Tangent cones to positive-$(1,1)$ De Rham currents

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Abstract: *We consider positive-*$(1,1)$ *De Rham currents in arbitrary almost complex manifolds and prove the uniqueness of the tangent cone at any point where the density does not have a jump with respect to all of its values in a neighbourhood. Without this assumption, counterexamples to the uniqueness of tangent cones can be produced already in* $\mathbb{C}^n$*, hence our result is optimal. The key idea is an implementation, for currents in an almost complex setting, of the classical blow up of curves in algebraic or symplectic geometry. Unlike the classical approach in* $\mathbb{C}^n$*, we cannot rely on plurisubharmonic potentials.*

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1 Introduction

In many problems from analysis one is naturally led to study possibly non-smooth objects: $W^{1,2}$-harmonic maps between manifolds, volume-minimizing currents and weak solutions to equations are a few important examples. In order to understand the behaviour of the object around a singular point, the first study that is typically done is the *blow-up analysis*. We look at the object inside smaller and smaller balls $B_{r_n}(x)$ centered at the chosen point x and dilate to a reference size (e.g. the unit ball). For any sequence $r_n \downarrow 0$ of radii the rescaled objects converge, up to a subsequence, to what is called *a tangent* (tangent maps, tangent cones...). Of course we ask the question: will we get different tangents by choosing different sequences of radii for the blow-up analysis? If not, then we say that the object under investigation has a *unique tangent* at the chosen point. This uniqueness is a very important regularity property, which has been widely investigated in several problems using different techniques. Without hoping to do justice to the vast literature, we present a short overview (see also the survey [15]).

Regarding tangent cones at a point x of a mass-minimizing current it is known that the masses of the rescaled currents converge in a non-increasing fashion towards the so-called density at x: the speed of convergence is called *rate of decay* of the mass ratio at x. An approach often used to prove uniqueness of the tangent cone at x is to show that this rate of decay is fast enough (see [12] 5.4.3). In [34] B. White proved the uniqueness of tangent cone at all points of a 2-dimensional mass-minimizing integral cycle by showing, via a comparison method, an epiperimetric inequality, from which the desired decay followed. In [24] D. Pumberger and T. Rivière proved, also by showing the "fast decay property", that at any point of a semi-calibrated integral 2-cycle the tangent cone is unique.

In other works on (semi-)calibrated 2-cycles alternative proofs have been given by using techniques of slicing with positive intersection: this is the case of integral pseudo-holomorphic 2-cycles in dimension 4 (C. H. Taubes in [31], T. Rivière and G. Tian in [25]) and integral Special Legendrian 2-cycles in dimension 5 (the author and T. Rivière in [2], [3]).

In [26] the uniqueness for pseudo holomorphic integral 2-dimensional cycles is achieved in arbitrary codimension by means of a lower-epiperimetric inequality.

In [29] L. Simon proved that if a tangent cone to a minimal integral current has multiplicity one and has an isolated singularity, then it is unique. This proof applies to tangents at isolated singular points for harmonic maps taking values into an analytic manifold and is based on the Lojaciewicz inequality, again leading to a rate of decay (for the energy) which implies the uniqueness. On the other hand, White showed in [35] that tangent maps at isolated singularities of harmonic maps might fail to be unique if the assumption of analiticity on the target manifold is dropped.

Negative answers to the uniqueness of tangent cones have also been obtained in the case of non-rectifiable mass-minimizing currents: this failure was proved for positive-(p, p) normal cycles in a complex manifold by C. O. Kiselman in [18]. In further works, e.g. [6] and [7], necessary and sufficient conditions on the rate of decay of the mass ratio were given, under which the uniqueness holds (these works are closely related to the issue of tangent maps to plurisubharmonic maps).

The problems described so far are of elliptic type, the use of blow-up techniques goes however much further. For example in [1] the authors address a rectifiability issue for a measure arising in the context of conservation laws for hyperbolic PDEs and employ for the proof a delicate blow-up analysis. Turning our attention to a parabolic problem, the classification of possible singularities arising after finite time for a Mean-Curvature Flow is again built upon a blow-up analysis.

In the present work, an announcement of which appeared in [4], we will be dealing with a a first order elliptic problem: we address the issue of the uniqueness of blow-ups for positive-$(1, 1)$ normal cycles in almost complex manifolds. We present a new technique, which does not require the understanding of the rate of decay. An analogous approach can be used (see [5]) to yield the uniqueness of tangent cones to pseudo-holomorphic integral cycles of arbitrary dimension and codimension.

We will now describe the setting and the connections to other problems, after which a sketch of the proof will be provided.

Setting. Let $(\mathcal{M}, J)$ be a smooth almost complex manifold of dimension $2n+2$ (with $n \in \mathbb{N}^*$), endowed with a non-degenerate 2-form ω compatible with J. If $d\omega = 0$ then we have a symplectic form, but we will not need to assume closedness. Let g be the associated Riemannian metric, $g(\cdot,\cdot) := \omega(\cdot, J\cdot)$.

The form ω is a semi-calibration on $\mathcal{M}$ for the metric g, i.e. the comass $\|\omega\|^*$ is 1; recall that the comass of ω is defined to be

$$||\omega||^* := \sup\{\langle \omega_x, \xi_x \rangle : x \in \mathcal{M},\ \xi_x \text{ is a unit simple 2-vector at } x\},$$

where the metric that we are using on $T_x\mathcal{M}$ is naturally g_x. Then $\|\omega\|^* = 1$ follows from $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$, recalling that J is an orthogonal endomorphism. If ω is closed, then we have a classical calibration, as in [16].

Among the oriented 2-dimensional planes of the Grassmannians $G(x, T_x\mathcal{M})$, we pick those that, represented as unit simple 2-vectors, realize the equality $\langle \omega_x, \xi_x \rangle = 1$. Define the set $\mathcal{G}(\omega)$ of *2-planes calibrated by* ω as

$$\mathcal{G}(\omega) := \cup_{x\in\mathcal{M}}\ \mathcal{G}_x := \cup_{x\in\mathcal{M}}\{\xi_x \in G(x, T_x\mathcal{M}) : \langle \omega_x, \xi_x \rangle = 1\}.$$

Before turning to the main object of these work, let us recall a few facts from Geometric Measure Theory.

Currents were first introduced by De Rham as the dual space of smooth and compactly supported differential forms (see [8]). Some distinguished classes of currents have, since the sixties, played a key role in Geometric Measure Theory (see [13], [12], [23], [14] or [28]).

For De Rham currents we have the notions of boundary and mass, which we now recall in the case of interest, i.e. a 2-dimensional De Rham current C (the case of general dimension is completely analogous).

The boundary ∂C of C is the 1-dimensional current characterized by its action on an arbitrary compactly supported one-form α as follows:

$$(\partial C)(\alpha) := C(d\alpha) = 0.$$

The mass of C is

$$M(C) := \sup\{C(\beta) : \beta \text{ compactly supported 2-form}, ||\beta||^* \leq 1\}.$$

A De Rham current C such that $M(C)$ and $M(\partial C)$ are finite is called a *normal current.* Any current C of finite mass is representable by integration (see [14] pages 125-126), i.e. there exist

(i) a positive Radon measure $\|C\|$,

(ii) a generalized tangent space $\vec{C}_x \in \Lambda_2\ (T_x\mathcal{M})$, that is defined for $\|C\|$-a.a. points x, is $\|C\|$-measurable and has[1] mass-norm 1,

such that the action of C on any 2-form β with compact support is expressed as follows

$$C(\beta) = \int_{\mathcal{M}} \langle \beta, \vec{C} \rangle d\|C\|.$$

A current with zero boundary is shortly called a *cycle*. We will consider a ω-**positive** normal 2-cycle T. Equivalent notions of ω-positiveness (see [16] or [17]) are

- $\vec{T} \in$ convex hull of $\mathcal{G}(\omega)$ $\quad \|T\|$-a.e.
- $\langle \omega, \vec{T} \rangle = 1 \quad \|T\|$-a.e.

The last condition is clearly equivalent to the important equality

$$T(\omega) = \int_{\mathcal{M}} \langle \omega, \vec{T} \rangle d\|T\| = M(T). \tag{1}$$

Remark that for arbitrary currents $M(C) := \sup\{C(\beta) : ||\beta||^* \leq 1\}$ and in general this sup need not be achieved. Also remark that for currents of finite mass the action can be extended to forms with non-compact support (actually to forms with merely bounded Borel coefficients, see [14] page 127). So $T(\omega)$ in (1) makes sense.

In the case when ω is closed, from (1) one also gets the important fact that a ω-positive T is (locally) homologically mass-minimizing (see [16]). In the case of a non-closed ω, the same argument shows that a ω-positive cycle T is locally an almost-minimizer of the mass (also called λ-minimizer). When the normal cycle is actually rectifiable (see [12] or [14] for definitions) a common term used, instead of ω-positive, is ω-(semi)calibrated.

In the case we are investigating there is a useful equivalent characterization for the fact that a unit simple 2-vector at x is in $\mathcal{G}_x$, i.e. it is ω_x-calibrated. Indeed, testing on $w_1 \wedge w_2$ such that w_1 and w_2 are unit orthogonal vectors at x for g_x and recalling that J is an othogonal endomorphism of the tangent space we get

$$\omega_x(w_1 \wedge w_2) = 1 \Leftrightarrow g_x(J_x(w_1), w_2) = 1 \Leftrightarrow J_x(w_1) = w_2. \tag{2}$$

Thus a 2-plane is in $\mathcal{G}_x$ if an only if it is J_x-invariant or, in other words, if an only if it is J_x-holomorphic.

So an equivalent way to express ω-positiveness is that $\|T\|$-a.e. $\vec{T}$ belongs to the convex hull of J-holomorphic simple unit 2-vectors, in particular $\vec{T}$

[1] The mass-norm for 2-vectors is defined in duality with the comass on two-forms. The unit ball for the mass-norm on $\Lambda_2\mathbb{R}^{2n+2}$ is the convex envelope of unit simple 2-vectors.

itself is J-invariant. For this reason ω-positive normal cycles are also called positive-$(1,1)$ normal cycles[2]. Remarkably the $(1,1)$-condition **only depends** on J, so a positive-$(1,1)$ cycle is ω-positive for any J-compatible couple (ω, g). This fact will turn out to be a key ingredient in our argument.

Positive cycles satisfy an important *almost monotonicity property*: at any point x_0 the mass ratio $\frac{M(T \llcorner B_r(x_0))}{\pi r^2}$ is an almost-increasing function of r, i.e. it can be expressed as a weakly increasing function of r plus an infinitesimal of r. The precise statement can be found in Section 2.

Monotonicity yields a well-defined limit

$$\nu(x_0) := \lim_{r \to 0} \frac{M(T \llcorner B_r(x_0))}{\pi r^2}.$$

This is called the (two-dimensional) **density** of the current T at the point x_0 (*Lelong number* in the classical literature, see [20]). The almost monotonicity property also yields that the density is an upper semi-continuous function.

Consider a dilation of T around x_0 of factor r which, in normal coordinates around x_0, is expressed by the push-forward of T under the action of the map $\dfrac{x - x_0}{r}$:

$$(T_{x_0,r} \llcorner B_1)(\psi) := \left[\left(\frac{x - x_0}{r} \right)_* T \right] (\chi_{B_1} \psi) = T \left(\chi_{B_r(x_0)} \left(\frac{x - x_0}{r} \right)^* \psi \right). \tag{3}$$

The current $T_{x_0,r}$ is positive for the semi-calibration

$$\omega_{x_0,r} := \frac{1}{r^2} (r(x - x_0))^* \omega,$$

with respect to the metric

$$g_{x_0,r}(X, Y) := \frac{1}{r^2} g \left((r(x - x_0))_* X, (r(x - x_0))_* Y \right).$$

We thus have the equality $M(T_{x_0,r} \llcorner B_1) = \frac{M(T \llcorner B_r(x_0))}{r^2}$, where the masses are computed respectively with respect to $g_{x_0,r}$ and g.

The fact that $\dfrac{M(T \llcorner B_r(x_0))}{r^2}$ is monotonically almost-decreasing as $r \downarrow 0$ gives that, for $r \le r_0$ (for a small enough r_0), we are dealing with a family of currents $\{T_{x_0,r} \llcorner B_1\}$ that satisfy the hypothesis of Federer-Fleming's

[2]We are using the term *dimension* for a current as it is customary in Geometric Measure Theory, i.e. the dimension of a current is the degree of the forms it acts on. Remark however that in the classical works on positive currents and plurisubharmonic functions, e.g. [20] or [30], our 2-cycle in $\mathbb{C}^{n+1}$ would actually be called a current of bidimension $(1,1)$ and bidegree (n,n).

compactness theorem (see [14] page 141) with respect to the flat metric (the metrics $g_{x_0,r}$ converge, as $r \to 0$, uniformly to the flat metric g_0).

Thus there exist a sequence $r_n \to 0$ and a boundaryless current T_∞ such that

$$T_{x_0,r_n} \llcorner B_1 \to T_\infty.$$

This procedure is called the *blow up limit* and the idea goes back to De Giorgi [10]. Any such limit T_∞ turns out to be a cone (a so called **tangent cone** to T at x_0) with density at the origin the same as the density of T at x_0. Moreover T_∞ is ω_{x_0}-positive.

The main issue regarding tangent cones is whether the limit T_∞ depends or not on the sequence $r_n \downarrow 0$ yielded by the compactness theorem, i.e. whether T_∞ is **unique or not**. It is not hard to check that any two sequences $r_n \to 0$ and $\rho_n \to 0$ fulfilling $a \leq \frac{r_n}{\rho_n} \leq b$ for $a, b > 0$ must yield the same tangent cone, so non-uniqueness can arise for sequences with different asymptotic behaviours.

The fact that a current possesses a unique tangent cone is a symptom of regularity, roughly speaking of regularity at infinitesimal level. It is generally expected that currents minimizing (or almost-minimizing) functionals such as the mass should have fairly good regularity properties. This issues are however hard in general.

The uniqueness of tangent cones is known for some particular classes of integral currents, namely for mass-minimizing integral cycles of dimension 2 ([34]) and for general semi-calibrated integral 2-cycles ([24]).

Passing more generally to normal currents, things get harder. Many examples of ω-positive normal 2-cycles can be given by taking a family of pseudoholomorphic curves and assigning a positive Radon measure on it (this can be made rigorous). However ω-positive normal 2-cycles need not be necessarily of this form, as the following example shows.

Example 1.1. In $\mathbb{R}^4 \cong \mathbb{C}^2$, with the standard complex structure, consider the unit sphere S^3 and the standard contact form γ on it.

The 2-dimensional current C_1 supported in S^3 and dual to γ, i.e. defined by $C_1(\beta) := \int_{S^3} \gamma \wedge \beta \; d\mathcal{H}^3$, is positive-$(1,1)$ and its boundary is given by $\partial C_1(\alpha) := \int_{S^3} d\gamma \wedge \alpha \; d\mathcal{H}^3$, i.e. the boundary is the 1-current given by the uniform Hausdorff measure on S^3 and the Reeb vector field.

Now consider the positive-$(1,1)$ cone C with vertex at the origin, obtained by assigning the uniform measure $\frac{1}{4\pi}\mathcal{H}^2$ on $\mathbb{CP}^1$, i.e. C is obtained by taking the family of holomorphic disks through the origin and endowing it with a unifom measure of total mass 1. The current $C_2 := C \llcorner (\mathbb{R}^4 \setminus \overline{B_1^4(0)})$ has boundary $\partial C_2 = -\partial C_1$, therefore $C_1 + C_2$ is a positive-$(1,1)$ cycle.

This construction shows that a ω-positive normal 2-cycle T is not very rigid and it is not true that, restricting for example to a ball B, the current

$T \llcorner B$ is the unique minimizer for its boundary (which is instead true for integral cycles). Indeed it is not even the unique ω-positive normal current with that boundary. This can be interpreted as a *lack of unique continuation* for these currents.

This issue reflects (as will be shown in Section 6) into the fact that the uniqueness of tangent cones to ω-positive normal 2-cycles **fails** in general, already in the case of the complex manifold $(\mathbb{C}^n, J_0)$, where J_0 is the standard complex structure: this was proven by Kiselman [18]. Further works extended the result to arbitrary dimension and codimension (see [6] and [7], where conditions on the rate of convergence of the mass ratio are given, under which uniqueness holds).

While in the integrable case $(\mathbb{C}^n, J_0)$ positive cycles have been studied quite extensively, there are no results avaliable when the structure J is *almost complex*.

In this work we prove the following result:

Theorem 1.1. *Given an almost complex $(2n+2)$-dimensional manifold $(\mathcal{M}, J, \omega, g)$ as above, let T be a positive-$(1,1)$ normal cycle, or equivalently a ω-positive normal 2-cycle.*

Let x_0 be a point of positive density $\nu(x_0) > 0$ and assume that there is a sequence $x_m \to x_0$ of points $x_m \neq x_0$ all having positive densities $\nu(x_m)$ and such that $\nu(x_m) \to \nu(x_0)$.

Then the tangent cone at x_0 is unique and is given by $\nu(x_0)[\![D]\!]$ for a certain J_{x_0}-invariant disk D.

The notation $[\![D]\!]$ stands for the current of integration on D. Our proof actually yields the stronger result stated in Theorem 2.1.

In the integrable case $(\mathbb{C}^n, J_0)$, Siu [30] proved a beautiful and remarkable regularity theorem, which in our situation states the following: given $c > 0$, the set of points of a positive-$(1,1)$ cycle of density $\geq c$ is made of analytic varieties each carrying a positive, real, constant multiplicity. Therefore, in the integrable case, Theorem 1.1 follows from Siu's result.

In the non-integrable case, on the other hand, there are no regularity results avaliable at the moment. The proofs of Siu's theorem given in the integrable case, see [30], [19], [21], [9], strongly rely on a connection with a plurisubharmonic potential for the current, which is not avaliable in the almost complex setting.

In addition to the interest for tangent cones themselves, Theorems 1.1 and 2.1 are a first step towards a regularity result analogous to the one in [30], this time in the non-integrable setting (they can be seen as an infinitesimal version of that). The quest for such a regularity result is strongly motivated by several geometric issues, problems where the structure must be perturbed from a complex to almost complex one, in order to ensure some transversality conditions. Some of these are discussed in [11], [25], [32], [33]. We give here an example related to the study of pseudo-holomorphic maps into algebraic varieties, as those analyzed in [25]. Indeed, if $u : M^4 \to \mathbb{CP}^1$ is pseudoholomorphic and locally strongly approximable as in [25], with M^4 a compact closed 4-dimensional almost-complex manifold, denoting by ϖ the symplectic form on $\mathbb{CP}^1$, then the 2-current U defined by $U(\beta) := \int_{M^4} u^*\varpi \wedge \beta$ is a positive-$(1,1)$ normal cycle in M^4. As explained in [25], as a consequence of the fact that such a map is stationary harmonic, the singular set of u is of zero $\mathcal{H}^2$-measure and coincides with the set of points where the density of U is $\geq \epsilon$, for a positive ϵ depending on M^4 (this is a so-called ε-regularity result, see [27]). Then we would be reduced, in order to understand singularities of u, to the study of points of density $\geq \epsilon$ of U. Knowing that such a set is made of pseudoholomorphic subvarieties, together with the fact that it is $\mathcal{H}^2$-null, would imply that the singular set is made of isolated points, the same result achieved in [25] with different techniques. Let us now go a bit further in this example and roughly describe how we can also face compactness issues. Consider a sequence $\{u_n\}$ of maps as in [25] and assume $u_n \rightharpoonup u_\infty$ weakly in $W^{1,2}$. For the associated currents $U_n(\beta) := \int_{M^4} u_n^*\varpi \wedge \beta$ we have the convergence $U_n \rightharpoonup U$, where U is a positive-$(1,1)$ current but it does not necessarily hold that $U(\beta) = \int_{M^4} u_\infty^*\varpi \wedge \beta$ because bubbling phenomena can occur. Indeed we will have a concentration set (where bubbling occurs) and the limiting map u_∞ is a priori not necessarily stationary harmonic (from the general theory of harmonic maps). This failure of stationarity is due to the possible appearance of topological singularities for u_∞. However topological singularities must be the boundary of the concentration set and the concentration set is characterized as the set of points where the density for U is $\geq \epsilon$. Then a result analogous to Siu's would give that the concentration set is a cycle, thus ruling out the appearence of topological singularities and yielding that u_∞ is also stationary harmonic.

The strategy might then be applied to other dimensions. Positive-$(1,1)$ cycles, or more generally other calibrated currents, might also serve for other kind of problems, in which ε-regularity results play a role, for example when dealing with some Yang-Mills fields for high dimensional Gauge Theory (see for example the case of anti-self-dual instantons in Section 5 of the survey [33]).

Sketch of the proof. The key idea for the proof of our result is to realize for our current a sort of "*algebraic blow up*".

This is a well-known construction in Algebraic and Symplectic Geometry, with the name *"blow up"*. To avoid confusion we will call it *algebraic blow up*, since we have already introduced the notion of blow up as limit of dilations, as customary in Geometric Measure Theory. We now briefly recall the *algebraic blow up* in the complex setting (see figure 2).

Algebraic blow up (or *proper transform*), (see [22]). Define $\widetilde{\mathbb{C}}^{n+1}$ to be the submanifold of $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ made of the pairs $(\ell, (z_0, ...z_n))$ such that $(z_0, ...z_n) \in \ell$.

$\widetilde{\mathbb{C}}^{n+1}$ is a complex submanifold and inherits from $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ the standard complex structure, which we denote I_0. The metric $\mathbf{g}_0$ on $\widetilde{\mathbb{C}}^{n+1}$ is inherited from the ambient $\mathbb{CP}^n \times \mathbb{C}^{n+1}$, that is endowed with the product of the Fubini-Study metric on $\mathbb{CP}^n$ and of the flat metric on $\mathbb{C}^{n+1}$. Let $\Phi : \widetilde{\mathbb{C}}^{n+1} \to \mathbb{C}^{n+1}$ be the projection map $(\ell, (z_0, ...z_n)) \to (z_0, ...z_n)$. Φ is holomorphic for the standard complex structures J_0 on $\mathbb{C}^{n+1}$ and I_0 on $\widetilde{\mathbb{C}}^{n+1}$ and is a diffeomorphism between $\widetilde{\mathbb{C}}^{n+1} \setminus (\mathbb{CP}^n \times \{0\})$ and $\mathbb{C}^{n+1} \setminus \{0\}$. Moreover the inverse image of $\{0\}$ is $\mathbb{CP}^n \times \{0\}$.

$\widetilde{\mathbb{C}}^{n+1}$ is a complex line bundle on $\mathbb{CP}^n$ but we will later view it as an orientable manifold of (real) dimension $2n + 2$. The transformation Φ^{-1} (called *proper transform*) sends the point $0 \neq (z_0, ...z_n) \in \mathbb{C}^{n+1}$ to the point $([z_0, ...z_n], (z_0, ...z_n)) \in \widetilde{\mathbb{C}}^{n+1} \subset \mathbb{CP}^n \times \mathbb{C}^{n+1}$. With the almost complex structures J_0 and I_0, the J_0-holomorphic planes through the origin are sent to the fibers of the line bundle, which are I_0-holomorphic planes.

Outline of the argument. We have a positive-$(1, 1)$ normal cycle T in $\mathbb{C}^{n+1}$, at the moment with reference to the standard complex structure J_0, and we want to to understand the tangent cones at the origin, that we assume to be a point of density 1. By assumption we have a sequence of points $x_m \to 0$ with densities converging to 1. Take a subsequence x_{m_k} such that $\frac{x_{m_k}}{|x_{m_k}|} \to y$ for a point $y \in \partial B_1$.

We can make sense (section 4) of the proper transform $\left(\Phi^{-1}\right)_* T$, although the map Φ^{-1} degenerates at the origin, and prove that $\left(\Phi^{-1}\right)_* T$ is a positive-$(1, 1)$ normal cycle in $(\widetilde{\mathbb{C}}^{n+1}, I_0, \mathbf{g}_0)$.

The densities of points different than the origin are preserved under the proper transform (see the appendix), therefore the current $\left(\Phi^{-1}\right)_* T$ has a sequence of points converging to a certain y_0 (that lives in $\mathbb{CP}^n \times \{0\} \subset \widetilde{\mathbb{C}}^{n+1}$) and the densities of these points converge to 1. More precisely $y_0 = H(y)$, where $H : S^{2n+1} \to \mathbb{CP}^n$ is the Hopf projection.

$\left(\Phi^{-1}\right)_* T$ is a positive-$(1, 1)$ cycle in $(\widetilde{\mathbb{C}}^{n+1}, I_0, \mathbf{g}_0)$, so by upper semicontinuity of the density y_0 is also a point of density ≥ 1.

Turning now to a sequence T_{0,r_n} of dilated currents, with a limiting cone T_∞, we can take the proper transforms $\left(\Phi^{-1}\right)_* T_{0,r_n}$ and find that all of them share the features just described, with the same y_0 (because radial dilations do not affect the fact that there is a sequence of points of density 1 whose normalizations converge to y). But going to the limit we realize that $\left(\Phi^{-1}\right)_* T_{0,r_n}$ weakly converge to the proper transform $\left(\Phi^{-1}\right)_* T_\infty$, which is also positive-$(1, 1)$.

The mass is continuous under weak convergence of positive (or calibrated) currents, therefore y_0 is a point of density ≥ 1 for $\left(\Phi^{-1}\right)_* T_\infty$. This limit, however, is of a very peculiar form, being the transform of a cone. Recall that the fibers of $\widetilde{\mathbb{C}}^{n+1}$ are holomorphic planes coming from holomorphic planes through the origin of $\mathbb{C}^{n+1}$. Since T_∞ is a positive-$(1,1)$ cone, it is made of a weighted family of holomorphic disks through the origin, as described in (4), and the weight is a positive measure. Then $\left(\Phi^{-1}\right)_* T_\infty$ is made of a family of fibers of the line bundle $\widetilde{\mathbb{C}}^{n+1}$ with a positive weight. Then the fact that y_0 has density ≥ 1 implies that the whole fiber L^{y_0} at y_0 is counted with a weight ≥ 1. Transforming back, T_∞ must contain the plane $\Phi(L^{y_0})$ with a weight ≥ 1.

But the density of T at the origin is 1, so there is no space for anything else and T_∞ **must be** the disk $\Phi(L^{y_0})$ with multiplicity 1. Since we started from an arbitrary sequence r_n, the proof is complete, and it is also clear that $H\left(\frac{x_m}{|x_m|}\right)$ cannot have accumulation points other than y_0.

In the almost complex setting we need to adapt the algebraic blow up, respecting the almost complex structure.

In the next section we recall some facts on monotonicity and tangent cones for ω-positive cycles and state the stronger Theorem 2.1.

In Section 3 we construct suitable coordinates, used in Section 4 for the almost complex implementation of the algebraic blow up. In Section 4 we also face the hard analysis aspects of our technique, namely we prove that the proper transform actually yields a current of finite mass and without boundary. Appendix B contains two lemmas: pseudo holomorphic maps preserve both the $(1,1)$-condition and the densities. With all this, in Section 5 we conclude the proof. In Section 6 we revisit some essential aspects of the construction in [18] by means of our blow up technique, which sheds new light on the geometry of Kiselman's counterexample.

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2 Tangent cones to positive-$(1,1)$ cycles.

Given an almost complex $(2n+2)$-dimensional manifold $(\mathcal{M}, J, \omega, g)$, let T be a ω-positive normal 2-cycle. Tangent cones are a local matter, it suffices then to work in a chart around the point under investigation.

One of the key properties of positive currents is the following *almost monotonicity property* for the mass-ratio. The statement here follows from proposition B.1 in the appendix, which is in turn borrowed from [24].

Proposition 2.1. *Let T be a ω-positive normal cycle in an open and bounded set of $\mathbb{R}^{2n+2}$, endowed with a metric g and a semicalibration ω. We assume that g and ω are L-Lipschitz for some constant $L > 1$ and that $\frac{1}{5}\mathbb{I} \leq g \leq 5\mathbb{I}$, where $\mathbb{I}$ is the identity matrix, representing the flat metric.*

Let $B_r(x_0)$ be the ball of radius r around x_0 with respect to the metric g_{x_0} and let M be the mass computed with respect to the metric g. There exists $r_0 > 0$ depending only on L such that, for any x_0 and for $r \leq r_0$ the mass ratio $\frac{M(T \llcorner B_r(x_0))}{\pi r^2}$ is an almost-increasing function in r, i.e. $\frac{M(T \llcorner B_r(x_0))}{\pi r^2} = R(r) + o_r(1)$ for a function R that is monotonically non-increasing as $r \downarrow 0$ and a function $o_r(1)$ which is infinitesimal of r.

Independently of x_0, the perturbation term $o_r(1)$ is bounded in modulus by $C \cdot L \cdot r$, where C is a universal constant.

The fact that r_0 and C do not depend on the point yield that the density $\nu(x)$ of T is an upper semi-continuous function; the proof is rather standard.

Another very important consequence of monotonicity is that the mass is continuous and not just lower semi-continuous under weak convergence of semicalibrated or positive cycles. Basically this is due to the fact that computing mass for a ω-positive cycle amounts to testing it on the form ω, as described in (1); testing on forms is exactly how weak convergence is defined. This fact is of key importance for this work and will be formally proved when needed (see (27) in Section 5).

Let us now focus on tangent cones. If we perform the blow up procedure around a point of density 0, then the limiting cone is unique and is the zero-current. So in this situation there is no issue about the uniqueness of the tangent cone.

We are therefore interested in the limiting behaviour around a point x_0 of strictly positive density $\nu(x_0) > 0$.

From [6] we know that any normal positive 2-cone in $\mathbb{C}^{n+1}$ is a positive Radon measure on $\mathbb{CP}^n$. Combining[3] this with the fact that a tangent cone T_∞ at x_0 to a ω-positive cycle is ω_{x_0}-positive and has density $\nu(x_0)$ at the vertex, we get that T_∞ is represented by a Radon measure, with total measure $\nu(x_0)$, on the set of ω_{x_0}-calibrated 2-planes. Precisely, there exists a positive Radon measure τ on $\mathbb{CP}^n$ such that, denoting by D^X the J_{x_0}-holomorphic unit disk in $B_1^{2n+2}(0)$ corresponding to $X \in \mathbb{CP}^n$, the action of T_∞ on any two-form β is expressed as

[3]As explained in [18] and [6], the family of possible tangent cones at a point x_0 must be a convex and connected subset of the space of ω_{x_0}-positive cones with density $\nu(x_0)$.

$$T_\infty(\beta) = \int_{\mathbb{CP}^n} \left\{ \int_{D^X} \langle \beta, \vec{D}^X \rangle \, d\mathcal{L}^2 \right\} d\tau(X). \tag{4}$$

Let x_0 be a point of positive density $\nu(x_0) > 0$ and assume that there is a sequence $x_m \to x_0$ of points of positive density $\nu(x_m) \geq \kappa > 0$ for a fixed $\kappa > 0$. By upper-semicontinuity of ν it must be $\nu(x_0) \geq \kappa$.

Blow up around x_0 for the sequence of radii $|x_m - x_0|$: up to a subsequence we get a tangent cone T_∞. What can we immediately say about this cone?

With these dilations, the currents $T_{x_0, |x_m - x_0|}$ always have a point $y_m := \frac{x_m - x_0}{|x_m - x_0|}$ on the boundary of B_1 with density $\nu(y_m) \geq \kappa$. By compactness we can assume $y_m \to y \in \partial B_1$. By monotonicity, for any fixed $\delta > 0$, localizing to the ball $B_\delta(y)$ we find, using (1) and recalling from (3) that T_∞ and $T_{x_0,r}$ are positive respectively for $\omega_{x_0} := \omega(x_0)$ and $\omega_{x_0,r} := \frac{1}{r^2}(r(x - x_0))^*(\omega)$,

$$M(T_\infty \llcorner B_\delta(y)) = T_\infty(\chi_{B_\delta(y)} \omega_{x_0}) = \lim_m T_{x_0, |x_m - x_0|}(\chi_{B_\delta(y)} \omega_{x_0}) =$$
$$\lim_m T_{x_0, |x_m - x_0|} \left[\frac{\chi_{B_\delta(y)}}{|x_m - x_0|^2} \left(|x_m - x_0| (x - x_0) \right)^* \omega \right] =$$
$$= \lim_m M(T_{x_0, |x_m - x_0|} \llcorner B_\delta(y)) \geq \kappa \pi \delta^2,$$

which[4] implies that y has density $\nu(y) \geq \kappa$.

Therefore T_∞ "must contain" $\kappa [\![D]\!]$, where D is the holomorphic disk through 0 and y; i.e. $T_\infty - \kappa [\![D]\!]$ is a ω_{x_0}-positive cone having density $\nu(x_0) - \kappa$ at the vertex.

More precisely, what we have just shown the following well-known lemma. In the sequel $H : S^{2n+1} \to \mathbb{CP}^n$ denotes the standard Hopf projection.

Lemma 2.1. *Let x_0 be a point of positive density $\nu(x_0) > 0$ and assume that there is a sequence $x_m \to x_0$, $x_m \neq x_0$, of points of positive density $\nu(x_m) \geq \kappa > 0$ for a fixed $\kappa > 0$. Let $\{y_\alpha\}_{\alpha \in A}$ be the set of accumulation points on $\mathbb{CP}^n$ for the sequence $y_m := H\left(\frac{x_m - x_0}{|x_m - x_0|} \right)$. Let D_α be the J_{x_0}-holomorphic disk in $T_{x_0} \mathcal{M}$ containing 0 and $H^{-1}(y_\alpha)$. Then for every $\alpha \in A$ there is at least a tangent cone to T at x_0 of the form $\kappa [\![D_\alpha]\!] + \tilde{T}_\alpha$, for a ω_{x_0}-positive cone $\tilde{T}_\alpha$.*

In other words, each $\kappa [\![D_\alpha]\!]$ "must appear" in at least one tangent cone. What about all other (possibly different) tangent cones that we get by choosing different sequences of radii?

The following result shows that **any** tangent cone to T at x_0 "must contain" <u>each</u> disk $\kappa [\![D_\alpha]\!]$, for all $\alpha \in A$.

[4] This computation is an instance of the fact that the mass is continuous under weak convergence of positive currents, unlike the general case when it is just lower semi-continuous.

Theorem 2.1. *Given an almost complex $(2n+2)$-dimensional manifold $(\mathcal{M}, J, \omega, g)$, let T be a ω-positive normal 2-cycle.*

Let x_0 be a point of positive density $\nu(x_0) > 0$ and assume that there is a sequence of points $\{x_m\}$ such that $x_m \to x_0$, $x_m \neq x_0$ and the x_m have positive densities satisfying $\liminf_{m\to\infty} \nu(x_m) \geq \kappa$ for a fixed $\kappa > 0$.

Let $\{y_\alpha\}_{\alpha\in A}$ be the set of accumulation points on $\mathbb{CP}^n$ for the sequence $y_m := H\left(\frac{x_m - x_0}{|x_m - x_0|}\right)$. Let D_α be the J_{x_0}-holomorphic disk in $T_{x_0}\mathcal{M}$ containing 0 and $H^{-1}(y_\alpha)$.

Then the points y_α's are finitely many and $\underline{\text{any}}$ tangent cone T_∞ to T at x_0 is such that $T_\infty - \oplus_\alpha \kappa [\![D_\alpha]\!]$, is a ω_{x_0}-positive cone.

Remark 2.1. It follows that the cardinality of the y_α's is bounded by $\left\lfloor \frac{\nu(x_0)}{\kappa} \right\rfloor$. In particular, Theorem 1.1 follows from this result.

3 Pseudo holomorphic polar coordinates

T is ω-positive 2-cycle of finite mass in a $(2n+2)$-dimensional almost complex manifold endowed with a compatible metric and form, $(\mathcal{M}, J, \omega, g)$; T is shortly called a positive-$(1,1)$ normal cycle.

Since tangent cones to T at a point x_0 are a local issue it suffices to work in a chart. We can assume straight from the beginning to work in the geodesic ball of radius 2, in normal coordinates centered at x_0; for this purpose it is enough to start with the current T already dilated enough around x_0. Always up to a dilation, without loss of generality we can actually start with the following situation.

T is a ω-positive normal cycle in the unit ball $B_2^{2n+2}(0)$, the coordinates are normal, J is the standard complex structure at the origin, ω is the standard symplectic form ω_0 at the origin, $\|\omega - \omega_0\|_{C^{2,\nu}(B_2^{2n+2})}$ and $\|J - J_0\|_{C^{2,\nu}(B_2^{2n+2})}$ are small enough.

The dilations needed for the blow up are expressed by the map $\dfrac{x}{r}$ for $r > 0$ (we are in a normal chart centered at the origin). So in these coordinates we need to look at the family of currents

$$T_{0,r} := \left(\frac{x}{r}\right)_* T.$$

It turns out effective, however, to work in coordinates adapted to the almost-complex structure, as we are going to explain in this section.

With coordinates $(z_0, ... z_n)$ in $\mathbb{C}^{n+1}$, we use the notation (ε is a small positive number)

$$\tilde{\mathcal{S}}_\varepsilon := \{(z_0, z_1, ... z_n) \in B_{1+\varepsilon}^{2n+2} \subset \mathbb{C}^{n+1} : |(z_1, ..., z_n)| < (1+\varepsilon)|z_0|\}. \qquad (5)$$

We have a canonical identification of $X = [z_0, z_1, ..., z_n] \in \mathbb{CP}^n$ with the 2-dimensional plane $D^X = \{\zeta(z_0, z_1, ..., z_n) : \zeta \in \mathbb{C}\}$, which is complex for the standard structure J_0.

As X ranges in the open ball

$$\mathcal{V}_\varepsilon \subset \mathbb{CP}^n, \;\; \mathcal{V}_\varepsilon := \{[z_0, z_1, ..., z_n] : |(z_1, ..., z_n)| < (1+\varepsilon)|z_0|\},$$

the planes D^X foliate the sector $\tilde{\mathcal{S}}_\varepsilon$. We thus canonically get a *polar foliation* of the sector, by means of holomorphic disks.

Let the ball (of radius 2) $B_2^{2n+2} \subset \mathbb{R}^{2n+2}$ be endowed with an almost complex structure J. The same set as in (5), this time thought of as a subset of (B_2^{2n+2}, J), will be denoted by $\mathcal{S}_\varepsilon$.

We can get a *polar foliation* of the sector $\mathcal{S}_0$, by means of J-pseudo holomorphic disks; this is achieved by perturbing the canonical foliation exhibited for $\tilde{\mathcal{S}}_\varepsilon$. The case $n = 1$ is Lemma A.2 in the appendix of [25], the proof is however valid for any n; the reader can refer to our Appendix A for the construction in arbitrary dimension. Here is the statement as we will need it in the sequel.

Existence of a J-pseudo holomorphic polar foliation. There exists $\alpha_0 > 0$ small enough such that, if $\|J - J_0\|_{C^{2,\nu}(B_2^{2n+2})} < \alpha_0$ and $J = J_0$ at the origin, then the following holds.

There exists a $C^{2,\nu}$-map

$$\Psi : \tilde{\mathcal{S}}_\varepsilon \to (B_2^{2n+2}, J) \quad , \tag{6}$$

that is a diffeomorphism with its image and that extends continuously up to the origin, with $\Psi(0) = 0$, with the following properties (see top picture of figure 1):

(i) Ψ sends the 2-disk $D^X \cap \tilde{\mathcal{S}}_\varepsilon$ represented by $X = [z_0, z_1, ...z_n] \in \mathbb{CP}^n$ to an embedded J-pseudo holomorphic disk through 0 with tangent D^X at the origin;

(ii) the image of Ψ contains $\mathcal{S}_0 = B_1^{2n+2} \cap \{|(z_1, ..., z_n)| < |z_0|\}$;

(iii) $\|\Psi - Id\|_{C^{2,\nu}(\mathcal{S}_\varepsilon)} < C_0$, where C_0 is a positive constant that can be made as small as wished by assuming α_0 small enough.

The collection $\{\Psi\left(D^Y\right) : Y \in \mathcal{V}_\varepsilon\}$ of these embedded J-pseudo holomorphic disks foliates a neighbourhood of the sector $\mathcal{S}_0$; we will call it a *J-pseudo holomorphic polar foliation*.

The proof (see [25]) also shows that, in order to foliate $\mathcal{S}_0$, the ε needed in (6) can be made small by taking α_0 small enough.

Rescale the foliation. We are now going to use this *polar foliation* to construct coordinates adapted to J.

The result in [25] actually shows that there exists α_0 such that for all $\alpha \in [0, \alpha_0]$, if $\|J - J_0\|_{C^{2,\nu}(B_2^{2n+2})} = \alpha$ and $J = J_0$ at the origin, then there is a map Ψ_α yielding a polar foliation with $\|\Psi_\alpha - Id\|_{C^{2,\nu}(\mathcal{S}_\varepsilon)} < o_\alpha(1)$ (an infinitesimal of α).

We make use however only of the result for α_0, as we are about to explain. When we dilate the current T in normal coordinates with a factor r and look at the dilated current in the new ball B_2^{2n+2}, we find that it is positive-$(1,1)$ for J_r, where $J_r := (\lambda_r^{-1})^* J$, i.e. $J_r(V) := (\lambda_r)_*[J\left((\lambda_r^{-1})_* V\right)]$.

As $r \to 0$ it holds $\|J_r - J_0\|_{C^{2,\nu}(B_2^{2n+2})} \to 0$. Once we have applied the existence result of the J-pseudo holomorphic polar foliation to the ball B_2^{2n+2} endowed with J (assuming $\|J - J_0\|_{C^2} < \alpha_0$), then we get a J_r-pseudo holomorphic polar foliation of (B_2, J_r) just as follows.

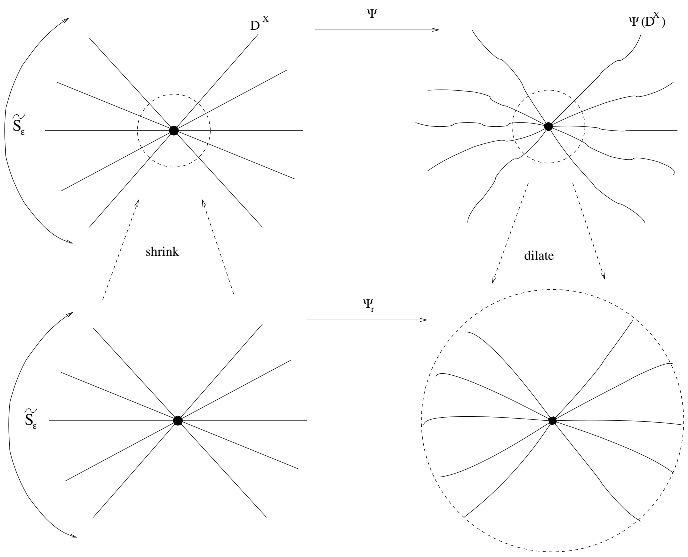

Figure 1: J-pseudo holomorphic polar foliation via Ψ and J_r-pseudo holomorphic polar foliation via Ψ_r.

Let $\tilde{\lambda}_r$ be the dilation (in Euclidean coordinates) $x \to \frac{x}{r}$; we use the tilda to remind that we are in $\tilde{\mathcal{S}}_\varepsilon$. The same dilation in normal coordinates in $\Psi(\mathcal{S}_\varepsilon) \subset (B_2^{2n+2}, J)$ is denoted by λ_r. Introduce the map (see figure 1)

$$\begin{array}{rccc} \Psi_r : & \tilde{\mathcal{S}}_\varepsilon & \to & \left(B_2^{2n+2}, J_r\right) \\ & x & \to & \lambda_r \circ \Psi \circ \tilde{\lambda}_r^{-1}(x). \end{array} \tag{7}$$

Ψ_r clearly yields a J_r-pseudo holomorphic polar foliation for the ball B_2^{2n+2} endowed with J_r. Remark, in view of (11), that Ψ_r can actually be defined on the sector $\tilde{\lambda}_r(\tilde{\mathcal{S}}_\varepsilon)$.

From the proof in [25] we get that[5] $\Psi_r \to Id$ in $C^1(\mathcal{S}_\varepsilon)$ as $r \to 0$.

Adapted coordinates. The aim is to pull back the problem on $\tilde{\mathcal{S}}_\varepsilon$ via Ψ. Endow for this purpose $\tilde{\mathcal{S}}_\varepsilon$ with the almost complex structure $\Psi^* J$.

Recall that we have in mind to look at $T_{0,r}$ in $\left(B_2^{2n+2}, J_r\right)$ as $r \to 0$. So we are going to study the family

$$\left(\Psi_r^{-1}\right)_* \left[T_{0,r} \llcorner \left(\Psi_r(\tilde{\mathcal{S}}_\varepsilon)\right)\right]$$

as $r \to 0$. For each $r > 0$ these currents are positive-$(1,1)$ normal cycles in $\tilde{\mathcal{S}}_\varepsilon$ endowed with the almost complex structure $\Psi_r^* J_r$, as proved in Lemma B.1.

It is elementary to check that

$$\Psi_r^* J_r = (\tilde{\lambda}_r^{-1})^* \Psi^* \lambda_r^* J_r = (\tilde{\lambda}_r^{-1})^* \Psi^* J,$$

so we can equivalently look, for $r > 0$, at $\tilde{\mathcal{S}}_\varepsilon$ with the almost complex structure $(\tilde{\lambda}_r^{-1})^* \Psi^* J$. The latter is obtained from $(\tilde{\mathcal{S}}_\varepsilon, \Psi^* J)$ by dilation. Remark that $\Psi_r^* J_r \to J_0$ in C^0 as $r \to 0$; moreover, assuming α_0 small enough, the fact that $D\Psi$ is C^0-close to $\mathbb{I}$ yields $|\nabla\left(\Psi^* J\right)| \leq 2|\nabla J|$.

We are looking, in normal coordinates, at a sequence $T_{0,r_n} := (\lambda_{r_n})_* T = \left(\frac{x}{r_n}\right)_* T \to T_\infty$. Restricting to $\Psi_{r_n}(\tilde{\mathcal{S}}_\varepsilon)$, i.e. $T_{0,r_n} \llcorner \Psi_{r_n}(\tilde{\mathcal{S}}_\varepsilon)$, we pull back the problem on $\tilde{\mathcal{S}}_\varepsilon$ and look at

$$\left(\Psi_{r_n}^{-1}\right)_* \left(T_{0,r_n} \llcorner \Psi_{r_n}(\tilde{\mathcal{S}}_\varepsilon)\right). \tag{8}$$

Recalling that $\Psi_r \to Id$ in C^1 and that T_{0,r_n} have equibounded masses we have, for any two-form β,

$$\left(\Psi_{r_n}^{-1}\right)_* \left(T_{0,r_n} \llcorner \Psi_{r_n}(\tilde{\mathcal{S}}_\varepsilon)\right)(\beta) - (Id)_* \left(T_{0,r_n} \llcorner \Psi_{r_n}(\tilde{\mathcal{S}}_\varepsilon)\right)(\beta) \to 0. \tag{9}$$

This follows with a proof as in step 2 of lemma B.2, by writing the difference $(\Psi_{r_n}{}^{-1})^* \beta - Id^* \beta$ in terms of the coefficients of β. Then from (8) and (9) we get

$$\lim_{n\to\infty} \left(\Psi_{r_n}^{-1}\right)_* \left(T_{0,r_n} \llcorner \Psi_{r_n}(\tilde{\mathcal{S}}_\varepsilon)\right) = \left(\lim_{n\to\infty} (\lambda_{r_n})_* T\right) \llcorner \mathcal{S}_\varepsilon . \tag{10}$$

[5]This follows, with reference to the notation in [25], by observing that the map Ξ_q on page 84 (associated to the diffeomorphism that we called Ψ) satisfies $\Xi_q \to Id$ uniformly as $q \to 0$, by the condition that above we called (i). Then the $C^{1,\nu}$ bounds there and Ascoli-Arzelà's theorem (applied to $D\Psi_r$) yield that $\Psi_r \to Id$ in C^1.

In the last equality we are identifying the space with the tilda and the one without (because they can be naturally identifies after taking limits). On the other hand by (6) we have

$$\begin{aligned}\left(\Psi_{r_n}^{-1}\right)_* \left(T_{0,r_n} \llcorner \Psi_{r_n}(\tilde{\mathcal{S}}_\varepsilon)\right) &= \left[\left(\Psi_{r_n}^{-1}\right)_* \lambda_{r_n *} \left(T \llcorner \Psi(\tilde{\mathcal{S}}_\varepsilon)\right)\right] \llcorner \tilde{\mathcal{S}}_\varepsilon \\ &= \left[\left(\tilde{\lambda}_{r_n}\right)_* \left(\Psi^{-1}\right)_* \left(T \llcorner \Psi(\tilde{\mathcal{S}}_\varepsilon)\right)\right] \llcorner \tilde{\mathcal{S}}_\varepsilon .\end{aligned} \tag{11}$$

What we have obtained with (10) and (11) is that, using Ψ, we can just pull back T to $\tilde{\mathcal{S}}_\varepsilon$ endowed with $\Psi^* J$, $\Psi^* g$ and $\Psi^* \omega$ and dilate with $\tilde{\lambda}_r$ and observe what happens in the limit. All the possible limits of this family are cones, namely all the possible tangent cones to the original T, restricted to the sector $\mathcal{S}_\varepsilon$.

All the information we need about the family $T_{0,r} \llcorner \mathcal{S}_0$ can be obtained in this way. So **we are substituting** the blow up in normal coordinates with a different one, that behaves well with respect to J and has the same asymptotic behaviour, i.e. it yields the same cones.

Remark that lemmas B.1 and B.2 tell us that $\left(\Psi^{-1}\right)_* \left(T \llcorner \Psi(\tilde{\mathcal{S}}_\varepsilon)\right)$ is still positive-$(1,1)$ and the densities are preserved. Observe that we cannot use the monotonicity formula for $\left(\Psi^{-1}\right)_* \left(T \llcorner \Psi(\tilde{\mathcal{S}}_\varepsilon)\right)$ at the origin, since 0 is now a boundary point. However the monotonicity for T reflects into the following

Lemma 3.1. *For the current $\left(\Psi^{-1}\right)_* \left(T \llcorner \Psi(\tilde{\mathcal{S}}_\varepsilon)\right)$, with respect to the flat metric in $\mathcal{S}_\varepsilon$, it holds*

$$\frac{M\left(\left(\Psi^{-1}\right)_* \left(T \llcorner \Psi(\tilde{\mathcal{S}}_\varepsilon)\right) \llcorner (B_r \cap \tilde{\mathcal{S}}_\varepsilon)\right)}{\pi r^2} \leq K \tag{12}$$

with a constant K independent of r.

proof of lemma 3.1. We denote, only for this proof, by C the current $\left(\Psi^{-1}\right)_* \left(T \llcorner \Psi(\tilde{\mathcal{S}}_\varepsilon)\right)$. Since $|D\Psi - \mathbb{I}| \leq c r^\nu$ (where $\mathbb{I} = D(Id)$ is the identity matrix) and $g = g_0 + O(r^2)$ (where g_0 is the flat metric), we also get $\Psi^* g = g_0 + O(r^\nu)$.

Comparing the masses of C with respect to g_0 and $\Psi^* g$ we get

$$M_{g_0}\left(C \llcorner (B_r \cap \tilde{\mathcal{S}}_\varepsilon)\right) \leq (1 + |O(r^\nu)|) M_{\Psi^* g}\left(C \llcorner (B_r \cap \tilde{\mathcal{S}}_\varepsilon)\right),$$

where B_r is always Euclidean. Now recall that, by the positiveness of the currents,

$$M_{\Psi^* g}\left(C \llcorner (B_r \cap \tilde{\mathcal{S}}_\varepsilon)\right) = \left(C \llcorner (B_r \cap \tilde{\mathcal{S}}_\varepsilon)\right)(\Psi^* \omega) = M_g\left(T \llcorner \Psi(B_r \cap \tilde{\mathcal{S}}_\varepsilon)\right).$$

The condition $|\Psi - Id| \leq cr^{1+\nu}$ implies that $\Psi(B_r \cap \tilde{\mathcal{S}}_\varepsilon) \subset B_{r+cr^{1+\nu}} \cap \mathcal{S}_\varepsilon$. In $\mathcal{S}_\varepsilon$ coordinates are normal, so, putting all together:

$$\frac{M_{g_0}\left(C\llcorner(B_r \cap \tilde{\mathcal{S}}_\varepsilon)\right)}{r^2} \leq (1 + |O(r^\nu)|)\, \frac{(r+cr^{1+\nu})^2}{r^2}\, \frac{M_g\left(T\llcorner B_{r+cr^{1+\nu}}\right)}{(r+cr^{1+\nu})^2},$$

which is equibounded in r by almost monotonicity (Proposition 2.1).

□

So we **restate our problem** in the following terms, where we drop the tildas and the pull-backs (resp. push-forwards) via Ψ (resp. Ψ^{-1}), since there will be no more confusion arising. It is also clear that we can take $\varepsilon = 0$ and just work in $\mathcal{S} := \mathcal{S}_0$ rather than $\mathcal{S}_\varepsilon$, which will make the notation lighter and the exposition easier to follow.

New setting: pseudo holomorphic polar coordinates.

Endow $\mathcal{S} := \mathcal{S}_0 \subset B_1^{2n+2}(0)$ with a smooth almost complex structure J such that, denoting by J_0 the standard complex structure,

- there is $Q > 0$ such that for any $0 < r < 1$, $|J - J_0|_{C^0(\mathcal{S}\cap B_r)} < Q \cdot r$ and $|\nabla J| < Q$ (and Q can be assumed to be small);
- the 2-planes D^X (for $X \in \mathcal{V} := \mathcal{V}_0$) foliating the sector $\mathcal{S}$ are J-pseudo holomorphic.

Let ω and g be respectively a compatible non-degenerate two-form and the associated Riemannian metric such that $\|\omega - \omega_0\|_{C^0(\mathcal{S}\cap B_r)} < Q \cdot r$ and $\|g - g_0\|_{C^0(\mathcal{S}\cap B_r)} < Q \cdot r$, where ω_0 and g_0 are the standard ones.

Let T be a positive-$(1,1)$ normal cycle in $\mathcal{S}$.

Study the asymptotic behaviour as $r \to 0$ of the family $(\lambda_r)_* T$, where $\lambda_r = \frac{Id}{r}$ in Euclidean coordinates. More precisely we can restate Theorem 2.1 as follows; in Theorem 2.1 we can assume, up to a rotation and passing to a subsequence, that $y_m = \frac{x_m}{|x_m|} \to (1, 0, ..., 0)$.

Proposition 3.1. *With the assumptions just made on J and T, assume that there exists a sequence $x_m \to 0$, $0 \neq x_m \in \mathcal{S}$, of points all having densities satisfying* $\liminf_{m\to\infty} \nu(x_m) \geq \kappa$ *for a fixed $\kappa > 0$ and such that $y_m := \frac{x_m}{|x_m|} \to (1, 0, ..., 0)$. Then any limit*

$$\lim_{r_n \to 0} (\lambda_{r_n})_* T$$

is a positive-$(1,1)$ cone (for J_0) of the form $\kappa[\![D^{[1,0,...,0]}]\!] + \tilde{T}$, where $\tilde{T}$ is also a positive-$(1,1)$ cone for J_0 ($\tilde{T}$ possibly depending on $\{r_n\}$).

Remark 3.1. As observed in (12), our new T satisfies, with respect to the flat metric, $\frac{M(T\llcorner(B_r\cap\mathcal{S}))}{r^2} \leq K$ for a constant independent of r.

4 Algebraic blow up

The classical symplectic (or algebraic) blow up was recalled in the introduction (maps Φ and Φ^{-1} in figure 2). More details can be found in [22]. $\widetilde{\mathbb{C}}^{n+1}$ is a complex line bundle over $\mathbb{CP}^n$, that we view as an embedded sumbanifold in $\mathbb{CP}^n \times \mathbb{C}^{n+1}$. We use standard coordinates on $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ coming from the product, so we have $2n$ "horizontal variables" and $2n+2$ "vertical variables". The standard symplectic form on $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ is given by the two form $\vartheta_{\mathbb{CP}^n} + \vartheta_{\mathbb{C}^{n+1}}$, where $\vartheta_{\mathbb{CP}^n}$ is the standard symplectic form[6] on $\mathbb{CP}^n$ extended to $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ (so independent of the "vertical variables") and $\vartheta_{\mathbb{C}^{n+1}}$ is the symplectic two-form on $\mathbb{C}^{n+1}$, extended to $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ (so independent of the "horizontal variables"). To $\vartheta_{\mathbb{CP}^n} + \vartheta_{\mathbb{C}^{n+1}}$ we associate the standard metric, i.e. the product of the Fubini-Study metric on $\mathbb{CP}^n$ and the flat metric on $\mathbb{C}^{n+1}$. The associated complex structure is denoted I_0.

As a complex submanifold, $\widetilde{\mathbb{C}}^{n+1}$ inherits from the ambient space a complex structure, still denoted I_0, and the restricted symplectic form $\vartheta_0 := \mathcal{E}^*\left(\vartheta_{\mathbb{CP}^n} + \vartheta_{\mathbb{C}^{n+1}}\right)$, where $\mathcal{E}$ is the embedding in $\mathbb{CP}^n \times \mathbb{C}^{n+1}$. Let further $\mathbf{g}_0$ denote the ambient metric restricted to $\widetilde{\mathbb{C}}^{n+1}$: $\mathbf{g}_0$ is then compatible with I_0 and ϑ_0, i.e. $\vartheta_0(\cdot,\cdot) := \mathbf{g}_0(\cdot, -I_0\cdot)$.

We now turn to the almost complex situation and will adapt the previous construction by building on the results of Section 3.

Implementation in the almost complex setting. We make use of the notation

$$\mathcal{S} := \mathcal{S}_0 = \{(z_0, z_1, ...z_n) \in B_1^{2n+2} \subset \mathbb{C}^{n+1} : |(z_1, ..., z_n)| < |z_0|\}$$

as in (5). Also recall that $\mathcal{V} := \mathcal{V}_0 := \left\{\sum_{j=1}^n \frac{|z_j|^2}{|z_0|^2} < 1\right\} \subset \mathbb{CP}^n$.

The inverse image $\Phi^{-1}(\mathcal{S})$ is given by $\{(\ell, z) \in \mathcal{V} \times \mathbb{C}^{n+1} : 0 < |z| < 1\}$. The union $\Phi^{-1}(\mathcal{S}) \cup (\mathcal{V} \times \{0\})$ will be denoted by $\mathcal{A}$.

$\mathcal{A}$ is an open set in $\widetilde{\mathbb{C}}^{n+1}$ but we will endow it with other almost complex structures, different from I_0, so $\mathcal{A}$ should be thought of just as an oriented manifold and the structure on it will be specified in every instance.

We will keep using the same letters Φ^{-1} and Φ to denote the restricted maps

$$\begin{aligned} \Phi^{-1} :\ & \mathcal{S} \to \mathcal{A} \\ \Phi :\ & \mathcal{A} \to \mathcal{S} \cup \{0\} \end{aligned} \tag{13}$$

[6] In the chart $\mathbb{C}^n \equiv \{z_0 \neq 0\}$ of $\mathbb{CP}^n$, the form $\vartheta_{\mathbb{CP}^n}$ is expressed, using coordinates $Z = (Z_1, ..., Z_n)$, by $\partial\overline{\partial} f$, where $f = \frac{i}{2}\log(1 + |Z|^2)$ (see [22]). The metric g_{FS} associated to $\vartheta_{\mathbb{CP}^n}$ and to the standard complex structure is called Fubini-Study metric and it fulfils $\frac{1}{4}\mathbb{I} \leq g_{\mathrm{FS}} \leq 4\mathbb{I}$ when we compare it to the flat metric on the domain $\{|Z| < 1\}$.

also when we look at these spaces just as oriented manifolds (not complex ones). We will make use of the notation

$$\mathcal{S}^\rho := \mathcal{S} \cap B^{2n+2}_\rho \text{ and } \mathcal{A}^\rho := \Phi^{-1}(\mathcal{S}^\rho) \cup (\mathcal{V} \times \{0\}) .$$

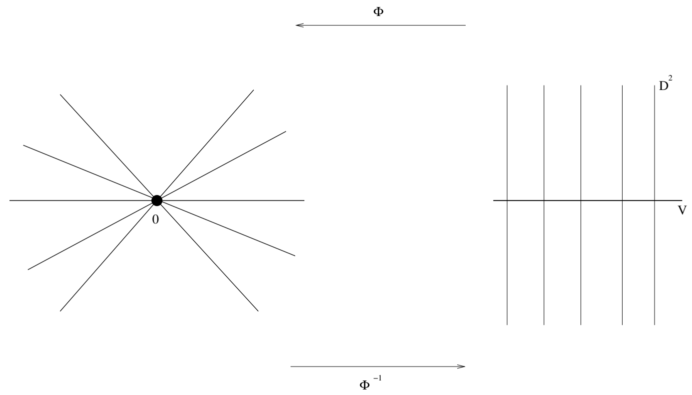

Figure 2: Blowing up the origin. The maps $\Phi^{-1} : \mathcal{S} \to \mathcal{A}$ and $\Phi : \mathcal{A} \to \mathcal{S} \cup \{0\}$.

Define on $\mathcal{A} \setminus (\mathbb{CP}^n \times \{0\})$:

- the almost complex structure $I := \Phi^* J$, i.e. $I(\cdot) := \left(\Phi^{-1}\right)_* J \Phi_*(\cdot)$,
- the metric $\mathbf{g}(\cdot,\cdot) := \frac{1}{2}\left(\mathbf{g}_0(\cdot,\cdot) + \mathbf{g}_0(I\cdot, I\cdot)\right)$,
- the non-degenerate two-form $\vartheta(\cdot,\cdot) := \mathbf{g}(I\cdot,\cdot) = \frac{1}{2}\left(\mathbf{g}_0(I\cdot,\cdot) - \mathbf{g}_0(\cdot, I\cdot)\right)$.

The triple $(I, \mathbf{g}, \vartheta)$ is smooth on $\mathcal{A} \setminus (\mathbb{CP}^n \times \{0\})$ and makes it an almost Hermitian manifold. We do not know yet, however, the behaviour of $(I, \mathbf{g}, \vartheta)$ as we approach $\mathcal{V} \times \{0\}$.

Lemma 4.1 (the new structure is Lipschitz). *The almost complex structure I fulfils*

$$|I - I_0|(\cdot) \leq c\,dist_{\mathbf{g}_0}(\,\cdot\,,\ \mathbb{CP}^n \times \{0\}),$$

for $c = C \cdot Q$, where C is a dimensional constant and Q is as in the hypothesis on J (paragraph "new setting", just before Proposition 3.1). The almost complex structure I can thus be extended continuously across $\mathcal{V} \times \{0\}$.

Analogously we have $|\mathbf{g} - \mathbf{g}_0|(\cdot) \leq c\,dist_{\mathbf{g}_0}(\,\cdot\,,\ \mathbb{CP}^n \times \{0\})$ and $|\vartheta - \vartheta_0|(\cdot) \leq c\,dist_{\mathbf{g}_0}(\,\cdot\,,\ \mathbb{CP}^n \times \{0\})$. The triple $(I, \mathbf{g}, \vartheta)$ can be extended across $\mathbb{CP}^n \times \{0\}$ to the whole of $\mathcal{A}$ by setting it to be the standard $(I_0, \mathbf{g}_0, \vartheta_0)$ on $\mathbb{CP}^n \times \{0\}$. The structures $I, \mathbf{g}, \vartheta$ so defined are globally Lipschitz-continuos on $\mathcal{A}$, with Lipschitz constant $L + C \cdot Q$, where $L > 0$ is an upper bound for the Lipschitz constants of I_0, $\mathbf{g}_0$ and ϑ_0 (with respect to Euclidean coordinates on $\mathcal{V} \times D^2$).

proof of lemma 4.1. Recall that Φ is holomorphic for the standard structures J_0 and I_0. With respect to the flat metric on $\mathcal{S}$, we can choose an orthonormal basis at any point $q \neq 0$ made as follows:

$$\{L_1, J_0(L_1), L_2, J_0(L_2), ..., L_n, J_0(L_n), W, J_0(W)\},$$

where W and $J_0(W)$ span the J_0-complex 2-plane through the origin and q. The map $\left(\Phi^{-1}\right)_*$ is holomorphic and sends this basis to one at $\left(\Phi^{-1}\right)(q) \in \mathcal{A}$, sending W and $I_0(W)$ to a pair of vectors spanning the fiber through $\left(\Phi^{-1}\right)(q)$. On the vertical vectors $\left(\Phi^{-1}\right)_*$ is length preserving, while for the others $|\left(\Phi^{-1}\right)_* L_j| = |\left(\Phi^{-1}\right)_* J_0(L_j)| = \frac{1}{|q|}$, as one can compute from the explicit expression of the Fubini-Study metric[7].

Reversing this construction we can choose two basis, respectively at p and $q = \Phi(p)$, as follows:

$$\{H_1, I_0(H_1), ..., H_n, I_0(H_n), V, I_0(V)\}$$

made of $\mathbf{g}_0$-unit vectors with scalar products w.r.t. $\mathbf{g}_0$ bounded by $|q|$, and

$$\left\{\frac{1}{|q|}K_1, \frac{1}{|q|}J_0(K_1), ..., \frac{1}{|q|}K_n, \frac{1}{|q|}J_0(K_n), W, J_0(W)\right\},$$

orthonormal at $q = \Phi(p)$, such that:

(i) $K_j := \Phi_* H_j$ and $W := \Phi_* V$;

(ii) V and $I_0(V)$ are vertical, i.e. they span the vertical fiber through p: by **(i)**, W and $J_0(W)$ span the J_0-complex 2-plane through the origin and q.

By the assumption that J is close to J_0 in B_1 we can write the action of J on K_1 as

$$\begin{aligned} J(K_1) = (1+\lambda)J_0(K_1) + \sum_{j=1}^{n} \mu_j K_j + \sum_{j=2}^{n} \tilde{\mu}_j J_0(K_j) + \\ + |q|\sigma W_1 + |q|\tilde{\sigma} J_0(W_1). \end{aligned} \tag{14}$$

[7]Recall that the Kähler form associated to the Fubini-Study metric on the complex manifold $\mathbb{CP}^n$ in the chart $z_0 \neq 0$ is

$$\frac{\sum_{j=1}^{n} dz^j \wedge d\overline{z}^j}{|z_0|^2 \left(1 + \sum_{j=1}^{n} \frac{|z_j|^2}{|z_0|^2}\right)} - \sum_{1 \leq j,m \leq n}^{j \neq m} \frac{z_j \overline{z}_m \, d\overline{z}^j \wedge dz^m}{|z_0|^4 \left(1 + \sum_{j=1}^{n} \frac{|z_j|^2}{|z_0|^2}\right)^2}.$$

By $SU(n+1)$-invariance it is enough to work on the complex line corresponding to $[1, 0, ..., 0] \in \mathbb{CP}^n$ in order to compute $|\left(\Phi^{-1}\right)_* L_j|$ and $|\left(\Phi^{-1}\right)_* J_0(L_j)|$. On this line the form reads $\frac{\sum_{j=1}^{n} dz^j \wedge d\overline{z}^j}{|z|^2}$ and thus the lengths of the vectors L_j and $J_0(L_j)$ get multiplied by $\frac{1}{|z|}$ under the map $\left(\Phi^{-1}\right)$.

Here λ, μ_j, $\tilde{\mu}_j$, σ and $\tilde{\sigma}$ are functions on $\mathcal{S}$ depending on $J - J_0$, evaluated at q, so their moduli are bounded by $|J - J_0|(q) < Q|q|$.

Let us write the action of I on H_1 explicitly: by definition of I, using (14),

$$I(H_1) := \left(\Phi^{-1}\right)_* J\Phi_*(H_1) = \left(\Phi^{-1}\right)_* J(K_1) =$$

$$\begin{aligned} = ((1+\lambda)\circ\Phi)I_0(H_1) + \sum_{j=1}^{n}(\mu_j\circ\Phi)H_j + \sum_{j=2}^{n}(\tilde{\mu}_j\circ\Phi)I_0(K_j) + \\ +|q|(\sigma\circ\Phi)V_1 + |q|(\tilde{\sigma}\circ\Phi)I_0(V_1). \end{aligned} \tag{15}$$

Similar expressions are obtained for the actions on H_j and $I_0(H_j)$ for all j. Now

$$J(W) = \sigma W + (1+\tilde{\sigma})J_0(W),$$

since the 2-plane spanned by W and $J_0(W)$ is J-pseudo holomorphic by hypothesis.

Here σ and $\tilde{\sigma}$ are functions on $\mathcal{S}$ depending on $J - J_0$, evaluated at q, and their moduli are bounded by $|J - J_0| < Q|q|$.

So the action of I on V is explicitly given by

$$I(V) := \left(\Phi^{-1}\right)_* J\Phi_*(V) = \left(\Phi^{-1}\right)_* J(W) =$$
$$= (\sigma\circ\Phi)\left(\Phi^{-1}\right)_*(W) + ((1+\tilde{\sigma})\circ\Phi)\left(\Phi^{-1}\right)_* J_0(W)$$

$$= (\sigma\circ\Phi)V + ((1+\tilde{\sigma})\circ\Phi)I_0(V). \tag{16}$$

So we have, from (15) and (16) that there exists $c = C\cdot Q$ (for some dimensional constant C) such that $(I - I_0)$ at the point $p = \left(\Phi^{-1}\right)(q)$ has norm $\leq c|q| = c\,\mathrm{dist}_{\mathbf{g}_0}(\,\cdot\,,\,\mathbb{CP}^n\times\{0\})$.

The analogous estimates on $\mathbf{g}$ and ϑ follow by their definition. So we can extend the triple $(I, \mathbf{g}, \vartheta)$ across $\mathbb{CP}^n\times\{0\}$ in a Lipschitz continuous fashion.

From (15) and (16) we also get that I is, globally in $\mathcal{A}$, a Lipschitz continuous perturbation of I_0, and the same goes for $\mathbf{g}$ and ϑ: indeed the Lipschitz constants of λ, μ_j, $\tilde{\mu}_j$, σ and $\tilde{\sigma}$ are controlled by $C\cdot Q$, for some dimensional constant C (which can be taken the same as the C we had above, by choosing the larger of the two).

$\square$

Remark 4.1. The importance of working with coordinates adapted to J, as chosen in Section 3, relies in the fact that this allows to obtain the Lipschitz extension across $\mathbb{CP}^n\times\{0\}$, which could fail on the vertical vectors if coordinates were taken arbitrary.

The aim is now to **translate our problem** in the new space $(\mathcal{A}, I, g, \vartheta)$. The trouble is that the push-forward of T via Φ^{-1} can only be done away from the origin and the map Φ^{-1} degenerates as we get closer to 0.

For any $\rho > 0$ we can take the *proper transform* of $T \llcorner (\mathcal{S} \setminus \mathcal{S}^\rho)$ by pushing forward via Φ^{-1}, since this is a diffeomorphism away from the origin:

$$P_\rho := \left(\Phi^{-1}\right)_* (T \llcorner (\mathcal{S} \setminus \mathcal{S}^\rho)) .$$

What happens when $\rho \to 0$? The following two lemmas yield the answers.

Lemma 4.2. *The current $P := \lim_{\rho \to 0} P_\rho = \lim_{\rho \to 0} \left(\Phi^{-1}\right)_* (T \llcorner (\mathcal{S} \setminus \mathcal{S}^\rho))$ is well-defined as the limit of currents of equibounded mass to be a current of finite mass in $\mathcal{A}$.*

The mass of P, both with respect to $\mathbf{g}$ and to $\mathbf{g}_0$, is bounded by a dimensional constant C times the mass of T.

Lemma 4.3. *The current $P := \lim_{\rho \to 0} P_\rho = \lim_{\rho \to 0} \left(\Phi^{-1}\right)_* (T \llcorner (\mathcal{S} \setminus \mathcal{S}^\rho))$ is a ϑ-positive normal cycle in the open set $\mathcal{A}$ (ϑ is a semi-calibration with respect to $\mathbf{g}$).*

A little notation before the proofs. For any ρ consider the dilation $\lambda_\rho(\cdot) := \frac{\cdot}{\rho}$, sending B_ρ to B_1, and the map

$$\Lambda_\rho : \mathcal{A}^\rho \to \mathcal{A}, \;\; \Lambda_\rho := \Phi^{-1} \circ \lambda_\rho \circ \Phi, \tag{17}$$

which in the coordinates of $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ (the ambient space in which $\mathcal{A}$ is embedded) reads $\Lambda_\rho(\ell, z) = \left(\ell, \frac{z}{\rho}\right)$.

proof of Lemma 4.2. The map Φ^{-1} is pseudo holomorphic with respect to J and I by definition of I; thus each $P_\rho = \left(\Phi^{-1}\right)_* (T \llcorner (\mathcal{S} \setminus \mathcal{S}^\rho))$ is ϑ-positive by construction (see lemma B.1), so $M(P_\rho) = P_\rho(\vartheta)$, where the mass is computed here with respect to $\mathbf{g}$, the metric defined before Lemma 4.1. The currents P_ρ and $P_{\rho'}$, for $\rho > \rho'$, coincide on $\mathcal{A} \setminus \overline{\mathcal{A}^\rho}$, therefore for any two sequences $\rho_k \to 0$ and $\rho'_k \to 0$ such that P_{ρ_k} and $P_{\rho'_k}$ have limits, these limits must coincide (as currents). So we will prove that P_ρ have equibounded masses and thus the compactness theorem will yield the existence of a unique limit P for the whole family P_ρ as $\rho \to 0$.

We use in $\mathcal{A}$ standard coordinates inherited from $\mathbb{CP}^n \times \mathbb{C}^{n+1}$, i.e. we have $2n$ horizontal variables (from $\mathbb{CP}^n$) and $2n+2$ vertical variables.

The standard symplectic form ϑ_0 is $\mathcal{E}^*(\vartheta_{\mathbb{CP}^n} + \vartheta_{\mathbb{C}^{n+1}})$, as in the beginning of Section 4. We want to estimate $M(P_\rho) = P_\rho(\vartheta) = P_\rho(\vartheta_0) + P_\rho(\vartheta - \vartheta_0)$.

Now let us first consider $|P_\rho(\vartheta - \vartheta_0)|$. Thanks to the Lipschitz control from Lemma 4.1, i.e. $|\vartheta - \vartheta_0|(\cdot) \leq c\,\mathrm{dist}_{\mathbf{g}_0}(\cdot, \mathbb{CP}^n \times \{0\})$, the two-form

$\left(\Phi^{-1}\right)^*(\vartheta-\vartheta_0)$ in $\mathcal{S}$ has comass $\leq \frac{c\cdot C}{|z|} \leq \frac{C}{|z|}$, where $|z|$ is the distance from the origin and C is a dimensional constant (c can be assumed to be smaller than 1).

We can then decompose $\mathcal{S} = \cup_{j=0}^{\infty} A_j$, where $A_j = \mathcal{S} \cap \left(B_{\frac{1}{2^j}} \setminus B_{\frac{1}{2^{j+1}}}\right)$. As observed in Remark 3.1 it holds $M(T \llcorner A_j) \leq K \frac{1}{2^{2j}}$. On the other hand the comass of $\left(\Phi^{-1}\right)^*(\vartheta-\vartheta_0)$ in A_j is $\leq C\, 2^{j+1}$.

Therefore summing on all j's we can bound

$$
\begin{aligned}
|P_\rho(\vartheta-\vartheta_0)| &= \left|(T \llcorner \mathcal{S})\left(\left(\Phi^{-1}\right)^*(\vartheta-\vartheta_0)\right)\right| \leq \\
&\leq K\,C \sum_{j=0}^{\infty} 2^{j+1}\frac{1}{2^{2j}} = K\,C\sum_{j=0}^{\infty} 2^{1-j} = 4K\,C,
\end{aligned}
\tag{18}
$$

so $|P_\rho(\vartheta-\vartheta_0)|$ is equibounded independently of ρ.

Let us next deal with $P_\rho(\mathcal{E}^*\vartheta_{\mathbb{CP}^n}) = T_{0,\rho}\llcorner(\mathcal{S}\setminus\mathcal{S}^\rho)(\left(\Phi^{-1}\right)^*\mathcal{E}^*\vartheta_{\mathbb{CP}^n})$. Using the chart $z_0 \neq 0$ on $\mathcal{V} \subset \mathbb{CP}^n$ we find that the map $\mathcal{E}\circ\Phi^{-1} : \mathcal{S} \to \mathcal{A}$ has the coordinate expression $(z_0, ...z_n) \to \left(\left(\frac{z_1}{z_0}, ..., \frac{z_n}{z_0}\right), (z_0, ...z_n)\right) \in \mathcal{V}\times\mathbb{C}^{n+1}$.

Using the explicit expression of $\vartheta_{\mathbb{CP}^n}$ (see [22] or the beginning of this section) we can write in the domain $\mathcal{S}$, where $z_0 \neq 0$,

$$
\left(\Phi^{-1}\right)^*\mathcal{E}^*(\vartheta_{\mathbb{CP}^n}) = -\frac{i}{2}\partial\overline{\partial}\log\left(1+\sum_{j=1}^{n}\frac{|z_j|^2}{|z_0|^2}\right).
$$

This two-form is explicitly computed to be $\frac{(\omega_0)_t}{|z|^2}$, where ω_0 is the standard Kähler form of $\mathbb{C}^{n+1}$ restricted to $\mathcal{S}$, $(\omega_0)_t := \frac{\partial}{\partial r}\lrcorner(dr\wedge\omega_0)$ denotes its tangential part (as in [16], [24]) and $|z|$ is the distance from the origin. This can be seen by explicitly computing the $\partial\overline{\partial}\log\left(1+\sum_{j=1}^{n}\frac{|z_j|^2}{|z_0|^2}\right)$ which gives

$$
\frac{\sum_{j=1}^{n} dz^j\wedge d\overline{z}^j}{|z_0|^2\left(1+\sum_{j=1}^{n}\frac{|z_j|^2}{|z_0|^2}\right)} - \sum_{1\leq j,m\leq n}^{j\neq m}\frac{z_j\overline{z}_m\, d\overline{z}^j\wedge dz^m}{|z_0|^4\left(1+\sum_{j=1}^{n}\frac{|z_j|^2}{|z_0|^2}\right)^2}.
$$

Evaluating this expression on the 2-plane $\{z_1 = ... = z_n = 0\}$ we get $\dfrac{\sum_{j=1}^{n} dz^j\wedge d\overline{z}^j}{|z_0|^2}$, so that $\left(\Phi^{-1}\right)^*\mathcal{E}^*\theta_{\mathbb{CP}^n}$ and $\dfrac{(\omega_0)_t}{|z|^2}$ agree on this 2-plane. The $SU(n+1)$ invariance of both $\left(\Phi^{-1}\right)^*\mathcal{E}^*\theta_{\mathbb{CP}^n}$ and $\dfrac{(\omega_0)_t}{|z|^2}$ then implies that the two forms agree everywhere.

The two 2-forms $(\omega_0)_t$ and $\omega_t := \frac{\partial}{\partial r}\lrcorner(dr\wedge\omega)$ differ in L^∞-norm by a term bounded in modulus with $C|z|$, (for a constant $C = \|\omega\|_\infty$). Therefore with the same diadic decomposition used in estimate (21) we can obtain that

$\dfrac{\left|\langle(\omega_0)_t - \omega_t, \vec{T}\rangle\right|}{|z|^2}$ is summable with respect to the measure $\|T\| \llcorner (\mathcal{S}_\varepsilon \setminus B_\rho)$ with a bound from above independent of ρ. By the monotonicity formula (Proposition B.1) we know that $\dfrac{\omega_t}{|z|^2}$ is summable with respect to the measure $\|T\| \llcorner (\mathcal{S}_\varepsilon \setminus B_\rho)$ with a bound independent of ρ, therefore so is $\dfrac{(\omega_0)_t}{|z|^2}$. This means that, independently of ρ,

$$|P_\rho\left(\mathcal{E}^*\theta_{\mathbb{CP}^n}\right)| \leq CK. \tag{19}$$

The estimate

$$|P_\rho(\mathcal{E}^*\vartheta_{\mathbb{C}^{n+1}})| = |T_{0,\rho} \llcorner (\mathcal{S} \setminus \mathcal{S}^\rho)(\left(\Phi^{-1}\right)^* \mathcal{E}^*\vartheta_{\mathbb{C}^{n+1}})| \leq K \tag{20}$$

follows easily since Φ^{-1} is lenght-preserving in the vertical coordinates and thus $(\mathcal{E} \circ \Phi^{-1})^*$ preserves the comass of $\vartheta_{\mathbb{C}^{n+1}}$.

Putting (18), (19) and (20) together, we obtain that $M(P_\rho)$ are uniformly bounded by K times a dimensional constant C. By compactness there exists a current P in $\mathcal{A}$ such that $P_\rho \rightharpoonup P$.

So far we were taking the mass with respect to $\mathbf{g}$. Since $\mathbf{g}$ is c-close to $\mathbf{g}_0$, for a small constant c, an analogous bound holds, up to doubling the constant C, for the mass of P computed with respect to $\mathbf{g}_0$. This observation is needed later in Section 5.

$\square$

Our next aim is to prove that the current P just obtained is in fact a cycle in the open set $\mathcal{A}$. A priori this is not clear, for in the limit $\rho \to 0$ some boundary could be created on $\mathbb{CP}^n \times \{0\}$.

proof of Lemma 4.3. *Step 1: choice of the sequence.* The currents T and $T_{0,r} := (\lambda_r)_*(T \llcorner B_r)$ are defined in $\mathcal{S}$ and by Remark 3.1, i.e. by the almost monotonicity formula, we have a uniform bound on the masses: $M(T_{0,r}) \leq K$. Denote by $\langle T, |z| = r\rangle$ the slice of a current T with the sphere ∂B_r. Choose a sequence $\rho_k \to 0$ so to ensure

- (i) $T_{\rho_k} \rightharpoonup T_\infty$ in $\mathcal{S}$ for a certain cone T_∞,
- (ii) $M(\langle T_{\rho_k}, |z| = 1\rangle)$ are equibounded by $4K$.

This is achieved as follows: take a sequence ρ'_k fulfilling (i); Remark 3.1 tells us that $M(T_{\rho'_k})$ are equibounded by a constant K independent of k. By slicing theory (see [14])

$$\int_{\frac{1}{2}}^{1} M(\langle T_{\rho'_k}, |z| = r\rangle)dr \leq M(T_{\rho'_k} \llcorner (B_1 \setminus B_{\frac{1}{2}})) \leq K,$$

thus at least half of the slices $\langle T_{\rho'_k}, |z| = r\rangle_{r\in[\frac{1}{2},1]}$ have masses $\leq 2K$. For every k we can choose $\frac{1}{2} \leq s_k \leq 1$ such that all the slices $\langle T_{\rho'_k}, |z| = s_k\rangle$ exist and have mass $\leq 2K$. Then with $\rho_k = s_k\rho'_k$ it holds

$$M(\langle T_{\rho_k}, |z| = 1\rangle) = M\left(\left(\lambda_{\frac{\rho_k}{\rho'_k}}\right)_* \left\langle T_{\rho'_k}, |z| = s_k = \frac{\rho_k}{\rho'_k}\right\rangle\right) \leq 2 \cdot 2K$$

and since $\frac{\rho'_k}{2} \leq \rho_k \leq \rho'_k$ the sequence T_{ρ_k} also converges to the same T_∞. Since $\langle T_{\rho_k}, |z| = 1\rangle = (\lambda_{\rho_k})_* \langle T, |z| = \rho_k\rangle$, condition (ii) also reads

$$M(\langle T, |z| = \rho_k\rangle) \leq 4K\rho_k. \tag{21}$$

Step 2. We are viewing P as a current in the open set $\mathcal{A}$ in the manifold $\widetilde{\mathbb{C}}^{n+1}$, so the same should be done for the currents $P_\rho := \left(\Phi^{-1}\right)_* (T \llcorner (\mathcal{S} \setminus \mathcal{S}^\rho))$. Given the sequence $\rho_k \to 0$ chosen in step 1, we will observe the boundaries ∂P_{ρ_k} and prove that they converge to zero. It is enough to work with the sequence chosen in step 1 in order to obtain $\partial P = 0$ because, by the previous lemma, we know that the limit P is well-determined, independently of the chosen sequence ρ_k.

The boundaries ∂P_{ρ_k} satisfy (see condition (i) in step 1), as $k \to \infty$, by the definition (17) of Λ_{ρ_k}:

$$(\Lambda_{\rho_k})_*(\partial P_{\rho_k}) = -\left(\Phi^{-1}\right)_* \langle T_{0,\rho_k}, |z| = 1\rangle \rightharpoonup -\left(\Phi^{-1}\right)_* \langle T_\infty, |z| = 1\rangle. \tag{22}$$

Recall that we are viewing P_{ρ_k} as currents in the open set $\mathcal{A}$, so also $T \llcorner (\mathcal{S} \setminus \mathcal{S}^\rho)$ should be thought of as a current in the open set $\mathcal{S}$: this is why the only boundary comes from the slice of T with $|z| = \rho_k$.

Moreover we know (condition (ii) in step 1) that $(\Lambda_{\rho_k})_*(\partial P_{\rho_k})$ have equibounded masses, since so do $\partial(T_{0,\rho_k})$ and Φ^{-1} is a diffeomorphism on ∂B_1.

The current T_∞ has a special form: it is a $(1,1)$-cone, so the 1-current $\langle T_\infty, |z| = 1\rangle$ has an associated vector field that is always tangent to the Hopf fibers[8] of S^{2n+1}.

Step 3. We want to show that P is a cycle in $\mathcal{A}$, i.e. that $\partial P_{\rho_k} \to 0$ as $n \to \infty$. The boundary in the limit could possibly appear on $\mathbb{CP}^n \times \{0\}$ and we can exclude that as follows.

Let α be a 1-form of comass one with compact support in $\mathcal{A}$ and let us prove that $\partial P_{\rho_k}(\alpha) \to 0$. Since $\mathcal{A}$ is a submanifold in $\mathbb{CP}^n \times \mathbb{C}^{n+1}$, we can extend α to be a form in $\mathbb{CP}^n \times \mathbb{C}^{n+1}$. Let us write, using horizontal

[8] Recall that the Hopf fibration is defined by the projection $H : S^{2n+1} \subset \mathbb{C}^{n+1} \to \mathbb{CP}^n$, $H(z_0, ..., z_n) = [z_0, ..., z_n]$. The Hopf fibers $H^{-1}(p)$ for $p \in \mathbb{CP}^n$ are maximal circles in S^{2n+1}, namely the links of complex lines of $\mathbb{C}^{n+1}$ with the sphere.

coordinates $\{t_j\}_{j=1}^{2n}$ on $\mathbb{CP}^n$ and vertical ones $\{s_j\}_{j=1}^{2n+2}$ for $\mathbb{C}^{n+1}$, $\alpha = \alpha_h + \alpha_v$, where α_h is a form in the dt_j's, α_v in the ds_j's. Rewrite, viewing P_{ρ_k} as currents in $\mathbb{CP}^n \times \mathbb{C}^{n+1}$,

$$\partial P_{\rho_k}(\alpha) = [(\Lambda_{\rho_k})_*(\partial P_{\rho_k})] \left((\Lambda_{\rho_k}^{-1})^*\alpha\right).$$

The map $\Lambda_{\rho_k}^{-1}$ is expressed in our coordinates by $(t_1, ..., t_{2n}, s_1, ...s_{2n+2}) \to (t_1, ..., t_{2n}, \rho_k s_1, ...\rho_k s_{2n+2})$, therefore

$$(\Lambda_{\rho_k}^{-1})^*\alpha = \alpha_h^k + \alpha_v^k,$$

where the decomposition is as above and with $\|\alpha_h^k\|^* \approx \|\alpha_h\|^*$ and $\|\alpha_v^k\|^* \lesssim \rho_k \|\alpha_v\|^*$. The signs $\approx$ and $\lesssim$ mean respectively equality and inequality of the comasses up to a dimensional constant, so independently of the index k of the sequence.

As $k \to \infty$ it holds $\alpha_h^k \to \alpha_h^\infty$ in some C^ℓ-norm, where $\|\alpha_h^\infty\|^* \lesssim 1$ and α_h^∞ is a form in the dt_j's. More precisely α_h^∞ coincides with the restriction of α_h to $\mathbb{CP}^n \times \{0\}$, extended to $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ independently of the s_j-variables. We can write

$$\left|[(\Lambda_{\rho_k})_*(\partial P_{\rho_k})] (\alpha_h^k)\right| \leq \left|[(\Lambda_{\rho_k})_*(\partial P_{\rho_k})] (\alpha_h^k - \alpha_h^\infty)\right| + |[(\Lambda_{\rho_k})_*(\partial P_{\rho_k})] (\alpha_h^\infty)|$$

and both terms on the r.h.s. go to 0. The first, since $M((\Lambda_{\rho_k})_*(\partial P_{\rho_k}))$ are equibounded (as we said in step 1) and $|\alpha_h^k - \alpha_h^\infty| \to 0$; the second because we can use (22) and $\left(\Phi^{-1}\right)_* \partial(T_\infty)$ has zero action on a form that only has the dt_j's components, as remarked in step 1.

Moreover

$$\left|[(\Lambda_{\rho_k})_*(\partial P_{\rho_k})] (\alpha_v^k)\right| \to 0,$$

because $(\Lambda_{\rho_k})_*(\partial P_{\rho_k}) = -\left(\Phi^{-1}\right)_* \langle T_{0,\rho_k}, |z| = 1\rangle$ have equibounded masses by the choice of ρ_k, while $\|\alpha_v^k\|^* \lesssim \rho_k \|\alpha_v\|^*$ have comasses going to 0.

Therefore no boundary appears in the limit and P is a normal cycle in $\mathcal{A}$. The fact that it is ϑ-positive follows easily by the fact that so are the currents P_ρ, as remarked in the beginning of the proof of Lemma 4.2.

□

Summarizing, we define the current P just constructed to be the **proper transform** of the positive-$(1,1)$ normal cycle $T \llcorner \mathcal{S}$. P is a normal and ϑ-positive cycle in $\mathcal{A}$, where the semicalibration ϑ is Lipschitz (and actually smooth away from $\mathbb{CP}^n \times \{0\}$). Therefore the almost monotonicity formula holds true for P. Observe that the metric $\mathbf{g}$ on $\mathcal{A}$ fulfils the hypothesis $\frac{1}{5}\mathbb{I} \leq \mathbf{g} \leq 5\mathbb{I}$ of Proposition 2.1, because it is a perturbation of $\mathbf{g}_0$, which is in turn built from the Fubini-Study metric.

5 Proof of the result

With the assumptions in Proposition 3.1, we have to observe the family $T_{0,r} = (\lambda_r)_* T$ as $r \to 0$. These currents have equibounded masses by (12).

Take any converging sequence $T_{0,r_k} := (\lambda_{r_k})_* T \rightharpoonup T_\infty$ for $r_k \to 0$. Take the proper transform of each T_{0,r_k} and denote it by P_k. Lemmas 4.2 and 4.3 yield that P_k is a ϑ_k-positive cycle, for a semicalibration ϑ_k in the manifold $\mathcal{A}$. The form ϑ_k is smooth away from $\mathcal{V} \times \{0\}$ and it is Lipschitz-continuous, with $|\vartheta_k - \vartheta_0| < c_k \text{dist}_{\mathbf{g}_0}(\cdot, \mathbb{CP}^n \times \{0\})$. Recalling Lemma 4.1 we can see that, since the almost complex structure J_{r_k} on $\mathcal{S}$ fulfils $|J_{r_k} - J_0| < (Q\, r_k) \cdot r$ in $\mathcal{S}$ (by dilation), then the constants c_k go to 0 as $k \to \infty$. Analogously we get that the Lipschitz constants of ϑ_k are uniformly bounded by $2L$.

By Lemma 4.2 the masses of P_k are uniformly bounded in k (with respect to $\mathbf{g}_0$), since so are the masses of T_{0,r_k}, $M(T_{0,r_k}) \leq K$.

So by compactness, up to a subsequence that we do not relabel, we can assume $P_k \rightharpoonup P_\infty$ as $k \to \infty$ for a normal cycle P_∞.

Lemma 5.1. *P_∞ is a ϑ_0-positive cycle; more precisely it is the proper transform of T_∞.*

Proof. ϑ_0-positiveness follows straight from the ϑ_k-positiveness of P_k and $|\vartheta_k - \vartheta_0| < c_k \text{dist}_{\mathbf{g}_0}(\cdot, \mathbb{CP}^n \times \{0\})$, $c_k \to 0$.

Recall that $\vartheta_0 = \mathcal{E}^*(\vartheta_{\mathbb{CP}^n} + \vartheta_{\mathbb{C}^{n+1}})$; we want to estimate (notation from Section 4)

$$M(P_\infty \llcorner \mathcal{A}^\rho) = (P_\infty \llcorner \mathcal{A}^\rho)(\vartheta_0) = \lim_{k\to\infty} (P_k \llcorner \mathcal{A}^\rho)(\vartheta_0).$$

Write

$$(P_k \llcorner \mathcal{A}^\rho)(\vartheta_0) = (P_k \llcorner \mathcal{A}^\rho)(\mathcal{E}^* \vartheta_{\mathbb{CP}^n}) + (P_k \llcorner \mathcal{A}^\rho)(\mathcal{E}^* \vartheta_{\mathbb{C}^{n+1}}). \tag{23}$$

Let us bound the second term on the r.h.s.

$$(P_k \llcorner \mathcal{A}^\rho)(\mathcal{E}^* \vartheta_{\mathbb{C}^{n+1}}) = (\Lambda_\rho)_*(P_k \llcorner \mathcal{A}^\rho) \left((\Lambda_\rho^{-1})^*(\mathcal{E}^* \vartheta_{\mathbb{C}^{n+1}})\right).$$

The current $(\Lambda_\rho)_*(P_k \llcorner \mathcal{A}^\rho)$ is the proper transform of $T_{0,\rho r_k}$, therefore (Lemma 4.2) $M\left((\Lambda_\rho)_*(P_k \llcorner \mathcal{A}^\rho)\right) \leq K\,C$ independently of k; the form $(\Lambda_\rho^{-1})^*(\mathcal{E}^* \vartheta_{\mathbb{C}^{n+1}})$ has comass bounded by ρ^2. Altogether

$$(P_k \llcorner \mathcal{A}^\rho)(\mathcal{E}^* \vartheta_{\mathbb{C}^{n+1}}) \leq K\,C\,\rho^2.$$

To bound the first term on the r.h.s. of (23), let P be the proper transform of T; using that $(\Lambda_{r_k})^* \mathcal{E}^* \vartheta_{\mathbb{CP}^n} = \mathcal{E}^* \vartheta_{\mathbb{CP}^n}$ we can write

$$(P_k \llcorner \mathcal{A}^\rho)(\mathcal{E}^* \vartheta_{\mathbb{CP}^n}) = (P \llcorner \mathcal{A}^{r_k \rho})(\mathcal{E}^* \vartheta_{\mathbb{CP}^n}) \leq M\left(P \llcorner \mathcal{A}^{r_k \rho}\right) \leq M\left(P \llcorner \mathcal{A}^{\rho}\right),$$

which goes to 0 as $\rho \to 0$ by Lemma 4.2. Summarizing we get that there exists a function $o_\rho(1)$ that is infinitesimal as $\rho \to 0$, such that $|(P_k \llcorner \mathcal{A}^\rho)(\vartheta_0)| \leq o_\rho(1)$ (the point is that $o_\rho(1)$ can be chosen independently of k).

Therefore also $M(P_\infty \llcorner \mathcal{A}^\rho) = \lim_{k\to\infty}(P_k \llcorner \mathcal{A}^\rho)(\vartheta_0) \leq o_\rho(1)$, which means that

$$P_\infty = \lim_{\rho\to 0} P_\infty \llcorner (\mathcal{A} \setminus \mathcal{A}^\rho). \tag{24}$$

Recall now that the proper transform is a diffeomorphism away from the origin, thus

$$P_\infty \llcorner (\mathcal{A} \setminus \mathcal{A}^\rho) = \lim_n \left(\Phi^{-1}\right)_* T_{0,r_n} \llcorner (\mathcal{S} \setminus \mathcal{S}^\rho) = \left(\Phi^{-1}\right)_* T_\infty \llcorner (\mathcal{S} \setminus \mathcal{S}^\rho),$$

which concludes, together with (24), the proof that P_∞ is the proper transform of $\left(\Phi^{-1}\right)_* T_\infty$.

□

Recalling (4), the previous lemma tells us that P_∞ is of a very special form. Denoting $\mathcal{V} := \left\{\sum_{j=1}^n \frac{|z_j|^2}{|z_0|^2} < 1\right\} \subset \mathbb{CP}^n$ and, for each disk D^X in $\mathcal{S}$, L^X the disk such that $\Phi(L^X) = D^X$, we have

$$P_\infty(\beta) = \int_{\mathcal{V}} \left\{ \int_{L^X} \langle \beta, \vec{L^X} \rangle \, d\mathcal{L}^2 \right\} d\tau|_{\mathcal{V}}(X). \tag{25}$$

When we take the proper transform the density is preserved going from $\mathcal{S}$ to $\Phi^{-1}(\mathcal{S})$, since Φ^{-1} is a diffeomorphism on $\mathcal{S}$ (see Lemma B.2).

We are ready to conlcude the proof of Proposition 3.1, and therefore of Theorems 1.1 and 2.1.

proof of Proposition 3.1. The points $\frac{x_m}{|x_m|}$ converge to the point $(1, 0, ..., 0)$ in $D \cap S^{2n+1}$, where D is the disk $D = D^{[1,0,...0]}$.

We want to show that any converging sequence $T_{0,r_k} := (\lambda_{r_k})_* T \rightharpoonup T_\infty$ is such that the cone T_∞ contains $\kappa[\![D]\!]$.

Let us apply the proper transform to T_{0,r_k} and get P_k as in Lemma 5.1. Fix k: there is a sequence $\{x_m\}$ tending to the origin of points with densities such that $\liminf_{m\to\infty} \nu(x_m) \geq \kappa$. By Lemma B.2 the points $p_m := \left(\Phi^{-1}\right)(x_m)$ also have densities fulfilling that their lim inf is $\geq \kappa$ for P_k.

It easily seen that it holds $p_m \to p_0 = ([1, 0, ...0], 0) \in \mathbb{CP}^n \times \mathbb{C}^{n+1}$.

By upper semi-continuity of the density (which follows from the almost monotonicity formula for P_k) we get that p_0 also has density $\geq \kappa$ for P_k.

Doing this for every k we get that we are dealing with a sequence of normal cycles P_k all having the point p_0 as a point of density $\geq \kappa$. We wish to prove that, being the cycles P_k positive, then the point p_0 is also of density $\geq \kappa$ for the limit P_∞.

The cycles P_k are ϑ_k-positive so for any $\delta > 0$ it holds

$$M(P_k \llcorner B_\delta(p_0)) = (P_k \llcorner B_\delta(p_0))(\vartheta_k).$$

By weak convergence

$$\begin{aligned} M(P_\infty \llcorner B_\delta(p_0)) &= (P_\infty \llcorner B_\delta(p_0))(\vartheta_0) = \\ &= \lim_{k\to\infty} (P_k \llcorner B_\delta(p_0))(\vartheta_0). \end{aligned}$$

We can split

$$(P_k \llcorner B_\delta(p_0))(\vartheta_0) = (P_k \llcorner B_\delta(p_0))(\vartheta_0 - \vartheta_k) + (P_k \llcorner B_\delta(p_0))(\vartheta_k). \tag{26}$$

The semi-calibrations ϑ_k have uniform bounds on their Lipschitz constants, say $2L$. The metrics at p_0 coincide with $\mathbf{g}_0$ independently of k. We can therefore use the almost monotonicity formula for P_k at p_0 (Proposition 2.1) to get

$$(P_k \llcorner B_\delta(p_0))(\vartheta_k) = M(P_k \llcorner B_\delta(p_0)) \geq \pi(\kappa - C2L\delta)\delta^2,$$

where C is a universal constant. The forms ϑ_k fulfil $|\vartheta_k - \vartheta_0| < c_k$ in $B_\delta(p_0)$ and $c_k \to 0$ as $k \to \infty$. Therefore we can bound, from (26),

$$|(P_k \llcorner B_\delta(p_0))(\vartheta_0)| \geq -c_k K\, C + M(P_k \llcorner B_\delta(p_0)) \geq -c_k K\, C + \pi\kappa\delta^2 - 2CL\delta^3.$$

Since $c_k \to 0$ we can conclude

$$M(P_\infty \llcorner B_\delta(p_0)) \geq \pi\kappa\delta^2 - 2CL\delta^3 \tag{27}$$

independently of δ, which means that p_0 is a point of density $\geq \kappa$ for the ϑ_0-positive cycle P_∞.

Recall the structure of P_∞ from (25): it is made by the holomorphic disks L^X weighted with the positive measure τ, so if y_0 has density $\geq \kappa$, then the disk $L^{[1,0,\ldots 0]}$ must be weighted with a mass $\geq \kappa$, in other words the measure τ must have an atom of mass $\geq \kappa$ at y_0.

So P_∞ is of the form $\kappa[\![L^{[1,0,\ldots 0]}]\!] + \tilde{P}$, for a ϑ_0-positive current $\tilde{P}$. Transforming back via Φ, T_∞ contains the disk $\kappa[\![D]\!]$, as required.

$\square$

6 The uniqueness can fail without the assumption on the densities.

In this final section we show the importance of the assumption on the densities in Theorem 1.1. The algebraic blow up introduced in the paper and used in the proof turns out to be very useful again and allows us to provide a geometric picture of Kiselman's counterexample [18]. From now on, for sake of simplicity, we assume to be working in the integrable case $(\mathbb{C}^2, J_0, \omega_0)$, with coordinates $z = (z_1, z_2)$. The word positive will be used instead of ω_0-positive.

We briefly recall the connection between plurisubharmonic functions and positive currents.

A plurisubharmonic (psh) function f is, by definition, an upper semi-continuous function whose restriction to complex lines is subharmonic. Equivalently, the Levi form $L_{ij} := \frac{\partial^2 f}{\partial z_i \partial \overline{z}_j}$, for $i, j \in \{1, 2\}$, is positive definite. This last condition automatically implies that $\frac{\partial^2 f}{\partial z_i \partial \overline{z}_j}$ are Radon measures.

For $i \in \{1, 2\}$, denote by $\partial_i : \Lambda^k \to \Lambda^{k+1}$ (respectively $\overline{\partial}_i$) the operator on forms whose action on a function f is given by $\partial_i(f) := \frac{\partial f}{\partial z_i} dz^i$ (resp. $\overline{\partial}_i(f) := \frac{\partial f}{\partial \overline{z}_i} d\overline{z}^i$). So the two-form

$$\partial\overline{\partial} f = (\partial_1 + \partial_2)\left(\overline{\partial}_1 + \overline{\partial}_2\right) f = \sum_{ij} \frac{\partial^2 f}{\partial z_i \partial \overline{z}_j} dz^i \wedge d\overline{z}^j$$

has measure coefficients and gives rise to a normal positive cycle T_f, which acts on a two-form β as follows

$$T_f(\beta) := \int_{\mathbb{C}^2} \partial\overline{\partial} f \wedge \beta.$$

Recall that $\partial^2 = \overline{\partial}^2 = 0$. Since the standard differential d equals $\partial + \overline{\partial}$, the two-form $\partial\overline{\partial} f$ is closed and T_f is easily seen to be a cycle: for any one-form α

$$\partial T_f(\alpha) := \int_{\mathbb{C}^2} \partial\overline{\partial} f \wedge d\alpha = \int_{\mathbb{C}^2} d(\partial\overline{\partial} f) \wedge \alpha = 0.$$

With this in mind, every positive cone with density ν at the vertex (assume without loss of generality that the vertex is at the origin) is representable as a plurisubharmonic function h with the following homogeneity property (see [18])

$$h(tz) = \nu \log|t| + h(z), \text{ for any } t \in \mathbb{C}, z \in \mathbb{C}^2.$$

More precisely (see [6]), denoting $\pi : \mathbb{C}^2 \to \mathbb{CP}^1$ the standard projection, h is of the form $\nu \log|z| + f \circ \pi$ for some $f : \mathbb{CP}^1 \to \mathbb{R}$.

Let us see a concrete example, where we translate the question from $\mathbb{C}^2$ to $\widetilde{\mathbb{C}}^2$ using the algebraic blow up. Let us fix notations first.

We send the point $0 \neq (z_1, z_2) \in \mathbb{C}^2$ to the point $([z_1, z_2], z_1, z_2) \in \widetilde{\mathbb{C}}^2 \subset \mathbb{CP}^1 \times \mathbb{C}^2$. Using the chart $\frac{z_1}{z_2}$ on $\mathbb{CP}^1$, we can identify $([z_1, z_2], z_1, z_2)$ with $(\frac{z_1}{z_2}, \frac{z_1}{z_2} z_2, z_2) = (a, \lambda a, \lambda) \in \mathbb{C} \times \mathbb{C}^2$ with $a, \lambda \in \mathbb{C}$. Therefore we can locally identify the complex line bundle $\widetilde{\mathbb{C}}^2$ with $\mathbb{C} \times \mathbb{C}$ with coordinates (a, λ). The holomorphic planes through the origin are sent to the holomorphic planes $\{p\} \times \mathbb{C}$. The image of a sphere of radius r in $\mathbb{C}^2$ is, after the blow-up, the hypersurface $|\lambda| = \frac{r}{\sqrt{1+|a|^2}}$.

A positive cone in $\mathbb{C}^2$, with density ν at the vertex (placed at the origin) is given by assigning a positive Radon measure on $\mathbb{CP}^1$, having total mass ν. Let us blow-up the origin of $\mathbb{C}^2$ and move to $\mathbb{C} \times \mathbb{C}$. We have a measure μ on the first $\mathbb{C}$-factor, i.e. a measure on the set of "vertical" holomorphic planes. Consider the psh function $\nu \log |\lambda| + f(a)$ (with $\nu = |\mu|$), where $f(a) = \int_{\mathbb{C}} \log |\zeta - a| d\mu(\zeta)$, i.e. f is the convolution of μ with the fundamental solution of the Laplace operator in $\mathbb{C}$. In particular, $\Delta f = \mu$.

The current $\partial\overline{\partial}(\nu \log |\lambda| + f(a))$ is the sum of the current associated to the 2-surface $\mathbb{C} \times \{0\}$ with multiplicity ν and the current associated to integration on the vertical 2-planes, weighted with μ. [9]

To fix ideas, let $\nu = 1$. Let us assume that, for $r_1 < R_2$, in the domain $\frac{R_2}{\sqrt{1+|a|^2}} < |\lambda| < \frac{R_1}{\sqrt{1+|a|^2}}$ the current is very close to the plane $\nu[\![\{p_1\} \times \mathbb{C}]\!]$ and for $\frac{r_2}{\sqrt{1+|a|^2}} < |\lambda| < \frac{r_1}{\sqrt{1+|a|^2}}$ it is very close to the plane $\nu[\![\{p_2\} \times \mathbb{C}]\!]$, where $\{p_1\} \times \mathbb{C}$ and $\{p_2\} \times \mathbb{C}$ are distinct "vertical" holomorphic planes. To fix ideas, let f, g be such that Δf and Δg are positive measures of total mass 1, very close to Dirac deltas placed respectively in correspondence of the points p_1 and p_2.

Question: how can we extend the current across the intermediate domain $\frac{r_1}{\sqrt{1+|a|^2}} < |\lambda| < \frac{R_2}{\sqrt{1+|a|^2}}$ in such a way that it is globally positive and boundaryless?

Consider the currents T_1 and T_2 representable as integration of the actions of the vertical planes with weights $c_1 \Delta f$ and $c_2 \Delta g$, with $c_1 > c_2 > 1$. In terms of psh functions, $T_1 = c_1 \partial\overline{\partial} f$ and $T_2 = c_2 \partial\overline{\partial} g$. Consider the function

$$E(a, \lambda) := c_2 g(a) - c_1 f(a) - (c_1 - c_2) \log |\lambda|.$$

Its level sets $E_\eta = \{E = \eta\}$ are the hypersurfaces $|\lambda| = e^{\frac{c_2 g - c_1 f - \eta}{c_1 - c_2}}$. Choose a positive value of η, large enough in modulus, so to ensure that the corresponding level set E_η is contained in the intermediate domain $\frac{r_1}{\sqrt{1+|a|^2}} <$

[9] By going back to the coordinates $z_1 = a\lambda, z_2 = \lambda$, we can recover the psh function on $\mathbb{C}^2$: for this purpose it is convenient to rewrite $\nu \log |\lambda| + f(a) = \nu \log |\lambda| + \nu \log \sqrt{1+|a|^2} - \nu \log \sqrt{1+|a|^2} + f(a)$, so that the inverse transform of $\nu \log |\lambda| + \nu \log \sqrt{1+|a|^2}$ is $\nu \log |z|$.

$|\lambda| < \frac{R_2}{\sqrt{1+|a|^2}}$.

Set the current $\chi_{\{E\geq\eta\}}T_1 + \chi_{\{E\leq\eta\}}T_2$, which equals T_1 above E_η and T_2 below E_η. Such a current develops a boundary on E_η given by the slices of T_1 and T_2 with the level set E_η. Precisely the boundary is given by the 1-current of integration along the "vertical" circles $E_\eta \cap (\{a\} \times \mathbb{C})$ weighted with $-c_1\Delta f + c_2\Delta g$.

Claim: there exists a positive 2-current, supported in E_η, whose boundary is the same as the boundary just described with opposite sign.

This can be seen explicitly as follows. Let us consider the one form $\iota_{\nabla E}\omega_0$, where ω_0 is the standard symplectic form on $\mathbb{C}^2$. Its differential is given, through Cartan's formula, by $d(\iota_{\nabla E}\omega_0) = \mathcal{L}_{\nabla E}\omega_0$, the Lie derivative of ω_0 along the gradient field.

It holds, as we are about to see, on the set $\{\lambda \neq 0\}$,

$$\mathcal{L}_{\nabla E}\omega_0 = (c_1\Delta f - c_2\Delta g)\left(\frac{-i}{2}da \wedge d\bar{a}\right). \tag{28}$$

Let us compute this Lie derivative. For the sake of notation, only for this computation we take coordinates x_1, x_2, x_3, x_4 so that $a = x_1 + ix_2$ and $\lambda = x_3 + ix_4 = 0$ and denote by ∂_j the derivations w.r.t. x_j. So we write $\omega_0 = dx^1 \wedge dx^2 + dx^3 \wedge dx^4$ and $\nabla E = (\partial_i E)\partial_i$.

$$\begin{aligned}\mathcal{L}_{\nabla E}dx^1 \wedge dx^2 &= (\mathcal{L}_{\nabla E}dx^1) \wedge dx^2 + dx^1 \wedge (\mathcal{L}_{\nabla E}dx^2) = \\ &= (d\mathcal{L}_{\nabla E}x^1) \wedge dx^2 + dx^1 \wedge (d\mathcal{L}_{\nabla E}x^2) = (d(\partial_1 E)) \wedge dx^2 + dx^1 \wedge (d(\partial_2 E)) \\ &= \sum_{k=1}^4 (\partial_k\partial_1 E)dx^k \wedge dx^2 + \sum_{k=1}^4 (\partial_k\partial_2 E)dx^1 \wedge dx^k = (\partial_1\partial_1 E + \partial_2\partial_2 E)dx^1 \wedge dx^2.\end{aligned}$$

In the last equality we used the specific form of E. An analogous computation yields

$$\mathcal{L}_{\nabla E}dx^3 \wedge dx^4 = (\partial_3\partial_3 E + \partial_4\partial_4 E)dx^3 \wedge dx^4.$$

Away from $\{\lambda = x_3 + ix_4 = 0\}$, the Laplacian $(\partial_3\partial_3 E + \partial_4\partial_4 E) = \Delta(\log\lambda)$ vanishes, so (28) is proven.

If we take, in the boundaryless 3-surface E_η, the 2-current corresponding in duality to the one-form $\iota_{\nabla\eta}\omega_0$, this is positive and its boundary is the 1-current dual to $d(\iota_{\nabla E}\omega_0)$, i.e. its boundary erases exactly that of $\chi_{\{E\geq\eta\}}T_1 + \chi_{\{E\leq\eta\}}T_2$. So the claim is proved: this current fills the gap between T_1 and T_2 along E_η.

This operation seems a bit magical, but is nothing else than the geometric picture corresponding to the operation of taking the supremum of the psh functions $c_1 f$ and $c_2 g - \eta$. Since psh functions are closed under this operation, the current $\partial\bar{\partial}(\sup\{c_1 f, c_2 g - \eta\})$ is guaranteed to be positive and boundaryless: E_η is the set where the two functions are equal and the operation of taking the sup amounts (from the geometric point of view when considering the associated currents) to filling the gap along E_η with the current $\iota_{\nabla E}\omega_0$.

The construction described in this section is exacly what Kiselman does, although the geometric picture in [18] is a bit hidden by the fact that everything is expressed in terms of psh functions. The current he constructs is made by iterating the construction: with suitable choices of the parameters involved he alternates currents defined in suitable domains $\frac{r_2}{\sqrt{1+|a|^2}} < |\lambda| < \frac{r_1}{\sqrt{1+|a|^2}}$ and ensures that the measures that we called Δf and Δg have different limits.

As we can see from the explicit construction just exhibited, the current constructed in [18] has the property that the point x_0 where it fails to have a unique tangent cone is an isolated point of strictly positive density. The proof given in the present work therefore breaks down in that case.

More precisely, the failure of uniqueness of tangent cone for a positive $(1,1)$-cycle must happen at a point x_0 where there exists $\delta > 0$ so that the Lelong number ν fulfils $\nu(x_0) \geq \nu(x) + \delta$ for all points x in a neghbourhood of x_0. This fact follows, in the complex setting, from Siu's result [30], while for an almost complex setting it is yielded by Theorem 1.1.

Indeed, the current constructed in [18] has the property that the point x_0 where it fails to have a unique tangent cone is an isolated point of strictly positive density. Therefore the proof given in this paper for Theorem 1.1 fails in that case.

A Appendix

We give here a brief sketch of the main ideas involved in the construction of the *pseudo holomorphic polar foliation* used in the present paper. As mentioned in Section 3 the construction is exactly as in [25], the only extra care required is that we must perform it in $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$ for arbitrary n, while in [25] only the case $n = 1$ is addressed. We will keep notations as close to [25] as possible in order to facilitate the comparison.

Let J be an almost complex structure in B_2^{2n+2} that agrees with the standard J_0 at the origin and assume that $\|J - J_0\|_{C^{2,\nu}(B_2^{2n+2})} \leq \alpha_0$, where α_0 is small enough. For $X \in \mathbb{CP}^n$ let $D_{0,X}$ denote the (flat) J_0-holomorphic disk through the origin such that the image of $D_{0,X} \setminus \{0\}$ via the standard projection $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ is X.

The first essential step requires the following: given X construct in $B_{1+\varepsilon}^{2n+2}$ an embedded J-pseudo holomorphic disk that goes through the origin with tangent D_X.

Using coordinates $(w_1, ..., w_n, z)$ on $\mathbb{C}^{n+1}$, for $X = [W_1, ...W_n, Z] \in \mathbb{CP}^n$ the disk $D_{0,X}$ is represented by

$$D_{0,X} = \left\{ (w_1, ..., w_n, z) : w_j = \frac{W_j}{Z} z \text{ for } j = 1, ..., n \right\},$$

i.e. we have a representation of $D_{0,X}$ as a graph on the complex line spanned by the z-coordinate. We let

$$h_{0,X}^0(z) = \left(\frac{W_1}{Z} z, ..., \frac{W_n}{Z} z \right)$$

and $D_{0,X}$ is the graph of $h_{0,X}^0$. We will denote by $(h_{0,X}^0)^j$ the j-th component of the function $h_{0,X}^0$. We can also think of $D_{0,X}$ as the image of the parametrized curve from $D^2 \in \mathbb{C}$ into B_2^{2n+2} given by

$$H_{0,X}^0(z) = \left(h_{0,X}^0(z), z \right).$$

In a first moment we will look for a perturbation of $D_{0,X}$ of the form

$$\hat{H}_X(z) = \left((h_{0,X}^0)^1(z) + \lambda^1(z), ..., (h_{0,X}^0)^n(z) + \lambda^n(z), z + \mu(z) \right),$$

with $\lambda^1, ..., \lambda^n, \mu$ functions of z to be determined. What we require on $\hat{H}_X(z)$ is that it must represent a J-pseudo holomorphic curve, but in general it will neither pass through the origin nor be a graph on the z-coordinate line. As a parametrized J-pseudo holomorphic curve from $D^2 \in \mathbb{C}$ into (B_2^{2n+2}, J) it must satisfy the equation (we denote by x, y the real coordinates on D^2)

$$\frac{\partial \hat{H}_X}{\partial y} = J \frac{\partial \hat{H}_X}{\partial x}.$$

Splitting $\hat{H}_X(z) = H^0_{0,X}(z) + T_X(z)$, with $T_X(z) = \big(\lambda^1(z), ..., \lambda^n(z), \mu(z)\big)$, we rewrite the latter equation as

$$\frac{\partial T_X}{\partial \bar{z}} = i(J - J_0)\frac{\partial H^0_{0,X}}{\partial x}(z) + i(J - J_0)\frac{\partial T_X}{\partial x}(z). \tag{29}$$

We can uniquely solve (29). Indeed, denote by $(T_X)^j$ the j-th component of T_X. Recall at this stage that the following is a well-posed elliptic problem from $C^{l,\nu}$ to $C^{l+1,\nu}$:

$$\left\{ \begin{array}{c} \frac{\partial (T_X)^j}{\partial \bar{z}}(z) = F_j(z) \\ (T_X)^j|_{\partial D^2} \in \operatorname{span}\{e^{-ik\theta} : k \in \mathbb{N} \setminus \{0\}\} \end{array} \right. . \tag{30}$$

For $\|J - J_0\|_{C^{2,\nu}(B_2^{2n+2})}$ small enough a fixed point argument in $C^{2,\nu} \times ... \times C^{2,\nu}$ yields a unique solution for $T_X(z)$, which belongs to $C^{3,\nu}$. We thus have the parametrized J-pseudo holomorphic curve $\hat{H}_X(z)$. Since $\mu(z)$ is small, the function $z \to \zeta(z) = z + \mu(z)$ is invertible (it is a perturbation of the identity). Then we set

$$\tilde{H}(z) := \hat{H}_X \circ \zeta^{-1}(z) \tag{31}$$

and this curve is now a J-pseudo holomorphic graph on the z-coordinate line. In general however this is not a graph through the origin: therefore we need to extend the previous construction in the following way.

For $p = (w_{1p}, ..., w_{np}, z_p) \in \mathbb{C}^{n+1}$ and $X = [W_1, ...W_n, Z] \in \mathbb{CP}^n$, let $D^0_{p,X}$ be

$$D^0_{p,X} = \left\{(w_1, ..., w_n, z) : w_j - w_{j_p} = (h^0_{p,X})^j(z) := \frac{W_j}{Z}(z - z_p) \text{ for } j = 1, ..., n\right\},$$

i.e. the flat J_0-holomorphic disk through p with direction given by X. Following the strategy used before, we first of all perturb $D^0_{p,X}$ to a parametrized J-pseudo holomorphic curve $\hat{H}_{p,X}(z)$ of the form

$$\hat{H}_{p,X}(z) = \big((h^0_{p,X})^1(z) + \lambda^1_{p,X}(z), ..., (h^0_{0,X})^n(z) + \lambda^n_{p,X}(z), z + \mu_{p,X}(z)\big)$$

by imposing

$$\frac{\partial \hat{H}_{p,X}}{\partial y} = J \frac{\partial \hat{H}_{p,X}}{\partial x}.$$

Again by a fixed point argument, using the well-posedness of the elliptic problem (30), we find a unique solution. By inverting the function $\zeta_{p,X}(z) = z + \mu_{p,X}(z)$ we then obtain

$$\tilde{H}_{p,X}(z) := \hat{H}_{p,X} \circ \zeta_{p,X}^{-1}(z) = (\tilde{h}_{p,X}(z), z),$$

so that the image of the curve $\tilde{H}_{p,X}$ is the J-pseudo holomorphic graph of $\tilde{h}_{p,X}$.

The graph of $\tilde{h}_{p,X}$ does not necessarily pass through the point p. However, the fact that it is a $C^{l+1,\nu}$-graph over the complex line spanned by the coordinate z, allows us to perform the following operation: given $(p, X) \in \mathbb{C}^{n+1} \times \mathbb{CP}^n$ as above, we can construct $\tilde{H}_{p,X}$ and take the intersection of the complex hyperplane $\{z = 0\}$ with this curve. In other words we are taking the point $\tilde{h}_{p,X}(0) \in \mathbb{C}^n \cong \{z = 0\}$. Moreover by the $C^{l+1,\nu}$-regularity we can take the element of $\mathbb{CP}^n$ obtained as (homogeneous) $(n+1)$-tuple

$$\left[\frac{\partial \left(\hat{H}_{p,X}\right)^j}{\partial z}(\zeta_{p,X}^{-1}(0))\right].$$

To sum up we can assign to (p, X) a point in $\{z = 0\} \cong \mathbb{C}^n$ and an element of $\mathbb{CP}^n$. Restrict these two assignments to the couples (p, X) such that $p = (w_{1p}, ..., w_{np}, 0)$. Identifying $\{z = 0\} \cong \mathbb{C}^n$ we thus get a map[10] $\psi : \mathbb{C}^n \times \mathbb{CP}^n \to \mathbb{C}^n \times \mathbb{CP}^n$. The elliptic estimates coming from problem (30) imply that (since $\|J - J_0\|_{C^{2,\nu}(B_2^{2n+2})}$ is small) ψ is a $C^{2,\nu}$-perturbation of the identity and thus invertible. This means that, given the couple $(0, X)$ we can find a point $q \in \{z = 0\}$ and a direction $Y \in \mathbb{CP}^n$ such that $\psi(q, Y) = (0, X)$, i.e. that graph of $\tilde{h}_{q,Y}$ goes through 0 with direction X.

At this stage we know how to construct a J-pseudo holomorphic disk through the origin with a given tangent. Now with a bit more effort we can see that, as X ranges over an open ball U of $\mathbb{CP}^n$, the disks that we constructed foliate a punctured cone with vertex at the origin.

Consider for example the punctured cone $\{|w_j| < \frac{|z|}{2}$ for $j = 1, ..., n\}$. For any point $q = (w_{1q}, ..., w_{nq}, z_q)$ in the cone define the map

$$\Xi_q : U \to \mathbb{CP}^n$$

in the following way: given $X \in U$, after constructing (using the recipe illustrated above) the unique $\tilde{h}_{p,Y}$ such that the J-pseudo holomorphic graph of $\tilde{h}_{p,Y}$ goes through 0 with direction X, we take the (homogeneous) $(n+1)$-tuple $[\tilde{h}_{p,Y}(z_q), z_q]$. Still by means of elliptic estimates it can be checked

[10]This map, that we denote by ψ, is denoted by Ψ in [25]. We keep however the capital letter for the map defined further below, in order to remain coherent with the notation introduced in Section 3.

(compare [25]) that Ξ_q is (independently of q) a perturbation of the identity, which in turn yields that we have a foliation of the punctured cone. Remark that there is nothing special about our choice of the cone, we could do the same for an arbitrary ball U of $\mathbb{CP}^n$: the only care that should be taken, is that the choice of U can influence how small $\|J - J_0\|_{C^{2,\nu}(B_2^{2n+2})}$ needs to be, in order to ensure the existence of the polar foliation.

To avoid confusion, remark that in Section 3 and throughout the paper we use coordinates $(z_0, ..., z_n)$ in $\mathbb{C}^{n+1}$ and construct a polar foliation in a punctured cone that is a neighbourhood of the (real) 2-dimensional plane $\{z_1 = ... = z_n = 0\}$. On the other hand in this appendix we used coordinates $(w_1, ..., w_n, z)$ and constructed a polar foliation in a neighbourhood of $\{w_1 = ... = w_n = 0\}$. This should not give rise to confusion and was only done in order to keep the notation in the appendix as close as possible to [25].

Te map Ψ that we use in the paper is given by doing (in words) the following: the input of Ψ is a point $(z_0, ..., z_n)$. Consider the complex line $[z_0, z_n]$ and determine, by applying ψ^{-1} the unique couple (q, Y) such that $\tilde{h}_{q,Y}$ is a J-pseudo holomorphic graph that passes through the origin with tangent $[z_0, z_n]$. Then

$$\Psi(z_0, ..., z_n) = \left(\tilde{h}_{q,Y}(z_n), z_n\right).$$

With a slight abuse of notation, due to the fact that we are now going to identify $\psi^{-1}(0, [z_0, z_n])$ (which is a couple in $\mathbb{C}^n \times \mathbb{CP}^n$) with the corresponding point in $\mathbb{C}^{n+1} \times \mathbb{CP}^n$ (recall that $\mathbb{C}^n \cong \{z = 0\} \subset \mathbb{C}^{n+1}$) we can write:

$$\Psi(z_0, ..., z_n) := \tilde{h}_{\psi^{-1}(0,[z_0,...,z_n])}.$$

B Appendix

The following almost-monotonicity formula for positive or semi-calibrated cycles is proved in [24], Proposition 1, for a C^1 semi-calibration: the same proof works as well for a form with Lipschitz-continous coefficients, so we only give the statement.

Let the ball of radius 2 in $\mathbb{R}^d$ be endowed with a metric g and a two-form ω such that both g and ω have Lipschitz-continuous coefficients (with respect to the standard coordinates on $\mathbb{R}^n$) and ω has unit comass for g. The metric g is represented by a matrix and we further assume that $\frac{1}{5}\mathbb{I} \leq g \leq 5\mathbb{I}$, where $\mathbb{I}$ is the identity matrix. So g is a Lipshitz perturbation of the flat metric.

Let T be a ω-positive normal cycle. Then we have a 2-vector field $\vec{T}(x)$, of unit mass with respect to g. This means that for $\|T\|$-a.a. x, $\vec{T}(x) = \sum_{k=1}^{N(x)} \lambda_k(x)\vec{T}_k(x)$, a convex combination of ω_x-calibrated unit simple 2-vectors. The mass refers pointwise to the metric g_x.

Proposition B.1. *In the previous hypothesis, there exists $r_0 > 0$ and $C > 0$, depending only on the Lipschitz constants of g and ω such that, given an arbitrary point $x_0 \in B_1(0)$, the following holds.*

Denote by $B_r(x_0)$ (respectively $B_s(x_0)$) the ball around x_0 of radius r (respectively s) with respect to the metric g_{x_0}; let $|\cdot|$ be the distance for g_{x_0} and $|\cdot|_g$ the mass-norm with respect to g. Let $\frac{\partial}{\partial r}$ be the unit radial vector field with respect to x_0 and g_{x_0}.

For any $0 < s < r < r_0$, we have

$$\begin{aligned}
&\frac{e^{Cr}+Cr}{r^2}\left(T \llcorner B_r(x_0)\right)(\omega) - \frac{e^{Cs}+Cs}{s^2}\left(T \llcorner B_s(x_0)\right)(\omega) \\
&\geq \int_{B_r \setminus B_s(x_0)} \frac{1}{|x-x_0|^2} \sum_{k=1}^{N(x)} \lambda_k(x) \left|\vec{T}_k(x) \wedge \frac{\partial}{\partial r}\right|^2_{g(x)} d\|T\| = \\
&\qquad = \int_{B_r \setminus B_s(x_0)} \frac{\langle \omega_t(x), \vec{T}(x)\rangle}{|x-x_0|^2} d\|T\|
\end{aligned} \tag{32}$$

and

$$\begin{aligned}
&\frac{e^{Cr}-Cr}{r^2}\left(T \llcorner B_r(x_0)\right)(\omega) - \frac{e^{Cs}-Cs}{s^2}\left(T \llcorner B_s(x_0)\right)(\omega) \\
&\leq \int_{B_r \setminus B_s(x_0)} \frac{1}{|x-x_0|^2} \sum_{k=1}^{N(x)} \lambda_k(x) \left|\vec{T}_k(x) \wedge \frac{\partial}{\partial r}\right|^2_{g(x)} d\|T\| = \\
&\qquad = \int_{B_r \setminus B_s(x_0)} \frac{\langle \omega_t(x), \vec{T}(x)\rangle}{|x-x_0|^2} d\|T\|,
\end{aligned} \tag{33}$$

where $\omega_t := \frac{\partial}{\partial r} \lrcorner (dr \wedge \omega)$ is the tangential part of ω (as in [16] and [24]).

The following two lemmas are used in the paper when pushing forward a positive cycle under a diffeomorphism.

Lemma B.1. *[the pushforward of a positive-$(1,1)$ current via a pseudoholomorphic diffeomorphism is positive-$(1,1)$]*

Let C be a positive-$(1,1)$ normal current in an open set $U \subset \mathbb{R}^{2N}$, endowed with an almost complex structure J_1, a compatible metric g_1 and a two-form ω_1. Let $f : U \to \mathbb{R}^{2N}$ be a smooth pseudoholomorphic diffeomorphism, where $\mathbb{R}^{2N}$ is endowed with an almost complex structure J_2 and compatible metric and semi-calibration g_2 and ω_2. Then $f_* C$ is a positive-$(1,1)$ normal current in $(\mathbb{R}^{2N}, J_2, g_2)$.

proof of lemma B.1. The current C is represented by a couple $(\mu_C, \vec{C})$, where μ_C is a Radon measure and $\vec{C}$ is a unit 2-vector field, well defined μ_C-a.e. The $(1,1)$-condition can be expressed by the fact that $\vec{C} = \sum_{j=1}^M \lambda_j \vec{C}_j$, with $\sum_{j=1}^M \lambda_j = 1$, $\lambda_j \geq 0$ and $\vec{C}_j$ are unit simple J_1-invariant.

The push-forward $f_* C$ can be represented by the Radon measure $f_* \mu_C$ and the 2-vector field (defined $f_*\mu_C$-a.e.) $f_* \vec{C}$, the latter is however not of unit mass. Denoting by $\|\cdot\|$ the mass norm on 2-vectors with respect to g_2, we rewrite it as

$$f_* \vec{C} = \sum_{j=1}^M \lambda_j f_* \vec{C}_j = \sum_{j=1}^M \lambda_j \cdot \|f_* \vec{C}_j\| \frac{f_* \vec{C}_j}{\|f_* \vec{C}_j\|} =$$

$$= \left(\sum_{j=1}^M \lambda_j \cdot \|f_* \vec{C}_j\| \right) \sum_{j=1}^M \frac{\lambda_j \cdot \|f_* \vec{C}_j\|}{\left(\sum_{j=1}^M \lambda_j \cdot \|f_* \vec{C}_j\| \right)} \frac{f_* \vec{C}_j}{\|f_* \vec{C}_j\|},$$

where each simple 2-vector $\frac{f_* \vec{C}_j}{\|f_* \vec{C}_j\|}$ is of unit mass and J_2-invariant (by the hypothesis on f).

We can then represent $f_* C$ by the Radon measure

$$\left(\sum_{j=1}^M \lambda_j \cdot \|f_* \vec{C}_j\| \right) f_* \mu_C$$

and the 2-vector field of unit mass

$$\sum_{j=1}^M \frac{\lambda_j \cdot \|f_* \vec{C}_j\|}{\left(\sum_{j=1}^M \lambda_j \cdot \|f_* \vec{C}_j\| \right)} \frac{f_* \vec{C}_j}{\|f_* \vec{C}_j\|},$$

which is a convex combination of unit simple J_2-holomorphic 2-vectors.

$\square$

Lemma B.2. *[the density is preserved]*

Let U, V be open sets in $\mathbb{R}^{2n+2}$, ω be a calibration in U, T be a normal ω-positive 2-cycle in U, $f : U \to V$ be a diffeomorphism. Be $\nu(p) \geq 0$ the density of T at $p \in U$. Then the current $f_* T$ has 2-density equal to $\nu(p)$ at the point $f(p) \in V$.

proof of lemma B.2. Up to translations, which do not affect densities, we may assume $p = f(p) = 0$, the origin of $\mathbb{R}^{2n+2}$. We use coordinates $q = (q_1, q_2, ..., q_{2n+2})$.

Step 1. Assume that f is linear. Choose any sequence of radii $R_n \downarrow 0$ and dilate the current $f_* T$ around 0 with the chosen factors, i.e. observe the sequence:

$$\left(\frac{Id}{|R_n|}\right)_* (f_* T) = \left(\frac{Id}{|R_n|} \circ f\right)_* T.$$

By the linearity of f this is the same as

$$\left(f \circ \frac{Id}{|R_n|}\right)_* T = f_* \left(\frac{Id}{|R_n|}\right)_* T.$$

The assumptions yield a subsequence R_{n_j} such that $\left(\frac{Id}{|R_{n_j}|}\right)_* T \rightharpoonup T_\infty$ for a cone T_∞, whose density at the vertex is $\nu(0)$. So

$$f_* \left(\frac{Id}{|R_{n_j}|}\right)_* T \rightharpoonup f_* T_\infty.$$

Recall that T_∞ is represented by a positive Radon measure on the 2-planes, with total mass $\nu(0)$. The linearity of f gives that $f_* T_\infty$ is still a cone with the same density $\nu(0)$ at the vertex, so we have found a subsequence R_{n_j} such that $\left(\frac{Id}{|R_{n_j}|}\right)_* (f_* T)$ weakly converges to a cone with density $\nu(0)$. Since the sequence R_n was arbitrary, we get in particular that $f_* T$ has 2-density equal to $\nu(0)$ at the point $f(0) = 0$.

Step 2. For a general f, write $f(q) = Df(0) \cdot q + o(|q|)$.

As before, we have to observe $\left(\frac{Id}{|R_n|}\right)_* (f_* T)$. We show that this sequence has the same limiting behaviour as $\left(\frac{Id}{|R_n|}\right)_* ((Df(0) \cdot q)_* T)$, for which *Step 1* applies.

We estimate the difference of the actions on a two-form β supported in the unit ball B_1:

$$\left(\frac{Id}{|R_n|}\right)_* [f_* T - (Df(0) \cdot q)_* T] (\beta) =$$

$$= T \left(f^* \left(\frac{Id}{|R_n|}\right)^* \beta - (Df(0) \cdot q)^* \left(\frac{Id}{|R_n|}\right)^* \beta \right).$$

Writing explicitly $\beta = \sum_I \beta_I dq^I$, where $dq^I = dq^i \wedge dq^j$ for $i \neq j \in \{1, 2, ..., 2n+2\}$, the difference in brackets reads[11]

$$\sum_I \frac{\beta_I \circ \frac{Id}{|R_n|} \circ f - \beta_I \circ \frac{Id}{|R_n|} \circ (Df(0) \cdot q)}{R_n^2} df^I.$$

This form is supported, for n large enough, in a ball of radius $\leq \frac{1}{2|Df(0)|} R_n$ around 0. Moreover, for each I, we can estimate from above, for n large enough:

$$|df^I| \left| \frac{\beta_I \circ \frac{Id}{|R_n|} \circ f - \beta_I \circ \frac{Id}{|R_n|} \circ (Df(0) \cdot q)}{R_n^2} \right| \leq$$

$$\leq \frac{\|f\|_{C^1(B_1)} \|\beta_i\|_{C^1(B_1)}}{R_n^3} \cdot |o(|q|)| \leq \frac{|o(1)|}{R_n^2},$$

for a function $o(1)$, infinitesimal as $n \to \infty$, depending on β and $\|f\|_{C^2}$. Using monotonicity, we get a constant $K > 0$, depending on $\nu(0)$ and $\|f\|_{C^1}$, such that $M\left(T \llcorner B_{\frac{1}{2|Df(0)|} R_n}\right) \leq K R_n^2$ for n large enough. These estimates imply

$$T\left(f^* \left(\frac{Id}{|R_n|}\right)^* \beta - (Df(0) \cdot q)^* \left(\frac{Id}{|R_n|}\right)^* \beta\right) \to 0 \text{ as } n \to \infty,$$

so the limiting behaviour of $\left(\frac{Id}{|R_n|}\right)_* (f_* T)$ must be the same as that of $\left(\frac{Id}{|R_n|}\right)_* ((Df(0) \cdot q)_* T)$. In particular the density of $f_* T$ at the point $f(0) = 0$ is $\nu(0)$.

□

[11] Writing $f = (f^1, f^2, ..., f^{2n+2})$ and $I = (i, j)$, the notation df^I stands for $d(f^i) \wedge d(f^j)$, as in [14] (page 120).

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