Embeddedness of Min-Max CMC Hypersurfaces in Manifolds with Positive Ricci Curvature

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Abstract

We prove that on a compact Riemanninan manifold of dimension 3 or higher, with positive Ricci curvature, the Allen–Cahn min-max scheme in [6], with prescribing function taken to be a non-zero constant λ, produces an embedded hypersurface of constant mean curvature λ (λ-CMC). More precisely, we prove that the interface arising from said min-max contains no even-multiplicity minimal hypersurface and no quasi-embedded points (both of these occurrences are in principle possible in the conclusions of [6]). The immediate geometric corollary is the existence (in ambient manifolds as above) of embedded, closed λ-CMC hypersurfaces (with Morse index 1) for any prescribed non-zero constant λ, with the expected singular set when the ambient dimension is 8 or higher.

Theorem 1. *For any $\lambda \in \mathbb{R} \setminus \{0\}$, and compact Riemannian manifold (N, g), with positive Ricci curvature and $dim\, N = n+1 \geq 3$, there exists a smooth, embedded, two-sided hypersurface M, with constant mean curvature λ (λ-CMC), and*

1. *M is closed when $2 \leq n \leq 6$,*
2. *$\overline{M} \setminus M$ consists of finitely many points when $n = 7$,*
3. *$dim_{\mathcal{H}}(\overline{M} \setminus M) \leq n - 7$, when $n \geq 8$.*

In Theorem 1 the emphasis is on the fact that M is embedded: this appears to be a new result. The statement of Theorem 1 with embedded replaced by (the weaker notion of) quasi-embedded was on the other hand known, as detailed below (with two methods available). We recall that quasi-embedded means that the hypersurface is a smooth immersion, with any self-intersections being tangential, and with local structure around any point of tangential intersection being that of two embedded disks lying on opposite sides of each other (see [6, Definition 8]).

As it will be important for our arguments, we begin by recalling that the existence result in Theorem 1, with embedded replaced by quasi-embedded, follows from the work by the first author and N. Wickramasekera in [6]. In fact, [6, Theorem 1.1] proves the following more general result. Given a compact Riemannian manifold, (N, g), $\dim N \geq 3$ (without any curvature assumptions) and a non-negative Lipschitz function $h : N \to \mathbb{R}$, there exists a quasi-embedded, two-sided C^2 hypersurface M_h such that, for each $x \in M_h$, the scalar mean curvature of M_h at x is given by $h(x)$; the singular set $\overline{M_h} \setminus M_h$ satisfies the dimensional estimates listed in Theorem 1. The construction of M_h is carried out in the Allen–Cahn min-max framework, and serves as a starting point for the present work. We briefly recall it here in the case $h = \lambda$ constant, with further details in Section 1.1.

Consider a sequence of functions $\{u_i\}$ in $W^{1,2}(N)$, where each u_i is the solution of the appropriate ε_i-scaled inhomogeneous Allen–Cahn equation, with $\varepsilon_i \to 0$. Assuming uniform energy bound, the works of J. Hutchinson–Y. Tonegawa [12] and M. Röger–Y. Tonegawa [15] give, in the $\varepsilon_i \to 0$ limit, an integral varifold V (a "limit interface"), with generalised mean curvature $H_V \in L^\infty(\text{supp}\|V\|)$, along with a Caccioppoli set E, with $\partial^* E \subset \text{supp}\,\|V\|$, such that,

$$\begin{cases} H_V(x) = \lambda,\, \theta_V(x) = 1, & \mathcal{H}^n - a.e.\, x \in \partial^* E, \\ H_V(x) = 0,\, \theta_V(x) \in 2\mathbb{Z}_{\geq 1}, & \mathcal{H}^n - a.e.\, x \in \text{supp}\,\|V\| \setminus \partial^* E. \end{cases}$$

In the presence of such a sequence $\{u_i\}$, the two major roadblocks to an existence result for a λ-CMC are (i) $\partial^* E$ may be empty, in which case the limit interface is actually minimal (ii) even if $\partial^* E \neq \emptyset$, it may not have sufficient regularity ([6, Figure 1] illustrates how lack of regularity could prevent $\partial^* E$ from being an admissible candidate). In [6] a (first) sequence u_i is produced by means of a classical mountain pass lemma; the Morse index of u_i is at most 1 (as a consequence of the fact that the min-max has one parameter). It is moreover shown (see [6, Remark 6.7]) that in the case of ambient manifold with positive Ricci curvature (and with $h = \lambda$ constant), occurrence (i) cannot arise, that is, $\partial^* E$ is non-trivial when u_i is the sequence obtained from the min-max. For arbitrary ambient manifolds, in the event that u_i leads to occurrence (i), [6] implements a gradient flow that yields a (second) sequence $\{v_i\}$, for which $\partial^* E \neq \emptyset$ and with Morse index 0. The matter is thus reduced to a regularity question for the limit interface arising from a sequence u_i with uniformly bounded Morse index. This index control is used in a key way to obtain regularity ([6, Theorem 1.2]), whose proof relies on extensions of Y. Tonegawa's work [18] and Y. Tonegawa–N. Wickramasekera's work [19], and crucially on the (non-variational) varifold regularity result [5, Theorem 9.1] (see also [6, Theorem 3.2]). In conclusion, [6] obtains that $V = V_\lambda + V_0$, where $\text{supp}\,\|V_\lambda\| = \partial E = \overline{M_\lambda}$ and $\text{supp}\,\|V_0\| = \overline{M_0}$; here M_λ is a two-sided, quasi-embedded λ-CMC hypersurface, and M_0 an embedded minimal hypersurface, both satisfying the dimensional estimates listed in Theorem 1. Furthermore, any intersections between M_h and M_0, and self-intersections of M_h, are always tangential intersections of C^2 graphs lying on one side of each other.

With this as a starting point, our first step in establishing Theorem 1 is to show that when $\text{Ric}_g > 0$, the one-parameter Allen–Cahn min-max just recalled does not produce any minimal components in the limit interface, i.e. $V_0 = 0$. (As mentioned earlier, in this case [6] establishes already that $V_\lambda \neq 0$ for the u_i produced by min-max.)

Theorem 2. *Let (N, g) be a compact Riemannian manifold of dimension ≥ 3, with positive Ricci curvature, and $\lambda > 0$. The one-parameter Allen–Cahn min-max in [6], with prescribing function set to λ, produces a two-sided λ-CMC hypersurface and no minimal hypersurface.*

Theorem 2 is achieved by exhibiting a suitable continuous path, admissible in the min-max construction (which employs paths that are continuous in $W^{1,2}(N)$). This path will move through functions that are each modelled on a level set of the signed distance to M_λ. The idea is to try to place a 1-dimensional Allen–Cahn profile along the normal direction to a given level set and thus produce a function (a point in the path). This might appear problematic due to the presence of points where the level sets are not smoothly embedded in N (which, for example, may be caused by the presence of the singular set $\overline{M_\lambda} \setminus M_\lambda$, or by the fact that M_λ has quasi-embedded points). We handle this after observing that all "problematic points" are contained in a closed n-rectifiable set. The open complement (in N) of this n-rectifiable set is described (via a diffeomorphism) as an open subset of $\tilde{M} \times \mathbb{R}$, where $\tilde{M}$ is a (abstract) n-manifold whose immersion into N gives M_λ. We will refer to this open subset as the Abstract Cylinder (which is endowed with a metric pulled back from N). Each level set of the distance function becomes a subset of $\tilde{M} \times \{s\}$, where s is the chosen distance value. The sought path is then defined by "sliding" the 1-dimensional Allen–Cahn profiles in the $\mathbb{R}$-direction in the whole cylinder $\tilde{M} \times \mathbb{R}$, then restricting these functions to the Abstract Cylinder, and passing them to N. We check that this indeed produces a continuous path in $W^{1,2}(N)$. Furthermore, performing the energy calculations on the Abstract Cylinder, we see that the potentially "problematic points" do not cause any issues. The sliding argument yields a path with the (key) property that the relevant Allen-Cahn energy attains a maximum (along the path) at the function obtained in correspondence of M_λ (signed distance equal to 0); this relies on the positivity of the Ricci curvature. This property of the path easily implies that $V_0 = 0$ (no minimal component), for otherwise the min-max characterisation of V would be contradicted. Theorem 1 is then proven by showing that the λ-CMC hypersurface arising in Theorem 2 is, in fact, embedded. This is again done by exhibiting a suitable path (admissible in the min-max). This path is constructed by editing the previous one about its maximum, under the contradiction assumption that a non-embedded point exists in M_λ. The modification requires the identification of suitable hypersurfaces obtained by deforming M_λ about the non-embedded point. This construction ensures that the modified path attains a maximum that is strictly smaller than the maximum obtained for the path used in the proof of Theorem 2. This contradicts the min-max characterisation. We stress that these path constructions capitalise on the a priori knowledge (from [6]) that M_λ and M_0 are sufficiently regular.

We remark that Theorem 2 is somewhat interesting in its own sake: it is an open question whether (and under what assumptions) a sequence of solutions to the inhomogeneous Allen–Cahn equation with nowhere vanishing inhomogeneous term, and with a uniform bound on the Morse index, can produce minimal components. (The regularity result in [6] recalled earlier allows us to refer to the minimal and prescribed-mean-curvature components as hypersurfaces that are separately smooth, except for a possible small singular set when the ambient dimension is 8 or higher.) Theorem 2 rules out minimal components in the special instance in which the solutions come from a one-parameter min-max (in N compact with $\mathrm{Ric}_N > 0$) and the inhomogeneous term is constant.

The absence of minimal components and of non-embedded points established by Theorem 1 has, among its consequences, a Morse index estimate:

Corollary 1. *The λ-CMC hypersurface in Theorem 2 has Morse index equal to 1.*

This follows directly from C. Mantoulidis [14]. Alternatively, the arguments of F. Hiesmayr [11] apply verbatim. (We refer to Section 9 for the definition of Morse index.)

As we recalled, [6] employs an Allen–Cahn approximation scheme to construct the λ-CMC quasi-embedded hypersurface. The statement of Theorem 1 with embedded replaced by quasi-embedded can also be obtained (without any curvature assumption on N) using the so-called Almgren–Pitts method for the min-max, see the combined works of X. Zhou–J. Zhu [22] ($2 \le n \le 6$) and A. Dey in [7] (for $n \ge 7$, relying on the compactness theory in [4, 5]).

Regardless of the method used for the min-max construction, and without the need of curvature assumptions, if $2 \leq n \leq 6$ the λ-CMC hypersurface obtained is closed and immersed (completely smooth). In B. White's work [20, Theorem 35] it is proven that for each $\lambda \in \mathbb{R}$, there exists a generic set (in the sense of Baire category) of smooth metrics on the ambient manifold such that any closed, codimension-1 (completely smooth) immersion with constant mean curvature λ, is self-transverse. Therefore, combining the existence of quasi-embedded λ-CMC ([6] or [22]) with [20, Theorem 35], one obtains: when $2 \leq n \leq 6$, for any λ, there exists a generic set of metrics on N, such that each admits an embedded λ-CMC hypersurface.[1]

This argument relies however on the complete smoothness of the λ-CMC hypersurface, which is not available for $n \geq 7$ in the existence results. The flavour of Theorem 1 differs from the statement just given in that it allows a singular set and can handle all dimensions; moreover the class of metrics (Ricci positive metrics) is the same for all $\lambda \in \mathbb{R}$. We also stress that the proof of embeddedness in Theorem 1 exploits the min-max characterisation of the λ-CMC, while one can apply [20, Theorem 35] to any smooth CMC immersion, not necessarily one coming from a min-max. Theorem 1 and 2 may also hold with other assumptions on the metric on N, or other choices on the set of prescribing functions. (In these different scenarios an alternative approach to the sliding argument mentioned above could be a gradient flow, for example, along the lines of [3, Section 5.4] and [6, Section 6.9].)

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Contents

[1]A more general version of this statement is available for h-PMC hypersurfaces, $2 \leq n \leq 6$, by again combining [20, Theorem 35] with either [6] or [23]. Note that the class of prescribing functions, h, is different in these two results.

1 Preliminaries

1.1 Allen-Cahn and Construction of CMC Immersion

We recall the min-max construction in [6], of critical points to the inhomogeneous Allen–Cahn energy,

$$\mathcal{F}_{\varepsilon,\lambda}(u) = \int_N \frac{\varepsilon}{2}|\nabla u|^2 + \frac{W(u)}{\varepsilon} - \sigma \int_N \lambda u, \qquad \varepsilon \in (0,1),\ u \in W^{1,2}(N). \tag{1}$$

Where W is a smooth function on $\mathbb{R}$, with $W(\pm 1) = 0$ being non-degenerate minima, and $W(t) > 0$, for $t \in \mathbb{R} \setminus \{\pm 1\}$. Furthermore, we impose that W has only three critical points, $t = 0, \pm 1$, and quadratic growth outside some compact interval. For example $W(t) = (1-t^2)^2/4$, for $t \in [-2,2]$ and is quadratic outside $[-3,3]$. The constant σ is given by,

$$\sigma = \int_{-1}^{1} \sqrt{W(s)/2}\, ds.$$

Moreover, we take $\lambda > 0$.

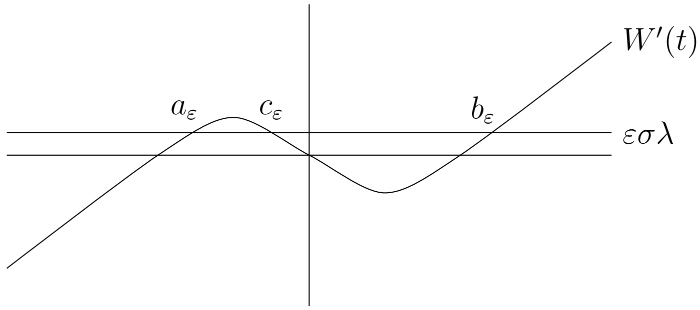


Figure 1: Intersection points, a_ε, b_ε, and c_ε, are the solutions to $W'(t) = \varepsilon\sigma\lambda$.

Consider the first and second variations of (1) with respect to $\varphi \in C^\infty(N)$,

$$\delta\mathcal{F}_{\varepsilon,\lambda}(u)(\varphi) = \int_N \varepsilon\nabla u \cdot \nabla\varphi + \left(\frac{W'(u)}{\varepsilon} - \sigma\lambda\right)\varphi, \tag{2}$$

$$\delta^2\mathcal{F}_{\varepsilon,\lambda}(u)(\varphi,\varphi) = \int_N \varepsilon|\nabla\varphi|^2 + \frac{W''(u)}{\varepsilon}\varphi^2. \tag{3}$$

We say that u is a critical point of (1), if $\delta\mathcal{F}_{\varepsilon,\lambda}(u)(\varphi) = 0$, for all $\varphi \in C^\infty(N)$, and then by standard elliptic theory we have that $u \in C^\infty(N)$, and strongly solves,

$$\varepsilon\Delta u = \frac{W'(u)}{\varepsilon} - \sigma\lambda. \tag{4}$$

If $\delta^2\mathcal{F}_{\varepsilon,\lambda}(u)(\varphi,\varphi) \geq 0$, for all $\varphi \in C^\infty(N)$, then we say that u is a stable solution to (4). By Figure 1, we see that there exists two stable constant solutions, $a_\varepsilon > -1$, and $b_\varepsilon > 1$. Furthermore, as $\varepsilon \to 0$, we have that $a_\varepsilon \to -1$, and $b_\varepsilon \to 1$. As $\mathrm{Ric}_g > 0$, [2, Proposition 7.1] shows that these are the only stable critical points of (1).

The existence of these isolated, stable solutions permits us to find non-trivial critical points of (2) via a min-max argument.

Proposition 1. *(Existence of Min-Max Solution, [6, Proposition 5.1]) For $\varepsilon > 0$, let Γ denote the collection of all continuous paths $\gamma\colon [-1,1] \to W^{1,2}(N)$, such that $\gamma(-1) = a_\varepsilon$, and $\gamma(1) = b_\varepsilon$. Then there exists an $\varepsilon_0 > 0$, such that for all $\varepsilon < \varepsilon_0$,*

$$\inf_{\gamma\in\Gamma} \sup_{u\in\gamma([-1,1])} \mathcal{F}_{\varepsilon,\lambda} = \beta_\varepsilon > \mathcal{F}_{\varepsilon,\lambda}(a_\varepsilon) > \mathcal{F}_{\varepsilon,\lambda}(b_\varepsilon), \tag{5}$$

is a critical value, i.e. there exists $u_\varepsilon \in W^{1,2}(N)$, critical point of $\mathcal{F}_{\varepsilon,\lambda}$, with $\mathcal{F}_{\varepsilon,\lambda}(u_\varepsilon) = \beta_\varepsilon$; moreover, u_ε has Morse index ≤ 1.

In our Ricci positive setting, as a_ε and b_ε are the only stable critical points we actually have that u_ε has Morse index equal to 1.

Now taking a sequence $\{\varepsilon_i\}_{i\in\mathbb{N}} \subset (0, \varepsilon_0)$, with $\varepsilon_i \to 0$, and associated critical points from Proposition 1, $\{u_i = u_{\varepsilon_i}\}$, we associate the following Radon measures,

$$\mu_i := (2\sigma)^{-1}\left(\frac{\varepsilon_i}{2}|\nabla u_i|^2 + \frac{W(u_i)}{\varepsilon_i}\right) d\mu_g. \tag{6}$$

Where μ_g is the volume measure of (N, g). Moreover there exists constants K, $L > 0$, such that for all i,

$$\sup_N |u_i| + \mu_i(N) \leq K, \tag{7}$$

and

$$\mu_i(N) \geq L. \tag{8}$$

By the bounds of (7) and (8), there exists a subsequence $\{u_{i'}\} \subset \{u_i\}$, along with a $u_0 \in BV(N)$, with $u_0(y) \in \{+1, -1\}$ for all $y \in N$, and a non-zero Radon measure μ, such that $u_{i'} \to u_0$ in $L^1(N)$, and $\mu_{i'} \rightharpoonup \mu$ as Radon measures. By [12, Theorem 1] and [15, Theorem 3.2], we have that μ is the weight measure of an integral n-varifold V, with the following properties:

1. V, is an integral n-varifold with bounded generalised mean curvature H_V, and first variation $\delta V = -H_V \mu_V$.
2. The set $E := \{u_0 = +1\}$ is a Caccioppoli set, with reduced boundary $\partial^* E \subseteq \operatorname{spt} V \subset N \setminus E \neq \emptyset$.
3. For $\mathcal{H}^n$-a.e. $x \in \partial^* E$, $\Theta(\mu_V, x) = 1$, and $H_V(x) \cdot \nu(x) = \lambda$; where ν is the inward pointing unit normal to $\partial^* E$, i.e. $\nu = \nabla u_0 / |\nabla u_0|$.
4. For $\mathcal{H}^n$ a.e. $x \in \operatorname{spt} V \setminus \partial^* E$, $\Theta(\mu_V, x)$ is an even integer ≥ 2, and $H_V(x) = 0$.

Optimal regularity of V was then proven in [6].

1. $V = V_0 + V_\lambda$
2. V_0 is a (possibly zero) stationary integral n-varifold with singular set of Hausdorff dimension at most $n - 7$.
3. $V_\lambda = |\partial^* E| \neq \emptyset$, and $\partial^* E$ is a quasi-embedded hypersurface with constant mean curvature λ, with respect to unit normal pointing into E. The singular set of $\partial^* E$ has Hausdorff dimension at most $n - 7$.
4. V has a $(\lambda, 0)$-CMC structure.

By $(\lambda, 0)$-CMC structure we mean that for each point on the support of V, potentially away from a closed set of Hausdorff dimension at most $n - 7$, the local picture is one of the following,

1. There is a single embedded λ-CMC disk.
2. There are two embedded λ-CMC disks that lie on either side of each other and only touch tangentially.
3. There is a single embedded minimal disk
4. There is a single embedded λ-CMC disk and a single embedded minimal disk that only touch tangentially.
5. There are two embedded λ-CMC disks that lie on either side of each other, along with an embedded minimal disk, such that all three disks only touch tangentially.

For a detailed definition of a $(\lambda, 0)$-CMC structure, see [6, Definition 8]. We define the set gen-reg V, to be the set of points on supp $\|V\|$, which satisfy one of the local pictures of 1 to 5. For a detailed definition of gen-reg V see [6, Definition 5].

Therefore, we have the following

Theorem 3. *(Theorem 1.1 [6]) Let N be a closed Riemannian manifold of dimension $n+1 \geq 3$, with positive Ricci curvature, and let $\lambda \in (0, \infty)$ be a fixed constant. There exists a smooth, quasi-embedded hypersurface $M \subset N$, with;*

1. *$\overline{M} \setminus M = \emptyset$, if $2 \leq n \leq 6$;*
2. *$\overline{M} \setminus M$ is finite if $n = 7$;*
3. *$dim_{\mathcal{H}}(\overline{M} \setminus M) \leq n - 7$, if $n \geq 8$.*

Moreover M is the image of a two sided immersion with mean curvature $H_M = \lambda\nu$, for a choice ν of continuous unit normal to the immersion.

We restate Theorems 1 and 2 with our new notation.

Theorem 4. *Consider a closed Riemannian manifold (N, g), with positive Ricci curvature and $\dim N = n+1 \geq 3$. Take $\lambda \in (0, +\infty)$. The limiting varifold $V = V_\lambda + V_0$ from Section 1.1 has the following properties*

1. *$M \coloneqq$ gen-regV_λ is embedded, connected and has index 1.*
2. *$V_0 = 0$.*

This says that only case 1 can occur.

1.2 One Dimensional Allen–Cahn Solution

We refer to [2, Section 2.2] as a reference for this Section.

We define the function $\mathbb{H}$ on $\mathbb{R}$ to denote the monotonically increasing solution to the ODE $u'' - W'(u) = 0$, with the conditions $\mathbb{H}(0) = 0$ and $\lim_{t\to\pm\infty} \mathbb{H}(t) = \pm 1$. We then define $\mathbb{H}_\varepsilon(\,\cdot\,) = \mathbb{H}(\varepsilon^{-1}\,\cdot)$, which solves the ODE $\varepsilon u'' - \varepsilon^{-1}W'(u) = 0$.

We define an approximation for $\mathbb{H}_\varepsilon$. Start by considering the following bump function

$$\begin{cases} \chi \in C_c^\infty(\mathbb{R}), & \\ \chi(t) = 1, & t \in (-1, 1), \\ \chi(t) = 0, & t \in \mathbb{R} \setminus (-2, 2), \\ \chi(t) = \chi(-t), & t \in \mathbb{R}, \\ \chi'(t) \leq 0, & t \geq 0. \end{cases}$$

For $\varepsilon \in (0, 1)$, we define the truncation of $\mathbb{H}_\varepsilon$ by

$$\overline{\mathbb{H}}_\varepsilon(t) \coloneqq \begin{cases} \chi((\varepsilon\Lambda)^{-1}t)\mathbb{H}_\varepsilon(t) + 1 - \chi((\varepsilon\Lambda)^{-1}t), & t > 0, \\ \chi((\varepsilon\Lambda)^{-1}t)\mathbb{H}_\varepsilon(t) - 1 + \chi((\varepsilon\Lambda)^{-1}t), & t < 0, \end{cases}$$

where $\Lambda = 3|\log \varepsilon|$. There exists a constant $\beta = \beta(W) < +\infty$, such that for all $\varepsilon \in (0, 1/4)$,

$$2\sigma - \beta\varepsilon^2 < \int_{\mathbb{R}} \frac{\varepsilon}{2}|(\overline{\mathbb{H}}_\varepsilon)'(t)|^2 + \frac{W(\overline{\mathbb{H}}_\varepsilon(t))}{\varepsilon}\, dt < 2\sigma + \beta\varepsilon^2.$$

2 Idea of Proof

We first prove Theorem 1 for the case $\lambda > 0$. To then prove for $\lambda < 0$, we take $\tilde{\lambda} = -\lambda > 0$, and reverse the direction of the unit normal on the resulting $\tilde{\lambda}$-CMC hypersurface. From here on we take $\lambda > 0$.

For Caccioppoli sets $\Omega \subset N$, we define the following functional,

$$\mathcal{F}_\lambda(\Omega) := \mathcal{H}^n(\partial^*\Omega) - \lambda \mu_g(\Omega).$$

Recall our converging sequence of critical points $\{u_{\varepsilon_j}\}$, along with our limiting varifold $V = V_\lambda + V_0$, and Caccioppoli set E from Section 1.1. We have, as $\varepsilon_j \to 0$,

$$\mathcal{F}_{\varepsilon_j,\lambda}(u_{\varepsilon_j}) \to 2\sigma\mathcal{F}_\lambda(E) + 2\sigma\mathbb{M}(V_0) + \sigma\lambda\mu_g(N)$$

Therefore constructing minimising paths between $\emptyset$ and N for $\mathcal{F}_\lambda$, may provide insight to minimising paths from a_ε to b_ε for $\mathcal{F}_{\varepsilon,\lambda}$.

As N is compact one obvious path that includes E, is $\{E_t\}$ for $t \in [-2\operatorname{diam}(N), 2\operatorname{diam}(N)]$, where,

$$E_t := \{y\colon \tilde{d}(y) > t\}.$$

Here $\tilde{d}$ is the signed distance function to $M := \partial^* E$, taking positive values in E, and negative values in $N \setminus E$. We also denote,

$$\Gamma_t := \{y\colon \tilde{d}(y) = t\} = \partial E_t.$$

Assuming sufficent regularity on the sets Γ_t and E_t, and the functions $t \mapsto \mathcal{H}^n(\Gamma_t)$ and $t \mapsto \mu_g(E_t)$, we have for $t > 0$,

$$\begin{aligned} \mathcal{F}_\lambda(E_t) - \mathcal{F}_\lambda(E) &= \int_0^t \frac{d}{ds}\mathcal{H}^n(\Gamma_s)\,ds - \lambda\int_0^t \frac{d}{ds}\mu_g(E_s)\,ds, \\ &= \int_0^t \int_{\Gamma_s} \lambda - H_{\Gamma_s}(x)\,d\mathcal{H}^n(x)\,ds, \end{aligned} \tag{9}$$

where H_{Γ_s} is the scalar mean curvature of Γ_s with respect to unit normal $\nabla\tilde{d}$. Recalling that $H_{\Gamma_0} = \lambda$, a straightforward calculation yields the following inequalities.

$$\begin{cases} H_{\Gamma_t} \geq \lambda + mt, & t \geq 0, \\ H_{\Gamma_t} \leq \lambda + mt, & t \leq 0, \end{cases}$$

where $m = \min_{|X|=1} Ric_g(X,X) > 0$. Therefore by (9) for $t \geq 0$,

$$\mathcal{F}_\lambda(E_t) \leq \mathcal{F}_\lambda(E).$$

The same inequality holds for $t \leq 0$. Here we see the importance the assumption on the Ricci curvature. Therefore,

$$\gamma\colon t \in [-2\operatorname{diam}(N), 2\operatorname{diam}(N)] \mapsto E_{-t} \in \{\text{Caccioppoli sets of } N\},$$

is a path from $\emptyset$ to N, that has maximum height $\mathcal{F}_\lambda(E)$.

We look to replicate this path in $W^{1,2}(N)$. Consider the Lipschitz function on N,

$$v_\varepsilon^t = \overline{\mathbb{H}}_\varepsilon(\tilde{d}(x) - t),$$

which can be thought of as placing the truncated one dimensional Allen-Cahn solution from Section 1.2 along the normal profile of Γ_t. By the Co-Area formula we have,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^t) = \int_{\mathbb{R}} Q_\varepsilon(s-t)\mathcal{H}^n(\Gamma_s)\, ds - \sigma\lambda \int_{\mathbb{R}} \overline{\mathbb{H}}_\varepsilon(s-t)\mathcal{H}^n(\Gamma_s)\, ds,$$

were,

$$Q_\varepsilon(t) = \frac{\varepsilon}{2}|\left(\overline{\mathbb{H}}_\varepsilon\right)'(t)|^2 + \frac{W\left(\overline{\mathbb{H}}_\varepsilon(t)\right)}{\varepsilon}$$

The functions

$$t \mapsto \int_{\mathbb{R}} Q_\varepsilon(s-t)\mathcal{H}^n(\Gamma_s)\, ds, \quad \textit{and} \quad t \mapsto \sigma\lambda \int_{\mathbb{R}} \overline{\mathbb{H}}_\varepsilon(s-t)\mathcal{H}^n(\Gamma_s)\, ds,$$

act as a smooth approximations to $t \mapsto 2\sigma\mathcal{H}^n(\Gamma_t)$, and $t \mapsto 2\sigma\lambda\mu_g(E_t) - \sigma\lambda\mu_g(N)$, respectively.

We say that v_ε^0 is an Allen–Cahn approximation of M as,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^0) \to 2\sigma\mathcal{H}^n(M) - \sigma\lambda\mu_g(E) + \sigma\lambda\mu_g(N \setminus E) =: A_2,$$

as $\varepsilon \to 0$, Section 3.6. Carrying out a calculation which replicates the previous one, we deduce that for all $\tau > 0$, there exists an $\varepsilon_\tau > 0$, such that for all $\varepsilon \in (0, \varepsilon_\tau)$,

$$\max_{t\in[-2\operatorname{diam}(N), 2\operatorname{diam}(N)]} \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^t) < A_2 + \tau = A_1 - 2\sigma\mathbb{M}(V_0) + \tau,$$

where $A_1 := 2\sigma\mathcal{H}^n(M) + 2\sigma\mathbb{M}(V_0) - \sigma\mu_g(E) + \sigma\mu_g(N \setminus E)$. Connecting $v_\varepsilon^{2\operatorname{diam}(N)} = -1$ to a_ε, and $v_\varepsilon^{-2\operatorname{diam}(N)} = +1$ to b_ε, by constant functions, we see that we have an appropriate min-max path in $W^{1,2}(N)$.

This path proves that we cannot have a minimal piece V_0. We also get criterion for M. Indeed, as there exists a 'Wall', [6, Lemma 5.1], that all min-max paths must climb over, we have that

$$2\sigma\lambda\mathcal{H}^n(M) - \sigma\lambda\mu_g(E) + \sigma\lambda\mu_g(N \setminus E) > \sigma\lambda\mu_g(N).$$

Rearranging yields,

$$\mathcal{H}^n(M) > \lambda\mu_g(E).$$

We note that the above path can be constructed for any suitable λ-CMC hypersurface which encloses a volume. Therefore for any such pair (M, E), the above inequality holds, and our min-max must choose the pair that minimises the positive quantity $\mathcal{H}^n(M) - \lambda\mu_g(E)$. From this we can deduce that E must be connected.

We turn our attention to proving that M is embedded. We prove by contradiction, exploiting the min-max characterisation of M. We now know that, given our sequence of critical points $\{u_{\varepsilon_j}\}$, and potentially after taking a subsequence,

$$\mathcal{F}_{\varepsilon_j,\lambda}(u_{\varepsilon_j}) \to 2\sigma\mathcal{H}^n(M) - \sigma\lambda\mu_g(E) + \sigma\lambda\mu_g(N \setminus E) = A_2,$$

as $\varepsilon_j \to 0$. Assume that M has a non-embedded point z_0. Then for every $\varepsilon_j > 0$, we construct a continuous, connected path,

$$\gamma_{\varepsilon_j} : [-1, 1] \to W^{1,2}(N),$$

were, $\gamma_{\varepsilon_j}(-1) = a_{\varepsilon_j}$, and $\gamma_{\varepsilon_j}(1) = b_{\varepsilon_j}$. This path satisfies the following, there exists a J in $\mathbb{N}$, and $\varsigma > 0$, independent of j, such that for all $j \geq J$,

$$\max_{t \in [-1,1]} \mathcal{F}_{\varepsilon_j,\lambda}(\gamma_{\varepsilon_j}(t)) < 2\sigma \mathcal{H}^n(M) - \sigma\lambda\mu_g(E) + \sigma\lambda\mu_g(N \setminus E) - \varsigma,$$

This is a contradiction of the min-max characterisation of u_{ε_j}.

We sketch the main ideas of the path in the ε-limit, Figure 2.

The picture at z_0 is Figure 3a. The limiting energy for this structure is A_2. The starting point for building this path is to construct a competitor with lower limiting energy. Then we wish to connect this competitor to $+1$ and -1, with energy always remaining a fixed amount below A_2.

***Step 1*: ** Construction of Competitor, (1) $\to$ (2) in Figure 2, Section 5

The structure at z_0 is two smooth, embedded CMC disks, that touch tangentially at z_0 and lie either side of each other. To construct the competitor, we push these disks together, and delete portions of the disks that are pushed past each other. This reduces the area of our structure while also increasing the size of E, leading to a drop in energy.

***Idea 1*: ** Push the whole of M by some fixed distance ρ.

This equates to pushing M to the level $\Gamma_{-\rho}$. As seen previously, this will lead to a drop in energy. Furthermore there is an obvious path to $+1$, namely we keep pushing along level sets, Γ_{-r} for r in $[\rho, 2\,\mathrm{diam}(N)]$. However there is no obvious path to -1. Pushing $\Gamma_{-\rho}$ in the direction of E, will increase the energy and bring us back to M, undoing the energy drop that the competitor created.

***Idea 2*: ** Push the disks together locally.

Consider open balls $B_1 \subset\subset B_2$ about z_0. We smoothly bump the disks at z_0 such that inside B_1 we move the disks of distance $\rho > 0$, and outside B_2 we remain fixed. The balls B_1 and B_2 along with ρ, are chosen so that the area inside B_1 gets deleted, Figure 3b. Letting,

$$\varsigma = \frac{\sigma}{2}\mathcal{H}^n(B_1 \cap M),$$

we see that our competitor has energy lying below, $A_2 - \varsigma$.

***Step 2*: ** Path to $+1$, Section 7

To connect to the competitor $+1$ we look to copy the successful path to $+1$ of the competitor in ***Idea 1***. To construct the competitor we only edited M locally about z_0. Therefore pushing the competitor to the level set $\Gamma_{-\rho}$ will correspond to a similar drop in energy from pushing M to $\Gamma_{-\rho}$. This is (2) $\to$ (6) in Figure 2. See Figures 3b and 3f for local pictures about z_0. From $\Gamma_{-\rho}$ we can easily connect to $+1$ by pushing along level sets Γ_{-r}, as previously discussed.

***Step 3*: ** Path to -1, Section 6

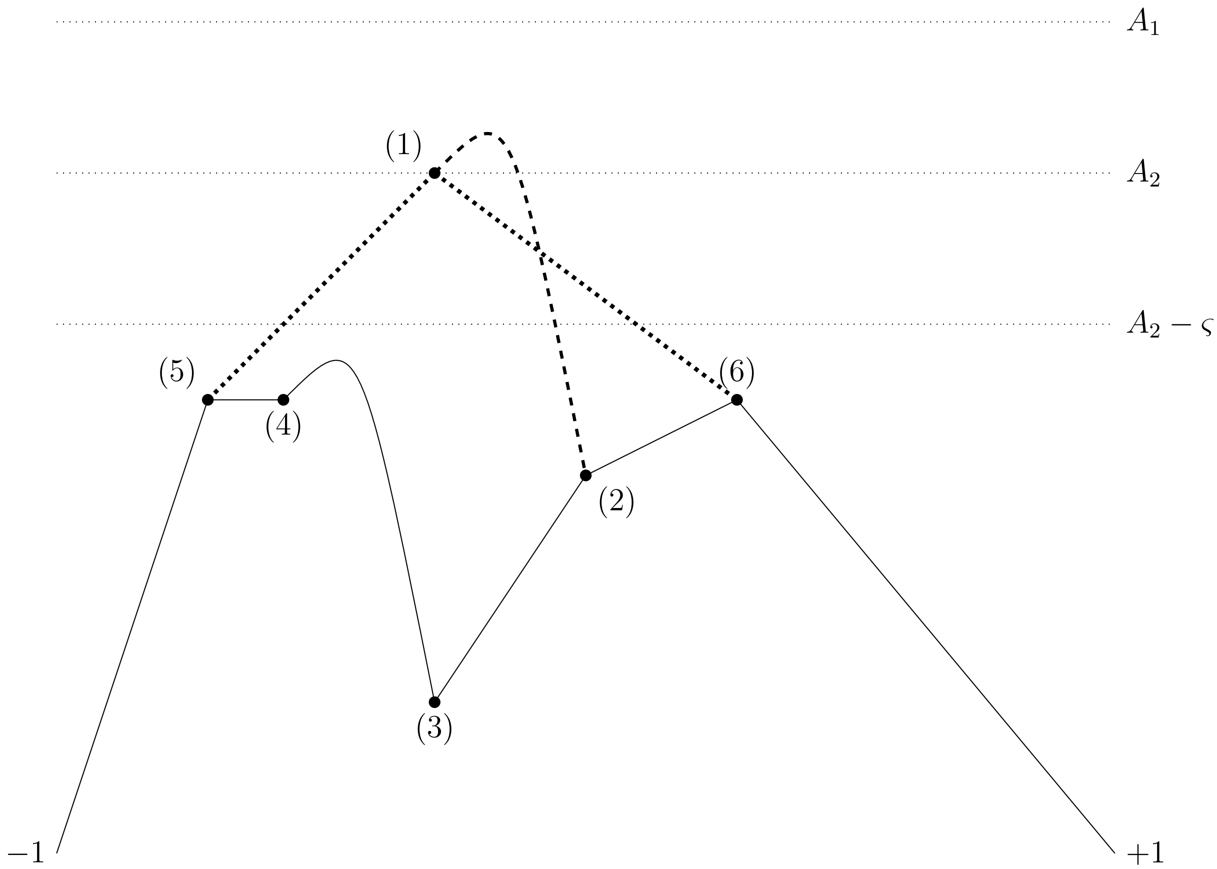
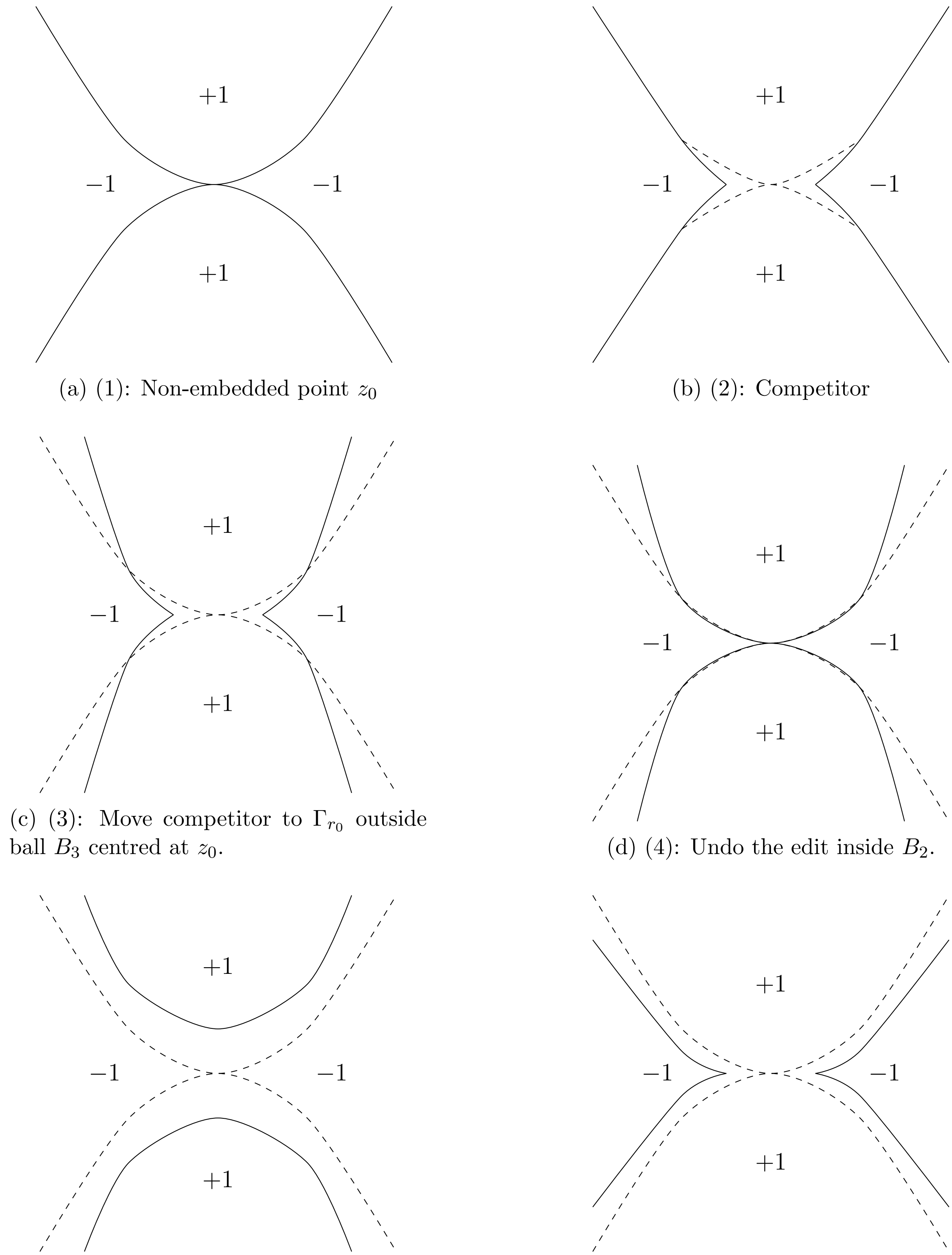


Figure 2: The Paths. To prove $V_0 = 0$, we follow the path from -1 to (5), then the dotted line to (1), dotted line to (6), then complete the path to $+1$. The dashed line from (1) to (2) is the construction of the competitor. Then to prove that M is embedded we follow the path from -1 to $+1$ given by the solid lines. Refer to Figure 3 for the local picture about non-emebedded point z_0 at each numbered stage on the paths.

We look to follow a similar method as in ***Step 2*** by connecting our competitor to a level set Γ_{r_0}, for $r_0 > 0$, then push this along level sets Γ_r for r in $[r_0, 2\,\mathrm{diam}(N)]$ to connect it to -1. By pushing our competitor straight to Γ_{r_0} we run the risk of pushing through M and increasing our energy back up to A_2. Therefore we carry out our path in stages, again making use of the fact that our edit about z_0 was local.

The first stage is (2) $\to$ (3) in Figure 2. We fix our competitor in B_2, and outside we push forward, so that outside some larger ball B_3, we line up with Γ_{r_0}. See Figures 3b and 3c for local pictures about z_0. Again as our edit is local about z_0, this corresponds to a similar drop in energy of pushing M to Γ_{r_0}, and the drop will be of order r_0^2. For large enough r_0 this will give us a large enough energy drop to be able to undo the edit inside B_2, and still have our energy remain below $A_2 - \varsigma$. This is the second stage from (3) $\to$ (4), in Figure 2. See Figure 3d, for local picture about non-embedded point. From here we push up inside B_3 to line up with Γ_{r_0}, (4) $\to$ (5) in Figure 2, Figure 3e. Finally we connect to -1 by sliding along level sets as previously stated.

Path at ε Level



(a) (1): Non-embedded point z_0

(b) (2): Competitor

(c) (3): Move competitor to Γ_{r_0} outside ball B_3 centred at z_0.

(d) (4): Undo the edit inside B_2.

(e) (5): Push up in B_3 to come into line with Γ_{r_0}.

(f) (6): Push Competitor to come in line with $\Gamma_{-\rho}$.

Figure 3: Stages of the Path at the non-embedded point z_0. In each image, the dashed lines represent the original λ-CMC disks, as a reference to what we are changing at each step. Furthermore, in each image it is the solid lines that are the boundaries between the '+1' and '−1' regions.

We carry out this 'pushing', on what we refer to as our abstract cylinder, $\tilde{M} \times \mathbb{R}$. See Section 3.3. Here $\tilde{M}$ is an n-dimensional manifold and $\iota\colon \tilde{M} \to M$ is a smooth immersion. We define the following map,

$$\begin{aligned} F\colon \tilde{M} \times \mathbb{R} &\to N, \\ (x,t) &\mapsto \exp_{\iota(x)}(t\nu(x))), \end{aligned}$$

with ν being a smooth choice of unit normal to immersion, pointing into E. Therefore we view points (x,t) on our cylinder $\tilde{M} \times \mathbb{R}$ as having base point $\iota(x)$ and moving length t along the geodesic with initial direction $\nu(x)$. See Figure 4.

Recall our function $v_\varepsilon^0 = \overline{\mathbb{H}}_\varepsilon \circ \tilde{d}$, then by the Co-Area formula,

$$\begin{aligned} \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^0) &= \int_{\mathbb{R}} \left(\frac{\varepsilon}{2}|(\overline{\mathbb{H}}_\varepsilon)'(t)|^2 + \frac{W(\overline{\mathbb{H}}_\varepsilon(t))}{\varepsilon} - \sigma\lambda\overline{\mathbb{H}}_\varepsilon(t) \right) \mathcal{H}^n(\Gamma_t)\, dt, \\ &= \int_{\mathbb{R}} \int_{\tilde{M}} \left(\frac{\varepsilon}{2}|(\overline{\mathbb{H}}_\varepsilon)'(t)|^2 + \frac{W(\overline{\mathbb{H}}_\varepsilon(t))}{\varepsilon} - \sigma\lambda\overline{\mathbb{H}}_\varepsilon(t) \right) \theta_t(x)\, d\mathcal{H}^n(x)\, dt, \end{aligned}$$

where $\theta_t\colon \tilde{M} \to \mathbb{R}$, is defined by the Area Formula to be such that for a.e $t \in \mathbb{R}$, and any $\mathcal{H}^n$-measurable function on N,

$$\int_{\Gamma_t} g\, d\mathcal{H}^n = \int_{\tilde{M}} (g \circ F_t)\, \theta_t\, d\mathcal{H}^n,$$

with $F_t(\,\cdot\,) = F(\,\cdot\,,t)$. Then we carry out the relevant 'pushings' by considering a continuous family of functions $\{g_r\}_{r\in[0,r']} \subset C(\tilde{M})$,

$$\begin{aligned} \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^r) = \int_{\mathbb{R}} \int_{\tilde{M}} \Bigg(&\frac{\varepsilon}{2}|(\overline{\mathbb{H}}_\varepsilon)'(t - g_r(x))|^2 + \frac{W(\overline{\mathbb{H}}_\varepsilon(t - g_r(x)))}{\varepsilon} \\ &-\sigma\lambda\overline{\mathbb{H}}_\varepsilon(t - g_r(x)) \Bigg) \theta_t(x)\, d\mathcal{H}^n(x,t)\, dt. \end{aligned}$$

See Figure 5.

2.1 Structure of the Paper

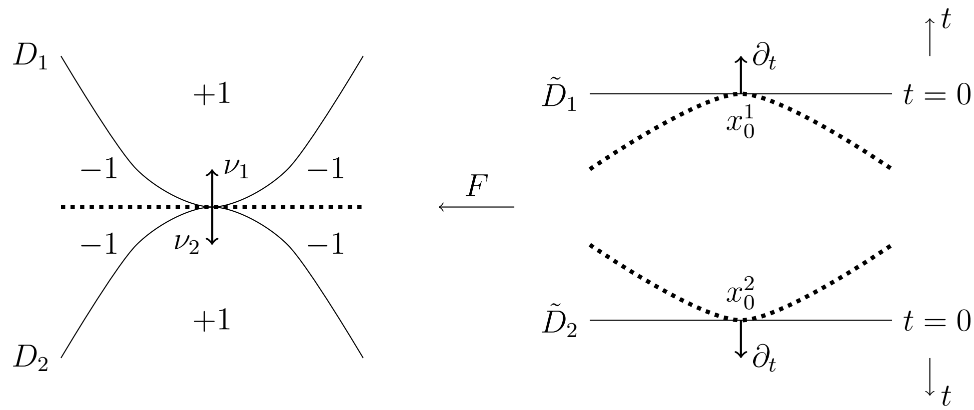
The paper is organised as follows. We start with setup:

- Section 3 is devoted to setup of objects used in the main computation.
- In Section 4 we carry out the main computation. The constructions that follow are carried out by plugging explicitly defined functions into this computation.

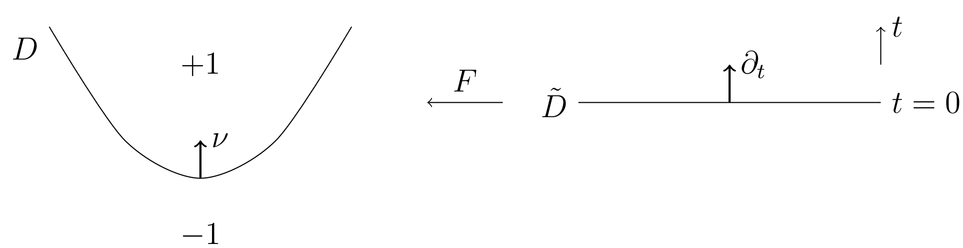
To prove Theorem 2:

- In Section 8.2 we build the dotted path (5) → (1) → (6) in Figure 2. Theorem 2 then follows upon combining this with computations in Sections 6.4 and 7.2; in these sections we build the paths (5) → −1, and (6) → +1, in Figure 2.

To prove Theorem 1 we argue by contradiction, assuming that M has a non-embedded point z_0:



(a) On the left we have a local picture about a non-emebedded point z_0 of M. On the right the two local pictures about x_0^1 and x_0^2 in $\tilde{M} \times \mathbb{R}$, where $\iota(x_0^1) = z_0 = \iota(x_0^2)$. We have, $F(\tilde{D}_i) = D_i$, and $dF_{x_0^i}(\partial_t) = \nu_i$, for $i = 1$ and 2. The dotted line on the left picture respresents points in N which are of equal distance to D_1 and D_2. The dotted lines on the right hand picture are the preimages of the dotted line on the left, under the map F, and these can be seen as acting as the boundary to the open set $\tilde{T}$ in $\tilde{M} \times \mathbb{R}$.



(b) On the left a local picture about an embedded point of M. On the right is it's preimage in $\tilde{T}$ under the map F.

Figure 4: Local pictures about points in M

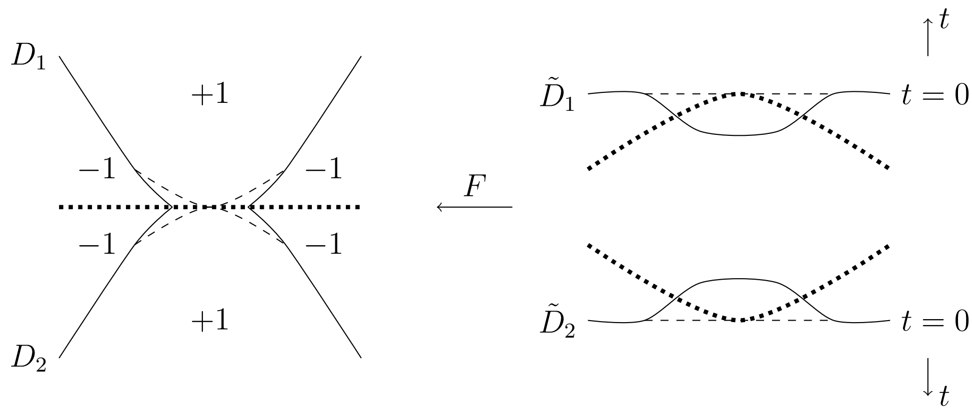


Figure 5: How the competitor is constructed as the graph of bump functions about points x_0^1 and x_0^2 over $\tilde{M}$. Whatever is bumped out beyond the dotted line on the right hand side, is not considered in N. In other words it is deleted.

- In Section 5 we construct our competitor about z_0. This is the dashed path (1) $\to$ (2) in Figure 2.
- In Section 6 we construct a path from the competitor to the stable constant a_ε. This is the solid path (2) $\to$ (6) $\to$ +1, in Figure 2.
- In Section 7 we construct a path from the competitor to the stable constant b_ε. This is the solid path (2) $\to$ (3) $\to$ (4) $\to$ (5) $\to$ -1 in Figure 2.
- In Section 8.3 we piece together this continuous path from a_ε to b_ε, in $W^{1,2}(N)$. The energy $\mathcal{F}_{\varepsilon,\lambda}$, is less than $A_2 - \varsigma$ for every point along this path, Figure 2. This contradicts the min-max construction, proving that M is embedded.

Finally, in Section 9 we prove Corollary 1 (the Morse index of M is equal to 1, which also implies that M must be connected).

2.2 A Note on Choice of Constants

The biggest subtlety in the Construction of the path in Sections 5, 6 and 7 is the choice of constants, and the order that we choose them in. We explicitly list the order of choices here, and reference where they have been chosen.

1. We first choose a non-embedded point z_0
2. We choose $\delta = \delta(z_0, N, M, g, \lambda, W) > 0$, in Remarks 5, 6, 7, 8, 16.
3. We choose $L = L(z_0, N, M, g, \delta, \lambda, W) > 0$, in Remarks 12, 21.
4. We choose $k = k(z_0, N, M, g, \delta, L, \lambda, W)$, in Remark 22.
5. We choose $r_0 = r_0(z_0, N, M, g, \delta, L, k, \lambda, W) > 0$, in Remarks 12, 21, 24.
6. We choose $\rho = \rho(z_0, N, M, g, \delta, L, k, r_0, \lambda, W) > 0$, in Remarks 9, 10, 11, 14, 18, 23.

7. We define $l = l(\rho)$ in (10).

8. We choose $\tau > 0$.

9. We finally choose $\varepsilon_\tau = \varepsilon_\tau(z_0, N, M, g, \delta, L, k, r_0, \rho, \tau, \lambda, W) > 0$, in Remarks 15, 19, 20 and Sections 8.2 and 8.3.

3 Construction of Objects

3.1 Signed Distance Function

Let $d_{\overline{M}}\colon N \to \mathbb{R}$ be the distance function to the closed set $\overline{M} \subset N$. As $\overline{M}$ is closed, and N is complete, Hopf–Rinow tells us that, for each z in N, the value, $d_{\overline{M}}(z)$, is obtained by a geodesic from z to a point on $\overline{M}$. Furthermore, $d_{\overline{M}}$ is Lipschitz, with Lipschitz constant 1.

The set $E = \{u_0 = 1\}$ is an open in N, and $\overline{M} = \partial E$. This allows us to define the signed distance function, $\tilde{d}\colon N \to \mathbb{R}$, to $\overline{M}$, which takes positive values in E, and negative values in $N \setminus E$,

$$\tilde{d}(y) = \begin{cases} d_{\overline{M}}(z), & x \in E, \\ -d_{\overline{M}}(z), & x \notin E. \end{cases}$$

This is a 1-Lipschitz function on N.

3.2 Abstract Surface

M is a quasi-embedded λ-CMC hypersurface, [6, Definition 8].

Remark 1. *For a point $z \in M$, there exists an n-dimensional linear subspace $T = T_z \subset T_zN$, and a unit vector $\nu_z \in T^\perp$, along with $r = r(z) > 0$, $s = s(z) > 0$, and $S = S(z) > 0$, such that $S < inj(N)$. We define the cylinder*

$$C_{z,T,r,s} := \exp_z\left(\{X + t\nu_z \colon X \in B_r^{T_zN}(0) \cap T,\, t \in (-s, s)\}\right) \subset B_S^N(z),$$

and, one of the following holds:

1. *(See Figure 4b) There exists a smooth function,*

$$f\colon B_{z,T,r} := B_r^{T_zN}(0) \cap T \to (-s, s),$$

which satisfies,

$$\begin{cases} f(0) = 0, \\ \nabla^T f(0) = 0, \\ \Delta_T f(0) = \lambda, \end{cases}$$

and,

$$\overline{M} \cap C_{z,T,r,s} = \exp_z(Graph(f)) = \exp_z(\{X + f(X)\nu_z \colon X \in B_{z,T,r}\})$$

Furthermore, we have that,

$$E \cap C_{z,T,r,s} = \exp_z(\{X + t\nu_z \colon X \in B_{z,T,r},\, f(X) < t < s\}),$$

and we can define a smooth choice of unit normal to $\exp_z(Graph(f))$,

$$\nu\colon \exp_z(Graph(f)) \to T(\exp_z(Graph(f)))^{\perp},$$

such that $\nu(z) = \nu_z$.

2. *(See Figure 4a) There exists two smooth functions,*

$$f_1,\ f_2\colon B_{z,T,r} \to (-s, s),$$

which satisfy,

$$\begin{cases} f_1(0) = 0 = f_2(0), \\ f_1 \geq f_2, \\ \nabla^T f_1(0) = 0 = \nabla^T f_2(0), \\ \Delta_T f_1(0) = \lambda = -\Delta_T f_2(0), \end{cases}$$

and,

$$\overline{M} \cap C_{z,T,r,s} = \bigcup_{i=1,2} \exp_z(Graph(f_i)) = \bigcup_{i=1,2} \exp_z(\{X + f_i(X)\nu_z \colon X \in B_{z,T,r}\}).$$

Furthermore, we have that,

$$\begin{aligned} E \cap C_{z,T,r,s} &= \exp_z(\{X + t\nu_z \colon X \in B_{z,T,r},\ f_1(X) < t < s\}) \\ &\quad \cup \exp_z(\{X + t\nu_z \colon X \in B_{z,T,r},\ -s < t < f_2(X)\}), \end{aligned}$$

and we can define smooth choices of unit normals,

$$\nu_i\colon \exp_z(Graph(f_i)) \to T(\exp_z(Graph(f)))^{\perp},$$

such that $\nu_1(z) = \nu_z$, *and* $\nu_2(z) = -\nu_z$.

If Case 1 holds then we say that z *is an embedded point of* M. *Alternatively, if Case 2 holds we say that* z *is a non-embedded point of* M. *In either case the tangent space of* M *at* z *is given by,* $T_zM \coloneqq T_z$.

Claim 1. *(Remark 2.6 of [4]) The set of non-embedded points of* M *has* $\mathcal{H}^n$*-measure* 0.

We define our abstract surface $\tilde{M}$ by

$$\tilde{M} = \{(z, \nu)\colon z \in M,\ \nu \in T_zM^{\perp}, \text{ with } |\nu| = 1, \text{ and points into } E\}.$$

Locally $\tilde{M}$ is a smooth, embedded CMC disk in N, therefore $\tilde{M}$ is a smooth n-dimensional manifold.

3.3 Abstract Cylinder

Consider x in $\tilde{M}$, then $x = (z, X)$, for some z in M and X in $T_zM^{\perp}$. We define two, smooth projections, first from $\tilde{M}$ to $TM^{\perp}$,

$$\nu\colon (z, X) \mapsto X,$$

and secondly from $\tilde{M}$ to M,

$$\iota\colon (z, X) \mapsto z.$$

From these we define the following map,

$$\begin{array}{rcl} F\colon \tilde{M}\times\mathbb{R} & \to & N, \\ (x,t) & \mapsto & \exp_{\iota(x)}(t\nu(x)), \end{array}$$

which, as N is complete, is well defined. For a fixed x in $\tilde{M}$, F is a unit parametrisation of a geodesic which, at time 0, passes through $\iota(x)$, with velocity $\nu(x)$. The set $\{t\colon d_{\overline{M}}(F(x,t)) = |t|\}$, is the set of times t, at which this geodesic achieves the shortest distance from $F(x,t)$ to $\overline{M}$. Consider the subset $\{t\colon \tilde{d}(F(x,t)) = t\} \subset \{t\colon d_{\overline{M}}(F(x,t)) = |t|\}$, and its endpoints,

$$\begin{array}{rcl} \sigma^+(x) & = & \sup\{t\colon \tilde{d}(F(x,t)) = t\} \geq 0, \\ \sigma^-(x) & = & \inf\{t\colon \tilde{d}(F(x,t)) = t\} \leq 0. \end{array}$$

These are uniformly bounded functions on $\tilde{M}$, and in fact as the next claim shows, $\{t\colon \tilde{d}(F(x,t)) = t\}$ is a closed and connected interval on $\mathbb{R}$.

Claim 2. *We have that*

$$[\sigma^-(x),\sigma^+(x)] = \{t : \tilde{d}(F(x,t)) = t\}.$$

Proof. Consider the geodesic, $\gamma : t \mapsto F(x,t)$, and define the following function,

$$f : t \mapsto \tilde{d}(F(x,t)).$$

This is a 1-Lipschitz function with $f(0) = 0$. Indeed,

$$|f(t_1) - f(t_2)| \leq |d(F(x,t_1),F(x,t_2))| \leq \mathrm{Length}(\gamma_{|[t_1,t_2]}) = |t_1 - t_2|.$$

For $t_0 \geq 0$, such that $f(t_0) \neq t_0$, we must have $f(t_0) < t_0$. By Lipschitz constant 1, for any $t > t_0$,

$$\begin{array}{rcl} f(t) & = & f(t) - f(t_0) + f(t_0), \\ & \leq & t - t_0 + f(t_0), \\ & < & t. \end{array}$$

Similarly if we have $t_0 \leq 0$, such that $f(t_0) \neq t_0$, then $f(t) \neq t$, for all $t < t_0$.

By continuity we have that $\tilde{d}(F(x,\sigma^+(x))) = \sigma^+(x)$, and therefore by above, for all $t \in [0,\sigma^+(x)]$, we must have that $\tilde{d}(F(x,t)) = t$. By definition of $\sigma^+(x)$, for all $t > \sigma^+(x)$, $\tilde{d}(F(x,t)) < t$. Therefore

$$[0,\sigma^+(x)] = \{t \geq 0\colon \tilde{d}(F(x,t)) = t\}.$$

Similarly $[\sigma^-(x),0] = \{t \leq 0\colon \tilde{d}(F(x,t)) = t\}$. □

We define the abstract cylinder,

$$\tilde{T} = \{(x,t)\colon x \in \tilde{M},\, t \in (\sigma^-(x),\sigma^+(x))\} \subset \tilde{M}\times\mathbb{R}.$$

Defining the projection map from $\tilde{M}\times\mathbb{R}$ onto $\mathbb{R}$, $p\colon (x,t) \mapsto t$, then on $\tilde{T}$ we have that $\tilde{d}\circ F = p$.

We wish to work on $\tilde{T}$ instead of N. The following Lemma is crucial in that respect.

Lemma 1. *(Geodesic Touching Lemma) For all y in $N \setminus \overline{M}$, there exists a geodesic from y to $\overline{M}$ that achieves the length of $d_{\overline{M}}(y)$. The end point of this geodesic on $\overline{M}$ must in fact be a quasi-embedded point of M, and the geodesic will hit M orthogonally.*

Proof. Identical argument to [2, Lemma 3.1], except we replace the Sheeting Theorem of [21] with the Sheeting Theorem of [4]. □

From this Lemma the following result is immediate,

Proposition 2. *For all y in $N \setminus (\overline{M} \setminus M)$, there exists an x in $\tilde{M}$, such that $F(x, \tilde{d}(y)) = y$.*

Understanding the regularity of σ^+ and σ^-, will be helpful in our analysis of $\tilde{T}$.

Claim 3. *The functions, σ^+, $\sigma^- \colon \tilde{M} \to \mathbb{R}$, are continuous.*

Proof. We prove by contradiction. Suppose there exists an $\hat{x} \in \tilde{M}$ such that, $\liminf_{x \to \hat{x}} \sigma^+(x) = \alpha < \sigma^+(\hat{x})$. Choose $0 < \delta < \sigma^+(\hat{x}) - \alpha$, then there exists $x_n \to \hat{x}$ in $\tilde{M}$ such that $\sigma^+(x_n) < \alpha + \delta$. Now consider the points,

$$z_n = F(x_n, \alpha + \delta) \to z \coloneqq F(\hat{x}, \alpha + \delta).$$

By Claim 2, $\tilde{d}(z_n) < \alpha + \delta$. By Proposition 2 there exists a sequence $\tilde{x}_n$, such that,

$$F(\tilde{x}_n, \tilde{d}(z_n)) = z_n.$$

After potentially taking a subsequence and renumerating we have that there exists an $y \in \overline{M}$, such that $\iota(\tilde{x}_n) \to y$, then note $d(y, z) = \tilde{d}(z) = \alpha + \delta$. Therefore, by Lemma 1, $y \in M$, and as $t \mapsto F(\hat{x}, t)$ is the unique length minimising geodesic from M to z, $\tilde{x}_n \to \hat{x}$ in $\tilde{M}$. Now we have that,

$$F(x_n, \alpha + \delta) = z_n = F(\tilde{x}_n, \tilde{d}(z_n)).$$

However, $(x_n, \alpha + \delta) \neq (\tilde{x}_n, \tilde{d}(z_n))$, and

$$\lim_{n \to \infty} (x_n, \alpha + \delta) = (\hat{x}, \alpha + \delta) = \lim_{n \to \infty} (\tilde{x}_n, \tilde{d}(z_n)).$$

This implies that F is not a diffeomorphism about the point $(\hat{x}, \alpha + \delta)$, and therefore by classical theory of geodesics, [16, Lemma 2.11], $t \mapsto F(\hat{x}, t)$ is no longer length minimising to M beyond time $t = \alpha + \delta$. This contradicts $\alpha + \delta < \sigma^+(\hat{x})$.

Now suppose that $\sigma^+(\hat{x}) < \limsup_{x \to \hat{x}} \sigma^+(x) = \beta < +\infty$. Choose $0 < \delta < \beta - \sigma^+(\hat{x})$, and sequence $x_n \to \hat{x}$, such that,

$$\sigma^+(x_n) > \sigma^+(\hat{x}) + \delta.$$

Define,

$$z_n = F(x_n, \sigma^+(\hat{x}) + \delta),$$

then $\tilde{d}(z_n) = \sigma^+(\hat{x}) + \delta$. By continuity of F,

$$z_n \to z \coloneqq F(\hat{x}, \sigma^+(\hat{x}) + \delta).$$

However, by definition of $\sigma^+(\hat{x})$, $\tilde{d}(z) < \sigma^+(\hat{x}) + \delta = \tilde{d}(z_n)$. This contradicts continuity of $\tilde{d}$.

Similar arguments show that σ^- is also continuous. □

We define the Cut Locus of M to be the following points in N,

$$\mathrm{Cut}\,(M) = \{F(x, \sigma^+(x)) \colon x \in \tilde{M}\} \cup \{F(x, \sigma^-(x)) \colon x \in \tilde{M}\} \subset N.$$

and by Proposition 2, we have that,

$$N \setminus (\overline{M} \setminus M) = F(\tilde{T}) \cup \mathrm{Cut}\,(M).$$

Proposition 3. *Cut(M) is an n-rectifiable set.*

To prove Proposition 3, we first classify points in $\mathrm{Cut}(M)$,

Proposition 4. *A point y in $N \setminus (\overline{M} \setminus M)$, lies in Cut$(M)$ if and only if at least one of the following conditions holds:*

1. *y lies in $N \setminus \overline{M}$, and there exists an x in $\tilde{M}$ such that $F(x, \tilde{d}(y)) = y$, and $dF_{(x,\tilde{d}(y))} \colon T_x\tilde{M} \times \mathbb{R} \to T_yN$, is non-invertible.*

2. *y lies in $N \setminus \overline{M}$, and there exists at least two unique geodesics from y to $\overline{M}$ which achieve the length $d_{\overline{M}}(y)$.*

3. *y is a non-embedded point of M.*

Proof. Consider a point $y = F(x, 0) \in M$. If y is an embedded point of M, then case 1 of Remark 1 holds, and there exists an $S > 0$, such that $\overline{M} \cap B_S(y)$ is a smooth, embedded CMC disk. Therefore ([13, Proposition 4.2]) there exists an r in $(0, S/2)$, such that for all t in $(-r, r)$, $\tilde{d}(F(x, t)) = t$. Therefore if $y \in M \cap \mathrm{Cut}(M)$, then y must be a non-embedded point.

Alternatively, if y is a non-embedded point then case 2 of Remark 1 holds, and (y, ν) and $(y, -\nu)$ both lie in $\tilde{M}$. Moreover, for $t \in (-s, 0)$, $t < f_2(0)$, implying that $F((y, \nu), t) = \exp_y(t\nu)$ lies in E. Therefore, $\tilde{d}(F((y, \nu), t)) \geq 0$, implying that $\sigma^-(y, \nu) = 0$, and thus y is a point in $\mathrm{Cut}(M)$.

For $y \in N \setminus \overline{M}$, the conclusion follows from standard theory of geodesics, see [16]. We can use this classical theory in our setting by Lemma 1. This observation is seen [2, Proposition 3.1]. ☐

Remark 2. *By point 2 of Proposition 4, $F(\tilde{T})$ and Cut(M) must be disjoint. Therefore by point 1 of Proposition 4, F must be a local diffeomorphism on $\tilde{T}$. Moreover by point 2, $F \colon \tilde{T} \to F(\tilde{T})$ is a bijection.*

Proof. (of Proposition 3) As $\mathrm{Cut}(M) \cap M$ consists of non-embedded points of M, by Claim 1 we have $\mathcal{H}^n(\mathrm{Cut}(M) \cap M) = 0$. Therefore to prove that $\mathrm{Cut}(M)$ is rectifiable we just need to concern ourselves with $\mathrm{Cut}(M) \setminus M$. This follows from the observation made in the proof of [2, Proposition 3.1], that as Lemma 1 holds, then the arguments in [13, Theorem 4.10] hold verbatim. ☐

Remark 3. *As M is smooth we have that $\tilde{d}$ is smooth in $F(\tilde{T})$, [13, Proposition 4.2].*

Denoting $h = F^*g$, we have that $F \colon (\tilde{T}, h) \to (F(\tilde{T}), g)$, is a bijective, local isometry.

Consider the projection map,

$$\begin{aligned} p \colon \tilde{M} \times \mathbb{R} &\to \mathbb{R}, \\ (x, t) &\mapsto t. \end{aligned}$$

In $\tilde{T}$, we have that $p = \tilde{d} \circ F$, and

$$|\nabla p(x,t)|_h = |\nabla \tilde{d}(F(x,t))|_g = 1.$$

We denote the sets,

$$\tilde{\Gamma}_t = p^{-1}(t) \cap \tilde{T},$$

and,

$$\Gamma_t = \tilde{d}^{-1}(t) \subset N.$$

Note,

$$F(\tilde{\Gamma}_t) = \begin{cases} \Gamma_t \cap F(\tilde{T}) = \Gamma_t \setminus \mathrm{Cut}(M), & t \neq 0, \\ \{\text{embedded points of } M\}, & t = 0. \end{cases}$$

Denote $H_{\tilde{\Gamma}_t}(x,t)$ as the scalar mean curvature of $\tilde{\Gamma}_t$, at (x,t), with respect to unit normal $\nabla p(x,t)$, and define the following function,

$$\begin{aligned} H_t \colon \tilde{M} &\to \mathbb{R}, \\ x &\mapsto \begin{cases} H_{\tilde{\Gamma}_t}(x,t), & (x,t) \in \tilde{T}, \\ 0, & (x,t) \notin \tilde{T}, \end{cases} \end{aligned}$$

For (x,t) in $\tilde{T}$, we have,

$$H_t(x) = -\mathrm{tr}_{T_{(x,t)}\tilde{\Gamma}_t} h(\nabla .\, \nabla p(x,t), \,\cdot\,) = -\Delta_{\tilde{\Gamma}_t} p(x,t).$$

However, as ∇p is a geodesic vector field

$$\nabla_{\nabla p} \nabla p = 0,$$

and as $|\nabla p| = 1$,

$$h(\nabla_X \nabla p, \nabla p) = \frac{1}{2} X(|\nabla p|) = 0.$$

Therefore, $\Delta_{\tilde{\Gamma}_t} p(x,t) = \Delta_{\tilde{T}} p(x,t)$, and thus for (x,t) in $\tilde{T}$,

$$H_t(x) = -\Delta p(x,t).$$

Proposition 5. *([10, Corollary 3.6]) For (x,t) in $\tilde{T}$,*

$$\partial_t H_t(x) = -\nabla p(\Delta p)(x,t) \geq m,$$

where $m = \inf_{|X|=1} Ric_g(X,X) > 0$.

Remark 4. *Consider fixed x in $\tilde{M}$. For $\sigma^-(x) < 0$, we have $H_0(x) = \lambda$. If $\sigma^-(x) = 0$, we still have,*

$$\lim_{t \searrow 0} H_t(x) = \lambda.$$

Thus by Proposition 5, we have for $(x,t) \in \tilde{T}$,

$$\begin{cases} H_t(x) \geq \lambda + m\,t, & t > 0, \\ H_0(x) = \lambda, \\ H_t(x) \leq \lambda + m\,t, & t < 0. \end{cases}$$

3.4 Area Element

We define the function on $\tilde{M}$,

$$\theta_t(x) = \begin{cases} J_{\Pi_t}(x), & (x,t) \in \tilde{T}, \\ 0, & (x,t) \notin \tilde{T}, \end{cases}$$

where J_{Π_t} is the Jacobian of the map $\Pi_t \colon x \in \tilde{M} \mapsto (x,t) \in \tilde{M} \times \mathbb{R}$. By the Area Formula,

$$\int_{\tilde{M}} \theta_t \, d\mathcal{H}^n = \mathcal{H}^n(\tilde{\Gamma}_t).$$

Proposition 6. *([10, Theorem 3.11]) For (x_0, t_0) in $\tilde{T}$,*

$$\partial_t \log(\theta_t)(x_0)_{|t=t_0} = -H_{t_0}(x_0).$$

Consider a fixed point (x_0, t_0) in $\tilde{T}$. First consider $t_0 \geq 0$. For all t in $(0, t_0]$, (x_0, t) lies in $\tilde{T}$, which implies that the function $t \mapsto \theta_t(x_0)$ is smooth on the interval $(0, t_0]$. Furthermore $\lim_{t \to 0^+} \theta_t(x_0) = 1$, and applying Fundamental Theorem of Calculus,

$$\log(\theta_{t_0}(x_0)) = -t_0 \left(\lambda + \frac{1}{2} m t_0 \right).$$

Therefore,

$$\theta_{t_0}(x_0) \leq e^{-t_0\left(\lambda + \frac{1}{2} m t_0\right)}.$$

Identical inequality holds for $t_0 \leq 0$.

The term $-t(\lambda + \frac{1}{2}mt)$ achieves a global maximum at $t = -\frac{\lambda}{m}$. Noting that for (x_0, t_0) not in $\tilde{T}$, $\theta_{t_0}(x_0) = 0$, we have that,

$$0 \leq \theta_{t_0}(x_0) \leq e^{\frac{\lambda^2}{2m}},$$

for all (x_0, t_0) in $\tilde{M} \times \mathbb{R}$.

3.5 Construction About Non-Embedded point

Let z_0 in M be a non-embedded point.

Remark 5. *(Choice of $\delta = \delta(z_0, N, M, g)$ to define $\tilde{d}_1$ and $\tilde{d}_2$) We are in case 2 of Remark 1. We can choose $\delta = \delta(z_0, M, N, g)$ such that,*

$$B_{2\delta}(z_0) \subset C_{z_0,T,r,s}.$$

We have three disjoint sets,

$$\begin{aligned} E_1 &:= \exp_z(\{X + t\nu \colon X \in B_{z_0,T,r},\ f_1(X) < t < s\}) \cap B^N_{2\delta}(z_0), \\ F &:= \exp_z(\{X + t\nu \colon X \in B_{z_0,T,r},\ f_2(X) \leq t \leq f_1(X)\}) \cap B^N_{2\delta}(z_0), \\ E_2 &:= \exp_z(\{X + t\nu \colon X \in B_{z_0,T,r},\ -s < t < f_2(X)\}) \cap B^N_{2\delta}(z_0). \end{aligned}$$

As $\partial E_i \cap B^N_{2\delta}(z_0) = \exp_z(\{Graph(f_i)\}) \cap B^N_{2\delta}(z_0) =: D_i$*, the following signed distance functions are well defined for* $i = 1, 2$*,*

$$\begin{array}{rcl} \tilde{d}_i\colon B^N_{2\delta}(z_0) & \to & \mathbb{R}, \\ y & \mapsto & \begin{cases} d_{D_i}(y), & y \in E_i, \\ -d_{D_i}(y), & y \in B^N_{2\delta}(z_0) \setminus E_i. \end{cases} \end{array}$$

For y *in* $B^N_\delta(z_0) \subset\subset B^N_{2\delta}(z_0)$*,*

$$\tilde{d}(y) = \max\{\tilde{d}_1(y), \tilde{d}_2(y)\}.$$

Furthermore by [13, Proposition 4.2], we may choose $\delta > 0$*, such that* $\tilde{d}_1$ *and* $\tilde{d}_2$ *will be smooth on* $B^N_{2\delta}(z_0)$*.*

For $i = 1$, 2, we define

$$\tilde{D}_i \coloneqq \{(z, \nu_i(z))\colon z \in D_i\} \subset \tilde{M},$$

and points $x_0^i = (z_0, \nu_i(z_0))$.

Remark 6. *We make a choice of* $\delta = \delta(N, M, g, z_0) > 0$ *small enough such that, for each* $i = 1$, 2*, we have open sets* $\tilde{V}_i \subset \tilde{M} \times \mathbb{R}$*, and maps,*

$$F_i\colon \tilde{V}_i \to B^N_{2\delta}(z_0),$$

such that $\tilde{D}_i = \tilde{V}_i \cap \{t = 0\}$*, and* $F_i = F_{|\tilde{V}_i}$*, is a diffeomorphism. We also insist that* $\delta = \delta(N, M, g, z_0) > 0$*, is chosen small enough such that* $Cut(D_1)$ *and* $Cut(D_2)$ *are empty in* $B_{2\delta}(z_0)$*. We know we can pick such a* $\delta > 0$ *by [13, Proposition 4.2]*

By choice of $\delta > 0$ in Remark 6, and Proposition 4,

$$\mathrm{Cut}(M) \cap B^N_\delta(z_0) = \{y \in B^N_\delta(z_0)\colon \tilde{d}_1(y) = \tilde{d}_2(y)\} \subset B^N_\delta(z_0) \setminus E.$$

Remark 7. *Denote the set,*

$$A = \{y \in B^N_{2\delta}(z_0)\colon \tilde{d}_1(y) = \tilde{d}_2(y)\}.$$

For $i = 1$, 2*, we consider the functions,*

$$\begin{array}{rcl} \psi_i\colon \tilde{V}_i & \to & \mathbb{R}, \\ (x, t) & \mapsto & \tilde{d}_1(F_i(x, t)) - \tilde{d}_2(F_i(x, t)). \end{array}$$

Therefore, $A = F_i(\{\psi_i = 0\})$*. Moreover,*

$$\partial_t \psi_i(x_0^i, 0) = dF_i^{-1}(\nabla \tilde{d}_1(z_0)) - dF_i^{-1}(\nabla \tilde{d}_2(z_0)) = 2\partial_t \neq 0.$$

Thus, by Implicit Function Theorem we may choose $\delta = \delta(z_0, N, M, g) > 0$*, such that set* $A = Cut(M) \cap B^N_{2\delta}(z_0)$ *is a smooth* n*-submanifold in* $B^N_{2\delta}(z_0)$*, and* σ^- *is smooth on* $\tilde{D}_1 \cup \tilde{D}_2$*.*

We now look to define the push out function to construct our competitor, Figure 5.

We wish to determine the amount we want to push out by, and the set we wish to push out on. Fix $\rho > 0$, and we set $l = l(\rho)$, to be,

$$l(\rho) = \sup\{t\colon\ for\,all\,x\,in\,B_t^{\tilde{M}}(x_0^1),\ |\sigma^-(x)| < \rho\}. \tag{10}$$

Here, $B_t^{\tilde{M}}(x)$ is the geodesic ball in $\tilde{M}$, about point x, of radius t. As σ^- is smooth about x_0^1, and $\sigma^-(x_0^1) = 0$, this implies that $l(\rho) > 0$ for all $\rho > 0$. Furthermore, $l(\rho)$ is increasing in ρ, implying that the limit of $l(\rho)$, as $\rho \to 0$, exists. Therefore as $\sigma^-(x) = 0$ if and only if $\iota(x)$ is a non-embedded point, and such points have $\mathcal{H}^n$-measure 0 in $\tilde{M}$, we have that this limit must be 0.

Remark 8. *As σ^- is smooth on $\tilde{D}_1$, $\sigma^- \leq 0$, and $\sigma^-(x_0^1) = 0$, then there exists a $C_1 = C_1(N, M, g, z_0) < +\infty$, and a $\delta = \delta(N, M, g, z_0)$, such that for all x in $\tilde{D}_1$,*

$$\sigma^-(x) \geq -C_1 d^2_{\tilde{M}}(x, x_0^1).$$

As $l(\rho) \to 0$, as $\rho \to 0$, this implies that we can choose $\rho > 0$, Remark 9, such that

$$B^{\tilde{M}}_{l(\rho)}(x_0^1) \subset\subset \tilde{D}_1.$$

There exists an x' in $\tilde{D}_1$, such that $d_{\tilde{M}}(x', x_0^1) = l$, and $\sigma^-(x') = -\rho$. Therefore, by Remark 8,

$$\rho \leq C_1 l^2.$$

Remark 9. *Note that we have made our first choice of $\rho = \rho(z_0, N, M, g, \delta)$.*

We push out on disks D_1 and D_2 equally, so that they meet on the Cut Locus in $B^N_\delta(z_0)$, which is our previously denoted set A, as seen in Figure 5. We consider the open sets $\tilde{W}_i \subset \tilde{D}_i$, defined by,

$$\tilde{W}_i = \{x \colon F_i(x, \sigma^-(x)) \in B_\delta(z_0)\}.$$

Clearly x_0^i lies in $\tilde{W}_i$, therefore these sets are non-empty. We can then define a diffeomorphism between $\tilde{W}_1$, and $\tilde{W}_2$.

$$\begin{aligned} \Psi \colon \tilde{W}_1 &\to \tilde{W}_2, \\ x &\mapsto (\pi \circ F_2^{-1} \circ F_1 \circ (\mathrm{id}, \sigma^-))(x), \end{aligned}$$

where we define, π by,

$$\begin{aligned} \pi \colon \tilde{M} \times \mathbb{R} &\to \tilde{M}, \\ (x, t) &\mapsto x, \end{aligned}$$

and (id, σ^-), by

$$\begin{aligned} (\mathrm{id}, \sigma^-) \colon \tilde{M} &\to \tilde{M} \times \mathbb{R}, \\ x &\mapsto (x, \sigma^-(x)). \end{aligned}$$

The function Ψ is smooth and has smooth inverse given by

$$\begin{aligned} \Psi^{-1} \colon \tilde{W}_2 &\to \tilde{W}_1, \\ x &\mapsto (\pi \circ F_1^{-1} \circ F_2 \circ (\mathrm{id}, \sigma^-))(x). \end{aligned}$$

We note that, $d\Psi_{x_0^1} = Id$.

Remark 10. *We choose $\rho = \rho(z_0, N, M, g, \delta) > 0$, such that,*

$$B^{\tilde{M}}_{2l}(x_0^1) \subset\subset \tilde{W}_1.$$

Consider a *push out function*, which lies in $C_c^\infty(\tilde{D}_1)$, and has the following properties,

$$f_1(x) = \begin{cases} -1, & x \in B_l^{\tilde{M}}(x_0^1), \\ [-1,0], & x \in B_{2l}^{\tilde{M}}(x_0^1) \setminus B_l^{\tilde{M}}(x_0^1), \\ 0, & x \in \tilde{D}_1 \setminus B_{2l}^{\tilde{M}}(x_0^1). \end{cases}$$

We further impose the condition,

$$|\nabla f_1| \leq \frac{2}{l}.$$

Define f_2 in $C_c^\infty(\tilde{D}_2)$, by

$$f_2(x) = \begin{cases} (f_1 \circ \Psi^{-1})(x), & x \in \tilde{W}_2, \\ 0, & x \in \tilde{D}_2 \setminus \tilde{W}_2. \end{cases}$$

The support of f_2 will lie in $\Psi(B_{2l}^{\tilde{M}}(x_0^1)) \subset\subset \Psi(\tilde{W}_1) = \tilde{W}_2$. We then define the function f in $C_c^\infty(\tilde{M})$, by $f = f_1 + f_2$.

Define the sets,

$$\begin{aligned} B_{2l} &= B_{2l}^{\tilde{M}}(x_0^1) \cup \Psi(B_{2l}^{\tilde{M}}(x_0^1)), \\ B_l &= B_l^{\tilde{M}}(x_0^1) \cup \Psi(B_l^{\tilde{M}}(x_0^1)), \\ A_l &= B_{2l} \setminus B_l. \end{aligned}$$

We will similarly define the sets,

$$B_t = B_t^{\tilde{M}}(x_0^1) \cup \Psi(B_t^{\tilde{M}}(x_0^1)),$$

for $t > 0$, such that $B_t^{\tilde{M}}(x_0^1) \subset \tilde{W}_1$.

Remark 11. *We choose $\rho = \rho(z_0, M, N, g, \delta, W, \lambda) > 0$, such that*

$$F_1\left(B_{2l} \times (-2\rho, 2\rho)\right) \subset\subset B_\delta^N(z_0)$$

We now look to define the function that will 'push out away from non-embedded point'. This function will define the path from (2) to (3) in Figure 2.

Remark 12. *(Choice of L and r_0) We choose $L = L(z_0, N, M, g, \delta) > 0$ and $r_0 = r_0(z_0, N, M, g, \delta) > 0$, such that,*

$$B_L^{\tilde{M}}(x_0^1) \subset\subset \tilde{W}_1,$$

and,

$$F\left(B_L \times (-2r_0, 2r_0)\right) \subset\subset B_\delta^N(z_0).$$

For a sets $\tilde{\Omega}$ and Ω, were Ω is open and $\tilde{\Omega} \subset\subset \Omega$, we define the 2-Capacity of $\tilde{\Omega}$ in Ω as the value,

$$\mathrm{Cap}_2(\tilde{\Omega}, \Omega) = \inf\left\{\int_\Omega |\nabla\varphi|^2 \, d\mathcal{H}^n \colon \varphi \in C_c^\infty(\Omega),\, \varphi \geq \chi_{\tilde{\Omega}}\right\}.$$

For $n \geq 3$ [8, Theorem 4.16, Section 4.7.2],

$$\mathrm{Cap}_2(\{x_0^1\}, B_L^{\tilde{M}}(x_0^1)) = 0.$$

Applying this to [8, Theorem 4.15 (ix), Section 4.7.1],

$$\lim_{k\to\infty} \mathrm{Cap}_2(B_{\frac{L}{k}}^{\tilde{M}}(x_0^1), B_L^{\tilde{M}}(x_0^1)) = \mathrm{Cap}_2(\{x_0^1\} \cup \{x_0^2\}, B_L) = 0.$$

Identical proofs show that this also holds for $n = 2$. Therefore, for all $\gamma > 0$, there exists a function $\varphi_{\gamma,k}$, such that,

$$\begin{cases} \varphi_{\gamma,k} \in C_c^\infty(B_L^{\tilde{M}}(x_0^1)), \\ \varphi_{\gamma,k} \colon \tilde{M} \to [0,1], \\ \varphi_{\gamma,k}(x) = 1,\ x \in B_{\frac{L}{k}}^{\tilde{M}}(x_0^1), \end{cases}$$

and, defining $\tilde{\varphi}_{\gamma,k} = \varphi_{\gamma,k} + \varphi_{\gamma,k} \circ \Psi^{-1}$, we have

$$\int_{\tilde{M}} |\nabla \tilde{\varphi}_{\gamma,k}|^2 \, d\mathcal{H}^n(x) < \gamma.$$

We consider the function $\tilde{f} = 1 - \tilde{\varphi}_{\gamma,k}$ in $C^\infty(\tilde{M})$, and $\|\nabla \tilde{f}\|_{L^2(\tilde{M})}^2 < \gamma$.

Remark 13. *(Later Choices of r_0, L, γ, and k) We will later make fixed choices for $L = L(z_0, M, N, g, \delta, W, \lambda)$, $r_0 = r_0(z_0, M, N, g, \delta, W, \lambda, L)$, $\gamma = \gamma(z_0, N, M, g, \delta, r_0, L)$, and $k = k(z_0, N, M, g, \delta, L, \gamma$*

Remark 14. *(Choice of ρ based on r_0 and L) We make a further choice of $\rho = \rho(z_0, N, M, g, \delta, L, r_0, k)$, such that,*

$$B_{2l} \subset\subset B_{\frac{L}{k}},$$

We will make a further choice of ρ later on, so that $\rho = \rho(z_0, N, M, g, \delta, L, k, r_0)$.

3.6 Approximating Function for CMC

We use the tools we have constructed to give a simple proof that function,

$$v_\varepsilon(y) = \overline{\mathbb{H}}_\varepsilon(\tilde{d}(y)),$$

is suitable approximation of M, i.e.

$$\lim_{\varepsilon\to 0} \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) = 2\sigma\mathcal{H}^n(M) - \sigma\lambda\mathcal{H}^{n+1}(E) - \sigma\lambda\mathcal{H}^{n+1}(N \setminus E) = F_\lambda(E).$$

By the Co-Area formula on the function $\tilde{d}$,

$$\begin{aligned} \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) &= \int_N \frac{\varepsilon}{2}|\nabla v_\varepsilon|^2 + \frac{W(v_\varepsilon)}{2} - \sigma\lambda \int_N v_\varepsilon, \\ &= \int_{\mathbb{R}} \int_{\Gamma_t} Q_\varepsilon(t) \, d\mathcal{H}^n \, dt - \sigma\lambda \int_{\mathbb{R}} \int_{\Gamma_t} \overline{\mathbb{H}}_\varepsilon(t) \, d\mathcal{H}^n \, dt, \end{aligned}$$

where,

$$Q_\varepsilon(t) = \frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t))^2 + \frac{W(\overline{\mathbb{H}}_\varepsilon(t))}{\varepsilon}.$$

Using the fact that $N \setminus F(\tilde{T})$ is a set of 0 $\mathcal{H}^{n+1}$-measure, and that $F\colon (\tilde{T}, h) \to (F(\tilde{T}), g)$, is a bijective local isometry, we have,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) = \int_{\mathbb{R}} Q_\varepsilon(t)\, \mathcal{H}^n(\tilde{\Gamma}_t)\, dt - \sigma\lambda \int_{\mathbb{R}} \overline{\mathbb{H}}_\varepsilon(t)\, \mathcal{H}^n(\tilde{\Gamma}_t)\, dt.$$

From Analysis of $\overline{\mathbb{H}}_\varepsilon$, we have that, $\operatorname{supp} Q_\varepsilon \subset [-2\varepsilon\Lambda, 2\varepsilon\Lambda]$, and

$$2\sigma - \beta\varepsilon^2 \leq \int_{\mathbb{R}} Q_\varepsilon(t)\, dt \leq 2\sigma + \beta\varepsilon^2.$$

Furthermore,

$$\overline{\mathbb{H}}_\varepsilon(t) \leq \begin{cases} 1, & t > -2\varepsilon\Lambda, \\ -1, & t \leq -2\varepsilon\Lambda, \end{cases}$$

and,

$$\overline{\mathbb{H}}_\varepsilon(t) \geq \begin{cases} 1, & t > 2\varepsilon\Lambda, \\ -1, & t \leq 2\varepsilon\Lambda. \end{cases}$$

Therefore,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \leq (2\sigma + \beta\varepsilon^2) \operatorname*{ess\,sup}_{t\in[-2\varepsilon\Lambda, 2\varepsilon\Lambda]} \mathcal{H}^n(\tilde{\Gamma}_t) - \sigma\lambda \int_{2\varepsilon\Lambda}^{+\infty} \mathcal{H}^n(\tilde{\Gamma}_t)\, dt + \sigma\lambda \int_{-\infty}^{2\varepsilon\Lambda} \mathcal{H}^n(\tilde{\Gamma}_t)\, dt.$$

Similarly,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \geq (2\sigma - \beta\varepsilon^2) \operatorname*{ess\,inf}_{t\in[-2\varepsilon\Lambda, 2\varepsilon\Lambda]} \mathcal{H}^n(\tilde{\Gamma}_t) - \sigma\lambda \int_{-2\varepsilon\Lambda}^{+\infty} \mathcal{H}^n(\tilde{\Gamma}_t)\, dt + \sigma\lambda \int_{-\infty}^{2\varepsilon\Lambda} \mathcal{H}^n(\tilde{\Gamma}_t)\, dt.$$

We have,

$$\mathcal{H}^n(\tilde{\Gamma}_t) = \int_{\tilde{M}} \theta_t(x)\, d\mathcal{H}^n(x),$$

and by applying Dominated Convergence Theorem to θ_t, we have that,

$$\lim_{t\to 0} \mathcal{H}^n(\tilde{\Gamma}_t) = \lim_{t\to 0} \int_{\tilde{M}} \theta_t(x)\, d\mathcal{H}^n(x) = \mathcal{H}^n(\tilde{M} \cap \tilde{T}) = \mathcal{H}^n(\tilde{M}).$$

This implies that,

$$\lim_{\varepsilon\to 0} \operatorname*{ess\,sup}_{t\in[-2\varepsilon\Lambda, 2\varepsilon\Lambda]} \mathcal{H}^n(\tilde{\Gamma}_t) = \mathcal{H}^n(\tilde{M}) = \mathcal{H}^n(M),$$

and,

$$\lim_{\varepsilon\to 0} \operatorname*{ess\,inf}_{t\in[-2\varepsilon\Lambda, 2\varepsilon\Lambda]} \mathcal{H}^n(\tilde{\Gamma}_t) = \mathcal{H}^n(\tilde{M}) = \mathcal{H}^n(M).$$

The function $t \mapsto \mathcal{H}^n(\tilde{\Gamma}_t)$ is measurable, implying that,

$$\lim_{\varepsilon\to 0} \int_{\pm 2\varepsilon\Lambda}^{+\infty} \mathcal{H}^n(\tilde{\Gamma}_t)\, dt = \int_0^{+\infty} \mathcal{H}^n(\tilde{\Gamma}_t)\, dt = \mathcal{H}^{n+1}(\{y \in N \colon \tilde{d}(y) > 0\}) = \mathcal{H}^{n+1}(E),$$

and,

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\pm 2\varepsilon\Lambda}\mathcal{H}^n(\tilde{\Gamma}_t)\,dt=\int_{-\infty}^{0}\mathcal{H}^n(\tilde{\Gamma}_t)\,dt=\mathcal{H}^{n+1}(\{y\in N\colon \tilde{d}(y)<0\})=\mathcal{H}^{n+1}(N\setminus E).$$

Therefore we have,

$$\lim_{\varepsilon\to 0}\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon)=2\sigma\mathcal{H}^n(M)-\sigma\lambda\mathcal{H}^{n+1}(E)+\sigma\lambda\mathcal{H}^{n+1}(N\setminus E).$$

4 Base Computation

Consider a smooth function,

$$g\colon \mathbb{R}\times\tilde{M}\to\mathbb{R}.$$

and define the following

$$\begin{array}{rcl} v_\varepsilon^{r,g}\colon \tilde{M}\times\mathbb{R} & \to & \mathbb{R}, \\ (x,t) & \mapsto & \overline{\mathbb{H}}_\varepsilon(t-g(r,x)). \end{array}$$

By Gauss Lemma,

$$|\nabla v_\varepsilon^{r,g}(x,t)|^2=((\overline{\mathbb{H}}_\varepsilon)'(t-g(r,x)))^2(1+|\nabla_x g(r,x)|^2_{(x,t)}).$$

By the co-area formula on p,

$$\begin{aligned}
\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g}) &= \int_{\tilde{T}}\frac{\varepsilon}{2}|\nabla v_\varepsilon^{r,g}|^2+\frac{W(v_\varepsilon^{r,g})}{\varepsilon}-\sigma\lambda v_\varepsilon^{r,g}\,d\mu_h,\\
&= \int_{\mathbb{R}}\int_{\tilde{\Gamma}_t}\frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t-g(r,x)))^2|\nabla_x g(r,x)|^2_{(x,t)}\,d\mathcal{H}^n(x,t)\,dt\\
&\quad+\int_{\mathbb{R}}\int_{\tilde{\Gamma}_t}\frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t-g(r,x)))^2+\frac{W(\overline{\mathbb{H}}_\varepsilon(t-g(r,x)))}{\varepsilon}-\sigma\lambda\overline{\mathbb{H}}_\varepsilon(t-g(r,x))\,d\mathcal{H}^n(x,t)\,dt,\\
&= \int_{\tilde{M}}\int_{\sigma^-(x)}^{\sigma^+(x)}\frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t-g(r,x)))^2|\nabla_x g(r,x)|^2_{(x,t)}\,\theta_t(x)\,dt\,d\mathcal{H}^n(x)\\
&\qquad+\int_{\tilde{M}}\int_{\sigma^-(x)}^{\sigma^+(x)}\left(\frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t-g(r,x)))^2+\frac{W(\overline{\mathbb{H}}_\varepsilon(t-g(r,x)))}{\varepsilon}\right.\\
&\qquad\qquad\left.-\sigma\lambda\overline{\mathbb{H}}_\varepsilon(t-g(r,x))\right)\theta_t(x)\,dt\,d\mathcal{H}^n(x),
\end{aligned}$$

In the last equality we use Fubini's Theorem to switch the integrals.

We have,

$$\begin{aligned}
\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g})-\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{0,g}) &= \int_{\tilde{M}}\int_{\sigma^-(x)}^{\sigma^+(x)}\frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t-g(r,x)))^2|\nabla_x g(r,x)|^2_{(x,t)}\,\theta_t(x)\,dt\,d\mathcal{H}^n(x)\\
&\qquad-\int_{\tilde{M}}\int_{\sigma^-(x)}^{\sigma^+(x)}\frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t-g(r,x)))^2|\nabla_x g(0,x)|^2_{(x,t)}\,\theta_t(x)\,dt\,d\mathcal{H}^n(x),\\
&\quad+\int_{\tilde{M}}\int_{\sigma^-(x)}^{\sigma^+(x)}(Q_\varepsilon(t-g(r,x))-Q_\varepsilon(t-g(0,x)))\theta_t(x)\,dt\,d\mathcal{H}^n(x)\\
&\quad-\int_{\tilde{M}}\int_{\sigma^-(x)}^{\sigma^+(x)}\sigma\lambda(\overline{\mathbb{H}}_\varepsilon(t-g(r,x))-\overline{\mathbb{H}}_\varepsilon(t-g(0,x)))\,\theta_t(x)\,dt\,d\mathcal{H}^n(x),
\end{aligned}$$

We have the following two terms,

$$
\begin{aligned}
I_\varepsilon^{r,g} &= \int_{\tilde M}\int_{\sigma^-(x)}^{\sigma^+(x)} \frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t-g(r,x)))^2|\nabla_x g(r,x)|^2_{(x,t)}\,\theta_t(x)\,dt\,d\mathcal{H}^n(x)\\
&\qquad -\int_{\tilde M}\int_{\sigma^-(x)}^{\sigma^+(x)} \frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t-g(r,x)))^2|\nabla_x g(0,x)|^2_{(x,t)}\,\theta_t(x)\,dt\,d\mathcal{H}^n(x),
\end{aligned}
$$

and, by Fundamental Theorem of Calculus and Fubini's Theorem,

$$
\begin{aligned}
II_\varepsilon^{r,g} &= \int_{\tilde M}\int_{\sigma^-(x)}^{\sigma^+(x)} (Q_\varepsilon(t-g(r,x))-Q_\varepsilon(t-g(0,x)))\theta_t(x)\,dt\,d\mathcal{H}^n(x)\\
&\quad -\int_{\tilde M}\int_{\sigma^-(x)}^{\sigma^+(x)} \sigma\lambda(\overline{\mathbb{H}}_\varepsilon(t-g(r,x))-\overline{\mathbb{H}}_\varepsilon(t-g(0,x)))\,\theta_t(x)\,dt\,d\mathcal{H}^n(x),\\
&= -\int_0^r\int_{\tilde M}\partial_s g(s,x)\int_{\sigma^-(x)}^{\sigma^+(x)} Q_\varepsilon'(t-g(s,x))\,\theta_t(x)\,dt\,d\mathcal{H}^n(x)\,ds\\
&\quad +\int_0^r\int_{\tilde M}\partial_s g(s,x)\int_{\sigma^-(x)}^{\sigma^+(x)} \sigma\lambda(\overline{\mathbb{H}}_\varepsilon)'(t-g(s,x))\,\theta_t(x)\,dt\,d\mathcal{H}^n(x)\,ds,\\
&= -\int_0^r\int_{\tilde M}\partial_s g(s,x)Q_\varepsilon(\sigma^+(x)-g(s,x))\,\theta^+(x)\,d\mathcal{H}^n(x)\,ds\\
&\quad +\int_0^r\int_{\tilde M}\partial_s g(s,x)Q_\varepsilon(\sigma^-(x)-g(s,x))\,\theta^-(x)\,d\mathcal{H}^n(x)\,ds\\
&\quad +\int_0^r\int_{\tilde M}\partial_s g(s,x)\int_{\sigma^-(x)}^{\sigma^+(x)} Q_\varepsilon(t-g(s,x))\,\partial_t\theta_t(x)\,dt\,d\mathcal{H}^n(x)\,ds\\
&\quad +\int_0^r\int_{\tilde M}\partial_s g(s,x)\int_{\sigma^-(x)}^{\sigma^+(x)} \sigma\lambda(\overline{\mathbb{H}}_\varepsilon)'(t-g(s,x))\,\theta_t(x)\,dt\,d\mathcal{H}^n(x)\,ds,\\
&= -\int_0^r\int_{\tilde M}\partial_s g(s,x)Q_\varepsilon(\sigma^+(x)-g(s,x))\,\theta^+(x)\,d\mathcal{H}^n(x)\,ds\\
&\quad +\int_0^r\int_{\tilde M}\partial_s g(s,x)Q_\varepsilon(\sigma^-(x)-g(s,x))\,\theta^-(x)\,d\mathcal{H}^n(x)\,ds\\
&\quad +\int_0^r\int_{\tilde M}\partial_s g(s,x)\int_{\sigma^-(x)}^{\sigma^+(x)} Q_\varepsilon(t-g(s,x))\,(\lambda-H_t(x))\,\theta_t(x)\,dt\,d\mathcal{H}^n(x)\,ds\\
&\quad +\lambda\int_0^r\int_{\tilde M}\Theta^1_{\varepsilon,g}(s,x)-\Theta^2_{\varepsilon,g}\,d\mathcal{H}^n(x)\,ds,
\end{aligned}
$$

Where,

$$
\begin{aligned}
\theta^+(x) &= \lim_{t\nearrow\sigma^+(x)}\theta_t(x),\\
\theta^-(x) &= \lim_{t\searrow\sigma^-(x)}\theta_t(x),\\
\Theta^1_{\varepsilon,g}(s,x) &= \sigma\int_{\sigma^-(x)}^{\sigma^+(x)}\partial_s g(s,x)(\overline{\mathbb{H}}_\varepsilon)'(t-g(s,x))\theta_t(x)\,dt,\\
\Theta^2_{\varepsilon,g}(s,x) &= \int_{\sigma^-(x)}^{\sigma^+(x)}\partial_s g(s,x)Q_\varepsilon(t-g(s,x))\theta_t(x)\,dt.
\end{aligned}
$$

For the last equality of $II_\varepsilon^{r,g}$ we are using $\partial_t\theta_t(x) = -H_t(x)\theta_t(x)$, for t in $(\sigma^-(x), \sigma^+(x))$.

5 Competitor

5.1 Calculation on $\tilde{M} \times \mathbb{R}$

Here we construct the path in Figure 2 from (1) to (2).

Set $g_1(r,x) = rf(x)$, take r in $[0,\rho]$, where $\rho \in (0,1)$ will be chosen later and $f\colon \tilde{M} \to \mathbb{R}$ is defined in Section 3.5.

Remark 15. *(Choice in ε_1) We choose $\varepsilon_1 = \varepsilon_1(\rho) \in (0, 1/4)$, such that,*

$$2\varepsilon_1\Lambda = 6\varepsilon_1|\log\varepsilon_1| << \rho.$$

From here we consider ε in $(0,\varepsilon_1)$.

We have,

$$\begin{aligned} II_\varepsilon^{r,g_1} = &- \int_0^r \int_{\tilde{M}} f(x) Q_\varepsilon(\sigma^+(x) - sf(x))\,\theta^+(x)\,d\mathcal{H}^n(x)\,ds \\ &+ \int_0^r \int_{\tilde{M}} f(x) Q_\varepsilon(\sigma^-(x) - sf(x))\,\theta^-(x)\,d\mathcal{H}^n(x)\,ds \\ &+ \int_0^r \int_{\tilde{M}} f(x) \int_{\sigma^-(x)}^{\sigma^+(x)} Q_\varepsilon(t - sf(x))\,(\lambda - H_t(x))\,\theta_t(x)\,dt\,d\mathcal{H}^n(x)\,ds \\ &+ \lambda \int_0^r \int_{\tilde{M}} \Theta^1_{\varepsilon,g}(s,x) - \Theta^2_{\varepsilon,g}\,d\mathcal{H}^n(x)\,ds, \end{aligned} \tag{11}$$

Concentrate on the second term of the right hand side of (11). As the integrand is non-positive, $f = -1$ on B_l and $\operatorname{supp} Q_\varepsilon \subset [-2\varepsilon\Lambda, 2\varepsilon\Lambda]$, we have

$$\begin{aligned} &\int_0^r \int_{\tilde{M}} f(x) Q_\varepsilon(\sigma^-(x) - sf(x))\,\theta^-(x)\,d\mathcal{H}^n(x)\,ds \\ &\qquad \le -(2\sigma - \beta\varepsilon^2) \int_{B_l \cap \{-r + 2\varepsilon\Lambda \le \sigma^-(x) \le -2\varepsilon\Lambda\}} \theta^-(x)\,d\mathcal{H}^n(x) \end{aligned}$$

We look for lower bounds on θ^-.

Remark 16. *Choose $\delta = \delta(z_0, N, M, g) > 0$, such that,*

$$\min_{y \in B^N_\delta(z_0)} \{\Delta\tilde{d}_1(y), \Delta\tilde{d}_2(y)\} \ge \frac{\lambda}{2},$$

and,

$$\max_{y \in B^N_\delta(z_0)} \{\Delta\tilde{d}_1(y), \Delta\tilde{d}_2(y)\} \le \frac{3\lambda}{2}.$$

Therefore, for (x,t) in $\tilde{T}$, such that, $F(x,t)$ lies in $B^N_\delta(z_0)$, we have that,

$$\frac{\lambda}{2} \leq H_t(x) \leq \frac{3\lambda}{2}.$$

Thus by similar calculations carried out in Section 3.4, for all (x,t) in $\tilde{T}$, such that $F(x,t)$ lies in $B^N_\delta(z_0)$, we have,

$$\theta_t(x) \geq \begin{cases} e^{-\frac{3\lambda t}{2}}, & t \geq 0, \\ e^{-\frac{\lambda t}{2}}, & t \leq 0. \end{cases}$$

For x in B_l, we have $\sigma^-(x) > -\rho$, and by choice of ρ in Remark 11, we have that $F(\{x\} \times (\sigma^-(x), 0)) \subset B^N_\delta(z_0)$. Thus

$$\theta^-(x) = \lim_{t \searrow \sigma^-(x)} \theta_t(x) \geq e^{-\frac{\lambda\,\sigma^-(x)}{2}} \geq 1,$$

for all x in B_l. Therefore,

$$\begin{aligned} &\int_0^r \int_{\tilde{M}} f(x) Q_\varepsilon(\sigma^-(x) - s f(x))\, \theta^-(x)\, d\mathcal{H}^n(x)\, ds \\ &\qquad \leq -2\sigma \mathcal{H}^n(\{x \colon x \in B_l,\, -r + 2\varepsilon\Lambda \leq \sigma^-(x) \leq -2\varepsilon\Lambda\}) + C_2 \varepsilon^2, \end{aligned}$$

for $C_2 = C_2(N, M, g, \lambda, W) < +\infty$. This is a lower bound for the area deleted in pushing the disks together.

Concentrate on First term on the right hand side of (11). By choice of $\delta > 0$ in Remark 6 and $\rho > 0$ in Remark 11 we have that for x in $\operatorname{supp}(f) \subset B_{2l}$, $\sigma^+(x) > 2\rho >> 2\varepsilon\Lambda$. Thus,

$$\int_0^r \int_{\tilde{M}} f(x) Q_\varepsilon(\sigma^+(x) - s f(x))\, \theta^+(x)\, d\mathcal{H}^n(x)\, ds = 0.$$

Concentrate on the third term on the right hand side of (11). Consider $s > 0$, and x in $\tilde{M}$, such that $s f(x) < -2\varepsilon\Lambda$. Using the fact that $\operatorname{supp} Q_\varepsilon \subset [-2\varepsilon\Lambda, 2\varepsilon\Lambda]$, and the inequalities on H_t in Remark 4,

$$\begin{aligned} \int_{\sigma^-(x)}^{\sigma^+(x)} Q_\varepsilon(t - s f(x))\, (\lambda - H_t(x))\, \theta_t(x)\, dt &= \int_{-2\varepsilon\Lambda}^{2\varepsilon\Lambda} Q_\varepsilon(\xi)(\lambda - H_{\xi + s f(x)}) \theta_{\xi + s f(x)} d\xi, \\ &\geq 0. \end{aligned}$$

For $s f(x) \geq -2\varepsilon\Lambda$, we have,

$$\begin{aligned} \int_{\sigma^-(x)}^{\sigma^+(x)} Q_\varepsilon(t - s f(x))\, (\lambda - H_t(x))\, \theta_t(x)\, dt &= \int_{-2\varepsilon\Lambda}^{2\varepsilon\Lambda} Q_\varepsilon(\xi)(\lambda - H_{\xi + s f(x)}) \theta_{\xi + s f(x)} d\xi, \\ &\geq C_2 \min_{t \in [-4\varepsilon\Lambda, 2\varepsilon\Lambda]} (\lambda - H_t(x)) \theta_t(x), \end{aligned}$$

potentially rechoosing $C_2 = C_2(M, N, g, \lambda, W)$. Therefore we have that for all r in $[0, \rho]$,

$$\begin{aligned} II^{r, g_1}_\varepsilon \leq\; & -2\sigma \mathcal{H}^n(\{x \colon x \in B_l,\, -r + 2\varepsilon\Lambda \leq \sigma^-(x) \leq -2\varepsilon\Lambda\}) \\ & + C_2 \left(r \int_{B_{2l}} q^1_\varepsilon(x)\, d\mathcal{H}^n(x) + \int_0^r \int_{\tilde{M}} \Theta^1_{\varepsilon, g}(s, x) - \Theta^2_{\varepsilon, g}\, d\mathcal{H}^n(x)\, ds + \varepsilon^2 \right). \end{aligned}$$

where,

$$q^1_\varepsilon(x) = \max_{t\in[-4\varepsilon\Lambda,2\varepsilon\Lambda]} (H_t(x) - \lambda)\theta_t(x) \geq 0,$$

and we have potentially rechosen $C_2 = C_2(M, N, g, \lambda, W)$. Therefore, for $r < 4\varepsilon\Lambda$,

$$II^{r,g_1}_\varepsilon = C_2\left(r\int_{B_{2l}} q^1_\varepsilon(x)\, d\mathcal{H}^n(x) + \int_0^r \int_{\tilde{M}} \Theta^1_{\varepsilon,g}(s,x) - \Theta^2_{\varepsilon,g}\, d\mathcal{H}^n(x)\, ds + \varepsilon^2\right),$$

and for $r \geq 4\varepsilon\Lambda$,

$$\begin{aligned} II^{r,g_1}_\varepsilon \ &\leq\ -2\sigma\mathcal{H}^n(\{x\colon x \in B_l,\ -r + 2\varepsilon\Lambda \leq \sigma^-(x) \leq 0\}) \\ &\quad +C_2\Bigg(\mathcal{H}^n(\{x\colon x \in B_l, -2\varepsilon\Lambda < \sigma^-(x) \leq 0\}) \\ &\quad +\int_{B_{2l}} q^1_\varepsilon(x)\, d\mathcal{H}^n(x) + \int_0^r \int_{\tilde{M}} \Theta^1_{\varepsilon,g}(s,x) - \Theta^2_{\varepsilon,g}\, d\mathcal{H}^n(x)\, ds + \varepsilon^2\Bigg). \end{aligned}$$

Again we have potentially rechoosing $C2 = C_2(M, N, g, \lambda, W)$.

We now turn our attention to the term,

$$\begin{aligned} I^{r,g_1}_\varepsilon \ &=\ \int_{\tilde{M}}\int_{\sigma^-(x)}^{\sigma^+(x)} \frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t - rf(x)))^2|r\nabla f(x)|^2_{(x,t)}\, \theta_t(x)\, dt\, d\mathcal{H}^n(x), \\ &=\ \int_{\tilde{M}}\int_{-2\varepsilon\Lambda}^{2\varepsilon\Lambda} \frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(\xi))^2|r\nabla f(x)|^2_{(x,\xi+rf(x))}\, \theta_{\xi+rf(x)}(x)\, d\xi\, d\mathcal{H}^n(x), \end{aligned}$$

with,

$$|\nabla f(x)|^2_{(x,t)} = g_{\exp_{\iota(x)}(t\nu(x))}(d\exp_{\iota(x)}(t\nu(x))(\iota_*(\nabla f(x))), d\exp_{\iota(x)}(t\nu(x))(\iota_*(\nabla f(x))))$$

Remark 17. *We may choose $\delta = \delta(z_0, N, g) > 0$, such that $B^N_{2\delta}(z_0)$ is a totally normal neighbourhood, and the following holds*

$$C_3 = C_3(z_0, N, g, \delta) = \sup\{|d\exp_y(X)|^2\colon y \in B^N_\delta(z_0),\ X \in B^{T_yN}_{2\delta}(0)\} \leq 100n^2.$$

By choices of ρ in Remark 11, and ε in Remark 15, for all x in A_l, r in $[0, \rho]$, and ξ in $[-2\varepsilon\Lambda, 2\varepsilon\Lambda]$,

$$|r\nabla f(x)|^2_{(x,\xi+rf(x))} \leq C_3|r^2\nabla f(x)|^2_{(x,0)} \leq 2C_3\frac{r^2}{l^2}.$$

Note that for x in $\tilde{M} \setminus A_l$, $\nabla f(x) = 0$, therefore, $|\nabla f(x)|_{(x,t)} = 0$, for all t. We have,

$$I^{r,g_1}_\varepsilon \leq C_3\mathcal{H}^n(A_l)\frac{r^2}{l^2},$$

where we have potentially rechosen $C_3 = C_3(z_0, N, M, g, \delta, \lambda, W)$.

For r in $[0, 4\varepsilon\Lambda)$, we have,

$$\begin{aligned} I^{r,g}_\varepsilon + II^{r,g}_\varepsilon \ &\leq\ C_3\frac{(\varepsilon\Lambda)^2}{l^2} + C_2\Bigg(\varepsilon\Lambda\int_{B_{2l}} q^1_\varepsilon(x)\, d\mathcal{H}^n(x) \\ &\quad + \int_0^r \int_{\tilde{M}} \Theta^1_{\varepsilon,g}(s,x) - \Theta^2_{\varepsilon,g}\, d\mathcal{H}^n(x)\, ds + \varepsilon^2\Bigg). \end{aligned}$$

Again, we are potentially rechoosing $C_3 = C_3(z_0, N, M, g, \delta) < +\infty$.

For r in $[4\varepsilon\Lambda, \rho]$ we define the following non-decreasing function,

$$P_\varepsilon(r) := \frac{\mathcal{H}^n(\{x \colon x \in B_l,\, -r + 2\varepsilon\Lambda \leq \sigma^-(x) \leq 0\})}{\mathcal{H}^n(A_l)},$$

and we have,

$$\begin{aligned} I_\varepsilon^{r,g_1} + II_\varepsilon^{r,g_1} &\leq \mathcal{H}^n(A_l)\left(C_3\frac{r^2}{l^2} - 2\sigma P_\varepsilon(r)\right) \\ &\quad + C_2\Bigg(\mathcal{H}^n(\{x \colon x \in B_l, -2\varepsilon\Lambda < \sigma^-(x) \leq 0\}) + \int_{B_{2l}} q_\varepsilon^1(x)\, d\mathcal{H}^n(x) \\ &\quad + \int_0^r \int_{\tilde{M}} \Theta_{\varepsilon,g}^1(s,x) - \Theta_{\varepsilon,g}^2\, d\mathcal{H}^n(x)\, ds + \varepsilon^2\Bigg). \end{aligned}$$

We now define the following function on $[0,1]$,

$$\kappa_\varepsilon(s) = \begin{cases} 0, & s \in [0, (4\varepsilon\Lambda)/\rho), \\ C_2\frac{\rho^2}{l^2}s^2 - 2\sigma P_\varepsilon(s\rho), & s \in [(4\varepsilon\Lambda)/\rho, 1]. \end{cases}$$

Note that,

$$P_\varepsilon(\rho) \xrightarrow{\varepsilon\to 0} \frac{\mathcal{H}^n(B_l)}{\mathcal{H}^n(A_l)} \xrightarrow{\rho\to 0} \frac{1}{2^n - 1},$$

and furthermore, recalling the bound $\rho \leq C_1 l^2$, $C_1 = C_1(z_0, N, M, g, \delta) < +\infty$, we have,

$$0 < \frac{\rho^2}{l^2} \leq C_1\rho \xrightarrow{\rho\to 0} 0.$$

Remark 18. *Choose $\rho = \rho(z_0, N, M, g, \delta, \lambda, W) > 0$, such that*

$$C_3\frac{\rho^2}{l^2} < \frac{\sigma}{2(2^n - 1)},$$

and,

$$\frac{\mathcal{H}^n(B_l)}{\mathcal{H}^n(A_l)} > \frac{7}{8(2^n - 1)}.$$

Remark 19. *(Choice of ε_2) There exists an $\varepsilon_2 = \varepsilon_2(z_0, M, N, g, \delta, W, \lambda, \rho) > 0$, such that, $\varepsilon_2 \leq \varepsilon_1$, and for all ε in $(0, \varepsilon_2)$,*

$$P_\varepsilon(\rho) > \frac{3}{4(2^n - 1)}.$$

From here we always choose ε in $(0, \varepsilon_2)$.

We have that,

$$\max_{s\in[0,1]} \kappa_\varepsilon(s) \le C_2\frac{\rho^2}{l^2} < \frac{\sigma}{2(2^n-1)},$$

and,

$$\kappa_\varepsilon(1) < -\frac{\sigma}{2^n-1}.$$

We have, for r in $[0, 4\varepsilon\Lambda)$,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g_1}) \le \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) + III_\varepsilon^{1,r},$$

where,

$$III_\varepsilon^{1,r} = C_4\left(\varepsilon\Lambda\int_{B_{2l}} q_\varepsilon^1(x)\,d\mathcal{H}^n(x) + \int_0^r\int_{\tilde{M}} \Theta^1_{\varepsilon,g}(s,x) - \Theta^2_{\varepsilon,g}\,d\mathcal{H}^n(x)\,ds + (\varepsilon\Lambda)^2\right),$$

and $C_4 = C_4(z_0, M, N, g, \delta, W, \lambda, \rho) < +\infty$.

For r in $[4\varepsilon\Lambda, \rho]$,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g_1}) \le \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) + \mathcal{H}^n(A_l)\kappa_\varepsilon\left(\frac{r}{\rho}\right) + III_\varepsilon^{2,r},$$

where,

$$\begin{aligned} III_\varepsilon^{2,r} &= C_2\bigg(\mathcal{H}^n(\{x\colon x\in B_l, -2\varepsilon\Lambda < \sigma^-(x) \le 0\}) + \int_{B_{2l}} q_\varepsilon^1(x)\,d\mathcal{H}^n(x) \\ &\quad + \int_0^r\int_{\tilde{M}} \Theta^1_{\varepsilon,g}(s,x) - \Theta^2_{\varepsilon,g}\,d\mathcal{H}^n(x)\,ds + \varepsilon^2\bigg). \end{aligned}$$

Furthermore,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\rho,g_1}) \le \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\sigma\mathcal{H}^n(A_l)}{2^n-1} + III_\varepsilon^{2,\rho}.$$

5.2 Appropriate Function on Manifold

We wish to show that for every r in $[0,\rho]$, there exists an $\tilde{v}_\varepsilon^{r,g_1}$, in $W^{1,\infty}(N) \subset W^{1,2}(N)$, such that, for every (x,t) in $\tilde{T}$,

$$\tilde{v}_\varepsilon^{r,g_1}(F(x,t)) = v_\varepsilon^{r,g_1}(x,t).$$

This implies that $\mathcal{F}_{\varepsilon,\lambda}(\tilde{v}_\varepsilon^{r,g_1})(N) = \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g_1})(\tilde{T})$. Indeed, this follows from the fact that $\mathcal{H}^{n+1}(N \setminus F(\tilde{T})) = \mathcal{H}^{n+1}(\mathrm{Cut}(M) \cup (\overline{M}\setminus M)) = 0$, and $F\colon \tilde{T} \to F(\tilde{T})$ is a bijection between open sets, Remark 2,

$$\begin{aligned} \mathcal{F}_{\varepsilon,\lambda}(\tilde{v}_\varepsilon^{r,g_1})(N) &= \mathcal{F}_{\varepsilon,\lambda}(\tilde{v}_\varepsilon^{r,g_1})(N\setminus(\mathrm{Cut}(M)\cup(\overline{M}\setminus M))), \\ &= \mathcal{F}_{\varepsilon,\lambda}(\tilde{v}_\varepsilon^{r,g_1})(F(\tilde{T})), \\ &= \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g_1})(\tilde{T}). \end{aligned}$$

We have the following,

$$B_\delta^N(z_0) = \Upsilon_1 \sqcup A \sqcup \Upsilon_2,$$

where,

$$\Upsilon_1 = \{y \in B_\delta^N(z_0)\colon \tilde{d}_1(y) > \tilde{d}_2(y)\},$$

and,

$$\Upsilon_2 = \{y \in B_\delta^N(z_0)\colon \tilde{d}_2(y) > \tilde{d}_1(y)\}.$$

Recall Remark 7,

$$A = \{y \in B_\delta^N(z_0)\colon \tilde{d}_1(y) = \tilde{d}_2(y)\},$$

is a smooth n-submanifold in $B_\delta^N(z_0)$. Recall the diffeomorphisms, for $i = 1, 2$, defined in Remark 6,

$$F_i\colon \tilde{V}_i \subset \tilde{M} \times \mathbb{R} \to B_{2\delta}^N(z_0).$$

We then define, $\tilde{v}_\varepsilon^{r,g_1}$,

$$\tilde{v}_\varepsilon^{r,g_1}(y) = \begin{cases} \overline{\mathbb{H}}_\varepsilon(\tilde{d}(y)), & y \notin B_\delta^N(z_0), \\ v_\varepsilon^{r,g_1}(F_1^{-1}(y)), & y \in \overline{\Upsilon_1} \cap B_\delta^N(z_0), \\ v_\varepsilon^{r,g_1}(F_2^{-1}(y)), & y \in \overline{\Upsilon_2} \cap B_\delta^N(z_0). \end{cases}$$

For (x,t) in $\tilde{T}$, we have $\tilde{v}_\varepsilon^{r,g_1}(F(x,t)) = v_\varepsilon^{r,g_1}(x,t)$. Indeed, first we consider the case that $F(x,t)$ lies in $\Upsilon_1 \cup \Upsilon_2$. In Υ_i, $F = F_i$, and we have,

$$\tilde{v}_\varepsilon^{r,g_1}(F(x,t)) = v_\varepsilon^{r,g_1}(F_i^{-1}(F(x,t))) = v_\varepsilon^{r,g_1}(x,t).$$

As $A \subset \mathrm{Cut}(M)$, we know that $F(x,t)$ cannot lie on A. Last case to consider case is $F(x,t)$ lies in $N \setminus B_\delta^N(z_0)$. By Remark 11 (x,t) must lie in $\tilde{T} \setminus (B_{2l} \times (-2\rho, 2\rho))$. If x lies in $\tilde{M} \setminus B_{2l}$, then $f(x) = 0$, and,

$$v_\varepsilon^{r,g_1}(x,t) = \overline{\mathbb{H}}_\varepsilon(t) = \overline{\mathbb{H}}_\varepsilon(\tilde{d}(F(x,t))) = \tilde{v}_\varepsilon^{r,g_1}(F(x,t)).$$

If x lies in B_{2l}, then $|t| \geq 2\rho > r|f(x)| + 2\varepsilon\Lambda$, and therefore,

$$v_\varepsilon^{r,g_1}(x,t) = \overline{\mathbb{H}}_\varepsilon(t - rf(x)) = \begin{cases} 1, & t \geq 2\rho > rf(x) + 2\varepsilon\Lambda, \\ -1, & t \leq -2\rho < rf(x) - 2\varepsilon\Lambda. \end{cases}$$

Also, $\tilde{d}(F(x,t)) = t$, implies that,

$$\tilde{v}_\varepsilon^{r,g_1}(F(x,t)) = \overline{\mathbb{H}}_\varepsilon(t) = \begin{cases} 1, & t \geq 2\rho > 2\varepsilon\Lambda, \\ -1, & t \leq -2\rho < -2\varepsilon\Lambda. \end{cases}$$

Therefore, for all (x,t) in $\tilde{T}$, we have that $v_\varepsilon^{r,g_1}(x,t) = \tilde{v}_\varepsilon^{r,g_1}(F(x,t))$.

We now just look to show that $\tilde{v}_\varepsilon^{r,g_1}$ lies in $W^{1,\infty}(N)$. First consider y in $N \setminus F(B_{2l} \times (-2\rho, 2\rho))$. There exists an x in $\tilde{M}$, such that, $F(x, \tilde{d}(y)) = y$, and $(x, \tilde{d}(y))$ lies in $(\tilde{M} \times \mathbb{R}) \setminus (B_{2l} \times (-2\rho, 2\rho))$. By previous argument we see that,

$$\tilde{v}_\varepsilon^{r,g_1}(y) = \overline{\mathbb{H}}_\varepsilon(\tilde{d}(y)).$$

and therefore, $\tilde{v}_\varepsilon^{r,g_1}$ is Lipschitz on the set $N \setminus F(B_{2l} \times (-2\rho, 2\rho))$.

As

$$F(B_{2l} \times (-2\rho, 2\rho)) \subset\subset B_\delta^N(z_0),$$

showing that $\tilde{v}_\varepsilon^{r,g_1}$ is Lipschitz on $B_\delta^N(z_0)$, implies that it is Lipschitz on N.

Consider y on A, then $\tilde{d}_1(y) = \tilde{d}_2(y)$, and by construction of f and Ψ,

$$f(\pi(F_1^{-1}(y))) = f(\pi(F_2^{-1}(y))).$$

Therefore,

$$v_\varepsilon^{r,g_1}(F_1^{-1}(y)) = v_\varepsilon^{r,g_1}(F_2^{-1}(y)),$$

and $\tilde{v}_\varepsilon^{r,g_1}$ is well defined and continuous across the smooth n-submanifold A. Thus we have that $\tilde{v}_\varepsilon^{r,g_1}$ lies in $W^{1,\infty}(B_\delta^N(z_0))$.

5.3 Continuity of the Path

We show that the path,

$$\begin{array}{rcl} \gamma\colon [0,\rho] & \to & W^{1,2}(N), \\ r & \mapsto & \tilde{v}_\varepsilon^{r,g_1}, \end{array}$$

is continuous in $W^{1,2}(N)$.

Take r and s in $[0,\rho]$. Recalling that $\mathcal{H}^{n+1}(N \setminus F(\tilde{T})) = \mathcal{H}^{n+1}(\mathrm{Cut}(M) \cup (\overline{M} \setminus M)) = 0$,

$$\begin{array}{rcl} \|\tilde{v}_\varepsilon^{r,g_1} - \tilde{v}_\varepsilon^{s,g_1}\|_{L^2(N)}^2 & = & \displaystyle\int_{F(\tilde{T})} |\tilde{v}_\varepsilon^{r,g_1} - \tilde{v}_\varepsilon^{s,g_1}|^2, \\ & = & \displaystyle\int_{\mathbb{R}}\int_{\tilde{M}} |\overline{\mathbb{H}}_\varepsilon(t - rf(x)) - \overline{\mathbb{H}}_\varepsilon(t - sf(x))|^2 \theta_t(x)\, d\mathcal{H}^n(x)\, dt, \\ & \xrightarrow{s\to r} & 0, \end{array}$$

by Dominated Convergence Theorem.

Noting that, $\tilde{v}_\varepsilon^{r,g_1} = \tilde{v}_\varepsilon^{r,g_1}$ on $N \setminus B_\delta^N(z_0)$, for all r in $[0,\rho]$, and $\mathcal{H}^{n+1}(B_\delta^N(z_0) \setminus (\Upsilon_1 \cup \Upsilon_2)) = \mathcal{H}^{n+1}(A) = 0$,

$$\|\nabla\tilde{v}_\varepsilon^{r,g_1} - \nabla\tilde{v}_\varepsilon^{s,g_1}\|_{L^2(N)}^2 = \int_{\Upsilon_1\cup\Upsilon_2} |\nabla\tilde{v}_\varepsilon^{r,g_1} - \nabla\tilde{v}_\varepsilon^{s,g_1}|\, d\mu_g.$$

As $F_i^{-1}\colon (\Upsilon_i, g) \to (F_i^{-1}(\Upsilon_i), h)$ is an isometry, we have,

$$\begin{array}{rcl} \|\nabla\tilde{v}_\varepsilon^{r,g_1} - \nabla\tilde{v}_\varepsilon^{s,g_1}\|_{L^2(N)}^2 & = & \displaystyle\int_{F_1^{-1}(\Upsilon_1)\cup F_2^{-1}(\Upsilon_2)} |\nabla v_\varepsilon^{r,g_1}(x,t) - \nabla v_\varepsilon^{s,g_1}(x,t)|^2, \\ & = & \displaystyle\int_{F_1^{-1}(\Upsilon_1)\cup F_2^{-1}(\Upsilon_2)} (\overline{\mathbb{H}}_\varepsilon'(t - rf(x)) - \overline{\mathbb{H}}_\varepsilon'(t - sf(x)))^2 \\ & & \qquad + |\nabla_x f(x)|^2 (r\overline{\mathbb{H}}_\varepsilon'(t - rf(x)) - s\overline{\mathbb{H}}_\varepsilon'(t - sf(x)))^2, \\ & \xrightarrow{s\to r} & 0, \end{array}$$

by Dominated Convergence Theorem.

6 Path to a_ε

6.1 Fixed Energy Gain Away from Non-Embedded Point

We construct the path from (2) to (3) in Figure 2.

Recall $\tilde{f}$ from Section 3.5 and set,

$$g_2(r,x) = \rho f(x) + r\tilde{f}(x),$$

for r in $[0, r_0]$, where $r_0 \in (0, \min\{1, \operatorname{diam}(N)/2\})$, will be chosen later. Denote, $A_L^k = B_L \setminus B_{\frac{L}{k}}$.

Remark 20. *(Choice of ε_3) We choose $0 < \varepsilon_3 \leq \varepsilon_2$, such that $2\varepsilon_3\Lambda = 6\varepsilon_3|\log \varepsilon_3| << r_0$. From here on we consider ε on $(0, \varepsilon_3)$.*

We slightly edit the Base Computation in Section 4. Consider $r > 2\varepsilon\Lambda$,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g_2}) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g_2}) = I_\varepsilon^{r,g_2} + (II_\varepsilon^{r,g_2} - II_\varepsilon^{2\varepsilon\Lambda,g_2}) + II_\varepsilon^{2\varepsilon\Lambda,g_2}.$$

We have,

$$\begin{aligned} II_\varepsilon^{r,g_2} - II_\varepsilon^{2\varepsilon\Lambda,g_2} = &- \int_{2\varepsilon\Lambda}^r \int_{\tilde{M}\setminus B_{\frac{L}{k}}} \tilde{f}(x) Q_\varepsilon(\sigma^+(x) - s\tilde{f}(x))\, \theta^+(x)\, d\mathcal{H}^n(x)\, ds \\ &+ \int_{2\varepsilon\Lambda}^r \int_{\tilde{M}\setminus B_{\frac{L}{k}}} \tilde{f}(x) Q_\varepsilon(\sigma^-(x) - s\tilde{f}(x))\, \theta^-(x)\, d\mathcal{H}^n(x)\, ds \\ &+ \int_{2\varepsilon\Lambda}^r \int_{\tilde{M}\setminus B_{\frac{L}{k}}} \tilde{f}(x) \int_{\sigma^-(x)}^{\sigma^+(x)} Q_\varepsilon(t - s\tilde{f}(x))\, (\lambda - H_t(x))\, \theta_t(x)\, dt\, d\mathcal{H}^n(x)\, ds \\ &+ \lambda \int_{2\varepsilon\Lambda}^r \int_{\tilde{M}} \Theta^1_{\varepsilon,g_2}(s,x) - \Theta^2_{\varepsilon,g_2}(s,x)\, d\mathcal{H}^n(x)\, ds. \end{aligned} \tag{12}$$

Considering the first term on the right hand side of (12),

$$- \int_{2\varepsilon\Lambda}^r \int_{\tilde{M}\setminus B_{\frac{L}{k}}} \tilde{f}(x) Q_\varepsilon(\sigma^+(x) - s\tilde{f}(x))\, \theta^+(x)\, d\mathcal{H}^n(x)\, ds \leq 0.$$

Considering the second term on the right hand side of (12), and by applying similar arguments for when we considered the corresponding term on the right hand side of (11) in Section 5.1,

$$\begin{aligned} &\int_{2\varepsilon\Lambda}^r \int_{\tilde{M}\setminus B_{\frac{L}{k}}} \tilde{f}(x) Q_\varepsilon(\sigma^-(x) - s\tilde{f}(x))\, \theta^-(x)\, d\mathcal{H}^n(x)\, ds \\ &\qquad\qquad \leq C_2 \mathcal{H}^n(\{x \colon x \in \tilde{M} \setminus B_{\frac{L}{k}},\ \sigma^-(x) \geq 2\varepsilon\Lambda(\tilde{f}(x) - 1)\}), \end{aligned}$$

where we are potentially rechoosing $C_2 = C_2(M, N, g, W, \lambda) < +\infty$.

Considering the third term on the right hand side of (12). Applying similar arguments in A_L^k from when we considered the corresponding term on the right hand side of (11) in Section 5.1,

$$\begin{aligned}\int_{2\varepsilon\Lambda}^{r}\int_{\tilde{M}\setminus B_{\frac{L}{k}}}\tilde{f}(x)\int_{\sigma^-(x)}^{\sigma^+(x)} Q_\varepsilon(t-s\tilde{f}(x))\,(\lambda-H_t(x))\,\theta_t(x)\,dt\,d\mathcal{H}^n(x)\,ds\\ \leq \int_{2\varepsilon\Lambda}^{r}\int_{\tilde{M}\setminus B_L}\int_{-2\varepsilon\Lambda}^{2\varepsilon\Lambda} Q_\varepsilon(\xi)(\lambda-H_{\xi+s}(x))\theta_{\xi+s}(x)\,d\xi\,d\mathcal{H}^n(x)\,ds\\ +C_2\int_{A_L^k} q_\varepsilon^2(x)\,d\mathcal{H}^n(x),\end{aligned}\tag{13}$$

where, $q_\varepsilon^2(x) := \max_{t\in[-2\varepsilon\Lambda,4\varepsilon\Lambda]}(\lambda-H_t(x))\theta_t(x)$, and we are potentially rechoosing $C_2 = C_2(M,N,g,\lambda,W)$.

Define the following measurable set,

$$\Omega_r = \{x\in\tilde{M}:\ \sigma^+(x) > 2r\}.$$

Remark 21. *We choose $L = L(z_0,N,M,g,\delta) > 0$, such that,*

$$\mathcal{H}^n(\tilde{M}\setminus B_L) > \frac{3}{4}\mathcal{H}^n(\tilde{M}).$$

Then we can find an $r_0 = r_0(z_0,\tilde{M},N,g,\delta,L) > 0$, such that, for all r in $[0,r_0]$,

$$\mathcal{H}^n(\{(x,2r):\ x\in\Omega_r\setminus B_L\}) > \frac{1}{2}\mathcal{H}^n(\tilde{M}).$$

For all x in Ω_r, s in $(2\varepsilon\Lambda,r)$, and ξ in $[-2\varepsilon\Lambda,2\varepsilon\Lambda]$, $s+\xi$ lies in $(0,\sigma^+(x))$. Therefore, recalling bounds on H_t and θ_t from Remark 4 and Claim 6, we have,

$$(\lambda-H_{\xi+s}(x))\theta_{\xi+s}(x) < -m(s+\xi)\theta_{\xi+s} \leq -m(s-2\varepsilon\Lambda)\theta_{2r}(x),$$

Then for r in $(2\varepsilon\Lambda,r_0]$, we compute an energy decrease from the first term on the right hand side of (13),

$$\int_{2\varepsilon\Lambda}^{r}\int_{\tilde{M}\setminus B_L}\int_{-2\varepsilon\Lambda}^{2\varepsilon\Lambda} Q_\varepsilon(\xi)(\lambda-H_{\xi+s}(x))\theta_{\xi+s}(x)\,d\xi\,d\mathcal{H}^n(x)\,ds \leq -\frac{m\sigma}{2}\mathcal{H}^n(\tilde{M})r^2 + C_2\varepsilon\Lambda,$$

potentially rechoosing $C_2 = C_2(N,M,g,\lambda,W) < +\infty$.

For r in $[0,2\varepsilon\Lambda]$, by repeating of arguments similar to those in Section 3.6, yields,

$$II_\varepsilon^{r,g_2} \;\leq\; C_2\Big(\int_{\tilde{M}\setminus B_{\frac{L}{k}}} m_\varepsilon^1(x)\,d\mathcal{H}^n(x) + \varepsilon\Lambda\Big),$$

where we are potentially rechoosing $C_2 = C_2(N,M,g,W,\lambda)$, and $m_\varepsilon^1(x) = \max_{t\in[-2\varepsilon\Lambda,4\varepsilon\Lambda]}\theta_t(x) - \min_{t\in[-2\varepsilon\Lambda,4\varepsilon\Lambda]}\theta_t(x)$.

For r in $[0, r_0]$, consider the term,

$$I_\varepsilon^{r,g_2} = \int_{A_L^k} \int_{-2\varepsilon\Lambda}^{2\varepsilon\Lambda} \frac{\varepsilon}{2}((\overline{\mathbb{H}})'(\xi))^2 |r\nabla \tilde{f}(x)|^2_{(x, r\tilde{f}(x)+\xi)} \theta_{r\tilde{f}(x)+\xi}(x)\, d\xi\, d\mathcal{H}^n(x).$$

By choice of L and r_0 in Remark 12, and constant $C_3 = C_3(z_0, M, N, g, \delta, \lambda, W)$ from Remark 17, we have, for all x in A_L^k, r in $[0, r_0]$, and ξ in $[-2\varepsilon\Lambda, 2\varepsilon\Lambda]$,

$$|\nabla \tilde{f}(x)|_{(x, r\tilde{f}(x)+\xi)} \leq C_3^{1/2} |\nabla \tilde{f}(x)|_{(x,0)}$$

Thus we have,

$$I_\varepsilon^{r,g_2} \leq C_3 \|\nabla \tilde{f}\|^2_{L^2(\tilde{M})} r^2.$$

Again we are potentially rechoosing $C_3 = C_3(z_0, M, N, g, \delta, W, \lambda)$.

Remark 22. *Choose $k = k(z_0, M, N, g, \delta, W, \lambda, L)$ such that*

$$\|\nabla \tilde{f}\|^2_{L^2(\tilde{M})} < C_3^{-1} \frac{m\sigma}{4} \mathcal{H}^n(\tilde{M}).$$

Therefore, after potentially rechoosing $C_3 = C_3(z_0, M, N, g, \delta, W, \lambda)$,

$$I_\varepsilon^{r,g_2} \leq \frac{m\sigma}{4} \mathcal{H}^n(\tilde{M}) r^2.$$

For r in $(0, 2\varepsilon\Lambda]$,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g_2}) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{0,g_2}) \leq III_\varepsilon^{3,r},$$

where,

$$III_\varepsilon^{3,r} = C_2 \left(\int_{\tilde{M} \setminus B_{\frac{L}{k}}} m_\varepsilon(x)\, d\mathcal{H}^n(x) + \varepsilon\Lambda \right),$$

where we are potentially rechoosing $C_2 = C_2(N, M, g, W, \lambda) < +\infty$. For r in $(2\varepsilon\Lambda, r_0]$,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g_2}) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{0,g_2}) \leq -\frac{m\sigma}{4} \mathcal{H}^n(\tilde{M}) r^2 + III_\varepsilon^{4,r}$$

where,

$$\begin{aligned} III_\varepsilon^{4,r} = C_2 \Big(& \mathcal{H}^n(\{x \colon x \in \tilde{M} \setminus B_{\frac{L}{k}},\, \sigma^-(x) \geq 2\varepsilon\Lambda(\tilde{f}(x) - 1)\}) \\ & + \int_{A_L^k} q_\varepsilon^2(x)\, d\mathcal{H}^n(x) + \int_{2\varepsilon\Lambda}^{r} \int_{\tilde{M}} \Theta^1_{\varepsilon,g_2}(s,x) - \Theta^2_{\varepsilon,g_2}\, d\mathcal{H}^n(x)\, ds \\ & + \int_{\tilde{M} \setminus B_{\frac{L}{k}}} m_\varepsilon(x)\, d\mathcal{H}^n(x) + \varepsilon\Lambda \Big), \end{aligned}$$

again, we are potentially rechoosing $C_2 = C_2(M, N, g, W, \lambda)$.

As $g_2(0,x) = g_1(\rho,x)$, we have, for r in $(0, 2\varepsilon\Lambda]$,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g_2}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\sigma \mathcal{H}^n(A_l)}{2^n - 1} + III_\varepsilon^{\rho,2} + III_\varepsilon^{r,3},$$

and for r in $(2\varepsilon\Lambda, r_0]$,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g_2}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\sigma \mathcal{H}^n(A_l)}{2^n - 1} - \frac{m\sigma}{4}\mathcal{H}^n(\tilde{M})r^2 + III_\varepsilon^{\rho,2} + III_\varepsilon^{r,4}.$$

We may define the appropriate function on N, for r in $[0, r_0]$,

$$\tilde{v}_\varepsilon^{r,g_2}(y) = \begin{cases} \overline{\mathbb{H}}_\varepsilon(\tilde{d}(y) - r), & y \notin B_\delta^N(z_0), \\ v_\varepsilon^{r,g_2}(F_1^{-1}(y)), & y \in \overline{\Upsilon_1} \cap B_\delta^N(z_0), \\ v_\varepsilon^{r,g_2}(F_2^{-1}(y)), & y \in \overline{\Upsilon_2} \cap B_\delta^N(z_0). \end{cases}$$

Following similar arguments to Sections 5.2, and 5.3, we may show that $\tilde{v}_\varepsilon^{r,g_2}$ lies in $W^{1,\infty}(N)$, and that the path $\tilde{v}_\varepsilon^{0,g_2} \to \tilde{v}_\varepsilon^{r_0,g_2}$ is continuous in $W^{1,2}(N)$.

6.2 Reversing Construction of Competitor

We construct the path from (3) to (4) in Figure 2.

For r in $[0, \rho]$, we set,

$$g_3(r,x) = r_0\tilde{f}(x) + (\rho - r)f(x).$$

For x in B_{2l},

$$g_3(r,x) = (\rho - r)f(x) = g_1(\rho - r, x),$$

and for x in $\tilde{M} \setminus B_{2l}$,

$$g_3(r,x) = r_0\tilde{f}(x) = g_3(0,x).$$

Therefore,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g_3}) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{0,g_3}) = \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\rho-r,g_1}) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\rho,g_1}).$$

As $g_1(\rho,x) = g_2(0,x)$, and $g_3(0,x) = g_2(r_0,x)$, we have,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g_3}) = \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\rho-r,g_1}) + \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r_0,g_2}) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{0,g_2}).$$

Remark 23. *(Choice of $\rho = \rho(z_0, M, N, g, \delta, W, \lambda, L, r_0)$) We choose $\rho > 0$, such that,*

$$\frac{\sigma \mathcal{H}^n(A_l)}{2^n - 1} < \frac{m\sigma}{4}\mathcal{H}^n(\tilde{M})r_0^2.$$

Therefore, we have that

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g_3}) < \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{\rho-r,g_1}) - \frac{\sigma \mathcal{H}^n(A_l)}{2^n - 1} + III_\varepsilon^{4,r_0}.$$

Furthermore, for r in $[0, \rho - 4\varepsilon\Lambda]$, we have,

$$\begin{aligned}\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g_3}) &< \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) + \frac{\sigma\mathcal{H}^n(A_l)}{2(2^n-1)} + III_\varepsilon^{\rho-r,2} - \frac{\sigma\mathcal{H}^n(A_l)}{2^n-1} + III_\varepsilon^{4,r_0}, \\ &= \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\sigma\mathcal{H}^n(A_l)}{2(2^n-1)} + III_\varepsilon^{\rho-r,2} + III_\varepsilon^{4,r_0}.\end{aligned}$$

For r in $(\rho - 4\varepsilon\Lambda, \rho]$, we similarly have,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g_3}) < \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\sigma\mathcal{H}^n(A_l)}{2(2^n-1)} + III_\varepsilon^{\rho-r,1} + III_\varepsilon^{4,r_0}.$$

We define the appropriate function on N. For r in $[0, \rho]$,

$$\tilde{v}_\varepsilon^{r,g_3}(y) = \begin{cases} \overline{\mathbb{H}}_\varepsilon(\tilde{d}(y) - r_0), & y \notin B_\delta^N(z_0), \\ v_\varepsilon^{r,g_3}(F_1^{-1}(y)), & y \in \overline{\Omega_1} \cap B_\delta^N(z_0), \\ v_\varepsilon^{r,g_3}(F_2^{-1}(y)), & y \in \overline{\Omega_2} \cap B_\delta^N(z_0). \end{cases}$$

Following similar arguments to Sections 5.2, and 5.3, we may show that $\tilde{v}_\varepsilon^{r,g_3}$ lies in $W^{1,\infty}(N)$, and that the path $\tilde{v}_\varepsilon^{0,g_3} \to \tilde{v}_\varepsilon^{\rho,g_3}$ is continuous in $W^{1,2}(N)$.

6.3 Lining Up With Level Set Γ_{r_0}

We construct path from (4) to (5) in Figure 2

For r in $[0, r_0]$, consider,

$$g_4(r,x) = r_0\tilde{f}(x) + r(1 - \tilde{f}(x)) = (r_0 - r)\tilde{f}(x) + r \geq r.$$

By applying similar arguments to those in Section 5.1, we have

$$\begin{aligned} II_\varepsilon^{r,g_4} \;\leq\; C_3\Big(&\mathcal{H}^n(\{x \in B_L \colon \sigma^-(x) \geq -2\varepsilon\Lambda\}) + \varepsilon\Lambda \\ &+ \int_0^r \int_{\tilde{M}} \Theta^1_{\varepsilon,g_4}(s,x) - \Theta^2_{\varepsilon,g_4}(s,x)\, d\mathcal{H}^n(x)\, ds\Big), \end{aligned}$$

where we are potentially rechoosing $C_3 = C_3(z_0, M, N, g, \delta, W, \lambda) < +\infty$.

We turn our attention to the term,

$$\begin{aligned} I_\varepsilon^{r,g_4} \;=\; &\int_{A_L^k}\int_{\sigma^-(x)}^{\sigma^+(x)} \frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t - (r_0 - r)\tilde{f}(x) - r))^2 (r_0 - r)^2 |\nabla_x \tilde{f}(x)|^2\, \theta_t(x)\, dt\, d\mathcal{H}^n(x) \\ &- \int_{A_L^k}\int_{\sigma^-(x)}^{\sigma^+(x)} \frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t - r_0\tilde{f}(x)))^2 r_0^2 |\nabla_x \tilde{f}(x)|^2\, \theta_t(x)\, dt\, d\mathcal{H}^n(x). \end{aligned}$$

For r in $[0,r_0]$,

$$\begin{aligned}
&\int_{\sigma^-(x)}^{\sigma^+(x)} \frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t-(r_0-r)\tilde{f}(x)-r))^2|\nabla_x\tilde{f}(x)|^2_{(x,t)}\,\theta_t(x)\,dt\\
&\qquad\le(\sigma+\beta\varepsilon^2)\max_{t\in[-2\varepsilon\Lambda,2\varepsilon\Lambda]}|\nabla\tilde{f}(x)|^2_{(x,t+g_4(r,x))}\theta_{t+g_4(r,x)}(x),\\
&\qquad=\sigma|\nabla\tilde{f}(x)|^2_{(x,g_4(r,x))}\theta_{g_4(r,x)}(x)+\beta\varepsilon^2\max_{t\in[-2\varepsilon\Lambda,2\varepsilon\Lambda]}|\nabla\tilde{f}(x)|^2_{(x,t+g_4(r,x))}\theta_{t+g_4(r,x)}(x)\\
&\qquad\quad+\sigma\left(\max_{t\in[-2\varepsilon\Lambda,2\varepsilon\Lambda]}|\nabla\tilde{f}(x)|^2_{(x,t+g_4(r,x))}\theta_{t+g_4(r,x)}(x)-|\nabla\tilde{f}(x)|^2_{(x,g_4(r,x))}\theta_{g_4(r,x)}(x)\right)
\end{aligned}$$

Denote the functions,

$$q^3_\varepsilon(x,r)=\max_{t\in[-2\varepsilon\Lambda,2\varepsilon\Lambda]}|\nabla\tilde{f}(x)|^2_{(x,t+g_4(r,x))}\theta_{t+g_4(r,x)}(x)-|\nabla\tilde{f}(x)|^2_{(x,g_4(r,x))}\theta_{g_4(r,x)}(x),$$

and,

$$p^1_\varepsilon(r)=\int_{A^k_L}q^3_\varepsilon(x,r)\,d\mathcal{H}^n(x).$$

We have,

$$\begin{aligned}
&\int_{A^k_L}\int_{\sigma^-(x)}^{\sigma^+(x)}\frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t-(r_0-r)\tilde{f}(x)-r))^2(r_0-r)^2|\nabla_x\tilde{f}(x)|^2\,\theta_t(x)\,dt\,d\mathcal{H}^n(x)\\
&\qquad\le\sigma\int_{A^k_L}(r_0-r)^2|\nabla\tilde{f}(x)|^2_{(x,g_4(r,x))}\theta_{g(r,x)}(x)\,d\mathcal{H}^n(x)+C_5(\varepsilon^2+p^1_\varepsilon(r)).
\end{aligned}$$

where $C_5=C_5(z_0,N,M,g,\delta,L,r_0,k,W,\lambda)<+\infty$. Similarly we have,

$$\begin{aligned}
&\int_{A^k_L}\int_{\sigma^-(x)}^{\sigma^+(x)}\frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t-r_0\tilde{f}(x)))^2r_0^2|\nabla_x\tilde{f}(x)|^2\,\theta_t(x)\,dt\,d\mathcal{H}^n(x)\\
&\qquad\ge\sigma\int_{A^k_L}r_0^2|\nabla\tilde{f}(x)|^2_{(x,g_4(0,x))}\theta_{g_4(0,x)}(x)\,d\mathcal{H}^n(x)-C_5(\varepsilon^2+p^2_\varepsilon(0)),
\end{aligned}$$

where,

$$p^2_\varepsilon(0)=\int_{A^k_L}q^4_\varepsilon(x,0)\,d\mathcal{H}^n(x),$$

and

$$q^4_\varepsilon(x,r)=\min_{t\in[-2\varepsilon\Lambda,2\varepsilon\Lambda]}|\nabla\tilde{f}(x)|^2_{(x,t+g_4(r,x))}\theta_{t+g_4(r,x)}(x)-|\nabla\tilde{f}(x)|^2_{(x,g_4(r,x))}\theta_{g_4(r,x)}(x)\le 0$$

Therefore we have, for r in $[0,r_0]$,

$$\begin{aligned}
I^{r,g_3}_\varepsilon\ \ \le\ \ &\sigma\int_{A^k_L}(r_0-r)^2|\nabla\tilde{f}(x)|^2_{(x,g(r,x))}\theta_{g(r,x)}(x)-r_0^2|\nabla\tilde{f}(x)|^2_{(x,g(0,x))}\theta_{g(0,x)}(x)\,d\mathcal{H}^n(x)\\
&+C_5(\varepsilon^2+p^1_\varepsilon(r)-p^2_\varepsilon(0)).
\end{aligned}$$

Claim 4. *There exists an $r_0>0$, such that for all r in $[0,r_0]$,*

$$\int_{A^k_L}(r_0-r)^2|\nabla\tilde{f}(x)|^2_{(x,g(r,x))}\theta_{g(r,x)}(x)-r_0^2|\nabla\tilde{f}(x)|^2_{(x,g(0,x))}\theta_{g(0,x)}(x)\,d\mathcal{H}^n(x)\le 0.$$

Proof. For $(x,t) \in \tilde{M} \times \mathbb{R}$, denote,

$$\zeta(x,t) := |\nabla \tilde{f}(x)|^2_{(x,t)} \theta_t(x).$$

We note that $\zeta \in C^\infty(\tilde{T})$, and $\zeta(x,t) = 0$, for $(x,t) \notin \tilde{T}$.

For r in $[0, r_0]$, denote,

$$G(r) := \int_{A_L^k} \zeta(x, g_4(r,x))\, d\mathcal{H}^n(x) = \int_{A_L^k \cap \{\tilde{f} \neq 0\}} \zeta(x, g_4(r,x))\, d\mathcal{H}^n(x).$$

For $r \in [0, r_0]$, and $\tilde{f}(x) \neq 0$, we have that,

$$g_4(r,x) \in (\sigma^-(x), \sigma^+(x)),$$

i.e. $(x, g_4(r,x)) \in \tilde{T}$. Therefore, G lies in $C^\infty([0,r_0])$, and

$$G'(r) = \int_{A_L^k \cap \{\tilde{f} \neq 0\}} (\partial_t \zeta)(x, g_4(r,x))(1 - \tilde{f}(x))\, dx.$$

We may obtain a bound $|G'(r)| \leq C_6 = C_6(z_0, M, N, g, \delta, W, \lambda, L, k) < +\infty$, for $r_0 \leq R_0 < +\infty$, $R_0 = R_0(z_0, M, N, g, \delta, W, \lambda, L, k)$ fixed. We also have that $G(0) = \|\nabla \tilde{f}\|^2_{L^2(\tilde{M})} > 0$, and we may choose $r_0 > 0$, small enough such that,

$$\min_{r \in [0, r_0]} G(r) \geq \frac{1}{2} G(0) > 0.$$

Denoting,

$$F(r) := (r_0 - r)^2 G(r).$$

Differentiating we obtain,

$$F'(r) = (r - r_0)(2G(r) + (r - r_0)G'(r)).$$

Thus setting, $r_0 < G(0)/C_6$, we have that $F'(r) \leq 0$, for all r in $[0, r_0]$. This completes the proof. □

Remark 24. *Claim 4 is a further choice of* $r_0 = r_0(z_0, M, N, g, \delta, W, \lambda, L, k) > 0$.

Therefore, there exists an r_0, such that for all r in $[0, r_0]$,

$$\begin{aligned} \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g_4}) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{0,g_4}) &= I_\varepsilon^{r,g_4} + II_\varepsilon^{r,g_4}, \\ &\leq III_\varepsilon^{5,r}, \end{aligned}$$

where

$$\begin{aligned} III_\varepsilon^{5,r} \leq\ & C_5 \Big(p_\varepsilon^1(r) - p_\varepsilon^2(0) + \mathcal{H}^n(\{x \in B_{2L} \colon \sigma^-(x) \geq -2\varepsilon\Lambda\}) + \varepsilon\Lambda \\ & + \lambda \int_0^r \int_{\tilde{M}} \Theta^1_{\varepsilon,g_4}(s,x) - \Theta^2_{\varepsilon,g_4}(s,x)\, d\mathcal{H}^n(x)\, ds \Big), \end{aligned}$$

where we are potentially rechoosing $C_5 = C_5(z_0, M, N, g, k, L, \delta, r_0, W, \lambda)$.

As $g_4(0, x) = g_3(\rho, x)$, we have, for r in $[0, r_0]$,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g_4}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\sigma\mathcal{H}^n(x)}{2(2^n-1)} + III_\varepsilon^{1,0} + III_\varepsilon^{4,r_0} + III_\varepsilon^{5,r}.$$

Consider the following function on N, for r in $[0, r_0]$,

$$\tilde{v}_\varepsilon^{r,g_4}(y) = \begin{cases} \overline{\mathbb{H}}_\varepsilon(\tilde{d}(y) - r_0), & y \notin B_\delta^N(z_0), \\ v_\varepsilon^{r,g_4}(F_1^{-1}(y)), & y \in \overline{\Upsilon_1} \cap B_\delta^N(z_0), \\ v_\varepsilon^{r,g_4}(F_2^{-1}(y)), & y \in \overline{\Upsilon_2} \cap B_\delta^N(z_0). \end{cases}$$

We can show, as in Section 5.2 and 5.3, that $\tilde{v}_\varepsilon^{r,g_3}$ lies in $W^{1,\infty}(N)$, $\mathcal{F}_{\varepsilon,\lambda}(\tilde{v}_\varepsilon^{r,g_4}) = \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g_4})$, and that, $r \mapsto \tilde{v}_\varepsilon^{r,g_4}$ is a continuous path in $W^{1,2}(N)$.

6.4 Completing Path to a_ε

We construct the path from (5) to '-1' in Figure 2.

Consider, for r in $[r_0, 2\,\mathrm{diam}(N)]$,

$$g_5(r, x) = r.$$

By repeating similar arguments to those in Section 5.1, we have

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g_5}) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r_0,g_5}) \leq III_\varepsilon^{6,r},$$

where

$$III_\varepsilon^{6,r} = \lambda \int_\rho^r \Theta^1_{\varepsilon,g_5}(s,x) - \Theta^2_{\varepsilon,g_5}(s,x)\, d\mathcal{H}^n(x)\, ds.$$

Recalling that, $g_5(r_0, x) = g_4(r_0, x)$, we have, for all r in $[r_0, 2\,\mathrm{diam}(N)]$,

$$\begin{aligned} \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g_5}) &\leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r_0,g_4}) + III_\varepsilon^{6,r}, \\ &< \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\sigma\mathcal{H}^n(A_l)}{2(2^n-1)} + III_\varepsilon^{1,0} + III_\varepsilon^{4,r_0} + III_\varepsilon^{5,r_0} + III_\varepsilon^{6,r}. \end{aligned}$$

Define the function, $\tilde{v}_\varepsilon^{r,g_5}(y) = \overline{\mathbb{H}}_\varepsilon(\tilde{d}(y) - r)$, in N. This function lies in $W^{1,\infty}(N)$. We have that, $\mathcal{F}_{\varepsilon,\lambda}(\tilde{v}_\varepsilon^{r,g_5}) = \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g_5})$, and $r \mapsto \tilde{v}_\varepsilon^{r,g_5}$ is a continuous path in $W^{1,2}(N)$.

As $|\tilde{d}(y)| \leq \mathrm{diam}(N)$, we have that,

$$\tilde{d}(y) - 2\,\mathrm{diam}(N) \leq -\,\mathrm{diam}(N) < -2\varepsilon\Lambda.$$

Therefore,

$$\tilde{v}_\varepsilon^{2\,\mathrm{diam}(N),g_5}(y) = \overline{\mathbb{H}}_\varepsilon(\tilde{d}(x) - 2\,\mathrm{diam}(N)) = -1.$$

Recall that our end point is $a_\varepsilon > -1$. We connect -1 to a_ε, by constant functions,

$$u_\varepsilon^r(y) = r$$

for r in $[-1, a_\varepsilon]$. Then,

$$\mathcal{F}_{\varepsilon,\lambda}(u_\varepsilon^r) = \int_N \frac{W(r)}{\varepsilon} - \sigma\lambda r \, d\mu_g \leq \mathcal{F}_{\varepsilon,\lambda}(u_\varepsilon^{-1}).$$

As $u_\varepsilon^{-1} = \tilde{v}_\varepsilon^{2\,\mathrm{diam}(N), g_5}$, we have that, for all r in $[-1, a_\varepsilon]$,

$$\mathcal{F}_{\varepsilon,\lambda}(u_\varepsilon^r) < \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\sigma\mathcal{H}^n(A_l)}{2(2^n-1)} + III_\varepsilon^{1,0} + III_\varepsilon^{4,r_0} + III_\varepsilon^{5,r_0} + III_\varepsilon^{6,2\,\mathrm{diam}(N)}.$$

7 Path to b_ε

7.1 Lining Up With Level Set $\Gamma_{-\rho}$

We construct the path from (2) to (6) in Figure 2

We consider, for r in $[0, \rho]$, and x in $\tilde{M}$,

$$g_6(r, x) = \rho f(x) - r(1 + f(x)).$$

First consider r, in $(2\varepsilon\Lambda, \rho]$,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g_6}) - \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{0,g_6}) = I_\varepsilon^{r,g_6} + (II_\varepsilon^{r,g_6} - II_\varepsilon^{2\varepsilon\Lambda,g_6}) + II_\varepsilon^{2\varepsilon\Lambda,g_6}$$

Similar to Section 6.1 we have,

$$II_\varepsilon^{r,g_6} - II_\varepsilon^{2\varepsilon\Lambda,g_6} \leq \lambda \int_{2\varepsilon\Lambda}^{r} \int_{\tilde{M}} \Theta^1_{\varepsilon,g_6}(s,x) - \Theta^2_{\varepsilon,g_6}(s,x)\, d\mathcal{H}^n(x)\, ds.$$

For r in $[0, 2\varepsilon\Lambda]$, again by similar arguments to those in Section 6.1

$$\begin{aligned} II_\varepsilon^{r,g_6} \quad \leq \quad & C_2\Bigg(\int_0^r \int_{\{\rho f < -2\varepsilon\Lambda\}} \Theta^1_{\varepsilon,g_6}(s,x) - \Theta^2_{\varepsilon,g_6}(s,x)\, d\mathcal{H}^n(x)\, ds \\ & + \int_{\{\rho f \geq -2\varepsilon\Lambda\}} m_\varepsilon^2(x)\, d\mathcal{H}^n(x) + \varepsilon\Lambda\Bigg), \end{aligned}$$

where,

$$m_\varepsilon^2(x) \coloneqq \max_{t\in[-6\varepsilon\Lambda.2\varepsilon\Lambda]} \theta_t(x) - \min_{t\in[-6\varepsilon\Lambda,2\varepsilon\Lambda]} \theta_t(x),$$

and we are potentially rechoosing $C_2(M, N, g, W, \lambda)$.

For r in $[0, \rho]$, we consider,

$$\begin{aligned} I_\varepsilon^{r,g_6} \quad = \quad & \int_{A_l}\int_{\sigma^-(x)}^{\sigma^+(x)} \frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t - g_4(r,x)))^2(\rho - r)^2|\nabla f(x)|^2_{(x,t)}\theta_t(x)\, dt\, d\mathcal{H}^n(x) \\ & - \int_{A_l}\int_{\sigma^-(x)}^{\sigma^+(x)} \frac{\varepsilon}{2}((\overline{\mathbb{H}}_\varepsilon)'(t - g_4(0,x)))^2\rho^2|\nabla f(x)|^2_{(x,t)}\theta_t(x)\, dt\, d\mathcal{H}^n(x) \end{aligned}$$

Following similar arguments to Section 5.1, and after potentially rechoosing $C_3 = C_3(z_0, M, N, g, \delta, W, \lambda)$, we have that,

$$I_\varepsilon^{r,g_6} \leq C_3 \mathcal{H}^n(A_l)\frac{\rho^2}{l^2}.$$

Therefore, recalling our choice of $\rho > 0$ in Remark 18, we have

$$I_\varepsilon^{r,g_6} \leq \frac{\sigma \mathcal{H}^n(A_l)}{2(2^n-1)}.$$

Thus, for r in $[0, 2\varepsilon\Lambda]$,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g_6}) - \mathcal{F}_\varepsilon(v_\varepsilon^{0,g_6}) \quad < \quad \frac{\sigma \mathcal{H}^n(A_l)}{2(2^n-1)} + III_\varepsilon^{7,r},$$

where,

$$\begin{aligned} III_\varepsilon^{7,r} \quad &= \quad C_2\bigg(\int_0^r \int_{\{\rho f < -2\varepsilon\Lambda\}} \Theta^1_{\varepsilon,g_6}(s,x) - \Theta^2_{\varepsilon,g_6}(s,x)\, d\mathcal{H}^n(x)\, ds \\ &\quad + \int_{\{\rho f \geq -2\varepsilon\Lambda\}} m_\varepsilon^2(x)\, d\mathcal{H}^n(x) + \varepsilon\Lambda\bigg). \end{aligned}$$

For r in $(2\varepsilon\Lambda, \rho]$,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g_6}) - \mathcal{F}_\varepsilon(v_\varepsilon^{0,g_6}) \quad < \quad \frac{\sigma \mathcal{H}^n(A_l)}{2(2^n-1)} + III_\varepsilon^{8,r},$$

where

$$\begin{aligned} III_\varepsilon^{8,r} \quad &= \quad C_2\bigg(\int_0^{2\varepsilon\Lambda} \int_{\{\rho f < -2\varepsilon\Lambda\}} \Theta^1_{\varepsilon,g_6}(s,x) - \Theta^2_{\varepsilon,g_6}(s,x)\, d\mathcal{H}^n(x)\, ds \\ &\quad + \int_{2\varepsilon\Lambda}^r \int_{\tilde{M}} \Theta^1_{\varepsilon,g_6}(s,x) - \Theta^2_{\varepsilon,g_6}(s,x)\, d\mathcal{H}^n(x) \\ &\quad + \int_{\{\rho f \geq -2\varepsilon\Lambda\}} m_\varepsilon^2(x)\, d\mathcal{H}^n(x) + \varepsilon\Lambda\bigg). \end{aligned}$$

As $g_6(0,x) = g_1(\rho, x)$, we have, for r in $[0, 2\varepsilon\Lambda]$,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g_6}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\sigma \mathcal{H}^n(A_l)}{2(2^n-1)} + III_\varepsilon^{2,\rho} + III_\varepsilon^{7,r},$$

and for r in $(2\varepsilon\Lambda, \rho]$, we have,

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g_6}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\sigma \mathcal{H}^n(A_l)}{2(2^n-1)} + III_\varepsilon^{2,\rho} + III_\varepsilon^{8,r}.$$

For r in $[0, \rho]$, we define the following function on N,

$$\tilde{v}_\varepsilon^{r,g_6}(y) = \begin{cases} \overline{\mathbb{H}}_\varepsilon(\tilde{d}(y) + r), & y \notin B_\delta^N(z_0), \\ v_\varepsilon^{r,g_6}(F_1^{-1}(y)), & y \in \overline{\Upsilon}_1 \cap B_\delta^N(z_0), \\ v_\varepsilon^{r,g_6}(F_2^{-1}(y)), & y \in \overline{\Upsilon}_2 \cap B_\delta^N(z_0). \end{cases}$$

We can show, as in Section 5.2 and 5.3 that, $\tilde{v}_\varepsilon^{r,g_6}$ lies in $W^{1,\infty}(N)$, $\mathcal{F}_{\varepsilon,\lambda}(\tilde{v}_\varepsilon^{r,g_6}) = \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon^{r,g_6})$, and the path $r \mapsto \tilde{v}_\varepsilon^{r,g_6}$ is continuous in $W^{1,2}(N)$.

7.2 Completing Path to b_ε

We construct the path from (6) to '+1' in Figure 2. This is done in an identical way to Section 6.4.

For r in $[\rho, 2\operatorname{diam}(N)]$, we define the following function on N,

$$\tilde{v}_\varepsilon^{r,g_7}(y) := \overline{\mathbb{H}}_\varepsilon(\tilde{d}(y)+r).$$

Similar to arguments in Section 6.4 we have,

$$\mathcal{F}_{\varepsilon,\lambda}(\tilde{v}_\varepsilon^{r,g_7}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\sigma\mathcal{H}^n(A_l)}{2(2^n-1)} + III_\varepsilon^{2,\rho} + III_\varepsilon^{8,\rho} + III_\varepsilon^{9,r},$$

where,

$$III_\varepsilon^{9,r} = \lambda\int_\rho^r \Theta^1_{\varepsilon,g_6}(s,x) - \Theta^2_{\varepsilon,g_6}(s,x)\, d\mathcal{H}^n(x)\, ds.$$

Again as in Section 6.4, we connect $\tilde{v}_\varepsilon^{2\operatorname{diam}(N),g_7} = 1$, to b_ε, by constant functions, $u_\varepsilon^r = r$, for r in $[1, b_\varepsilon]$. We have that for all r in $[1, b_\varepsilon]$,

$$\mathcal{F}_{\varepsilon,\lambda}(u_\varepsilon^r) \leq \mathcal{F}_{\varepsilon,\lambda}(\tilde{v}_\varepsilon^{2\operatorname{diam}(N),g_7}) \leq \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) - \frac{\sigma\mathcal{H}^n(A_l)}{2(2^n-1)} + III_\varepsilon^{2,\rho} + III_\varepsilon^{8,\rho} + III_\varepsilon^{9,2\operatorname{diam}(N)}.$$

Both $\tilde{v}_\varepsilon^{r,g_7}$, and u_ε^r give continuous paths in $W^{1,2}(N)$ with respect to r.

8 Conclusion of the Paths

8.1 Error Terms

8.1.1 Theta Error Terms

Consider the term,

$$\begin{aligned}\Theta^1_{\varepsilon,g}(s,x) - \Theta^2_{\varepsilon,g}(s,x) \quad &= \quad \sigma\int_{\sigma^-(x)}^{\sigma^+(x)} \partial_s g(s,x)(\overline{\mathbb{H}}_\varepsilon)'(t-g(s,x))\theta_t(x)\, dt \\ &\quad - \int_{\sigma^-(x)}^{\sigma^+(x)} \partial_s g(s,x) Q_\varepsilon(t-g(s,x))\theta_t(x)\, dt.\end{aligned}$$

Assuming that g is monotone in the first variable, we have,

$$|\Theta^1_{\varepsilon,g}(s,x) - \Theta^2_{\varepsilon,g}(s,x)| \leq 2\sigma|\partial_s g(s,x)| m_\varepsilon(g(s,x),x) + C_7\varepsilon^2.$$

where,

$$m_\varepsilon(T,x) = \max_{t\in[T-2\varepsilon\Lambda, T+2\varepsilon\Lambda]} \theta_t(x) - \min_{t\in[T-2\varepsilon\Lambda, T+2\varepsilon\Lambda]} \theta_t(x).$$

and $C_7 = C_7(N, m, \lambda, W, |g|_{C^1}) < +\infty$.

Now we assume that $\partial_s g \geq 0$, and $|\partial_s g|_{C^0(\mathbb{R}\times\tilde{M})} < +\infty$. Apply Fubini's Theorem to swap integrals,

$$\int_0^r \int_{\tilde{M}} \left|\Theta^1_{\varepsilon,g}(s,x) - \Theta^2_{\varepsilon,g}(s,x)\right| \, d\mathcal{H}^n(x)\, ds \leq 2\sigma \int_{\tilde{M}} \int_{g(0,x)}^{g(r,x)} m_\varepsilon(T,x)\, dT\, \mathcal{H}^n(x) + C_7 r\varepsilon^2.$$

Fixing x in $\tilde{M}$ and r in $\mathbb{R}$, we see that for all T in $\mathbb{R}\setminus\{\sigma^-(x), \sigma^+(x)\}$,

$$m_\varepsilon(T,x) \to 0, \; as\, \varepsilon \to 0,$$

and furthermore, we have the following bounds, $0 \leq m_\varepsilon(T,x) \leq e^{\frac{\lambda^2}{2m}}$. Therefore we can apply Dominated Convergence Theorem for fixed x in $\tilde{M}$ and r in $[0,\infty)$,

$$\int_{g(0,x)}^{g(r,x)} m_\varepsilon(T,x)\, dT \to 0, \; as\, \varepsilon \to 0.$$

Furthermore, as $0 \leq g(r,x) - g(0,x) \leq |\partial_s g|_{C^0(\mathbb{R}\times\tilde{M})}\, r$, we have the bounds,

$$0 \leq \int_{g(0,x)}^{g(r,x)} m_\varepsilon(T,x)\, dT \leq |\partial_s g|_{C^0(\mathbb{R}\times\tilde{M})} r e^{\frac{\lambda^2}{2m}}.$$

Therefore, again by Dominated Convergence Theorem, we have, for fixed r in $[0,\infty)$

$$\int_{\tilde{M}} \int_{g(0,x)}^{g(r,x)} m_\varepsilon(T,x)\, dT\, \mathcal{H}^n(x) \to 0, \; as\, \varepsilon \to 0.$$

Define the following continuous function on $[0,+\infty)$,

$$M^g_\varepsilon(r) = \int_{\tilde{M}} \int_{g(0,x)}^{g(r,x)} m_\varepsilon(T,x)\, dT\, \mathcal{H}^n(x).$$

We have that $M^g_\varepsilon(r) \to 0$, pointwise, as $\varepsilon \to 0$, and furthermore, as

$$0 \leq m_{\varepsilon_1}(T,x) \leq m_{\varepsilon_2}(T,x),$$

for all T in $\mathbb{R}$, x in $\tilde{M}$, and $0 < \varepsilon_1 < \varepsilon_2$, this implies that,

$$0 \leq M^g_{\varepsilon_1}(r) \leq M^g_{\varepsilon_2}(r),$$

for all r in $[0,+\infty)$. Therefore, by Dini's Theorem, we have that,

$$M^g_\varepsilon \to 0, \; as\, \varepsilon \to 0,$$

uniformly on compact sets of $[0,+\infty)$. Thus,

$$\int_0^r \int_{\tilde{M}} \left|\Theta^1_{\varepsilon,g}(s,x) - \Theta^2_{\varepsilon,g}(s,x)\right| \, d\mathcal{H}^n(x)\, ds \to 0 \tag{14}$$

as $\varepsilon \to 0$, uniformly in r, on compact sets of $[0,+\infty)$. The same holds assuming that g satisfies $\partial_s g \leq 0$, on $\mathbb{R}\times\tilde{M}$, and $|\partial_s g|_{C^0(\mathbb{R}\times\tilde{M})} < +\infty$.

For $i = 1,\ldots,7$ our g_i's are monotone in the first variable and $|\partial_s g_i|_{C^0(\mathbb{R}\times\tilde{M})} < +\infty$. Therefore (14) holds for each i.

8.1.2 The Other Error Terms

We first consider,

$$\int_{B_{2l}} q_\varepsilon^1(x)\, d\mathcal{H}^n(x),$$

with,

$$q_\varepsilon^1(x) = \max_{t\in[-4\varepsilon\Lambda, 2\varepsilon\Lambda]} (H_t(x) - \lambda)\theta_t(x).$$

By choice of $\varepsilon > 0$, in Remark 15, $2\varepsilon\Lambda << \rho$. Therefore by choice of $\rho > 0$, in Remark 11, and $\delta > 0$, from Remark 16, we have

$$0 \leq \max_{x\in B_{2l}} q_\varepsilon(x) \leq \frac{\lambda}{2} e^{\frac{\lambda^2}{2m}}.$$

Fixing x' in $B_{2l} \setminus \{x \colon \sigma^-(x) = 0\}$, we see that there exists an $\varepsilon' = \varepsilon'(x') > 0$, such that for all $0 < \varepsilon \leq \varepsilon'$,

$$[-4\varepsilon\Lambda, 2\varepsilon\Lambda] \subset (\sigma^-(x'), \sigma^+(x')).$$

Therefore, $(H_t(x') - \lambda)\theta_t(x')$, is a smooth function in t on $[-4\varepsilon\Lambda, 2\varepsilon\Lambda]$, and clearly,

$$\max_{t\in[-4\varepsilon\Lambda, 2\varepsilon\Lambda]} (H_t(x') - \lambda)\theta_t(x') \to 0,\ as\, \varepsilon \to 0.$$

Thus $q_\varepsilon^1 \to 0$, $\mathcal{H}^n$–a.e in B_{2l}, and we can apply Dominated Convergence Theorem to say that

$$\int_{B_{2l}} q_\varepsilon^1(x)\, d\mathcal{H}^n(x) \to 0,\ as\, \varepsilon \to 0.$$

Identically we also have,

$$\int_{A_L^k} q_\varepsilon^2(x)\, d\mathcal{H}^n(x) \to 0,\ as\, \varepsilon \to 0,$$

recalling $q_\varepsilon^2(x) = \max_{t\in[-2\varepsilon\Lambda, 4\varepsilon\Lambda]}(\lambda - H_t(x))\theta_t(x)$.

Now considering

$$q_\varepsilon^3(r, x) = \max_{t\in[-2\varepsilon\Lambda, 2\varepsilon\Lambda]} \zeta(x, t + g_4(r, x)) - \zeta(x, g_4(r, x))$$

where we are recalling the function

$$\zeta(x, t) = |\nabla \tilde{f}(x)|^2_{(x,t)}\theta_t(x)$$

from Claim 4. For x in B_{2L} such that $\tilde{f}(x) = 0$, we have that $\zeta(x, t) = 0$, for all t. Considering x in $A_L^k \cap \{\tilde{f} \neq 0\}$, such that $r_0\tilde{f}(x) > 2\varepsilon\Lambda$, then

$$[-2\varepsilon\Lambda + g_4(r, x), 2\varepsilon\Lambda + g_4(r, x)] \subset (0, 2r_0) \subset (\sigma^-(x), \sigma^+(x)).$$

Thus, for t in $[-2\varepsilon\Lambda + g_4(r, x), 2\varepsilon\Lambda + g_4(r, x)]$,

$$(x, t) \in \tilde{T} \cap (B_L \times (-2r_0, 2r_0)) \subset\subset \tilde{V}_1 \cup \tilde{V}_2,$$

where we are recalling sets $\tilde{V}_1$ and $\tilde{V}_2$ from Remark 6. Therefore, ζ is differentiable at (x, t) and

$$|\partial_t\zeta(x, t)| \leq C(|F_i|_{C^2(B_L\times(-2r_0, 2r_0))}, |\tilde{f}|_{C^1}, \lambda) \leq C_6.$$

Where we are potentially rechoosing $C_6 = C_6(z_0, M, N, g, k, L, \delta, W, \lambda)$. Therefore, for x in A_L^k such that $r_0\tilde{f}(x) > 2\varepsilon\Lambda$, we have that $0 \leq q_\varepsilon^3(r,x) \leq C_6\varepsilon\Lambda$. Furthermore, for all x in A_L^k,

$$|q_\varepsilon^3(r,x)| \leq \max_{(x,t)\in B_L\times(-2r_0,2r_0)} \zeta(x,t) \leq C_6.$$

Again we are potentially rechoosing $C_6 = C_6(z_0, M, N, g, k, L, \delta, W, \lambda)$.

Therefore,

$$\begin{aligned} p_\varepsilon^1(r) &= \int_{A_L^k\cap\{r_0\tilde{f}>2\varepsilon\Lambda\}} q_\varepsilon^3(r,x)\, d\mathcal{H}^n(x) + \int_{A_L^k\cap\{0<r_0\tilde{f}\leq 2\varepsilon\Lambda\}} q_\varepsilon^3(r,x)\, d\mathcal{H}^n(x), \\ &\leq C_6\left(\varepsilon\Lambda + \mathcal{H}^n(\{x\in A_L^k\colon 0 < r_0\tilde{f}(x) \leq 2\varepsilon\Lambda\})\right). \end{aligned}$$

Thus

$$\max_{r\in[0,r_0]} p_\varepsilon^1(r) \to 0$$

as $\varepsilon \to 0$. Similarly, $p_\varepsilon^2(0) \to 0$, as $\varepsilon \to 0$.

For the remaining error terms, as $\mathcal{H}^n(\{x \in \tilde{M}\colon \sigma^-(x) = 0\}) = 0$, by Dominated Convergence Theorem, we have that,

$$\mathcal{H}^n(\{x\in\tilde{M}\colon \sigma^-(x) \geq -2\varepsilon\Lambda\}) \to 0,$$

and

$$\int_{\tilde{M}} m_\varepsilon^i(x)\, d\mathcal{H}^n(x) \to 0,$$

where,

$$\begin{aligned} m_\varepsilon^1(x) &= \max_{t\in[-2\varepsilon\Lambda,4\varepsilon\Lambda]} \theta_t(x) - \min_{t\in[-2\varepsilon\Lambda,4\varepsilon\Lambda]} \theta_t(x), \\ m_\varepsilon^2(x) &= \max_{t\in[-6\varepsilon\Lambda,2\varepsilon\Lambda]} \theta_t(x) - \min_{t\in[-6\varepsilon\Lambda,2\varepsilon\Lambda]} \theta_t(x). \end{aligned}$$

8.2 Path for Theorem 2

Consider the following continuous path in $W^{1,2}(N)$, for $\varepsilon > 0$,

$$\gamma_\varepsilon(t) = \begin{cases} -1 - 2\mathrm{diam}\,(N) - t, & t\in[-2\,\mathrm{diam}\,(N) - a_\varepsilon - 1, 2\,\mathrm{diam}\,(N)], \\ \overline{\mathbb{H}}_\varepsilon(\tilde{d} - t), & t\in[-2\,\mathrm{diam}\,(N), 2\,\mathrm{diam}\,(N)], \\ 1 - 2\,\mathrm{diam}\,(N) + t, & t\in[2\,\mathrm{diam}\,(N), 2\,\mathrm{diam}\,(N) + b_\varepsilon - 1]. \end{cases}$$

which satisfies $\gamma_\varepsilon(-1 - 2\mathrm{diam}\,(N) - a_\varepsilon) = a_\varepsilon$, and $\gamma_\varepsilon(1 - 2\,\mathrm{diam}\,(N) + b_\varepsilon) = b_\varepsilon$.

Replacing $r_0 = 2\varepsilon\Lambda$, in Section 6.4, and $\rho = 2\varepsilon\Lambda$, in Section 7.2, we see that, for all ε in $(0,\tilde{\varepsilon})$, for some $\tilde{\varepsilon} = \tilde{\varepsilon}(N, M, g, \lambda, W) > 0$, fixed,

$$\begin{cases} \mathcal{F}_{\varepsilon,\lambda}(\gamma_\varepsilon(t)) < \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) + III_\varepsilon^{6,2\,\mathrm{diam}\,(N)}, & t\in[-2\,\mathrm{diam}\,(N) - a_\varepsilon - 1, 2\,\mathrm{diam}\,(N)], \\ \mathcal{F}_{\varepsilon,\lambda}(\gamma_\varepsilon(t)) < \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) + III_\varepsilon^{6,-t}, & t\in[-2\mathrm{diam}\,(N), -2\varepsilon\Lambda], \\ \mathcal{F}_{\varepsilon,\lambda}(\gamma_\varepsilon(t)) < \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) + III_\varepsilon^{9,t}, & t\in[2\varepsilon\Lambda, 2\mathrm{diam}\,(N)], \\ \mathcal{F}_{\varepsilon,\lambda}(\gamma_\varepsilon(t)) < \mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) + III_\varepsilon^{9,2\,\mathrm{diam}\,(N)}, & t\in[2\,\mathrm{diam}\,(N), 2\,\mathrm{diam}\,(N) + b_\varepsilon - 1]. \end{cases}$$

Recalling from Section 3.6

$$\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) \to 2\sigma\mathcal{H}^n(M) - \sigma\lambda\mathcal{H}^{n+1}(E) + \sigma\lambda\mathcal{H}^{n+1}(N \setminus E),$$

as $\varepsilon \to 0$, and Section 8.1.1,

$$\max_{t\in[2\varepsilon\Lambda, 2\operatorname{diam}(N)]} \left(III_\varepsilon^{6,t} + III^{9,t}\right) \to 0,$$

as $\varepsilon \to 0$. Therefore, for $\tau > 0$, there exists a $0 < \varepsilon_\tau = \varepsilon_\tau(N, M, g, \lambda, W) \leq \tilde{\varepsilon}$, such that for all ε in $(0, \varepsilon_\tau)$ and t in $[-2\operatorname{diam}(N) - a_\varepsilon - 1, 2\operatorname{diam}(N) + b_\varepsilon - 1] \setminus (-2\varepsilon\Lambda, 2\varepsilon\Lambda)$,

$$\mathcal{F}_{\varepsilon,\lambda}(\gamma_\varepsilon(t)) < 2\sigma\mathcal{H}^n(M) - \sigma\lambda\mathcal{H}^n(E) + \sigma\lambda\mathcal{H}^{n+1}(N \setminus E) + \tau.$$

Furthermore by similar arguments to those in Section 3.6, and after potentially rechoosing $\varepsilon_\tau > 0$, we have that for all ε in $(0, \varepsilon_\tau)$

$$\max_{t\in[-2\varepsilon\Lambda, 2\varepsilon\Lambda]} \mathcal{F}_{\varepsilon,\lambda}(\gamma_\varepsilon(t)) < 2\sigma\mathcal{H}^n(M) - \sigma\lambda\mathcal{H}^{n+1}(E) + \sigma\lambda\mathcal{H}^{n+1}(N \setminus E) + \tau.$$

Therefore this is an admissible path in $W^{1,2}(N)$, that proves that the limiting Allen-Cahn varifold can not have a minimal piece.

Remark 25. *Note that we can build the path γ_ε, for any suitable Caccioppoli set E. The suitable properties are the following:*

1. *$\partial^* E \neq \emptyset$, has a quasi embedded λ-CMC structure, with respect to unit normal pointing into E.*
2. *$\partial^* E$ satisfies the Geodesic Touching Lemma (Lemma 1).*

From Remark 25 we can deduce that E must be a single connected component and minimises the value

$$F_\lambda(E) = \mathcal{H}^n(\partial^* E) - \lambda\mathcal{H}^{n+1}(E) > 0,$$

among all suitable competitors.

8.3 Contradiction Path for Theorem 1

Recall all the error terms from Sections 5, 6 and 7. By Section 3.6 and 8.1, for $\tau > 0$, there exists an $\varepsilon_\tau = \varepsilon(z_0, M, N, g, \delta, W, \lambda, L, k, r_0, \rho, \tau) \in (0, \varepsilon_3)$, such that for all ε in $(0, \varepsilon_\tau)$, we have that

$$\begin{aligned}
\mathcal{F}_{\varepsilon,\lambda}(v_\varepsilon) &+ \max_{r\in[0,4\varepsilon\Lambda)} III_\varepsilon^{1,r} + \max_{r\in[4\varepsilon\Lambda,\rho]} III_\varepsilon^{2,r} + \max_{r\in[0,2\varepsilon\Lambda]} III_\varepsilon^{3,r} \\
&+ \max_{r\in(2\varepsilon\Lambda,r_0]} III_\varepsilon^{4,r} + \max_{r\in[0,r_0]} III_\varepsilon^{5,r} + \max_{r\in[r_0,2\operatorname{diam}(N)]} III_\varepsilon^{6,r} \\
&+ \max_{r\in[0,2\varepsilon\Lambda]} III_\varepsilon^{7,r} + \max_{r\in(2\varepsilon\Lambda,\rho]} III_\varepsilon^{8,r} + \max_{r\in[\rho,2\operatorname{diam}(N)]} III_\varepsilon^{9,r} \\
&< 2\sigma\mathcal{H}^n(M) - \sigma\lambda\mathcal{H}^{n+1}(E) + \sigma\lambda\mathcal{H}^{n+1}(N \setminus E) + \tau.
\end{aligned}$$

Therefore, for any $\tau > 0$, there exists an $\varepsilon_\tau > 0$, such that for any ε in $(0, \varepsilon_\tau)$, we can define a continuous path,

$$\gamma_\varepsilon \colon [-1 - a_\varepsilon, 4\operatorname{diam}(N) + r_0 + \rho + b_\varepsilon - 1] \to W^{1,2}(N),$$

by

$$\gamma_\varepsilon(t) = \begin{cases} -1-t, & t \in [-1-a_\varepsilon, 0], \\ \overline{\mathbb{H}}_\varepsilon(\tilde{d} - 2\operatorname{diam}(N) + t), & [0, 2\operatorname{diam}(N) - r_0], \\ \tilde{v}_\varepsilon^{2\operatorname{diam}(N)-t, g_4}, & [2\operatorname{diam}(N) - r_0, 2\operatorname{diam}(N)], \\ \tilde{v}_\varepsilon^{2\operatorname{diam}(N)+\rho-t, g_3}, & [2\operatorname{diam}(N), 2\operatorname{diam}(N)+\rho], \\ \tilde{v}_\varepsilon^{2\operatorname{diam}(N)+\rho+r_0-t, g_2}, & [2\operatorname{diam}(N)+\rho, 2\operatorname{diam}(N)+\rho+r_0], \\ \tilde{v}_\varepsilon^{2\operatorname{diam}(N)+\rho+r_0+t, g_6}, & [2\operatorname{diam}(N)+\rho+r_0, 2\operatorname{diam}(N)+2\rho+r_0], \\ \overline{\mathbb{H}}_\varepsilon(\tilde{d} + t - 2\operatorname{diam}(N) - \rho - r_0), & [2\operatorname{diam}(N)+2\rho+r_0, 4\operatorname{diam}(N)+\rho+r_0], \\ 1 + t - 4\operatorname{diam}(N) + r_0 + \rho, & [4\operatorname{diam}(N)+r_0+\rho, 4\operatorname{diam}(N)+r_0+\rho+b_\varepsilon-1] \end{cases}$$

This path satisfies the following; $\gamma_\varepsilon(-1-a_\varepsilon) = a_\varepsilon$, $\gamma_\varepsilon(4\operatorname{diam}(N)+r_0+\rho+b_\varepsilon-1) = b_\varepsilon$, and

$$\gamma_\varepsilon(t) < 2\sigma\mathcal{H}^n(M) - \sigma\lambda\mathcal{H}^{n+1}(E) + \sigma\lambda\mathcal{H}^{n+1}(N \setminus E) - \frac{\sigma\mathcal{H}^n(A_l)}{2(2^n-1)} + \tau,$$

for all t in $[-1-a_\varepsilon, 4\operatorname{diam}(N)+r_0+\rho+b_\varepsilon-1]$. This contradicts the min-max construction of M, implying that M must be embedded.

9 Morse Index

Recall the functional defined on Caccioppoli sets of N,

$$F_\lambda(\Omega) = \mathcal{H}^n(\partial^*\Omega) - \lambda\mu_g(\Omega).$$

For a C^2 vector field X, we may take variations in direction X by considering its flow $\{\Phi_t\}$. We define the first variation of F_λ by,

$$\delta F_\lambda(\Omega)(X) = \frac{d}{dt} F_\lambda(\Phi_t(\Omega))_{|t=0}. \tag{15}$$

and the second variation by,

$$\delta^2 F_\lambda(\Omega)(X) = \frac{d^2}{dt^2} F_\lambda(\Phi_t(\Omega))_{|t=0}. \tag{16}$$

We have that $\delta F_\lambda(E)(X) = 0$, for all C^1 vector fields X. Note that we require M to be embedded and orientable for the following to be well defined. Consider the class of vector fields $X \in C_c^2(N \setminus (\overline{M} \setminus M))$, such that $X_{|M} = \varphi\nu$, where $\varphi \in C_c^2(M)$. By [1, Proposition 2.5],

$$\delta^2 F_\lambda(E)(X) = \int_M |\nabla^M\varphi|^2 - (|A_M|^2 + \operatorname{Ric}(\nu,\nu))\varphi^2 \, d\mathcal{H}^n. \tag{17}$$

We extend the expression on the right hand side to all functions in $W_0^{1,2}(M)$, and define the following quadratic form,

$$B_M(\varphi,\varphi) := \int_M |\nabla^M\varphi|^2 - (|A_M|^2 + \operatorname{Ric}(\nu,\nu))\varphi^2 \, d\mathcal{H}^n, \qquad \varphi \in W_0^{1,2}(M).$$

After integrating by parts we obtain the second order elliptic operator on M,

$$L_M \coloneqq \Delta_M + |A_M|^2 + \mathrm{Ric}(\nu, \nu).$$

We restrict ourselves to a set $W \subset\subset N \setminus (\overline{M} \setminus M)$, to avoid our curvature term $|A_M|$, from potentially blowing up. A value $\kappa = \kappa(W) \in \mathbb{R}$ is said to be an eigenvalue of L_M in W, if there exists an $\varphi \in W_0^{1,2}(W \cap M)$ such that

$$L_M\varphi + \kappa\varphi = 0.$$

By standard elliptic theory, see [9], the spectrum of L_M in $W \cap M$,

$$\kappa_1(W) \leq \kappa_2(W) \leq \cdots \to +\infty,$$

is discrete and bounded from below. We then define the index of M in W by,

$$\mathrm{ind}_W(M) = |\{p \colon \kappa_p(W) < 0\}|,$$

or equivalently, it is the maximum dimension of a linear subspace of $W_0^{1,2}(W \cap M)$ on which B_M is negative definite. If $\mathrm{ind}_W(\mathrm{ind}\, M) = 0$, then we say that M is stable in W and $\kappa_p(W) \geq 0$, for all p in $\mathbb{N}$, and

$$B_M(\varphi, \varphi) \geq 0,\ for\, all\, \varphi \in W_0^{1,2}(W \cap M).$$

We define,

$$\mathrm{ind}(M) = \sup_{W \subset\subset N \setminus (\overline{M} \setminus M)} (\mathrm{ind}_W(M)).$$

As M is embedded, and our sequence of critical points $\{u_i\}$ from Section 1.1 has $\mathrm{ind}\, u_i \leq 1$, by [14, Theorem 1a.], we have that $\mathrm{ind} M \leq 1$.

Remark 26. *As M is two-sided and embedded, and the inhomogeneous term is a constant, we may also apply the ideas and arguments of [11] verbatim to conclude that $indM \leq 1$.*

Claim 5. *$ind\, M = 1$.*

Proof. We only need to show a lower bound, which follows from the Ricci positivity on N. We construct an appropriate function on M, using a similar argument to [2, Lemma 5.1].

We wish to prove that we can find a set $W \subset\subset N \setminus (\overline{M} \setminus M)$, and a function φ in $W_0^{1,2}(M \cap W)$ such that,

$$B_M(\varphi, \varphi) < 0.$$

By the Ricci positivity of N, for any $W \subset\subset N \setminus (\overline{M} \setminus M)$, and φ in $W_0^{1,2}(M \cap W)$, we have

$$B_M(\varphi, \varphi) \leq \int_M |\nabla^M \varphi|^2 - |A_M|^2 \varphi^2 \, d\mathcal{H}^n.$$

If $\overline{M} \setminus M = \emptyset$, we set $W = N$, and $\varphi = 1$,

$$B_M(\varphi, \varphi) \leq -\int_M |A_M|^2 \, d\mathcal{H}^n < 0.$$

For $\overline{M} \setminus M \neq \emptyset$, we first we note that we must have $n \geq 7$, and $\mathcal{H}^{n-1}(\overline{M} \setminus M) = 0$. Therefore, the 2-capacity of $\overline{M} \setminus M$ is 0, [8, Section 4.7.2, Theorem 3], implying that for all $\delta > 0$, there exists a function f_δ such that,

$$\begin{cases} f_\delta \in C_c^\infty(N \setminus (\overline{M} \setminus M)), \\ f_\delta(y) \in [0,1],\ y \in N, \\ \int_N |\nabla f_\delta|^2 d\mu_g < \delta, \\ \mu_g(\{f_\delta = 1\}) > \mu_g(N) - \delta. \end{cases}$$

Furthermore, as $|\overline{M}|$ is a multiplicity 1 integral varifold with uniformly bounded generalised mean curvature, we have a monotonicity formula [17, Corollary 17.8]. The existence of such a monotonicity formula implies Euclidean volume growth about each point in $\overline{M}$. Therefore, there exists a constant $C = C(N, M, g)$, such that, by the construction of f_δ as in [8, Section 4.7.2, Theorem 3],

$$\int_M |\nabla^M f_\delta|^2 \, d\mathcal{H}^n \leq C\delta.$$

Taking $W_\delta = \operatorname{supp} f_\delta \subset\subset N \setminus (\overline{M} \setminus M)$, we have that $(f_\delta)_{|M} \in W_0^{1,2}(M \cap W_\delta)$, and,

$$B_M(f_\delta, f_\delta) \leq C\delta - n^{-2}\lambda^2 \mathcal{H}^n(\{f_\delta = 1\} \cap M).$$

We have that as we send $\delta \to 0$, $\mathcal{H}^n(\{f_\delta = 1\} \cap M) \to \mathcal{H}^n(M)$. Therefore for small enough $\delta > 0$, we have that $B_M(f_\delta, f_\delta) < 0$. This implies that $\operatorname{ind} M \geq 1$. $\square$

The fact that M is connected immediately follows from this, as on each connected component we could construct a function as in Claim 5. Therefore each connected component adds atleast 1 to the index.

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