\mathcal{O}_{EK_\chi, \Pi(EK_\chi)}, \mathcal{T}_\chi) \\ &\rightarrow \bigcap_{\mathcal{R}_\chi[\mathcal{G}_E]^r} H^1(\mathcal{O}_{E, \Pi(E)}, \mathcal{T}_\chi). \end{aligned}$$

Here the first isomorphism follows from Kummer theory, the second from the fact \mathcal{R}_χ is a flat $\mathbb{Z}_p[\mathcal{G}_\chi]$ -module and the last map is induced by $\text{cores}_{EK_\chi/E}$. As E varies over $\Omega(\mathcal{L}/k)$, these maps $\text{Tw}_{E,\chi}^r$ combine to give a morphism of the required form (71).

The key point now is that the present hypotheses imply \mathcal{T}_χ validates Hypothesis (A.3): the required conditions (i)-(v) are satisfied as a consequence of [BSS19b, Lem. 5.3], whilst condition (vi) (with $t = 0$) is clear in this case as k_∞ is the cyclotomic \mathbb{Z}_p -extension. We may therefore apply Theorem (A.5) to the Euler system c^χ and thereby obtain an inclusion of Λ_χ -ideals

$$\text{im}(c_{k_\infty}^\chi)^{**} \subseteq \text{Fitt}_{\Lambda_\chi}^0(H_{\mathcal{F}_{\text{can}}}^1(\mathcal{O}_{k,\Pi}, \mathcal{T}_{\chi,k_\infty}^\vee(1))^\vee)^{**}.$$

Given this, the claimed inclusion follows directly from the existence of a canonical isomorphism $\text{Cl}_{K_\infty, S(K_\infty)}^{p,\chi} \cong H_{\mathcal{F}_{\text{can}}}^1(\mathcal{O}_{k,\Pi}, \mathcal{T}_{\chi,k_\infty}^\vee(1))^\vee$ (as in (70)). \square

(6.17) Remark. Under the stated conditions, the second assertion of Theorem (6.16) combines with Lemma (6.6) (d) to reduce the verification of (46) to showing, for each $\chi \in \widehat{\mathcal{G}}_K$, that

$$\text{im}(c_{K_\infty}^\chi)^{**} \subseteq \text{Fitt}_{\Lambda_\chi}^0(X_{K_\infty, S(K_\infty)}^\chi)^{**}.$$

This inclusion is, in effect, a lower bound on the ‘order of vanishing at χ ’ of the values of an Euler system in terms of the ramification behaviour of p -adic places in K_∞ and can sometimes be interpreted very explicitly. For example, if k is not totally-complex, $\widehat{\mathcal{G}}_K$ contains no totally-odd characters, $p > 3$ and both $\text{Cl}_{k_\infty, S(K_\infty)}^p$ and, for each $n > 0$, also $(\text{Cl}_{K_\infty, S(K_\infty)}^p)^{\Gamma_k^n}$ are finite, then the argument of Lemma (6.6) (a) implies the above inclusion is valid provided that

$$(\gamma - 1)^{m(\chi)} \cdot \text{im}(c_{K_\infty}^\chi)^{**} \subseteq \prod_{v \in S(k_\infty, \chi)} \Lambda_\chi \cdot (\gamma_v - 1)$$

for every $\chi \in \widehat{\mathcal{G}}_K$, one has where $m(\chi) = 1$ if $\chi = \mathbf{1}_K$ and $m(\chi) = 0$ otherwise, $S(k_\infty, \chi)$ denotes the set of p -adic places of k that ramify in k_∞ and are totally split in K^χ and γ_v is a generator of the decomposition subgroup of v in Γ_k . Further, the last displayed inclusion is easily seen to be valid if, for example, k has only one p -adic place and so one obtains a full verification of (46) in any such case.

A. Appendix: Euler systems for p -adic representations

In this section we generalise one of the main results of Mazur and Rubin in [MR04] concerning the Iwasawa theory of rank-one Euler systems. This result is stated as Theorem (A.5) and shows that higher-rank Euler systems for a wide class of p -adic representations control the structure of Selmer groups in precisely the manner predicted by ‘Main Conjectures’ in this setting.

A.1. Statement of the main result

We fix a prime number p and a finite extension \mathcal{Q} of \mathbb{Q}_p with ring of integers \mathcal{R} , uniformiser ϖ , and residue field $\mathbb{k} := \mathcal{R}/\varpi$. We also assume to be given an abelian pro- p extension \mathcal{L} of k that validates the following hypothesis.

(A.1) Hypothesis (Hyp(\mathcal{L})).

- (i) for almost all primes \mathfrak{q} of k , the maximal p -power degree extension of k in the ray class field $k(\mathfrak{q})$ is contained in \mathcal{L} ;
- (ii) there exists a \mathbb{Z}_p -extension k_∞ of k in \mathcal{L} in which no finite place splits completely.

We write $\Omega(\mathcal{L}/k)$ for the collection of all finite extensions of k in \mathcal{L} , and for each field E in $\Omega(\mathcal{L}/k)$ we consider the algebras $\mathcal{R}_\mathcal{L} := \mathcal{R}[[\mathcal{G}_\mathcal{L}]]$, $\Lambda := \mathcal{R}[[\mathcal{G}_{k_\infty}]]$ and $\Lambda_E := \mathcal{R}[[\mathcal{G}_{Ek_\infty}]]$.

We fix a finitely generated free \mathcal{R} -module \mathcal{T} that admits a action of G_k that is continuous (with

respect to the canonical compact topology on \mathcal{T}) and such that the set $S_{\text{ram}}(\mathcal{T})$ of places of k at which \mathcal{T} is ramified is finite. We also fix a finite set of places Π of k containing

$$S_{\text{min}}(\mathcal{T}) := S_{\infty}(k) \cup S_p(k) \cup S_{\text{ram}}(\mathcal{T}),$$

and for each field E in $\Omega(\mathcal{L}/k)$ we set $\Pi(E) := \Pi \cup S_{\text{ram}}(E/k)$.

(A.2) Definition. For each non-negative integer r , the $\mathcal{R}_{\mathcal{L}}$ -module $\text{ES}^r(\mathcal{T}, \mathcal{L}) = \text{ES}_{\Pi}^r(\mathcal{T}, \mathcal{L})$ of (Π -imprimitive) Euler systems of rank r for \mathcal{T} over \mathcal{L} is the collection of families

$$(\eta_E)_E \in \prod_{E \in \Omega(\mathcal{L}/k)} \bigcap_{\mathcal{R}[\mathcal{G}_E]}^r H^1(\mathcal{O}_{E, \Pi(E)}, \mathcal{T})$$

with the property that for every pair $E \subseteq E'$ in $\Omega(\mathcal{L}/k)$ one has

$$\text{cores}_{E'/E}(\eta_{E'}) = \left(\prod_{v \in \Pi(E') \setminus \Pi(E)} P_v(\mathcal{T}, \text{Frob}_v^{-1}) \right) (\eta_E), \quad (72)$$

where $P_v(\mathcal{T}, X)$ denotes the characteristic polynomial $\det(1 - \text{Frob}_v^{-1} X \mid \mathcal{T}^*(1)) \in \mathcal{R}[X]$.

For a G_k -module A we write $k(A)$ for the minimal Galois extension of k such that $G_{k(A)}$ acts trivially on A . In the sequel we assume p is odd and for each non-negative integer m we consider the fields

$$\begin{aligned} k_{p^m} &:= k(1; p)k(\mu_{p^m}, (\mathcal{O}_k^{\times})^{1/p^m}), & k_{p^{\infty}} &:= \bigcup_{m > 0} k_{p^m}, \\ k(\mathcal{T})_{p^m} &:= k_{p^m}k(\mathcal{T}/p^m\mathcal{T}), & k(\mathcal{T})_{p^{\infty}} &:= k_{p^{\infty}}k(\mathcal{T}), \end{aligned}$$

where $k(1; p)$ is the maximal p -extension inside the Hilbert class field of k and $(\mathcal{O}_k^{\times})^{1/p^m}$ the subgroup of $\overline{\mathbb{Q}}^{\times}$ comprising all elements whose p^m -th powers belong to \mathcal{O}_k^{\times} .

In terms of this notation, we assume in the sequel that \mathcal{T} satisfies the following hypotheses.

(A.3) Hypothesis ($\text{Hyp}(\mathcal{T})$). All of the following conditions are satisfied.

- (i) The \mathcal{R} -module $Y_k(\mathcal{T}) := \bigoplus_{v \in S_{\infty}(k)} H^0(k_v, \mathcal{T})$ is non-zero and free.
- (ii) The residual representation $\overline{\mathcal{T}} := \mathcal{T} \otimes_{\mathcal{R}} \mathbb{k}$ is an irreducible $\mathbb{k}[G_k]$ -module.
- (iii) There exists $\tau \in G_{(k_{p^{\infty}}k_{\infty})}$ such that $\mathcal{T}/(\tau - 1)\mathcal{T}$ is a free \mathcal{R} -module of rank one.
- (iv) The cohomology groups $H^1(k(\mathcal{T})_{p^{\infty}}/k, \overline{\mathcal{T}})$ and $H^1(k(\mathcal{T})_{p^{\infty}}/k, \overline{\mathcal{T}}^{\vee}(1))$ vanish.
- (v) If $p = 3$, then $\overline{\mathcal{T}}$ and $\overline{\mathcal{T}}^{\vee}(1)$ have no non-zero isomorphic $\mathcal{R}[G_k]$ -subquotients.
- (vi) There exists a (finite) filtration $\{0\} = \mathcal{T}_t \subset \mathcal{T}_{t-1} \subset \cdots \subset \mathcal{T}_0 = \mathcal{T}$ of $\mathcal{R}[[G_k]]$ -modules where each module $\mathcal{T}_i/\mathcal{T}_{i+1}$ is free over \mathcal{R} and such that the induced action of G_k factors through $\text{Gal}(L_i/k)$ for some finite extension L_i of k_{∞} that is Galois over k .

(A.4) Remark. The conditions (i)-(v) in Hypothesis (A.3) are standard in the theory of Euler and Kolyvagin systems (following Mazur and Rubin). Condition (vi) is sufficient for our present purposes and allows an easier approach than is possible for more general representations.

We write $\mathcal{T}_{k_{\infty}}$ for the induction from $G_{k_{\infty}}$ to G_k of the representation \mathcal{T} . We recall that, in each degree $m \geq 0$, the Iwasawa cohomology module of \mathcal{T} over a field E in $\Omega(\mathcal{L}/k)$ is then the (finitely generated) Λ_E -module obtained by setting

$$H_{\text{Iw}}^m(\mathcal{O}_{E, \Pi(E)}, \mathcal{T}) := H^m(\mathcal{O}_{E, \Pi(E)}, \mathcal{T}_{k_{\infty}}) = \varprojlim_{n \in \mathbb{N}} H^m(\mathcal{O}_{E_n, \Pi(E)}, \mathcal{T}),$$

where the second equality follows from Shapiro's Lemma (cf. [MR04, Lem 5.3.1]), the limit taken over layers E_n of the \mathbb{Z}_p -extension $E_{\infty} := Ek_{\infty}$ of E with respect to corestriction maps. In the sequel we will use the fact that, if \mathcal{T} satisfies $\text{Hyp}(\mathcal{T})$, then for each non-negative integer r , there exists a canonical isomorphism of Λ_E -modules of the form

$$\bigcap_{\Lambda_E}^r H_{\text{Iw}}^1(\mathcal{O}_{E, \Pi(E)}, \mathcal{T}) \cong \varprojlim_{n \in \mathbb{N}} \bigcap_{\mathcal{R}[\mathcal{G}_{E_n}]}^r H^1(\mathcal{O}_{E_n, \Pi(E)}, \mathcal{T}), \quad (73)$$

where the limit is with respect to the maps on exterior biduals that are induced by corestriction. (This isomorphism is obtained by applying the general result of [Sak20, Lem. B.15] to the complex (77) defined below).

In the sequel we will also use, without further explicit comment, the theory of Selmer structures that is recalled in [BSS19b, § 2]. In particular, we write \mathcal{F}_{can} for the canonical Selmer structure on \mathcal{T}_{k_∞} .

Taking account of Hypothesis $\text{Hyp}(\mathcal{T})$ (i) we set

$$a = a_{\mathcal{T}} := \text{rank}_{\mathcal{R}}(Y_k(\mathcal{T})).$$

Then the next result extends to arbitrary values of a the (rank-one) results established by Mazur and Rubin in [MR04, § 5.3]. Its proof will occupy the remainder of the appendix.

(A.5) Theorem. *Assume that the extension \mathcal{L}/k and representation \mathcal{T} respectively satisfy the hypotheses $\text{Hyp}(\mathcal{L})$ and $\text{Hyp}(\mathcal{T})$. Assume also that there exists an Euler system η in $\text{ES}^a(\mathcal{T}, \mathcal{L})$ for which the element η_{k_∞} that corresponds, via the isomorphism (73), to the family $(\eta_{k_n})_n$ is non-zero. Then both of the following claims are valid.*

- (a) *The Λ -module $H_{\text{Iw}}^2(\mathcal{O}_{k,\Pi}, \mathcal{T})$ is torsion.*
- (b) *The Λ -module $H_{\mathcal{F}_{\text{can}}}^1(\mathcal{O}_{k,\Pi}, \mathcal{T}_{k_\infty}^\vee(1))^\vee$ is both finitely generated and torsion, and there is an inclusion of Λ -ideals $\text{im}(\eta_{k_\infty})^{**} \subseteq \text{char}_\Lambda(H_{\mathcal{F}_{\text{can}}}^1(\mathcal{O}_{k,\Pi}, \mathcal{T}_{k_\infty}^\vee(1))^\vee)$.*

(A.6) Remark. Claim (a) of Theorem (A.5) asserts that, under the stated hypotheses, the ‘weak Leopoldt conjecture’ of Perrin-Riou [PR98, § 1.3] is valid for \mathcal{T} over k_∞ . In addition, Lemma (A.10) below implies that if η_{k_∞} is non-zero, then the inclusion in claim (b) is equivalent to an inclusion of characteristic ideals

$$\text{im}(\eta_{k_\infty})^{**} = \text{char}_\Lambda\left(\left(\bigcap_{\Lambda}^a H_{\text{Iw}}^1(\mathcal{O}_{k,\Pi}, \mathcal{T})\right)/(\Lambda\eta_{k_\infty})\right) \subseteq \text{char}_\Lambda(H_{\mathcal{F}_{\text{can}}}^1(\mathcal{O}_{k,\Pi}, \mathcal{T}_{k_\infty}^\vee(1))^\vee).$$

This observation implies that Theorem (A.5) (b) improves upon earlier results in the literature that depend upon imposing restrictive hypotheses in order, for example, to rule out trivial zeroes of p -adic L -functions and, at least for representations of the form $\mathcal{R}(1)(\chi)$ with a Dirichlet character χ , also assume the validity of Leopoldt’s conjecture (see, for example, the results of Büyükboduk in [Büy09], [Büy10], and [Büy11], and of Mazigh in [Maz17] and [Maz19]).

A.2. Twisting representations

In this section we reduce the proof of Theorem (A.5) to consideration of a family of natural twists of \mathcal{T} . To describe these twisted representations, we fix a distinguished polynomial f in $\mathcal{R}[X]$ that is either constant (and hence equal to ϖ) or of degree one. For each natural number n , we then define a polynomial

$$f_n = \begin{cases} f + \varpi^n & \text{if } f = X + a_0 \text{ (with } a_0 \in \mathcal{R}), \\ X^n + \varpi & \text{if } f = \varpi, \end{cases}$$

fix a root α_n of f_n in $\overline{\mathcal{Q}}$ and consider the ring $\mathcal{R}_n := \mathcal{R}[\alpha_n]$. We also fix a topological generator γ of $\text{Gal}(k_\infty/k) \cong \mathbb{Z}_p$, define a character

$$\psi_n: \mathcal{G}_{k_\infty} \rightarrow \mathcal{R}_n, \quad \gamma \mapsto 1 + \alpha_n,$$

and consider the associated representation

$$\mathcal{T}(\psi_n) := \mathcal{T} \otimes_{\mathcal{R}} \mathcal{R}_n(\psi_n) \cong \mathcal{T} \otimes_{\mathcal{R}} (\Lambda/(\Lambda f_n)), \quad (74)$$

where the isomorphism (of $\mathcal{R}[[G_k]]$ -modules) sends each element $t \otimes \alpha_n^m$ to $t \otimes (X^m \bmod (f_n))$. We note that $\mathcal{T}(\psi_n)$ is free of finite rank as an \mathcal{R} -module and endowed with a continuous action of G_k that is unramified outside a finite set of places of k .

(A.7) Lemma. *The following claims are valid:*

- (a) *For every n the representation $\mathcal{T}(\psi_n)$ satisfies the hypotheses $\text{Hyp}(\mathcal{T}(\psi_n))$ (i) – (v);*

(b) *There exists an unbounded set of n such that, for every prime \mathfrak{q} of k outside Π , and every non-negative integer i , the endomorphism $\text{Frob}_{\mathfrak{q}}^{p^i} - 1$ acts injectively on $\mathcal{T}(\psi_n)$.*

Proof. Since p is odd, the functor $(-)\otimes_{\mathcal{R}}(\Lambda/(\Lambda f_n))$ commutes with the functor $H^0(k_v, -)$ for each $v \in S_{\infty}(k)$, and the validity of $\text{Hyp}(\mathcal{T}(\psi_n))$ (i) follows easily from this. In addition, since the element τ given by $\text{Hyp}(\mathcal{T})$ (iii) acts as the identity on k_{∞} , the element $\psi_n(\tau)$ acts as the identity on $\mathcal{T}(\psi_n)$ and so τ also validates $\text{Hyp}(\mathcal{T}(\psi_n))$ (iii).

We next note that, since f is a distinguished polynomial, one has $\psi_n \equiv 1 \pmod{\varpi}$ and so the residual representations $\overline{\mathcal{T}}$ and $\overline{\mathcal{T}(\psi_n)}$ coincide. The validity of $\text{Hyp}(\mathcal{T}(\psi_n))$ (ii), (iv) and (v) therefore follows from the assumed validity of $\text{Hyp}(\mathcal{T})$. This proves claim (a).

To prove claim (b), it is enough to verify the stated claim for the twisted base-change representation $\mathcal{V}(\psi_n) := (\mathcal{T} \otimes_{\mathcal{R}} \mathbb{C}_p)(\psi_n)$. By inducting on the length of the filtration of \mathcal{T} that is given by $\text{Hyp}(\mathcal{T})$ (vi), we can then also reduce to the case that the action of G_k factors through \mathcal{G}_L for a finite extension L of k_{∞} that is Galois over k .

To deal with this case we set $\mathcal{G} := \mathcal{G}_L$ and fix a pre-image σ of γ under the projection map $\mathcal{G} \rightarrow \mathcal{G}_{k_{\infty}}$. We write s for the order of the finite normal subgroup $H := \ker(\pi)$ of \mathcal{G} and t for the order of the automorphism $h \mapsto h^{\sigma} := \sigma h \sigma^{-1}$ of H . Then, for each element g of \mathcal{G} one has $g = h\sigma^c$ for unique elements $h \in H$ and $c \in \mathbb{Z}_p$, and hence, setting $m := st$, one has

$$g^m = (h\sigma^c)^m = \left(\prod_{i=0}^{m-1} h^{\sigma^{ci}} \right) \sigma^{cm} = \left(\prod_{i=0}^{t-1} h^{\sigma^{ci}} \right)^s \sigma^{cm} = \sigma^{cm},$$

where the last equality is true since $\prod_{i=0}^{t-1} h^{\sigma^{ci}}$ belongs to H .

We now fix a prime ideal \mathfrak{q} of k outside Π and a non-negative integer i , write the image in \mathcal{G} of the element $\kappa := \text{Frob}_{\mathfrak{q}}^{p^i}$ as $h_0\sigma^{c_0}$ and assume to be given a non-zero element v of $\mathcal{V}(\psi_n)$ that lies in the kernel of $\kappa - 1$.

Then, after fixing an isomorphism $\mathcal{V}(\psi_n) \cong \mathbb{C}_p^a$ of \mathbb{C}_p -vector spaces that gives rise to a morphism $\rho: G_k \rightarrow \text{Aut}_{\mathbb{C}_p}(\mathcal{V}(\psi_n)) \cong \text{GL}_a(\mathbb{C}_p)$ induced by the action of G_k on $\mathcal{V}(\psi_n)$, the above computation shows that

$$\rho(\sigma)^{c_0 m} \cdot v = \rho(\kappa)^m \cdot v = \psi_n^{-1}(\gamma^{c_0})^m \cdot v = \psi_n^{-1}(\gamma)^{c_0 m} \cdot v, \quad (75)$$

and so $\psi_n^{-1}(\gamma)^{c_0 m}$ is an eigenvalue of $\rho(\sigma)^{c_0 m}$. Hence, since $\rho(\sigma) = \nu \cdot A \cdot \nu^{-1}$ for some invertible matrix ν and a matrix A in Jordan normal form, it follows that $\rho(\sigma)$ has an eigenvalue λ such that $\lambda^{c_0 m} = \psi_n^{-1}(\gamma)^{c_0 m}$. Thus, we have $\lambda = \xi \cdot \psi_n^{-1}(\gamma)$ for some root of unity $\xi \in \overline{\mathcal{Q}}^{\times}$.

If $f \neq \varpi$, then $\mathcal{R}_n = \mathcal{R}$ and so both λ and $\psi_n(\gamma)$, and hence also ξ , are contained in a finite extension of \mathcal{R} that only depends on the eigenvalues of $\rho(\sigma)$. On the other hand, if $f = \varpi$ and n is prime to p , then \mathcal{R}_n is the ring of integers of a totally ramified extension of degree n of \mathcal{Q} and so the order of any root of unity (such as ξ) in the extension of \mathcal{Q} generated by \mathcal{R}_n and λ is bounded independently of n .

In particular, as $\rho(\sigma)$ has at most a distinct eigenvalues, and the $\psi_n(\gamma)$ are distinct as n ranges over all natural numbers, this argument shows that there exists a natural number n_0 that is independent of \mathfrak{q} and such that the equality (75) cannot be valid for any n that is prime to p and greater than n_0 .

This implies that for any such n , any $i \geq 0$ and any $\mathfrak{q} \notin \Pi$, the kernel of $\text{Frob}_{\mathfrak{q}}^{p^i} - 1$ on $\mathcal{V}(\psi_n)$ must vanish, thereby proving claim (b). \square

We obtain Theorem (A.5) by applying the theory of higher-rank Euler systems developed in [BSS19b] to the representations $\mathcal{T}(\psi_n)$. To prepare for this, we first prove two technical results.

(A.8) Lemma. *For each natural number n and each field E in $\Omega(\mathcal{L}/k)$, there exists a natural map of Λ_E -modules*

$$j_{E,n}: \left(\bigcap_{\Lambda_E}^a H_{\text{Iw}}^1(\mathcal{O}_{E,\Pi(E)}, \mathcal{T}) \right) \otimes_{\Lambda_E} \mathcal{R}_n[\mathcal{G}_E] \rightarrow \bigcap_{\mathcal{R}_n[\mathcal{G}_E]}^a H^1(\mathcal{O}_{E,\Pi(E)}, \mathcal{T}(\psi_n)), \quad (76)$$

where the tensor product is defined using the ring homomorphism $\psi_{E,n,*} : \Lambda_E \rightarrow \mathcal{R}_n[\mathcal{G}_E]$ that is induced by the composite of ψ_n and the diagonal map $\mathcal{G}_{E k_\infty} \hookrightarrow \mathcal{G}_E \times \mathcal{G}_{k_\infty}$. These maps have the following properties.

(a) Each map $j_{k,n}$ is injective.

(b) The maps $\{j_{E,n}\}_{E \in \Omega(\mathcal{L}/k)}$ induce a natural map $(-)^{\psi_n} : \text{ES}^a(\mathcal{T}, \mathcal{L}) \rightarrow \text{ES}^a(\mathcal{T}(\psi_n), \mathcal{L})$ of $\mathcal{R}_{\mathcal{L}}$ -modules.

Proof. For each E in $\Omega(\mathcal{L}/k)$ we define a complex of $\mathcal{R}[\mathcal{G}_E]$ -modules by setting

$$C_E^\bullet(\mathcal{T}) = R\text{Hom}_{\mathcal{R}[\mathcal{G}_E]}(R\Gamma_c(\mathcal{O}_{E,\Pi(E)}, \mathcal{T}), \mathcal{R}[\mathcal{G}_E]).$$

Under hypothesis $\text{Hyp}(\mathcal{T})$, the result of [BSS19b, Lem. 3.11] implies that $H^0(E, \mathcal{T})$ vanishes and the module $H^1(E, \mathcal{T})$ is \mathcal{R} -torsion-free. These facts combine with the general result of [BS21a, Prop. 2.21] to imply that $C_E^\bullet(\mathcal{T})$ is isomorphic in $D(\mathcal{R}[\mathcal{G}_E])$ to a complex of the form $P_E \rightarrow P_E$, where P_E is a free $\mathcal{R}[\mathcal{G}_E]$ -module of finite rank and the first term is placed in degree zero. By a standard argument of derived homological algebra (as, for example, in [BH21, Prop. 3.2]), it then follows that the derived limit complex

$$C_{E_\infty}^\bullet(\mathcal{T}) := R\varprojlim_{n \in \mathbb{N}} C_{E_n}^\bullet(\mathcal{T}) \quad (77)$$

is isomorphic in $D(\Lambda_E)$ to a complex $P_\infty \rightarrow P_\infty$, where P_∞ is a free Λ_E -module of finite rank (and the first term is placed in degree zero).

Next we note that the homomorphism $\psi_{E,n,*}$ combines with the general result of [FK06, Prop. 1.6.5] to induce an isomorphism in $D(\Lambda_E)$ of the form

$$[P_\infty \otimes_{\Lambda_E} \mathcal{R}_n[\mathcal{G}_E] \rightarrow P_\infty \otimes_{\Lambda_E} \mathcal{R}_n[\mathcal{G}_E]] \cong C_{E_\infty}^\bullet(\mathcal{T}) \otimes_{\Lambda_E}^{\mathbb{L}} \mathcal{R}_n[\mathcal{G}_E] \cong C_E^\bullet(\mathcal{T}(\psi_n)),$$

and hence also an identification $H^1(C_{E_\infty}^\bullet(\mathcal{T}) \otimes_{\Lambda_E}^{\mathbb{L}} \mathcal{R}_n[\mathcal{G}_E]) \cong H^1(\mathcal{O}_{E,\Pi(E)}, \mathcal{T}(\psi_n))$. On the other hand, the complexes (where in the first case the first term is placed in degree zero)

$$\begin{aligned} C_{E_\infty}^{a,\bullet}(\mathcal{T}) &:= \left[\bigwedge_{\Lambda_E}^a P_\infty \rightarrow P_\infty \otimes_{\Lambda_E} \bigwedge_{\Lambda_E}^{a-1} P_\infty \right] \\ C_E^{a,\bullet}(\mathcal{T}(\psi_n)) &:= C_{E_\infty}^{a,\bullet}(\mathcal{T}) \otimes_{\Lambda_E}^{\mathbb{L}} \mathcal{R}_n[\mathcal{G}_E] \end{aligned}$$

are acyclic in degrees less than zero and, moreover, [Sak20, Lem. B.12] implies that there are canonical isomorphisms

$$\begin{aligned} H^0(C_{E_\infty}^{a,\bullet}(\mathcal{T})) &\cong \bigcap_{\Lambda_E}^a H_{\text{Iw}}^1(\mathcal{O}_{E,\Pi(E)}, \mathcal{T}) \\ H^0(C_E^{a,\bullet}(\mathcal{T}(\psi_n))) &\cong \bigcap_{\mathcal{R}_n[\mathcal{G}_E]}^a H^1(\mathcal{O}_{E,\Pi(E)}, \mathcal{T}(\psi_n)). \end{aligned}$$

These facts combine with an explicit analysis of the second page of the spectral sequence of Λ_E -modules

$$E_2^{i,j} = \text{Tor}_{-i}^{\Lambda_E}(H^j(C_{E_\infty}^{a,\bullet}(\mathcal{T})), \mathcal{R}_n[\mathcal{G}_E]) \implies H^{i+j}(C_E^{a,\bullet}(\mathcal{T}(\psi_n))). \quad (78)$$

to imply the existence of a homomorphism $j_{E,n}$ of the required sort.

Moreover, if $E = k$ then (78) degenerates on its second page (since the Λ -module $\mathcal{R}_n \cong \Lambda/(\Lambda f_n)$ has projective dimension one) to yield a short exact sequence

$$0 \rightarrow \left(\bigcap_{\Lambda}^a H_{\text{Iw}}^1(\mathcal{O}_{k,\Pi(k)}, \mathcal{T}) \right) \otimes_{\Lambda} \mathcal{R}_n \rightarrow \bigcap_{\mathcal{R}_n}^a H^1(\mathcal{O}_{k,\Pi(k)}, \mathcal{T}(\psi_n)) \rightarrow H^1(C_{k_\infty}^{a,\bullet}(\mathcal{T}))[f_n] \rightarrow 0,$$

thereby proving claim (a).

To prove claim (b) we note $\text{ES}^a(\mathcal{T}, \mathcal{L})$ identifies with the $\mathcal{R}_{\mathcal{L}}$ -module comprising all elements of $\prod_{E \in \Omega(\mathcal{L}/k)} \bigcap_{\Lambda_E}^a H_{\text{Iw}}^1(\mathcal{O}_{E,\Pi}, \mathcal{T})$ that satisfy the (Iwasawa-theoretic analogue of the) relation (72). It is therefore clear that the maps $\{j_{E,n}\}_E$ combine to induce a map of $\mathcal{R}_{\mathcal{L}}$ -modules

$$\text{ES}^a(\mathcal{T}, \mathcal{L}) \rightarrow \prod_{E \in \Omega(\mathcal{L}/k)} \bigcap_{\mathcal{R}_n[\mathcal{G}_E]}^a H^1(\mathcal{O}_{E,\Pi(E)}, \mathcal{T}(\psi_n)).$$

To see that this induces a map $(-)^{\psi_n}$ of the required sort, it is enough to show its image is contained in $\text{ES}^a(\mathcal{T}(\psi_n), \mathcal{L})$. This is true since, for every field E in $\Omega(\mathcal{L}/k)$, and every place v of k outside $S_{\min}(\mathcal{T}(\psi_n)) \cup S(E)$, the image of $P_v(\mathcal{T}, \text{Frob}_v^{-1})$ in Λ_E is sent by $\psi_{E,n,*}$ to the element $P_v(\mathcal{T}, \psi_n(\text{Frob}_v^{-1}) \cdot \text{Frob}_v^{-1}) = P_v(\mathcal{T}(\psi_n), \text{Frob}_v^{-1})$ of $\mathcal{R}_n[\mathcal{G}_E]$. \square

Before stating the next result we recall that the global duality theorem implies the existence of a canonical exact sequence of Λ -modules

$$0 \longrightarrow H_{\mathcal{F}_{\text{can}}^*}^1(\mathcal{O}_{k,\Pi}, \mathcal{T}_{k_\infty}^\vee(1))^\vee \longrightarrow H_{\text{Iw}}^2(\mathcal{O}_{k,\Pi}, \mathcal{T}) \longrightarrow \bigoplus_{v \in \Pi} H^2(k_v, \mathcal{T}_{k_\infty}). \quad (79)$$

(Since \mathcal{F}_{can} agrees with the relaxed Selmer structure on \mathcal{T}_{k_∞} (by [MR04, Lem. 5.3.1 (ii)]), this sequence follows, for example, from the argument of Perrin-Riou in [PR98, Prop. A.3.2].)

The above exact sequence implies $H_{\mathcal{F}_{\text{can}}^*}^1(\mathcal{O}_{k,\Pi}, \mathcal{T}_{k_\infty}^\vee(1))^\vee$ is a finitely generated Λ -module and hence that its Λ -torsion submodule has a well-defined characteristic ideal.

(A.9) Lemma. *For all sufficiently large n , the following claims are valid:*

- (a) *The (injective) map $j_{k,n}$ constructed in Lemma (A.8) is a pseudo-isomorphism and the order of $\text{coker}(j_{k,n})$ is bounded independently of n .*
- (b) *If η is any system as in Theorem (A.5), then the component $j_{k,n}(\eta_{k_\infty})$ of η^{ψ_n} at k is non-zero and, in \mathcal{R}_n , one has*

$$\text{im}(j_{k,n}(\eta_{k_\infty})) \subseteq \text{Fitt}_{\mathcal{R}_n}^0(H_{\mathcal{F}_{\text{can}}^*}^1(\mathcal{O}_{k,\Pi}, \mathcal{T}(\psi_n)^\vee(1))^\vee).$$

In particular, $H_{\mathcal{F}_{\text{can}}^}^1(\mathcal{O}_{k,\Pi}, \mathcal{T}(\psi_n)^\vee(1))^\vee$ is finite.*

- (c) *The kernel and cokernel of the natural map (of \mathcal{R}_n -modules)*

$$s_n: H_{\mathcal{F}_{\text{can}}^*}^1(\mathcal{O}_{k,\Pi}, \mathcal{T}_{k_\infty}^\vee(1))^\vee \otimes_{\Lambda} \mathcal{R}_n \rightarrow H_{\mathcal{F}_{\text{can}}^*}^1(\mathcal{O}_{k,\Pi}, \mathcal{T}(\psi_n)^\vee(1))^\vee$$

are finite and of orders bounded independently of n .

- (d) *The polynomial f_n does not divide the characteristic polynomial of the Λ -torsion submodule of $H_{\mathcal{F}_{\text{can}}^*}^1(\mathcal{O}_{k,\Pi}, \mathcal{T}_{k_\infty}^\vee(1))^\vee$.*

Proof. The proof of Lemma (A.8) (a) shows that the cokernel of $j_{k,n}$ is isomorphic to $Q[f_n]$ with $Q := H^1(C_{k_\infty}^{a,\bullet}(\mathcal{T}))$. In particular, if n is chosen large enough so that the prime ideal Λf_n is not contained in the support of the Λ -torsion submodule of Q , then $Q[f_n]$ is pseudo-null and so its order is at most the cardinality of the maximal finite Λ -submodule of Q . This proves claim (a).

Next we note that, since the element η_{k_∞} is (by assumption) non-zero, and a non-zero element of a unique factorisation domain can only have finitely many irreducible factors, neither of the principal ideals that are given by $\text{im}(\eta_{k_\infty})^{**}$ and the characteristic ideal of the Λ -torsion submodule of $H_{\mathcal{F}_{\text{can}}^*}^1(\mathcal{O}_{k,\Pi}, \mathcal{T}_{k_\infty}^\vee(1))^\vee$ can be contained in infinitely many of the ideals Λf_n . In particular, for any large enough n , the polynomial f_n does not divide the characteristic polynomial of the Λ -torsion submodule of $H_{\mathcal{F}_{\text{can}}^*}^1(\mathcal{O}_{k,\Pi}, \mathcal{T}_{k_\infty}^\vee(1))^\vee$ and, in addition, the element $j_{k,n}(\eta_{k_\infty})$ is non-zero. The first of these properties immediately implies the property in claim (d), whilst the second property implies that the image η^{ψ_n} of η inside $\text{ES}^a(\mathcal{T}(\psi_n), \mathcal{L})$ is non-zero. Now Lemma (A.7) implies that the representation $\mathcal{T}(\psi_n)$ satisfies all the hypotheses (H_0) through (H_4) that are listed in [BSS19b, § 3.1.3], whilst $\mathcal{T}(\psi_n)$ tautologically satisfies the hypothesis (H_5^\dagger) in loc. cit. We may therefore apply the general result of [BSS19b, Thm. 3.6 (iii) (c)] to the non-zero Euler system η^{ψ_n} to directly obtain the inclusion in claim (b).

Since $j_{k,n}(\eta_{k_\infty})$ is non-zero, the ideal $\text{im}(j_{k,n}(\eta_{k_\infty}))$ is not the zero ideal and hence of finite index in \mathcal{R}_n . The second assertion of claim (b) then follows upon noting that, since \mathcal{R}_n is a discrete valuation ring, a finitely generated \mathcal{R}_n -module M is finite if and only if its Fitting ideal is of finite index in \mathcal{R}_n .

Finally we note that the validity of the property in claim (c) (for sufficiently large n) follows upon taking Pontryagin dual of the result of Mazur and Rubin in [MR16, Prop. 5.3.14]. \square

A.3. The proof of Theorem (A.5)

To prove claim (a) of Theorem (A.5) we fix any distinguished linear polynomial f of Λ , for example we may take $f = X$. We then let n be a natural number that is large enough to ensure that all of the claims in Lemma (A.9) are valid with respect to the polynomial f_n .

Then, by Lemma (A.9) (b), the Fitting ideal $\text{Fitt}_{\mathcal{R}_n}^0(H_{\mathcal{F}_{\text{can}}^*}^1(\mathcal{O}_{k,\Pi}, \mathcal{T}(\psi_n)^\vee(1))^\vee)$ contains a non-zero element and so the (finitely generated) \mathcal{R}_n -module $H_{\mathcal{F}_{\text{can}}^*}^1(\mathcal{O}_{k,\Pi}, \mathcal{T}(\psi_n)^\vee(1))^\vee$ is finite. From the result of Lemma (A.9) (c), it then follows that $H_{\mathcal{F}_{\text{can}}^*}^1(\mathcal{O}_{k,\Pi}, \mathcal{T}_{k_\infty}^\vee(1))^\vee \otimes_\Lambda \mathcal{R}_n$ is finite and hence, as a consequence of the structure theorem for finitely generated Λ -modules, that the Λ -module $H_{\mathcal{F}_{\text{can}}^*}^1(\mathcal{O}_{k,\Pi}, \mathcal{T}_{k_\infty}^\vee(1))^\vee$ is torsion.

This observation directly implies the first assertion of Theorem (A.5) (b) and also combines with the exact sequence (79), and the fact that no place in $\Pi \cap S_{\text{fin}}(k)$ splits completely in k_∞ , to imply that the Λ -module $H_{\text{Iw}}^2(\mathcal{O}_{k,\Pi}, \mathcal{T})$ is torsion, thereby proving Theorem (A.5) (a).

Before proving the rest of Theorem (A.5) (b), we record two general results that will be used (the second of which is well-known but we include its proof for lack of a better reference).

We recall that, for any finitely generated module M over a Noetherian ring R the group $\text{Ext}_R^1(M, R)$ is naturally a module over the endomorphism ring R^* of R .

(A.10) Lemma. *Let R be a Gorenstein ring, M a finitely generated R -module and r a non-negative integer. If a non-zero element m of $\bigcap_R^r M$ generates a free R -module, then one has*

$$\text{im}(m)^{**} = \text{Fitt}_{R^*}^0(\text{Ext}_R^1((\bigcap_R^r M)/(Rm), R))^{**}.$$

Proof. Since reflexive ideals of R are uniquely determined by their localisations at primes of height at most one, we may, and will, assume R is a Gorenstein ring of dimension at most one. In this case the module $\text{Ext}_R^1(\bigcap_R^r M, R)$ vanishes since the exterior bidual $\bigcap_R^r M$ is reflexive. Upon dualising the tautological exact sequence

$$0 \longrightarrow Rm \longrightarrow \bigcap_R^r M \longrightarrow (\bigcap_R^r M)/(Rm) \longrightarrow 0$$

we therefore obtain an exact commutative diagram

$$\begin{array}{ccccccc} (\bigcap_R^r M)^* & \longrightarrow & (Rm)^* & \longrightarrow & \text{Ext}_R^1((\bigcap_R^r M)/(Rm), R) & \longrightarrow & 0 \\ \theta \rightarrow \theta(m) \downarrow & & \theta \rightarrow \theta(m) \downarrow & & \vdots \downarrow & & \\ 0 & \longrightarrow & \text{im}(m) & \longrightarrow & R & \longrightarrow & R/\text{im}(m) \longrightarrow 0. \end{array}$$

Here the second vertical map is bijective as m generates a free module of rank one, and the first vertical map is surjective since the present hypotheses on R imply the natural map $\bigwedge_R^r M^* \rightarrow (\bigcap_R^r M)^*$ is surjective (cf. [NSW08, Prop. (5.4.9) (iii)]). Upon applying the Snake Lemma to the diagram, one therefore finds that the dotted vertical map (that is induced by the commutativity of the diagram) is an isomorphism. This isomorphism leads directly to the claimed description of $\text{im}(m)$. \square

(A.11) Lemma. *Let M be a finitely generated torsion Λ -module that is pseudo-isomorphic to*

$$E_M := \bigoplus_{j=1}^s \Lambda/(\Lambda g_j^{m_j}),$$

where each g_j is either ϖ or an irreducible distinguished polynomial and each m_j a natural number. Then, for any irreducible element f of Λ that does not divide $\text{char}_\Lambda(M)$, one has

$$\text{ord}_p(|M/(fM)|) - \text{ord}_p(|M_{\text{fin}}|) \leq \text{ord}_p(|E_M/(fE_M)|) \leq \text{ord}_p(|M/(fM)|),$$

where M_{fin} denotes the maximal finite Λ -submodule of M .

Proof. The existence of a map of Λ -modules from M to E_M (or in the reverse direction) that has finite kernel and cokernel combines with a calculation of Herbrand quotients to show that

$$|E_M[f]|/|E_M/(fE_M)| = |M[f]|/|M/(fM)|.$$

The claimed inequalities follow from this equality, the obvious inequality $|M[f]| \leq |M_{\text{fin}}|$ and the fact that $E_M[f]$ vanishes as f is coprime to each g_j . \square

Turning now to the proof of Theorem (A.5) (b), we note that, since $H_{\text{Iw}}^2(\mathcal{O}_{k,\Pi}, \mathcal{T})$ is a torsion Λ -module, the argument of [BD21, Thm. 3.8 (a)] implies $\bigcap_{\Lambda}^a H_{\text{Iw}}^1(\mathcal{O}_{k,\Pi}, \mathcal{T})$ is a free Λ -module of rank one. Hence, since $\eta_{k_{\infty}}$ generates a free Λ -module, the quotient Λ -module

$$Q_{\eta} := \left(\bigcap_{\Lambda}^a H_{\text{Iw}}^1(\mathcal{O}_{k,\Pi}, \mathcal{T}) \right) / (\Lambda \eta_{k_{\infty}})$$

is torsion. Upon applying Lemma (A.10) with $M = H_{\text{Iw}}^1(\mathcal{O}_{k,\Pi}, \mathcal{T})$, $r = a$ and $m = \eta_{k_{\infty}}$, and identifying Λ^* with $\Lambda^{\#}$ (where, as before, $\#$ indicates that the $\mathcal{G}_{k_{\infty}}$ -action has been inverted using the involution $\sigma \mapsto \sigma^{-1}$), we may therefore deduce that

$$\text{im}(\eta_{k_{\infty}})^{**} = \text{Fitt}_{\Lambda^{\#}}^0(\text{Ext}_{\Lambda}^1(Q_{\eta}, \Lambda))^{**} = \text{Fitt}_{\Lambda^{\#}}^0(Q_{\eta}^{\#})^{**} = \text{Fitt}_{\Lambda}^0(Q_{\eta})^{**} = \text{char}_{\Lambda}(Q_{\eta}),$$

where the second equality follows from [NSW08, Prop. (5.5.13)] and the last from the fact that $\text{Fitt}_{\Lambda}^0(M)^{**} = \text{char}_{\Lambda}(M)$ for any finitely generated torsion Λ -module M . The proof of claim (b) is therefore reduced to the verification of an inclusion of characteristic ideals

$$\text{char}_{\Lambda}(Q_{\eta}) \subseteq \text{char}_{\Lambda}(H_{\mathcal{F}_{\text{can}}^*}^1(\mathcal{O}_{k,\Pi}, \mathcal{T}_{k_{\infty}}^{\vee}(1))^{\vee}). \quad (80)$$

To check this, we fix generating elements z_{η} and z_{Sel} of the (principal) Λ -ideals $\text{char}_{\Lambda}(Q_{\eta})$ and $\text{char}_{\Lambda}(H_{\mathcal{F}_{\text{can}}^*}^1(\mathcal{O}_{k,\Pi}, \mathcal{T}_{k_{\infty}}^{\vee}(1))^{\vee})$, respectively. We note that z_{η} and z_{Sel} can both be factored as a product of irreducible distinguished polynomials and a power of the uniformiser ϖ of \mathcal{R} . In addition, since we are free to verify (80) after a faithfully flat base change, we may, and will, assume that all occurring irreducible distinguished polynomials are linear.

In particular, to verify (80), it is now enough for us to fix a factor f of z_{Sel} that is equal to either ϖ or to $X + a_0$ (for some a_0 in the maximal ideal of \mathcal{R}) and to show that

$$\text{ord}_f(z_{\eta}) \geq \text{ord}_f(z_{\text{Sel}}). \quad (81)$$

To do this, we let n be a natural number for which all the claims of Lemma (A.9) are valid with respect to the fixed choice of f . We then choose a pseudo-isomorphism of the form

$$H_{\mathcal{F}_{\text{can}}^*}^1(\mathcal{O}_{k,\Pi}, \mathcal{T}_{k_{\infty}}^{\vee}(1))^{\vee} \rightarrow \bigoplus_{i=1}^t \Lambda/(\Lambda f^{m_i}) \oplus \bigoplus_j \Lambda/(\Lambda g_j) \quad (82)$$

in which t is a non-negative integer, each m_i a natural number and each g_j a (possibly reducible) distinguished polynomial that is independent of n and also, since n validates Lemma (A.9) (d), coprime to f_n .

Then, for each index j and every such n , the quotient module $\Lambda/(\Lambda g_j + \Lambda f_n)$ is finite and we claim that its cardinality is bounded independently of n . To show this, we note that, because f_n is an irreducible distinguished polynomial, the Weierstrass Preparation Theorem gives an isomorphism $\Lambda/(\Lambda g_j + \Lambda f_n) \cong \mathcal{R}_n/(\mathcal{R}_n g_j(\alpha_n))$. It is then enough to note that the \mathcal{R}_n -valuation of $g_j(\alpha_n)$ is bounded since, for all large enough n , the strong triangle inequality implies that

$$\text{ord}_{\mathcal{R}_n}(g_j(\alpha_n)) = \begin{cases} \text{ord}_{\mathcal{R}}(g_j(a_0)) & \text{if } f = X + a_0, \\ \text{ord}_{\mathcal{R}_n}(\alpha_n^{\deg(g_j)}) = \deg(g_j) & \text{if } f = \varpi. \end{cases}$$

This observation implies the existence of constants κ_1 and κ_2 that are independent of n and

such that

$$\begin{aligned}
\text{ord}_p \left(\left| H_{\mathcal{F}_{\text{can}}}^1(\mathcal{O}_{k,\Pi}, \mathcal{T}(\psi_n)^\vee(1))^\vee \right| \right) &= \text{ord}_p \left(\left| H_{\mathcal{F}_{\text{can}}}^1(\mathcal{O}_{k,\Pi}, \mathcal{T}_{k_\infty}^\vee(1))^\vee \otimes_{\Lambda} \mathcal{R}_n \right| \right) + \kappa_1 \quad (83) \\
&\geq \text{ord}_p \left(\prod_{i=1}^t |\Lambda/(\Lambda f^{m_i} + \Lambda f_n)| \right) + \kappa_2 \\
&= \sum_{i=1}^t \text{ord}_p \left(|\mathcal{R}_n/(\mathcal{R}_n f(\alpha_n)^{m_i})| \right) + \kappa_2 \\
&= n \cdot \sum_{i=1}^t m_i + \kappa_2 \\
&= n \cdot \text{ord}_f(z_{\text{Sel}}) + \kappa_2,
\end{aligned}$$

where the first equality follows from an application of Lemma (A.9) (c), the inequality from the second inequality of Lemma (A.11) and the final equality from the pseudo-isomorphism (82) and choice of element z_{Sel} .

In a similar way, after fixing a pseudo-isomorphism of the form

$$Q_\eta \rightarrow \bigoplus_{i=1}^{t'} \Lambda/(\Lambda f^{l_i}) \oplus \bigoplus_j \Lambda/(\Lambda h_j), \quad (84)$$

in which t' is a non-negative integer, each l_i a natural number and each h_j a (possibly reducible) polynomial that is both independent of n and coprime to f_n , one finds that there exist constants κ_3 and κ_4 that are independent of n and such that

$$\begin{aligned}
&\text{ord}_p \left(\left| \left(\prod_{\mathcal{R}_n}^a H^1(\mathcal{O}_{k,\Pi}, \mathcal{T}(\psi_n)) \right) / (\mathcal{R}_n j_{k,n}(\eta_{k_\infty})) \right| \right) \quad (85) \\
&= \text{ord}_p \left(|Q_\eta \otimes_{\Lambda} \mathcal{R}_n| \right) + \kappa_3 \\
&\leq \text{ord}_p \left(\sum_{i=1}^{t'} |\Lambda/(\Lambda f^{l_i} + \Lambda f_n)| \right) + \kappa_4 \\
&= n \cdot \sum_{i=1}^{t'} l_i + \kappa_4 \\
&= n \cdot \text{ord}_f(z_\eta) + \kappa_4.
\end{aligned}$$

Here the first equality follows from Lemma (A.9) (a), the inequality from the first inequality of Lemma (A.11), and the final equality from the pseudo-isomorphism (84) and choice of z_η .

Now, Lemma (A.9) (b) combines with Lemma (A.10) to imply an inclusion

$$\text{Fitt}_{\mathcal{R}_n}^0 \left(\left(\prod_{\mathcal{R}_n}^a H^1(\mathcal{O}_{k,\Pi}, \mathcal{T}(\psi_n)) \right) / (\mathcal{R}_n j_{k,n}(\eta_{k_\infty})) \right) \subseteq \text{Fitt}_{\mathcal{R}_n}^0 \left(H_{\mathcal{F}_{\text{can}}}^1(\mathcal{O}_{k,\Pi}, \mathcal{T}(\psi_n)^\vee(1))^\vee \right).$$

Since \mathcal{R}_n is a discrete valuation ring with finite residue field, this inclusion is therefore equivalent to an inequality

$$\left| \left(\prod_{\mathcal{R}_n}^a H^1(\mathcal{O}_{k,\Pi}, \mathcal{T}(\psi_n)) \right) / (\mathcal{R}_n j_{k,n}(\eta_{k_\infty})) \right| \geq \left| H_{\mathcal{F}_{\text{can}}}^1(\mathcal{O}_{k,\Pi}, \mathcal{T}(\psi_n)^\vee(1))^\vee \right|.$$

Upon combining this inequality with those of (83) and (85), we derive an inequality

$$n \cdot \text{ord}_f(z_\eta) + \kappa_4 \geq n \cdot \text{ord}_f(z_{\text{Sel}}) + \kappa_2.$$

By taking n sufficiently large, this then implies the inequality (81) and hence completes the proof of Theorem (A.5). \square

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