# Distribution of Resonances in Scattering by Thin Barriers 

by<br>Jeffrey Eric Galkowski<br>A dissertation submitted in partial satisfaction of the requirements for the degree of Doctor of Philosophy<br>in<br>Mathematics<br>in the<br>Graduate Division of the<br>University of California, Berkeley<br>Committee in charge:<br>Professor Maciej R. Zworski, Chair<br>Professor Daniel I. Tataru<br>Professor Oliver M. O'Reilly

Spring 2015

# Distribution of Resonances in Scattering by Thin Barriers 

Copyright 2015
by
Jeffrey Eric Galkowski

Abstract<br>Distribution of Resonances in Scattering by Thin Barriers<br>by<br>Jeffrey Eric Galkowski<br>Doctor of Philosophy in Mathematics<br>University of California, Berkeley<br>Professor Maciej R. Zworski, Chair

This thesis contains a detailed study of the rates of wave decay for scattering by thin barriers. Thin barriers are systems in which, except for a narrow region, waves do not interact. This type of behavior is observed in physical systems including concert halls and quantum corrals. A quantum corral is constructed by configuring individual atoms or molecules to form a barrier which partially confines electrons to its interior. Here, the atoms produce a potential which plays the role of the thin barrier. In the setting of concert halls, the walls play the role of the barrier and produce partial confinement of sound waves.

Rather than studying thin barriers as systems with a finite width interaction region, we imagine that the region is reduced to a single hypersurface in $\mathbb{R}^{d}$ by taking a limit of barriers whose width is decreasing and intensity is increasing. Specifically, we are interested in wave equations

$$
\begin{equation*}
\left(\partial_{t}^{2}+P\right) u=0 \tag{0.0.1}
\end{equation*}
$$

where $P$ is an operator of the form

$$
P=-\Delta_{\partial \Omega, \delta}:=-\Delta+\delta_{\partial \Omega} \otimes V \quad \text { or } P=-\Delta_{\partial \Omega, \delta^{\prime}}:=-\Delta+\partial_{\nu}\left(\delta_{\partial \Omega}\right) \otimes\left(V \partial_{\nu}\right),
$$

$\Omega \Subset \mathbb{R}^{d}$ and $V$ an operator acting on $L^{2}(\partial \Omega)$ and varying with frequency. These operators are used as models for leaky quantum graphs [26] and quantum corrals [5, 6, 18].

We approach the study of (0.0.1) from the point of view of scattering theory introduced by Lax-Phillips [46] and Vainberg [78]. Heuristically, one expects to have an expansion of solutions to 0.0.1) of the form

$$
\begin{equation*}
u(t, x) \sim \sum_{\lambda \in \operatorname{Res}} e^{-i t \lambda} u_{\lambda}(x) \tag{0.0.2}
\end{equation*}
$$

where $\lambda$ runs over a discrete set of scattering resonances, Res $\subset \mathbb{C}$. Hence the (negative) imaginary parts of the resonances control the decay rate of solutions to (0.0.1). There are two major steps in our analysis of $-\Delta_{\partial \Omega, \delta}$ and $-\Delta_{\partial \Omega, \delta^{\prime}}$ :

1. identify the set Res with the presence of non-trivial solutions to certain transmission problems and generalized non-selfadjoint eigenvalue problems.
2. understand the behavior of $\lambda \in \operatorname{Res}$ as $|\operatorname{Re} \lambda| \rightarrow \infty$.

The first step is accomplished using methods similar to those for scattering by $L^{\infty}$ potentials (see for example [21, Chapter 2, 3]). The additional requirement is to understand the free resolvent,

$$
R_{0}(\lambda):=\left(-\Delta-\lambda^{2}\right)^{-1}
$$

(meromorphically continued from $\operatorname{Im} \lambda \gg 1$ ) after restrictions to hypersurfaces that correspond to the single, double, and derivative double boundary layer operators; respectively,

$$
\begin{gathered}
G(\lambda) f(x):=\int_{\partial \Omega} R_{0}(\lambda)(x, y) f(y) d S(y) \quad \tilde{N}(\lambda) f(x):=\int_{\partial \Omega} \partial_{\nu_{y}} R_{0}(\lambda)(x, y) f(y) d S(y) \\
\partial_{\nu} \mathcal{D} \ell(\lambda) f(x):=\int_{\partial \Omega} \partial_{\nu_{x}} \partial_{\nu_{y}} R_{0}(\lambda)(x, y) f(y) d S(y)
\end{gathered}
$$

where $x \in \partial \Omega$.
The second step relies on understanding the trapping properties of transmission problems, that is, properties of light rays that are trapped in a fixed compact set for all time. One should notice that unlike in the case of scattering with smooth coefficients, one expects light rays to split into transmitted and reflected rays after interacting with $\partial \Omega$. This behavior frequently results in the presence of rays that are strongly trapped geometrically. However, even trapped rays decay because, depending on the precise nature of the transmission, varying proportions of the wave may be transmitted and reflected at each intersection with $\partial \Omega$. The precise understanding of these phenomena in a transmission problem leads to a description of the location of $\lambda \in$ Res.

As discussed above, the identification of $\lambda \in$ Res with the existence of solutions to certain transmission problems is accomplished via a precise understanding of the boundary layer operators $G, \tilde{N}$, and $\partial_{\nu} \mathcal{D} \ell$ at high energies. We first use restriction estimates for eigenfunctions of the Laplacian to prove estimates on the boundary layer operators when $\lambda$ has $|\lambda| \gg 1$. We also show that the estimates are sharp modulo a loss of $\log |\lambda|$. These estimates are enough to prove that the resonances of $-\Delta_{\partial \Omega, \delta}$ coincide with the existence of nontrivial solutions to a transmission problem as well as the solutions to the generalized eigenvalue problem

$$
\begin{equation*}
(I+G(\lambda) V) \varphi=0 \tag{0.0.3}
\end{equation*}
$$

Using a semiclassical adaptation of intersecting Lagrangian distributions from [49] and the Melrose-Taylor parametrix from [47] we then give a complete microlocal description of the boundary layer operators $G$ and $\partial_{\nu} \mathcal{D} \ell$ in the case $\partial \Omega$ is smooth and strictly convex. This allows us to remove the $\log$ loss from our high energy estimates for $G$ and $\partial_{\nu} \mathcal{D} \ell$ and to identify the resonances for $-\Delta_{\partial \Omega, \delta^{\prime}}$ with the existence of solutions to a transmission problem as well as the problem

$$
\left(I-\partial_{\nu} \mathcal{D} \ell(\lambda) V\right) \varphi=0
$$

When discussing the distribution of resonances for $-\Delta_{\partial \Omega, \delta}$ and $-\Delta_{\partial \Omega, \delta^{\prime}}$, we work with $\lambda=z / h$ with $\operatorname{Re} z \sim 1$ and $0<h \ll 1$. We then obtain the following results on the distribution of resonances.

## The case of $-\Delta_{\partial \Omega, \delta}$

For very general $\Omega$, we show that there exists $C>0$ such that resonances satisfy

$$
\operatorname{Im} z \geq-C h \log h^{-1}
$$

provided $\|V\|_{L^{2} \rightarrow L^{2}} \leq C h^{-\alpha}$ for some $\alpha<2 / 3$. This allows us to prove an expansion of the type (0.0.2) for solutions to (0.0.1) with $P=-\Delta_{\partial \Omega, \delta}$. We then turn our attention to the case $\partial \Omega$ is smooth and strictly convex where we can use the microlocal description of $G$ along with 0.0 .3 to understand transmission and reflection through the boundary. This understanding yields a dynamical characterization of the size of the resonance free region for $V \in C^{\infty}(\partial \Omega)$ with $|V| \ll h^{-2 / 3}$ that can be thought of as a Sabine law [61] and is of the form

$$
\begin{equation*}
\operatorname{Im} z \geq-R_{V, \Omega} h \log h^{-1}, \quad \operatorname{Re} z \sim 1 \tag{0.0.4}
\end{equation*}
$$

We next show that the constant in $(0.0 .4)$ is optimal for generic $V \in C^{\infty}(\partial \Omega)$ and generic $\Omega$ in the sense that for any constant $r>R_{V, \Omega}$ the number of resonances with

$$
\begin{equation*}
\operatorname{Im} z \geq-r h \log h^{-1}, \quad \operatorname{Re} z \sim 1 \tag{0.0.5}
\end{equation*}
$$

is unbounded as $h \rightarrow 0$. Moreover, the bound 0.0 .4 is sharp for $V \equiv V_{0} h^{-\alpha}$ and $\Omega=$ $B(0,1) \subset \mathbb{R}^{2}$. Finally, we give some upper bounds on the number of resonances in regions given in 0.0.5).

## The case of $-\Delta_{\partial \Omega, \delta^{\prime}}$

In this case, we only consider $\partial \Omega$ smooth and strictly convex. Then, for $V \in C^{\infty}(\partial \Omega)$ with $c h^{\alpha}<V<C h^{\alpha}$ and $\alpha>5 / 6$, we give a dynamical characterization of the resonance free region that is of the form

$$
\operatorname{Im} z \geq \begin{cases}-R_{V, \Omega, \alpha} h^{3-2 \alpha} & \alpha \leq 1  \tag{0.0.6}\\ -R_{V, \Omega, \alpha} h \log h^{-1} & \alpha>1\end{cases}
$$

Again, this bound can be thought of as a Sabine law for the $\delta^{\prime}$ potential and we show that it is sharp in the case that $\Omega=B(0,1) \subset \mathbb{R}^{2}$ and $V \equiv V_{0} h^{\alpha}>0$. As far as the author is aware, the example $-\Delta_{\partial \Omega, \delta^{\prime}}$ is the only general class known to have resonances converging to the real axis at a fixed polynomial rate.

## To Opa

For guiding our family through the American dream.

## Contents

Contents ..... ii
List of Figures ..... iv
1 Introduction ..... 1
2 Model Cases ..... 11
2.1 Introduction ..... 11
2.2 Asymptotics for Airy and Bessel Functions ..... 17
2.3 The $\delta$ Potential ..... 21
2.4 Resonance Free Regions ..... 22
2.5 Construction of Resonances ..... 28
2.6 The $\delta^{\prime}$ Potential ..... 34
2.7 Construction of Resonances ..... 39
2.8 The 1- $d$ case for the $\delta^{\prime}$ potential ..... 40
3 Geometric Preliminaries ..... 42
3.1 Local Symplectic Geometry ..... 42
3.2 The Billiard Ball Flow and Map ..... 54
4 Semiclassical Analysis ..... 59
4.1 Pseudodifferential Operators on $\mathbb{R}^{d}$ ..... 60
4.2 Pseudodifferential Operators on Manifolds ..... 73
4.3 Microlocalization ..... 82
4.4 Lagrangrian Distributions and Fourier Integral Operators ..... 91
4.5 Shymbol ..... 105
4.6 Semiclassical Intersecting Lagrangian Distributions ..... 107
5 The Semiclassical Melrose-Taylor Parametrix ..... 118
5.1 Semiclassical Melrose-Taylor Parametrix for Complex Energies ..... 119
5.2 Eikonal and Transport Equations ..... 121
5.3 Microlocal description of $H_{d}, H_{g}$ and the Airy multipliers ..... 141
5.4 Parametrix for diffractive points ..... 144
5.5 Relation with exact operators in gliding case ..... 146
5.6 Wave equation parametrices ..... 147
5.7 Semiclassical Fourier integral operators with singular phase ..... 148
6 Boundary Layer Operators ..... 150
6.1 Classical Layer Potential Theory ..... 150
6.2 Quasimode Estimates ..... 155
6.3 Estimates on the Single, Double and Derivative Double Layer Operators ..... 160
6.4 Sharpness of the Estimates for $\lambda \in \mathbb{R}$ ..... 171
6.5 Microlocal Description of the Free Resolvent ..... 175
6.6 Microlocal Decomposition of $G$ and $\partial_{\nu} \mathcal{D} \ell$ Away from Glancing ..... 179
6.7 Boundary layer operators and potentials near glancing ..... 188
7 Harmonic Analysis of $-\Delta_{\Gamma, \delta}$ ..... 198
7.1 Formal Definition of the Operator ..... 203
7.2 Meromorphy of the resolvent and Relation with Outgoing Solutions ..... 205
7.3 Approximation by Regular Potentials ..... 208
7.4 Resonance Expansion for the Wave Equation ..... 211
7.5 The Transmission Property for $C^{1,1}$ Domains ..... 219
8 Microlocal Analysis of $-\Delta_{\partial \Omega, \delta}$ ..... 222
8.1 Conjectures and Numerical Computation of Resonances ..... 226
8.2 Outline of the Proof and Organization of the Chapter ..... 229
8.3 Resonance Free Regions - Analysis of the Boundary Equation ..... 230
9 Existence Resonances for the Delta Potential ..... 238
9.1 Existence for generic domains and potentials. ..... 239
10 Analysis of $-\Delta_{\partial \Omega, \delta^{\prime}}$ ..... 251
10.1 Definition of the Operator and Identification of its Domain ..... 255
10.2 Dynamical Resonance Free Regions for the Delta Prime Potential ..... 258
Bibliography ..... 265
A Notation ..... 271
A. 1 Basic Notation ..... 271
A. 2 Calculus Notations ..... 272
A. 3 Function Spaces ..... 273
A. 4 Operators ..... 274
A. 5 Estimates ..... 274
A. 6 Symbol Classes ..... 275
A. 7 Semiclassical and Microlocal Operators ..... 275

## List of Figures

1.1 Image of a quantum corral ..... 2
1.2 The left hand image shows the original interaction region shaded in grey. On the ..... right, we reduce to the case that the interaction region is a hypersurface. . . . . 2
1.3 Model for calculating reflection coefficients ..... 7
1.4 Schematic of a wave packet inside a thin barrier ..... 7
2.1 Resonances computed for a circle ..... 13
2.2 Numerical computation of resonances for the disk with highly frequency depen- dent potential small Re $\lambda$. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 14
2.3 Numerical computation of resonances for the disk with highly frequency depen-dent potential large Re $\lambda$15
2.4 Numerical computation of resonance bands for the disk (log-log plot) ..... 16
3.1 Billiard ball map ..... 55
5.1 Trajectories for the Friedlander Mode ..... 120
6.1 Example for sharpness of the double layer operator estimates ..... 174
6.2 Wavefront sets for the decomposition of $G$ and $\partial_{\nu} \mathcal{D} \ell$ ..... 189
7.1 Examples of a finite union of compact subsets of strictly convex hypersurfaces, and of the boundary of a domain of $C^{1,1}$ regularity. ..... 199

| 7.2 | The various contours used in Section | 7.4 | to obtain the resonance expansion in |
| :--- | :--- | :--- | :--- |odd dimensions.216

8.1 Resonance free regions for $-\Delta_{\Omega, \delta}$ ..... 227
8.2 Numerically computed resonance free regions for the ellipse and Bunimovich sta- dium ..... 228

## Acknowledgments

First of all, I want to thank Maciej Zworski for suggesting the original problem and instructing me in the techniques necessary to approach it. His mentorship, both mathematical and otherwise, has been an invaluable source of inspiration and support throughout the last five years. I am grateful also to Semyon Dyatlov for many valuable mathematical discussions as well as his encouragement and advice, to Hart Smith for his interest in my work and for introducing me to many new techniques appearing in [31], and to Eric Heller for allowing me to include Figure 1.1 from [6].

I would like to thank Benjamin Harrop-Griffiths, Oran Gannot, and Long Jin for their interest in my work and for many interesting discussions of other problems in mathematics as well as my friends more generally for their camaraderie throughout my Ph.D. I am also thankful to the anonymous referees of my papers for suggesting many improvements and helping me to learn to write more clearly. This research was partially supported by the National Science Foundation through the National Science Foundation Graduate Research Fellowship Grant No. DGE 1106400 and grant DMS-1201417.

Finally, I would like to thank my family for their never ending support, encouragement, and belief in my ability to succeed. Special thanks to my father for helpful comments on a preliminary version of this thesis.

## Chapter 1

## Introduction

In this thesis, we seek to understand the scattering properties of thin barriers. One motivation for this work is to describe the long-term behavior of waves in quantum corrals and systems with similar properties. A quantum corral is a physical system that is assembled by using a scanning tunneling microscope to move individual atoms into a corral shape which partially confines electrons (see Figure 1.1). From our point of view, the important features of quantum corrals are:

1. Electrons propagate with little interaction away from the narrow region where atoms are placed.
2. The potential produced by the confining atoms is intense and localized to a thin region hereafter referred to as the boundary.
3. The potential can vary along the boundary.

Another physical motivation for our study is propagation of sound waves in a concert hall. Just as in the case of a quantum corral, sound propagation in a concert hall enjoys the above properties. Moreover, the strength of the interaction with materials inside walls varies as a function of the frequency of the interacting wave.

In order to model these systems, we imagine that, rather than a potential with support inside a narrow boundary of finite width, the potential is actually supported on a hypersurface (see Figure 1.2). We replace the physical potential by a model potential $V_{\bmod } \otimes \delta_{\Gamma}$ where $\delta_{\Gamma}$ is the Hausdorff $d-1$ measure on some hypersurface $\Gamma \Subset \mathbb{R}^{d}$ as done by Heller $|6|$ and Crommie [18]. We then study the decay of solutions to

$$
\begin{equation*}
\left(\partial_{t}^{2}+\left(-\Delta+\delta_{\Gamma} \otimes V_{\mathrm{mod}}\right)\right) u=0 \tag{1.0.1}
\end{equation*}
$$

That is, we study a delta function potential supported on a hypersurface. In section 7.3 , we show that this model is an accurate approximation of the physical potential. This model is also used to study so-called leaky quantum graphs. (See for example the summary article of Exner [26].)


Figure 1.1: This figure shows an image of a quantum corral taken using a scanning tunneling microscope. The atoms produce the large spikes in the potential around the boundary of a Bunimovich stadium. The smaller ripples are the wavefunction of an electron. One can see that while most of the wavefunction is confined inside the corral, there are smaller ripples in the exterior. This image is included from [6] with the permission of the authors.


Figure 1.2: The left hand image shows the original interaction region shaded in grey. On the right, we reduce to the case that the interaction region is a hypersurface.

In one dimension $-\Delta$ has a four dimensional family of self-adjoint extensions from $C_{c}^{\infty}(\mathbb{R} \backslash$ $\{0\}$ ) (see for example the work of Seba [64]). There is a two parameter family of such extensions for which the corresponding operator decouples into the the sum of a self-adjoint realization of $-\Delta$ on $(-\infty, 0)$ and one on $(0, \infty)$, and so does not produce interesting new behavior. The other two parameters correspond formally to

$$
\begin{equation*}
-\Delta+a \delta(x)+b \delta^{\prime}(x) \partial_{x} \tag{1.0.2}
\end{equation*}
$$

Thus, all 'transmissive' self-adjoint realizations of single point interactions are given formally by (1.0.2). Motivated by this in addition to the interest in $\delta^{\prime}$ interactions in mathematical physics $[2,28,54,64]$ and spectral theory $[3,44]$, as well as another model of leaky quantum graphs [26], we study scattering resonances for the operator

$$
\begin{equation*}
-\Delta_{\partial \Omega, \delta^{\prime}}:=-\Delta+\delta_{\partial \Omega}^{\prime} \otimes V_{\bmod } \partial_{\nu} \quad \delta_{\partial \Omega}^{\prime}(u):=\int_{\partial \Omega}-\partial_{\nu} u d S \tag{1.0.3}
\end{equation*}
$$

Because (1.0.2) represents all possible 'transmissive' single point interactions in 1 dimension, we expect that combinations of the $\delta_{\partial \Omega}$ interaction and the $\delta_{\partial \Omega}^{\prime}$ interaction represent all possible 'thin barriers' supported on $\partial \Omega$ in higher dimensions.

Solutions to 1.0.1 have resonance expansions of the form

$$
\begin{equation*}
u(t, x) \sim \sum_{z \in \text { Res }} e^{-i t z} u_{z}(x) \tag{1.0.4}
\end{equation*}
$$

where Res $\subset \mathbb{C}$ is a discrete set called the scattering resonances of the operator

$$
\begin{equation*}
-\Delta_{\Gamma, \delta}:=-\Delta+V_{\bmod } \otimes \delta_{\Gamma} \tag{1.0.5}
\end{equation*}
$$

(See Section 7.4 for a more precise statement.) Notice that solutions to the wave equation

$$
\left(\partial_{t}^{2}-\Delta\right) u=0,(t, x) \in \mathbb{R} \times\left.\Omega \quad u\right|_{\mathbb{R} \times \partial \Omega}=0
$$

with $\Omega$ a compact subset of $\mathbb{R}^{d}$ also have expansions of the form 1.0 .4 where $z \in \operatorname{Res}$ is replaced by $z$ an eigenvalue of the Dirichlet problem on $\Omega$. Thus, scattering resonances in the non-compact setting are analogous to eigenvalues in the compact setting.

The real and (negative) imaginary part of $z \in$ Res respectively give the frequency and decay rate of the associated resonant state and hence, resonances close to the real axis give information about the long term behavior of solutions to (1.0.1). In their seminal works, LaxPhillips [46] and Vainberg [78] understood the relation between propagation of singularities for the wave equation and the presence of scattering poles near the real axis. Through (1.0.4), this gives control over the long term decay of waves. We use this relation in Chapter 9 to demonstrate the existence of resonance with prescribed decay rates.

The scattering resonances of an operator, $P$, are defined to be the poles of the meromorphic continuation of the resolvent

$$
R_{P}(\lambda):=\left(P-\lambda^{2}\right)^{-1}
$$

from $\operatorname{Im} \lambda \gg 1$. In order to give an expansion of the form (1.0.4) (and hence prove exponential decay for waves), we need to find a region free of resonances near the real axis. Since the set of poles of the resolvent is discrete, it suffices to study resonances with $|\operatorname{Re} \lambda| \geq C$ and hence to studying high frequencies. Because of this, our main intuition comes from the quantumclassical correspondence: high energy waves inherit many properties of the corresponding classical dynamics.

To describe the classical dynamics of a system we use the Hamiltonian formalism. In this formalism, we let the Hamiltonian, $p(x, \xi)$, give the energy of a particle at a given position, $x$, and momentum, $\xi$. The flow

$$
\left\{\begin{array}{l}
\partial_{t} x(t)=\partial_{\xi} p(x, \xi) \\
\partial_{t} \xi(t)=-\partial_{x} p(x, \xi)
\end{array}\right.
$$

then describes the motion of a particle. In the theory of scattering by smooth compactly supported potentials, the energy of the system can be described as the sum of the kinetic energy, given by the momentum squared, plus the potential energy, given by the value of the potential. (We have assumed that the particle has unit mass.) That is,

$$
\begin{equation*}
p(x, \xi):=|\xi|^{2}+V(x) \tag{1.0.6}
\end{equation*}
$$

In such situations, it is easy to see that if the energy of a particle, $E$, is such that $\{V(x)<E\}$ has a component which is isolated from infinity, then there are particles with energy $E$ that never escape to infinity. Such particles are referred to as trapped particles. Since such a system produces confinement on the classical level, one expects decay of waves to result only from tunneling effects and hence for the decay to be very slow. There has been an extensive study of resonances for systems with various kinds of trapping (see for example the book of Dyatlov-Zworski [21, Chapter 7] or the paper of Nonnemacher-Zworski [55] and references therein).

When we work with a genuine wave or quantum system rather than the particle model, there is no notion of exact momentum or position. Instead, we think of observables as operators that come from quantizing classical properties. In particular,

$$
x_{i} \mapsto M_{x_{i}} \quad \xi_{i} \mapsto h D_{x_{i}}, \quad D_{x_{i}}=-i \partial_{x_{i}}
$$

where $h$ represents the inverse of frequency and $M_{x_{i}}$ multiplication by $x_{i}$. (For more precise details on this quantization procedure see Chapter 4.) In particular, the quantization of the Hamiltonian 1.0.6 is

$$
-h^{2} \Delta+V(x) .
$$

We are interested $P-\lambda^{2}$ for $|\operatorname{Re} \lambda| \gg 1$ and hence it is convenient to write $\lambda=z / h$ with $h \ll 1$. This converts problems of the form

$$
-\Delta+V(x)-(z / h)^{2} \rightarrow-h^{2} \Delta+h^{2} V(x)-z^{2} .
$$

Since we want to study operators of the form

$$
-\Delta+\delta_{\Gamma} \otimes V_{\mathrm{mod}}-\lambda^{2} \quad \text { and } \quad-\Delta-\delta_{\partial \Omega}^{\prime} \otimes V_{\bmod } \partial_{\nu}-\lambda^{2}
$$

we replace $V(x)$ in (1.0.6) with $h^{2} V_{\bmod } \otimes \delta_{\Gamma} \quad\left(\mathrm{or}-\delta_{\partial \Omega}^{\prime}\right)$. Then we can think of a potential of the form $h^{2} V_{\bmod } \otimes \delta_{\Gamma}$ as the distributional limit of a sequence of potentials $\left\{h^{2} V_{n}\right\} \subset C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ (see Section 7.3 for a precise version of this idea). As $n$ increases, $V_{n}$ narrows and increases in intensity. Because of the $h^{2}$ scaling, for each fixed $n$, the potential will not produce confinement at any positive energy $E$. However, as $V_{n}$ increases without bound, we expect the corresponding classical dynamics to approach the billiard ball flow (see Section 3.2). Thus, if $\mathbb{R}^{d} \backslash \Gamma$ has a bounded component, we expect classical confinement at any energy $E$. Using this naive analysis, we might expect very slow decay of waves at any frequency. However, as the potential $V_{n}$ narrows, tunneling effects decrease the strength of confinement. In fact, the precise analysis of scattering by delta functions, $\Delta_{\Gamma, \delta}$, presented in this thesis shows that if $V_{\text {mod }}$ grows mildly with frequency, then the confinement produced is only slightly stronger than that for $V \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$. However, if $V_{\mathrm{mod}}$ is allowed to depend strongly on frequency, then we demonstrate that as a result of effects coming from paths $x(t)$ nearly tangent to the submanifold $\Gamma$, confinement can become much stronger than that for $V \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$. Similarly, if the potential is more singular than $\delta_{\Gamma}$, then confinement becomes stronger than that for $V \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$.

The main goal of this thesis is to understand the precise nature of the distribution of resonances near the real axis for thin barriers and, as a by-product, to prove an expansion of the form $(1.0 .4)$ for $-\Delta_{\Gamma, \delta}$. A key step in doing so is to relate the poles of $R_{P}(\lambda)$ to the existence of nonzero $\lambda$-outgoing solutions to

$$
\begin{equation*}
\left(P-\lambda^{2}\right) u=0 . \tag{1.0.7}
\end{equation*}
$$

By $\lambda$-outgoing we mean that there exist $M>0$ and $\varphi \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ with

$$
u(x)=\left(R_{0}(\lambda) \varphi\right)(x), \quad \text { for }|x| \geq M
$$

where, $R_{0}(\lambda)$, the free resolvent, is the meromorphic continuation of

$$
R_{0}(\lambda):=\left(-\Delta-\lambda^{2}\right)^{-1}
$$

from $\operatorname{Im} \lambda \gg 1$. For the case of $-\Delta_{\partial \Omega, \delta}$ this is equivalent to solving

$$
\begin{cases}\left(-\Delta-\lambda^{2}\right) u_{1}=0 & \text { in } \Omega  \tag{1.0.8}\\ \left(-\Delta-\lambda^{2}\right) u_{2}=0 & \text { in } \mathbb{R}^{d} \backslash \bar{\Omega} \\ \left.u_{1}\right|_{\partial \Omega}=\left.u_{2}\right|_{\partial \Omega} & \\ \partial_{\nu} u_{1}-\partial_{\nu} u_{2}+V u_{1}=0 & \text { on } \partial \Omega \\ u_{2} \text { is } \lambda \text {-outgoing } & \end{cases}
$$

and for the case of $-\Delta_{\partial \Omega, \delta^{\prime}}$ it is equivalent to solving

$$
\begin{cases}\left(-\Delta-\lambda^{2}\right) u_{1}=0 & \text { in } \Omega  \tag{1.0.9}\\ \left(-\Delta-\lambda^{2}\right) u_{2}=0 & \text { in } \mathbb{R}^{d} \backslash \bar{\Omega} \\ \left.\partial_{\nu} u_{1}\right|_{\partial \Omega}=\left.\partial_{\nu} u_{2}\right|_{\partial \Omega} & \\ u_{1}-u_{2}+V \partial_{\nu} u_{1}=0 & \text { on } \partial \Omega \\ u_{2} \text { is } \lambda \text {-outgoing } & \end{cases}
$$

We postpone the proof of these results to Chapters 7 and 10 . Equations of the form (1.0.8) and $(1.0 .9)$ are called transmission problems and resonances for such systems have been considered in other cases. For example, Popov-Vodev [58] and Cardoso-Popov-Vodev [13, 14 consider the case of a transparent obstacle having differing wave speeds inside and outside $\Omega$.

In order to gain some heuristic understanding of how resonances behave for $-\Delta_{\partial \Omega, \delta^{\prime}}$ and $-\Delta_{\Gamma, \delta}$, we look to the case where $\partial \Omega=\left\{x_{1}=0\right\} \subset \mathbb{R}^{d}$. We consider a plane wave with frequency $h^{-1}$, $e^{\frac{i}{h}\langle x, \xi\rangle}$, approaching $x_{1}=0$ from the left. (See Figure 1.3 for a depiction of the setup.) We are then interested in what fraction of the wave is reflected by the barrier and what fraction is transmitted. Let $R_{\delta}$ and $R_{\delta^{\prime}}$ denote the reflection coefficients and $T_{\delta}$, $T_{\delta^{\prime}}$ the transmission coefficients.

By a formal computation, one can see that the appropriate transmission condition for $V \delta\left(x_{1}\right)$ is

$$
u_{+}\left(0, x^{\prime}\right)=u_{-}\left(0, x^{\prime}\right) \quad \partial_{x_{1}} u_{-}\left(0, x^{\prime}\right)-\partial_{x_{1}} u_{+}\left(0, x^{\prime}\right)+V u_{+}\left(0, x^{\prime}\right)=0
$$

This leads to

$$
\begin{equation*}
R_{\delta}=\frac{h V}{2 i \xi_{1}-h V}, \quad T_{\delta}=\frac{2 i \xi_{1}}{2 i \xi_{1}-h V} \tag{1.0.10}
\end{equation*}
$$

By a similar formal computation, one can see that the appropriate transmission condition for $-V \delta^{\prime}\left(x_{1}\right)$ is given by

$$
\partial_{x_{1}} u_{+}\left(0, x^{\prime}\right)=\partial_{x_{1}} u_{-}\left(0, x^{\prime}\right) \quad u_{-}\left(0, x^{\prime}\right)-u_{+}\left(0, x^{\prime}\right)+V \partial_{x_{1}} u_{-}^{\prime}\left(0, x^{\prime}\right)=0
$$

which leads to

$$
\begin{equation*}
R_{\delta^{\prime}}=\frac{V i \xi_{1}}{V i \xi_{1}-2 h}, \quad T_{\delta^{\prime}}=\frac{2 h}{2 h-V i \xi_{1}} . \tag{1.0.11}
\end{equation*}
$$

Since we want to consider waves with frequency equal to $h^{-1}$, we have that $\xi \in S^{d-1}$. When $\xi_{1}$ is near 0 , the plane wave travels nearly tangent to $x_{1}=0$. Our first observation is that as $\xi_{1} \rightarrow 0, R_{\delta} \rightarrow 1$ while $R_{\delta^{\prime}} \rightarrow 0$. This reflects the fact that the normal derivative to $\Gamma=\left\{x_{1}=0\right\}$ does not see frequencies that are tangent to $\Gamma$. Thus, we expect glancing (tangent) trajectories to contribute less to the resonances of $-\Delta_{\partial \Omega, \delta^{\prime}}$ than to $-\Delta_{\Gamma, \delta}$.

To get a more quantitative heuristic for the resonances, we imagine solving the wave equation

$$
\left(\partial_{t}^{2}-P\right) u=0,\left.\quad u\right|_{t=0}=u_{0},\left.\quad u_{t}\right|_{t=0}=0
$$

\[

\]

Figure 1.3: The setup for plane wave interactions. Here $R_{\delta}$ and $R_{\delta^{\prime}}$ are the reflection coefficients and $T_{\delta}$ and $T_{\delta^{\prime}}$ are the transmission coefficients.


Figure 1.4: The figure shows the path of a wave packet along with the lengths between each intersection $\left(l_{i}\right)$ and the reflection coefficient at each point of intersection with the boundary $\left(R_{i}\right)$. After each reflection with the boundary, the amplitude of the wave packet inside $\Omega$ decays by a factor of $R_{i}$. The time between reflections is given by $l_{i}$.
where $P$ is either $-\Delta_{\partial \Omega, \delta^{\prime}}$ or $-\Delta_{\partial \Omega, \delta}$ with initial data $u_{0}$ a wave packet (that is a function localized in frequency and space up to the scale allowed by the uncertainty principle) localized at position $x_{0} \in \Omega$ and momentum $\xi_{0} \in S^{d-1}$. Then our heuristic computations giving (1.0.10) and (1.0.11) suggest that at each intersection of the billiard flow starting from $\left(x_{0}, \xi_{0}\right)$ with $x_{0} \in \Omega$, the amplitude inside of $\Omega$ will decay by a factor of $R$. Suppose that the billiard flow from $\left(x_{0}, \xi_{0}\right)$ intersects the boundary at $\left(x_{n}, \xi_{n}\right) n>0$. Let $l_{n}=\left|x_{n+1}-x_{n}\right|$ be the distance between two consecutive intersections with the boundary (see Figure 1.4). Then the amplitude of the wave decays by a factor $\prod_{i=1}^{n} R_{i}$ in time $\sum_{i=1}^{n} l_{i}$ where $R_{i}=R\left(x_{i}, \xi_{i}\right)$. The energy scales as amplitude squared and since the imaginary part of a resonance gives the exponential decay rate of $L^{2}$ norm, this leads us to the heuristic that resonances should occur at

$$
\begin{equation*}
\operatorname{Im} z=\frac{h}{2} \frac{\overline{\log |R|^{2}}}{\bar{l}} \tag{1.0.12}
\end{equation*}
$$

where the map ${ }^{-}$is defined by $\bar{f}=\frac{1}{N} \sum_{i=1}^{N} f_{i}, \lambda=z h^{-1}$, and $\operatorname{Re} z=1$. In the early 1900 s , Sabine [61] postulated that the decay rate of acoustic waves in a region with leaky walls is determined by the average decay over billiards trajectories. The expression (1.0.12) is a precise description of this statement. In Chapters 8 and 10 we show that a version of (1.0.12) and hence a Sabine type law holds for both $-\Delta_{\partial \Omega, \delta^{\prime}}$ and $-\Delta_{\partial \Omega, \delta}$ under certain conditions on the potential $V$ and the domain $\Omega$.

Equation 1.0.12 suggest that the resonances of $-\Delta_{\partial \Omega, \delta}$ lie in regions with $\operatorname{Im} z \sim$ $h \log h^{-1}$. On the other hand, if we assume that $V \sim h^{\alpha}$ for $\alpha<1$ then we obtain for $-\Delta_{\partial \Omega, \delta^{\prime}}$ that $\operatorname{Im} z \sim h^{3-2 \alpha}$. Thus, the resonances for $-\Delta_{\partial \Omega, \delta^{\prime}}$ are much closer to the real axis than those for $-\Delta_{\partial \Omega, \delta}$. Indeed, when written in terms of $\lambda$, the resonances for $-\Delta_{\partial \Omega, \delta^{\prime}}$ converge to the real axis at a fixed polynomial rate while those for $-\Delta_{\partial \Omega, \delta}$ diverge logarithmically from the real axis.

## Outline of the Thesis

We begin in Chapter 2 by analyzing the model case of $\Gamma=\partial B(0,1) \subset \mathbb{R}^{2}$ and $V_{\text {mod }}$ constant. In this case, we are able to separate variables and solve 1.0.7) in terms of Bessel functions for both $P=-\Delta_{\partial \Omega, \delta^{\prime}}$ and $P=-\Delta_{\partial \Omega, \delta}$. This reduces the study of resonances to asymptotic analysis of certain transcendental equations. The heuristic (1.0.12) and the fact that $R_{\delta^{\prime}} \rightarrow 0$ as $\xi_{1} \rightarrow 0$, suggest that the slowest decay rates for $-\Delta_{\partial \Omega, \delta^{\prime}}$ should come from non-glancing wave packets. Thus, we also consider the 1 dimensional case for $-\Delta_{\partial \Omega, \delta^{\prime}}$. Many of the results in Chapter 2 are special cases of the more general theorems that we present in later chapters. However, since we work with models where separation of variables is possible, we are able to explore some regimes where the more general techniques fail to give satisfactory analyses.

Chapters 3 and 4 are devoted to a review of the geometric and analytical tools that are used in the analysis of $-\Delta_{\Gamma, \delta}$ and $-\Delta_{\partial \Omega, \delta^{\prime}}$. In addition to this review, Chapter 4 develops a notion of a sheaf-valued symbol that is sensitive to local changes of semiclassical order. Finally, it adapts the Melrose-Uhlmann [49] notion of an intersecting Lagrangian distribution to the semiclassical setting.

One of our main goals in Chapters 3 and 4 is to give a self-contained presentation of the theory of semiclassical Fourier integral operators and Lagrangian distributions. We start by reviewing local symplectic geometry in Chapter 3. We then review the basics of semiclassical analysis in Chapter 4. In addition to this, Chapter 3 contains the necessary background on the billiard ball map and flow.

The next major tool that is used in our analysis is the Melrose-Taylor parametrix [47]. The parametrix was developed to understand the wave equation near curved boundaries and was adapted by Gerard and Stefanov-Vodev for use in the semiclassical Dirichlet problem outside a strictly convex obstacle in $\sqrt[32]{ }, 69]$. In Chapter 5 we adapt this construction to the Dirichlet problem in the interior and exterior of a convex domain and to perturbative ( $\operatorname{Im} z \leq$ $M h \log h^{-1}$ ) complex energies. We then use the semiclassical Melrose-Taylor parametrix to give microlocal models for the exterior the Dirichlet to Neumann map near a glancing point as well as for boundary layer potentials.

With these tools in place, we begin to analyze $-\Delta_{\Gamma, \delta}$ and $-\Delta_{\partial \Omega, \delta^{\prime}}$. We show in Chapter 7 that resonances of $-\Delta_{\Gamma, \delta}$ occur at $\lambda$ for which there exist nontrivial solutions $\varphi \in L^{2}(\Gamma)$ to

$$
\begin{equation*}
(I+G(\lambda) V) \varphi=0 \tag{1.0.13}
\end{equation*}
$$

where $G$ is the single layer operator. That is, the operator given by

$$
G(\lambda) f(x)=\int_{\Gamma} R_{0}(\lambda)(x, y) f(y) d S(y), \quad x \in \partial \Omega
$$

Moreover, in Chapter 10 we will see that (except for $d=1$ and $\lambda=0$ ) the resonances of $-\Delta_{\partial \Omega, \delta^{\prime}}$ occur at $\lambda$ for which there exist nontrivial solutions $\varphi \in H^{1}(\partial \Omega)$ to

$$
\begin{equation*}
\left(I-\partial_{\nu} \mathcal{D} \ell(\lambda) V\right) \varphi=0 \tag{1.0.14}
\end{equation*}
$$

where $\partial_{\nu} \mathcal{D} \ell$ is the derivative double layer operator. That is, the operator given by

$$
\partial_{\nu} \mathcal{D} \ell(\lambda) f(x)=\int_{\partial \Omega} \partial_{\nu_{x}} \partial_{\nu_{y}} R_{0}(\lambda)(x, y) f(y) d S(y), \quad x \in \partial \Omega
$$

Thus, our first step is to analyze the boundary layer operators $G, \tilde{N}$, and $\partial_{\nu} \mathcal{D} \ell$, which we do in Chapter 6. Here, we write $\tilde{N}$ for the double layer operator given by

$$
\tilde{N}(\lambda) f(x)=\int_{\partial \Omega} \partial_{\nu_{y}} R_{0}(\lambda)(x, y) f(y) d S(y), \quad x \in \partial \Omega
$$

We first prove high energy estimates for these operators using restriction bounds for eigenfunctions and their derivatives. We then show that these bounds are nearly sharp (i.e. sharp modulo a $\log \lambda \operatorname{loss}$ ).

Our next task is to give a microlocal description of $G$ and $\partial_{\nu} \mathcal{D} \ell$. To do this we use the semiclassical intersecting Lagrangians developed in Chapter 4 to give a microlocal description of the free resolvent. With this in hand, we are able to use the calculus of semiclassical Fourier integral operators to give a microlocal description of $G$ and $\partial_{\nu} \mathcal{D} \ell$ away from glancing i.e. away from momenta $\xi$ that are tangent to the boundary. In the case that $\Omega$ is strictly convex, we use the semiclassical Melrose-Taylor parametrix to understand $G$ and $\partial_{\nu} \mathcal{D} \ell$ near glancing. Finally, we use this microlocal model to remove the log loss from the estimates for $G$ and $\partial_{\nu} \mathcal{D} \ell$ in the case that $\Omega$ is strictly convex.

In Chapters 7, 8, and 9, we analyze the distribution of resonances for $-\Delta_{\Gamma, \delta}$. Chapter 7 gives the formal definition of $-\Delta_{\Gamma, \delta}$ when $\Gamma$ is a finite union of subsets of $C^{1,1}$ embedded hypersurfaces. We then prove the meromorphic continuation of the resolvent for such an operator and use the estimates on $G$ from Chapter 6 to give a rough bound for the size of the resonance free region. This simple bound along with some additional estimates on $R_{-\Delta_{\Gamma, \delta}}$ are enough to give a resonance expansion of the form (1.0.4) for $d$ odd ${ }^{11}$. In Chapter 7. we also show that $-\Delta_{\Gamma, \delta}$ is a good approximation to narrow but intense potentials.

In Chapter 8, we restrict our attention to $\Omega$ strictly convex with smooth boundary and perform a microlocal analysis of 1.0 .13 ) to give a dynamical characterization of the size of the resonance free region for $-\Delta_{\partial \Omega, \delta}$. We also give some conjectures and numerical results.

[^0]Then, in Chapter 9, we show that the dynamical resonance free region from Chapter 8 is generically sharp.

Finally, in Chapter 10, we consider $-\Delta_{\partial \Omega, \delta^{\prime}}$. We first give the formal definition of the operator along with a proof of the meromorphic continuation of its resolvent. It is already necessary to have some microlocal understanding of $\partial_{\nu} \mathcal{D} \ell$ to give a proof of the meromorphic continuation and so we restrict our attention to $\Omega$ with smooth boundary. Finally, further restricting to $\Omega$ strictly convex, we give a dynamical characterization of the size of the resonance free region for $-\Delta_{\partial \Omega, \delta^{\prime}}$. This characterization is sharp when $\Omega$ is the unit disk $\mathbb{R}^{2}$ and $V$ is constant.

Resonance free regions for $-\Delta_{\partial \Omega, \delta^{\prime}}$ are of the form $\operatorname{Im} \lambda \geq-C(\operatorname{Re} \lambda)^{-\gamma}$ for some fixed $\gamma>0$. As far as the author is aware, the operator $-\Delta_{\partial \Omega, \delta^{\prime}}$ with $\partial \Omega$ smooth and strictly convex is the only general class of examples known to exhibit such behavior. The only other specific example known is that of $-\Delta_{B(0,1), \delta}$ when $V$ depends strongly on frequency (see Chapter 2).

Appendix A contains a list of some of the notation used throughout this thesis.
Remark: Much of the work pertaining to $-\Delta_{\Gamma, \delta}$ is contained in the author's previous papers. The analysis of $-\Delta_{\Gamma, \delta}$ in Chapter 2 comes from [30]. Much of Chapter 7 comes from Galkowski-Smith 31]. The estimates on $G$ and $\tilde{N}$ can be found in 31 and Galkowski-Han-Tacy [37]. Much of the material pertaining to $G$ in Chapter 6 as well as the material in Chapters 5, 8, and 9 comes from Galkowski [29].

## Chapter 2

## Model Cases

### 2.1 Introduction

In the present chapter, we seek to understand resonances for a model case. In particular, we consider $-\Delta_{\partial \Omega, \delta},-\Delta_{\partial \Omega, \delta^{\prime}}$, when $\Omega=B(0,1) \subset \mathbb{R}^{2}$ and $V$ is a constant depending on $h$. In this case, we are able to separate variables and avoid most of the microlocal analysis involved in obtaining the more general results. Separating variables reduces the existence of resonances to the existence of a solution to one of an infinite family of transcendental equations. The symbols of the operators involved in the general analyses appear as asymptotic limits of the Bessel and Airy functions in these equations.

## Statement of results for the $\delta$ potential

For the purposes of this section, we define the resonances of $-\Delta_{\partial \Omega, \delta}$ as follows: We say that $z / h$ is a resonance for $-\Delta_{\partial \Omega, \delta}$ if there exists a nonzero $z / h$-outgoing solution, $\left(u_{1}, u_{2}\right) \in$ $H^{2}(\Omega) \oplus H_{\mathrm{loc}}^{2}\left(\mathbb{R}^{d} \backslash \Omega\right)$ to

$$
\begin{cases}\left(-h^{2} \Delta-z^{2}\right) u_{1}=0 & \text { in } \Omega  \tag{2.1.1}\\ \left(-h^{2} \Delta-z^{2}\right) u_{2}=0 & \text { in } \mathbb{R}^{d} \backslash \bar{\Omega} \\ u_{1}=u_{2} & \text { on } \partial \Omega \\ \partial_{\nu} u_{1}+\partial_{\nu^{\prime}} u_{2}+V \gamma u_{1}=0 & \text { on } \partial \Omega\end{cases}
$$

where, $\partial_{\nu}$ and $\partial_{\nu^{\prime}}$ are respectively the interior and exterior normal derivatives of $u$ at $\partial \Omega$. In Chapter 7, we show that having such a solution corresponds to having a pole $R_{-\Delta_{\partial \Omega, \delta}}$ and hence that these are indeed the resonances for $-\Delta_{\partial \Omega, \delta}$.

Denote the set of rescaled resonances for $-\Delta_{\partial \Omega, \delta}$ by

$$
\begin{align*}
\Lambda(h, \delta) & :=\left\{z \in B(h): z / h \text { is a resonance of }-\Delta_{\partial \Omega, \delta}\right\}  \tag{2.1.2}\\
B(h) & :=\left[1-c h^{3 / 4}, 1+c h^{3 / 4}\right]+i\left[-M h \log h^{-1}, 0\right]
\end{align*}
$$

Remark: The power $3 / 4$ can be taken to be any power $>0$.
We assume throughout that $V \equiv h^{-\alpha} V_{0}$ for $\alpha \leq 1$, and $V_{0}>0$ a constant independent of $h$. The first theorem proves the existence resonance free regions for $\alpha \leq 1$ and bands of resonance free regions for $1 \geq \alpha \geq 5 / 6$.

Theorem 2.1. Let $\Omega=B(0,1) \subset \mathbb{R}^{2}$ and $V \equiv h^{-\alpha} V_{0}>0$. Then for $z \in \Lambda(h, \delta)$ and all $\epsilon>0$ there exists $h_{\epsilon}>0$ such that for $0<h<h_{\epsilon}$, when $\alpha<5 / 6$, there exists $C_{\alpha, V_{0}}$ such that

$$
\begin{equation*}
-\operatorname{Im} z \geq\left(C_{\alpha, V_{0}}-\epsilon\right) h \log h^{-1} \tag{2.1.3}
\end{equation*}
$$

Moreover, when $\alpha \geq 5 / 6$ then for all $M>0$, there exists $h_{M, \epsilon}>0$ such that for $0<h<h_{\epsilon, M}$, either there exists $N>0$ such that

$$
\left|-\operatorname{Im} z h^{2 / 3-2 \alpha}-C_{V_{0}, N}\right|<\epsilon, \quad \text { or } \quad-\operatorname{Im} z \geq M h^{2 \alpha-2 / 3} .
$$

where

$$
C_{V_{0}, N}:=\frac{\sqrt[3]{2}}{8 \pi^{2} V_{0}^{2}\left|A_{-}\left(-\zeta_{N}\right)^{3} A i^{\prime}\left(-\zeta_{N}\right)\right|}
$$

and $-\zeta_{N}$ is the $N^{\text {th }}$ zero of $\operatorname{Ai}(s)$.
Remark: For $\alpha<5 / 6$, the constant in 2.1.3 will be computed using the more general methods in chapter 8 . It is equal to $(1-\alpha) / 2$.

The next theorem shows that the resonance free regions above are sharp.
Theorem 2.2. For all $N>0$, there exists $h_{0}>0$ such that for $h<h_{0}$, there exist $z(h) \in \Lambda$ with

$$
-\operatorname{Im} z(h)= \begin{cases}\frac{1-\alpha}{2} h \log h^{-1}-\frac{h}{2} \log \frac{V_{0}}{2}+O\left(h^{7 / 4}\right) & \alpha<1 \\ \frac{h}{4} \log \left(1+\frac{4}{V_{0}^{2}}\right)+O\left(h^{7 / 4}\right) & \alpha=1 \\ C_{V_{0}, N} h^{2 \alpha-2 / 3}+O\left(h^{3 \alpha-4 / 3}\right) & 2 / 3<\alpha \leq 1\end{cases}
$$

The proofs of Theorems 2.1 and 2.2 show that when $\alpha<5 / 6$ the resonances closest to the real axis come from modes concentrating away from glancing, while those for $\alpha \geq 5 / 6$ come from modes concentrating near glancing. Thus, the theorems show that glancing modes decay slower than non-glancing modes for $\alpha \geq 5 / 6$ while the opposite is true for $\alpha<5 / 6$ and gives a quantitative rate of decay for each type of mode.
Remark: When $B(0,1)$ is replaced by $B(0, R)$ we can use the same arguments that prove Theorems 2.1 and 2.2 to find that the resonance free region for $\Omega=B(0, R)$ and $\alpha \geq 5 / 6$ is given by $-\operatorname{Im} z \geq\left(C_{R^{2 / 3} V_{0}}-\epsilon\right) h^{2 \alpha-2 / 3}$. Hence the imaginary part of resonances from glancing modes scale as $\kappa^{4 / 3}$ where $\kappa$ is the curvature.

We also give a lower bound on the number of resonances.


Figure 2.1: When $\Omega=B(0,1) \subset \mathbb{R}^{2}$, the boundary values of resonance states can be expanded in a Fourier series $\sum a_{n} e^{i n x}$. We show the resonances for $V \equiv 1$ corresponding to the $n=0,10,100$, and 500 modes. The solid line shows the bound given by Theorem 2.1.

Theorem 2.3. For $M$ large enough, there exists $c>0$ such that

$$
\#\left\{z \in[1-\epsilon, 1+\epsilon]+i\left[-M h \log h^{-1}, 0\right]: z / h \text { is a resonance of }-\Delta_{V, \partial \Omega}\right\} \geq c h^{-2} .
$$

Remark:We have an upper bound of the form $C h^{-2}$ by [66], 81], 82], and 83] together with [31, Lemma 7.1](see also Lemma 7.4.1).

We present the proofs of Theorems 2.1, 2.2, and 2.3 in Section 2.3.

## Statement of results for the $\delta^{\prime}$ potential

As for the case of the $\delta$ potential, we make a preliminary definition of resonances in order to present simple arguments in the case of the disk. In particular, we say that $z / h$ is a resonance of $-\Delta_{\partial \Omega, \delta^{\prime}}$ if there exists a nonzero $z / h$-outgoing solution $\left(u_{1}, u_{2}\right) \in H^{2}(\Omega) \oplus H_{\mathrm{loc}}^{2}\left(\mathbb{R}^{d} \backslash \Omega\right)$ to

$$
\begin{cases}\left(-h^{2} \Delta-z^{2}\right) u_{1}=0 & \text { in } \Omega  \tag{2.1.4}\\ \left(-h^{2} \Delta-z^{2}\right) u_{2}=0 & \text { in } \mathbb{R}^{d} \backslash \bar{\Omega} \\ \partial_{\nu_{1}} u_{1}=-\partial_{\nu_{2}} u_{2} & \text { on } \partial \Omega \\ u_{1}-u_{2}+V \partial_{\nu_{1}} u_{1}=0 & \text { on } \partial \Omega\end{cases}
$$

As for the $\delta$ potential, we show in Chapter 7 that having such a solution at $z_{0}$ corresponds to $R_{-\Delta_{\partial \Omega, \delta^{\prime}}}$ having a pole at $z_{0}$.


Figure 2.2: We show resonances for the circle with $\operatorname{Re} \lambda \sim 10^{2}, V_{0}=1$ and several $\alpha$. The plots show $\operatorname{Im} \lambda v \operatorname{Re} \lambda$ in each case. The red line shows the bound coming from nonglancing modes. It is difficult to see the transition at $\alpha=5 / 6$ from logarithmic resonance free regions to polynomial size resonance free regions because the change does not happen until $\operatorname{Re} \lambda \sim 10^{6}$ (see Figure 2.3).

Denote the set of rescaled resonances by

$$
\begin{aligned}
\Lambda\left(h, \delta^{\prime}\right) & :=\left\{z \in B(h): z / h \text { is a resonance of }-\Delta_{\partial \Omega, \delta^{\prime}}\right\} \\
B(h) & :=\left[1-c h^{3 / 4}, 1+c h^{3 / 4}\right]+i\left[-M h \log h^{-1}, 0\right]
\end{aligned}
$$

We assume throughout the analysis of the $\delta^{\prime}$ potential that $V \equiv h^{\alpha} V_{0}$ for $0 \leq \alpha$, and $V_{0}>0$ a constant independent of $h$. The first theorem proves the existence of resonance free regions.

Theorem 2.4. Let $\Omega=B(0,1) \subset \mathbb{R}^{2}$ and $V \equiv h^{\alpha} V_{0}>0, \alpha>2 / 3$. There exists $C_{V_{0}}$ such that for all $z \in \Lambda\left(h, \delta^{\prime}\right)$ and $\epsilon>0$ there exists $h_{\epsilon}>0$ such that for $0<h<h_{\epsilon}$,

$$
\begin{equation*}
-\operatorname{Im} z \geq\left(C_{V_{0}}-\epsilon\right) h^{3-2 \alpha} \tag{2.1.5}
\end{equation*}
$$



Figure 2.3: We show resonances for the circle with $\operatorname{Re} \lambda \sim 10^{6}, V_{0}=1$ and several $\alpha$. The plots show $\operatorname{Im} \lambda$ vs. Re $\lambda$ in each case. The dashed red line shows the (logarithmic) bound for resonances coming from non-glancing trajectories and the black lines show the first few (polynomial) bands of resonances from near glancing trajectories. Since the dashed red line is above the black lines at $\alpha=5 / 6$, it is necessary to go to still larger Re $\lambda$ to see the transition. However, at $\alpha>5 / 6$, we start to see better agreement with the bands of resonances predicted in Theorem 2.1.

Remark: For $\alpha>3 / 4$, the constant in 2.1 .5 will be computed using the more general methods in Chapter 10. It is equal to $V_{0}^{-2}$.

The next theorem shows that the resonance free regions above are sharp.
Theorem 2.5. Let $\Omega$ and $V$ be as in Theorem 2.4. Then there exists $h_{0}>0$ such that for $h<h_{0}$, there exist $z(h) \in \Lambda\left(h, \delta^{\prime}\right)$ with

$$
-\operatorname{Im} z(h)=\left\{\begin{array}{ll}
\left(V_{0}^{-2}+o(1)\right) h^{3-2 \alpha} & 1 / 2<\alpha<1 \\
(1+o(1)) \frac{h}{4} \log \left(1+4 h^{2-2 \alpha} V_{0}^{-2}\right) & \alpha \geq 1
\end{array} .\right.
$$

The proofs of Theorems 2.4 and 2.5 show that the resonances closest to the real axis come from modes concentrating away from glancing. This is consistent with the fact that as discussed after equations (1.0.10) and (1.0.11) in the introduction to this thesis, we expect that the strongest confining effects for the $\delta^{\prime}$ potential come from directions transverse to


Figure 2.4: We show a plot of $\log (\operatorname{Re} \lambda)$ vs. $\log (-\operatorname{Im} \lambda)$ for $\operatorname{Re} \lambda \sim 10^{6}$ when $\alpha=1$ and $V_{0} \equiv 1$. The bands predicted by Theorem 2.1 are shown by the black lines and the bound for the non-glancing modes by the top red line.
the boundary. As such, we may also use a 1 dimensional model to understand the behavior of resonances for a wider range of parameters.

Theorem 2.6. Let $\Omega=(-1,1) \subset \mathbb{R}$ and $V=h^{\alpha} V_{0}>0$. Then for all $z \in \Lambda\left(h, \delta^{\prime}\right)$ and $\epsilon>0$ there exists $h_{0}>0$ such that for $h<h_{0}$,

$$
-\operatorname{Im} z \geq \begin{cases}\left(V_{0}^{-2}-\epsilon\right) h^{3-2 / \alpha} & \alpha<1 \\ (1-\epsilon) \frac{h}{4} \log \left(1+\frac{4 h^{2-2 \alpha}}{V_{0}^{2}}\right) & \alpha \geq 1\end{cases}
$$

Moreover, there exist $z(h) \in \Lambda\left(h, \delta^{\prime}\right)$ with

$$
-\operatorname{Im} z(h)=\left\{\begin{array}{ll}
\left(V_{0}^{-2}+o(1)\right) h^{3-2 \alpha} & \alpha<1 \\
(1+o(1)) \frac{h}{4} \log \left(1+\frac{4 h^{2-2 \alpha}}{V_{0}^{2}}\right) & \alpha \geq 1
\end{array} .\right.
$$

We present the proofs of Theorems 2.4, 2.5, in Section 2.6.

### 2.2 Asymptotics for Airy and Bessel Functions

We collect here some properties of the Airy and Bessel functions that are used in the analysis of case case of the unit disk. These formulae can be found in, for example [56, Chapter 9,10] and [84].

Recall that the Bessel of order $n$ functions are solutions to

$$
z^{2} y^{\prime \prime}+z y^{\prime}+\left(z^{2}-n^{2}\right) y=0
$$

We consider the two independent solutions $H_{n}^{(1)}(z)$ and $J_{n}(z)$. The Wronskian of the two solutions is given by

$$
\begin{equation*}
W\left(J_{n}, H_{n}^{(1)}\right)=J_{n} H_{n}^{(1)^{\prime}}(z)-J_{n}^{\prime} H_{n}^{(1)}(z)=\frac{2 i}{\pi z} \tag{2.2.1}
\end{equation*}
$$

We now record some asymptotic properties of Bessel functions. Consider $n$ fixed and $z \rightarrow \infty$

$$
\begin{align*}
J_{n}(z) & =\left(\frac{1}{2 \pi z}\right)^{1 / 2}\left(e^{i\left(z-\frac{n}{2} \pi-\frac{1}{4} \pi\right)}+e^{-i\left(z-\frac{n}{2} \pi-\frac{1}{4} \pi\right)}+\mathcal{O}\left(|z|^{-1} e^{|\operatorname{Im} z|}\right)\right) \\
H_{n}^{(1)}(z) & =\left(\frac{2}{\pi z}\right)^{1 / 2}\left(e^{i\left(z-\frac{n}{2} \pi-\frac{1}{4} \pi\right)}+\mathcal{O}\left(|z|^{-1} e^{|\operatorname{Im} z|}\right)\right) \\
J_{n}^{\prime}(z) & =i\left(\frac{1}{2 \pi z}\right)^{1 / 2}\left(e^{i\left(z-\frac{n}{2} \pi-\frac{1}{4} \pi\right)}-e^{-i\left(z-\frac{n}{2} \pi-\frac{1}{4} \pi\right)}+\mathcal{O}\left(|z|^{-1} e^{|\operatorname{Im} z|}\right)\right) \\
H_{n}^{(1)^{\prime}}(z) & =i\left(\frac{2}{\pi z}\right)^{1 / 2}\left(e^{i\left(z-\frac{n}{2} \pi-\frac{1}{4} \pi\right)}+\mathcal{O}\left(|z|^{-1} e^{|\operatorname{Im} z|}\right)\right) \\
J_{n}(z) H_{n}(z) & =\frac{1}{\pi z}\left(e^{i\left(2 z-n \pi-\frac{1}{2} \pi\right)}+1+\mathcal{O}\left(|z|^{-1} e^{2|\operatorname{Im} z|}\right)\right)  \tag{2.2.2}\\
J_{n}^{\prime}(z) H_{n}^{\prime}(z) & =-\frac{1}{\pi z}\left(e^{i\left(2 z-n \pi-\frac{1}{2} \pi\right)}-1+\mathcal{O}\left(|z|^{-1} e^{2|\operatorname{Im} z|}\right)\right) \tag{2.2.3}
\end{align*}
$$

Consider $\alpha$ fixed $|\operatorname{Arg} \alpha|<\pi / 2, z=\operatorname{sech} \alpha$, and $n \rightarrow \infty$

$$
\begin{align*}
J_{n}(n z) & \left.=\frac{e^{n(\tanh \alpha-\alpha)}}{(2 \pi n \tanh (\alpha))^{1 / 2}}\left(1+O\left(n^{-1}\right)\right)\right) \\
J_{n}^{\prime}(n z) & \left.=\frac{(\sinh (2 \alpha))^{1 / 2}}{(4 \pi n)^{1 / 2}} e^{n(\tanh \alpha-\alpha)}\left(1+O\left(n^{-1}\right)\right)\right) \\
H_{n}^{(1)}(n z) & =-i \frac{e^{n(\alpha-\tanh \alpha)}}{\left(\frac{1}{2} \pi n \tanh \alpha\right)^{1 / 2}}\left(1+O\left(n^{-1}\right)\right) \\
H_{n}^{(1)^{\prime}}(n z) & =i\left(\frac{\sinh (2 \alpha)}{\pi n}\right)^{1 / 2} e^{n(\alpha-\tanh (\alpha))}\left(1+O\left(n^{-1}\right)\right) \\
J_{n}(n z) H_{n}^{(1)}(n z) & =-\frac{i}{\pi n \tanh (\alpha)}\left(1+O\left(n^{-1}\right)\right)  \tag{2.2.4}\\
J_{n}^{\prime}(n z) H_{n}^{(1)^{\prime}}(n z) & =i \frac{\sinh (2 \alpha)}{2 \pi n}\left(1+O\left(n^{-1}\right)\right) \tag{2.2.5}
\end{align*}
$$

Next, we record asymptotics that are uniform in $n$ and $z$ as $n \rightarrow \infty$. Let $\zeta=\zeta(z)$ be the unique smooth solution on $0<z<\infty$ to

$$
\begin{equation*}
\left(\frac{d \zeta}{d z}\right)^{2}=\frac{1-z^{2}}{\zeta z^{2}} \tag{2.2.6}
\end{equation*}
$$

with

$$
\lim _{z \rightarrow 0} \zeta=\infty, \quad \lim _{z \rightarrow 1} \zeta=0, \quad \lim _{z \rightarrow \infty} \zeta=-\infty
$$

Then

$$
\begin{array}{rlrl}
\frac{2}{3}(-\zeta)^{3 / 2} & =\sqrt{z^{2}-1}-\operatorname{arcsec}(z) & 1<z<\infty \\
\frac{2}{3}(\zeta)^{3 / 2} & =\log \left(\frac{1+\sqrt{1-z^{2}}}{z}\right)-\sqrt{1-z^{2}} & 0<z<1 \\
\frac{1-z^{2}}{\zeta z^{2}} & \rightarrow \sqrt[3]{2} & & z \rightarrow 0 \tag{2.2.8}
\end{array}
$$

Let

$$
A i(s)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{i\left(\frac{1}{3} t^{3}+s t\right)} d t
$$

for $s \in \mathbb{R}$ be the Airy function solving

$$
A i^{\prime \prime}(z)-z A i(z)=0
$$

Then, $A_{-}(z)=A i\left(e^{2 \pi i / 3} z\right)$ is another solution of the Airy equation.

For $z$ fixed as $n \rightarrow \infty$

$$
\begin{align*}
J_{n}(n z) & =\left(\frac{4 \zeta}{1-z^{2}}\right)^{1 / 4}\left(\frac{A i\left(n^{2 / 3} \zeta\right)}{n^{1 / 3}}+O(E i(5 / 3,7 / 3))\right) \\
H_{n}^{(1)}(n z) & =2 e^{-\pi i / 3}\left(\frac{4 \zeta}{1-z^{2}}\right)^{1 / 4}\left(\frac{A_{-}\left(n^{2 / 3} \zeta\right)}{n^{1 / 3}}+O\left(E_{-}(5 / 3,7 / 3)\right)\right) \\
J_{n}^{\prime}(n z) & =-\frac{2}{z}\left(\frac{1-z^{2}}{4 \zeta}\right)^{1 / 4}\left(\frac{A i^{\prime}\left(n^{2 / 3} \zeta\right)}{n^{2 / 3}}+O(E i(8 / 3,4 / 3))\right) \\
H_{n}^{(1)^{\prime}}(n z) & =\frac{4 e^{2 \pi i / 3}}{z}\left(\frac{1-z^{2}}{4 \zeta}\right)^{1 / 4}\left(\frac{A_{-}^{\prime}\left(n^{2 / 3} \zeta\right)}{n^{2 / 3}}+O\left(E_{-}(8 / 3,4 / 3)\right)\right) \\
J_{n}(n z) H_{n}^{(1)}(n z) & =2 e^{-\pi i / 3}\left(\frac{4 \zeta}{1-z^{2}}\right)^{1 / 2}\left(\frac{A i\left(n^{2 / 3} \zeta\right) A_{-}\left(n^{2 / 3} \zeta\right)}{n^{2 / 3}}+O\left(E i_{-}\left(\frac{8}{3}, 2, \frac{10}{3}\right)\right)\right)  \tag{2.2.9}\\
J_{n}^{\prime}(n z) H_{n}^{(1)^{\prime}}(n z) & =\frac{8 e^{-\pi i / 3}}{z^{2}}\left(\frac{1-z^{2}}{4 \zeta}\right)^{1 / 2}\left(\frac{A i^{\prime}\left(n^{2 / 3} \zeta\right) A_{-}^{\prime}\left(n^{2 / 3} \zeta\right)}{n^{4 / 3}}+O\left(E i_{-}\left(\frac{8}{3}, 2, \frac{10}{3}\right)\right)\right) \tag{2.2.10}
\end{align*}
$$

where

$$
\begin{gathered}
E_{-}(\alpha, \beta)=\left|A_{-}^{\prime}\left(n^{2 / 3} \zeta\right)\right| n^{-\alpha}+\left|A_{-}\left(n^{2 / 3} \zeta\right)\right| n^{-\beta} \\
E i(\alpha, \beta)=\left|A i^{\prime}\left(n^{2 / 3} \zeta\right)\right| n^{-\alpha}+\left|A i\left(n^{-2 / 3} \zeta\right)\right| n^{-\beta} \\
E i_{-}(\alpha, \beta, \gamma)=\left|A i A_{-}\right|\left(n^{2 / 3} \zeta\right) n^{-\alpha}+\left(\left|A i^{\prime} A_{-}\right|+\left|A i A_{-}^{\prime}\right|\right)\left(n^{2 / 3} \zeta\right) n^{-\beta}+\left|A i^{\prime} A_{-}^{\prime}\left(n^{2 / 3} \zeta\right)\right| n^{-\gamma}
\end{gathered}
$$

Finally, we record some double asymptotic properties for fixed $n$ and $z \rightarrow \infty$ with $|\operatorname{Arg} z|<$ $\pi-\delta$,

$$
\begin{align*}
J_{n}(n z) & =\left(\frac{4 \zeta}{1-z^{2}}\right)^{1 / 4}\left(\frac{A i\left(n^{2 / 3} \zeta\right)}{n^{1 / 3}}+O\left(E i^{z}(1,5 / 3,3,1 / 3)\right)\right) \\
H_{n}^{(1)}(n z) & =2 e^{-\pi i / 3}\left(\frac{4 \zeta}{1-z^{2}}\right)^{1 / 4}\left(\frac{A_{-}\left(n^{2 / 3} \zeta\right)}{n^{1 / 3}}+O\left(E_{-}^{z}(1,5 / 3,3,1 / 3)\right)\right) \\
J_{n}^{\prime}(n z) & =-\frac{2}{z}\left(\frac{1-z^{2}}{4 \zeta}\right)^{1 / 4}\left(\frac{A i^{\prime}\left(n^{2 / 3} \zeta\right)}{n^{2 / 3}}+O\left(E i^{z}(1,2 / 3,0,4 / 3)\right)\right) \\
H_{n}^{(1){ }^{\prime}}(n z) & =\frac{4 e^{2 \pi i / 3}}{z}\left(\frac{1-z^{2}}{4 \zeta}\right)^{1 / 4}\left(\frac{A_{-}^{\prime}\left(n^{2 / 3} \zeta\right)}{n^{2 / 3}}+O\left(E_{-}^{z}(1,2 / 3,0,4 / 3)\right)\right) \\
J_{n}(n z) H_{n}^{(1)}(n z) & =2 e^{-\pi i / 3}\left(\frac{4 \zeta}{1-z^{2}}\right)^{1 / 2}\left(\frac{A i\left(n^{2 / 3} \zeta\right) A_{-}\left(n^{2 / 3} \zeta\right)}{n^{2 / 3}}+O\left(E i_{-}^{z}\left(3, \frac{2}{3}, 1,2,2, \frac{10}{3}\right)\right)\right)  \tag{2.2.11}\\
J_{n}^{\prime}(n z) H_{n}^{(1)^{\prime}}(n z) & =\frac{8 e^{-\pi i / 3}}{z^{2}}\left(\frac{1-z^{2}}{4 \zeta}\right)^{1 / 2}\left(\frac{A i^{\prime}\left(n^{2 / 3} \zeta\right) A_{-}^{\prime}\left(n^{2 / 3} \zeta\right)}{n^{4 / 3}}+O\left(E i_{-}^{z}\left(0, \frac{8}{3}, 0,2,1, \frac{4}{3}\right)\right)\right) \tag{2.2.12}
\end{align*}
$$

$$
\begin{aligned}
E_{-}^{z}(\alpha, \gamma, \beta, \delta)= & \left|A_{-}^{\prime}\left(n^{2 / 3} \zeta\right)\right||z|^{-3 \alpha / 2} n^{-\gamma}+\left|A_{-}\left(n^{2 / 3} \zeta\right)\right||\zeta|^{-\beta} n^{-\delta} \\
E i^{z}(\alpha, \gamma, \beta, \delta)= & \left|A i^{\prime}\left(n^{2 / 3} \zeta\right)\right||\zeta|^{-\alpha} n^{-\gamma}+\left|A i\left(n^{-2 / 3} \zeta\right)\right||\zeta|^{-\beta} n^{-\delta} \\
E i_{-}^{z}(\alpha, \delta, \beta, \epsilon, \gamma, \rho)= & \left|A i A_{-}\right|\left(n^{2 / 3} \zeta\right)|\zeta|^{-\alpha} n^{-\delta}+\left(\left|A i^{\prime} A_{-}\right|+\left|A i A_{-}^{\prime}\right|\right)\left(n^{2 / 3} \zeta\right)|\zeta|^{-\beta} n^{-\epsilon} \\
& +\left|A i^{\prime} A_{-}^{\prime}\left(n^{2 / 3} \zeta\right)\right||\zeta|^{-\gamma} n^{-\rho}
\end{aligned}
$$

We now record some facts about the Airy functions $A i$ and $A_{-}$. The Wronskian of these two solutions is given by

$$
\begin{equation*}
W\left(A_{i}, A_{-}\right)=A i A_{-}^{\prime}(z)-A i^{\prime} A_{-}(z)=\frac{e^{-\pi i / 6}}{2 \pi} \tag{2.2.13}
\end{equation*}
$$

Furthermore, for $s \in \mathbb{R}$,

$$
A i(s)=e^{-\pi i / 3} A_{-}(s)+e^{\pi i / 3} \overline{A_{-}(s)}
$$

and hence

$$
\begin{equation*}
\operatorname{Im}\left(e^{-5 \pi i / 6} A_{-}(s)\right)=-\frac{A i(s)}{2} \tag{2.2.14}
\end{equation*}
$$

The zeros of $A i(z)$ and $A i^{\prime}(z)$ all lie on $(-\infty, 0]$. We use the notation $-\zeta_{k}$ and $-\zeta_{k}^{\prime}$ to denote the $k^{\text {th }}$ zero of $A i$ and $A i^{\prime}$ respectively.

Finally, we record asymptotics for Airy functions as $z \rightarrow \infty$ in the sector $|\operatorname{Arg} z|<$ $\pi / 3-\delta$. Many of these asymptotic formulae hold in larger regions, but we restrict our attention to this sector. Let $\eta=2 / 3 z^{3 / 2}$ where we take principal branch of the square root. Then

$$
\begin{align*}
& A_{-}(z)= \frac{e^{-\pi i / 6} e^{\eta}}{2 \sqrt{\pi} z^{1 / 4}}\left(1+O\left(|z|^{-3 / 2}\right)\right) \\
& A_{-}^{\prime}(z)=\frac{e^{-\pi i / 6} z^{1 / 4} e^{\eta}}{2 \sqrt{\pi}}\left(1+O\left(|z|^{-3 / 2}\right)\right) A_{-}(-z)=\frac{e^{\pi i / 12} e^{i \eta}}{2 \sqrt{\pi} z^{1 / 4}} \\
& A i(z)=\frac{z^{-1 / 4} e^{-\eta}}{2 \sqrt{\pi}}\left(1+O\left(|z|^{-3 / 2}\right)\right) \\
& A i(-z)=\frac{e^{-5 \pi i / 12} z^{1 / 4} e^{i \eta}}{2 \sqrt{\pi}} \\
& 2 \sqrt{\pi}\left(e^{i \eta-i \pi / 4}+e^{-i \eta+i \pi / 4}+O\left(|z|^{-3 / 2} e^{|\operatorname{Im} \eta|}\right)\right)  \tag{2.2.15}\\
& A i^{\prime}(z)=-\frac{z^{1 / 4} e^{-\eta}}{2 \sqrt{\pi}}\left(1+O\left(|z|^{-3 / 2}\right)\right) z^{1 / 4}\left(e^{i \eta-i \pi / 4}-e^{-i \eta+i \pi / 4}+O\left(|z|^{-3 / 2} e^{|\operatorname{Im} \eta|}\right)\right)
\end{align*}
$$

$$
\begin{align*}
A i(z) A_{-}(z) & =\frac{1}{4 \pi z^{1 / 2}}\left(1+O\left(|z|^{-3 / 2}\right)\right)  \tag{2.2.16}\\
A i(-z) A_{-}(-z) & =\frac{e^{\pi i / 3}}{4 \pi z^{1 / 2}}\left(1-i e^{2 i \eta}+O\left(|z|^{-3 / 2} e^{2|\operatorname{Im} \eta|}\right)\right)  \tag{2.2.17}\\
A i^{\prime}(z) A_{-}^{\prime}(z) & =\frac{e^{5 \pi i / 6} z^{1 / 2}}{4 \pi}\left(1+O\left(|z|^{-3 / 2}\right)\right)  \tag{2.2.18}\\
A i^{\prime}(-z) A_{-}^{\prime}(-z) & =\frac{e^{\pi i / 3} z^{1 / 2}}{4 \pi}\left(1+i e^{2 i \eta}+O\left(|z|^{-3 / 2} e^{2|\operatorname{Im} \eta|}\right)\right) \tag{2.2.19}
\end{align*}
$$

### 2.3 The $\delta$ Potential

This section is organized as follows. In section 2.3 we reduce the problem of the existence of resonances to finding solutions of a transcendental equation. In section 2.4, we demonstrate the existence of the various resonance free regions in Theorem 2.1. Finally, in Section 2.5, we show the existence of the resonances in Theorem 2.2 and prove Theorem 2.3 .

## Reduction to Transcendental Equations on the Circle

We now consider (2.1.1 with $\Omega=B(0,1) \subset \mathbb{R}^{2}$ and $V \equiv h^{-\alpha} V_{0}$ on $\partial \Omega$. Then for $i=1,2$,

$$
\begin{cases}\left(-h^{2} \partial_{r}^{2}-\frac{h^{2}}{r} \partial_{r}-\frac{h^{2}}{r^{2}} \partial_{\theta}^{2}-z^{2}\right) u_{1}=0 & \text { in } B(0,1)  \tag{2.3.1}\\ \left(-h^{2} \partial_{r}^{2}-\frac{h^{2}}{r} \partial_{r}-\frac{h^{2}}{r^{2}} \partial_{\theta}^{2}-z^{2}\right) u_{2}=0 & \text { in } \mathbb{R}^{2} \backslash B(0,1) \\ u_{1}(1, \theta)=u_{2}(1, \theta) & \\ \partial_{r} u_{1}(1, \theta)-\partial_{r} u_{2}(1, \theta)+V u_{1}(1, \theta)=0 & \\ u_{2} \text { is } z \text { outgoing } & \end{cases}
$$

Expanding in Fourier series, write $u_{i}(r, \theta):=\sum_{n} u_{i, n}(r) e^{i n \theta}$. Then, $u_{i, n}$ solves

$$
\left(-h^{2} \partial_{r}^{2}-h^{2} \frac{1}{r} \partial_{r}+h^{2} \frac{n^{2}}{r^{2}}-z^{2}\right) u_{i, n}(r)=0 .
$$

Multiplying by $r^{2}$ and rescaling by $x=z h^{-1} r$, we see that $u_{i, n}(r)$ solves the Bessel equation with parameter $n$ in the $x$ variables. Then, using that $u_{2}$ is outgoing and $u_{1} \in L^{2}$, we obtain that

$$
u_{1, n}(r)=K_{n} J_{n}\left(z h^{-1} r\right) \quad \text { and } \quad u_{2, n}(r)=C_{n} H_{n}^{(1)}\left(z h^{-1} r\right)
$$

where $J_{n}$ is the $n^{\text {th }}$ Bessel function of the first kind, and $H_{n}^{(1)}$ is the $n^{\text {th }}$ Hankel function of the first kind.

To solve (2.3.1) and hence find a resonance, we only need to find $z$ such that the boundary conditions hold. Using the boundary condition $u_{1}(1, \theta)=u_{2}(1, \theta)$, we have $K_{n} J_{n}\left(z h^{-1}\right)=$
$C_{n} H_{n}^{(1)}\left(z h^{-1}\right)$. Hence,

$$
C_{n}=\frac{K_{n} J_{n}\left(z h^{-1}\right)}{H_{n}^{(1)}\left(z h^{-1}\right)}
$$

Next, we rewrite the second boundary condition in (2.3.1) and use that $V \equiv h^{-\alpha} V_{0}$ to get

$$
\sum_{n}\left(K_{n} z h^{-1} J_{n}^{\prime}\left(z h^{-1}\right)-C_{n} z h^{-1} H_{n}^{(1)^{\prime}}\left(z h^{-1}\right)+h^{-\alpha} V_{0} K_{n} J_{n}\left(z h^{-1}\right)\right) e^{i n \theta}=0
$$

Then, since $e^{i n \theta}$ are $L^{2}$ orthogonal, we have

$$
K_{n}\left(z h^{-1} J_{n}^{\prime}\left(z h^{-1}\right)-z h^{-1} \frac{J_{n}\left(z h^{-1}\right)}{H_{n}^{(1)}\left(z h^{-1}\right)} H_{n}^{(1)^{\prime}}\left(z h^{-1}\right)+h^{-\alpha} V_{0} J_{n}\left(z h^{-1}\right)\right)=0, \quad n \in \mathbb{Z}
$$

Thus

$$
K_{n} h^{-\alpha} V_{0}=K_{n} z h^{-1}\left(\frac{H_{n}^{(1)^{\prime}}\left(z h^{-1}\right)}{H_{n}^{(1)}\left(z h^{-1}\right)}-\frac{J_{n}^{\prime}\left(z h^{-1}\right)}{J_{n}\left(z h^{-1}\right)}\right)
$$

which can be written

$$
\begin{equation*}
h^{-\alpha} V_{0} K_{n}=K_{n} z h^{-1} \frac{W\left(J_{n}, H_{n}^{(1)}\right)}{J_{n}\left(z h^{-1}\right) H_{n}^{(1)}\left(z h^{-1}\right)}=\frac{2 i K_{n}}{\pi J_{n}\left(z h^{-1}\right) H_{n}^{(1)}\left(z h^{-1}\right)} \tag{2.3.2}
\end{equation*}
$$

where $W(f, g)$ is the Wronskian of $f$ and $g$.
Then, without loss, we assume $K_{n}=1$ or $K_{n}=0$. Hence, we seek solutions $z(h, n)$ to

$$
\begin{equation*}
1-\frac{\pi h^{-\alpha} V_{0}}{2 i} J_{n}\left(h^{-1} z(h, n)\right) H_{n}^{(1)}\left(h^{-1} z(h, n)\right)=0 \tag{2.3.3}
\end{equation*}
$$

The quantity $n h^{-1}$ is the tangential frequency of the mode $u_{i, n} e^{i n \theta}$. In particular, the wave front set, denoted $\mathrm{WF}_{\mathrm{h}}$ (see Chapter 4 or 87, Chapter 4]), of $e^{i n \theta}$ has

$$
\mathrm{WF}_{\mathrm{h}}\left(e^{i n \theta}\right) \subset\left\{\xi^{\prime}=n h \quad \bmod o(1)\right\}
$$

Thus, $|n|<(1-\epsilon) h^{-1}$ corresponds to modes concentrating near directions transverse to the boundary, $|n| \sim h^{-1}$ are the glancing frequencies, that is directions tangent to the boundary, and $|n|>(1+\epsilon) h^{-1}$ corresponds to elliptic frequencies.

### 2.4 Resonance Free Regions

In this section, we demonstrate the existence of resonance free regions. In particular, we prove Theorem 2.1. We write $n=m h^{-1}$ and assume that

$$
|\operatorname{Im} z| \leq M_{0} \min \left(h \log h^{-1}, h^{2 \alpha-2 / 3}\right) .
$$

## Analysis for $m \gg 1$

We use the asymptotics (2.2.4) in (2.3.3). It then reads for $m^{-1} z=\operatorname{sech}(\alpha)$,

$$
1+\frac{h^{-\alpha} V_{0}}{2 m h^{-1} \tanh (\alpha)}\left(1+O\left(m^{-1} h\right)\right)=0
$$

but for $m$ large enough (independent of $h$ when $h$ is small enough), this clearly has no solution since the second term has positive real part.

## Analysis for $m \ll 1$

We use the asymptotics 2.2.11) in 2.3.3). It then reads for $\zeta=\zeta\left(m^{-1} z\right)$,

$$
1-\pi h^{-\alpha} V_{0} e^{-5 \pi i / 6}\left(\frac{4 \zeta}{1-m^{-2} z^{2}}\right)^{1 / 2}\left(\frac{A i\left(n^{2 / 3} \zeta\right) A_{-}\left(n^{2 / 3} \zeta\right)}{n^{2 / 3}}+O\left(E i_{-}^{z}\left(3, \frac{2}{3}, 1,2,2, \frac{10}{3}\right)\right)=0\right.
$$

Now, since $m \ll 1, m^{-1} z \rightarrow \infty$ and hence $\zeta \rightarrow-\infty$ so we use (2.2.17) to obtain

$$
1+\frac{h^{1-\alpha} V_{0} i}{2 m\left(m^{-2} z^{2}-1\right)^{1 / 2}}\left(1-i e^{\frac{4 m}{3 h} i(-\zeta)^{3 / 2}}+O\left(n^{-1}|\zeta|^{-1}\left(|\zeta|^{-1 / 2} e^{\frac{4 m}{3 h}\left|\operatorname{Im}(-\zeta)^{3 / 2}\right|}+1\right)\right)\right)=0
$$

Since $h \ll 1$ and $m\left(m^{-2} z^{2}-1\right)^{1 / 2}$ is bounded above and below as $m \rightarrow 0$, such a solution must have $e^{\frac{4 m}{3 h} i(-\zeta)^{3 / 2}}$ comparable to $h^{\alpha-1}$ and hence

$$
-\operatorname{Im}(-\zeta)^{3 / 2}=\frac{3 h}{8 m} \log \left|1+\frac{\left.4\left(|z|^{2}-m^{2}\right)\right)}{h^{2-2 \alpha} V_{0}^{2}}+O\left(\left(|\zeta|^{-1} n^{-1}+\operatorname{Im} z\right) h^{\alpha-1}\right)\right|
$$

Then, using (2.2.7), we have

$$
\operatorname{Im}(-\zeta)^{3 / 2}=\frac{3}{2 m} \operatorname{Im} z\left(1+o_{m \rightarrow 0}(1)\right)
$$

which gives

$$
-\operatorname{Im} z=\frac{h}{4} \log \left(1+\frac{4}{h^{2-2 \alpha} V_{0}^{2}}\right)\left(1+o_{m \rightarrow 0}(1)\right)
$$

Hence, choosing $m$ small enough in a manner depending on $\delta$ gives
Lemma 2.4.1. For all $\delta>0$ there exists $M, \epsilon>0$ and $h_{0}>0$ such that for $0<h<h_{0}$, $\operatorname{Re} z \in\left[1-C h^{3 / 4}, 1+C h^{3 / 4}\right]$, and $n=m h^{-1}$ with $m<\epsilon$ or $m>M$ there are no solutions to (2.3.3 with

$$
-\operatorname{Im} z \leq(1-\delta) \frac{h}{4} \log \left(1+\frac{4}{h^{2-2 \alpha} V_{0}^{2}}\right)
$$

Analysis for $\epsilon \leq m \leq M$
In this section, we consider the remaining values of $m$. First, we use 2.2.9) in (2.3.3) to write

$$
\begin{equation*}
1-2 \pi h^{-\alpha} V_{0} e^{-5 \pi i / 6}\left(\frac{\zeta}{1-z^{2}}\right)^{1 / 2}\left(\frac{A i\left(n^{2 / 3} \zeta\right) A_{-}\left(n^{2 / 3} \zeta\right)}{n^{2 / 3}}+O\left(E i_{-}\left(\frac{8}{3}, 2, \frac{10}{3}\right)\right)\right)=0 \tag{2.4.1}
\end{equation*}
$$

where $\zeta=\zeta\left(m^{-1} h\right)$. We first ignore the error term in (2.4.1) and show that there are no solutions with the appropriate bounds on $\operatorname{Im} \zeta$. In particular, define $h_{1}:=n^{-1}$ and

$$
\Phi:=h_{1}^{2 / 3} h^{-\alpha}\left(\frac{\zeta}{1-\left(h_{1} h^{-1} z\right)^{2}}\right)^{1 / 2} V_{0}=O_{C^{\infty}}\left(h_{1}^{2 / 3} h^{-\alpha}\right)
$$

The fact that $\Phi$ has uniform bounds for $\zeta$ in the relevant region comes from the fact that $h_{1} h^{-1}=m$ and $\epsilon<m<M$. Then, rewriting (2.4.1) without the lower order terms, we have

$$
\begin{equation*}
1-2 \pi e^{-5 \pi i / 6} \Phi(\zeta) A_{-}\left(h_{1}^{-2 / 3} \zeta\right) A i\left(h_{1}^{-2 / 3} \zeta\right)=0 \tag{2.4.2}
\end{equation*}
$$

Notice that if $\alpha \geq 2 / 3$ and $M h^{2-2 \alpha} \leq|\operatorname{Re} \zeta| \leq C h^{\delta}$ or $\alpha<2 / 3$ and $|\operatorname{Re} \zeta| \leq C h^{\delta}$ for any $\delta>0$, then the second term in (2.4.1) is bounded above by $1-\epsilon$. Hence, 2.4.1) has no solutions and we need only consider the remaining $\operatorname{Re} \zeta$.

## Analysis at glancing ( $m \sim 1$ )

We next analyze $|\zeta|<M \max \left(h_{1}^{2 / 5(3-2 \alpha)}, h_{1}^{2 / 3}\right)$. Let $s=h_{1}^{-2 / 3} \operatorname{Re} \zeta$. then,

$$
0 \leq|s|<M \max \left(h_{1}^{2 / 5(3-2 \alpha)-2 / 3}, 1\right)
$$

and

$$
\zeta=h_{1}^{2 / 3} s+\operatorname{Im} \zeta=h_{1}^{2 / 3} s+O\left(\min \left(h \log h^{-1}, h^{2 \alpha-2 / 3}\right)\right)
$$

Thus,

$$
\begin{aligned}
\left|\Phi(\zeta) A i A_{-}\left(h_{1}^{-2 / 3} \zeta\right)-\Phi\left(h_{1}^{2 / 3} s\right) A i A_{-}(s)-\Phi\left(h_{1}^{2 / 3} s\right)\left(A i A_{-}\right)^{\prime}(s) i \operatorname{Im} h_{1}^{-2 / 3} \zeta\right| \leq \\
O\left(h_{1}^{2 / 3}\langle s\rangle^{1 / 2} h^{-\alpha}\left(\operatorname{Im} h_{1}^{-2 / 3} \zeta\right)^{2}\right)+O\left(h_{1}^{2 / 3} h^{-\alpha} \operatorname{Im} \zeta A i A_{-}\left(h_{1}^{-2 / 3} \zeta\right)\right)
\end{aligned}
$$

We obtain lower bounds on

$$
f\left(s, h, h_{1}\right):=1-2 \pi e^{-5 \pi i / 6} \Phi\left(h_{1}^{2 / 3} s\right)\left(A_{-} A i(s)+\left(A_{-} A i\right)^{\prime}(s) i \operatorname{Im} h_{1}^{-2 / 3} \zeta\right)
$$

Letting $\alpha:=e^{-5 \pi i / 6}$, we have by 2.2.14 that

$$
\alpha A_{-}(s) A i(s)=\operatorname{Re}\left(\alpha A_{-}(s)\right) A i(s)-i \frac{A i^{2}(s)}{2}
$$

and

$$
\begin{aligned}
\left(\alpha A_{-} A i\right)^{\prime}(s) i \operatorname{Im} h_{1}^{-2 / 3} \zeta= & \left(A i(s) A i^{\prime}(s)+,\right. \\
& \left.i\left[A i^{\prime}(s) \operatorname{Re}\left(\alpha A_{-}(s)\right)+A i(s) \operatorname{Re}\left(\alpha A_{-}^{\prime}(s)\right)\right]\right) \operatorname{Im} h_{1}^{-2 / 3} \zeta
\end{aligned}
$$

Thus,

$$
\operatorname{Im} f=-2 \pi \Phi\left(h_{1}^{2 / 3} s\right)\left(-\frac{A i^{2}(s)}{2}+\left(A i^{\prime}(s) \operatorname{Re}\left(\alpha A_{-}(s)\right)+A i(s) \operatorname{Re}\left(\alpha A_{-}^{\prime}(s)\right)\right) \operatorname{Im} h_{1}^{-2 / 3} \zeta\right)
$$

and

$$
\operatorname{Re} f=1-2 \pi \Phi\left(h_{1}^{2 / 3} s\right) A i(s)\left(\operatorname{Re}\left(\alpha A_{-}(s)\right)+A i^{\prime}(s) \operatorname{Im} h_{1}^{-2 / 3} \zeta\right)
$$

So, when

$$
|A i(s)| \leq \frac{1-\delta}{2 \pi \Phi\left(h_{1}^{2 / 3} s\right) \operatorname{Re}\left(\alpha A_{-}(s)\right)}, \quad \text { or } \quad|A i(s)| \geq \frac{1-\delta}{2 \pi \Phi\left(h_{1}^{2 / 3} s\right) \operatorname{Re}\left(\alpha A_{-}(s)\right)}
$$

then $|f| \geq \delta$. Note that for $\alpha<2 / 3$, this condition is never satisfied. Thus, we need only consider

$$
\begin{equation*}
\frac{1-\delta}{2 \pi \Phi\left(h_{1}^{2 / 3} s\right) \operatorname{Re}\left(\alpha A_{-}(s)\right)} \leq|A i(s)| \leq \frac{1-\delta}{2 \pi \Phi\left(h_{1}^{2 / 3} s\right) \operatorname{Re}\left(\alpha A_{-}(s)\right)} \tag{2.4.3}
\end{equation*}
$$

That is, using the fact that $\left|A i^{\prime}(-s)\right| \sim c|s|^{1 / 4}$ and $\left|A_{-}(-s)\right| \sim c|s|^{-1 / 4}$,

$$
\begin{equation*}
s=-\zeta_{k}+O\left(h^{\alpha} h_{1}^{-2 / 3}\right) \tag{2.4.4}
\end{equation*}
$$

where $-\zeta_{k}$ is the $k^{\text {th }}$ zero $A i(s)$.
Remark: For $\alpha \leq 2 / 3$, notice that (2.4.4) does not give us any additional information on the location of $s$. However, it is easy to see that in this situation $\operatorname{Im} f \geq C h^{\alpha-2 / 3}$. Since we need only consider small $\operatorname{Re} \zeta$ when $\alpha \geq 2 / 3$, this implies that in the relevant region $|\operatorname{Im} f| \geq c$ and hence there are no solutions to (2.3.3) in this region.

Now, $|\operatorname{Im} z| \leq M_{0} \min \left(h \log h^{-1}, h^{2 \alpha-2 / 3}\right)$ implies that $|\operatorname{Im} \zeta| \leq M_{1} h^{2 \alpha-2 / 3}$. So, using the fact that $A_{-}(-s)=O\left(|s|^{-1 / 4}\right)$ and $A i^{\prime}(-s)=O\left(|s|^{1 / 4}\right)$ we see that there exists $K=K\left(M_{1}\right)$ such that if

$$
\inf _{k \leq K\left(M_{1}\right)}\left|\operatorname{Im} \zeta-\frac{h_{1}^{2 / 3}}{8 \pi^{2} \Phi\left(h_{1}^{2 / 3}\left(-\zeta_{k}\right)\right)^{2} \operatorname{Re}\left(\alpha A_{-}\left(-\zeta_{k}\right)\right)^{3} A i^{\prime}\left(-\zeta_{k}\right)}\right| \geq \delta h^{2 \alpha} h_{1}^{-2 / 3}
$$

then

$$
|\operatorname{Im} f| \geq \epsilon h^{\alpha} h_{1}^{-2 / 3}
$$

Finally, we account for the error terms. We have suppressed terms of the form

$$
O\left(\max \left(1, h_{1}^{1 / 5(3-2 \alpha)-1 / 3}\right) \min \left(h_{1}^{4 \alpha-8 / 3}, h^{2 / 3}\left(\log h^{-1}\right)^{2}\right)+h_{1}^{\alpha-2 / 3} A i A_{-}\left(h^{-2 / 3} \zeta\right)\right) .
$$

Together with 2.4.3) the estimate $|f| \geq \epsilon h^{\alpha} h_{1}^{2 / 3}$ implies that there are no solutions to (2.3.3) for $|\operatorname{Re} \zeta|<M \max \left(h_{1}^{2 / 5(3-2 \alpha)}, h^{2 / 3}\right),|\operatorname{Im} \zeta|<M_{1} \min \left(h^{2 \alpha-2 / 3}, h \log h^{-1}\right)$, satisfying (2.4).

## Asymptotic analysis in the hyperbolic ( $\epsilon<m<1$ ) and elliptic

 ( $M>m>1$ ) regionsWe need to analyze $K \geq|\operatorname{Re} \zeta| \geq h_{1}^{2 / 5(3-2 \alpha)} M$. To do this, let $G_{\Delta}=\frac{i h_{1}^{1 / 3}}{2(-\zeta)^{1 / 2}}$, and $b:=$ $2 \pi e^{-5 \pi i / 6}$. Then if $\zeta$ solves (2.4.1)

$$
G_{\Delta}^{1 / 2} \Phi=-G_{\Delta}^{1 / 2} \Phi G_{\Delta}^{1 / 2}\left(-G_{\Delta}^{-1 / 2} b A i\left(h_{1}^{-2 / 3} \zeta\right) A_{-}\left(h_{1}^{-2 / 3} \zeta\right) G_{\Delta}^{-1 / 2}+o\left(h^{2}\right)\right) G_{\Delta}^{1 / 2} \Phi
$$

and hence

$$
\begin{aligned}
\left(1+G_{\Delta}^{1 / 2} \Phi G_{\Delta}^{1 / 2}\right) & G_{\Delta}^{1 / 2} \Phi \\
& =-G_{\Delta}^{1 / 2} \Phi G_{\Delta}^{1 / 2}\left(-G_{\Delta}^{-1 / 2} b A i\left(h_{1}^{-2 / 3} \zeta\right) A_{-}\left(h_{1}^{-2 / 3} \zeta\right) G_{\Delta}^{-1 / 2}-1+O\left(h^{2}\right)\right) G_{\Delta}^{1 / 2} \Phi
\end{aligned}
$$

Using (2.2.17) for $\operatorname{Re} \zeta<-M h^{2 / 3}$, we have

$$
\begin{gathered}
\left(1+G_{\Delta}^{1 / 2} \Phi G_{\Delta}^{1 / 2}\right) G_{\Delta}^{1 / 2} \Phi=-G_{\Delta}^{1 / 2} \Phi G_{\Delta}^{1 / 2}\left(-i e^{\frac{4 i}{3 h_{1}}(-\zeta)^{3 / 2}}\left(1+\mathcal{O}\left(h_{1} \zeta^{-3 / 2}\right)\right)\right) G_{\Delta}^{1 / 2} \Phi \\
G_{\Delta}^{1 / 2} \Phi=-\left(I+G_{\Delta}^{1 / 2} \Phi G_{\Delta}^{1 / 2}\right)^{-1} G_{\Delta}^{1 / 2} \Phi G_{\Delta}^{1 / 2}\left(-i e^{\frac{4 i}{h_{1}}(-\zeta)^{3 / 2}}\left(1+\mathcal{O}\left(h_{1} \zeta^{-3 / 2}\right)\right)\right) G_{\Delta}^{1 / 2} \Phi .
\end{gathered}
$$

For $\zeta>M h_{1}^{2 / 3}$, we use (2.2.16) to obtain

$$
\left(I+G_{\Delta}^{1 / 2} \Phi G_{\Delta}^{1 / 2}\right) G_{\Delta}^{1 / 2} \Phi=-G_{\Delta}^{1 / 2} \Phi G_{\Delta}^{1 / 2}\left(O\left(h_{1} \zeta^{-3 / 2}\right)\right) G_{\Delta}^{1 / 2} \Phi
$$

Hence,

$$
G_{\Delta}^{1 / 2} \Phi=-\left(I+G_{\Delta}^{1 / 2} \Phi G_{\Delta}^{1 / 2}\right)^{-1} G_{\Delta}^{1 / 2} \Phi G_{\Delta}^{1 / 2} O\left(h_{1} \zeta^{-3 / 2}\right) G_{\Delta}^{1 / 2} \Phi
$$

Remark: The analog of reflection operator in this setting is given by

$$
-\left(I+G_{\Delta}^{1 / 2} \Phi G_{\Delta}^{1 / 2}\right)^{-1} G_{\Delta}^{1 / 2} \Phi G_{\Delta}^{1 / 2}
$$

To see that $I+G_{\Delta}^{1 / 2} \Phi G_{\Delta}^{1 / 2} \neq 0$ observe that when $\operatorname{Re} \zeta<-M h^{2 / 3}$,

$$
\left|\operatorname{Re} \frac{i h_{1}^{1 / 3} \Phi}{2(-\zeta)^{1 / 2}}\right|=h_{1}^{1-\alpha} O\left(\frac{\operatorname{Im} \zeta}{|\operatorname{Re} \zeta|^{3 / 2}}\right)=o(1)
$$

and when $\operatorname{Re} \zeta>M h^{2 / 3}$,

$$
\operatorname{Re} \frac{h_{1}^{1 / 3} \Phi}{2 \zeta^{1 / 2}} \geq 0
$$

Now, since $|\operatorname{Re} \zeta|>M h_{1}^{2 / 3}, O\left(h_{1} \zeta^{-3 / 2}\right) \ll 1$ for $M$ large. Hence, there are no zeros for $\operatorname{Re} \zeta>0$. For $\operatorname{Re} \zeta<0$, there are no zeros of (2.4.2) when

$$
\begin{equation*}
\left|\left(I+G_{\Delta}^{1 / 2} \Phi G_{\Delta}^{1 / 2}\right)^{-1} G_{\Delta}^{1 / 2} \Phi G_{\Delta}^{1 / 2}\left(1+O\left(h_{1} \zeta^{-3 / 2}\right)\right) e^{\frac{4 i}{3 h_{1}}(-\zeta)^{3 / 2}}\right|<1 \tag{2.4.5}
\end{equation*}
$$

Let $\zeta=s+i \operatorname{Im} \zeta$. Then

$$
(-\zeta)^{3 / 2}=(-s)^{3 / 2}\left(1-i \operatorname{Im} \zeta(-s)^{-1}\right)^{3 / 2}=(-s)^{3 / 2}\left(1-\frac{3}{2} i \operatorname{Im} \zeta(-s)^{-1}+O\left((\operatorname{Im} \zeta)^{2} s^{-2}\right)\right)
$$

and

$$
(-\zeta)^{1 / 2}=(-s)^{1 / 2}\left(1+O\left(\operatorname{Im} \zeta s^{-1}\right)\right)
$$

So,

$$
\left|e^{\frac{4 i}{3 h_{1}}(-\zeta)^{3 / 2}}\right|=e^{\frac{2 \operatorname{Im} \zeta(-s)^{1 / 2}}{h_{1}}+O\left((\operatorname{Im} \zeta)^{2}|s|^{-1 / 2} h_{1}^{-1}\right)} .
$$

Taking logarithms of 2.4.5,

$$
\frac{2 \operatorname{Im} \zeta(-s)^{1 / 2}}{h_{1}}+O\left((\operatorname{Im} \zeta)^{2} h_{1}^{-1}|s|^{-1 / 2}\right)+\log \left|\frac{h_{1}^{1 / 3} \Phi}{2 i(-\zeta)^{1 / 2}-h_{1}^{1 / 3} \Phi}\right|+O\left(h_{1} \zeta^{-3 / 2}\right)<0
$$

Thus, for $-K \leq \operatorname{Re} \zeta=s \leq-M \max \left(h_{1}^{2 / 5(3-2 \alpha), h_{1}^{2 / 3}}\right.$, there are no solutions with

$$
\begin{aligned}
\operatorname{Im} \zeta<\inf _{-K<s<-M \max \left(h_{1}^{2 / 5(3-2 \alpha)}, h_{1}^{2 / 3}\right)} \frac{h_{1}}{4(-s)^{1 / 2}} & \log \left|1+4(-s) h_{1}^{-2 / 3} \Phi^{-2}\right| \\
& +O\left((\operatorname{Im} \zeta)^{2}|s|^{-1}+\operatorname{Im} \zeta|s|^{-1} h_{1}\right)+O\left(h_{1}^{2}|s|^{-2}\right)
\end{aligned}
$$

|  | Main Term | Error |
| :---: | :---: | :---: |
| $\|s\|<h^{2-2 \alpha}$ | $h_{1}^{1 / 3} h^{2 \alpha-4 / 3}(-s)^{1 / 2}$ | $\|s\|^{-1}\binom{h_{1}^{2}\|s\|^{-1}+}{\min \left(h^{2}\left(\log h^{-1}\right)^{2}, h_{1} h^{2 \alpha-2 / 3}\right.}$ |
| $\|s\| \geq h^{2-2 \alpha}$ | $\frac{h_{1}}{(-s)^{1 / 2}} \log \left(1-s h^{2 \alpha-4 / 3} h_{1}^{-2 / 3}\right)$ | $h_{1}^{2} s^{-2}+O\left(h^{4 / 3}\left(\log h^{-1}\right)^{2}\right)$ |

Thus, since we have $|s|>M \max \left(h_{1}^{2 / 5(3-2 \alpha)}, h_{1}^{2 / 3}\right)$, the error terms are lower order and hence

$$
\operatorname{Im} \zeta<\inf _{-K<s<-M h_{1}^{2 / 5(3-2 \alpha)}} \frac{h_{1}}{4(-s)^{1 / 2}} \log \left|1+4(-s) h_{1}^{-2 / 3} \Phi^{-2}\right| .
$$

So, for $\epsilon \leq m \leq K$, and $|s|<h^{2 / 5(3-2 \alpha)}$ there are no zeros of 2.3.3) for

$$
\begin{equation*}
\operatorname{Im} \zeta \leq C \min \left(M^{1 / 2} h^{2 \alpha-2 / 3}, C h \log h^{-1}\right) \tag{2.4.6}
\end{equation*}
$$

Taking $M$ large enough and $h$ small enough, $C M^{1 / 2} h^{2 \alpha-2 / 3}$ is larger than $|\operatorname{Im} \zeta|$.
Our last task is to relate the imaginary part of $z$ to that of $\zeta$ when $|\zeta|<K$. By (2.2.6) and (2.2.8), we have that

$$
\begin{equation*}
z=h_{1}^{-1} h-h_{1}^{-1} h \frac{\zeta}{\sqrt[3]{2}}+O\left(\zeta^{2}\right), \quad \operatorname{Im} z=-h_{1}^{-1} h \frac{\operatorname{Im} \zeta}{\sqrt[3]{2}}+O(\operatorname{Re} \zeta \operatorname{Im} \zeta) \tag{2.4.7}
\end{equation*}
$$

More generally $\operatorname{Im} z \sim C \operatorname{Im} \zeta+O\left(|\operatorname{Im} \zeta|^{2}\right)$ for $|\zeta|<M$. Since we assume $\operatorname{Re} z \in[1-$ $\left.C h^{3 / 4}, 1+C h^{3 / 4}\right]$, we have $h_{1}=h+O\left(h^{1+\delta}\right)$ when $|s|<h^{\delta}$. Together with (2.4), (2.4.6), 2.4.7, Lemma 2.4.1, and the fact that

$$
\lim _{w \rightarrow 1} \Phi^{2}(w)=(\sqrt[3]{2})^{-2}
$$

this completes the proof of the existence of resonance free regions of the sizes given in Theorem 2.1.

### 2.5 Construction of Resonances

In this section, we demonstrate the existence of resonances. That is, we prove Theorem 2.2 . We first prove the following analog of Newton's method:

Lemma 2.5.1. Suppose that $z_{0} \in \mathbb{C}$. Let $\Omega:=\left\{z \in \mathbb{C}:\left|z-z_{0}\right| \leq \epsilon(h)\right\}$ and suppose $f: \Omega \rightarrow \mathbb{C}$ is analytic. Suppose that

$$
\left|f\left(z_{0}\right)\right| \leq a(h), \quad\left|\partial_{z} f\left(z_{0}\right)\right| \geq b(h), \quad \sup _{z \in \Omega}\left|\partial_{z}^{2} f(z)\right| \leq d(h)
$$

Then if

$$
\begin{equation*}
a(h)+d(h) \epsilon(h)^{2}<\epsilon(h) b(h)<c<1 \tag{2.5.1}
\end{equation*}
$$

there is a unique solution $z(h)$ to $f(z(h))=0$ in $\Omega$.
Proof. Let

$$
g(z):=z-\frac{f(z)}{\partial_{z} f\left(z_{0}\right)}
$$

Then,

$$
\left|\partial_{z} g(z)\right|=\left|1-\frac{\partial_{z} f(z)}{\partial_{z} f\left(z_{0}\right)}\right| \leq \frac{d(h) \epsilon(h)}{b(h)}
$$

and

$$
\left|g(z)-z_{0}\right| \leq\left|g\left(z_{0}\right)-z_{0}\right|+\sup _{\Omega}\left|\partial_{z} g(z)\right|\left|z-z_{0}\right| \leq\left|\frac{a(h)}{b(h)}\right|+\left|\frac{d(h) \epsilon(h)^{2}}{b(h)}\right|
$$

Thus under the condition (2.5.1), $g: \Omega \rightarrow \Omega$ and

$$
\left|g(z)-g\left(z^{\prime}\right)\right|<\sup _{w \in \Omega}\left|\partial_{z} g(w)\right|\left|z-z^{\prime}\right|<c\left|z-z^{\prime}\right|
$$

Hence, $g$ is a contraction mapping and by the contraction mapping theorem, there is a unique fixed point of $g$ in $\Omega$ and hence a zero of $f(z)$ in $\Omega$.

## Resonances at glancing

We now analyze $n \sim h^{-1}$ which correspond to glancing trajectories. In particular, for $\alpha>2 / 3$, we construct solutions to (2.3.3) for $0<h<h_{0}$ with

$$
\operatorname{Im} z \geq C h^{2 \alpha-2 / 3}
$$

Let $h_{1}=n^{-1}$. Then, suppressing terms of size $h_{1}^{2+2 / 3} h^{-\alpha}$, we seek solutions to (2.4.2). Our ansatz is

$$
\zeta=-h_{1}^{2 / 3} \zeta_{k}+\epsilon(h)
$$

where $-\zeta_{k}$ is the $k^{\text {th }}$ zero of $\operatorname{Ai}(s)$. Then,

$$
\begin{aligned}
& \Phi(\zeta) A_{-}\left(h_{1}^{-2 / 3} \zeta\right) A i\left(h_{1}^{-2 / 3} \zeta\right)=\left(\Phi\left(-h_{1}^{2 / 3} \zeta_{k}\right)+\sum_{k \geq 1} \frac{\Phi^{(k)}\left(-h_{1}^{2 / 3} \zeta_{k}\right)}{k!} \epsilon^{k}\right) \\
&\left(A_{-} A i^{\prime}\left(-\zeta_{k}\right) h_{1}^{-2 / 3} \epsilon+A_{-}^{\prime} A i^{\prime}\left(-\zeta_{k}\right) h_{1}^{-4 / 3} \epsilon^{2}+\sum_{k \geq 3} \frac{\left(A_{-} A i\right)^{(k)}\left(-\zeta_{k}\right)}{k!} h_{1}^{-2 k / 3} \epsilon^{k}\right) .
\end{aligned}
$$

Let $\epsilon=\epsilon_{0}+\epsilon_{1}$ where $\epsilon_{1}=o\left(\epsilon_{0}\right)$. Then, ignoring terms terms of size $\epsilon^{2}$ and letting $b:=2 \pi e^{-5 \pi i / 6}$, we have

$$
1-b \Phi\left(-h_{1}^{2 / 3} \zeta_{k}\right) A_{-} A i^{\prime}\left(-\zeta_{k}\right) h_{1}^{-2 / 3} \epsilon_{0}=0
$$

That is,

$$
\epsilon_{0}=\frac{h_{1}^{2 / 3}}{b \Phi\left(-h_{1}^{2 / 3} \zeta_{k}\right) A_{-}\left(-\zeta_{k}\right) A i^{\prime}\left(-\zeta_{k}\right)}=C h_{1}^{2 / 3} h^{\alpha} h_{1}^{-2 / 3}
$$

Then, using terms of size $\epsilon_{0}^{2}$ and $\epsilon_{1}$, we have

$$
\begin{aligned}
\Phi\left(-h_{1}^{2 / 3} \zeta_{k}\right) A_{-} A i^{\prime} & \left(-\zeta_{k}\right) h_{1}^{-2 / 3} \epsilon_{1} \\
& +\left(\Phi\left(-h_{1}^{2 / 3} \zeta_{k}\right) A_{-}^{\prime} A i^{\prime}\left(-\zeta_{k}\right) h_{1}^{-2 / 3}+\Phi^{\prime}\left(-h_{1}^{2 / 3} \zeta_{k}\right) A_{-} A^{\prime}\left(-\zeta_{k}\right)\right) h_{1}^{-2 / 3} \epsilon_{0}^{2}=0
\end{aligned}
$$

That is,

$$
\begin{aligned}
\epsilon_{1} & =-\frac{h_{1}^{2 / 3}\left(\Phi\left(-h_{1}^{2 / 3} \zeta_{k}\right) A_{-}^{\prime} A i^{\prime}\left(-\zeta_{k}\right) h_{1}^{-4 / 3}+\Phi^{\prime}\left(-h_{1}^{2 / 3} \zeta_{k}\right) A_{-} A i^{\prime}\left(-\zeta_{k}\right) h_{1}^{-2 / 3}\right) \epsilon_{0}^{2}}{\Phi\left(-h_{1}^{2 / 3} \zeta_{k}\right) A_{-} A i^{\prime}\left(-\zeta_{k}\right)} \\
& =-h_{1}^{-2 / 3} \epsilon_{0}^{2} \frac{A_{-}^{\prime}\left(-\zeta_{k}\right)}{A_{-}\left(-\zeta_{k}\right)}\left(1+O\left(\epsilon_{0} h_{1}^{-2 / 3}\right)\right) \\
& =-\frac{h_{1}^{2 / 3} A_{-}^{\prime}\left(-\zeta_{k}\right)}{\left(\Phi\left(-h_{1}^{2 / 3} \zeta_{k}\right)\right)^{2} 4 \pi^{2} e^{-5 \pi i / 3} A_{-}^{3}\left(-\zeta_{k}\right)\left(A i^{\prime}\left(-\zeta_{k}\right)\right)^{2}}\left(1+O\left(\epsilon_{0} h_{1}^{-2 / 3}\right)\right) .
\end{aligned}
$$

So, since by (2.2.14)

$$
\begin{gathered}
\operatorname{Im}\left(e^{-5 \pi i / 6} A_{-}(s)\right)=-\frac{A i(s)}{2} . \\
\operatorname{Im} \epsilon_{1}=-\frac{h_{1}^{2 / 3} \operatorname{Im}\left(e^{-5 \pi i / 6} A_{-}^{\prime}\left(-\zeta_{k}\right)\right)}{\left.\left(\Phi\left(-h_{1}^{2 / 3} \zeta_{k}\right)\right)\right)^{2} 4 \pi^{2}\left(e^{-5 \pi i / 6}\right)^{3} A_{-}^{3}\left(-\zeta_{k}\right)\left(A i^{\prime}\left(-\zeta_{k}\right)\right)^{2}}\left(1+O\left(\epsilon_{0} h_{1}^{-2 / 3}\right)\right) \\
=\frac{h_{1}^{2 / 3}}{\left(\Phi\left(-h_{1}^{2 / 3} \zeta_{k}\right)\right)^{2} 8 \pi^{2}\left(e^{-5 \pi i / 6}\right)^{3} A_{-}^{3}\left(-\zeta_{k}\right) A i^{\prime}\left(-\zeta_{k}\right)}\left(1+O\left(\epsilon_{0} h_{1}^{-2 / 3}\right)\right)
\end{gathered}
$$

Since $b \Phi\left(-h_{1}^{2 / 3} \zeta_{k}\right) A_{-} A i^{\prime}\left(-\zeta_{k}\right) \neq 0$, repeating in this way we obtain an asymptotic expansion for $\epsilon(h)$ in powers of $h^{\alpha} h_{1}^{-2 / 3}$ such that for $\zeta=-h_{1}^{2 / 3} \zeta_{k}+\epsilon(h)$,

$$
1-b \Phi(\zeta) A_{-}\left(h_{1}^{-2 / 3} \zeta\right) A i\left(h_{1}^{-2 / 3} \zeta\right)=O\left(h_{1}^{\infty}\right)
$$

Let

$$
f(\zeta)=1-b \Phi(\zeta) A_{-}\left(h_{1}^{-2 / 3} \zeta\right) A i\left(h_{1}^{-2 / 3} \zeta\right)
$$

Then, for $\zeta=-h_{1}^{2 / 3} \zeta_{k}+O\left(h_{1}^{\alpha}\right)$,

$$
\left|f^{\prime}(\zeta)\right| \geq c h^{-\alpha}
$$

and

$$
\left|f^{\prime \prime}(\zeta)\right| \leq C h^{-\alpha} h_{1}^{-2 / 3}
$$

Thus, letting $n=h^{-1}+O(1)$ and using Lemma 2.5.1, there is a solution $\zeta_{0}\left(h_{1}, h\right)$ to $f\left(\zeta_{0}\left(h_{1}, h\right)\right)=0$ with

$$
\zeta_{0}=-h_{1}^{2 / 3} \zeta_{k}+\epsilon(h)+O\left(h^{\infty}\right)
$$

Now, by the implicit function theorem (or Rouche's theorem) $f(\zeta)=a(\zeta)$ defines $\zeta$ in a neighborhood of $\zeta_{0}$ for $a$ small enough. Hence, since we suppressed terms of size $h_{1}^{8 / 3-\alpha}$ in (2.2.9), we have that there is a resonance with

$$
\zeta=\zeta_{0}+\frac{O\left(h_{1}^{8 / 3} h^{-\alpha}\right)}{\partial_{\zeta} f\left(\zeta_{0}\right)}=\zeta_{0}+O\left(h_{1}^{8 / 3}\right)
$$

## Resonances normal to the boundary

Next, we consider $n$ fixed relative to $h$. That is, we consider modes that concentrate normal to $\partial B(0,1)$.

Using asymptotics (2.2.2) in (2.3.3), we have

$$
\begin{equation*}
1-\frac{h^{1-\alpha} V_{0}}{2 i z(h, n)}\left(e^{2 i z(h, n) / h-\left(n+\frac{1}{2}\right) \pi i}\left(1+O\left(h z(h, n)^{-1}\right)\right)+1\right)=0 . \tag{2.5.2}
\end{equation*}
$$

Let

$$
F(\epsilon, k, n, h)=1-\frac{2 h^{1-\alpha} V_{0}}{i \pi h(4 k+2 n+1)}\left(e^{2 i \epsilon / h}+1\right)
$$

Then,

$$
\epsilon_{0}(k, n, h)=\frac{-i h}{2} \log \left[h^{\alpha-1} \frac{i \pi h(4 k+2 n+1)}{2 V_{0}}-1\right]
$$

has

$$
F\left(\epsilon_{0}(k, n, h), k, n, h\right)=0, \quad\left|\partial_{\epsilon} F\left(\epsilon_{0}(k, n, h), k, n, h\right)\right| \geq c h^{-1}
$$

Now, for $0<c$ and $c h^{-1}<k<C h^{-1}$ by 2.5.2), $z(h, k, n)$ can be defined by a solution $z(h, k, n)=\frac{\pi h}{4}(4 k+2 n+1)+\epsilon(k, n, h)$ where

$$
F(\epsilon, k, n, h)=O\left(e^{2 i \epsilon / h} h / z+\epsilon\right)
$$

So, by the implicit function theorem there is a solution $\epsilon$ satisfying

$$
\begin{aligned}
\epsilon(k, n, h) & =\epsilon_{0}(k, n, h)+\left(\partial_{\epsilon} F\left(\epsilon_{0}(k, n, h), k, n, h\right)\right)^{-1} O\left(h^{1-\alpha} e^{2 i \epsilon_{0} / h}\left(h / z+\epsilon_{0}\right)\right) \\
& =\epsilon_{0}(k, n, h)+O\left(h^{2}\right)
\end{aligned}
$$

Thus, for all $\epsilon>0$ and $0<h<h_{\epsilon}$, there exist $z(h) \in \Lambda$ with

$$
\frac{\operatorname{Im} z}{h} \sim \begin{cases}-\frac{(1-\alpha)}{2} \log h^{-1}+\frac{1}{2} \log \left(\frac{2}{V_{0}}\right)+O\left(h^{3 / 4}\right) & \alpha<1  \tag{2.5.3}\\ -\frac{1}{4} \log \left(1+\frac{4}{V_{0}^{2}}\right)+O\left(h^{3 / 4}\right) & \alpha=1\end{cases}
$$

Remark: Note that the size of the error terms in 2.5.3 comes from the fact that we allow $\operatorname{Re} z \in\left[1-C h^{3 / 4}, 1+C h^{3 / 4}\right]$.

This completes the proof of Theorem 2.2.

## Resonances Away from Glancing

Finally, we construct resonances coming from modes concentrating farther away from glancing but not normal to the boundary. In particular, we show the existence of modes concentrating $h^{2 / 3-2 \epsilon / 3}$ of glancing for $(3 \alpha-2) / 4<\epsilon \leq 1$. This will prove Theorem 2.3.

To do this, let $w=(n h)^{-1} z$ and $\zeta=\zeta(w)$. Then we first suppress the lower order terms in (2.2.9) and solve 2.4.2). Using the asymptotics 2.2.17), in 2.4.2 and letting $n=h_{1}^{-1}$ we have

$$
\begin{align*}
1-\frac{h_{1}^{1 / 3} \Phi i}{2(-\zeta)^{1 / 2}}\left(1+\sum_{j=1}^{N-1} \frac{c_{k} h_{1}^{k}}{(-\zeta)^{3 k / 2}}\right. & \left.-i e^{\frac{4}{3 h_{1}} i(-\zeta)^{3 / 2}}\left(1+\sum_{k=1}^{N-1} \frac{b_{k} h_{1}^{k}}{(-\zeta)^{3 k / 2}}\right)\right) \\
& +O\left(h_{1}^{N+1} h^{-\alpha}(-\zeta)^{-(3 N+1) / 2}\left(1+e^{\frac{4}{3 h_{1}} i(-\zeta)^{3 / 2}}\right)\right)=0 \tag{2.5.4}
\end{align*}
$$

where $c_{k}$ and $b_{k}$ are real.
We make the ansatz

$$
\begin{equation*}
(-\zeta)^{3 / 2}=\frac{3}{8} \pi h_{1}(4 k-1)+\epsilon=: m+\epsilon \tag{2.5.5}
\end{equation*}
$$

where we assume $\epsilon=O\left(m h^{\delta}\right)$ for some $\delta>0$. Then,

$$
\begin{equation*}
(-\zeta)^{1 / 2}=m^{1 / 3}\left(1+\frac{1}{3 m} \epsilon+O\left(\epsilon^{2} / m^{2}\right)\right), \quad(-\zeta)=m^{2 / 3}\left(1+\frac{2}{3 m} \epsilon+O\left(\epsilon^{2} / m^{2}\right)\right) . \tag{2.5.6}
\end{equation*}
$$

and $i e^{\frac{4}{3 h_{1}} i(-\zeta)^{3 / 2}}=e^{\frac{4}{3 h_{1}} \epsilon}$. Multiplying (2.5.4) by $(-\zeta)^{1 / 2}$ and using

$$
\Phi(\zeta)=\sum_{n=0}^{N-1} \frac{\Phi^{(n)}(m) \epsilon^{n}}{n!}+O\left(h_{1}^{2 / 3} h^{-\alpha} \epsilon^{N}\right)
$$

we have

$$
\begin{align*}
(-\zeta)^{1 / 2}-\frac{h_{1}^{1 / 3} \Phi(m) i}{2}\left(1+\sum_{k=1}^{N-1} \frac{c_{k} h_{1}^{k}}{(-\zeta)^{3 k / 2}}\right. & \left.-i e^{\frac{4}{3 h_{1}} i(-\zeta)^{3 / 2}}\left(1+\sum_{k=1}^{N-1} \frac{b_{k} h_{1}^{k}}{(-\zeta)^{3 k / 2}}\right)\right) \\
& +O\left(h_{1}^{1} h^{-\alpha}\left(h_{1}^{N} m^{-N}+\epsilon\right)\left(1+e^{\frac{4}{3 h_{1}} i \epsilon}\right)=0\right. \tag{2.5.7}
\end{align*}
$$

Then, let $\epsilon(h)=\epsilon_{0}+\epsilon_{1}$ where $\epsilon_{1}=O\left(\epsilon_{0} h^{\delta}\right)$ for some $\delta>0$. Using terms which do not involve $\epsilon$ and the exponential term,

$$
\epsilon_{0}=-\frac{3 h_{1} i}{4}\left[\log \left(\frac{2 m^{1 / 3} i}{h_{1}^{1 / 3} \Phi(m)}+1+\sum_{k=1}^{N-1} c_{k} h^{k} m^{-k}\right)-\log \left(1+\sum_{k=1}^{N-1} b_{k} h^{k} m^{-k}\right)\right] .
$$

Now, using

$$
e^{\frac{4 i}{3 h_{1}}\left(\epsilon_{0}+\epsilon_{1}\right)}=e^{\frac{4 i}{3 h_{1}} \epsilon_{0}}\left(1+\frac{4 i}{3 h_{1}} \epsilon_{1}+\mathcal{O}\left(\epsilon_{1}^{2} h^{-2}\right)\right) .
$$

we can solve for an asymptotic expansion for $\epsilon(h)$ in powers of $h_{1} m^{-1}$ so that for $\left(-\zeta_{0}\right)^{3 / 2}=$ $m+\epsilon(h)$,

$$
\left(-\zeta_{0}\right)^{1 / 2}-\frac{h^{1 / 3} \Phi\left(\zeta_{0}\right) i}{2}\left(1+\sum_{j=1}^{N-1} \frac{c_{k} h^{k}}{\left(-\zeta_{0}\right)^{3 k / 2}}-i e^{\frac{4}{3 h} i\left(-\zeta_{0}\right)^{3 / 2}}\left(1+\sum_{k=1}^{N-1} \frac{b_{k} h^{k}}{\left(-\zeta_{0}\right)^{3 k / 2}}\right)\right)=O\left(h^{\infty}\right)
$$

Then, since

$$
f(\eta)=\eta-\frac{h^{1 / 3} \Phi\left(\eta^{2}\right) i}{2}\left(1+\sum_{j=1}^{N-1} \frac{-\zeta_{k} h^{k}}{\eta^{3 k}}-i e^{\frac{4}{3 h} i \eta^{3}}\left(1+\sum_{k=1}^{N-1} \frac{b_{k} h^{k}}{\eta^{3 k}}\right)\right)
$$

has

$$
\begin{equation*}
\left|f^{\prime}(\eta)\right| \geq c\left|\zeta_{0}\right| h^{-\alpha}\left(1+\left|\zeta_{0}\right|^{1 / 2} h^{\alpha-1}\right), \quad\left|f^{\prime \prime}(\eta)\right| \leq c\left|\zeta_{0}\right|^{2} h^{-\alpha-1}\left(1+\left|\zeta_{0}\right|^{1 / 2} h^{\alpha-1}\right) \tag{2.5.8}
\end{equation*}
$$

when

$$
\left|\eta-\zeta_{0}^{1 / 2}\right| \leq C h
$$

Hence, Lemma 2.5.1 implies the existence of a solution to $f(\eta)=0$ that is $O\left(h^{\infty}\right)$ close to $\left(-\zeta_{0}\right)^{1 / 2}$. Next, by the implicit function theorem, $f(\eta)=a(\eta)$ defines $\eta$ as a function of $a$ for a sufficiently small. Thus, since

$$
\left.O\left(h_{1} h^{-\alpha}\left(h_{1}^{N} m^{-N}+\epsilon^{N}\right)\left(1+m^{1 / 2} h^{\alpha-1}\right)\right)+h_{1}^{2} h^{-\alpha} m^{2 / 3}\right)=O\left(h_{1}^{2} h^{-\alpha} m^{2 / 3}\right)
$$

there exists a solution, $z(k, h, n)$, to (2.3.3) with

$$
(-\zeta)^{1 / 2}=\left(-\zeta_{0}\right)^{1 / 2}+\frac{a\left(\left(-\zeta_{0}\right)^{1 / 2}, h\right)}{\partial_{\eta} f\left(\left(-\zeta_{0}\right)^{1 / 2}\right)}=\left(-\zeta_{0}\right)^{1 / 2}+\mathcal{O}\left(h_{1}^{2}\left(1+m^{1 / 3} h^{\alpha-1}\right)^{-1}\right)=\left(-\zeta_{0}\right)^{1 / 2}+O\left(h_{1}^{2}\right)
$$

Thus,

$$
\zeta=\zeta_{0}+O\left(\left(-\zeta_{0}\right)^{1 / 2} h_{1}^{2}\right)
$$

This shows that if $m \geq c h^{1-\delta}$, we can solve for $\zeta$ so that

$$
\zeta=\zeta_{0}+O\left(h^{2}\right)
$$

by choosing $N$ large enough. Now,

$$
\operatorname{Im}\left(-\zeta_{0}\right)=-\frac{3 h_{1}}{8 m^{1 / 3}} \log \left(\frac{4 m^{2 / 3}}{h_{1}^{2 / 3} \Phi^{2}(m)}+1\right)+O\left(\epsilon_{0} h_{1} m^{-4 / 3}\right)
$$

Hence, we have constructed resonances with

$$
\operatorname{Im} \zeta_{1}=\frac{3 h_{1}}{8 m^{1 / 3}} \log \left(\frac{4 m^{2 / 3}}{h_{1}^{2 / 3} \Phi^{2}(m)}+1\right)+O\left(\epsilon_{0} h_{1} m^{-4 / 3}+h_{1}^{2}\right)
$$

Because of the size of the lower order terms above, this construction only gives accurate estimates on $\operatorname{Im}\left(-\zeta_{0}\right)$ when $\delta>(3 \alpha-2) / 4$.

Thus, for $\delta \geq 0$, there exist resonances coming from modes concentrating $h^{2 / 3(1-\delta)}$ close to glancing with

$$
\operatorname{Im} z \sim \begin{cases}C h^{2 \alpha-2 / 3-\delta / 3} & (3 \alpha-2) / 4<\delta<3 \alpha-2 \\ C h & \delta=3 \alpha-2 \\ C h^{2 / 3+\delta / 3} \log h^{-1} & 3 \alpha-2<\delta \leq 1\end{cases}
$$

Moreover, for each $n$ with $(1-\epsilon) h^{-1} \leq|n| \leq(1+\epsilon) h^{-1}$, we have $\left(1-C h^{3 / 4}\right) n h^{-1} \leq$ $\operatorname{Re} w \leq n h^{-1}\left(1+C h^{3 / 4}\right)$. Hence, $\operatorname{Re} \zeta$ ranges over an interval of size $C h^{3 / 4}$. Together with the construction above, this implies that for each such $n$ we have at least $c h^{-1 / 4}$ resonances a fixed distance from glancing. Thus, for $M$ large enough

$$
\#\{z \in \Lambda(h)\} \geq C h^{-5 / 4}
$$

This implies Theorem 2.3.

### 2.6 The $\delta^{\prime}$ Potential

## Reduction to Transcendental Equations on the Circle $\delta^{\prime}$

We now consider (2.1.4) with $\Omega=B(0,1) \subset \mathbb{R}^{2}$ and $V \equiv h^{\alpha} V_{0}$ with $V_{0}>0$ and $0 \leq \alpha$. Then for $i=1,2$,

$$
\begin{cases}\left(-h^{2} \partial_{r}^{2}-\frac{h^{2}}{r} \partial_{r}-\frac{h^{2}}{r^{2}} \partial_{\theta}^{2}-z^{2}\right) u_{1}=0 & \text { in } B(0,1)  \tag{2.6.1}\\ \left(-h^{2} \partial_{r}^{2}-\frac{h^{2}}{r} \partial_{r}-\frac{h^{2}}{r^{2}} \partial_{\theta}^{2}-z^{2}\right) u_{2}=0 & \text { in } \mathbb{R}^{2} \backslash B(0,1) \\ \partial_{r} u_{1}(1, \theta)=\partial_{r} u_{2}(1, \theta) \\ u_{1}(1, \theta)-u_{2}(1, \theta)+V \partial_{r} u_{1}(1, \theta)=0 \\ u_{2} \text { is } z \text { outgoing }\end{cases}
$$

Expanding in Fourier series, write $u_{i}(r, \theta):=\sum_{n} u_{i, n}(r) e^{i n \theta}$. Then, $u_{i, n}$ solves

$$
\left(-h^{2} \partial_{r}^{2}-h^{2} \frac{1}{r} \partial_{r}+h^{2} \frac{n^{2}}{r^{2}}-z^{2}\right) u_{i, n}(r)=0
$$

Multiplying by $r^{2}$ and rescaling by $x=z h^{-1} r$, we see that $u_{i, n}(r)$ solves the Bessel equation with parameter $n$ in the $x$ variables. Then, using that $u_{2}$ is outgoing and $u_{1} \in L^{2}$, we obtain that $u_{1, n}(r)=K_{n} J_{n}\left(z h^{-1} r\right)$ and $u_{2, n}(r)=C_{n} H_{n}^{(1)}\left(z h^{-1} r\right)$ where $J_{n}$ is the $n^{\text {th }}$ Bessel function of the first kind, and $H_{n}^{(1)}$ is the $n^{\text {th }}$ Hankel function of the first kind.

To solve (2.6.1) and hence find a resonance, we only need to find $z$ such that the boundary conditions hold. Using the boundary condition $\partial_{r} u_{1}(1, \theta)=\partial_{r} u_{2}(1, \theta)$, we have $z h^{-1} K_{n} J_{n}^{\prime}\left(z h^{-1}\right)=z h^{-1} C_{n} H_{n}^{\prime(1)}\left(z h^{-1}\right)$. Hence,

$$
C_{n}=\frac{K_{n} J_{n}^{\prime}\left(z h^{-1}\right)}{H_{n}^{\prime(1)}\left(z h^{-1}\right)} .
$$

Next, we rewrite the second boundary condition in 2.3.1 and use that $V \equiv h^{\alpha} V_{0}$ to get

$$
\sum_{n}\left(K_{n} J_{n}\left(z h^{-1}\right)-C_{n} H_{n}^{\prime(1)}\left(z h^{-1}\right)+h^{\alpha} V_{0} K_{n} z h^{-1} J_{n}^{\prime}\left(z h^{-1}\right)\right) e^{i n \theta}=0
$$

Then, since $e^{i n \theta}$ are $L^{2}$ orthogonal, we have

$$
K_{n}\left(J_{n}\left(z h^{-1}\right)-\frac{J_{n}^{\prime}\left(z h^{-1}\right)}{H_{n}^{\prime(1)}\left(z h^{-1}\right)} H_{n}^{(1)}\left(z h^{-1}\right)+h^{\alpha} V_{0} z h^{-1} J_{n}^{\prime}\left(z h^{-1}\right)\right)=0, \quad n \in \mathbb{Z}
$$

Thus

$$
-K_{n} z h^{-1+\alpha} V_{0}=K_{n}\left(\frac{J_{n}\left(z h^{-1}\right)}{J_{n}^{\prime}\left(z h^{-1}\right)}-\frac{H_{n}^{(1)}\left(z h^{-1}\right)}{H_{n}^{\prime(1)}\left(z h^{-1}\right)}\right)
$$

which can be written

$$
\begin{equation*}
-h^{-1+\alpha} z V_{0} K_{n}=K_{n} \frac{W\left(J_{n}, H_{n}^{(1)}\right)\left(z h^{-1}\right)}{J_{n}^{\prime}\left(z h^{-1}\right) H_{n}^{\prime(1)}\left(z h^{-1}\right)}=\frac{2 i K_{n}}{\pi z h^{-1} J_{n}^{\prime}\left(z h^{-1}\right) H_{n}^{\prime(1)}\left(z h^{-1}\right)} \tag{2.6.2}
\end{equation*}
$$

where $W(f, g)$ is the Wronskian of $f$ and $g$.
Then, without loss, we assume $K_{n}=1$ or $K_{n}=0$. Hence, we seek solutions $z(h, n)$ to

$$
\begin{equation*}
1+\frac{\pi z^{2}(h, n) h^{-2+\alpha} V_{0}}{2 i} J_{n}^{\prime}\left(h^{-1} z(h, n)\right) H_{n}^{\prime(1)}\left(h^{-1} z(h, n)\right)=0 . \tag{2.6.3}
\end{equation*}
$$

## Resonance Free Regions for the Disk $\delta^{\prime}$

We write $n=m h^{-1}$ and assume that

$$
|\operatorname{Im} z| \leq M_{0} \min \left(h \log h^{-1}, h^{3-2 \alpha}\right)
$$

## Analysis for $m \gg 1$

We use the asymptotics (2.2.5) in (2.6.3). Equation 2.6.3) then reads for $m^{-1} z=\operatorname{sech}(\alpha)$,

$$
1+\frac{z^{2} h^{-1+\alpha} V_{0} \sinh (2 \alpha)}{4 m}\left(1+O\left(m^{-1} h\right)\right)=0
$$

but for $m$ large enough (independent of $h$ small enough), this clearly has no solution since the second term has positive real part.

## Analysis for $m \ll 1$

The asymptotics (2.2.12) are not quite strong enough to make the analysis go through for $m \ll 1$. Rather than proceeding to use higher order terms, we refer the reader to Chapter 10 where we treat the general case and, using the fact that $\mathrm{WF}_{\mathrm{h}}\left(e^{i m h^{-1} \theta}\right) \subset\left\{\left|\xi^{\prime}\right|=m\right\}$, we obtain

Lemma 2.6.1. For all $\delta>0$ there exists $M, \epsilon>0$ and $h_{0}>0$ such that for $0<h<h_{0}$, $\operatorname{Re} z \in\left[1-C h^{3 / 4}, 1+C h^{3 / 4}\right]$, and $n=m h^{-1}$ with $m<\epsilon$ or $m>M$ then there are no solutions to 2.6.3 with

$$
-\operatorname{Im} z \leq(1-\delta) \min \left(\frac{1}{V_{0}^{2}} h^{3-2 \alpha}, \frac{1}{2} h \log h^{-1}\right)
$$

## Analysis for $\epsilon<m<M$

In this section, we consider the remaining values of $m$. First, we use 2.2.10 in 2.6.3 to write

$$
\begin{align*}
1+\pi h^{-2 / 3+\alpha} V_{0} 2 e^{-5 \pi i / 6} m^{2 / 3} & \left(\frac{1-m^{-2} z^{2}}{\zeta}\right)^{1 / 2} \\
& \left(A i^{\prime}\left(n^{2 / 3} \zeta\right) A_{-}^{\prime}\left(n^{2 / 3} \zeta\right)+m^{4 / 3} h^{-4 / 3} O\left(E i_{-}\left(\frac{8}{3}, 2, \frac{10}{3}\right)\right)\right)=0 \tag{2.6.4}
\end{align*}
$$

where $\zeta=\zeta\left(m^{-1} h\right)$. The error term can be estimated by

$$
\left.\left\lvert\, h^{-4 / 3} E i_{-}\left(\frac{8}{3}, 2, \frac{10}{3}\right)\right.\right) \mid \leq C\left(\left(h^{4 / 3}\left\langle h^{-2 / 3} \zeta\right\rangle^{-1 / 2}+h^{2 / 3}+h^{2}\left\langle h^{-2 / 3} \zeta\right\rangle^{1 / 2}\right)=O\left(h^{2 / 3}\right)\right.
$$

We first ignore the error term in (2.6.4) and show that there are no solutions with the appropriate bounds on $\operatorname{Im} \zeta$. In particular, define $h_{1}=: n^{-1}$ and

$$
\Phi:=h_{1}^{-2 / 3} h^{\alpha}\left(\frac{1-\left(h_{1} h^{-1} z\right)^{2}}{\zeta}\right)^{1 / 2} V_{0}=O_{C^{\infty}}\left(h_{1}^{-2 / 3} h^{\alpha}\right)
$$

The fact that $\Phi$ has uniform bounds for $\zeta$ in the relevant region comes from the fact that $h_{1} h^{-1}=m$ and $\epsilon<m<M$. Then, rewriting 2.4.1 without the lower order terms, we have

$$
\begin{equation*}
1+2 \pi e^{-5 \pi i / 6} \Phi(\zeta) A_{-}^{\prime}\left(h_{1}^{-2 / 3} \zeta\right) A i^{\prime}\left(h_{1}^{-2 / 3} \zeta\right)=0 \tag{2.6.5}
\end{equation*}
$$

Then, ignoring lower order terms in (2.6.3), we show that there are no solutions to 2.6.5 with the appropriate bounds on $\operatorname{Im} \zeta$. Notice that for $\alpha \geq 2 / 3$, and any $\delta>0$ we need not consider the region $|\operatorname{Re} \zeta|<\min \left(\epsilon h^{2-2 \alpha}, h^{\delta}\right)$, since in this region $\left|h^{\alpha-2 / 3} A i^{\prime} A_{-}^{\prime}\right| \ll 1$.

## Analysis at glancing ( $m \sim 1$ )

We first analyze the region very close to glancing. In particular, $|\zeta|<\epsilon h_{1}^{1 / 6}$. By the observation at the end of the last section, we need only consider $\alpha \leq 1$.

Let $s=h_{1}^{-2 / 3} \operatorname{Re} \zeta$. then, $0 \leq|s|<\epsilon h^{-1 / 2}$ and

$$
\zeta=h_{1}^{2 / 3} s+\operatorname{Im} \zeta=h_{1}^{2 / 3} s+O\left(h^{3-2 \alpha}\right)
$$

where we have used the fact that $\alpha \leq 1$. Thus,

$$
\begin{aligned}
\left|\Phi(\zeta) A i^{\prime} A_{-}^{\prime}\left(h_{1}^{-2 / 3} \zeta\right)-\Phi\left(h_{1}^{2 / 3} s\right) A i^{\prime} A_{-}^{\prime}(s)-\Phi\left(h_{1}^{2 / 3} s\right)\left(A i^{\prime} A_{-}^{\prime}\right)^{\prime}(s) i \operatorname{Im} h_{1}^{-2 / 3} \zeta\right| \leq \\
O\left(h_{1}^{-2 / 3}\langle s\rangle^{3 / 2} h^{\alpha}\left(\operatorname{Im} h_{1}^{-2 / 3} \zeta\right)^{2}\right)+O\left(h_{1}^{-2 / 3} h^{\alpha} \operatorname{Im} \zeta\langle s\rangle^{1 / 2}\right)
\end{aligned}
$$

We obtain lower bounds on

$$
f\left(s, h, h_{1}\right):=1+2 \pi e^{-5 \pi i / 6} \Phi\left(h_{1}^{2 / 3} s\right)\left(A_{-}^{\prime} A i^{\prime}(s)+\left(A_{-}^{\prime} A i^{\prime}\right)^{\prime}(s) i \operatorname{Im} h_{1}^{-2 / 3} \zeta\right)
$$

Hence, letting $\alpha:=e^{-5 \pi i / 6}$, we have by 2.2 .14 that

$$
\alpha A_{-}^{\prime}(s) A i^{\prime}(s)=\operatorname{Re}\left(\alpha A_{-}^{\prime}(s)\right) A i^{\prime}(s)-i \frac{\left(A i^{\prime}\right)^{2}(s)}{2}
$$

and

$$
\begin{aligned}
\left(\alpha A_{-}^{\prime} A i^{\prime}\right)^{\prime}(s) i \operatorname{Im} h_{1}^{-2 / 3} \zeta= & s\left(A i(s) A i^{\prime}(s)\right. \\
& \left.+i\left[A i^{\prime}(s) \operatorname{Re}\left(\alpha A_{-}(s)\right)+A i(s) \operatorname{Re}\left(\alpha A_{-}^{\prime}(s)\right)\right]\right) \operatorname{Im} h_{1}^{-2 / 3} \zeta
\end{aligned}
$$

Thus,

$$
\operatorname{Im} f=2 \pi \Phi\left(h_{1}^{2 / 3} s\right)\left(-\frac{\left(A i^{\prime}\right)^{2}(s)}{2}+s\left(A i^{\prime}(s) \operatorname{Re}\left(\alpha A_{-}(s)\right)+A i(s) \operatorname{Re}\left(\alpha A_{-}^{\prime}(s)\right)\right) \operatorname{Im} h_{1}^{-2 / 3} \zeta\right)
$$

and

$$
\operatorname{Re} f=1+2 \pi \Phi\left(h_{1}^{2 / 3} s\right) A i^{\prime}(s)\left(\operatorname{Re}\left(\alpha A_{-}^{\prime}(s)\right)+s A i^{\prime}(s) \operatorname{Im} h_{1}^{-2 / 3} \zeta\right)
$$

So, when

$$
\left|A i^{\prime}(s)\right| \leq \frac{1-\delta}{2 \pi \Phi\left(h_{1}^{2 / 3} s\right) \operatorname{Re}\left(\alpha A_{-}^{\prime}(s)\right)}, \quad \text { or } \quad\left|A i^{\prime}(s)\right| \geq \frac{1-\delta}{2 \pi \Phi\left(h_{1}^{2 / 3} s\right) \operatorname{Re}\left(\alpha A_{-}^{\prime}(s)\right)}
$$

then $|f| \geq \delta$. Thus, we need only consider

$$
\begin{equation*}
\frac{1-\delta}{2 \pi \Phi\left(h_{1}^{2 / 3} s\right) \operatorname{Re}\left(\alpha A_{-}^{\prime}(s)\right)} \leq\left|A i^{\prime}(s)\right| \leq \frac{1-\delta}{2 \pi \Phi\left(h_{1}^{2 / 3} s\right) \operatorname{Re}\left(\alpha A_{-}^{\prime}(s)\right)} \tag{2.6.6}
\end{equation*}
$$

Using (2.6.6) together with (2.2.15), we obtain that

$$
|\operatorname{Im} f| \geq h_{1}^{2 / 3} h^{-\alpha}\langle s\rangle^{-1 / 2}
$$

provided

$$
\left.\left.h^{4 / 3-2 \alpha}\right\rangle s\right\rangle^{-1 / 2} \geq M h^{7 / 3-2 \alpha} .
$$

But, this is satisfied since we assume $s \leq \epsilon h^{-1 / 2}$. Moreover, the terms we ignored are of size

$$
O\left(h^{4-3 \alpha}\langle s\rangle^{3 / 2}+h^{7 / 3-\alpha}\langle s\rangle^{1 / 2}+h^{\alpha}\right)
$$

so this implies that there are no solutions to 2.6 .3 ) for $|\operatorname{Re} \zeta| \leq \epsilon h^{1 / 6}$.

Asymptotic analysis in the hyperbolic ( $\epsilon<m<1$ ) and elliptic ( $K>m>1$ ) regions
We need to analyze $K \geq|\operatorname{Re} \zeta| \geq h^{1 / 6} \epsilon$. To do this, let $D_{\Delta}=\frac{i h_{1}^{-1 / 3}(-\zeta)^{1 / 2}}{2}$, and $b:=2 \pi e^{-5 \pi i / 6}$. Then, if $\zeta$ solves 2.6.3)

$$
D_{\Delta}^{1 / 2} \Phi=D_{\Delta}^{1 / 2} \Phi D_{\Delta}^{1 / 2}\left(-D_{\Delta}^{-1 / 2} b A i^{\prime}\left(h_{1}^{-2 / 3} \zeta\right) A_{-}^{\prime}\left(h_{1}^{-2 / 3} \zeta\right) D_{\Delta}^{-1 / 2}+O\left(h^{4 / 3}\right)\right) D_{\Delta}^{1 / 2} \Phi
$$

and

$$
\begin{aligned}
& \left(1-D_{\Delta}^{1 / 2} \Phi D_{\Delta}^{1 / 2}\right) D_{\Delta}^{1 / 2} \Phi \\
& \quad=D_{\Delta}^{1 / 2} \Phi D_{\Delta}^{1 / 2}\left(-D_{\Delta}^{-1 / 2} b A i^{\prime}\left(h_{1}^{-2 / 3} \zeta\right) A_{-}^{\prime}\left(h_{1}^{-2 / 3} \zeta\right) D_{\Delta}^{-1 / 2}-1+O\left(h^{4 / 3}\right)\right) D_{\Delta}^{1 / 2} \Phi .
\end{aligned}
$$

Using 2.2.18 for $\operatorname{Re} \zeta<-M h^{2 / 3}$, we have

$$
\begin{gathered}
\left(1-D_{\Delta}^{1 / 2} \Phi D_{\Delta}^{1 / 2}\right) D_{\Delta}^{1 / 2} \Phi=D_{\Delta}^{1 / 2} \Phi D_{\Delta}^{1 / 2}\left(i e^{\frac{4 i}{3 h_{1}}(-\zeta)^{3 / 2}}\left(1+O\left(h_{1} \zeta^{-3 / 2}\right)\right)\right) D_{\Delta}^{1 / 2} \Phi \\
D_{\Delta}^{1 / 2} \Phi=\left(I-D_{\Delta}^{1 / 2} \Phi D_{\Delta}^{1 / 2}\right)^{-1} D_{\Delta}^{1 / 2} \Phi D_{\Delta}^{1 / 2}\left(i e^{\frac{4 i}{3 h_{1}}(-\zeta)^{3 / 2}}\left(1+O\left(h_{1} \zeta^{-3 / 2}\right)\right)\right) D_{\Delta}^{1 / 2} \Phi
\end{gathered}
$$

Remark: The analog of the reflection operator in this setting is given by

$$
\left(I-D_{\Delta}^{1 / 2} \Phi D_{\Delta}^{1 / 2}\right)^{-1} D_{\Delta}^{1 / 2} \Phi D_{\Delta}^{1 / 2}
$$

For $\zeta>M h_{1}^{2 / 3}$, we use (2.2.19) to obtain

$$
\left(I-D_{\Delta}^{1 / 2} \Phi D_{\Delta}^{1 / 2}\right) D_{\Delta}^{1 / 2} \Phi=D_{\Delta}^{1 / 2} \Phi D_{\Delta}^{1 / 2}\left(O\left(h_{1} \zeta^{-3 / 2}\right)\right) D_{\Delta}^{1 / 2} \Phi .
$$

Hence,

$$
D_{\Delta}^{1 / 2} \Phi=\left(I-D_{\Delta}^{1 / 2} \Phi D_{\Delta}^{1 / 2}\right)^{-1} D_{\Delta}^{1 / 2} \Phi D_{\Delta}^{1 / 2} O\left(h_{1} \zeta^{-3 / 2}\right) D_{\Delta}^{1 / 2} \Phi .
$$

To see that $I-D_{\Delta}^{1 / 2} \Phi D_{\Delta}^{1 / 2} \neq 0$ observe that when $\operatorname{Re} \zeta<-M h^{2 / 3}$,

$$
\left|\operatorname{Re} i h_{1}^{-1 / 3} \Phi(-\zeta)^{1 / 2}\right|=h_{1}^{-1} h^{\alpha} O(\operatorname{Im} \zeta)=o(1)
$$

and when $\operatorname{Re} \zeta>M h^{2 / 3}$,

$$
-\operatorname{Re} h_{1}^{-1 / 3} \Phi \zeta^{1 / 2} \geq 0
$$

Now, since $|\operatorname{Re} \zeta|>M h_{1}^{2 / 3}, O\left(h_{1} \zeta^{-3 / 2}\right) \ll 1$ for $M$ large. Hence, there are no zeros for $\operatorname{Re} \zeta>0$. For $\operatorname{Re} \zeta<0$, there are no zeros of (2.4.2) when

$$
\begin{equation*}
\left|\left(I-D_{\Delta}^{1 / 2} \Phi D_{\Delta}^{1 / 2}\right)^{-1} D_{\Delta}^{1 / 2} \Phi D_{\Delta}^{1 / 2}\left(1+O\left(h_{1} \zeta^{-3 / 2}\right)\right) e^{\frac{4 i}{3 h_{1}}(-\zeta)^{3 / 2}}\right|<1 . \tag{2.6.7}
\end{equation*}
$$

Let $\zeta=s+i \operatorname{Im} \zeta$. Then

$$
(-\zeta)^{3 / 2}=(-s)^{3 / 2}\left(1-i \operatorname{Im} \zeta(-s)^{-1}\right)^{3 / 2}=(-s)^{3 / 2}\left(1-\frac{3}{2} i \operatorname{Im} \zeta(-s)^{-1}+O\left((\operatorname{Im} \zeta)^{2} s^{-2}\right)\right)
$$

and

$$
(-\zeta)^{1 / 2}=(-s)^{1 / 2}\left(1+O\left(\operatorname{Im} \zeta s^{-1}\right)\right)
$$

So,

$$
\left|e^{\frac{4 i}{3 h_{1}}(-\zeta)^{3 / 2}}\right|=e^{\frac{2 \operatorname{Im} \zeta(-s)^{1 / 2}}{h_{1}}+O\left((\operatorname{Im} \zeta)^{2}|s|^{-1 / 2} h_{1}^{-1}\right)},
$$

and taking logarithms of (2.6.7),

$$
\frac{2 \operatorname{Im} \zeta(-s)^{1 / 2}}{h_{1}}+\mathcal{O}\left((\operatorname{Im} \zeta)^{2} h_{1}^{-1}|s|^{-1 / 2}\right)+\log \left|\frac{i h_{1}^{-1 / 3} \Phi(-\zeta)^{1 / 2}}{2-i h_{1}^{-1 / 3}(-\zeta)^{1 / 2} \Phi}\right|+\mathcal{O}\left(h_{1} \zeta^{-3 / 2}\right)<0
$$

Thus,

$$
\begin{aligned}
\operatorname{Im} \zeta<\inf _{-K<s<-\epsilon h_{1}^{1 / 6}} & \frac{h_{1}}{4(-s)^{1 / 2}} \log \left|1+4(-s)^{-1} h_{1}^{2 / 3} \Phi^{-2}\right| \\
& +O\left((\operatorname{Im} \zeta)^{2}|s|^{-1}+O\left(\left.\operatorname{Im} \zeta h_{1}^{2} h^{-\alpha}(-s)^{-1 / 2}\left\langle h_{1} h^{-\alpha}\right| s\right|^{1 / 2}\right\rangle^{-1}\right)+O\left(h_{1}^{2}|s|^{-2}\right)
\end{aligned}
$$

That is, since we have $K>|s|>\epsilon h_{1}^{1 / 6}$ and $\alpha>2 / 3$, the error terms are lower order and hence

$$
\operatorname{Im} \zeta<\min \left(C h^{3-2 \alpha}, C h \log h^{-1}\right)
$$

Together with the fact that $\operatorname{Im} m^{-1} z=-C \operatorname{Im} \zeta+O\left((\operatorname{Im} \zeta)^{2}\right)$, this completes the proof of the existence of resonance free regions of the sizes given in Theorem 2.4.

### 2.7 Construction of Resonances

## Resonances normal to the boundary

We consider $n$ fixed relative to $h$. That is, we consider modes that concentrate normal to $\partial B(0,1)$.

Using asymptotics (2.2.3) in 2.6.3), we have

$$
\begin{equation*}
1+\frac{i z h^{-1+\alpha} V_{0}}{2}\left(e^{i\left(2 z h^{-1}-n \pi-\pi / 2\right)}-1+\mathcal{O}\left(|z|^{-1} h e^{2|\operatorname{Im} z| h^{-1}}\right)\right)=0 \tag{2.7.1}
\end{equation*}
$$

Let $z=\frac{h}{4}\left(\pi(2 n+4 k+1)+4 \epsilon h^{-1}\right)$ with $\pi k=h^{-1}+\mathcal{O}(1)$. Substituting this in to 2.7.1) and ignoring the error term, as well as higher order terms in $\epsilon$, we obtain

$$
F(\epsilon, k, n, h)=1+i \frac{\pi(2 n+4 k+1) h^{\alpha} V_{0}}{8}\left(e^{2 i \epsilon / h}-1\right)
$$

Then,

$$
\epsilon_{0}(k, n, h)=-\frac{i h}{2} \log \left[1+i \frac{8 h^{-\alpha}}{\pi(2 n+4 k+1) V_{0}}\right]=\frac{-i h}{2} \log \left[1+i 2 h^{1-\alpha} V_{0}^{-1}(1+O(h))\right]
$$

has

$$
F\left(\epsilon_{0}(k, n, h), k, n, h\right)=0, \quad\left|\partial_{\epsilon} F\left(\epsilon_{0}(k, n, h), k, n, h\right)\right| \geq c h^{-1}
$$

Now, for $0<c$ and $c h^{-1}<k<C h^{-1}$ by (2.7.1), $z(h, k, n)$ can be defined by a solution $z(h, k, n)=\frac{\pi h}{4}(4 k+2 n+1)+\epsilon(k, n, h)$ where

$$
F(\epsilon, k, n, h)=O\left(e^{2 i \epsilon / h} h^{\alpha}|z|^{-1}+\epsilon h^{-1+\alpha}\right) .
$$

So, by the implicit function theorem there is a solution $\epsilon_{1}$ satisfying

$$
\begin{aligned}
\epsilon(k, n, h) & =\epsilon_{0}(k, n, h)+\left(\partial_{\epsilon} F\left(\epsilon_{0}(k, n, h), k, n, h\right)\right)^{-1} O\left(h^{-1+\alpha} e^{2 i \epsilon_{0} / h}\left(h|z|^{-1}+\epsilon_{0}\right)\right) \\
& =\epsilon_{0}(k, n, h)+O\left(h^{2}+\min \left(h^{2} \log h^{-1}, h^{3-\alpha}\right)=\epsilon_{0}+o\left(\operatorname{Im} \epsilon_{0}\right)\right.
\end{aligned}
$$

where the last equality follows from the fact that $\alpha>1 / 2$.
Thus, for all $\alpha>1 / 2 \epsilon>0$ and $0<h<h_{\epsilon}$, there exist $z(h) \in \Lambda$ with

$$
\operatorname{Im} z= \begin{cases}-(1+o(1)) V_{0}^{-2} h^{3-2 \alpha} & 1 / 2<\alpha<1  \tag{2.7.2}\\ -(1+o(1)) \frac{h}{4} \log \left(1+4 h^{2-2 \alpha} V_{0}^{-2}\right)+O\left(h^{3 / 4}\right) & \alpha \geq 1\end{cases}
$$

### 2.8 The 1- $d$ case for the $\delta^{\prime}$ potential

Consider $-\Delta_{\partial \Omega, \delta^{\prime}}$ with $\Omega=(-1,1) \subset \mathbb{R}$ with $v=h^{\alpha} V_{0}$. We first compute the reflection coefficient at a point $x=x_{0}$ from the left and the right. That is consider a solution to $u$ to (2.1.4). Then

$$
u= \begin{cases}e^{-i \lambda x} & x<-1 \\ a_{+} e^{i \lambda x}+a_{-} e^{-i \lambda x} & -1<x<1 \\ T_{+} e^{i \lambda x} & x>1\end{cases}
$$

where $\lambda=z h^{-1}$. If we consider only the interface at $x=1$, and assume for the moment that $a_{+}=1, a_{-}=R$, and $T_{+}=T$ then we must have

$$
e^{i \lambda}-R e^{-i \lambda}=T e^{i \lambda} \quad e^{i \lambda}+R e^{-2 i \lambda}+T(-1+i V \lambda) e^{i \lambda}=0 .
$$

Hence,

$$
R=\frac{-i \lambda V}{2-i \lambda} e^{2 i \lambda}, \quad T=\frac{2}{2-i \lambda V}
$$

That is, for the solution $u$, we must have $a_{-}=R a_{+}$and $T_{+}=T a_{+}$. On the other hand, if we consider the interface at $x=-1$, then under a change of variables $x \mapsto-x$, we see that $a_{+}=R a_{-}$, and $T_{-}=T a_{-}$. So, there is a resonance if and only if $R^{2}=1$. That is,

$$
\begin{equation*}
e^{4 i \lambda} \frac{-\lambda^{2} V^{2}}{(2-i \lambda V)^{2}}=1 \quad \Rightarrow \quad e^{4 i \lambda}=1+4 i \lambda^{-1} V^{-1}-4 \lambda^{-2} V^{-2} \tag{2.8.1}
\end{equation*}
$$

Now, a solution $z$ to this equation has

$$
\begin{aligned}
z & =-\frac{i h}{4} \log \left(1+\frac{4 i h^{1-\alpha}}{z V_{0}}-\frac{4 h^{2-2 \alpha}}{V_{0}^{2} z^{2}}\right) \\
& =-\frac{i h}{4} \log \left(1-\frac{4 h^{2-2 \alpha}}{V_{0}^{2}(\operatorname{Re} z)^{2}}+\frac{4 i h^{1-\alpha}}{\operatorname{Re} z V_{0}}\right)+\operatorname{Im} z O\left(h^{2-\alpha}\left\langle h^{1-\alpha}\right\rangle^{-1}\right) \\
\operatorname{Im} z\left(1+O\left(h^{2-\alpha}\left\langle h^{1-\alpha}\right\rangle^{-1}\right)\right) & =-\frac{h}{8} \log \left(1+\frac{4 h^{2-2 \alpha}}{(\operatorname{Re} z)^{2} V_{0}^{2}}\right)^{2} \\
& =-\frac{h}{4} \log \left(1+\frac{4 h^{2-2 \alpha}}{(\operatorname{Re} z)^{2} V_{0}^{2}}\right)
\end{aligned}
$$

This gives the resonance free region from Theorem 2.6.
To show the existence of resonances, let

$$
F(z)=e^{4 i z / h}-1-8 i h^{1-\alpha} V_{0}^{-1} z^{-1}+16 h^{2-2 \alpha} V_{0}^{-2} z^{-2} .
$$

Take

$$
\epsilon_{0}=-\frac{i h}{4} \log \left(1+8 i \frac{h^{1-\alpha}}{V_{0} \pi k h}-\frac{16 h^{2-2 \alpha}}{V_{0}^{2}(\pi h k)^{2}}\right)
$$

and let $z_{0}=\frac{\pi h k}{2}+\epsilon_{0}$ wiht $c h^{-1} \leq k \leq C h^{-1}$. Then, $F\left(z_{0}\right)=O\left(\operatorname{Im} \epsilon_{0} h^{1-\alpha}\left\langle h^{1-\alpha}\right\rangle^{-1}\left\langle h^{2-2 \alpha}\right\rangle\right)$, $\left|F^{\prime}\left(z_{0}\right)\right| \geq C h^{-1}\left\langle h^{2-2 \alpha}\right\rangle$ and for $\left|z-z_{0}\right| \leq C h\left(\log h^{-1}\right)^{-1},\left|F^{\prime \prime}\left(z_{0}\right)\right| \leq C h^{-2}\left\langle h^{2-2 \alpha}\right\rangle$. Hence, by Lemma 2.5.1 we have a resonance $z(k, h)$ with

$$
z(k, h)=\left\{\begin{array}{ll}
z_{0}(k)+O\left(h^{5-3 \alpha}\right) & \alpha \leq 1 \\
z_{0}(k)+O\left(h^{2} \log h^{-1}\right) & \alpha>1
\end{array} .\right.
$$

This completes the proof of Theorem 2.6 if we let $\pi h k / 2=1+O(h)$.

## Chapter 3

## Geometric Preliminaries

### 3.1 Local Symplectic Geometry

We give a brief review of the notation of symplectic geometry followed by a more detailed review of the theory of Lagrangian submanifolds. For our purposes, we need consider only symplectic geometry on $\mathbb{R}^{2 d}$ which we later identify with $T^{*} \mathbb{R}^{d}$. We follow [87, Chapter 2], [36, Chapter 5, 9], 41, Chapter 21] where one can find a more complete treatment.

## Notation

For a point $z \in \mathbb{R}^{d} \times \mathbb{R}^{d}$, we use coordinates $(x, \xi)$ where $x$ represents position and $\xi$ represents momentum. Similarly, we use $w=(y, \eta)$ for another typical point. We then make the following definitions

Definition 3.1.1. Let the one-forms $d x_{j}$ and $d \xi_{j}$ be dual to $\partial_{x_{j}}$ and $\partial_{\xi_{j}}$ respectively. Then we define the canonical one form by

$$
\omega:=\xi d x=\sum_{j} \xi_{j} d x_{j} .
$$

The symplectic form is given by

$$
\sigma:=d \omega=d \xi \wedge d x=\sum_{j} d \xi_{j} \wedge d x_{j}
$$

Then, letting $\langle\cdot, \cdot\rangle$ denote the usual inner product on $\mathbb{R}^{d}$, we have that

$$
\sigma(z, w)=\langle\xi, y\rangle-\langle x, \eta\rangle
$$

With this definition, $\sigma$ is a non-degenerate, closed, antisymmetric two form.

Definition 3.1.2. Let $U, V$ be open subsets of $\mathbb{R}^{2 d}$. We say that a diffeomorphism $\kappa: U \rightarrow V$ is a symplectomorphism if it preserves $\sigma$. That is,

$$
\kappa^{*} \sigma(z, w):=\sigma(d \kappa z, d \kappa w)=\sigma(z, w)
$$

If we write $\kappa(x, \xi)=(y(x, \xi), \eta(x, \xi))$, we sometimes write this as

$$
d \eta \wedge d y=d \xi \wedge d x
$$

Notice that if $\gamma: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ is a diffeomorphism, then it can be lifted to a symplectomorphism $\kappa: \mathbb{R}^{2 d} \rightarrow \mathbb{R}^{2 d}$ by letting

$$
\kappa(x, \xi)=\left(\gamma(x),\left(\partial \gamma(x)^{-1}\right)^{t} \xi\right)
$$

We next define the notion of Hamiltonian vector fields which was alluded to in the introduction.

Definition 3.1.3. Let $f \in C^{\infty}\left(\mathbb{R}^{2 d}\right)$. Then the corresponding Hamiltonian vector field, $H_{f}$ is given by

$$
\sigma\left(z, H_{f}\right)=d f(z) \quad \text { for all } z=(x, \xi)
$$

With this definition,

$$
H_{f}=\left\langle\partial_{\xi} f, \partial_{x}\right\rangle-\left\langle\partial_{x} f, \partial_{\xi}\right\rangle
$$

The Hamiltonian flow of $f$ is defined to be the flow of the vector field $H_{f}$.
Definition 3.1.4. For $f, g \in C^{\infty}\left(\mathbb{R}^{2 d}\right)$, the Poisson bracket of $f$ and $g,\{f, g\}$ is given by

$$
\{f, g\}:=H_{f} g=\sigma(\partial f, \partial g)=\left\langle\partial_{\xi} f, \partial_{x} g\right\rangle-\left\langle\partial_{x} g, \partial_{\xi} f\right\rangle
$$

Lemma 3.1.5. The Poisson bracket satisfies the following identities:
(i) Jacobi's identity

$$
\{f,\{g, h\}\}+\{g,\{h, f\}\}+\{h,\{f, g\}\}=0
$$

(ii)

$$
H_{\{f, g\}}=\left[H_{f}, H_{g}\right] .
$$

We are able to reduce our study of symplectic geometry to that where $\sigma$ is as above by Darboux's Theorem:

Lemma 3.1.6. Let $U$ be a neighborhood of $\left(x_{0}, \xi_{0}\right)$ and suppose $\eta$ is a non-degenerate, closed 2-form. Then near $\left(x_{0}, \xi_{0}\right)$, there exists a symplectomorphism $\kappa$ such that

$$
\kappa^{*} \eta=\sigma
$$

## Lagrangian Submanifolds

Definition 3.1.7. A Lagrangian submanifold, $\Lambda$, in $\mathbb{R}^{2 d}$ is a $d$-dimensional submanifold such that

$$
\left.\sigma\right|_{\Lambda}=0
$$

That is, for all $z \in \Lambda$ and $u_{1}, u_{2} \in T_{z} \Lambda$,

$$
\sigma\left(u_{1}, u_{2}\right)=0
$$

Our goal for the remainder of this section is to understand the local structure of Lagrangian submanifolds.

Lemma 3.1.8. Let $\Lambda$ be a Lagrangian submanifold of $\mathbb{R}^{2 d}$. For each point $z \in \Lambda$, there exists an open neighborhood, $U \subset \Lambda$ and a smooth function $\varphi: U \rightarrow \mathbb{R}$ such that in $U$,

$$
\left.\omega\right|_{\Lambda}=d \varphi .
$$

Proof. Fix $z \in \Lambda$. Then choose an open neighborhood $U$ of $z$ and a diffeomorphism $\gamma: U \rightarrow$ $B(0,1) \subset \mathbb{R}^{d}$. Define $\rho=\gamma^{-1}$ and $\alpha:=\rho^{*}\left(\left.\omega\right|_{\Lambda}\right)$ defined on the open unit ball. Then we have

$$
d \alpha=d\left(\left.\rho^{*} \omega\right|_{\Lambda}\right)=\rho^{*} d\left(\left.\omega\right|_{\Lambda}\right)=0
$$

Hence, by Poincaré's Theorem [87, Theorem B.5] $\alpha=d \psi$ for some $\psi: B(0,1) \rightarrow \mathbb{R}$. Now, let $\varphi:=\gamma^{*} \psi$. Then

$$
d \varphi=d\left(\gamma^{*} \psi\right)=\gamma^{*} d \psi=\gamma^{*} \alpha=\left.\omega\right|_{\Lambda} .
$$

Next, we show that for any $\left(x_{0}, \xi_{0}\right) \in \Lambda$, there is always a Lagrangian plane transverse to $T_{\left(x_{0}, \xi_{0}\right)} \Lambda$. A Lagrangian plane is a hyperplane that is also a submanifold.

Lemma 3.1.9. Suppose that $\Lambda_{0}$ is a Lagrangian plane. Then there exists a $d \times d$ matrix $H$ such that after changing coordinates $(x, \xi) \mapsto\left(H x,\left(H^{-1}\right)^{t} \xi\right), \Lambda_{0}$ takes the form

$$
\begin{equation*}
\Lambda_{0}=\left\{\left(0, x^{\prime \prime}, \xi^{\prime}, B \xi^{\prime}\right)\right\} \tag{3.1.1}
\end{equation*}
$$

where $B$ is a symmetric matrix and $\left(x^{\prime}, x^{\prime \prime}\right) \in \mathbb{R}^{k} \times \mathbb{R}^{d-k}$ is a splitting of coordinates for some $0 \leq k \leq d$. In particular, $\left\{\left(x^{\prime \prime}, \xi^{\prime}\right)=\right.$ constant $\}$ is transverse to $\Lambda_{0}$ in these coordinates.

Proof. Let $L \subset \mathbb{R}_{x}^{d}$ be the projection of $\Lambda_{0}$. Then, after a change of coordinates in $x$, with the corresponding change in $\xi, L=\left\{x^{\prime}=0\right\}$. Let $v_{0}=\left(x_{0}^{\prime}, 0,0, \xi_{0}^{\prime \prime}\right) \in \Lambda_{0}$ then by the form of $L$, we have that $x_{0}^{\prime}=0$. Hence,

$$
\sigma\left(v_{0},(x, \xi)\right)=\left\langle\xi_{0}^{\prime \prime}, x^{\prime \prime}\right\rangle=0, \quad(x, \xi) \in \Lambda_{0}
$$

and by the form of $L, x^{\prime \prime} \in \mathbb{R}^{d-k}$ is arbitrary. This implies $\xi_{0}^{\prime \prime}=0$. Therefore, the map $\Lambda_{0} \ni(x, \xi) \mapsto\left(x^{\prime \prime}, \xi^{\prime}\right) \in \mathbb{R}^{d}$ is bijective and

$$
\Lambda_{0}=\left\{\left(0, x^{\prime \prime}, \xi^{\prime}, B x^{\prime \prime}+C \xi^{\prime}\right\}\right.
$$

To see that $C=0$, observe that

$$
0=\sigma\left(\left(0,\left(\xi_{0}^{\prime}, C \xi_{0}^{\prime}\right)\right),(x, \xi)\right)=\left\langle C \xi_{0}^{\prime}, x^{\prime \prime}\right\rangle, \quad(x, \xi) \in \Lambda_{0}
$$

Finally, $B$ is symmetric since for all $x^{\prime \prime}, y^{\prime \prime} \in \mathbb{R}^{d-k}$,

$$
0=\sigma\left(\left(0, x^{\prime \prime}, 0, B x^{\prime \prime}\right),\left(0, y^{\prime \prime}, 0, B y^{\prime \prime}\right)\right)=\left\langle B x^{\prime \prime}, y^{\prime \prime}\right\rangle-\left\langle B y^{\prime \prime}, x^{\prime \prime}\right\rangle
$$

Remark: Let $\Lambda_{A}:=\{(x, A x)\}$. with $A$ a real symmetric matrix. Then $\Lambda_{A}$ is Lagrangian. Moreover, when $\Lambda_{0}$ is written as in (3.1.1), then we see that if $A x=\left(0, D x^{\prime \prime}\right)$, then $\Lambda_{A}$ is transverse to $\Lambda_{0}$ if $\operatorname{det}(D-B) \neq 0$. Hence there is always at least one $A$ such that $\Lambda_{A}$ is transverse to $\Lambda_{0}$.

We next show that a Lagrangian submanifold can locally be written in terms of a generating function.

Lemma 3.1.10. Let $\left(x^{\prime}, x^{\prime \prime}\right) \in \mathbb{R}^{k} \times \mathbb{R}^{d-k}$ and $\left(\xi^{\prime}, \xi^{\prime \prime}\right) \in \mathbb{R}^{k} \times \mathbb{R}^{d-k}$ be a splitting of coordinates. Suppose that the Lagrangian plane $\left\{\left(x^{\prime \prime}, \xi^{\prime}\right)=\left(x_{0}^{\prime \prime}, \xi_{0}^{\prime}\right)\right\}$ is transversal to $\Lambda$. Then there exists a neighborhood, $U \subset \mathbb{R}^{2 d}$, of $\left(x_{0}, \xi_{0}\right)$ and a smooth function $\varphi\left(x^{\prime \prime}, \xi^{\prime}\right)$ such that

$$
\begin{equation*}
\Lambda \cap U=\left\{\left(\partial_{\xi^{\prime}} \varphi, x^{\prime \prime}, \xi^{\prime},-\partial_{x^{\prime \prime}} \varphi\right): x^{\prime \prime} \in \mathbb{R}^{d-k}, \xi^{\prime} \in \mathbb{R}^{k}\right\} \cap U \tag{3.1.2}
\end{equation*}
$$

Definition 3.1.11. We say $\varphi$ a generating function for $\Lambda$ near $\left(x_{0}, \xi_{0}\right)$ if $\Lambda \cap U$ has the form (3.1.2).

Proof. Define $\pi: \mathbb{R}^{2 d} \rightarrow \mathbb{R}^{k} \times \mathbb{R}^{d-k}$ by $\pi\left(x^{\prime}, x^{\prime \prime}, \xi^{\prime}, \xi^{\prime \prime}\right)=\left(x^{\prime \prime}, \xi^{\prime}\right)$. Then $d\left(\left.\pi\right|_{\Lambda}\right): T_{\left(x_{0}, \xi_{0}\right)} \Lambda \rightarrow$ $T_{\left(x_{0}^{\prime \prime}, \xi_{0}^{\prime}\right)}\left(\mathbb{R}^{d-k} \times \mathbb{R}^{k}\right)$ is bijective. To see this, note that

$$
\operatorname{ker}\left(\left.d \pi\right|_{\left.T_{\left(x_{0}, \xi_{0}\right)} \mathbb{R}^{2 d}\right)}=T_{\left(x_{0}, \xi_{0}\right)}\left\{\left(x^{\prime \prime}, \xi^{\prime}\right)=\left(x_{0}^{\prime \prime}, \xi_{0}^{\prime}\right)\right\}\right.
$$

and by transversality

$$
T_{\left(x_{0}, \xi_{0}\right)} \mathbb{R}^{2 d}=T_{\left(x_{0}, \xi_{0}\right)} \Lambda+T_{\left(x_{0}, \xi_{0}\right)}\left\{\left(x^{\prime \prime}, \xi^{\prime}\right)=\left(x_{0}^{\prime \prime}, \xi_{0}^{\prime}\right)\right\}
$$

Hence, by the implicit function theorem $\left.\pi\right|_{\Lambda}$ is invertible near $\left(x_{0}, \xi_{0}\right)$ and there exists a neighborhood $U$ such that $\left(x^{\prime \prime}, \xi^{\prime}\right)$ can be used as coordinates pm $U$.

This implies that there exists a neighborhood $U$ of $\left(x_{0}, \xi_{0}\right)$ and smooth maps

$$
f: \mathbb{R}^{d-k} \times \mathbb{R}^{k} \rightarrow \mathbb{R}^{k}, \quad g: \mathbb{R}^{d-k} \times \mathbb{R}^{k} \rightarrow \mathbb{R}^{d-k}
$$

such that

$$
\Lambda \cap U=\left\{\left(f\left(x^{\prime \prime}, \xi^{\prime}\right), x^{\prime \prime}, \xi^{\prime}, g\left(x^{\prime \prime}, \xi^{\prime}\right)\right): x^{\prime \prime} \in \mathbb{R}^{d-k}, \xi^{\prime} \in \mathbb{R}^{k}\right\} \cap U
$$

Using Lemma 3.1.8 and the fact that $\omega=\xi d x$, we have that there exists $\psi=\psi\left(x^{\prime \prime}, \xi^{\prime}\right)$

$$
\begin{aligned}
\left\langle\partial_{x^{\prime \prime}} \psi, d x^{\prime \prime}\right\rangle+\left\langle\partial_{\xi^{\prime}} \psi, d \xi^{\prime}\right\rangle=\left.\omega\right|_{\Lambda} & =\left\langle g, d x^{\prime \prime}\right\rangle+\left\langle\xi^{\prime}, \partial_{x^{\prime \prime}} f d x^{\prime \prime}+\partial_{\xi^{\prime}} f d \xi^{\prime}\right\rangle \\
& =\left\langle g+\left(\partial_{x^{\prime \prime}} f\right)^{t} \xi^{\prime}, d x^{\prime \prime}\right\rangle+\left\langle\left(\partial_{\xi^{\prime}} f\right)^{t} \xi^{\prime}, d \xi^{\prime}\right\rangle \\
& =\left\langle g+\partial_{x^{\prime \prime}}\left\langle f, \xi^{\prime}\right\rangle, d x^{\prime \prime}\right\rangle+\left\langle\partial_{\xi^{\prime}}\left\langle f, \xi^{\prime}\right\rangle-f, d \xi^{\prime}\right\rangle
\end{aligned}
$$

So, putting

$$
\varphi\left(x^{\prime \prime}, \xi^{\prime}\right)=\left\langle f\left(x^{\prime \prime}, \xi^{\prime}\right), \xi^{\prime}\right\rangle-\psi\left(x^{\prime \prime}, \xi^{\prime}\right)
$$

gives the result.
Finally, we show that except at the intersection of a Lagrangian submanifold, $\Lambda$ with the 0 section, we can change coordinates to make $\xi=$ constant transversal to $\Lambda$.

Lemma 3.1.12. Suppose that $p \in \Lambda \subset \mathbb{R}^{d} \times \mathbb{R}^{d}$ does not lie in the zero section and that $\Lambda$ is a Lagrangian submanifold. Then there exist coordinates on $\mathbb{R}^{d}$ such that $\{\xi=$ constant $\}$ is transversal to $\Lambda$ at $p$.

Proof. Since $p$ does not lie in the zero section, we can choose coordinates $\left(x_{1}, \ldots x_{d}\right)$ at $\pi_{x}(p)$ so that $p=\left(x_{0}, \xi_{0}\right)=(0,(0, \ldots, 0,1))$. Then, $\Lambda_{0}=T_{\left(x_{0}, \xi_{0}\right)}(\Lambda)$ is a Lagrangian subspace of $\mathbb{R}^{d} \times \mathbb{R}^{d}$. Thus, by Lemma 3.1.9 and the following remark, there exists a symmetric matrix $A$ such that $\Lambda_{A}:=\{(x, A x)\}$ is transversal to $\Lambda_{0}$ at $\left(x_{0}, \xi_{0}\right)$. Now, $\Lambda_{A}=T_{x_{0}, \xi_{0}} \Lambda_{\varphi}$ where $\Lambda_{\varphi}=\left\{\left(x, \partial_{x} \varphi\right)\right\}$ and $\varphi(x)=x_{d}+\frac{1}{2}\langle A x, x\rangle$. Let $y_{i}=x_{i} 1 \leq i \leq d-1$ and $y_{d}=\varphi(x)$ be new coordinates centered at $x_{0}$. Then if the canonical coordinates associated to $y$ are $\left(y_{1}, \ldots y_{d}, \eta_{1}, \ldots \eta_{d}\right),\{\eta=$ constant $\}=\Lambda_{\varphi}$ is transverse to $\Lambda$ at $p$.

When we seek to understand Lagrangian distributions (Section 4.4), what we have done so far will allow us to handle the parts of the Lagrangian in compact subsets of the fiber. However, we also seek to understand distributions associated to certain unbounded Lagrangians. For this, we define the radial compactification of $T^{*} \mathbb{R}^{d}, \bar{T}^{*} \mathbb{R}^{d}:=T^{*} \mathbb{R}^{d} \sqcup S^{*} \mathbb{R}^{d}$ where

$$
S^{*} \mathbb{R}^{d}:=\left(T^{*} \mathbb{R}^{d} \backslash\{0\}\right) / \mathbb{R}_{+}
$$

and the $\mathbb{R}_{+}$action is given by $(t,(x, \xi)) \mapsto(x, t \xi)$. We identify $\bar{T}^{*} \mathbb{R}^{d}$ with $\mathbb{R}^{d} \times \overline{\mathbb{R}}^{d}$ where $\overline{\mathbb{R}}^{d}=$ $\mathbb{R}^{d} \sqcup S^{d-1}$. Here, a neighborhood of a point $\left(x_{0}, \xi_{0}\right) \in \mathbb{R}^{d} \times S^{d-1}$ is given by $V \times(U \cap|\xi| \geq K)$ where $V$ is a neighborhood of $x_{0}$ and $U$ is a conic neighborhood of $\xi_{0}$.

It is not hard to see that the symbol class which after multiplication by a suitable power of $|\xi|$ extends smoothly to $\bar{T}^{*} \mathbb{R}^{d}$ is given by

Definition 3.1.13. We say that $a(x, \theta ; h) \in S_{\delta}^{m}\left(\mathbb{R}^{d} \times \mathbb{R}^{N}\right)$ is a classical Kohn-Nirenberg symbol of order $m$, (denoted by either $S_{\delta, c l}^{m}\left(\mathbb{R}^{d} \times \mathbb{R}^{N}\right)$ or $\left.S_{\delta}^{m}\left(\mathbb{R}^{d} \times \overline{\mathbb{R}}^{N}\right)\right)$ if there exist $M>0$ and $a_{j}(x, \theta ; h) \in S_{\delta}^{j}\left(\mathbb{R}^{d} \times \mathbb{R}^{N}\right)$ homogeneous of degree $j$ for $|\theta|>M$ such that in a neighborhood of $\mathbb{R}^{d} \times S^{N-1}$, for all $\alpha, \beta, M$,

$$
\left|\partial_{x}^{\alpha} \partial_{\theta}^{\beta}\left(a(x, \theta ; h)-\sum_{j=-M}^{m} a_{j}(x, \theta ; h)\right)\right| \leq h^{-\delta(|\alpha|+|\beta|)}\langle\theta\rangle^{-M-1-|\beta|} .
$$

Remark: To see that there is indeed a difference between $S_{0}^{m}$ and $S_{0, c}^{m}$, consider the symbol $e^{i \chi(|\xi|) \log |\xi|}$ where $\chi \equiv 1$ for $|\xi| \geq 1$ and $\operatorname{supp} \chi \subset\{|\xi|>1 / 2\}$.

When defining Lagrangian distributions, we will want a more general notion than that of a generating function.

Definition 3.1.14. Let $\varphi(x, \theta) \in C^{\infty}\left(\mathbb{R}^{d} \times \mathbb{R}^{L} \backslash\{0\}\right)$ be a smooth real-valued, homogeneous degree 1 function on some open conic subset $U_{\varphi}$ of $\mathbb{R}^{d} \times \mathbb{R}^{L} \backslash\{0\}$. We call $x$ the base variable and $\theta$ the oscillatory variable. Similar to [41, Section 21.2], we say that $\varphi$ is a homogeneous clean phase function with excess $e$ if $|d \varphi|>0$ and

$$
C_{\varphi}:=\left\{(x, \theta) \mid \partial_{\theta} \varphi=0\right\} \subset U_{\varphi}
$$

is a $C^{\infty}$ manifold with tangent plane given by $d \varphi_{\theta}^{\prime}=0$. Then the number of linearly independent differentials $d\left(\partial_{\theta_{1}} \varphi\right), \ldots, d\left(\partial_{\theta_{L}} \varphi\right)$ on $C_{\varphi}$ is equal to $L-e$ where $e=\operatorname{dim} C_{\varphi}-d$. If $e=0$, We call $\varphi$ a non-degenerate phase function.

Let $\varphi \in S^{1}\left(\mathbb{R}^{d} \times \overline{\mathbb{R}}^{L}\right)$ be a smooth real-valued function on some open subset $U_{\varphi}$ of $\mathbb{R}^{d} \times \overline{\mathbb{R}}^{L}$, (possibly intersecting the boundary). We say that $\varphi$ is a clean phase function with excess $e$ if $C_{\varphi}$ is a smooth manifold with tangent plane given by $d \varphi_{\theta}^{\prime}=0$, the number of linearly independent differentials $d\left(\partial_{\theta_{1}} \varphi\right), \ldots, d\left(\partial_{\theta_{L}} \varphi\right)$ on $C_{\varphi}$ is equal to $L-e$ where $e=\operatorname{dim} C_{\varphi}-d$, and if

$$
\varphi \sim \sum_{j=-\infty}^{1} \varphi_{j}
$$

is the asymptotic expansion given in Definition 3.1.13, then $\varphi_{1}$ is a clean homogeneous phase function with excess $e$. If $e=0$, We call $\varphi$ a non-degenerate homogeneous phase function.

Lemma 3.1.15. Let $\varphi$ be a clean phase function with excess e and $j: C_{\varphi} \rightarrow \mathbb{R}^{2 d} j:(x, \theta) \mapsto$ $\left(x, \partial_{x} \varphi\right)$. Define

$$
\begin{equation*}
\Lambda_{\varphi}:=\left\{\left(x, \partial_{x} \varphi(x, \theta)\right) \mid(x, \theta) \in C_{\varphi}\right\} \tag{3.1.3}
\end{equation*}
$$

Then $j$ is a fibration of $C_{\varphi}$ with fibers of dimension $e$ and $\Lambda_{\varphi}$ is a Lagrangian submanifold.
Proof. Consider dj:TC $\rightarrow T \mathbb{R}^{2 d}$. Let $\left(\delta_{x}, \delta_{\theta}\right) \in T_{(x, \theta)} C_{\varphi}$. Then,

$$
\varphi_{x \theta}^{\prime \prime} \delta_{x}+\varphi_{\theta \theta}^{\prime \prime}\left(\delta_{\theta}\right)=0
$$

Also,

$$
d j\left(\delta_{x}, \delta_{\theta}\right)=\left(\delta_{x},\left(\varphi_{x x}^{\prime \prime} \delta_{x}+\varphi_{\theta x}^{\prime \prime} \delta_{\theta}\right)\right)
$$

So, if $d j\left(\delta_{x}, \delta_{\theta}\right)=0$, then $\delta_{x}=0$ and hence

$$
\delta_{\theta} \in \operatorname{ker}\binom{\varphi_{\theta x}^{\prime \prime}}{\varphi_{\theta \theta}^{\prime \prime}} .
$$

Hence, $\delta_{\theta}$ lies in an $e$-dimensional subspace. So, changing coordinates in $\theta$ if necessary, there exists a splitting of coordinates $\theta=\left(\theta^{\prime}, \theta^{\prime \prime}\right) \in \mathbb{R}^{L-e} \times \mathbb{R}^{e}$ such that the map

$$
\tilde{j}: C_{\varphi} \ni(x, \theta) \mapsto\left(x, \partial_{x} \varphi, \theta^{\prime \prime}\right) \in \mathbb{R}^{2 d}
$$

has injective differential. Hence, shrinking the neighborhood of a point $\left(x_{0}, \theta_{0}\right)$ if necessary, $\tilde{j}$ is a diffeomporphism from $C_{\varphi} \rightarrow \Lambda_{\varphi} \times \mathbb{R}^{e}$ as desired.

Now, using these new coordinates, we can define a new phase function $\varphi_{1}=\varphi\left(x, \theta^{\prime}, \theta_{0}^{\prime \prime}\right)$ such that $\Lambda_{\varphi}=\Lambda_{\varphi_{1}}$ near $\left(x_{0}, \theta_{0}\right)$. Then, $\varphi_{1}$ is a nondegenerate phase function and hence $j_{1}: C_{\varphi_{1}} \rightarrow \Lambda_{\varphi}$ is a diffeomorphism.

To see that $\Lambda_{\varphi}$ is Lagrangian, observe that, identifying $C_{\varphi_{1}}$ with $\Lambda_{\varphi}$, the canonical one form $\omega$ has

$$
\begin{aligned}
\left.\omega\right|_{C \varphi_{1}} & =\left.\sum_{i=1}^{d} \partial_{x_{j}} \varphi d x^{j}\right|_{C_{\varphi_{1}}}=\left.\left(\sum_{j} \partial_{x_{j}} \varphi d x^{j}+\sum \partial_{\theta_{j}^{\prime}} \varphi d \theta^{\prime j}\right)\right|_{C_{\varphi_{1}}} \\
& =\left.(d \varphi)\right|_{C_{\varphi_{1}}}=d\left(\left.\varphi_{1}\right|_{C_{\varphi_{1}}}\right)
\end{aligned}
$$

and hence $\left.d \omega\right|_{C \varphi_{1}}=0$.
Definition 3.1.16. If $\Lambda_{\varphi}$ is given by (3.1.3), then we say that the clean phase function $\varphi$ generates $\Lambda_{\varphi}$.

With the above definitions, it follows from Lemma 3.1.10 that the function $S\left(x, \xi^{\prime}\right):=$ $\left\langle x^{\prime}, \xi^{\prime}\right\rangle-\varphi\left(x^{\prime \prime}, \xi^{\prime}\right)$ is a non-degenerate phase function generating $\Lambda$ in $U$.

Finally, we define a class of Lagrangians to which we are able to associate distributions in Section 4.4.

Definition 3.1.17. We say that a Lagrangian submanifold $\Lambda \subset \mathbb{R}^{2 d}$ is admissible at $\left(x_{0}, \xi_{0}\right) \in \mathbb{R}^{d} \times S^{d-1}$ if there exists a neighborhood $U$ of $\left(x_{0}, \xi_{0}\right)$, a splitting of coordinates $\left(x^{\prime}, x^{\prime \prime}\right) \in \mathbb{R}^{k} \times \mathbb{R}^{d-k}$ and a function $H\left(x^{\prime \prime}, \xi^{\prime}\right) \in S^{1}\left(\mathbb{R}^{d-k} \times \overline{\mathbb{R}}^{k}\right)$ such that

$$
\Lambda \cap U=\left\{\left(-\partial_{\xi^{\prime}} H\left(x^{\prime \prime}, \xi^{\prime}\right), x^{\prime \prime}, \xi^{\prime},-\partial_{x^{\prime \prime}} H\left(x^{\prime \prime}, \xi^{\prime}\right)\right):\left(x^{\prime \prime}, \xi^{\prime}\right) \in W\right\} \cap U .
$$

We say that $\Lambda$ is admissible if it is admissible at $\left(x_{0}, \xi_{0}\right)$ for all $\left(x_{0}, \xi_{0}\right) \in \Lambda \cap\left(\mathbb{R}^{d} \times S^{d-1}\right)$.
Remark: Notice that any Lagrangian which is conic outside of a compact set is admissible.

## Canonical Relations

Suppose that $U_{1}, U_{2} \subset \mathbb{R}^{2 d}$ are open with symplectic forms $\sigma_{1}$ and $\sigma_{2}$ respectively. Let $\kappa: U_{2} \rightarrow U_{1}$ be a symplectomorphism with $\kappa(y, \eta)=(x, \xi)$. Then the graph

$$
\begin{equation*}
G=\{(x(y, \eta), \xi(y, \eta), y, \eta)\} \subset U_{1} \times U_{2} \tag{3.1.4}
\end{equation*}
$$

is a Lagrangian submanifold of $U_{1} \times U_{2}$ with respect to the 'twisted' symplectic form $\sigma_{12}:=$ $\pi_{1}^{*} \sigma_{1}-\pi_{2}^{*} \sigma_{2}$ where $\pi_{1}: U_{1} \times U_{2} \rightarrow U_{1}$ and $\pi_{2}: U_{1} \times U_{2} \rightarrow U_{1}$ are the two projections. We often simply write $\sigma_{i}$ for $\pi_{i}^{*} \sigma_{i}$ when no confusion will arise. In this case, it is clear that $\left.\pi_{i}\right|_{G}$ is a diffeomorphism.

Lemma 3.1.18. Suppose $U_{1}, U_{2} \subset \mathbb{R}^{2 d}$ and $\Lambda \subset U_{1} \times U_{2}$ is Lagrangian with respect to $\sigma_{1}-\sigma_{2}$ with the property that $\left.\pi_{i}\right|_{\Lambda}$ are diffeomorphisms near $(w, z)$. Then $\Lambda$ can be written in the form (3.1.4) near $(w, z)$.

Proof. The fact that $\left.\pi_{2}\right|_{\Lambda}$ is a diffeomorphism implies that $(y, \eta) \in U_{2}$ can be used as coordinates on $\Lambda$. Lemma 3.1.10 shows that there exists a splitting of coordinates $\left(x^{\prime},\left(x^{\prime \prime}, y\right)\right)$ or $\left(\left(x, y^{\prime}\right), y^{\prime \prime}\right) \in \mathbb{R}^{k} \times \mathbb{R}^{2 d-k}$ (without loss, we assume $\left.\left(x^{\prime},\left(x^{\prime \prime}, y\right)\right)\right)$ and a smooth function $H\left(x^{\prime \prime}, y, \xi^{\prime}\right) \in C^{\infty}\left(\mathbb{R}^{2 d}\right)$ such that (taking into acount the fact that $\Lambda$ is Lagrangian with respect to $\sigma_{1}-\sigma_{2}$ )

$$
\Lambda=\left\{\left(\partial_{\xi^{\prime}} H, x^{\prime \prime}, \xi^{\prime},-\partial_{x^{\prime \prime}} H, y, \partial_{y} H\right):\left(x^{\prime \prime}, y, \xi^{\prime}\right) \in \mathbb{R}^{2 d}\right\} \cap\left(U_{1} \times U_{2}\right)
$$

Then, since we know $(y, \eta)$ can be used as coordinates on $\Lambda$, the map

$$
\kappa: U_{2} \ni\left(y, \partial_{y} H\right) \mapsto\left(\partial_{\xi^{\prime}} H, x^{\prime \prime}, \xi^{\prime},-\partial_{x^{\prime \prime}} H\right) \in U_{1}
$$

is well defined. Similarly, $\left.\pi_{1}\right|_{\Lambda}$ a diffeomorphism shows that $(x, \xi)$ can be used as coordinates and hence that $\kappa$ is a diffeomorphism. To check that $\kappa$ is symplectic, we compute

$$
\begin{aligned}
d \eta \wedge d y & =\left(\partial_{y}^{2} H d y+\partial_{x^{\prime \prime} y}^{2} H d x^{\prime \prime}+\partial_{\xi^{\prime} y}^{2} H d \xi^{\prime}\right) \wedge d y \\
& =\left(\partial_{x^{\prime \prime} y}^{2} H d x^{\prime \prime}\right) \wedge d y+\left(\partial_{\xi^{\prime} y}^{2} H d \xi^{\prime}\right) \wedge d y
\end{aligned}
$$

and

$$
\begin{aligned}
d \xi \wedge d x= & d \xi^{\prime} \wedge\left(\partial_{\xi^{\prime}}^{2} H d \xi^{\prime}+\partial_{x^{\prime \prime}}^{2} d x^{\prime \prime}+\partial_{y, \xi^{\prime}}^{2} H d y^{\prime}\right) \\
& -\left(\partial_{x^{\prime \prime}}^{2} H d x^{\prime \prime}+\partial_{\xi^{\prime} x^{\prime \prime}}^{2} H d \xi^{\prime}+\partial_{y x^{\prime \prime}}^{2} H d y\right) \wedge d x^{\prime \prime} \\
= & \left(\left(\partial_{y \xi^{\prime}}^{2} H\right)^{t} d \xi^{\prime}\right) \wedge d y-d y \wedge\left(\left(\partial_{y x^{\prime \prime}}^{2} H\right)^{t} d x^{\prime \prime}\right)+d \xi^{\prime} \wedge\left(\partial_{x^{\prime \prime} \xi^{\prime}}^{2} d x^{\prime \prime}\right)-d \xi^{\prime} \wedge\left(\left(\partial_{\xi^{\prime} x^{\prime \prime}}^{2} H\right)^{t} d x^{\prime \prime}\right) \\
= & \left(\partial_{\xi^{\prime} y}^{2} H d \xi^{\prime}\right) \wedge d y-d y \wedge\left(\partial_{x^{\prime \prime} y}^{2} H d x^{\prime \prime}\right)=d \eta \wedge d y
\end{aligned}
$$

It is natural to consider more general Lagrangian submanifolds that do not have $\left.\pi_{i}\right|_{\Lambda}$ diffeomorphisms and indeed have $U_{1}$ and $U_{2}$ with different dimensions. To this end, we define:

Definition 3.1.19. Let $U_{1} \subset \mathbb{R}^{2 d_{1}}$ and $U_{2} \subset \mathbb{R}^{2 d_{2}}$ be open with symplectic forms $\sigma_{1}$ and $\sigma_{2}$ respectively. Then a Lagrangian submanifold $\Lambda \subset U_{1} \times U_{2}$ with respect to $\sigma_{1}-\sigma_{2}$ is called a canonical relation from $U_{2}$ to $U_{1}$. A canonical relation such that $\pi_{i}: \Lambda \rightarrow U_{i}$ are both diffeomporphsims is called a canonical graph. A canonical relation is called admissible if $\Lambda$ is admissible.

In order to generalize composition of functions, we can define a notion of composition of relations. First, suppose that $E \subset U_{2}$ and $R \subset U_{1} \times U_{2}$, then $R$ can be thought of as a relation mapping $E \rightarrow R(E)$ where

$$
R(E):=\left\{\left(\gamma_{1} \in U_{1}:\left(\gamma_{1}, \gamma_{2}\right) \in R \text { for some } \gamma_{2} \in E\right\}=\pi_{1}\left(R \cap \pi_{2}^{-1}(E)\right)\right.
$$

It is easy to see that if $R$ is the graph of $\kappa$, then $R(E)=\kappa(E)$. With this in mind, we see that if $R_{1} \subset U_{1} \times U_{2}$ and $R_{2} \subset U_{2} \times U_{3}$ are relations, then $R_{1} \circ R_{2}$ can be defined as

$$
R_{1} \circ R_{2}:=\left\{\left(\gamma_{1}, \gamma_{3}\right):\left(\gamma_{1}, \gamma_{2}\right) \in R_{1} \text { and }\left(\gamma_{2}, \gamma_{3}\right) \in R_{2} \text { for some } \gamma_{2} \in U_{2}\right\} .
$$

This can also be written

$$
\begin{equation*}
R_{1} \circ R_{2}=\pi\left(R_{1} \times R_{2} \cap\left(U_{1} \times \Delta\left(U_{2}\right) \times U_{3}\right)\right) \tag{3.1.5}
\end{equation*}
$$

where $\pi: U_{1} \times U_{2} \times U_{2} \times U_{3} \rightarrow U_{1} \times U_{3}$ is projection and $\Delta\left(U_{2}\right)$ is the diagonal in $U_{2} \times U_{2}$. We again note that if $R_{1}$ and $R_{2}$ are graphs, then $R_{1} \circ R_{2}$ is the graph of the composition.

Our next goal is to understand the composition of canonical relations. In general, such compositions will not be canonical relations or even smooth manifolds. However, under certain conditions on the intersection with the diagonal in (3.1.5), we can guarantee that the composition generates a new canonical relation. We first consider the linear case.

Lemma 3.1.20. Suppose that $S_{i}$ are symplectic vector spaces for $i=1, \ldots 3$. Suppose that $V_{1} \subset S_{1} \times S_{2}$ and $V_{2} \subset S_{2} \times S_{3}$ are Lagrangian subspaces. Then $V_{1} \circ V_{2}$ is a Lagrangian subspace.

We need the following simple lemma in the proof of Lemma 3.1.20
Lemma 3.1.21. Suppose $S$ is a symplectic vector space and $V \subset S$ is a linear subspace. Thn

$$
S^{\prime}=V+V^{\perp} / V \cap V^{\perp}
$$

is a symplectic vector space with dimension

$$
\operatorname{dim} S^{\prime}=\operatorname{dim} S-2 \operatorname{dim}\left(V \cap V^{\perp}\right)=2 \operatorname{dim}\left(V+V^{\perp}\right)-\operatorname{dim} S
$$

where $V^{\perp}$ is the symplectic complement of $V$.

Proof. Let $W=V+V^{\perp}$ then $W^{\perp}=V^{\perp} \cap\left(V^{\perp}\right)^{\perp}=V \cap V^{\perp} \subset W$. Now, $w \in W$ has $\sigma\left(w, w^{\prime}\right)=0$ for all $w^{\prime} \in W$ if and only if $w \in W^{\perp}$. So, $\sigma$ is nondegenerate when restricted to $S^{\prime}$ and hence $S^{\prime}$ is a symplectic vector space. To calculate the dimension, observe that

$$
\begin{aligned}
\operatorname{dim} S^{\prime} & =\operatorname{dim}\left(V+V^{\perp}\right)-\operatorname{dim}\left(V \cap V^{\perp}\right) \\
& =\operatorname{dim} V+\operatorname{dim} V^{\perp}-2 \operatorname{dim}\left(V \cap V^{\perp}\right)=\operatorname{dim} S-2 \operatorname{dim}\left(V \cap V^{\perp}\right)
\end{aligned}
$$

where we use the fact that $\operatorname{dim} V+\operatorname{dim} V^{\perp}=\operatorname{dim} S$ which follows from the nondegeneracy of $\sigma$.

Now, we prove Lemma 3.1.20
Proof. We endow $S_{1} \times S_{2} \times S_{2} \times S_{3}$ with the symplectic form $\sigma:=\sigma_{1}-\sigma_{21}+\sigma_{22}-\sigma_{3}$ where the $\sigma_{2 i}$ denote the lift of $\sigma_{2}$ from $S_{2}$ to act on the appropriate copy of $S_{2}$ in $S_{1} \times S_{2} \times S_{2} \times S_{3}$. That is,

$$
\sigma_{21}\left(v_{1}, v_{2}, v_{3}, v_{4}\right)=\sigma_{2}\left(v_{2}\right) \quad \sigma_{22}\left(v_{1}, v_{2}, v_{3}, v_{4}\right)=\sigma_{2}\left(v_{3}\right)
$$

Then

$$
\begin{aligned}
V_{1} \circ V_{2} & =\pi\left(\left(V_{1} \times V_{2}\right) \cap\left(S_{1} \times \Delta\left(S_{2}\right) \times S_{3}\right)\right) \\
& =\left(V_{1} \times V_{2}\right) \cap\left(S_{1} \times \Delta\left(S_{2}\right) \times S_{3}\right) /\left(V_{1} \times V_{2}\right) \cap\left(\{0\} \times \Delta\left(S_{2}\right) \times\{0\}\right) .
\end{aligned}
$$

Now, notice that $\Delta=\{0\} \times \Delta\left(S_{2}\right) \times\{0\}$ is isotropic and, moreover, $S_{1} \times \Delta\left(S_{2}\right) \times S_{3}$ is its symplectic complement. Therefore, by the previous Lemma $S^{\prime}=\Delta^{\perp} / \Delta$ is symplectic with symplectic form the restriction of $\sigma$ (i.e. $\sigma^{\prime}=\sigma_{1}-\sigma_{3}$ ). This implies that $V_{1} \circ V_{2}$ is isotropic since $V_{1} \times V_{2}$ is Lagrangian. To calculate the dimension, observe that

$$
\begin{aligned}
\operatorname{dim}\left(V_{1} \times V_{2}\right)+\operatorname{dim} \Delta^{\perp} & =\operatorname{dim}\left(V_{1} \times V_{2} \cap \Delta^{\perp}\right)+\operatorname{dim}\left(V_{1} \times V_{2}+\Delta^{\perp}\right) \\
& =\operatorname{dim}\left(\left(V_{1} \times V_{2}\right) \cap \Delta^{\perp}\right)+\operatorname{dim} S-\operatorname{dim}\left(\left(V_{1} \times V_{2}\right) \cap \Delta\right)
\end{aligned}
$$

where we have used the fact that $V_{1} \times V_{2}$ is Lagrangian to see that $\left(V_{1} \times V_{2}\right)^{\perp}=V_{1} \times V_{2}$. Thus,

$$
\begin{aligned}
\operatorname{dim}\left(V_{1} \circ V_{2}\right) & =\operatorname{dim}\left(\left(V_{1} \times V_{2}\right) \cap \Delta^{\perp}\right)-\operatorname{dim}\left(\left(V_{1} \times V_{2}\right) \cap \Delta\right) \\
& =\operatorname{dim}\left(V_{1} \times V_{2}\right)+\operatorname{dim} \Delta^{\perp}-\operatorname{dim} S \\
& =\frac{\operatorname{dim}(S)-2 \operatorname{dim}(\Delta)}{2}=\frac{\operatorname{dim}\left(S^{\prime}\right)}{2}
\end{aligned}
$$

by the previous Lemma. Hence, $V_{1} \circ V_{2}$ is Lagrangian.
Lemma 3.1.22. Suppose that $U_{i} \subset \mathbb{R}^{2 d_{i}}$ are open sets endowed with the symplectic structure. If $\Lambda_{1} \subset U_{1} \times U_{2}$ and $\Lambda_{2} \subset U_{2} \times U_{3}$ are canonical relations for the symplectic for $\sigma_{1}-\sigma_{2}$ and $\sigma_{2}-\sigma_{3}$ respectively, and $\Lambda_{1} \times \Lambda_{2}$ intersects $U_{1} \times \Delta\left(U_{2}\right) \times U_{3}$ cleanly with intersection $\Lambda$, then the projection $\pi$ from $\Lambda$ to $U_{1} \times U_{3}$ has rank $\left(\operatorname{dim} U_{1}+\operatorname{dim} U_{3}\right) / 2$ and the range of $\pi$ is locally a Lagrangian submanifold with respect to $\sigma_{1}-\sigma_{3}$.

Proof. The clean intersection property shows that at a point $\gamma=\left(\gamma_{1}, \gamma_{2}, \gamma_{2}, \gamma_{3}\right) \in \Lambda$,

$$
T_{\gamma} G=\lambda \cap T_{\gamma_{1}}\left(U_{1}\right) \times \Delta\left(T_{\gamma_{2}} U_{2}\right) \times T_{\gamma_{3}} U_{3} \quad \lambda=T_{\left(\gamma_{1}, \gamma_{2}\right)} \Lambda_{1} \times T_{\left(\gamma_{2}, \gamma_{3}\right)} \Lambda_{2}
$$

Then $\lambda$ is a Lagrangian subspace of

$$
T_{\gamma_{1}} U_{1} \oplus T_{\gamma_{2}} U_{2} \oplus T_{\gamma_{2}} U_{2} \oplus T_{\gamma_{3}} U_{3}
$$

with symplectic form $\sigma_{1}-\sigma_{21}+\sigma_{22}-\sigma_{3}$. Hence, Lemma 3.1 .20 applies to show that the range of the differential of $\pi$ is a Lagrangian subspace. Hence, $d \pi$ has constant rank and the range of $\pi$ is locally a Lagrangian submanifold.

Finally, we understand how phase functions associated to canonical relations can be combined. Throughout the discussion of relations we have been using the "twisted" symplectic form $\sigma_{1}-\sigma_{2}$. However, it is often more convenient to think of the standard symplectic structure on $U_{1} \times U_{2}$ given by $\sigma_{1}+\sigma_{2}$. If $\Lambda$ is a canonical relation, then

$$
\Lambda^{\prime}=\{(x, \xi, y,-\eta):(x, \xi, y, \eta) \in \Lambda\}
$$

is Lagrangian with respect to the standard symplectic form. If we take a clean phase function generating $\Lambda^{\prime}$, it generates $\Lambda$ by

$$
\begin{equation*}
\Lambda=\left\{\left(x, \varphi_{x}^{\prime}, y,-\varphi_{y}^{\prime}\right): \varphi_{\theta}^{\prime}=0\right\} \tag{3.1.6}
\end{equation*}
$$

When we say that a phase function defines a canonical relation $G$, we will refer to the formula (3.1.6).

Lemma 3.1.23. Let $V_{i} \subset \mathbb{R}^{d_{i}}$ be open and $\Lambda_{1}, \Lambda_{2}$ be canonical relations $\subset \bar{T}^{*} V_{1} \times \bar{T}^{*} V_{2}$ and $\bar{T}^{*} V_{2} \times \bar{T}^{*} V_{3}$ respectively parametrized locally by nondegenerate phase functions $\varphi(x, y, \eta)$, $\theta \in \overline{\mathbb{R}}^{N_{1}}$ and $\psi(y, z, \tau), \tau \in \overline{\mathbb{R}}^{N_{2}}$, defined in neighborhoods of $\left(x_{0}, y_{0}, \theta_{0}\right)$ and $\left(y_{0}, z_{0}, \tau_{0}\right)$ where $\varphi_{\theta}^{\prime}=\psi_{\tau}^{\prime}=0$ and $\varphi_{y}^{\prime}+\psi_{y}^{\prime}=0$. Suppose further that the composition $\Lambda_{1} \circ \Lambda_{2}$ is clean at the corresponding point, and

$$
\begin{equation*}
\left\{(z, \zeta):(x, 0, z, \zeta) \in \Lambda_{1}\right\} \cap\left\{(z, \zeta):(z, \zeta, y, 0) \in \Lambda_{2}\right\} \cap \mathbb{R}^{d} \times S^{d-1}=\emptyset \tag{3.1.7}
\end{equation*}
$$

then

$$
\Phi(x, z, y, \theta, \tau)=\varphi(x, y, \theta)+\psi(y, z, \tau)
$$

is a clean phase function defining the composition where $(y, \theta, \tau)$ are now the phase variables. The excess of $\Phi$ is equal to the excess in the clean intersection of $\Lambda_{1} \times \Lambda_{2}$ with $T^{*} V_{1} \times$ $\Delta\left(T^{*} V_{2}\right) \times T^{*} V_{3}$.

Notice that $y$ lies in a bounded open set, so asymptotic properties in $y$ are irrelevant. However, to match the definition of a phase function we use the parameters

$$
\left.\left(\left.y\langle | \theta\right|^{2}+|\tau|^{2}\right\rangle^{1 / 2}, \theta, \tau\right) \in \mathbb{R}^{d_{2}+N_{1}+N_{2}} .
$$

Then $\Phi \in S^{1}\left(\mathbb{R}^{d_{1}} \times \mathbb{R}^{d_{2}} \times \overline{\mathbb{R}}^{d_{2}+N_{1}+N_{2}}\right)$ follows from the corresponding facts for $\varphi$ and $\psi$.

Proof. The non-degeneracy of $\varphi$ and $\psi$ gives that

$$
C_{\varphi}=\left\{(x, y, \theta): \varphi_{\theta}^{\prime}=0\right\} \quad C_{\psi}=\left\{(y, z, \tau): \psi_{\tau}^{\prime}=0\right\}
$$

are locally manifolds with tangent planes defined by $d \varphi_{\theta}^{\prime}=0$ and $d \psi_{\tau}^{\prime}=0$. Then Lemma 3.1.15 shows that the maps

$$
\begin{align*}
& C_{\varphi} \ni(x, y, \theta) \mapsto\left(x, \varphi_{x}^{\prime}, y,-\varphi_{y}^{\prime}\right) \in \Lambda_{1}  \tag{3.1.8}\\
& C_{\psi} \ni(y, z, \tau) \mapsto\left(y, \psi_{y}^{\prime}, z,-\psi_{z}^{\prime}\right) \in \Lambda_{2} \tag{3.1.9}
\end{align*}
$$

are local diffeomorphisms. Hence we can use $\left(x, y^{\prime}, \theta, y^{\prime \prime}, z, \tau\right) \in C_{\varphi} \times C_{\psi}$ as coordinates on $\Lambda_{1} \times \Lambda_{2}$ and the tangent plane to $\Lambda_{1} \times \Lambda_{2}$ is given by

$$
\begin{equation*}
\left\{\left(d x, d \varphi_{x}^{\prime}, d y^{\prime},-d \varphi_{y^{\prime}}^{\prime}, d y^{\prime \prime}, d \psi_{y^{\prime \prime}}^{\prime}, d z, d \psi_{z}^{\prime}\right): d \varphi_{\theta}^{\prime}=d \psi_{\tau}^{\prime}=0\right\} \tag{3.1.10}
\end{equation*}
$$

Now, if $X$ and $Y$ intersect cleanly in a manifold $X \cap Y$ with excess $e$, then $\operatorname{codim}(X \cap$ $Y)+e=\operatorname{codim}(X)+\operatorname{codim}(Y)$ (see for example [41, Appendx C.3]). Thus the fact that $\Lambda_{1} \circ \Lambda_{2}$ is clean translates to the fact that

$$
\Lambda=\left\{\left(x, \xi, y^{\prime}, \eta^{\prime}, y^{\prime \prime}, \eta^{\prime \prime}, z, \zeta\right) \in \Lambda_{1} \times \Lambda_{2}: y^{\prime}=y^{\prime \prime} \eta^{\prime}=\eta^{\prime \prime}\right\}
$$

is a manifold of dimension $n_{1}+n_{3}-e$ where $e$ is the excess of the composition. Then using coordinates $\left(x, y^{\prime}, \theta, y^{\prime \prime}, z, \tau\right) \in C_{\varphi} \times C_{\psi}, \Lambda$ is defined by

$$
\begin{align*}
& \xi=\varphi_{x}^{\prime}\left(x, y^{\prime}, \theta\right), \quad y^{\prime}=y^{\prime \prime}, \eta^{\prime}=-\varphi_{y}^{\prime}\left(x, y^{\prime}, \theta\right), \quad \eta^{\prime}=\eta^{\prime \prime}  \tag{3.1.11}\\
& \eta^{\prime}=\psi_{y}^{\prime}\left(y^{\prime \prime}, z, \tau\right), \quad \zeta=-\psi_{z}^{\prime}\left(y^{\prime \prime}, z, \tau\right) . \tag{3.1.12}
\end{align*}
$$

Moreover, by the cleanness of composition together with (3.1.10) the tangent plane to $\Lambda$ is given by

$$
\begin{equation*}
\left\{\left(d x, d \varphi_{x}^{\prime}, d y^{\prime},-d \varphi_{y^{\prime}}, d y^{\prime}, d \psi_{y^{\prime \prime}}^{\prime}, d z, d \psi_{z}^{\prime}\right): d \varphi_{\theta}^{\prime}=d \psi_{\tau}^{\prime}=d \varphi_{y}^{\prime}+d \psi_{y^{\prime \prime}}^{\prime}=d y^{\prime}-d y^{\prime \prime}=0\right\} \tag{3.1.13}
\end{equation*}
$$

We just need to show that

$$
C_{\Phi}:=\left\{(x, y, z, \theta, \tau): \varphi_{\theta}^{\prime}=\psi_{\tau}^{\prime}=\varphi_{y}^{\prime}+\psi_{y}^{\prime}=0\right\}
$$

is a manifold with tangent plane defined by

$$
d \varphi_{\theta}^{\prime}=0 \quad d \psi_{\tau}^{\prime}=0 \quad d\left(\varphi_{y}^{\prime}+\psi_{y}^{\prime}\right)=0
$$

Then letting $\eta^{\prime}=-\varphi_{y}^{\prime}$ and $\eta^{\prime \prime}=\psi_{y}^{\prime}$ along with 3.1.11 3.1.12) 3.1.13) identifies $C_{\Phi}$ with $\Lambda$. Moreover, the excess is given by $\operatorname{dim} \Lambda-\operatorname{dim} V_{1}-\operatorname{dim} V_{3}=e$.

Let $\left.\omega=\left.y\langle | \theta\right|^{2}+|\tau|^{2}\right\rangle^{1 / 2}$ and

$$
\Phi \sim \sum_{j=-\infty}^{1} \Phi_{j}
$$

where $\Phi_{j}$ is homogeneous of degree $j$ in $(\omega, \theta, \tau)$. The final condition we need to check is that $\Phi_{1}$ is a homogeneous phase function. The fact that $C_{\Phi_{1}}$ is a smooth manifold with the appropriate tangent plane and $d\left(\partial_{\theta_{i}} \Phi_{1}\right)$ have the required properties follow from the previous arguments applied to $\varphi_{1}$ and $\psi_{1}$ where $\varphi_{1}$ and $\psi_{1}$ are respectively the homogeneous degree 1 parts of $\varphi$ and $\psi$.

Finally, we check $\left|d \Phi_{1}\right|>0$. Using the fact that $\varphi$ and $\psi$ are classical symbols, and $y$ lies in a bounded set, we can assume that $\left.\left.\langle | \theta\right|^{2}+|\tau|^{2}\right\rangle^{1 / 2}$ is large. Clearly, there is no difficulty if $\theta_{0} \notin S^{N_{1}-1}$ and $\tau_{0} \notin S^{N_{2}-1}$. Suppose without loss that $\theta_{0} \in S^{N_{1}-1}$. So,

$$
\varphi=\varphi_{1}+O_{S^{-1}}(1)
$$

We have

$$
\left.c|d \Phi| \geq\left|\partial_{x} \varphi\right|+\left|\partial_{\theta} \varphi\right|+\left.\left|\partial_{y} \varphi+\partial_{y} \psi\right|\langle | \theta\right|^{2}+|\tau|^{2}\right\rangle^{-1 / 2}+\left|\partial_{z} \psi\right|+\left|\partial_{\tau} \psi\right| .
$$

So, if at a point $(x, \theta), \partial_{x} \varphi_{1}=\partial_{\theta} \varphi_{1}=0$, then $\left|\partial_{z} \varphi_{1}\right|>c\langle\theta\rangle>0$. Now, if $|\tau| \ll|\theta|$, then clearly $|d \Phi| \geq C\langle\theta\rangle$, so we assume $|\tau| \geq C|\theta|$ and, reversing arguments, $c|\theta| \leq|\tau| \leq C|\theta|$. So, we also use

$$
\psi=\psi_{1}+O_{S^{-1}}(1)
$$

where $\psi_{i}$ is homogeneous degree $i$ in $\tau$. Then, by (3.1.7), if

$$
\partial_{x} \varphi_{i}=\partial_{z} \psi_{i}=\partial_{\theta} \varphi_{1}=\partial_{\tau} \psi_{1}=0
$$

then $\left.\left|\partial_{y} \psi+\partial_{y} \varphi\right| \geq\left. C\langle | \theta\right|^{2}+|\tau|^{2}\right\rangle^{1 / 2}$. Hence, $\left|\partial_{\omega} \Phi\right| \geq C$ which implies $\left|d \Phi_{1}\right|>c>0$.

### 3.2 The Billiard Ball Flow and Map

We need notation for the billiard ball flow and billiard ball map. Write

$$
\left.S^{*} \mathbb{R}^{d}\right|_{\partial \Omega}=\partial \Omega_{+} \sqcup \partial \Omega_{-} \sqcup \partial \Omega_{0}
$$

where $(x, \xi) \in \partial \Omega_{+}$if $\xi$ is pointing out of $\Omega,(x, \xi) \in \partial \Omega_{-}$if it points inward, and $(x, \xi) \in \partial \Omega_{0}$ if $(x, \xi) \in S^{*} \partial \Omega$. The points $(x, \xi) \in \partial \Omega_{0}$ are called glancing points. Let $B^{*} \partial \Omega$ be the unit coball bundle of $\partial \Omega$ and denote by $\pi_{ \pm}: \partial \Omega_{ \pm} \rightarrow B^{*} \partial \Omega$ and $\pi:\left.S^{*} \mathbb{R}^{d}\right|_{\partial \Omega} \rightarrow \overline{B^{*} \partial \Omega}$ the canonical projections onto $\overline{B^{*} \partial \Omega}$. Then the maps $\pi_{ \pm}$are invertible. Finally, write

$$
t_{0}(x, \xi)=\inf \left\{t>0:\left.\exp _{t}(x, \xi) \in T^{*} \mathbb{R}^{d}\right|_{\partial \Omega}\right\}
$$

where $\exp _{t}(x, \xi)$ denotes the lift of the geodesic flow to the cotangent bundle. That is, $t_{0}$ is the first positive time at which the geodesic starting at $(x, \xi)$ intersects $\partial \Omega$.

We define the broken geodesic flow as in [24, Appendix A]. Without loss of generality, we assume $t_{0}>0$. Fix $(x, \xi) \in S^{*} \mathbb{R}^{d}$ and denote $t_{0}=t_{0}(x, \xi)$. If $\exp _{t_{0}}(x, \xi) \in \partial \Omega_{0}$, then the


Figure 3.1: The figure shows how the billiard ball map is constructed. Let $q=(x, \xi) \in B^{*} \partial \Omega$. The solid black arrow on the left denotes the covector $\xi \in B_{x}^{*} \partial \Omega$ and that on the right $\xi(\beta(q)) \in B_{\pi_{x}(\beta(q))}^{*} \partial \Omega$. The center of the left circle is $x$ and that of the right is $\pi_{x}(\beta(q))$.
billiard flow cannot be continued past $t_{0}$. Otherwise there are two cases: $\exp _{t_{0}}(x, \xi) \in \partial \Omega_{+}$ or $\exp _{t_{0}}(x, \xi) \in \partial \Omega_{-}$. We let

$$
\left(x_{0}, \xi_{0}\right)= \begin{cases}\pi_{-}^{-1}\left(\pi_{+}\left(\exp _{t_{0}}(x, \xi)\right)\right) \in \partial \Omega_{-}, & \text {if } \exp _{t_{0}}(x, \xi) \in \partial \Omega_{+} \\ \pi_{+}^{-1}\left(\pi_{-}\left(\exp _{t_{0}}(x, \xi)\right)\right) \in \partial \Omega_{+}, & \text {if } \exp _{t_{0}}(x, \xi) \in \partial \Omega_{-}\end{cases}
$$

We then define $\varphi_{t}(x, \xi)$, the broken geodesic flow, inductively by putting

$$
\varphi_{t}(x, \xi)= \begin{cases}\exp _{t}(x, \xi) & 0 \leq t<t_{0} \\ \varphi_{t-t_{0}}\left(x_{0}, \xi_{0}\right) & t \geq t_{0}\end{cases}
$$

We introduce notation from [63] for the billiard flow. Let $K$ be the set of ternary fractions of the form $0 . k_{1} k_{2}, \ldots$, where $k_{j}=0$ or 1 and $S$ denote the left shift operator

$$
S\left(0 . k_{1} k_{2} \ldots\right)=0 . k_{2} k_{3} \ldots
$$

For $k \in K$, we define the billiard flow of type $k, G_{k}^{t}: S^{*} \mathbb{R}^{d} \rightarrow S^{*} \mathbb{R}^{d}$ as follows. For $0 \leq t \leq t_{0}$,

$$
G_{k}^{t}(x, \xi)= \begin{cases}\varphi_{t}(x, \xi) & \text { if } k_{1}=0  \tag{3.2.1}\\ \exp _{t}(x, \xi) & \text { if } k_{1}=1\end{cases}
$$

Then, we define $G_{k}^{t}$ inductively for $t>t_{0}$ by

$$
\begin{equation*}
G_{k}^{t}(x, \xi)=G_{S k}^{t-t_{0}}\left(G_{k}^{t_{0}}(x, \xi)\right) \tag{3.2.2}
\end{equation*}
$$

We call $G_{k}^{t}$ the billiard flow of type $k$. By [63, Proposition 2.1], $G_{k}^{t}$ is measure preserving.

## Remarks:

- In [63], geodesics could be of multiple types when total internal reflection occurred. However, in our situation, the metrics on either side of the boundary match, so there is no total internal reflection and geodesics are uniquely identified by their starting points and $k \in K$.
- In general, there exist situations where $G_{k}^{t}$ intersects the boundary infinitely many times in finite time. However, since we work in convex domains, we need not consider this situation.

Now, for $k \in K$ and $T>0$, we define the set $\mathcal{O}_{T, k} \subset S^{*} \mathbb{R}^{d}$ to be the complement of the set of $(x, \xi)$ such that one can define the flow $G_{k}^{t}$ for $t \in[0, T]$. That is, $\mathcal{O}_{T, k}$ is the set for which the billiard flow of type $k$ is glancing in time $0 \leq t \leq T$. Last, define the set

$$
\begin{equation*}
\mathcal{O}_{T}=\bigcup_{k \in K} \mathcal{O}_{k, T} \tag{3.2.3}
\end{equation*}
$$

The billiard ball map reduces the dynamics of $G_{0}^{k}$ to the boundary. We define the billiard ball map as in 34. Let $\left(x, \xi^{\prime}\right) \in B^{*} \partial \Omega$ and $(x, \xi)=\pi_{-}^{-1}\left(x, \xi^{\prime}\right) \in \partial \Omega_{-}$be the unique inward pointing covector with $\pi(x, \xi)=\left(x, \xi^{\prime}\right)$. Then, the billiard ball map $\beta: B^{*} \partial \Omega \rightarrow \overline{B^{*} \partial \Omega}$ maps $\left(x, \xi^{\prime}\right)$ to the projection onto $T^{*} \partial \Omega$ of the first intersection of the billiard flow with the boundary. That is,

$$
\begin{equation*}
\beta:\left(x, \xi^{\prime}\right) \mapsto \pi\left(\exp _{t_{0}(x, \xi)}(x, \xi)\right) \tag{3.2.4}
\end{equation*}
$$

## Remarks:

- Just like the billiard flow, the billiard ball map is not defined for $\left(x, \xi^{\prime}\right) \in \pi\left(\partial \Omega_{0}\right)=$ $S^{*} \partial \Omega$. However, since we consider convex domains, $\beta: B^{*} \Omega \rightarrow B^{*} \Omega$ and $\beta^{n}$ is well defined on $B^{*} \partial \Omega$.
- Figure 3.1 shows a how the process by which the billiard ball map is defined.

The billiard ball map is symplectic. This follows from the fact that the Euclidean distance function $\left|x-x^{\prime}\right|$ is locally a generating function for $\beta$; that is, the graph of $\beta$ in a neighborhood of $\left(x_{0}, \xi_{0}, y_{0}, \eta_{0}\right)$ is given by

$$
\begin{equation*}
\left\{\left(x,-d_{x}|x-y|, y, d_{y}|x-y|\right):(x, y) \in \partial \Omega \times \partial \Omega\right\} \tag{3.2.5}
\end{equation*}
$$

We denote the graph of $\beta$ by $C_{b}$. For strictly convex $\Omega, C_{b}$ is given globally by (3.2.5).

## Dynamics in Strictly Convex Domains

Let $\partial \Omega$ be strictly convex near a point $x_{0}$ and let $\gamma:[0, \delta) \rightarrow \partial \Omega$ be a smooth geodesic parametrized by arc length with $\gamma(0)=x_{0}$. We are interested in how $\left|\xi^{\prime}\right|_{g}$ changes under the billiard ball map for $\left|\xi^{\prime}\right|_{g}$ close to 1 . Our interest in this region comes from a desire to understand how the reflection coefficients $R_{\text {delta }}$ and $R_{\delta^{\prime}}$ from (1.0.10) and (1.0.11) behave when a wave travels nearly tangent to a strictly convex boundary.

We examine how the normal component to $\partial \Omega$ changes under the billiard ball map. Notice that for $\left|\xi^{\prime}\right|_{g}$ sufficiently close to 1 , the strict convexity of $\partial \Omega$ at $x_{0}$ implies that there is a geodesic connecting $x_{0}$ to $\pi_{x}\left(\beta\left(x_{0}\right)\right)$ which lies inside a small neighborhood of $x_{0}$. (Here $\pi_{x}$ denotes projection to the base.) Hence, we consider

$$
\Delta_{\xi_{d}}=\frac{((\gamma(s)-\gamma(0)) \cdot \nu(0)-(\gamma(0)-\gamma(s)) \cdot \nu(s))}{|\gamma(s)-\gamma(0)|}=\frac{(\gamma(s)-\gamma(0)) \cdot(\nu(0)+\nu(s))}{|\gamma(s)-\gamma(0)|} .
$$

Here $|\cdot|$ is the euclidean norm in $\mathbb{R}^{d}$ and $\nu$ is the inward pointing unit normal.
First, note that

$$
\gamma^{\prime \prime}(s)=k(s) \nu(s), \quad \nu^{\prime}(s) \cdot \gamma^{\prime}(s)=-k(s), \quad \gamma^{\prime}(s) \cdot \nu(s)=0, \quad\left\|\gamma^{\prime}(s)\right\|=\|\nu(s)\|=1
$$

where $k(s)$ is the curvature of $\gamma$. Then, expanding in Taylor series gives

$$
\begin{align*}
\Delta_{\xi_{d}}\left[s+\mathcal{O}\left(s^{2}\right)\right]= & {\left[\gamma^{\prime}(0) s+\gamma^{\prime \prime}(0) \frac{s^{2}}{2}+\gamma^{(3)}(0) \frac{s^{3}}{6}+\mathcal{O}\left(s^{4}\right)\right] . }  \tag{3.2.6}\\
& {\left[2 \nu(0)+\nu^{\prime}(0) s+\nu^{\prime \prime}(0) \frac{s^{2}}{2}+\mathcal{O}\left(s^{3}\right)\right] } \\
\Delta_{\xi_{d}}[1+\mathcal{O}(s)]= & {\left[2 \gamma^{\prime}(0) \cdot \nu(0)+\left(\gamma^{\prime} \cdot \nu\right)^{\prime}(0) s+\left(2 \gamma^{(3)}(0) \cdot \nu(0)+3\left(\gamma^{\prime} \cdot \nu^{\prime}\right)^{\prime}(0)\right) \frac{s^{2}}{6}+\mathcal{O}\left(s^{3}\right)\right] } \\
\Delta_{\xi_{d}}= & {\left[2\left(k^{\prime}(0) \nu(0)-k(0) \nu^{\prime}(0)\right) \cdot \nu(0)-3 \kappa^{\prime}(0)\right] \frac{s^{2}}{6}+\mathcal{O}\left(s^{3}\right) } \\
\Delta_{\xi_{d}}= & \left(2 k^{\prime}(0)-3 k^{\prime}(0)\right) \frac{s^{2}}{6}+\mathcal{O}\left(s^{3}\right)=-k^{\prime}(0) \frac{s^{2}}{6}+\mathcal{O}\left(s^{3}\right) . \tag{3.2.7}
\end{align*}
$$

Now, we have

$$
\frac{\gamma(s)-\gamma(0)}{|\gamma(s)-\gamma(0)|} \cdot \nu(0)=\frac{k(0)}{2} s+\mathcal{O}\left(s^{2}\right) .
$$

Thus, if $\Omega$ is strictly convex, $k(0)>c>0$ and hence, if $\sqrt{1-\left|\xi^{\prime}\right|_{g}^{2}}=r, c s \leq r \leq C s$. Using (3.2.7), we have $\sqrt{1-\left|\xi^{\prime}\left(\beta\left(x_{0}, \xi^{\prime}\right)\right)\right|_{g}^{2}}=r+\mathcal{O}\left(r^{2}\right)$. Summarizing, we have

Lemma 3.2.1. Let $\Omega \subset \mathbb{R}^{d}$ be strictly convex. Then, letting $q \in B^{*} \partial \Omega$ and denote

$$
r:=\sqrt{1-\left|\xi^{\prime}(q)\right|_{g}^{2}}
$$

we have

$$
\left|\pi_{x}(q)-\pi_{x}(\beta)(q)\right|=r+O\left(r^{2}\right)
$$

By the calculations above, the set of $O\left(h^{\epsilon}\right)$ near glancing points is stable under the billiard ball map. This also follows from the equivalence of glancing hypersurfaces [48]. Moreover, we have the following lemma:

Lemma 3.2.2. Fix $\epsilon>0$. Suppose that $\Omega$ is strictly convex and $\left(x^{\prime}, \xi^{\prime}\right) \in T^{*} \partial \Omega$ with $\left|1-\left|\xi^{\prime}\right|_{g}\right|=O\left(h^{\epsilon}\right)$. Then, for $N=O\left(h^{-\epsilon / 2}\right)$,

$$
\left|1-\left|\xi^{\prime}\left(\beta^{N}\left(x^{\prime}, \xi^{\prime}\right)\right)\right|_{g}\right|=O\left(h^{\epsilon}\right)
$$

Proof. Suppose that $\left|1-\left|\xi^{\prime}\right|_{g}\right|=r$. Then, by (3.2.7),

$$
\left|1-\left|\xi^{\prime}\left(\beta\left(x^{\prime}, \xi^{\prime}\right)\right)\right|\right| \leq r+C_{1} r^{2} \quad \text { for } r \text { small enough }
$$

where $C_{1}>0$ is uniform in $B^{*} \partial \Omega$. Let $a_{n}=\left|1-\left|\xi^{\prime}\left(\beta^{n}\left(x^{\prime}, \xi^{\prime}\right)\right)\right|_{g}\right|$. Then, $a_{n} \leq a_{n-1}+C_{1} a_{n-1}^{2}$. Therefore, we need only examine the sequence

$$
x_{n}=x_{n-1}+C_{1} x_{n-1}^{2}, \quad x_{1}=C h^{\epsilon}
$$

First, observe that if $x_{j} \leq C j h^{\epsilon}$, then,

$$
x_{j+1}=x_{j}\left(1+C_{1} x_{j}\right) \leq C j h^{\epsilon}\left(1+C C_{1} j h^{\epsilon}\right)=C(j+1) h^{\epsilon}\left(\frac{j}{j+1}+\frac{C C_{1} j h^{\epsilon}}{j+1}\right)
$$

Therefore, for $j \leq C^{-1} C_{1}^{-1} h^{-\epsilon}, x_{j+1} \leq C(j+1) h^{\epsilon}$.
Now, we have
$\frac{x_{n}}{x_{1}}=\prod_{j=1}^{n-1}\left(1+C_{1} x_{j}\right) \leq \prod_{j=1}^{n-1}\left(1+C C_{1} j h^{\epsilon}\right) \leq\left(1+C C_{1}(n-1) h^{\epsilon}\right)^{n-1}=\left(1+\frac{(n-1)^{2} C C_{1} h^{\epsilon}}{n-1}\right)^{n-1}$.
As long as $(n-1)^{2}=O\left(h^{-\epsilon}\right)$ and $n-1 \leq C_{1}^{-1} C^{-1} h^{-\epsilon}$, we have $x_{n}=x_{1} O(1)$.

## Chapter 4

## Semiclassical Analysis

In this thesis, we view semiclassical analysis as a tool for studying links between classical and quantum systems. In classical mechanics, observables take the form of functions $a(x, \xi) \in$ $C^{\infty}\left(T^{*} \mathbb{R}^{d}\right)$ where the $x$ and $\xi$ variables represent position and momentum respectively. When these variables are quantized, we have the following relations

$$
x \mapsto x, \quad \xi_{i} \mapsto \frac{h}{i} \partial_{x_{i}}=: h D_{x_{i}} .
$$

where $h$ represents the Planck constant and $x_{i}$ represents multiplication by $x_{i}$. In this setting, the simplest observables are those given by partial differential operators $a_{\alpha}(x)\left(h D_{x}\right)^{\alpha}$. Writing this using the Fourier transform gives

$$
\left(a_{\alpha}(x)\left(h D_{x}\right)^{\alpha} f\right)(x)=\mathcal{F}_{h}^{-1}\left(a_{\alpha}(x) \xi^{\alpha}\left(\mathcal{F}_{h}(f)\right)(\xi)\right)
$$

where

$$
\mathcal{F}_{h} f:=\int e^{-\frac{i}{h}\langle y, \xi\rangle} f(y) d y, \quad \mathcal{F}_{h}^{-1} f:=\frac{1}{(2 \pi h)^{d}} \int e^{\frac{i}{h}\langle x, \xi\rangle} f(\xi) d \xi
$$

are the semiclassical Fourier transform and inverse Fourier transform. In order to quantize more general observables, we can use the same formula, writing for $a \in C^{\infty}\left(T^{*} \mathbb{R}^{d}\right)$ (with some additional conditions which we suppress for now),

$$
\begin{equation*}
\left(\mathrm{Op}_{\mathrm{h}}(a) f\right)(x)=\mathcal{F}_{h}^{-1}\left(a(x, \xi)\left(\mathcal{F}_{h}(f)\right)(\xi)\right) \tag{4.0.1}
\end{equation*}
$$

Such operators are called semiclassical pseudodifferential operators. The classical-quantum correspondence states that the classical properties of systems should be correspond to high energy behavior the of the quantum systems. In our setting, high energy corresponds to $h \rightarrow 0$ and as such we work to obtain error bounds in terms of functions of $h$.

In this chapter, we review the methods of semiclassical analysis which are needed throughout the rest of the thesis. The theories of pseudodifferential operators, wavefront sets, and the local theory of Fourier integral operators are standard and our treatment follows that in [87] and [23]. We make a small generalization from conic Lagrangians to a certain class of

Lagrangians satisfying an admissibility condition at the fiber radially compactified boundary of $T^{*} M$. Instead we impose a certain admissibility condition on the Lagrangians at infinity. We consider the special case of semiclassical analysis on a compact manifold $M$. See [23] for a treatment of the non-compact case. We introduce the notion shymbol from [29] which is a notion of sheaf-valued symbol that is sensitive to local changes in semiclassical order of a symbol. Finally, we consider a semiclassical version of the Melrose-Uhlmann [49] intersecting Lagrangian distributions.

### 4.1 Pseudodifferential Operators on $\mathbb{R}^{d}$

We first define semiclassical pseudodifferential operators on $\mathbb{R}^{d}$ following [87, Chapter 4].

## Symbols and Quantization

In order for (4.0.1) to be well defined (even for $f \in \mathcal{S}$, the Schwartz class), we must place some assumptions on $a \in C^{\infty}\left(T^{*} \mathbb{R}^{d}\right)$. In fact, we allow $a$ to have some controlled dependence on $h$.

Definition 4.1.1. Let $a(x, \xi ; h) \in C^{\infty}\left(\mathbb{R}^{N} \times \mathbb{R}^{M}\right)$ depend smoothly on $h$. Define the symbol class $S_{\delta}^{m}\left(\mathbb{R}^{N} \times \mathbb{R}^{M}\right)$ for $m \in \mathbb{R}$ and $\delta \in[0,1 / 2]$ by

$$
\begin{equation*}
S_{\delta}^{m}\left(\mathbb{R}^{N} \times \mathbb{R}^{M}\right):=\left\{a(x, \xi ; h):\left|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} a(x, \xi ; h)\right| \leq C_{\alpha, \beta} h^{-\delta(|\alpha|+|\beta|)}\langle\xi\rangle^{m-|\beta|}\right\} \tag{4.1.1}
\end{equation*}
$$

We denote $S_{\delta}^{\infty}:=\cup_{m} S_{\delta}^{m}, S_{\delta}^{-\infty}:=\cap_{m} S_{\delta}^{m}$ and when one of the parameters $\delta$ or $m$ is 0 , we suppress it in the notation. When we write $S_{\delta}^{m}\left(T^{*} \mathbb{R}^{d}\right)$, we identify $T^{*} \mathbb{R}^{d}$ with $\mathbb{R}^{d} \times \mathbb{R}^{d}$. We define two other symbol classes. We write $a \in S_{\delta}^{\text {comp }}\left(\mathbb{R}^{N} \times \mathbb{R}^{M}\right)$ if $a$ satisfies 4.1.1) and is compactly supported in some $h$-independent set. For an open set $U \subset \mathbb{R}^{N}$, when we write $S_{\delta, \text { loc }}^{m}\left(U \times \mathbb{R}^{M}\right)$, we mean that the estimates in 4.1.1) hold uniformly on compact subsets of $U$.

Remark: This notion of a symbol is invariant under changes of variables and thus will be useful in defining pseudodifferential operators on manifolds. (See Lemma 4.2.9)

With this definition of symbol, we can formally define quantization. Although 4.0.1) gives one notion of such quantization, it is easy to see that it is not the only one. In particular, on the classical level $x \xi=\xi x$, but clearly $h D_{x} x \neq x h D_{x}$, so we have some choice in quantization procedure.

Definition 4.1.2. For $a \in S_{\delta}^{m}\left(T^{*} \mathbb{R}^{d}\right)$ and $f \in \mathcal{S}$, we define the semiclassical $t$-quantization by

$$
\mathrm{Op}_{\mathrm{h}, \mathrm{t}}(a) f=\frac{1}{(2 \pi h)^{d}} \iint e^{\frac{i}{h}\langle x-y, \xi\rangle} a(t x+(1-t) y, \xi) f(y) d y d \xi
$$

The integral is defined in the sense of an oscillatory integral (see Lemma 4.1.3)

Furthermore, we define the class of semiclassical pseudodifferential operators of order $m$ and class $\delta, \Psi_{\delta}^{m}\left(\mathbb{R}^{d}\right)$ by

$$
\Psi_{\delta}^{m}\left(\mathbb{R}^{d}\right):=\left\{A=\mathrm{Op}_{\mathrm{h}, \mathrm{t}}(a)+B: a \in S_{\delta}^{m}\left(T^{*} \mathbb{R}^{d}\right), 0 \leq t \leq 1, B=O_{\mathcal{D}^{\prime} \rightarrow H^{\infty}}\left(h^{\infty}\right)\right\}
$$

Two particularly convenient quantizations are $\mathrm{Op}_{\mathrm{h}, 1 / 2}$, called the Weyl quantization, where for real $a, \mathrm{Op}_{\mathrm{h}, 1 / 2}(a)$ is self-adjoint, and $\mathrm{Op}_{\mathrm{h}, 1}$, called the standard quantization, where formula 4.0.1 holds.

Lemma 4.1.3. For $a \in S_{\delta}^{m}\left(\mathbb{R}^{2 d} \times \mathbb{R}^{d}\right)$, the map

$$
\begin{equation*}
A: f \mapsto \frac{1}{(2 \pi h)^{d}} \iint e^{\frac{i}{h}\langle x-y, \xi\rangle} a(x, y, \xi ; h) f(y) d y d \xi \tag{4.1.2}
\end{equation*}
$$

is bounded from $\mathcal{S} \rightarrow \mathcal{S}$ in the sense that each seminorm of $A f$ can be controlled by a finite number of seminorms of $f$.

Proof. Let

$$
L=\frac{1-\left\langle h D_{y}, \xi\right\rangle+\left\langle h D_{\xi}, x-y\right\rangle}{1+|\xi|^{2}+|x-y|^{2}}
$$

Then $L\left(e^{\frac{i}{h}\langle x-y, \xi\rangle}\right)=e^{\frac{i}{h}\langle x-y, \xi\rangle}$ and

$$
L^{t}=-L+\frac{4 h}{i} \frac{\langle x-y, \xi\rangle}{\left(1+|\xi|^{2}+|x-y|^{2}\right)^{2}} .
$$

Hence, $L^{t}(a(x, y, \xi ; h) f(y)) \in S_{\delta}^{m-1}\left(\mathbb{R}^{2 d} \times \mathbb{R}^{d}\right)$ with seminorms bounded by those with a single derivative on $f$. So, integrating by parts finitely many times losing a finite number of derivatives on $f$, we may assume $a \in S_{\delta}^{-N}\left(\mathbb{R}^{2 d} \times \mathbb{R}^{d}\right)$ for any large $N$. Then, letting $K$ denote the kernel of $A$,

$$
K(x, y):=\mathcal{F}_{h}^{-1}(a(x, y, \cdot ; h))(x-y)
$$

Thus, fixing $|\alpha|$ and $|\beta|$ and choosing $N$ large enough, $\sup _{x, y}\left|(x-y)^{\alpha} \partial_{x}^{\beta} K(x, y)\right|<\infty$. Now, $\langle x\rangle \leq C\langle x-y\rangle\langle y\rangle$. So, for any $M$,

$$
\int\langle x\rangle^{|\alpha|}\left|\partial_{x}^{\beta} K(x, y) f(y)\right| d y \leq C_{M} \int\langle x-y\rangle^{|\alpha|}\langle y\rangle^{|\alpha|}\left|\partial_{x}^{\beta} K(x, y)\right|\langle y\rangle^{-M} d y \leq C_{\alpha, \beta, M}
$$

Remark: In section 4.4, we will use similar techniques to define semiclassical Fourier integral operators (FIOs) on $\mathcal{S}$.

The previous lemma gives
Corollary 4.1.4. For $a \in S_{\delta}^{m}\left(T^{*} \mathbb{R}^{d}\right), 0 \leq \delta \leq 1 / 2$ and $0 \leq t \leq 1, \mathrm{Op}_{\mathrm{h}, \mathrm{t}}(a): \mathcal{S} \rightarrow \mathcal{S}$ is continuous.

When we impose additional decay on $a, \mathrm{Op}_{\mathrm{h}, \mathrm{t}}(a)$ has better mapping properties.
Lemma 4.1.5. Suppose that $a \in \mathcal{S}$. Then $\mathrm{Op}_{\mathrm{h}, \mathrm{t}}(a): \mathcal{S}^{\prime} \rightarrow \mathcal{S}$ continuously. Moreover, the formal adjoint is given by $\operatorname{Op}_{\mathrm{h}, \mathrm{t}}(a)^{*}=\mathrm{Op}_{\mathrm{h}, 1 \mathrm{t}}(\bar{a})$. In particular, for real a $\mathrm{Op}_{\mathrm{h}, 1 / 2}(a)$ is formally self adjoint.

Proof. Observe that $\mathrm{Op}_{\mathrm{h}, \mathrm{t}}(a)$ has kernel

$$
K_{t}(x, y)=\mathcal{F}_{h}^{-1}(a(t x+(1-t) y, \cdot ; h))(x-y) \in \mathcal{S}
$$

Hence, $\mathrm{Op}_{\mathrm{h}, \mathrm{t}}(a) u=u\left(K_{t}(x, \cdot)\right) \in \mathcal{S}$ as desired.
To see the second claim, we simply write distributional kernel of $\mathrm{Op}_{\mathrm{h}, \mathrm{t}}(a)^{*}$.
Next we show that the definition of $\Psi_{\delta}^{m}$ could have used any particular $t \in[0,1]$ and that varying $t$ does not change $\mathrm{Op}_{\mathrm{h}, \mathrm{t}}(a)$ by a principal order term. To do this, we need the following useful lemma.

Lemma 4.1.6. Let $Q$ denote a nonsingular symmetric matrix. Define

$$
e^{\frac{i \hbar}{2}\langle Q D, D\rangle} u(x):=\mathcal{F}_{h}^{-1}\left(e^{\frac{i}{2 h}\langle Q \xi, \xi\rangle} \mathcal{F}_{h}(u)(\xi)\right)(x)
$$

Then, for $u \in \mathcal{S}$,

$$
e^{\frac{i h}{2}\langle Q D, D\rangle} u(x)=\frac{|\operatorname{det} Q|^{-1 / 2}}{(2 \pi h)^{d / 2}} e^{\frac{i \pi}{4} \operatorname{sgn}(Q)} \int e^{-\frac{i}{2 h}\left\langle Q^{-1} y, y\right\rangle} u(x+y) d y .
$$

Moreover, $e^{\frac{i h}{2}\langle Q D, D\rangle}$ extends to $S_{\delta}\left(T^{*} \mathbb{R}^{d}, g\right)$ where

$$
S_{\delta}\left(T^{*} \mathbb{R}^{d}, g\right):=\left\{a \in C^{\infty}\left(T^{*} \mathbb{R}^{d}\right):\left|\partial^{\alpha} a(x, \xi)\right| \leq C h^{-|\alpha| \delta} g(x, \xi)\right\}
$$

and for all $z, w, 0 \leq g(z) \leq C\langle w-z\rangle^{N} g(w)$.
Proof. First, assume that $u \in \mathcal{S}$. Then, using the Fourier transform of a complex exponential (see, for example, [87, Chapter 3]), we have

$$
\begin{aligned}
e^{\frac{i h}{2}\langle Q D, D\rangle} u(x) & =(2 \pi h)^{-d} \int e^{\frac{i}{h}(\langle x-y, \xi\rangle+\langle Q \xi, \xi\rangle)} u(y) d y d \xi \\
& =\frac{|\operatorname{det} Q|^{-1 / 2}}{(2 \pi h)^{-d / 2}} e^{\frac{i \pi}{4} \operatorname{sgn} Q} \int e^{-\frac{i}{2 h}\left\langle Q^{-1}(x-y), x-y\right\rangle} u(y) d y \\
& =\frac{|\operatorname{det} Q|^{-1 / 2}}{(2 \pi h)^{-d / 2}} e^{\frac{i \pi}{4} \operatorname{sgn} Q} \int e^{-\frac{i}{2 h}\left\langle Q^{-1} y, y\right\rangle} u(x+y) d y
\end{aligned}
$$

as desired.

To see that $e^{\frac{i h}{2}\langle Q D, D\rangle}$ extends to $S_{\delta}\left(T^{*} \mathbb{R}^{d}, g\right)$, we break the integral into two pieces. Let $\chi \in C_{c}^{\infty}\left(\mathbb{R}^{2 d}\right)$ have supp $\chi \subset B(0,2)$. Suppose that $a \in \mathcal{S} \cap S_{\delta}\left(T^{*} \mathbb{R}^{d}, g\right)$. Then, letting $w=(y, \eta), z=(x, \xi), \varphi(w)=\left\langle Q^{-1} w, w\right\rangle$ and

$$
C=(2 \pi)^{-d}|\operatorname{det} A|^{-1 / 2} e^{\frac{i \pi}{4} \operatorname{sgn} Q}
$$

we have,

$$
\begin{aligned}
e^{\frac{i h}{2}\langle Q D, D\rangle} a(z) & =C h^{-d} \int_{\mathbb{R}^{2 d}} e^{\frac{\varphi(w)}{h}} a(z+w) d w \\
& =C h^{-d} \int e^{\frac{i \varphi(w)}{h}}(\chi(w)+(1-\chi(w))) a(z+w) d w=: A+B
\end{aligned}
$$

We first consider $\delta<1 / 2$. To estimate $A$, we apply the principle of stationary phase to see that

$$
\begin{align*}
\left|\partial^{\alpha}\left(A-\sum_{k=0}^{N-1} \frac{h^{k}}{k!}\left(\frac{i}{2}\langle Q D, D\rangle\right)^{k} a\right)(z)\right| & \leq C_{1} h^{N} \sup _{|\gamma| \leq d+2 N+1}^{|w| \leq 2} \\
& \left|\partial^{\alpha} \partial^{\gamma} a(z+w)\right|  \tag{4.1.3}\\
& \leq C_{\alpha} h^{-|\alpha| / 2} g(z)
\end{align*}
$$

To estimate $B$, observe that $|\partial \varphi(w)| \geq C|w|$, So, letting $L:=|\partial \varphi|^{-2}\langle\partial \varphi, h D\rangle$, we have

$$
\left|\left(L^{t}\right)^{M}(1-\chi) a\right| \leq C_{M} h^{M}\langle w\rangle^{-M} \sup _{|\alpha| \leq M}\left|\partial^{\alpha} a\right| \leq C h^{M(1-\delta)}\langle w\rangle^{-M+N} g(z)
$$

So, using

$$
B=C h^{-d} \int e^{i \varphi / h}\left(L^{t}\right)^{M}((1-\chi) a) d w
$$

we have $B=O_{S_{\delta}\left(T^{*} \mathbb{R}^{d}, g\right)}\left(h^{\infty}\right)$.
Next, for $\delta=1 / 2$, we rescale with $\omega=h^{-1 / 2} w$. Then,

$$
\left|\partial^{\alpha} A\right| \leq C \sup _{|\omega| \leq 2}\left|\partial^{\alpha} a\left(z+h^{1 / 2} \omega\right)\right| \leq C_{\alpha} h^{-|\alpha| / 2} g(z) .
$$

To estimate $B$, we integrate by parts as above to obtain the same estimate on $B$.
Lemma 4.1.7. Suppose that $A \in \Psi_{\delta}^{m}$ and $A=\operatorname{Op}_{\mathrm{h}, \mathrm{t}}\left(a_{t}\right)$ for $0 \leq t \leq 1$, Then

$$
a_{t}(x, \xi)=e^{i h(t-s)\left\langle D_{x}, D_{\xi}\right\rangle} a_{s} .
$$

Moreover, $a_{t}-a_{s} \in h^{1-2 \delta} S_{\delta}^{m-1}$.

Proof. We compute the kernel of $\mathrm{Op}_{\mathrm{h}, \mathrm{t}}\left(a_{t}\right), K_{t}(x, y)$

$$
\begin{aligned}
K_{t}(x, y) & =(2 \pi h)^{-d} \int e^{\frac{i}{h}\langle x-y, \xi\rangle} a_{t}(t x+(1-t) y, \xi) d \xi \\
& =(2 \pi h)^{-3 d} \int e^{\frac{i}{h}\left(\langle x-y, \xi\rangle+\left\langle t x+(1-t) y+(t-s) \xi^{*}-w, x^{*}\right\rangle+\left\langle\xi, \xi^{*}\right\rangle-\left\langle\xi^{*}, \eta\right\rangle\right)} a_{s}(w, \eta) d w d x^{*} d \xi^{*} d \xi d \eta \\
& =(2 \pi h)^{-2 d} \int e^{\frac{i}{h}\left(\left\langle x-y+\xi^{*}, \xi\right\rangle-\left\langle\xi^{*}, \eta\right\rangle\right)} a_{s}\left(t x+(1-t) y+(t-s) \xi^{*}, \eta\right) d \xi^{*} d \xi d \eta \\
& =(2 \pi h)^{-d} \int e^{\frac{i}{h}\langle x-y, \eta\rangle} a_{s}(s x+(1-s) y, \eta) d \eta
\end{aligned}
$$

Next, observe that

$$
\begin{aligned}
h D_{t} a_{t} & =\left\langle h D_{x}, h D_{\xi}\right\rangle \mathcal{F}_{h}^{-1}\left(e^{\frac{i}{h}(t-s)\left\langle x^{*}, \xi^{*}\right\rangle} \mathcal{F}_{h}\left(a_{s}\right)\left(x^{*}, \xi^{*}\right)\right)(x, \xi) . \\
& =\left\langle h D_{x}, h D_{\xi}\right\rangle a_{t}=O_{S_{\delta}^{m-1}}\left(h^{2-2 \delta}\right)
\end{aligned}
$$

Hence, integrating we have $a_{t}-a_{s} \in h^{1-2 \delta} S_{\delta}^{m-1}$.
Now that we have this lemma in place, we focus on the Weyl quantization. By Lemma 4.1.7 the properties of other quantizations agree up to lower order terms.

## Composition of Symbols

The main lemmas of pseudodifferential calculus demonstrate how two elements, $A \in \Psi_{\delta}^{m_{1}}\left(\mathbb{R}^{d}\right)$ and $B \in \Psi_{\delta}^{m_{2}}\left(\mathbb{R}^{d}\right)$ behave when composed. In particular, up to lower order terms, the composition is the quantization of the product of the symbols.

In preparation for this, we need to prove Borel's Theorem for the asymptotic properties of the classes $S_{\delta}^{m}\left(T^{*} \mathbb{R}^{d}\right)$.

Definition 4.1.8. Suppose that $a_{j} \in S_{\delta}^{m-j}\left(T^{*} \mathbb{R}^{d}\right)$ for $j=0,1, \ldots$ and $r(h)=o(1)$. We say that $a \sim \sum r^{j} a_{j}$ in $S_{\delta}^{-\infty}\left(T^{*} \mathbb{R}^{d}\right)$ if

$$
a-\sum_{j=0}^{N-1} r^{j} a_{j}=O_{S_{\delta}^{m-N}}\left(r^{N}\right)
$$

In this case we call $a_{0}$ the principal symbol of $a$.
Lemma 4.1.9. Suppose that $a_{j} \in S_{\delta}^{m-j}\left(T^{*} \mathbb{R}^{d}\right)$ for $j=0,1, \ldots, r(h)=o(1)$. Then there exists $a \in S_{\delta}^{m}\left(T^{*} \mathbb{R}^{d}\right)$ such that

$$
a \sim \sum_{j=0}^{\infty} r^{j} a_{j} \quad \text { in } S_{\delta}^{-\infty}\left(T^{*} \mathbb{R}^{d}\right)
$$

Moreover, if $\hat{a} \sim \sum_{j} r^{j} a_{j}$ then $a-\hat{a}=O_{S_{\delta}^{-\infty}}\left(r^{\infty}\right)$.

Proof. Let $\chi \in C_{c}^{\infty}(\mathbb{R})$ such that $\chi \equiv 1$ on $|x| \leq 1$ and supp $\chi \subset\{|x|<2\}$. Let $\psi=1-\chi$. Let

$$
\begin{equation*}
a=\sum_{j=0}^{\infty} r^{j} \psi\left(\left(\lambda_{j} r\right)^{-1}\langle\xi\rangle\right) a_{j} \tag{4.1.4}
\end{equation*}
$$

where $\lambda_{j}, j=0,1, \ldots$ is some sequence with $\lambda_{j} \rightarrow \infty$ to be chosen later. Notice that since $\lambda_{j} \rightarrow \infty$, only finitely many terms in (4.1.4) are non-zero for any fixed $h$ and $\xi$.

First, we estimate

$$
\begin{aligned}
\left|\partial_{\xi}^{\beta} \psi\left(\left(\lambda_{j} r\right)^{-1} \xi\right) a_{j}\right| & \leq \sum_{\gamma_{1}+\gamma_{2}=\beta} \sum_{k=1}^{\left|\gamma_{2}\right|} \sum_{\substack{\alpha_{1}+\ldots \alpha_{k}=\gamma_{2} \\
\mid \alpha_{i} \geq 1}} C_{\alpha}\left(\lambda_{j} r\right)^{-k} \partial^{k} \psi\left(\left(\lambda_{j} r\right)^{-1} \xi\right) \partial_{\xi}^{\gamma_{1}} a_{j} \prod_{l=1}^{k} \partial_{\xi}^{\alpha_{l}}\langle\xi\rangle \\
& \leq C_{j, \beta} \psi\left(2\left(\lambda_{j} r\right)^{-1} \xi\right)\langle\xi\rangle^{m-j-|\beta|} h^{-\delta|\beta|}
\end{aligned}
$$

where we use the fact that $2\langle\xi\rangle^{-1} \geq\left(\lambda_{j} r\right)^{-1} \geq C\langle\xi\rangle^{-1}$ on $\operatorname{supp} \partial \psi$. Denote $\chi_{2}=1-\chi_{1}$. Next, we estimate for $|(\alpha, \beta)| \leq j$,

$$
\begin{align*}
r^{j}\left|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} \psi\left(\left(r \lambda_{j}\right)^{-1}\langle\xi\rangle\right) a_{j}\right| & \leq C_{j \alpha \beta} r^{j}\left(\left(\lambda_{j} r\right)^{-1}\langle\xi\rangle\right) \psi\left(2\left(\lambda_{j} r\right)^{-1}\langle\xi\rangle\right) \lambda_{j} r\langle\xi\rangle^{-1} h^{-(|\alpha|+|\beta|) \delta}\langle\xi\rangle^{m-j-|\beta|}  \tag{4.1.5}\\
& \leq 2 C_{j \alpha \beta} r^{j-1} h^{-(|\alpha|+|\beta|) \delta} \lambda_{j}^{-1}\langle\xi\rangle^{m-j-|\beta|+1} \\
& \leq 2^{-j} r^{j-1} h^{-(|\alpha|+|\beta|) \delta}\langle\xi\rangle^{m-j-|\beta|+1} \tag{4.1.6}
\end{align*}
$$

where we choose $\lambda_{j} \geq C_{j \alpha \beta} 2^{j+1}$ for $|(\alpha, \beta)| \leq j$.
Fix $|(\alpha, \beta)|$. For $N \geq|(\alpha, \beta)|$ we estimate

$$
\begin{gathered}
\left|\partial_{x}^{\alpha} \partial_{\xi}^{\beta}\left(a-\sum_{j=0}^{N} r^{j} a_{j}\right)\right| \leq \sum_{j=N+1}^{\infty} r^{j}\left|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} \psi\left(\left(\lambda_{j} r\right)^{-1}\langle\xi\rangle\right) a_{j}\right|+\sum_{j=0}^{N} r^{j}\left|\partial_{x}^{\alpha} \partial_{\xi}^{\beta}\left(1-\psi\left(\left(r \lambda_{j}\right)^{-1}\langle\xi\rangle\right)\right) a_{j}\right| \\
=: A+B
\end{gathered}
$$

Then, by (4.1.6)

$$
A \leq \sum_{j=N+1}^{\infty} 2^{-j} r^{j-1} h^{-(|\alpha|+|\beta|) \delta}\langle\xi\rangle^{m-j+1-|\beta|} \leq r^{N} h^{-(|\alpha|+|\beta|) \delta}\langle\xi\rangle^{m-N-|\beta|}
$$

To estimate $B$, observe that on $r^{-1}\langle\xi\rangle \geq 2 \lambda_{N}, B=0$. So, consider $r^{-1}\langle\xi\rangle \leq 2 \lambda_{N}$. Then using 4.1.5 and

$$
\langle\xi\rangle^{-1} 2 r \lambda_{N} \geq 1,
$$

we have

$$
B \leq \sum_{j=0}^{N} C_{j \alpha \beta} r^{j+N}\langle\xi\rangle^{-N} 2^{N} \lambda_{N}^{N} h^{-(|\alpha|+|\beta|) \delta}\langle\xi\rangle^{m-j} \leq C_{N \alpha \beta} h^{-(|\alpha|+|\beta|) \delta} r^{N}\langle\xi\rangle^{m-N-|\beta|}
$$

For $N \leq|(\alpha, \beta)|$, we need only estimate a finite number of additional terms in $A$ and hence we can obtain the result by adjusting constants.

We now prove the main composition theorem
Lemma 4.1.10. Let $A=\mathrm{Op}_{\mathrm{h}, 1 / 2}(a) \in \Psi_{\delta}^{m_{1}}\left(\mathbb{R}^{d}\right)$ and $B=\mathrm{Op}_{\mathrm{h}, 1 / 2}(b) \in \Psi_{\delta}^{m_{2}}\left(\mathbb{R}^{d}\right)$. Then, $A \circ B=\mathrm{Op}_{\mathrm{h}, 1 / 2}(a \# b) \in \Psi_{\delta}^{m_{1}+m_{2}}$ where

$$
a \# b=\left.e^{\frac{i h}{2} \sigma\left(D_{x}, D_{\xi}, D_{y}, D_{\eta}\right)}(b(x, \xi) b(y, \eta))\right|_{\substack{y=x \\ \eta=\xi}}
$$

and for $\delta<1 / 2$,

$$
\left.a \# b \sim \sum_{k=0}^{\infty} \frac{i^{k} h^{k}}{2^{k} k!} \sigma\left(D_{x}, D_{\xi}, D_{y}, D_{\eta}\right)^{k}(a(x, \xi) b(y, \eta))\right|_{\substack{y=x \\ \eta=\xi}} \quad \text { in } S_{\delta}^{-\infty}\left(T^{*} \mathbb{R}^{d}\right)
$$

In particular,

$$
\begin{gathered}
a \# b=a b+\frac{h}{2 i}\{a, b\}+O_{S_{\delta}^{m_{1}+m_{2}-2}}\left(h^{2(1-2 \delta}\right) \\
{[A, B]=\operatorname{Op}_{\mathrm{h}, 1 / 2}\left(\frac{h}{i}\{a, b\}+O_{S_{\delta}^{m_{1}+m_{2}-3}}\left(h^{3(1-2 \delta)}\right)\right)}
\end{gathered}
$$

Proof. We first assume that $a, b \in \mathcal{S}$. Then observe that the kernel of $A, K_{A}(x, y)$ has

$$
K_{A}(x+t / 2, y-t / 2)=(2 \pi h)^{-d} \int a(x, \xi) e^{\frac{i}{h}\langle t, \xi\rangle} d \xi
$$

Hence, by the Fourier inversion formula,

$$
a(x, \xi)=\int K_{A}(x+t / 2, y-t / 2) e^{-\frac{i}{h}\langle t, \xi\rangle} d t
$$

Thus $A \circ B=\mathrm{Op}_{\mathrm{h}, 1 / 2}(c)$ where

$$
c(x, \xi)=(2 \pi h)^{-2 d} \int e^{\frac{i}{h} \varphi} a\left(\frac{x+w+t / 2}{2}, \eta\right) b\left(\frac{w+y-t / 2}{2}, \tau\right) d \eta d w d \xi d t
$$

where

$$
\varphi=\langle x-w+t / 2, \eta\rangle+\langle w-x+t / 2, \tau\rangle-\langle t, \xi\rangle
$$

Then, using $\eta-\xi, \tau-\xi,(w-x+t / 2) / 2$, and $(w-x-t / 2) / 2$ as new variables, introduces a Jacobian factor of $2^{2 d}$. Hence,

$$
\begin{aligned}
c(x, \xi) & =(\pi h)^{-2 d} \int a(x+z, \xi+\eta) b(x+t, \xi+\tau) e^{\frac{i}{h} 2 \sigma(t, \tau ; z, \eta)} d z d \eta d t d \tau \\
& \left.=e^{\frac{i h}{2} \sigma\left(D_{x}, D_{\xi}, D_{y}, D_{\eta}\right)}(a(x, \xi) b(y, \eta)) \right\rvert\, \begin{array}{c}
y=x \\
\eta=\xi \\
\hline
\end{array}
\end{aligned}
$$

This is defined on $S_{\delta}^{m}\left(T^{*} \mathbb{R}^{d}\right)$ since $S_{\delta}^{m}\left(T^{*} \mathbb{R}^{d}\right) \subset S_{\delta}\left(T^{*} \mathbb{R}^{d},\langle\xi\rangle^{m}\right)$. Thus, we need to check that the image lies in $S_{\delta}^{m_{1}+m_{2}}\left(T^{*} \mathbb{R}^{d}\right)$. Let

$$
A(D)=\frac{1}{2} \sigma\left(D_{x}, D_{\xi}, D_{y}, D_{\eta}\right)
$$

Then

$$
\begin{equation*}
g(x, \xi)=c(x, \xi)-\left.\sum_{k=0}^{N} \frac{i^{k} h^{k}}{k!} A(D)^{K} a(x, \xi) b(y, \eta)\right|_{\substack{y=x \\ \eta=\xi}} \tag{4.1.7}
\end{equation*}
$$

Then

$$
g(x, \xi)=\left.C \int_{0}^{1}(1-t)^{N} \exp (t h A(D))(h A(D))^{k}(a(x, \xi) b(y, \eta))\right|_{\substack{y=x \\ \eta=\xi}} d t
$$

Now,

$$
(h A(D))^{N+1}(a(x, \xi) b(y, \eta)) \in h^{(N+1)(1-2 \delta)} \sum_{k=0}^{N+1} S_{\delta}\left(\langle\xi\rangle^{m_{1}-k}\langle\eta\rangle^{m_{2}-N-1+k}\right)
$$

and $\exp (t h A(D))$ preserves these symbol classes. Hence evaluation at $\xi=\eta$ proves the claim since $N$ is arbitrary.

The following useful corollary demonstrates the pseudolocality of pseudodifferential operators.

Corollary 4.1.11. Suppose that $\delta<1 / 2, a \in S_{\delta}^{m}\left(T^{*} \mathbb{R}^{d}\right), b \in S_{\delta}^{n}\left(T^{*} \mathbb{R}^{d}\right)$ and that $\operatorname{supp} a \cap$ $\operatorname{supp} b=\emptyset$. Then,

$$
\operatorname{Op}_{\mathrm{h}, 1 / 2}(a) \circ \mathrm{Op}_{\mathrm{h}, 1 / 2}(a)=O_{\Psi_{\delta}^{-\infty}}\left(h^{\infty}\right)
$$

Next, we prove a lemma that will be useful when understanding how symbols change under changes of coordinates

Lemma 4.1.12. Operators of the form (4.1.2) lie in $\Psi_{\delta}^{m}\left(\mathbb{R}^{d}\right)$. In particular, $A=\mathrm{Op}_{\mathrm{h}, 1 / 2}\left(a_{1}\right)$ for

$$
a_{1}(x, \xi)=\left.e^{i h\left\langle D_{z}, D_{\xi}\right\rangle} a\left(x-\frac{z}{2}, x+\frac{z}{2}, \xi\right)\right|_{z=0} \in S_{\delta}^{m}\left(T^{*} \mathbb{R}^{d}\right)
$$

Proof. For $a \in \mathcal{S}$, we have

$$
a_{1}(x, \xi)=(2 \pi h)^{-d} \iint a(x-z / 2, x+z / 2, \zeta+\xi) e^{-\frac{i}{h}\langle\zeta, z\rangle} d z d \zeta
$$

But this is the definition of

$$
\left.e^{i h\left\langle D_{z}, D_{\xi}\right\rangle} a\left(x-\frac{z}{2}, x+\frac{z}{2}, \xi\right)\right|_{z=0} .
$$

which maps $S_{\delta}^{m}\left(\mathbb{R}^{2 d} \times \mathbb{R}^{d}\right)$ to $S_{\delta}^{m}\left(T^{*} \mathbb{R}^{d}\right)$ by the same arguments used in the proof of Lemma 4.1.10.

## $L^{2}$ boundedness

The goal of this section is to prove that for $a \in S_{\delta}\left(T^{*} \mathbb{R}^{d}\right)$, $\mathrm{Op}_{\mathrm{h}, 1 / 2}(a)$ is bounded on $L^{2}$ and to give estimates on the norm of $\mathrm{Op}_{\mathrm{h}, 1 / 2}(a)$. As a first step, we show that

Lemma 4.1.13. Suppose that $a \in \mathcal{S}$. Then

$$
\left\|\mathrm{Op}_{\mathrm{h}, 1 / 2}(a)\right\|_{L^{2} \rightarrow L^{2}} \leq(2 \pi h)^{-2 d}\left\|\mathcal{F}_{h}(a)\right\|_{L^{1}\left(\mathbb{R}^{2 d}\right)}
$$

Proof. The kernel of $a$ is given by

$$
K(x, y)=(2 \pi h)^{-d} \int a\left(\frac{x+y}{2}, \xi\right) e^{\frac{i}{h}\langle x-y, \xi\rangle} d \xi=(2 \pi h)^{-2 d} \int \mathcal{F}_{h}(a)(\eta, y-x) e^{\frac{i}{h}\langle x+y, \eta / 2} d \eta
$$

Thus,

$$
\int|K(x, y)| d x \leq(2 \pi h)^{-2 d}\left\|\mathcal{F}_{h}(a)\right\|_{L^{1}}, \quad \int|K(x, y)| d y \leq(2 \pi h)^{-2 d}\left\|\mathcal{F}_{h}(a)\right\|_{L^{1}}
$$

and Schur's test proves the lemma.
To prove $L^{2}$ boundedness for symbols in $S_{\delta}\left(T^{*} \mathbb{R}^{d}\right)$, we use a partition of unity on $T^{*} \mathbb{R}^{d}$ combined with the Cotlar-Stein Lemma [87, Theorem C.5] to exploit oscillations in the kernel of $\mathrm{Op}_{\mathrm{h}, 1 / 2}(a)$.

Let $\chi \in C_{c}^{\infty}\left(T^{*} \mathbb{R}^{d}\right)$ such that $0 \leq \chi \leq 1, \operatorname{supp} \chi \subset B(0,2)$, and $\sum_{\alpha \in \mathbb{Z}^{2 d}} \chi_{\alpha} \equiv 1$ where $\chi_{\alpha}(\cdot)=\chi(\cdot-\alpha)$. Then, for a symbol $a$, define $a_{\alpha}=\chi_{\alpha} a$ and $b_{\alpha \beta}:=\bar{a}_{\alpha} \# a_{\beta}$. Following [87, Theorem 4.22], we start with $h=1$.

Lemma 4.1.14. For $h=1$ and all $N \geq 0$ and multiindeces $\gamma$,

$$
\begin{equation*}
\left|\partial^{\gamma} b_{\alpha \beta}(z)\right| \leq C_{\gamma, N}\langle\alpha-\beta\rangle^{-N}\left\langle z-\frac{\alpha+\beta}{2}\right\rangle^{-N} \tag{4.1.8}
\end{equation*}
$$

Moreover,

$$
\begin{equation*}
\left\|\mathrm{Op}_{1,1 / 2}\left(b_{\alpha \beta}\right)\right\|_{L^{2} \rightarrow L 2} \leq C_{N}\langle\alpha-\beta\rangle^{-N} \tag{4.1.9}
\end{equation*}
$$

Proof. Let $\zeta \in C_{c}^{\infty}\left(\mathbb{R}^{4 d}\right)$ with $\zeta \equiv 1$ on $B(0,1)$, and $\operatorname{supp} \zeta \subset B(0,2)$. We have that

$$
\begin{aligned}
b_{\alpha \beta}(z) & =(\pi)^{-2 d} \int_{\mathbb{R}^{2 d}} \int_{\mathbb{R}^{2 d}} e^{i \varphi(w)}(\zeta(w)+(1-\zeta(w))) \bar{a}_{\alpha}\left(z-w_{1}\right) a_{\beta}\left(z-w_{2}\right) d w_{1} d w_{2} \\
& =: A+B
\end{aligned}
$$

where $\varphi=2 \sigma\left(w_{1}, w_{2}\right)$. Note that on $\operatorname{supp} \zeta,\left|w_{1}\right|+\left|w_{2}\right|<2$. So, if $|\alpha-\beta|>8$ then $A=0$. Therefore $|\alpha-\beta|<8$. Moreover, $\left|z-w_{1}-\alpha\right|<2$ and $\left|z-w_{2}-\beta\right|<2$. So,

$$
|2 z-\alpha+\beta| \leq\left|z-w_{1}-\alpha\right|+\left|z-w_{2}-\beta\right|+\left|w_{1}\right|+\left|w_{2}\right| \leq 8
$$

Hence 4.1.8 is clear.
To estimate $B$, we use the same argument as in Lemma 4.1.6 to see that

$$
\left|\partial^{\gamma} B\right| \leq C_{M, \gamma} \int_{\mathbb{R}^{2 d}} \int_{\mathbb{R}^{2 d}}\langle w\rangle^{-M} c_{\alpha}\left(z-w_{1}\right) c_{\beta}\left(z-w_{2}\right) d w_{1} d w_{2}
$$

where $\operatorname{supp} c_{\alpha} \subset B(\alpha, 2)$ and $\operatorname{supp} c_{\beta} \subset B(\beta, 2)$ and

$$
\begin{equation*}
\sup \left|c_{\alpha} c_{\beta}\right| \leq C \sup _{|\alpha| \leq M}\left|\partial^{\alpha} a\right| \tag{4.1.10}
\end{equation*}
$$

Hence, on $\operatorname{supp} c_{\alpha} c_{\beta}$,

$$
\langle\alpha-\beta\rangle+\left\langle z-\frac{\alpha+\beta}{2}\right\rangle<C\langle w\rangle .
$$

Then the estimate (4.1.8) follows.
We use Lemma 4.1.13 to prove 4.1.9).

$$
\begin{aligned}
\left\|\mathcal{F}_{1}\left(b_{\alpha \beta}\right)\right\|_{L^{1}} & =\int\langle w\rangle^{-2 d-1}\left|\mathcal{F}_{1}\left(b_{\alpha \beta}\right)(w)\langle w\rangle^{2 d+1}\right| d z \\
& \leq C\left\|\langle w\rangle^{-2 d-1}\right\|_{L^{1}}\left\|\left\langle D_{z}\right\rangle^{2 d+1} b_{\alpha \beta}\right\|_{L^{\infty}} \leq C\langle\alpha-\beta\rangle^{-N}
\end{aligned}
$$

Finally, we prove $L^{2}$ boundedness for symbols in $S_{\delta}\left(T^{*} \mathbb{R}^{d}\right)$.
Lemma 4.1.15. Suppose that $a \in S_{\delta}\left(T^{*} \mathbb{R}^{d}\right)$ with $0 \leq \delta \leq 1 / 2$. Then,

$$
\left\|\mathrm{Op}_{\mathrm{h}, 1 / 2}(a)\right\|_{L^{2} \rightarrow L^{2}} \leq C \sum_{|\alpha| \leq M d} h^{|\alpha| / 2} \sup \left|\partial^{\alpha} a\right| .
$$

Proof. Let $A_{\alpha}=\mathrm{Op}_{\mathrm{h}, 1 / 2}\left(a_{\alpha}\right)$ and $B_{\alpha \beta}=\mathrm{Op}_{\mathrm{h}, 1 / 2}\left(b_{\alpha \beta}\right)$. Then, $B_{\alpha \beta}=A_{\alpha}^{*} A_{\beta}$. So, for $h=1$, Lemma 4.1.14 implies that

$$
\left\|A_{\alpha}^{*} A_{\beta}\right\|_{L^{2} \rightarrow L^{2}}+\left\|A_{\alpha} A_{\beta}^{*}\right\|_{L^{2} \rightarrow L^{2}} \leq C\langle\alpha-\beta\rangle^{N} .
$$

Hence,

$$
\sup _{\alpha} \sum_{\beta}\left\|A_{\alpha} A_{\beta}^{*}\right\|^{1 / 2}+\sup _{\alpha} \sum_{\beta}\left\|A_{\alpha}^{*} A_{\beta}\right\| \leq C
$$

and, since $\mathrm{Op}_{\mathrm{h}, 1 / 2}(a)=\sum_{\alpha} A_{\alpha}$, the Cotlar-Steim Theorem implies the result. Moreover, by (4.1.10), the constants depend only on a finite number of derivatives of $a$.

To prove the estimate for $h \neq 1$ we rescale to $h=1$. Let $\tilde{x}=h^{-1 / 2} x, \tilde{\xi}=h^{-1 / 2} \xi$, $\tilde{u}(\tilde{x})=h^{\frac{d}{4}} u\left(h^{1 / 2} \tilde{x}\right) \tilde{a}_{h}(\tilde{x}, \tilde{\xi})=a\left(h^{1 / 2} \tilde{x}, h^{1 / 2} \tilde{\xi}\right)$. Then, $\|\tilde{u}\|_{L^{2}}=\|u\|_{L^{2}}$. Moreover,

$$
\begin{aligned}
\mathrm{Op}_{\mathrm{h}, 1 / 2}(a) u & =\frac{h^{-\frac{d}{4}}}{(2 \pi)^{d}} \iint a_{h}\left(\frac{\tilde{x}+\tilde{y}}{2}, \tilde{\xi}\right) e^{i\langle\tilde{x}-\tilde{y}, \tilde{\xi}\rangle} \tilde{u}(\tilde{y}) d \tilde{y} d \tilde{\xi} \\
& =h^{-\frac{d}{4}} \mathrm{Op}_{1,1 / 2}\left(a_{h}\right) \tilde{u}(\tilde{x})
\end{aligned}
$$

So,

$$
\left\|\mathrm{Op}_{\mathrm{h}, 1 / 2}(a) u\right\|_{L_{x}^{2}}=\left\|\left(\mathrm{Op}_{1,1 / 2} a_{h} \tilde{u}\right)(\tilde{x})\right\|_{L_{\tilde{x}}^{2}} \leq C \sup _{|\alpha| \leq M d}\left|\partial^{\alpha} a_{h}\right|\|u\|_{L_{x}^{2}} \leq C \sup _{|\alpha| \leq M d} h^{|\alpha|}\left|\partial^{\alpha} a\right|\|u\|_{L_{x}^{2}}
$$

## Ellipticity

Throughout this thesis, we will be interested in inverting pseudodifferential operators. The first lemma used to understand this is

Lemma 4.1.16. Suppose that $m \geq 0, a \in S_{\delta}^{m}\left(T^{*} \mathbb{R}^{d}\right)$ has $|a| \geq \epsilon\langle\xi\rangle^{m}$. Then for $u \in \mathcal{S}$, $\delta<1 / 2$, and $h$ small enough,

$$
\|u\|_{L^{2}} \leq C\left\|\mathrm{Op}_{\mathrm{h}, 1 / 2}(a) u\right\|_{L^{2}} .
$$

Moreover, if $m=0$, then $\left(\mathrm{Op}_{\mathrm{h}, 1 / 2}(a)\right)^{-1} \in \Psi_{\delta}\left(\mathbb{R}^{d}\right)$.
Proof. Let $a$ as above. Then, $b:=a^{-1} \in S_{\delta}^{-m}\left(T^{*} \mathbb{R}^{d}\right)$ and hence $a \# b, b \# a \in S_{\delta}$ with

$$
a \# b=1+O_{S_{\delta}^{-1}\left(T^{*} \mathbb{R}^{d}\right)}\left(h^{1-2 \delta}\right), \quad b \# a=1+O_{S_{\delta}^{-1}\left(T^{*} \mathbb{R}^{d}\right)}\left(h^{1-2 \delta}\right)
$$

That is,

$$
\mathrm{Op}_{\mathrm{h}, 1 / 2}(a) \circ \mathrm{Op}_{\mathrm{h}, 1 / 2}(b)=I+R_{1}, \quad \mathrm{Op}_{\mathrm{h}, 1 / 2}(b) \circ \mathrm{Op}_{\mathrm{h}, 1 / 2}(a)=I+R_{2}
$$

where $R_{i}=O_{L^{2} \rightarrow L^{2}}\left(h^{1-2 \delta}\right)$. Now, we have that $I+R_{i}$ is invertible by Neumann series. Thus,

$$
\|u\|_{L^{2}}=\left\|\left(I+R_{2}\right)^{-1} \mathrm{Op}_{\mathrm{h}, 1 / 2}(b) \circ \mathrm{Op}_{\mathrm{h}, 1 / 2}(a) u\right\|_{L^{2}} \leq C\left\|\mathrm{Op}_{\mathrm{h}, 1 / 2}(a) u\right\|_{L^{2}}
$$

If $m=0$, then $a$ is bounded on $L^{2}$ and hence $\mathrm{Op}_{\mathrm{h}, 1 / 2}(a)$ has a left and a right inverse. Thus, it has an inverse $\left(\mathrm{Op}_{\mathrm{h}, 1 / 2}(a)\right)^{-1}=\left(I+R_{2}\right)^{-1} \mathrm{Op}_{\mathrm{h}, 1 / 2}(b)$. Now,

$$
\left(I+R_{2}\right)^{-1}=\sum_{k=0}^{\infty}(-1)^{k} R_{2}^{k}=\sum_{k=0}^{N-1}(-1)^{k} R_{2}^{k}+O_{\Psi_{\delta}^{-N}}\left(h^{N(1-2 \delta)}\right)
$$

Since $N$ is arbitrary, $\left(I+R_{2}\right)^{-1} \in \Psi_{\delta}$ and hence also $\left(\mathrm{Op}_{\mathrm{h}, 1 / 2}(a)\right)^{-1} \in \Psi_{\delta}^{-m}$.
Now, we prove the Sharp Gårding inequality as in [87, Theorem 4.32]
Lemma 4.1.17. Let $a \in S_{\delta}\left(T^{*} \mathbb{R}^{d}\right)$ with $a \geq \gamma \geq 0$. Then

$$
\left\langle\mathrm{Op}_{\mathrm{h}, 1 / 2}(a) u, u\right\rangle \geq\left(\gamma-C h^{1-2 \delta}\right)\|u\|_{L^{2}}^{2}
$$

for $0<h<h_{0}$ and $u \in L^{2}$.

Remark: In fact more is true. The Fefferman-Phong inequality states that for such an $a$

$$
\left\langle\mathrm{Op}_{\mathrm{h}, 1 / 2}(a) u, u\right\rangle \geq\left(\gamma-C h^{2(1-2 \delta)}\right)\|u\|_{L^{2}}^{2} .
$$

We follow [87, Theorem 4.32]. We first need the following calculus lemma.
Lemma 4.1.18. Suppose $a \in S_{\delta}\left(T^{*} \mathbb{R}^{d}\right)$ has $a \geq 0$. Then, $|\partial a| \leq C h^{-\delta} a^{1 / 2}$.
Proof. Write

$$
a(w+z)=a(w)+\langle\partial a(w), z\rangle+\int_{0}^{1}(1-t)\left\langle\partial^{2} a(w+t z) z, z\right\rangle d t
$$

Then, using $z=-\lambda \partial a(w)$ and $a \geq 0$,

$$
\begin{aligned}
\lambda|\partial a(w)|^{2} & \leq a(w)+\lambda^{2} \int_{0}^{1}(1-t)\left\langle\partial^{2} a(w-\lambda t \partial a(w)) \partial a(w), \partial a(w)\right\rangle d t \\
& \leq a(w)+\frac{\lambda^{2}}{2}|\partial a(w)|^{2} \sup \left|\partial^{2} a\right| \leq a(w)+C h^{-2 \delta} \frac{\lambda^{2}}{2}|\partial a(w)|^{2}
\end{aligned}
$$

So, choosing $\lambda=C^{-1} h^{2 \delta},|\partial a(w)|^{2} \leq 2 C h^{-2 \delta} a(w)$ as desired.
We now prove Lemma 4.1.17.
Proof. Rewriting $a=a-\gamma$, we assume that $\gamma=0$. The idea of this proof is to use an $\tilde{h}$ calculus. That is, we fix a small $\tilde{h}$ and give symbol estimates that are uniform in $\tilde{h}$ for $h$ small enough depending on $\tilde{h}$. In fact, we use slightly different symbol classes than those in (4.1.1). We write

$$
\tilde{S}_{\delta}^{m}\left(\mathbb{R}^{d} \times \mathbb{R}^{d}\right)=\left\{a \in C^{\infty}\left(\mathbb{R}^{d} \times \mathbb{R}^{d}\right):\left|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} a\right| \leq C_{\alpha \beta} h^{-\delta(|\alpha|+|\beta|)}\langle\xi\rangle^{m}\right\}
$$

The quantization and composition formulae for such symbols follow from slightly simpler versions of those above that can be found in [87, Chapter 4].

In particular, we show that for $\tilde{h}$ fixed small and $\lambda=h^{1-2 \delta} / \tilde{h}$,

$$
h^{1-2 \delta}(a+\lambda)^{-1} \in \tilde{h} \tilde{S}_{1 / 2}\left(T^{*} \mathbb{R}^{d}\right)
$$

With this estimate, we will be able to invert $a+\lambda$ modulo a lower order term. First, note that

$$
\partial^{\alpha}(a+\lambda)^{-1}=(a+\lambda)^{-1} \sum_{k=1}^{|\alpha|} \sum_{\substack{\alpha=\beta_{1}+\ldots+\beta_{k} \\\left|\beta_{j}\right| \geq 1}} C_{\beta_{1} \ldots \beta_{k}} \prod_{j=1}^{k}(a+\lambda)^{-1} \partial^{\beta_{j}} a .
$$

Now, for $|\beta|=1$, Lemma 4.1.18 implies

$$
\left|\partial^{\beta} a\right|(a+\lambda)^{-1} \leq C h^{-\delta} \lambda^{-1 / 2}\left(\lambda^{1 / 2} a^{1 / 2}\right)(a+\lambda)^{-1} \leq C h^{-\delta} \lambda^{-1 / 2}
$$

Also, $\left|\partial^{\beta} a\right|(a+\lambda)^{-1} \leq C h^{-\delta|\beta|} \lambda^{-1}$. Together, these imply that

$$
\mid \prod_{j=1}^{k}\left((a+\lambda)^{-1} \partial^{\beta_{j}} a \mid \leq C h^{-\delta|\alpha|} \lambda^{-|\alpha| / 2}\right.
$$

Hence, for $\lambda=h^{1-2 \delta} / \tilde{h}$,

$$
b:=(a+\lambda)^{-1} \in \tilde{h} / h^{1-2 \delta} \tilde{S}_{1 / 2}\left(T^{*} \mathbb{R}^{d}\right)
$$

Since $\left\{a+\lambda,(a+\lambda)^{-1}\right\}=0$, we have by Taylor's formula,

$$
\begin{aligned}
(a+\lambda) \# b(w) & =\left.e^{i h A(D)}(a(w)+\lambda) b(z)\right|_{z=w} \\
& =1+\left.\int_{0}^{1}(1-t) e^{i t h A(D)}(i h A(D))^{2}(a(w)+\lambda) b(z)\right|_{z=w} d t=: 1+r(w)
\end{aligned}
$$

Now, for $|\alpha|=2, h^{2} \partial^{\alpha} b \in h^{2 \delta} \tilde{h} \tilde{S}_{1 / 2}\left(T^{*} \mathbb{R}^{d}\right)$. Hence, using the fact that $e^{i t h A(D)}$ preserves symbol classes, and $\left|\partial^{\alpha} a\right| \leq C h^{-2 \delta}$, we have that for $\tilde{h}$ small,

$$
\left\|\mathrm{Op}_{\mathrm{h}, 1 / 2}(r)\right\|_{L^{2} \rightarrow L^{2}} \leq C \tilde{h}<1
$$

This implies that $\mathrm{Op}_{\mathrm{h}, 1 / 2}(b)$ is an approximate right inverse of $\mathrm{Op}_{\mathrm{h}, 1 / 2}(a)+\lambda$. Similarly, it is an approximate right inverse.

Together, this implies that for all $\tau \geq 0, \operatorname{Op}_{\mathrm{h}, 1 / 2}(a)+\tau$ has an inverse and hence that

$$
\operatorname{Spec}\left(\mathrm{Op}_{\mathrm{h}, 1 / 2}(a)\right) \subset[-\lambda, \infty)
$$

Hence, [87, Theorem C.8]

$$
\left\langle\mathrm{Op}_{\mathrm{h}, 1 / 2}(a) u, u\right\rangle \geq-\lambda\|u\|_{L^{2}}^{2}=-C h^{1-2 \delta}\|u\|_{L^{2}}^{2}
$$

as desired.
With this in place, we now improve the estimate in Lemma 4.1.15 (see also 21, Appendix E]).

Lemma 4.1.19. Suppose that $a \in S_{\delta}\left(T^{*} \mathbb{R}^{d}\right)$. Then there exists $C>0$ such that

$$
\left\|\mathrm{Op}_{\mathrm{h}, 1 / 2}(a)\right\|_{L^{2} \rightarrow L^{2}} \leq \sup _{T^{*} \mathbb{R}^{d}}|a|+C h^{\frac{1}{2}(1-2 \delta)}
$$

Moreover, if $\sup _{T^{*} \mathbb{R}^{d}}|a|>c>0$, then

$$
\left\|\mathrm{Op}_{\mathrm{h}, 1 / 2}(a)\right\|_{L^{2} \rightarrow L^{2}} \leq \sup _{T^{*} \mathbb{R}^{d}}|a|+C h^{1-2 \delta}
$$

Proof. Let $C_{0}=\sup _{T^{*} \mathbb{R}^{d}}|a|$. Write $A=\operatorname{Op}_{\mathrm{h}, 1 / 2}(a)$. Then $A^{*}=\mathrm{Op}_{\mathrm{h}, 1 / 2}(\bar{a})$. Hence, $A^{*} A=$ $\mathrm{Op}_{\mathrm{h}, 1 / 2}(\bar{a} \# a)$, Now,

$$
\bar{a} \# a=|a|^{2}+O_{S_{\delta}^{-1}\left(T^{*} \mathbb{R}^{d}\right)}\left(h^{1-2 \delta}\right)
$$

Next, we apply the Sharp Gårding inequality

$$
C_{0}^{2}\|u\|_{L^{2}}^{2}-\|A u\|_{L^{2}}^{2}=\left\langle\left(C_{0}^{2}-A^{*} A\right) u, u\right\rangle \geq-C h^{1-2 \delta}\|u\|_{L^{2}}^{2}
$$

Hence,

$$
\|A u\|_{L^{2}}^{2} \leq C_{0}^{2}\|u\|_{L^{2}}^{2}+C h^{(1-2 \delta)}
$$

which implies both statements.
Remark: In fact (see, for example [87, Chapter 13]) we have

$$
\sup |a|-C h^{1-2 \delta} \leq\left\|\mathrm{Op}_{\mathrm{h}, 1 / 2}(a)\right\|_{L^{2} \rightarrow L^{2}} \leq \sup |a|+C h^{1-2 \delta}
$$

### 4.2 Pseudodifferential Operators on Manifolds

Up to this point, we have defined pseudodifferential operators acting on functions. However, thinking of pseudodifferential operators acting on half densities leads to better invariance properties. Moreover, when we move to Lagrangian distributions the only way to define an invariant (even locally) symbol will be to use half densities.

## Densities

We first recall the notion of a vector bundle over a manifold $M$.
Definition 4.2.1. A vector bundle $(V, \pi)$ of dimension $N$ over a smooth manifold $M$ is a smooth manifold $V$ and a smooth map $\pi: V \rightarrow M$ such that

1. For $x \in M, V_{x}:=\pi^{-1}(x) \approx \mathbb{C}^{N}$.
2. For all $x \in M$, there exists $U_{x}$ a neighborhood of $x$ and a diffeomorphism $\varphi$, called a local trivialization such $\varphi: \pi^{-1}(U) \rightarrow U \times \mathbb{C}^{N}$ with

$$
\begin{equation*}
\pi_{1} \circ \varphi=\pi: \pi^{-1}(U) \rightarrow U \tag{4.2.1}
\end{equation*}
$$

To specify a vector bundle $V$, it is actually enough to choose a set of transition matrices (see for example [87, Section 14.1.2]). In particular, let $\left\{\left(\gamma_{i}, U_{i}\right): i \in \mathcal{I}\right\}$ denote an atlas for $M$ and $\left\{\varphi_{i}: i \in \mathcal{I}\right\}$ such that and $\varphi_{i}: \pi^{-1}\left(U_{i}\right) \rightarrow U_{i} \times \mathbb{C}^{N}$ with property 4.2.1). Then the transition matrices are given by

$$
\gamma_{i j}=\varphi_{i} \circ \varphi_{j}^{-1} \in C^{\infty}\left(U_{i} \cap U_{j} ; G L(N, \mathbb{C})\right)
$$

where $G L(N, \mathbb{C})$ denotes the set of $N \times N$ complex, invertible matrices. Next recall that a section of a vector bundle $V$ is a smooth map $u: M \rightarrow V$ such that $\pi \circ u=\mathrm{Id}$. We then write $u \in C^{\infty}(M ; V)$.

We can now define the notion of $s$-densities.
Definition 4.2.2. The s-density bundle over $M$ is the vector bundle of dimension 1 given by choosing the transition functions

$$
\begin{equation*}
\gamma_{i j}(x):=\left|\operatorname{det} \partial\left(\gamma_{j} \circ \gamma_{i}^{-1}\right)\right|^{s} \circ \gamma_{i}(x) \tag{4.2.2}
\end{equation*}
$$

It is denoted $\Omega^{s}(M)$.
Remark: As vector bundles $\Omega^{0}(M) \approx M \times \mathbb{C}$. That is, $C^{\infty}\left(M ; \Omega^{0}(M)\right)=C^{\infty}(M)$.
We call sections of $\Omega^{1}(M)$ densities, and sections of $\Omega^{1 / 2}(M)$, half densities. The important property of densities is that for $u \in C^{\infty}\left(M ; \Omega^{1}(M)\right)$, integrals of $u$ are invariantly defined.

The next lemma shows how $s$ and $t$ densities relate to one another.
Lemma 4.2.3. Suppose that $u \in C^{\infty}\left(M ; \Omega^{s}(M)\right)$ and $v \in C^{\infty}\left(M ; \Omega^{t}(M)\right)$. Then, $u v \in$ $C^{\infty}\left(M ; \Omega^{s+t}(M)\right)$.

Proof. We only need to check that $u v$ satisfies the correct transition relations. Let $\varphi_{s i}$ : $\pi_{s}^{-1}\left(U_{i}\right) \rightarrow U_{i} \times \mathbb{C}$ and $\varphi_{t i}: \pi_{t}^{-1}\left(U_{i}\right) \rightarrow U_{i} \times \mathbb{C}$ denote local trivializations of $\Omega^{s}(M)$ and $\Omega^{t}(M)$ respectively. Then, write $\varphi_{s i}(u)=\left(x, u_{i}(x)\right)$ and $\varphi_{t i}(v)=\left(x, v_{i}(x)\right)$. Define $(u v)_{i}(x)=$ $u_{i}(x) v_{i}(x)$. Then for $x \in U_{i} \cap U_{j}$,

$$
u_{j}(x)=\left|\operatorname{det} \partial\left(\gamma_{j} \circ \gamma_{i}^{-1}\right)\right|^{s} \circ \gamma_{i}(x) u_{i}(x) \quad v_{j}(x)=\left|\operatorname{det} \partial\left(\gamma_{j} \circ \gamma_{i}^{-1}\right)\right|^{t} \circ \gamma_{i}(x) v_{i}(x)
$$

Hence,

$$
(u v)_{j}(x)=\left|\operatorname{det} \partial\left(\gamma_{j} \circ \gamma_{i}^{-1}\right)\right|^{s+t} \circ \gamma_{i}(x)(u v)_{i}(x)
$$

We say that $u \in C^{\infty}\left(M ; \Omega^{1}(M)\right)$ is positive on $U \subset M$ if for all $W \Subset U$ open, $\int_{W} u>0$. By similar arguments to those above and the corresponding property for functions:

## Lemma 4.2.4.

1. For $u \in C^{\infty}\left(M ; \Omega^{1}(M)\right)$ a positive density, we can define its $r^{\text {th }}$ power

$$
u^{r} \in C^{\infty}\left(M ; \Omega^{r}(M)\right)
$$

2. Suppose that $v \in C^{\infty}\left(M ; \Omega^{1}(M)\right)$ is positive on $U$. Then for all $u \in C^{\infty}\left(M ; \Omega^{s}(M)\right)$, and $\chi \in C_{c}^{\infty}(U)$, there exists $f \in C_{c}^{\infty}(U)$ such that $\chi u=f v^{s}$.

Next, we show that for every coordinate patch $U$, there is a positive density.
Lemma 4.2.5. Let $U \subset M$ be a coordinate patch with coordinates $x^{i}$. Then $|d x|:=\mid d x^{1} \wedge$ $\cdots \wedge d x^{d} \mid$ is a positive density on $U$.

Proof. This follows from the fact that the transition functions for $d x^{1} \wedge \cdots \wedge d x^{d}$ are given by

$$
\gamma_{i j}(x)=\operatorname{det} \partial\left(\gamma_{j} \circ \gamma_{i}^{-1}\right) \circ \gamma_{i}(x)
$$

and that for $W \Subset U$,

$$
\int_{W}\left|d x^{1} \wedge \cdots \wedge d x^{d}\right|=\int_{\gamma(W)} d x_{1} \ldots d x_{d}=|\gamma(W)|>0
$$

The second property in Lemma 4.2 .4 together with Lemma 4.2 .5 show that to define an operator on $s$-densities, it is enough to define it on those of the type $f|d x|^{s}$.

A positive density, $v$, exists by combining a partition of unity argument with Lemma 4.2.5. Hence, after fixing such a $v$, we can identify $s$-densities with functions. In particular,

$$
C^{\infty}\left(M ; \Omega^{s}(M)\right)=\left\{u v^{s}: u \in C^{\infty}(M ; \mathbb{C})\right\}
$$

Finally, we define pull-backs for densities and functions.
Definition 4.2.6. Let $F: M \rightarrow N$ be a $C^{\infty}$ map between two manifolds.

1. We define the pull-back $F^{*}: C^{\infty}(N ; \mathbb{C}) \rightarrow C^{\infty}(M ; \mathbb{C})$ by $u \mapsto u \circ F$.
2. For $M$ and $N$ both of dimension $d$, we can define the pull-back $F^{*}: C^{\infty}\left(N ; \Omega^{s}(N)\right) \rightarrow$ $C^{\infty}\left(M ; \Omega^{s}(M)\right)$. Let $U$ and $V$ be coordinate patches on $N$ and $M$ respectively. Then, on for $u \in C^{\infty}(M)$ supported in $U \cap F^{-1}(V)$,

$$
u(y)|d y|^{s} \mapsto u(F(x))|\operatorname{det} \partial F|^{s}|d x|^{s} .
$$

where $|d y|$ and $|d x|$ denote the Lebesgue measure in the $x$ and $y$ coordinates respectively.

Remark: This definition of pull-backs for $s$-densities is consistent with their transition functions.

## Operators Acting on Half Densities on $\mathbb{R}^{d}$

For this section, we return to the case of $M=\mathbb{R}^{d}$ where there is a canonical density given by Lebesgue measure. Throughout this section, we identify $C^{\infty}\left(\mathbb{R}^{d} ; \Omega^{1 / 2}\left(\mathbb{R}^{d}\right)\right.$ with $C^{\infty}\left(\mathbb{R}^{d}\right)$ using this density. That is, we write $v=u(x)|d x|^{1 / 2}$.

We assume $\gamma: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ is a diffeomorphism equal to the identity outside a compact set and denote the new variables by $x_{1}=\gamma(x)$. We define the pull-back of a function $u$ by writing $u$ in the $x_{1}$ variables. That is, $\gamma^{*} u_{1}=u$ or

$$
u_{1}\left(x_{1}\right)=u_{1}(\gamma(x)):=u(x)
$$

The problem that we want to correct is the fact that $L^{2}$ norms of functions are not invariant under coordinate changes i.e.

$$
\int\left|u_{1}\left(x_{1}\right)\right|^{2} d x_{1} \neq \int|u(x)|^{2} d x
$$

To do this, we use the notion of a half-density from above.
If $u$ is a half-density, then $|u|^{2}$ is a density and hence can be integrated. Moreover, letting $u_{1}=\gamma^{*} u$,

$$
\int\left|u_{1}\right|^{2}=\int|u|^{2}
$$

and hence the $L^{2}$ norm of an element $u \in C_{c}^{\infty}\left(\mathbb{R}^{d} ; \Omega^{1 / 2}\left(\mathbb{R}^{d}\right)\right)$ is well-defined.
Definition 4.2.7. We define $L^{2}\left(\mathbb{R}^{d} ; \Omega^{1 / 2}\left(\mathbb{R}^{d}\right)\right)$ to be the completion of $C_{c}^{\infty}\left(\mathbb{R}^{d} ; \Omega^{1 / 2}\left(\mathbb{R}^{d}\right)\right)$ with respect to the norm

$$
\|u\|_{L^{2}\left(\mathbb{R}^{d} ; \Omega^{1 / 2}\left(\mathbb{R}^{d}\right)\right)}:=\left(\int_{\mathbb{R}^{d}}|u|^{2}\right)^{1 / 2} .
$$

It is then clear that $\gamma^{*}: L^{2}\left(\mathbb{R}^{d} ; \Omega^{1 / 2}\left(\mathbb{R}^{d}\right)\right) \rightarrow L^{2}\left(\mathbb{R}^{d} ; \Omega^{1 / 2}\left(\mathbb{R}^{d}\right)\right)$ is an isometry.
For an operator $A: C^{\infty}\left(\mathbb{R}^{d} ; \Omega^{1 / 2}\left(\mathbb{R}^{d}\right)\right) \rightarrow C^{\infty}\left(\mathbb{R}^{d} ; \Omega^{1 / 2}\left(\mathbb{R}^{d}\right)\right)$ or $A: L^{2}\left(\mathbb{R}^{d} ; \Omega^{1 / 2}\left(\mathbb{R}^{d}\right)\right) \rightarrow$ $L^{2}\left(\mathbb{R}^{d} ; \Omega^{1 / 2}\left(\mathbb{R}^{d}\right)\right)$ we define the pull-back by $\gamma^{-1}$ of $A$ by

$$
A_{\gamma^{-1}}:=\left(\gamma^{-1}\right)^{*} A \gamma^{*} .
$$

Integral kernels also fit nicely into the theory of operators acting on half densities. In particular, for $K \in \mathcal{S}^{\prime}\left(\mathbb{R}^{2 d}\right), K(x, y)|d x|^{1 / 2}|d y|^{1 / 2}$ acts as an integral kernel on half densities. That is,

$$
A\left(u|d x|^{1 / 2}\right)=\int_{\mathbb{R}^{d}} K(x, y)|d x|^{1 / 2}|d y|^{1 / 2} u(y)|d y|^{1 / 2}:=\left(\int_{\mathbb{R}^{2 d}} K(x, y) u(y) d y\right)|d x|^{1 / 2}
$$

Lemma 4.2.8. Let $K$ be the kernel of an operator $A$ acting on half densities. Then, the kernel of $A_{\gamma^{-1}}$ is given by $K_{\gamma^{-1}}\left(x_{1}, y_{1}\right)\left|d x_{1}\right|^{1 / 2}\left|d y_{1}\right|^{1 / 2}$ where

$$
K_{\gamma^{-1}}\left(x_{1}, y_{1}\right)=K(x, y)|\partial \gamma(x)|^{-1 / 2}|\partial \gamma(y)|^{-1 / 2}
$$

$x_{1}=\gamma(x), y_{1}=\gamma(y)$ and $|\partial \gamma(x)|=|\operatorname{det} \partial \gamma(x)|$.

Proof.

$$
\begin{aligned}
& A_{\gamma^{-1}}\left(u_{1}\left|d x_{1}\right|^{1 / 2}\right) \\
& =\left(\gamma^{-1}\right)^{*} \int K(x, y)|d x|^{1 / 2}|d y|^{1 / 2} \gamma^{*}\left(u_{1}\left|d y_{1}\right|^{1 / 2}\right) \\
& =\left(\gamma^{-1}\right)^{*} \int K(x, y)|d x|^{1 / 2}|d y|^{1 / 2} u_{1}(\gamma(y))|\partial \gamma(y)|^{1 / 2}|d y|^{1 / 2} \\
& =\left(\gamma^{-1}\right)^{*}\left(\int K(x, y) u_{1}(\gamma(y))|\partial \gamma(y)|^{1 / 2} d y\right)|d x|^{1 / 2} \\
& \left.=\left(\int K\left(\gamma^{-1}\left(x_{1}\right)\right), y\right) u_{1}(\gamma(y))|\partial \gamma(y)|^{1 / 2}\left|\partial \gamma^{-1}\left(x_{1}\right)\right|^{1 / 2} d y\right)\left|d x_{1}\right|^{1 / 2} \\
& \left.=\left(\int K\left(\gamma^{-1}\left(x_{1}\right)\right), \gamma^{-1}\left(y_{1}\right)\right) u_{1}\left(y_{1}\right)\left|\partial \gamma\left(\gamma^{-1}\left(y_{1}\right)\right)\right|^{-1 / 2}\left|\partial \gamma\left(\gamma^{-1}\left(x_{1}\right)\right)\right|^{-1 / 2} d y_{1}\right)\left|d x_{1}\right|^{1 / 2}
\end{aligned}
$$

Hence,

$$
K_{\gamma^{-1}}(\gamma(x), \gamma(y))=K(x, y)|\partial \gamma(y)|^{-1 / 2}|\partial \gamma(x)|^{-1 / 2}
$$

as desired.
It is often useful to think of operators acting on functions as acting on half densities. In particular, if $A$ has kernel $K(x, y)$ acting on functions, then we can think of $A$ as having kernel $K(x, y)|d x|^{1 / 2}|d y|^{1 / 2}$ when acting on half densities.

## Pseudodifferential Operators and Changing Variables

Our first step is in understanding how pseudodifferential operators behave under changes of variables is to see that the symbol classes $S_{\delta}^{m}\left(T^{*} \mathbb{R}^{d}\right)$ are invariant.

Lemma 4.2.9. Suppose that $\gamma: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ is a smooth diffeomorphism with

$$
\left|\partial^{\alpha} \gamma\right|+\left|\partial^{\alpha} \gamma^{-1}\right| \leq C_{\alpha}
$$

for all $\alpha$. Then if $a \in S_{\delta}^{m}\left(T^{*} \mathbb{R}^{d}\right)$, its pullback by the lift of $\gamma$ to $T^{*} \mathbb{R}^{d}$

$$
\gamma^{*} a(x, \xi):=a\left(\gamma^{-1}(x), \partial \gamma\left(\gamma^{-1}(x)\right)^{t} \xi\right)
$$

lies in $S_{\delta}^{m}\left(T^{*} \mathbb{R}^{d}\right)$.
Proof. Define $b(x, \xi):=a\left(\gamma^{-1}(x), \xi\right)$. Then,

$$
\partial_{x}^{\alpha} \gamma^{*} a=\sum c_{\gamma \sigma \rho}\left(\partial_{x}^{\gamma} \partial_{\xi}^{\sigma} b\right) \xi^{\rho}
$$

where $|\gamma|+|\sigma| \leq|\alpha|$ and $|\sigma|=|\rho|$. Hence,

$$
\partial_{x}^{\alpha} \partial_{\xi}^{\beta} \gamma^{*} a=\sum c_{\gamma \sigma \rho \nu \kappa \lambda}\left(\partial_{x}^{\gamma} \partial_{\xi}^{\sigma+\nu-\kappa} b\right) \xi^{\rho-\lambda}
$$

with $|\gamma|+|\sigma| \leq|\alpha|,|\sigma|=|\rho|,|\kappa|=|\lambda|,|\nu|=|\beta|, \nu \geq \kappa, \rho \geq \lambda$. Now, it is clear that $b \in S_{\delta}^{m}\left(T^{*} \mathbb{R}^{d}\right)$. So, we have the estimate

$$
\begin{aligned}
\left|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} \gamma^{*} a\right| & \leq \sum c_{\gamma \sigma \rho \nu \kappa \lambda} h^{-\delta(|\gamma|+|\sigma|+|\nu|-|\kappa|)}\langle\xi\rangle^{m+|\rho|-|\lambda|-|\sigma|-|\nu|+|\kappa|} \\
& \leq C_{\alpha \beta} h^{-\delta(|\alpha|+|\beta|)}\langle\xi\rangle^{m-|\beta|}
\end{aligned}
$$

Hence, $\gamma^{*} a \in S_{\delta}^{m}\left(T^{*} \mathbb{R}^{d}\right)$.
Next, we show that $\gamma^{*} a$ is the notion of pull-back that, up to lower order terms, corresponds to changing variables and applying a pseudodifferential operator. In particular,

Lemma 4.2.10. Suppose that $a \in S_{\delta}^{m}\left(T^{*} \mathbb{R}^{d}\right)$ and $\gamma$ as in Lemma 4.2.9. Then

$$
\mathrm{Op}_{\mathrm{h}, 1 / 2}(a)_{\gamma^{-1}}=\mathrm{Op}_{\mathrm{h}, 1 / 2}\left(\gamma^{*} a\right)+\mathrm{O}_{\Psi_{\delta}^{m-2}\left(\mathbb{R}^{d}\right)}\left(h^{2-3 \delta}\right)
$$

as operators acting on half densities. Moreover,

$$
\mathrm{Op}_{\mathrm{h}, 1 / 2}(a)_{\gamma^{-1}}=\mathrm{Op}_{\mathrm{h}, 1 / 2}\left(\gamma^{*} a\right)+O_{\Psi_{\delta}^{m-1}\left(\mathbb{R}^{d}\right)}\left(h^{1-2 \delta}\right)
$$

as operators on functions.
Proof. For simplicity of notation, we write from now on that $\gamma(x)=x_{1}$ and $\gamma(y)=y_{1}$. The kernel of $\mathrm{Op}_{\mathrm{h}, 1 / 2}(a)_{\gamma^{-1}}$ acting on half densities is given by

$$
K_{\gamma^{-1}}\left(x_{1}, y_{1}\right)=(2 \pi h)^{-d} \int a\left(\frac{x+y}{2}, \xi\right) e^{\frac{i}{h}\langle x-y, \xi\rangle} d \xi|\partial \gamma(x)|^{-1 / 2}|\partial \gamma(y)|^{-1 / 2}
$$

Lemma 4.1.12 shows that there exists $a_{1} \in S_{\delta}^{m}\left(T^{*} \mathbb{R}^{d}\right)$ such that

$$
K_{\gamma^{-1}}\left(x_{1}, y_{1}\right)=(2 \pi h)^{-d} \int a_{1}\left(\frac{x_{1}+y_{1}}{2}, \xi_{1}\right) e^{\frac{i}{h}\left\langle x_{1}-y_{1}, \xi_{1}\right\rangle} d \xi_{1}
$$

Let $\chi \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ with $\chi \equiv 1$ near 0 . Define

$$
K_{\gamma^{-1}}\left(x_{1}, y_{1}\right)=K_{\gamma^{-1}}\left(x_{1}, y_{1}\right)\left(\chi\left(x_{1}-y_{1}\right)+\left(1-\chi\left(x_{1}-y_{1}\right)\right)\right)=: A+B
$$

Letting $L=\frac{\left\langle x-y, h D_{\xi}\right\rangle}{|x-y|^{2}}$, using $L e^{\frac{i}{h}\langle x-y, \xi\rangle}=e^{\frac{i}{h}\langle x-y, \xi\rangle}$ together with the fact that $a \in S_{\delta}^{m}\left(T^{*} \mathbb{R}^{d}\right)$, we see that $B$ is a kernel of the form

$$
B=(2 \pi h)^{-d} \int b(x, y, \xi) e^{\frac{i}{h}\langle x-y, \xi\rangle} d \xi
$$

with $b \in h^{\infty} S_{\delta}^{-\infty}\left(T^{*} \mathbb{R}^{d}\right)$. Hence, Lemma 4.1.12 shows that $B$ is the kernel of an operator in $h^{\infty} \Psi_{\delta}^{-\infty}\left(\mathbb{R}^{d}\right)$. In particular, $A=\mathrm{Op} \mathrm{p}_{\mathrm{h}, 1 / 2}\left(a_{1}+O_{S_{\delta}^{-\infty}\left(\mathbb{R}^{d}\right)}\left(h^{\infty}\right)\right)$.

Thus, we need only consider $A$. We compute $a_{1}$ up to terms of $O_{S_{\delta}^{m-2}}\left(h^{2(1-2 \delta)}\right)$. To do this, write

$$
A\left(x_{1}, y_{1}\right)=(2 \pi h)^{-d} \int a_{1}\left(\frac{\gamma(x)+\gamma(y)}{2}, \xi_{1}\right) e^{\frac{i}{h}\left\langle\gamma(x)-\gamma(y), \xi_{1}\right\rangle} d \xi_{1}
$$

Since we are working near $x_{1}=y_{1}$, we are working near $x=y$ and hence Taylor expansion around $x=y$, give accurate representations in the region of interest. In particular, the Taylor expansion of $\gamma$ around $\frac{x+y}{2}=: z(x, y)$ gives

$$
\begin{aligned}
& \gamma(x)=\gamma(z)+\partial \gamma(z) \frac{x-y}{2}+\frac{1}{8}\left\langle\partial^{2} \gamma(z)(x-y), x-y\right\rangle+\langle F(x, y), x-y\rangle \\
& \gamma(y)=\gamma(z)-\partial \gamma(z) \frac{x-y}{2}+\frac{1}{8}\left\langle\partial^{2} \gamma(z)(x-y), x-y\right\rangle-\langle F(y, x), x-y\rangle
\end{aligned}
$$

where $F(x, y)=O\left(|x-y|^{2}\right)$. This implies that

$$
\gamma(x)-\gamma(y)=\left(\partial \gamma(z)+O\left(|x-y|^{2}\right)\right)(x-y)=: g(x, y)(x-y)
$$

and

$$
\gamma(x)+\gamma(y)=2 \gamma(z)+O\left(|x-y|^{2}\right)
$$

So, changing variables with $\xi_{1}=\left((g(x, y))^{-1}\right)^{t} \xi$,

$$
\begin{equation*}
A=(2 \pi h)^{-d} \int b(x, y, \xi)|\operatorname{det} g(x, y)|^{-1} e^{\frac{i}{h}\left\langle\gamma(x)-\gamma(y),\left(g(x, y)^{-1}\right)^{t} \xi\right\rangle} d \xi \tag{4.2.3}
\end{equation*}
$$

where

$$
b(x, y, \xi):=a_{1}\left(\gamma(z),\left((\partial \gamma(z))^{t}\right)^{-1} \xi\right)+O\left(h^{-\delta}|x-y|^{2}\right) c \quad c \in S_{\delta}^{m} .
$$

Moreover,

$$
|\operatorname{det} g(x, y)|^{2}=|\operatorname{det} \partial \gamma(z)|^{2}+O\left(|x-y|^{2}\right)
$$

and

$$
|\operatorname{det} \partial \gamma(z)|^{2}=|\operatorname{det} \partial \gamma(y) \| \operatorname{det} \partial \gamma(x)|+\langle A(z), x-y\rangle+\mathcal{O}\left(|x-y|^{2}\right)
$$

But since $z$ is symmetric under switching $x$ and $y$, and the term involving $A(z)$ is odd under switching $x$ and $y$. Thus $A(z)=0$. Finally, notice that

$$
g(x, y)^{-1}(\gamma(x)-\gamma(y))=x-y .
$$

All together this implies that

$$
A=(2 \pi h)^{-d} \int b(x, y, \xi) e^{\frac{i}{h}\langle x-y, \xi\rangle} d \xi|\operatorname{det} \partial \gamma(y)|^{-1 / 2}|\operatorname{det} \partial \gamma(x)|^{-1 / 2}
$$

where $b$ still has the same form. To verify that the $O\left(h^{-\delta}|x-y|^{2}\right) S_{\delta}^{m}\left(\mathbb{R}^{2 d} \times \mathbb{R}^{d}\right)$ term gives an error of $O_{S_{\delta}^{m-2}\left(T^{*} \mathbb{R}^{d}\right)}\left(h^{2-3 \delta}\right)$, we integrate by parts twice using $L$ from above.

Together, with Lemma 4.2.8 this implies that

$$
a_{1}\left(\gamma(z),\left((\partial \gamma(z))^{t}\right)^{-1} \xi\right)=a(z, \xi)+O_{S_{\delta}^{m-2}\left(T^{*} \mathbb{R}^{d}\right)}\left(h^{2-3 \delta}\right)
$$

as desired.
For operators on functions, we need to replace $|\operatorname{det} \partial \gamma(x)|^{-1 / 2}$ by $|\operatorname{det} \partial \gamma(y)|^{-1 / 2}$. This introduces an error of size $|x-y|$ and hence we have the second statement in the lemma.

## Definition of Pseudodifferential Operators on Manifolds

We now have the tools in place to define pseudodifferential operators on a compact manifold $M$ as in [87, Chapter 14].

Definition 4.2.11. For $\delta<1 / 2$, we say a linear operator $A: C^{\infty}(M) \rightarrow C^{\infty}(M)$ is a pseudodifferential operator of order $m$ and type $\delta$ and write $A \in \Psi_{\delta}^{m}(M)$ if

1. There exists an $m \in \mathbb{R}$ such that for each coordinate patch $U_{\gamma}$, there exists $a_{\gamma} \in$ $S_{\delta}^{m}\left(T^{*} \mathbb{R}^{d}\right)$ such that

$$
\varphi A(\psi u)=\varphi \gamma^{*} \mathrm{Op}_{\mathrm{h}, 1 / 2}\left(a_{\gamma}\right)\left(\gamma^{-1}\right)^{*}(\psi u)
$$

for all $\varphi, \psi \in C_{c}^{\infty}\left(U_{\gamma}\right)$ and $u \in C^{\infty}(M)$.
2. For all $\varphi_{1}, \varphi_{2} \in C^{\infty}(M)$ with $\operatorname{supp} \varphi_{1} \cap \operatorname{supp} \varphi_{2}=\emptyset$, and any $N$,

$$
\varphi_{1} A \varphi_{2}=O_{H^{-N}(M) \rightarrow H^{N}(M)}\left(h^{\infty}\right)
$$

Next, we show the existence of a symbol and sub-principal symbol for pseudodifferential operators acting on half densities. We say that $a \in S_{\delta}^{m}\left(T^{*} M\right)$ if for all coordinate maps $\gamma, \gamma^{*} a \in S_{\delta}^{m}\left(T^{*} \mathbb{R}^{d}\right)$. By Lemma 4.2.9, this does not depend on the particular choice of coordinates.

Lemma 4.2.12. There exist maps

$$
\sigma: \Psi_{\delta}^{m}(M) \rightarrow S_{\delta}^{m}\left(T^{*} M\right) / h^{1-2 \delta} S_{\delta}^{m}\left(T^{*} M\right)
$$

and

$$
\mathrm{Op}_{\mathrm{h}}: S_{\delta}^{m}\left(T^{*} M\right) \rightarrow \Psi_{\delta}^{m}(M)
$$

such that $\sigma\left(A_{1} A_{2}\right)=\sigma\left(A_{1}\right) \sigma\left(A_{2}\right)$ and

$$
\sigma\left(\mathrm{Op}_{\mathrm{h}}(a)\right)=[a] \in S_{\delta}^{m}\left(T^{*} M\right) / h^{1-2 \delta} S_{\delta}^{m}\left(T^{*} M\right)
$$

Moreover, there exists a map

$$
\sigma_{1}: \Psi_{\delta}^{M}(m) \rightarrow h^{1-2 \delta} S_{\delta}^{m-1}\left(T^{*} M\right) / h^{2-4 \delta} S_{\delta}^{m-2}\left(T^{*} M\right)
$$

such that

$$
\begin{equation*}
A-\mathrm{Op}_{\mathrm{h}}\left(\sigma(A)+\sigma_{1}(A)\right)=O_{\Psi_{\delta}^{m-2}}\left(h^{2-3 \delta}\right) \tag{4.2.4}
\end{equation*}
$$

The maps $\sigma$ and $\sigma_{1}$ are called respectively the principal and sub-principal symbol maps.
Proof. First, assume that $U \subset \mathbb{R}^{d}$ and $B: C_{c}^{\infty}(U) \rightarrow C^{\infty}(U)$ such that for $\varphi, \psi \in C_{c}^{\infty}(U)$, $\varphi B \psi \in \Psi_{\delta}^{m}\left(\mathbb{R}^{d}\right)$. Then, we show that

$$
\begin{equation*}
B=\mathrm{Op}_{\mathrm{h}, 1 / 2}(a)+B_{0} \tag{4.2.5}
\end{equation*}
$$

with $a \in S_{\delta, \text { loc }}^{m}\left(U \times \mathbb{R}^{d}\right)$ and for $V \Subset U$, and all $N, B_{0}: H^{-N}(V) \rightarrow H^{N}(U)=O\left(h^{\infty}\right)$.
Let $\psi_{j}$ be a locally finite partition of unity on $U$. Then, write $B=\sum_{j k} \psi_{j} B \psi_{k}$. We have that $\psi_{j} B \psi_{k}=\mathrm{Op}_{\mathrm{h}, 1 / 2}\left(a_{j k}\right)$ for some $a_{j k} \in S_{\delta}^{m}\left(T^{*} \mathbb{R}^{d}\right)$ with $a_{j k}(x, \xi)=0$ if $x \notin \operatorname{supp} \psi_{j}$.

Let

$$
\mathcal{J}:=\left\{(j, k): \operatorname{supp} \psi_{j} \cap \operatorname{supp} \psi_{k} \neq \emptyset\right\} .
$$

Then define $a:=\sum_{(j, k) \in \mathcal{J}} a_{j k}$. Then we have

$$
B_{0}=B-\mathrm{Op}_{\mathrm{h}, 1 / 2}(a)=\sum_{(j, k) \notin \mathcal{J}} \psi_{j} B \psi_{k}
$$

Hence, by the local finiteness of the partition, the fact that $\psi_{j} B \psi_{k} \in \Psi_{\delta}^{m}\left(\mathbb{R}^{d}\right)$, and Corollary 4.1.11 $B_{0}$ has the required property.

Now, for each coordinate chart $\left(\gamma, U_{\gamma}\right)$, and $\psi_{\gamma} \in C_{c}^{\infty}\left(U_{\gamma}\right), \psi_{\gamma} A$ has the same properties as $B$ above. Hence, 4.2.5) defines an $a_{\gamma} \in S_{\delta}^{m}\left(T^{*} U_{\gamma}\right)$. Now, if $U_{\gamma_{1}} \cap U_{\gamma_{2}} \neq \emptyset$, then as operators on half densities, Lemma 4.2.10 shows that

$$
\left.\left(a_{\gamma_{1}}-a_{\gamma_{2}}\right)\right|_{U_{\gamma_{1}} \cap U_{\gamma_{2}}} \in h^{2-3 \delta)} S_{\delta}^{m-2}\left(T^{*}\left(U_{\gamma_{1}} \cap U_{\gamma_{2}}\right)\right)
$$

To define the symbol map, we use a partition of unity $\varphi_{\gamma_{i}}$ subordinate to the coordinate charts $\left(\gamma_{i}, U_{\gamma_{i}}\right)$, and write $a=\sum_{i} \varphi_{\gamma_{i}} a_{\gamma_{i}}$. Then $a$ is well defined as an element of $S_{\delta^{m}\left(T^{*} M\right)} / h^{2(1-2 \delta)} S_{\delta}^{m-2}\left(T^{*} M\right)$ and hence we have defined $\sigma$ and $\sigma_{1}$.

Finally, we define the quantization procedure, $\mathrm{Op}_{\mathrm{h}}$. To do this, let $\sum_{i} \varphi_{i}^{2} \equiv 1$ be a partition of unity subordinate to $U_{\gamma_{i}}$. Then write

$$
\operatorname{Op}_{\mathrm{h}}(a)=\sum_{i} \psi_{i}\left(\gamma_{i}\right)^{*} \operatorname{Op}_{\mathrm{h}, 1 / 2}\left(\tilde{a}_{i}\right)\left(\gamma_{i}^{-1}\right)^{*} \psi_{i}
$$

where

$$
\tilde{a}_{i}(x, \xi)=a\left(\gamma_{i}^{-1}(x),\left(\partial \gamma_{i}^{-1}(x)\right)^{t} \xi\right)
$$

Then, 4.2.4) follows from Lemma 4.2 .10 and the composition properties follow from those of Weyl quantization.

Remark: If we use $\mathrm{Op}_{\mathrm{h}, \mathrm{t}}$ for $t \neq 1 / 2$, then the subprincipal symbol can still be defined, but the map $\sigma_{1}$ involves certain derivatives of the map $\sigma$.

It is now easy to check the analogs of Lemmas 4.1.16, 4.1.17, and 4.1.19 for pseudodifferential operators on manifolds.

### 4.3 Microlocalization

It is often useful in our analysis to think of decomposing functions and operators into pieces that are localized both in frequency and space. However, this is not strictly speaking possible. Instead, we develop the notions of wave-front set and microsupport in order to decompose into pieces that are almost localized to certain locations of phase space.

## Wave-front Sets and Microsupport

If $A: C^{\infty}(M) \rightarrow C^{\infty}(M)$ is a properly supported operator we say that $A=\mathcal{O}_{\Psi^{-\infty}}\left(h^{\infty}\right)$ if $A$ is smoothing and each of the $C^{\infty}(M \times M)$ seminorms of its Schwartz kernel is $\mathcal{O}\left(h^{\infty}\right)$.

We define the radial compactified contangent bundle $\bar{T}^{*} M:=T^{*} M \sqcup S^{*} M$ where

$$
S^{*} M:=\left(T^{*} M \backslash\{0\}\right) / \mathbb{R}_{+}
$$

and the $\mathbb{R}_{+}$action is given by $(t,(x, \xi)) \mapsto(x, t \xi)$. That is, $S^{*} M$ is the fiber at infinity of $T^{*} M$. Letting $|\cdot|_{g}$ denote the norm induced on $T^{*} M$ by the Riemannian metric $g$, a neighborhood of a point $\left(x_{0}, \xi_{0}\right) \in S^{*} M$ is given by $V \times\left(U \cap|\xi|_{g} \geq K\right)$ where $V$ is an open neighborhood of $x_{0}$ and $U$ is a conic neighborhood of $\xi_{0}$.

## Microlocalization of Pseudodifferential Operators

For each $A \in \Psi_{\delta}^{k}(M)$, we have $A=\mathrm{Op}_{\mathrm{h}}(a)+O_{\Psi^{-\infty}}\left(h^{\infty}\right)$ for some $a \in S_{\delta}^{k}\left(T^{*} M\right)$. Define the semiclassical wavefront set of $A, \mathrm{WF}_{\mathrm{h}, \Psi}(A) \subset \bar{T}^{*} M$, as follows.

Definition 4.3.1. A point $(x, \xi) \in \bar{T}^{*} M$ does not lie in $\mathrm{WF}_{\mathrm{h}, \Psi}(A)$, if there exists a neighborhood $U$ of $(x, \xi)$ in $\bar{T}^{*} M$ such that each $(x, \xi)$-derivative of $a$ is $O\left(h^{\infty}\langle\xi\rangle^{-\infty}\right)$ in $U \cap T^{*} M$. As in [4], we write $\mathrm{WF}_{\mathrm{h}, \Psi}(A)=: \mathrm{WF}_{\mathrm{h}, \Psi}^{\mathrm{f}}(A) \sqcup \mathrm{WF}_{\mathrm{h}, \Psi}^{\mathrm{i}}(A)$ where $\mathrm{WF}_{\mathrm{h}, \Psi}^{\mathrm{f}}(A)=\mathrm{WF}_{\mathrm{h}, \Psi}(A) \cap T^{*} M$ and $\mathrm{WF}_{\mathrm{h}, \Psi}^{\mathrm{i}}(A)=\mathrm{WF}_{\mathrm{h}, \Psi}(A) \cap S^{*} M$.

Remark: It is clear from the definition that $\mathrm{WF}_{\mathrm{h}, \Psi}(A)$ is closed.
Operators with compact wavefront sets in $T^{*} M$ are called compactly microlocalized; these are operators of the form $\operatorname{Op}_{\mathrm{h}}(a)+\mathcal{O}_{\Psi^{-\infty}}\left(h^{\infty}\right)$ for some $a \in S_{\delta}^{\text {comp }}\left(T^{*} M\right)$. We denote by $\Psi_{\delta}^{\text {comp }}(M)$ the class of all compactly microlocalized elements of $\Psi_{\delta}^{k}(M)$. As before, we put $\Psi^{\text {comp }}(M)=\Psi_{0}^{\text {comp }}(M)$. Compactly microlocalized operators should not be confused with compactly supported operators (operators whose Schwartz kernels are compactly supported).

We need a finer notion of microsupport on $h$-dependent sets.

Definition 4.3.2. An operator $A \in \Psi_{\delta}^{\text {comp }}(M)$ is said to be microsupported on an $h$ dependent family of sets $V(h) \subset T^{*} M$, if we can write $A=\mathrm{Op}_{\mathrm{h}}(a)+O_{\Psi^{-\infty}}\left(h^{\infty}\right)$, where for each compact set $K \subset T^{*} M$, each differential operator $\partial^{\alpha}$ on $T^{*} M$, and each $N$, there exists a constant $C_{\alpha N K}$ such that for $h$ small enough,

$$
\sup _{(x, \xi) \in K \backslash V(h)}\left|\partial^{\alpha} a(x, \xi ; h)\right| \leq C_{\alpha N K} h^{N} .
$$

We then write $\mathrm{MS}_{\mathrm{h}, \Psi}(A) \subset V(h)$.
Remark: Notice that since we are working with $A \in \Psi_{\delta}^{\text {comp }}(M)$ for $0 \leq \delta<1 / 2$ we have $a \in S_{\delta}^{\text {comp }}\left(T^{*} M\right)$ and $a$ can only vary on a scale $\sim h^{-\delta}$. This implies that the set $\mathrm{MS}_{\mathrm{h}, \Psi}(A)$ will respect the uncertainty principle.

Combining 4.2.3 with Lemma 4.1.12 we can see that change of variables formula for the full symbol of a pseudodifferential operator contains an asymptotic expansion in powers of $h$ consisting of derivatives of the original symbol. Thus definition 4.3.2 does not depend on the choice of coordinate maps in the quantization procedure $O p_{\mathrm{h}}$. Moreover, since we take $\delta<1 / 2$, if $A \in \Psi_{\delta}^{\text {comp }}(M)$ is microsupported inside some $V(h)$ and $B \in \Psi_{\delta}^{k}(M)$, then $A B$, $B A$, and $A^{*}$ are also microsupported inside $V(h)$. This implies the following.

Lemma 4.3.3. Suppose that $A, B \in \Psi_{\delta}^{\text {comp }}(M)$ and $\operatorname{MS}_{\mathrm{h}, \Psi}(A) \cap \mathrm{MS}_{\mathrm{h}, \Psi}(B)=\emptyset$. Then $\mathrm{WF}_{\mathrm{h}, \Psi}(A B)=\emptyset$.

It follows from the definition of the wavefront set that for $A \in \Psi_{\delta}^{\text {comp }}(x, \xi) \in T^{*} M$ does not lie in $\mathrm{WF}_{\mathrm{h}, \Psi}(A)$ if and only if there exists an $h$-independent neighborhood $U$ of $(x, \xi)$ such that $A$ is microsupported on the complement of $U$. However, $A$ need not be microsupported on $\mathrm{WF}_{\mathrm{h}}(A)$. It will be microsupported on any $h$-independent neighborhood of $\mathrm{WF}_{\mathrm{h}}(A)$. Finally, it can be seen by Taylor's formula that if $A \in \Psi_{\delta}^{\text {comp }}(M)$ is microsupported in $V(h)$ and $\delta^{\prime}>\delta$, then $A$ is also microsupported on the set of all points in $V(h)$ which are at least $h^{\delta^{\prime}}$ away from the complement of $V(h)$.

## Microlocal Ellipticity

In Section 4.1, we defined the notion of ellipticity for $A \in \Psi_{\delta}^{m}\left(\mathbb{R}^{d}\right)$. It is useful to have a notion of microlocal ellipticity for pseudodifferential operators.

Definition 4.3.4. We say that $(x, \xi) \in \bar{T}^{*} M$ is in the elliptic set of $P \in \Psi_{\delta}^{m}(M)$ if there exists a neighborhood $U$ of $(x, \xi)$ and a constant $C>0$ such that $|\sigma(P)| \geq C\langle\xi\rangle^{m}$ in $U$. We write $(x, \xi) \in \operatorname{ell}(P)$.

Remark: It is clear from the definition that ell $(P)$ is open.
We have the following analog of the estimates in 4.1 (see also [22, Section 2.2])
Lemma 4.3.5. Suppose that $P \in \Psi_{\delta}^{m}(M)$ and $A \in \Psi_{\delta}^{m^{\prime}}(M)$ with $\mathrm{WF}_{\mathrm{h}, \Psi}(A) \subset \operatorname{ell}(P)$. Then there exist $Q_{i} \in \Psi_{\delta}^{m^{\prime}-m}(M)$ such that

$$
A=Q_{1} P+O_{\Psi_{\delta}^{-\infty}}\left(h^{\infty}\right)=P Q_{2}+O_{\Psi^{-\infty}}\left(h^{\infty}\right) .
$$

In particular, for each $s \in \mathbb{R}$ and $u \in H_{h}^{s+m^{\prime}}$ there exists $C>0$ such that for all $N>0$,

$$
\|A u\|_{H_{h}^{s}} \leq C\|P u\|_{H_{h}^{s+m^{\prime}-m}}+O\left(h^{\infty}\right)\|u\|_{H_{h}^{-N}}
$$

Proof. By composing with appropriate powers of $\langle h D\rangle$, we may assume that $m=m^{\prime}=s=0$. Let $p \in S_{\delta}\left(T^{*} M\right)$ be such that $P=\operatorname{Op}_{\mathrm{h}}(p)+O_{\Psi^{-\infty}}\left(h^{\infty}\right)$. Define $q_{0}=\frac{a}{p}$. Then $q_{0} \in S_{\delta}\left(T^{*} M\right)$. Then, by Lemma 4.1.10 together with the definition of wave-front set,

$$
A=\mathrm{Op}_{\mathrm{h}}\left(q_{0}\right) P+h^{1-2 \delta} A_{1}
$$

where $A_{1} \in \Psi_{\delta}^{-1}(M)$ with $\mathrm{WF}_{\mathrm{h}}\left(A_{1}\right) \subset \ell(P)$. Then, by induction, there exist $q_{j} \in \Psi_{\delta}^{-j}$ such that

$$
A_{j}=\mathrm{Op}_{\mathrm{h}}\left(q_{j}\right) P+h^{1-2 \delta} A_{j+1}
$$

with $A_{j} \in \Psi_{\delta}^{-j}(M)$ and $\mathrm{WF}_{\mathrm{h}}\left(A_{j}\right) \subset \ell(P)$.
Now, by Borel's lemma 4.1.9, the exists $q \in S_{\delta}\left(T^{*} M\right)$ such that

$$
q \sim \sum_{j} h^{j(1-2 \delta)} q_{j} .
$$

Hence,

$$
A=\mathrm{Op}(q) P+O_{\Psi^{-\infty}}\left(h^{\infty}\right)
$$

The construction of $Q_{2}$ follows analogously.
To obtain the estimate, simply take $L^{2}$ norms of

$$
A u=Q P u+O_{\Psi_{\delta}^{-\infty}}\left(h^{\infty}\right) u
$$

to obtain

$$
\|A u\|_{L^{2}} \leq C\|P u\|_{L^{2}}+O\left(h^{\infty}\right)\|u\|_{H_{h}^{-N}}
$$

as desired.

## Microlocalization of Distributions and Operators

An $h$-dependent family $u(h) \in \mathcal{D}^{\prime}(M)$ is called $h$-tempered if for each open $U$ compactly contained in $M$, there exist constants $C$ and $N$ such that

$$
\|u(h)\|_{H_{h}^{-N}(U)} \leq C h^{-N} .
$$

Definition 4.3.6. For a tempered distribution $u$, we say that $\left(x_{0}, \xi_{0}\right) \in \bar{T}^{*} M$ does not lie in the wavefront set $\mathrm{WF}_{\mathrm{h}}(u)$, if there exists a neighborhood $V$ of $\left(x_{0}, \xi_{0}\right)$ such that for each $A \in \Psi(M)$ with $\mathrm{WF}_{\mathrm{h}}(A) \subset V$, we have $A u=O_{C^{\infty}}\left(h^{\infty}\right)$. As above, we write

$$
\mathrm{WF}_{\mathrm{h}}(u)=\mathrm{WF}_{\mathrm{h}}^{\mathrm{f}}(u) \sqcup \mathrm{WF}_{\mathrm{h}}^{\mathrm{i}}(u)
$$

where $\mathrm{WF}_{\mathrm{h}}^{\mathrm{i}}(u)=\mathrm{WF}_{\mathrm{h}}(u) \cap S^{*} M$.
By Lemma 4.3.5, $\left(x_{0}, \xi_{0}\right) \notin \mathrm{WF}_{\mathrm{h}}(u)$ if and only if there exists $A \in \Psi(M)$ elliptic at $\left(x_{0}, \xi_{0}\right)$ such that $A u=O_{C^{\infty}}\left(h^{\infty}\right)$. The wavefront set of $u$ is a closed subset of $\bar{T}^{*} M$. It is empty if and only if $u=O_{C^{\infty}(M)}\left(h^{\infty}\right)$.

Lemma 4.3.7. For $u$ tempered and $A \in \Psi_{\delta}^{k}(M), \mathrm{WF}_{\mathrm{h}}(A u) \subset \mathrm{WF}_{\mathrm{h}, \Psi}(A) \cap \mathrm{WF}_{\mathrm{h}}(u)$.
Proof. First, suppose that $\left(x_{0}, \xi_{0}\right) \notin \mathrm{WF}_{\mathrm{h}, \Psi}(A)$. Then let $\chi \in S\left(T^{*} M\right)$ have $\chi \equiv 1$ near $\left(x_{0}, \xi_{0}\right)$ and supp $\chi \cap \mathrm{WF}_{\mathrm{h}, \Psi}(A)=\emptyset$. Then $\mathrm{Op}_{\mathrm{h}} \chi A \in \Psi_{\delta}^{-\infty}(M)$ and hence $\mathrm{Op}_{\mathrm{h}} \chi(A u)=$ $O_{C \infty}\left(h^{\infty}\right)$ which implies $\left(x_{0}, \xi_{0}\right) \notin \mathrm{WF}_{\mathrm{h}}(A u)$.

Next, suppose that $\left(x_{0}, \xi_{0}\right) \notin \mathrm{WF}_{\mathrm{h}}(u)$ and let $\chi \in S\left(T^{*} M\right)$ have $\chi \equiv 1$ near $\left(x_{0}, \xi_{0}\right)$ and $\operatorname{supp} \chi \subset V$ where $V$ is as in the definition of $\mathrm{WF}_{\mathrm{h}}(u)$. Then $\mathrm{WF}_{\mathrm{h}, \Psi}\left(\mathrm{Op}_{\mathrm{h}}(\chi) A\right) \subset V$. Hence, $\mathrm{Op}_{\mathrm{h}}(\chi) A u=O_{C^{\infty}}\left(h^{\infty}\right)$ and $\left(x_{0}, \xi_{0}\right) \notin \mathrm{WF}_{\mathrm{h}}(A u)$.

Next, we prove that $C_{c}^{\infty}(M)$ is dense in $\mathcal{D}^{\prime}(M)$ in a way compatible with the wavefront set. Moreover, we give another characterization of wavefront set. (For the microlocal case, see [40, Section 8.3].)

Lemma 4.3.8. Let $V \subset \mathbb{R}^{d}$ be an open set. For a tempered distribution $u \in \mathcal{D}^{\prime}(V),\left(x_{0}, \xi_{0}\right) \notin$ $\mathrm{WF}_{\mathrm{h}}(u)$ if and only if there exists a neighborhood $U$ of $x_{0}$ and a neighborhood $W$ of $\xi_{0}$ such that for $\chi \in C_{c}^{\infty}(U)$,

$$
\begin{equation*}
\left|\mathcal{F}_{h}(\chi u)(\xi)\right| \leq C_{N} h^{N}\langle\xi\rangle^{-N}, \quad \xi \in W . \tag{4.3.1}
\end{equation*}
$$

Moreover, for all $\phi \in C_{c}^{\infty}(V)$ and $W$ closed neighborhood in $\overline{\mathbb{R}^{d}}$ (the radial compactification of $\mathbb{R}^{d}$ ) with

$$
\begin{equation*}
\mathrm{WF}_{\mathrm{h}}(u) \cap(\operatorname{supp} \phi \times W)=\emptyset \tag{4.3.2}
\end{equation*}
$$

there exists $u_{j} \in C_{c}^{\infty}(V)$ with $u_{j} \rightarrow u \in \mathcal{D}^{\prime}(V)$ and

$$
\sup _{j} \sup _{\xi \in W, h<h_{0}} h^{-N}\langle\xi\rangle^{N}\left|\mathcal{F}_{h}\left(\phi u_{j}\right)(\xi)\right|<\infty, \quad N=1,2, \ldots
$$

Proof. Suppose that $\left(x_{0}, \xi_{0}\right) \notin \mathrm{WF}_{\mathrm{h}}(u)$. Then there exists $W$ a neighborhood of $\left(x_{0}, \xi_{0}\right)$ such that for all $A$ with $\mathrm{WF}_{\mathrm{h}}(A) \subset W, A u=O_{C^{\infty}}\left(h^{\infty}\right)$. Take $\chi_{1} \in C_{c}^{\infty}(V)$ with $\chi_{1} \equiv 1$ near $x_{0}$ and $\chi_{2} \in C^{\infty}\left(\mathbb{R}^{d}\right)$ with $\chi_{2} \equiv 1$ in a neighborhood of $\xi_{0}$ and such that supp $\chi_{1}(x) \chi_{2}(\xi) \subset W$. Let $\chi(x, \xi)=\chi_{1}(x) \chi_{2}(\xi)$. Then,

$$
\mathcal{F}_{h}\left(\mathrm{Op}_{\mathrm{h}}(\chi) u\right)=\chi_{2}(\xi) \mathcal{F}_{h}\left(\chi_{1} u\right)(\xi)+O\left(h^{\infty}\langle\xi\rangle^{-\infty}\right)=O\left(h^{\infty}\langle\xi\rangle^{-\infty}\right)
$$

Hence (4.3.1) holds.
For the converse, suppose $\left(x_{0}, \xi_{0}\right)$ has (4.3.1). Suppose that $W_{1} \Subset W$ is a neighborhood of $\xi_{0}$ and $U_{1} \Subset U$ a neighborhood of $x_{0} . \mathrm{WF}_{\mathrm{h}}(A) \Subset U \times W_{1}$. Let $\chi_{1} \in C_{c}^{\infty}(U)$ and $\chi_{2} \in C^{\infty}\left(W_{1}\right)$ with $\chi_{1} \chi_{2} \equiv 1$ on $\mathrm{WF}_{\mathrm{h}}(A)$ and $\operatorname{supp} \chi_{1} \chi_{2} \subset U \times W_{1}$. Then,

$$
A u=A \operatorname{Op}_{\mathrm{h}}\left(\chi_{1} \chi_{2}\right) u+O_{C^{\infty}}\left(h^{\infty}\right)=O_{C^{\infty}}\left(h^{\infty}\right)
$$

Let $\phi$ and $W$ as in 4.3.2). Let $\chi_{j} \in C_{c}^{\infty}(V)$ with $\chi_{j} \equiv 1$ on any compact set in $V$ for $j$ large enough and $0 \leq \phi_{j} \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ with

$$
\int \phi_{j} d x=1
$$

and $\operatorname{supp} \phi_{j}+\operatorname{supp} \chi_{j} \subset V$ for $j$ large enough. Define $u_{j}:=\left(\chi_{j} u\right) * \phi_{j} \in C_{c}^{\infty}(V)$ for $j$ large enough. Then we have $u_{j} \rightarrow u \in \mathcal{D}^{\prime}(V)$ and $u_{j} \in C_{c}^{\infty}(V)$. Now, take $\psi \in C_{c}^{\infty}(V)$ and a neighborhood $W_{1}$ with interior containing $W$ such that

$$
\mathrm{WF}_{\mathrm{h}}(u) \cap\left(\operatorname{supp} \psi \times W_{1}\right)=\emptyset
$$

Then for $j$ large enough, $\phi u_{j}=\phi w_{j}$ with $w_{j}=\phi_{j} *(\psi u)$. Hence,

$$
\left|\mathcal{F}_{h} w_{j}\right|=\left|\mathcal{F}_{h}\left(\phi_{j}\right)\right|\left|\mathcal{F}_{h}(\psi u)\right| \leq\left|\mathcal{F}_{h}(\psi u)\right|=O\left(h^{\infty}\langle\xi\rangle^{-\infty}\right), \quad \xi \in W_{1}
$$

As for pseudodifferential operators, we need a finer notion of microlocalization on $h$ dependent sets

Definition 4.3.9. A tempered distribution $u$ is said to be microsupported on an $h$ - dependent family of sets $V(h) \subset T^{*} M$ if for $\delta \in[0,1 / 2), A \in \Psi_{\delta}^{\text {comp }}(M)$, and $\operatorname{MS}_{\mathrm{h}, \Psi}(A) \cap V=\emptyset$, $\mathrm{WF}_{\mathrm{h}}(A u)=\emptyset$.

Remark: As with $\mathrm{MS}_{\mathrm{h}, \Psi}, \mathrm{MS}_{\mathrm{h}}$ respects the uncertainty principle because we are testing with operators $A \in \Psi_{\delta}^{\text {comp }}(M)$ and these operators vary only on scales $\sim h^{-\delta}$.

Next, we define microlocalization for operators. Although this can be (almost) equivalently defined in terms of the microlocal properties of the distributional kernel of an operator we do not do so. Instead, we choose to define it in terms of how the operator transports the wave-front set of a distribution.

An $h$ - dependent family of operators $A(h): \mathcal{S}(M) \rightarrow \mathcal{S}^{\prime}\left(M^{\prime}\right)$ is called $h$-tempered if there exists $N \geq 0$ and $k \in \mathbb{Z}^{+}$, such that

$$
\|A(h)\|_{H_{h}^{k}(M) \rightarrow H_{h}^{-k}\left(M^{\prime}\right)} \leq C h^{-N}
$$

$\underline{D}^{*}$ Dinition 4.3.10. For an $h$-tempered family of operators, we say that $\left(x_{0}, \xi_{0}, y_{0}, \eta_{0}\right) \in$ $\bar{T}^{*} M^{\prime} \times \bar{T}^{*} M$ does not lie in the wavefront set $\mathrm{WF}_{\mathrm{h}}{ }^{\prime}(A)$ if there exists a neighborhood $V$ of $\left(x_{0}, \xi_{0}, y_{0}, \eta_{0}\right)$, such that for each $B_{1} \in \Psi\left(M^{\prime}\right)$ and $B_{2} \in \Psi(M)$ with $\mathrm{WF}_{\mathrm{h}, \Psi}\left(B_{1}\right) \times$ $\mathrm{WF}_{\mathrm{h}, \Psi}\left(B_{2}\right) \subset V$ and all $u(h)$ tempered distributions, we have $\mathrm{WF}_{\mathrm{h}}\left(B_{1} A B_{2} u\right)=\emptyset$.

Definition 4.3.11. A tempered operator $A$ is said to be microsupported on an $h$-dependent family of sets $V(h) \subset T^{*} M \times T^{*} M^{\prime}$, if for all $\delta \in[0,1 / 2)$ and each $B_{1} \in \Psi_{\delta}\left(M^{\prime}\right)$ and $B_{2} \in \Psi_{\delta}(M)$ with $\left(\mathrm{MS}_{\mathrm{h}, \Psi}\left(B_{1}\right) \times \mathrm{MS}_{\mathrm{h}, \Psi}\left(B_{2}\right)\right) \cap V=\emptyset$, we have $\mathrm{WF}_{\mathrm{h}}{ }^{\prime}\left(B_{1} A B_{2}\right)=\emptyset$. We then write $\mathrm{MS}_{\mathrm{h}}{ }^{\prime}(A) \subset V(h)$.

## Remarks:

- With the definitions above, we have for $A \in \Psi_{\delta}^{m}(M)$,

$$
\mathrm{WF}_{\mathrm{h}}{ }^{\prime}(A)=\left\{(x, \xi, x, \xi):(x, \xi) \in \mathrm{WF}_{\mathrm{h}, \Psi}(A)\right\} .
$$

In addition, we have that if $A \in \Psi_{\delta}^{\text {comp }}$, then $\mathrm{MS}_{\mathrm{h}, \Psi}(A) \subset V(h)$ if and only if

$$
\mathrm{MS}_{\mathrm{h}}{ }^{\prime}(A) \subset\{(x, \xi, x, \xi):(x, \xi) \in V(h)\} .
$$

Since there is a simple relationship between $\mathrm{WF}_{\mathrm{h}, \Psi}$ and $\mathrm{WF}_{\mathrm{h}}$, as well as $\mathrm{MS}_{\mathrm{h}, \Psi}$ and $\mathrm{MS}_{\mathrm{h}}$, we will only use the notation without $\Psi$ from this point forward and the correct object will be understood from context.

- Notice that if $K$ is the distributional kernel of an operator $A$, then

$$
\mathrm{WF}_{\mathrm{h}}(K)=\left\{(x, \xi, y, \eta):(x, \xi, y,-\eta) \in \mathrm{WF}_{\mathrm{h}}^{\prime}(A)\right\} .
$$

## Wavefront-set Calculus

In Section 4.4 we need some facts about the calculus of wave-front sets which we present here. Since the wave-front set is a local object, we restrict ourselves to considering $\mathbb{R}^{d}$. We follow [4. Section 3.1], [36, Section 7] and [40, Section 8.2] to develop the wavefront set calculus.

Lemma 4.3.12. Suppose that $X \subset \mathbb{R}^{n}$ and $Y \subset \mathbb{R}^{m}$ are open. Suppose that $u \in \mathcal{D}^{\prime}(X)$ and $v \in \mathcal{D}^{\prime}(Y)$ are tempered. Then the map $(u, v) \rightarrow u \otimes v$ from $\mathcal{D}^{\prime}(X) \times \mathcal{D}^{\prime}(Y) \rightarrow \mathcal{D}^{\prime}(X \times Y)$ is well defined and

$$
\mathrm{WF}_{\mathrm{h}}(u \otimes v) \subset \mathrm{WF}_{\mathrm{h}}(u) \times \mathrm{WF}_{\mathrm{h}}(v) \cup\left(\mathrm{WF}_{\mathrm{h}}^{\mathrm{i}}(u) \times \operatorname{supp} v \times\{0\}\right) \cup\left(\{0\} \times \operatorname{supp} u \times \mathrm{WF}_{\mathrm{h}}^{\mathrm{i}}(u)\right.
$$

Proof. For $u \in C_{c}^{\infty}(X)$ and $v \in C_{c}^{\infty}(Y)$, we define $u \otimes v$ by

$$
\langle u \otimes v, \varphi\rangle=\int u(x) v(y) \varphi(x, y) d x d y
$$

Hence,

$$
\begin{equation*}
\mathcal{F}_{h}(u \otimes v)(\xi, \eta)=\mathcal{F}_{h}(u)(\xi) \mathcal{F}_{h}(v)(\eta) \tag{4.3.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\langle u \otimes v, \varphi\rangle=(2 \pi h)^{-n+m} \int \mathcal{F}_{h}(u)(\xi) \mathcal{F}_{h}(v)(\eta) \mathcal{F}_{h}(\varphi)(\xi, \eta) d \xi d \eta \tag{4.3.4}
\end{equation*}
$$

Let $u \in \mathcal{D}^{\prime}(X)$ and $v \in \mathcal{D}^{\prime}(Y)$ with $u_{j} \rightarrow u$ and $v_{j} \rightarrow v, u_{j} \in C_{c}^{\infty}(X), v_{j} \in C_{c}^{\infty}(Y)$. Then $\left|\mathcal{F}_{h}\left(u_{j}\right)\right| \leq C h^{-M}\langle\xi\rangle^{-M}$ and $\left|\mathcal{F}_{h}\left(v_{2}\right)\right| \leq C h^{-M}\langle\eta\rangle^{-M}$. Hence, the right hand side of (4.3.4) with $u_{j}$ and $v_{j}$ replacing $u$ and $v$ converges by the dominated convergence theorem. Moreover, if we replace $u$ and $v$ by $\varphi u$ and $\psi v$ with $\varphi \in C_{c}^{\infty}(X)$ and $\psi \in C_{c}^{\infty}(Y)$, then the right and side of (4.3.4) is well defined and we see that the limit of $\left\langle u_{j} \otimes v_{j}, \varphi\right\rangle$ is independent of the sequences $u_{j}$ and $v_{j}$. Together with a partition of unity, this defines $u \otimes v$.

The wave front set condition follows from a partition of unity argument together with the fact that 4.3.3 holds for $u \in \mathcal{E}^{\prime}(X)$ and $v \in \mathcal{E}^{\prime}(Y)$.

First, we recall the definition of the pull-back of a function and the push-forward of a distribution.

Definition 4.3.13. Let $f: X \subset \mathbb{R}^{N} \rightarrow Y \subset \mathbb{R}^{M}$ be an $C^{\infty}$ map and $u \in C^{\infty}(Y)$. We define the pullback of $u$ by $f$, written $f^{*} u$ to be $u \circ f$. Then $f^{*}: C^{\infty}(Y) \rightarrow C^{\infty}(X)$.

Definition 4.3.14. Let $f: X \subset \mathbb{R}^{N} \rightarrow Y \subset \mathbb{R}^{M}$ be a proper $C^{\infty}$ map. For $u \in \mathcal{D}^{\prime}(X)$ we define the pushforward of $u$ by $f$, written $f_{*} u$, to be the distribution in $\mathcal{D}^{\prime}(Y)$ that has for $\varphi \in C_{c}^{\infty}(Y)$,

$$
\left\langle f_{*} u, \varphi\right\rangle:=\left\langle u, f^{*} \varphi\right\rangle .
$$

The next lemma extends the definition of pullbacks to certain classes of distributions. (We follow [40, Theorem 8.2.4])

Lemma 4.3.15. Let $f: X \subset \mathbb{R}^{N} \rightarrow Y \subset \mathbb{R}^{M}$ be an $C^{\infty}$ map. Denote the set of normals of the map by

$$
N_{f}=\left\{(f(x), \eta) \in Y \times S^{M-1} ;\left(d f_{x}\right)^{t} \eta=0\right\} .
$$

Then the definition of pullback can be extended to $u \in \mathcal{D}^{\prime}(Y)$ tempered with

$$
N_{f} \cap \mathrm{WF}_{\mathrm{h}}^{\mathrm{i}}(u)=\emptyset
$$

Moreover,

$$
\mathrm{WF}_{\mathrm{h}}\left(f^{*} u\right) \subset\left\{\left(x,\left(d f_{x}\right)^{t} \eta\right) \in \bar{T}^{*} X:(f(x), \eta) \in \mathrm{WF}_{\mathrm{h}}(u)\right\}=: f^{*} \mathrm{WF}_{\mathrm{h}}(u)
$$

Proof. We start by considering $u \in C_{c}^{\infty}(Y)$. Then, for $\chi \in C_{c}^{\infty}(X)$,

$$
\begin{equation*}
\left\langle f^{*} u, \chi\right\rangle=(2 \pi h)^{-M} \int \mathcal{F}_{h}(u)(\eta) I_{\chi}(\eta) d \eta \tag{4.3.5}
\end{equation*}
$$

where

$$
I_{\chi}(\eta)=\int \chi(x) e^{\frac{i}{h}\langle f(x), \eta\rangle}
$$

Let $x_{0} \in X, y_{0}=f\left(x_{0}\right)$, and $\Gamma_{y_{0}}=\left\{\eta:\left(y_{0}, \eta\right) \in \mathrm{WF}_{\mathrm{h}}^{\mathrm{i}}(u)\right\}$. Choose a closed neighborhood, $V$, of $\Gamma_{y_{0}}$ such that

$$
\begin{equation*}
\left(d f_{x}\left(x_{0}\right)\right)^{t} \eta \neq 0, \quad \eta \in V \tag{4.3.6}
\end{equation*}
$$

Next, let $Y_{0}$ be a compact neighborhood of $y_{0}$ such that $V$ is a neighborhood of $\Gamma_{y}$ for all $y \in Y_{0}$. Finally, choose $X_{0}$ a compact neighborhood of $x_{0}$ such that $f\left(X_{0}\right) \subset Y_{0}$ and 4.3.6) holds for all $x \in X_{0}$ and $\varphi \in C_{c}^{\infty}\left(Y_{0}\right)$ with $\varphi \equiv 1$ on $f\left(X_{0}\right)$.

With these choices 4.3.5 is valid for $u \in C^{\infty}(Y)$ when $u$ is replaced by $\varphi u$. Now, $d\langle f(x), \eta\rangle=\left\langle d x,\left(d f_{x}(x)\right)^{t} \eta\right\rangle$ and for $x \in \operatorname{supp} \chi$ and $\eta \in V,|\eta| \leq C\left|\left(d_{x} f(x)\right)^{t} \eta\right|$. Hence, integration by parts gives for all $N$ and $\eta \in V,\left|I_{\chi}(\eta)\right| \leq C_{N} h^{N}\langle\eta\rangle^{-N}$. Let $u_{j} \in C^{\infty}(Y)$ and $u_{j} \rightarrow u$ in $\mathcal{D}^{\prime}(Y)$ with

$$
\begin{equation*}
\sup _{j} \sup _{\xi \notin V h<h_{0}} h^{-N}\langle\xi\rangle^{N}\left|\mathcal{F}_{h}\left(\phi u_{j}\right)(\xi)\right|<\infty . \tag{4.3.7}
\end{equation*}
$$

(such a sequence exists by Lemma 4.3.8.) Since $u$ is tempered, $\left|\mathcal{F}_{h}\left(\phi u_{j}\right)\right| \leq C h^{-M}\langle\eta\rangle^{M}$. Thus, the dominated convergence theorem shows that

$$
\left\langle f^{*} u_{j}, \chi\right\rangle \rightarrow(2 \pi h)^{-M} \int \mathcal{F}_{h}(u)(\eta) I_{\chi}(\eta) d \eta
$$

independent of the sequence $u_{j}$. We define this limit to be $f^{*} u$.
Now, to get the wavefront set condition, let $v_{j}=\chi f^{*} u_{j}$. Then, pairing $v_{j}$ with $e^{-\frac{i}{h}\langle x, \xi\rangle}$ gives

$$
\mathcal{F}_{h}\left(v_{j}\right)(\xi)=(2 \pi h)^{-M} \int \mathcal{F}_{h}\left(u_{j}\right)(\eta) I_{\chi}(\eta, \xi) d \eta
$$

where

$$
I_{\chi}(\eta, \xi)=\int \chi(x) e^{\frac{i}{h}\langle f(x), \eta\rangle-\langle x, \xi\rangle} d x .
$$

Redefine $\Gamma_{y_{0}}=\left\{\eta:(y, \eta) \in \mathrm{WF}_{\mathrm{h}}(u)\right\}$. Then let $W$ be a neighborhood of $\left(d_{x} f\left(x_{0}\right)\right)^{t} \Gamma_{y_{0}}$. Then, adjusting $V$ and $X_{0}$ if necessary, we may assume for $x \in X_{0}$ and $\eta \in V, d_{x} f(x) \eta \in W$. Then,

$$
\left|\left(d_{x} f(x)\right)^{t} \eta-\xi\right| \geq c(|\xi|+|\eta|), \quad x \in X_{0}, \quad \eta \in V, \quad \xi \notin W
$$

So, integrating by parts, we have

$$
\left|I_{\chi}(\eta, \xi)\right| \leq C_{N} h^{N}\langle | \eta|+|\xi|\rangle^{-N}, \quad \xi \notin W, \quad \eta \in V .
$$

Next, use integration by parts with only $\langle x, \xi\rangle$ treated as a phase to obtain

$$
\left|I_{\chi}(\eta, \xi)\right| \leq C_{N}\langle\eta\rangle^{N}\langle\xi\rangle^{-N} \quad \eta \notin V .
$$

So, for $\xi \notin W$,

$$
\left|\mathcal{F}_{h}\left(v_{j}\right)(\xi)\right| \leq C_{N} h^{N}\left(\int_{V}\langle | \xi|+|\eta|\rangle^{M-N} d \eta+\langle\xi\rangle^{-N} \int_{\mathbb{R}^{d} \backslash v}\left|\mathcal{F}_{h}\left(\phi u_{j}\right)(\eta)\right|\langle\eta\rangle^{N} d \eta .\right.
$$

But, by (4.3.7),

$$
\sup _{j} \sup _{\xi \notin W h<h_{0}} h^{-N}\langle\xi\rangle^{N}\left|\mathcal{F}_{h}\left(v_{j}\right)(\xi)\right|<\infty
$$

which, together with a partition of unity implies the wavefront set condition.
Lemma 4.3.16. Let $f: X \subset \mathbb{R}^{N} \rightarrow Y \subset \mathbb{R}^{M}$ be a proper $C^{\infty}$ map. Then
$\mathrm{WF}_{\mathrm{h}}\left(f_{*} u\right) \subset\left\{\left(x^{\prime}, \eta\right) \in \bar{T}^{*} Y:\right.$ there exists $(x, \xi) \in \mathrm{WF}_{\mathrm{h}}(u)$ with $\left.x^{\prime}=f(x),\left(d f_{x}(x)\right)^{t} \eta=\xi\right\}$.
Proof. This lemma follows from arguments similar to those at the end of Lemma 4.3.15 together with the fact that for $u \in \mathcal{E}^{\prime}(X)$,

$$
\mathcal{F}_{h}\left(f_{*} u\right)(\xi)\left\langle f_{*} u, \chi(y) e^{-\frac{i}{\hbar}(y, \xi\rangle}\right\rangle=(2 \pi h)^{-N-M} \int \mathcal{F}_{h}(u)(\zeta) I_{\chi}(\zeta, \xi) d \zeta
$$

where

$$
I_{\chi}(\zeta, \xi)=\int \chi(y) e^{\frac{i}{h}(\langle x, \zeta\rangle-\langle y, \xi\rangle+\langle f(x)-y, \eta\rangle)} d y d \eta d x
$$

Suppose $A: \mathcal{S}(Y) \rightarrow \mathcal{S}^{\prime}(X)$. We denote

$$
\begin{aligned}
& \mathrm{WF}_{\mathrm{h} X}^{\mathrm{i}}(A):=\left\{(x, \xi, y, \eta): \text { there exists } y \in Y \text { with }(x, \xi, y, 0) \in \mathrm{WF}_{\mathrm{h}}^{\mathrm{i}}(A)\right\} \\
& \mathrm{WF}_{\mathrm{h} Y}^{\mathrm{i}}(A):=\left\{(x, \xi, y, \eta): \text { there exists } x \in X \text { with }(x, 0, y, \eta) \in \mathrm{WF}_{\mathrm{h}}^{\mathrm{i}}(A)\right\}
\end{aligned}
$$

Together, Lemma 4.3.12, 4.3.15, and 4.3.16 imply the following corollary

Corollary 4.3.17. Let the operator A have properly supported Schwartz kernel

$$
K \in \mathcal{D}^{\prime}(X \times Y)
$$

Suppose that $\mathrm{WF}_{\mathrm{h} Y}^{\mathrm{i}}(A) \cap \mathrm{WF}_{\mathrm{h}}^{\mathrm{i}}(u)=\emptyset$. Then $A u \in \mathcal{D}^{\prime}(X)$ is well defined and

$$
\left.\mathrm{WF}_{\mathrm{h}}(A u) \subset \mathrm{WF}_{\mathrm{h}}^{\prime}(A)\left(\mathrm{WF}_{\mathrm{h}}(u)\right) \cup \mathrm{WF}_{\mathrm{h} X}^{\mathrm{i}^{\prime}}(A)\right)
$$

where $\mathrm{WF}_{\mathrm{h}}{ }^{\prime}(A)$ acts as a relation on sets.
Proof. The Corollary follows from forming $K \otimes u$, pulling back to the diagonal, and pushing forward by the map $\pi_{y}: X \times \Delta_{Y} \rightarrow X$.

### 4.4 Lagrangrian Distributions and Fourier Integral Operators

We now define the notion of semiclassical Lagrangian distribution. Note that throughout this section manifolds are compact. This restriction can be removed by placing assumptions on properness of operators along with replacing Sobolev spaces by local Sobolev spaces.

## Definition of Lagrangian Distributions

We start by defining
Definition 4.4.1. Suppose that $\varphi$ is a clean phase function with excess $e$ (see definition 3.1.14 in an open neighborhood of $\left(x_{0}, \theta_{0}\right) \in \mathbb{R}^{d} \times \overline{\mathbb{R}}^{N}$ and $a \in S_{\delta}^{\infty}\left(\mathbb{R}^{d} \times \mathbb{R}^{N}\right)$ supported where $\varphi$ is defined. Then we define $I_{e}(a, \varphi)$ by

$$
I_{e}(a, \varphi):=(2 \pi h)^{-(d+2 N-2 e) / 4} \int e^{\frac{i}{h} \varphi(x, \theta)} a(x, \theta) d \theta
$$

where the integral is interpreted in the sense of oscillatory integrals (see Lemma 4.4.4). When $e=0$, we write $I(a, \varphi)$.

Then we have the following proposition
Proposition 4.4.2. Let $\Lambda \subset \bar{T}^{*} \mathbb{R}^{d}$ be an admissible Lagrangian submanifold and let $\gamma \in \Lambda$. Let $\varphi$ be a nondegenrate phase function in an open neighborhood of $\left(x_{0}, \theta_{0}\right) \in \mathbb{R}^{d} \times \overline{\mathbb{R}}^{N}$, $N \in \mathbb{N}_{0}$ such that $\Lambda=\Lambda_{\varphi}$ in a neighborhood of $\gamma$. If $a \in S_{\delta}^{m+(d-2 N) / 4}\left(\mathbb{R}^{d} \times \mathbb{R}^{N}\right)$ such that $\operatorname{supp} a \Subset V$, then define $u=I(a, \varphi)$. Then if $\left(x^{\prime}, x^{\prime \prime}\right),\left(\xi^{\prime}, \xi^{\prime \prime}\right) \in \mathbb{R}^{d-k} \times \mathbb{R}^{k}$ and $\Lambda=$
$\left\{\left(\partial_{\xi^{\prime}} H\left(x^{\prime \prime}, \xi^{\prime}\right), x^{\prime \prime}, \xi^{\prime},-\partial_{x^{\prime \prime}} H\left(x^{\prime \prime}, \xi^{\prime}\right)\right\}\right.$, there exists a constant $A$ depending only on $\varphi$ and $H$ such that

$$
\begin{align*}
e^{\frac{i}{h} H\left(x^{\prime \prime}, \xi^{\prime}\right)} & \mathcal{F}_{h, x^{\prime}}(u)\left(x^{\prime \prime}, \xi^{\prime}\right) \\
& -(2 \pi h)^{-(d-2 k) / 4} a(x, \theta) e^{\frac{i}{h} A} e^{i \frac{\pi}{4} \operatorname{sgn}(\Phi)}|\operatorname{det} \Phi|^{-\frac{1}{2}} \in h^{1-2 \delta-(d-2 k) / 4} S_{\delta}^{m+d / 4-k / 2-1} \tag{4.4.1}
\end{align*}
$$

where $\left(x^{\prime}\left(x^{\prime \prime}, \xi^{\prime}\right), \theta\left(x^{\prime \prime}, \xi^{\prime}\right)\right)$ is determined by $\partial_{\theta} \varphi(x, \theta)=0, \partial_{x^{\prime}} \varphi(x, \theta)=\xi^{\prime}, x^{\prime \prime}$, and

$$
\Phi=\left(\begin{array}{cc}
\varphi_{x^{\prime} x^{\prime}}^{\prime \prime} & \varphi_{x^{\prime} \theta}^{\prime \prime} \\
\varphi_{\theta x^{\prime}}^{\prime \prime} & \varphi_{\theta \theta}^{\prime \prime}
\end{array}\right)
$$

In particular,

$$
A=\varphi\left(x\left(x^{\prime \prime}, \xi^{\prime}\right), \theta\left(x^{\prime \prime}, \xi^{\prime}\right)\right)+H\left(x^{\prime \prime}, \xi^{\prime}\right)-\left\langle x^{\prime}\left(x^{\prime \prime}, \xi^{\prime}\right), \xi^{\prime}\right\rangle
$$

is constant in $\left(x^{\prime \prime}, \xi\right)$.
Moreover, if $\varphi_{1} \in S^{1}\left(\mathbb{R}^{d} \times \overline{\mathbb{R}}^{N_{1}}\right)$ is another nondegenerate phase function such that $\Lambda_{\varphi_{1}}=$ $\Lambda_{\varphi}$ in a neighborhood of $\gamma$, then there exists $b \in S_{\delta}^{m+\left(d-2 N_{1}\right) / 4}\left(\mathbb{R}^{d} \times \mathbb{R}^{N_{1}}\right)$ such that $I(a, \varphi)=$ $I\left(b, \varphi_{1}\right)+O_{C^{\infty}}\left(h^{\infty}\right)$. Finally, $\mathrm{WF}_{\mathrm{h}}(I(a, \varphi)) \subset \Lambda_{\varphi}$.

To prove this Proposition, we use a few lemmas on oscillatory integrals. First, we give a precise version of the principle of nonstationary phase.

Lemma 4.4.3. Suppose that $K \Subset X \subset \mathbb{R}^{d}$ and $u \in C_{c}^{\infty}(K), f \in C^{\infty}(X)$. Suppose further that $f$ is real valued with $|\partial f| \geq c>0$ on $K$. Then,

$$
\omega^{k}\left|\int u(x) e^{i \omega f(x)} d x\right| \leq C \sum_{|\alpha| \leq k} \sup \left|\partial^{\alpha} u\right||\partial f|^{|\alpha|-2 k}
$$

where $C$ depends on the compact set $K$ and $\|f\|_{C^{k+1}}$.
Proof. We proceed by induction. The statement for $k=0$ is clear. Let $L=\omega^{-1} \frac{\langle\partial f, D\rangle}{|\partial f|^{2}}$. Then, $L e^{i \omega f}=e^{i \omega f}$. Hence, integration by parts gives

$$
\int u(x) e^{i \omega f} d x=\int L^{t}(u(x)) e^{i \omega f} d x=i \omega^{-1} \int e^{i \omega f} \sum_{i} \partial_{i} \frac{u \partial_{i} \bar{f}}{|\partial f|^{2}} d x
$$

So, by the induction hypothesis,

$$
\omega^{k}\left|\int u(x) e^{i \omega f} d x\right| \leq C \sum_{\mu=0}^{k-1}\left\|u \partial f|\partial f|^{-2}\right\|_{C^{\mu}}|\partial f|^{|\mu|-2 k+2}
$$

To complete the proof, we show that

$$
|\partial f|\left\|u \partial \bar{f}|\partial f|^{-2}\right\|_{C^{\mu}} \leq \sum_{k=0}^{\mu}\|u\|_{C^{r}}|\partial f|^{k-\mu}
$$

For $\mu=0$, this is clear, so we proceed by induction. Let $N=|\partial f|^{2}$ and $w=u \partial \bar{f}|\partial f|^{-2}$. Then, $N w=u \partial \bar{f}$. Hence, applying $\partial^{\alpha}$ to this equation and estimating using the induction hypothesis, we have, using Lemma 4.1.18,

$$
\begin{aligned}
N\|w\|_{C^{\mu}} \leq & C_{\|f\|_{C^{\mu+1}}}\left(\|N\|_{C^{1}}\|u\|_{C^{\mu-1}}+\|u\|_{C^{\mu-2}}+\ldots\|u\|_{C^{0}}\right. \\
& \left.+\|f\|_{C^{1}}\|u\|_{C^{\mu}}+\|u\|_{C^{\mu-1}}+\cdots+\|u\|_{C^{0}}\right) \\
\leq & C_{\|f\|_{C^{\mu+1}}^{\prime}}\left(N^{1 / 2}\|u\|_{C^{\mu-1}}+\|u\|_{C^{\mu-2}}+\ldots\|u\|_{C^{0}}\right. \\
& \left.+N^{1 / 2}\|u\|_{C^{\mu}}+\|u\|_{C^{\mu-1}}+\cdots+\|u\|_{C^{0}}\right) \\
\leq & C_{\|f\|_{C^{\mu+1}}^{\prime \prime}} \sum_{k=0}^{\mu}\|u\|_{C^{k}} N^{k-\mu+1 / 2}
\end{aligned}
$$

Using the definitions of $N$ and $w$, this completes the proof of the lemma.
Lemma 4.4.4. Let $U \subset \mathbb{R}^{d} \times \overline{\mathbb{R}}^{L}$ be open. Suppose that $\varphi \in S^{1}(U)$ is a clean phase function with excess e as in Definition 3.1.14. Suppose that $V \Subset U$. Then the map

$$
a \mapsto I_{a, \varphi} \in \mathcal{D}^{\prime}\left(\mathbb{R}^{d}\right)
$$

where

$$
I_{a, \varphi}(u)=\int I_{e}(a, \varphi) u
$$

for $a \in S_{\delta}^{\text {comp }}(V)$ extends uniquely to a linear map from $\bigcup_{k, \delta} S_{\delta}^{k}(V) \rightarrow \mathcal{D}^{\prime}\left(\mathbb{R}^{d}\right)$.
Proof. Let $\chi \in C_{c}^{\infty}\left(\mathbb{R}^{L}\right)$ have $\chi \equiv 1$ on $B(0,1)$ and supp $\chi \subset B(0,2)$. Define

$$
\chi_{\nu}=\chi\left(2^{-\nu} \theta\right)-\chi\left(2^{-\nu+1} \theta\right), \quad \nu>0, \quad \chi_{0}=\chi
$$

Then, $\chi_{\nu}$ is a partition of unity with

$$
\operatorname{supp} \chi_{\nu} \subset\left\{2^{\nu-1}<|\theta|<2^{\nu+1}\right\} .
$$

Observe that for $a \in S_{\delta}^{k}(V)$

$$
\left|\partial_{x}^{\alpha} \partial_{\theta}^{\beta} \chi_{\nu}(\theta) a(x, \theta)\right| \leq h^{-\delta(|\alpha|+|\beta|)} C_{\alpha \beta}\langle\theta\rangle^{k-|\beta|}, \quad(x, \theta) \in V
$$

with uniform constant in $\nu$. This, together with the fact that at most two $\chi_{\nu}$ are nonzero imply that $\sum \chi_{\nu} a \rightarrow a$ in $S_{\delta}^{k^{\prime}}(V)$ for $k^{\prime}>k$. Thus, any extension with the required properties must have

$$
I_{a, \varphi}(u)=\sum_{\nu} I_{\chi_{\nu} a, \varphi}(u)
$$

and hence we consider

$$
\begin{aligned}
I_{\chi_{\nu} a}(u) & =(2 \pi h)^{-M} \int e^{\frac{i}{h} \varphi(x, \theta)} a(x, \theta) \chi_{\nu}(\theta) u(x) d \theta d x \\
& =(2 \pi h)^{-M} 2^{L \nu} \int e^{\frac{i \omega}{h} \frac{\varphi(x, \omega \theta)}{\omega}} a(x, \omega \theta) \chi(\theta) u(x) d \theta d x
\end{aligned}
$$

where $\omega=2^{\nu}$. Define $f_{\omega}(x, \theta)=\frac{\varphi(x, \omega \theta)}{\omega}$. Then, since $\varphi$ is a phase function (in particular since it has a polyhomogeneous expansion with top order term a homogeneous phase function),

$$
\sup _{\omega \geq 1 \operatorname{supp} \chi}\left|\partial_{x \theta}^{\alpha} f_{\omega}\right| \leq C_{|\alpha|}, \quad \inf _{\omega \geq 1 \operatorname{supp} \chi} \inf \left|\partial f_{\omega}\right| \geq c>0
$$

Now, we have that

$$
\left|\partial_{x \theta}^{\alpha} a\left(x, 2^{\nu} \theta\right)\right| \leq C M h^{-\delta|\alpha|} 2^{\nu m}
$$

for $1 / 2<|\theta|<2$. Hence, by Lemma 4.4.3

$$
\left|I_{\chi_{\nu} a}(u)\right| \leq C M 2^{\nu(L+m-k)} h^{(1-\delta) k} \sum_{|\alpha| \leq k} \sup \left|\partial^{\alpha} u\right| .
$$

Choosing $k$ large enough shows that the sum in $\nu$ converges and hence that $u \mapsto I_{a, \varphi}(u)$ is a distribution (of order $k$ ).

We now prove Proposition 4.4.2
Proof. Lemma 4.4.4 shows that $I(a, \varphi)$ is a well defined distribution and, if $a$ has compact support in $x$, then so does $I(a, \varphi)$.

Let $\gamma=\left(x_{0}, \xi_{0}\right)$. Then, by Lemma 3.1.10 or the definition of admissible, we can assume that

$$
\begin{equation*}
\Lambda_{\varphi}=\left\{\left(\partial_{\xi^{\prime}} H\left(x^{\prime \prime}, \xi^{\prime}\right), x^{\prime \prime}, \xi^{\prime},-\partial_{x^{\prime \prime}} H\left(x^{\prime \prime}, \xi^{\prime}\right):\left(x^{\prime \prime}, \xi^{\prime}\right) \in W\right\}\right. \tag{4.4.2}
\end{equation*}
$$

for some $W$ an open neighborhood of $\left(x_{0}^{\prime \prime}, \xi_{0}^{\prime}\right)$. Consider

$$
\begin{equation*}
e^{\frac{i}{h} H\left(x^{\prime \prime}, \xi^{\prime}\right)} \mathcal{F}_{h, x^{\prime}}(I(a, \varphi))\left(\xi^{\prime}\right)=(2 \pi h)^{-(d+2 N) / 4} \int e^{\frac{i}{h}\left(\varphi(x, \theta)+H\left(x^{\prime \prime}, \xi^{\prime}\right)-\left\langle x^{\prime}, \xi^{\prime}\right\rangle\right)} a(x, \theta) d \theta d x^{\prime} \tag{4.4.3}
\end{equation*}
$$

and use stationary phase to evaluate the integral. The exponent has a critical point if

$$
\varphi_{x^{\prime}}^{\prime}(x, \theta)=\xi^{\prime}, \quad \varphi_{\theta}^{\prime}=0
$$

Hence, $(x, \xi) \in \Lambda_{\varphi}$ and $x^{\prime}=H_{\xi^{\prime}}\left(x^{\prime \prime}, \xi^{\prime}\right), \xi^{\prime \prime}=-H_{x^{\prime \prime}}^{\prime}\left(x^{\prime \prime}, \xi^{\prime}\right)$. Since $\varphi$ is a nondegenerate phase function and $\Lambda$ has the form (4.4.2), the maps

$$
C=\left\{(x, \theta) ; \varphi_{\theta}^{\prime}=0\right\} \ni(x, \theta) \mapsto\left(x, \varphi_{x}^{\prime}\right) \in \Lambda \quad \text { and } \quad \Lambda \ni(x, \xi) \mapsto\left(x^{\prime \prime}, \xi^{\prime}\right)
$$

are diffeomorphisms (see Lemma 3.1.15). In particular, this implies that $j: C_{\varphi} \ni(x, \theta) \mapsto$ $\left(x^{\prime \prime}, \varphi_{x^{\prime}}^{\prime}\right)$ is a diffeomorphism and hence $d \varphi_{x^{\prime}}^{\prime}=d \varphi_{\theta}^{\prime}=d x^{\prime \prime}=0$ implies $d x=d \theta=0$. That is, $\Phi$ is nonsingular. Now, since $\varphi \in S^{1}\left(\mathbb{R}^{d} \times \overline{\mathbb{R}}^{N}\right)$, $|\operatorname{det} \Phi|^{-1 / 2} \in S^{(N-k) / 2}(U)$ where $U$ is a neighborhood of $C_{\varphi}$ and hence $a(x, \theta)|\operatorname{det} \Phi|^{-1 / 2} \in S^{m+d / 4-k / 2}(U)$. Since $\Lambda$ has the form (4.4.2), it is not hard to see that this remains true for the restriction to $C_{\varphi}$ regarded as a function of $\left(x^{\prime \prime}, \xi^{\prime}\right)$ (see, for example [42, Lemma 25.1.6]).

For $\left(x^{\prime \prime}, \xi^{\prime}\right) \notin W$, integration by parts shows that

$$
e^{\frac{i}{h} H\left(x^{\prime \prime}, \xi^{\prime}\right)} \mathcal{F}_{h, x^{\prime}}(I(a, \varphi))\left(\xi^{\prime}\right)=O\left(h^{\infty}\left\langle\xi^{\prime}\right\rangle^{-\infty}\right)
$$

Now, suppose $x^{\prime \prime} \in \pi W$ where $\pi: W \subset \mathbb{R}^{d-k} \times \mathbb{R}^{k} \rightarrow \mathbb{R}^{d-k}$ is the usual projection. Then if $\pi^{-1}\left(x^{\prime \prime}\right)$ is bounded the diffeomorphicity of $j$ together with integration by parts with respect to $\theta$ shows that the integral over $|\theta| \gg 1$, in $I(a, \varphi)$ is $O\left(h^{\infty}\left\langle\xi^{\prime}\right\rangle^{-\infty}\right)$. On the other hand, if $\pi^{-1}\left(x^{\prime \prime}\right)$ is unbounded, then

$$
\begin{equation*}
\left|\partial_{x^{\prime}} \varphi(x, \theta)\right| \geq C|\theta|, \quad|\theta|>M \tag{4.4.4}
\end{equation*}
$$

Define $\psi \in C^{\infty}(\mathbb{R})$ with $\psi \equiv 1$ on $\mathbb{R} \backslash((-2,2))$ and $\psi \equiv 0$ on $(-1,1)$. Then, letting $f_{x^{\prime \prime}}\left(x^{\prime}\right):=\left(\varphi\left(x^{\prime}, x^{\prime \prime}, \theta\right)-\left\langle x^{\prime}, \xi^{\prime}\right\rangle\right) /\left\langle\psi\left(\left|\xi^{\prime}\right|\right)\right| \xi^{\prime}|+\psi(|\theta|)| \theta| \rangle$. Then $f$ is bounded in $C^{\infty}$ and for $(x, \theta) \in \operatorname{supp} a, 4.4 .4$ shows that

$$
\begin{array}{ll}
\left|f^{\prime}\left(x^{\prime}\right)\right| \geq\left(\left|\xi^{\prime}\right|-C_{1}|\theta|\right) /\langle | \xi^{\prime}|+|\theta|\rangle & |\theta| /|\xi| \ll 1 \\
\left|f^{\prime}\left(x^{\prime}\right)\right| \geq\left(C_{2}|\theta|-\left|\xi^{\prime}\right|\right) /\langle | \xi^{\prime}|+|\theta|\rangle & |\xi| /|\theta| \ll 1 .
\end{array}
$$

Thus, applying Lemma 4.4.3 with $\omega=\langle | \xi^{\prime}|+|\theta|\rangle$. we have

$$
\begin{equation*}
\left|\int e^{\frac{i}{h}\left(\varphi(x, \theta)-\left\langle x^{\prime}, \xi^{\prime}\right\rangle\right)} a(x, \theta) d x\right| \leq C_{N} h^{N}\langle | \xi^{\prime}|+|\theta|\rangle^{-N} \quad|\theta|>C\left|\xi^{\prime}\right| \text { or }\left|\xi^{\prime}\right|>C|\theta| \tag{4.4.5}
\end{equation*}
$$

Now, choose $\chi \in C_{c}^{\infty}\left(\mathbb{R}^{N}\right)$. with $\chi \equiv 1$ on $|\theta|<C$. Then,

$$
e^{\frac{i}{h} H\left(x^{\prime \prime}, \xi^{\prime}\right)} \mathcal{F}_{h, \xi^{\prime}}(u)\left(x^{\prime \prime}, \xi^{\prime}\right)-U\left(x^{\prime \prime}, \xi^{\prime}\right)=O\left(h^{\infty}\left\langle\xi^{\prime}\right\rangle^{-\infty}\right)
$$

where

$$
U\left(x^{\prime \prime}, \xi^{\prime}\right)=(2 \pi h)^{-(d+2 N) / 4} \int e^{\frac{i}{h}\left(\varphi(x, \theta)+H\left(x^{\prime \prime}, \xi^{\prime}\right)-\left\langle x^{\prime}, \xi^{\prime}\right\rangle\right)} \chi\left(\theta /\left\langle\xi^{\prime}\right\rangle\right) a(x, \theta) d x d \theta
$$

Near the point $\left(x_{0}, \theta_{0}\right) \in C_{\varphi}, C_{\varphi} \ni(x, \theta) \mapsto\left(x^{\prime \prime}, \partial_{x^{\prime}} \varphi\right)$ is a diffeomorphism and hence write $x=x\left(x^{\prime \prime}, \xi^{\prime}\right)$ and $\theta=\theta\left(x^{\prime \prime}, \xi^{\prime}\right)$. Then, differentiating the phase with respect to $\left(x^{\prime \prime}, \xi^{\prime}\right)$,

$$
\partial_{x^{\prime \prime} \xi^{\prime}}\left(\varphi\left(x\left(x^{\prime \prime}, \xi^{\prime}\right), \theta\left(x^{\prime \prime}, \xi^{\prime}\right)\right)+H\left(x^{\prime \prime}, \xi^{\prime}\right)-\left\langle x^{\prime}\left(x^{\prime \prime}, \xi^{\prime}\right), \xi^{\prime}\right\rangle\right)=0
$$

Thus, on $C_{\varphi}$, the critical value of the phase is a constant $A$ depending on the choice of $\varphi$ and $H$ parametrizing $\Lambda$.

Changing variables, letting $t=\left\langle\xi^{\prime}\right\rangle$ and $\eta^{\prime}=\xi^{\prime} / t, \theta=t \theta$, we have

$$
U\left(x^{\prime \prime}, \xi\right):=(2 \pi h)^{-(d+2 N) / 4} \iint e^{\left.\frac{i t}{h} \frac{\varphi(x, t \theta)+H\left(x^{\prime \prime}, t \eta^{\prime}\right)}{t}-\left\langle x^{\prime}, \eta^{\prime}\right\rangle\right)} \chi(\theta) a(x, t \theta) t^{N} d x^{\prime} d \theta
$$

Now, there is only one critical point in the support of the integrand and using stationary phase, since $\chi \equiv 1$ in a neighborhood of the stationary point, the leading term is

$$
(2 \pi h)^{-(d-2 k) / 4} e^{\frac{i A}{h}} a(x, t \theta) e^{\frac{\pi i}{4} \operatorname{sgn}(\Phi)} t^{(N-k) / 2}|\operatorname{det} \Phi|^{-1 / 2} .
$$

The $n^{\text {th }}$ term will then have a factor of $h^{n} t^{-n}$ and a linear combination of up to $2 n$ derivatives of $a(x, t \theta)$ with respect to $x$ and $\theta$. Hence, it is in $h^{-(d-2 k) / 4+n(1-2 \delta)} S_{\delta}^{m+d / 4-k / 2-n}$ and we have an asymptotic expansion for $e^{\frac{i}{h} H\left(x^{\prime \prime}, \xi^{\prime}\right)} \mathcal{F}_{h, x^{\prime}}(u)\left(x^{\prime \prime}, \xi^{\prime}\right)$.

Now, let $\varphi_{1} \in S^{1}\left(\mathbb{R}^{d} \times \overline{\mathbb{R}}^{M}\right)$ be another nondegenerate phase function parametrizing $\Lambda$ in a neighborhood of $\gamma$. We seek to find a symbol $b$ so that $I(a, \varphi)=I(b, \varphi)+O_{C^{\infty}}\left(h^{\infty}\right)$. we use the first part of the proposition to write

$$
v=\mathcal{F}_{h, x^{\prime}}(u)\left(x^{\prime \prime}, \xi^{\prime}\right) e^{\frac{i}{h} H\left(x^{\prime \prime}, \xi^{\prime}\right)} \in h^{-(d+2 k) / 4} S_{\delta}^{m+d / 4-k / 2}\left(\mathbb{R}^{d}\right)
$$

having support in a small neighborhood of $\left(x_{0}^{\prime \prime}, \xi_{0}^{\prime}\right)$. Let $\psi \in S^{1}\left(\mathbb{R}^{d} \times \overline{\mathbb{R}}^{L}\right)$ be defined in a neighborhood of $\left(x_{0}, \theta_{0}\right)$ with $\psi(x, \theta)=\partial_{x} \varphi$ on $C_{\varphi}$. Then let

$$
b_{0}(x, \theta)=(2 \pi h)^{-d / 4} v \circ \psi(x, \theta) e^{-i A / h} e^{-\pi i / 4 \operatorname{sgn} \Phi}|\operatorname{det} \Phi|^{1 / 2} \in S_{\delta}^{m+(n-2 M) / 4}
$$

Then, define $u_{0}=I\left(b_{0}, \varphi_{1}\right)$. Then it follows that

$$
u-u_{0}=I\left(a_{1}, \varphi\right), \quad a_{1} \in h^{1-2 \delta} S^{m+(d-2 N) / 4-1}
$$

Repeating these arguments, we get an asymptotic sum

$$
b \sim \sum b_{j}, \quad b_{j} \in h^{j(1-2 \delta)} S_{\delta}^{m+(n-2 M) / 4-j}
$$

such that $I(a, \varphi)=I\left(b, \varphi_{1}\right)+O_{C \infty}\left(h^{\infty}\right)$.
To see the wavefront set condition, suppose $\left(x_{0}, \xi_{0}\right) \notin \Lambda_{\varphi}$ (thought of as a subset of $\left.\mathbb{R}^{d} \times \overline{\mathbb{R}}^{d}\right)$ and let $U \times V$ be a neighborhood of $\left(x_{0}, \xi_{0}\right)$ such that $U \times V \cap \Lambda_{\varphi}=\emptyset$. Then let $\chi \in C_{c}^{\infty}(U)$ and consider

$$
\mathcal{F}_{h}\left(\chi(x) I_{a, \varphi}\right)(\xi) \quad \xi \in V
$$

Then, by the same (even slightly simpler) arguments used to get 4.4.5), we can reduce to considering

$$
B(\xi)=(2 \pi h)^{-d+2 N / 4} \int e^{\frac{i}{h}(\varphi(x, \theta)-\langle x, \xi\rangle)} \chi(x) a(x, \theta) \chi(\theta /\langle\xi\rangle) d \theta d x
$$

Hence, integration by parts proves that

$$
B(\xi)=O\left(h^{\infty}\langle\xi\rangle^{-\infty}\right), \quad \xi \in V
$$

Next we give a lemma relating the microsupports of $I(a, \varphi)$ and $a$.
Lemma 4.4.5. Suppose that $a \in S_{\delta}^{\text {comp }}\left(U_{\varphi}\right)$ with $\operatorname{supp} a$ is contained in some $h$-dependent compact set $K(h) \Subset U_{\varphi}$ and $\gamma<1 / 2$. Then,

$$
\operatorname{MS}_{\mathrm{h}}(I(a, \varphi)) \subset E(h)
$$

$$
E(h):=\left\{\left(x, \partial_{x} \varphi(x, \theta)\right) \mid \text { there exists }(y, \omega) \in K(h) \cap C_{\varphi} \text { with } d((x, \theta),(y, \omega))<h^{\gamma}\right\} .
$$

Proof. Suppose that $\left(x_{0}, \xi_{0}\right) \notin E$. Then, consider

$$
\mathcal{F}_{h}\left(\chi\left(\left(x-x_{0}\right) h^{-\gamma}\right) I_{a, \varphi}\right)(\xi)=(2 \pi h)^{-d+2 N / 4} \int e^{\frac{i}{h}(\varphi(x, \theta)-\langle x, \xi\rangle)} a(x, \theta) \chi\left(\left(x-x_{0}\right) h^{-\gamma}\right) d \theta d x
$$

where $\chi \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$. But, since $\left(x_{0}, \xi_{0}\right) \in E$. $|d(\varphi-\langle x, \xi\rangle)| \geq h^{\gamma}$. Hence, integration by parts after adding a cutoff $\chi\left(\epsilon^{-1}\left(\xi-\xi_{0}\right) h^{-\gamma}\right)$, gives a gain of $h^{1-\gamma-\max (\gamma, \delta)}$. Hence, repeated integration by parts gives the result.
Definition 4.4.6. Let $\Lambda \subset \bar{T}^{*} M$ be an admissible Lagrangian. We say that $u \in I_{\delta}^{m}(\Lambda)$ if $\mathrm{WF}_{\mathrm{h}}(u) \subset \Lambda$ and if $\gamma \in \Lambda$ and $\varphi \in S^{1}(V)$ for $V \subset M \times \overline{\mathbb{R}}^{N}$ is a non-degenerate phase function so that $\Lambda_{\varphi}=\Lambda$ in a neighborhood, $U$ of $\gamma$ then there exists $a \in S_{\delta}^{m+(d-2 N) / 4}\left(M \times \mathbb{R}^{N}\right)$ such that $\mathrm{WF}_{\mathrm{h}}(u-I(a, \varphi)) \cap U=\emptyset$.

It will be important also to consider the case that $\varphi$ is only a clean phase function when we analyze the composition of Fourier integral operators. To this end, we prove
Proposition 4.4.7. Let $\Lambda \subset \bar{T}^{*} \mathbb{R}^{d}$ be an admissible Lagrangian submanifold and let $\gamma \in \Lambda$. Let $\varphi$ be a clean phase function with excess e in an open neighborhood of $\left(x_{0}, \theta_{0}\right) \in \mathbb{R}^{d} \times \overline{\mathbb{R}}^{d+N}$, $N \in \mathbb{N}_{0}$ such that $\Lambda=\Lambda_{\varphi}$ in a neighborhood of $\gamma$. If $a \in S_{\delta}^{m+(d-2 N-2 e) / 4}\left(\mathbb{R}^{d} \times \mathbb{R}^{N}\right)$ such that $\operatorname{supp} a \Subset V$, then

$$
\begin{equation*}
I(a, \varphi):=(2 \pi h)^{-(d+2 N-2 e) / 4} \int e^{\frac{i}{h} \varphi(x, \theta)} a(x, \theta) d \theta \tag{4.4.6}
\end{equation*}
$$

has $I(a, \varphi) \in I_{\delta}^{m}(\Lambda)$. Moreover, if $\left(x^{\prime}, x^{\prime \prime}\right),\left(\xi^{\prime}, \xi^{\prime \prime}\right) \in \mathbb{R}^{d-k} \times \mathbb{R}^{k}$, and

$$
\Lambda=\left\{\left(\partial_{\xi^{\prime}} H\left(x^{\prime \prime}, \xi^{\prime}\right), x^{\prime \prime}, \xi^{\prime},-\partial_{x^{\prime \prime}} H\left(x^{\prime \prime}, \xi^{\prime}\right)\right)\right\},
$$

then then there exists a constant $A$ depending only on $\varphi$ and $H$ such that

$$
\begin{align*}
& e^{\frac{i}{h} H\left(x^{\prime \prime}, \xi^{\prime}\right)} \mathcal{F}_{h, x^{\prime}}(u)\left(x^{\prime \prime}, \xi^{\prime}\right) \\
& \quad-(2 \pi h)^{-(d-2 k) / 4} \int_{C_{\left(x^{\prime \prime}, \xi^{\prime}\right)}} a(x, \theta) e^{\frac{i}{h} A} e^{i \frac{\pi}{4} \operatorname{sgn}(\Phi)}|\operatorname{det} \Phi|^{-\frac{1}{2}} d \theta^{\prime \prime} \in h^{1-2 \delta-(d-2 k) / 4} S_{\delta}^{m+d / 4-k / 2-1} . \tag{4.4.7}
\end{align*}
$$

Here $C_{\left(x^{\prime \prime}, \xi\right)}=\left\{\left(x^{\prime}, x^{\prime \prime}, \theta\right): \varphi_{\theta}^{\prime}=0, \partial_{x^{\prime}} \varphi(x, \theta)=\xi^{\prime}\right\}$ and $\theta=\left(\theta^{\prime}, \theta^{\prime \prime}\right)$ is a splitting of coordinates so that $C_{\left(x^{\prime \prime}, \xi^{\prime}\right)} \ni(x, \theta) \mapsto \theta^{\prime \prime}$ has bijective differential and

$$
\Phi=\left(\begin{array}{cc}
\varphi_{x^{\prime} x^{\prime}}^{\prime \prime} & \varphi_{x^{\prime} \theta^{\prime}}^{\prime \prime} \\
\varphi_{\theta^{\prime} x^{\prime}}^{\prime \prime} & \varphi_{\theta^{\prime} \theta^{\prime}}^{\prime \prime}
\end{array}\right)
$$

Moreover,

$$
A=\varphi\left(x\left(x^{\prime \prime}, \xi^{\prime}, \theta^{\prime \prime}\right), \theta^{\prime}\left(x^{\prime \prime}, \xi^{\prime}, \theta^{\prime \prime}\right), \theta^{\prime \prime}\right)+H\left(x^{\prime \prime}, \xi^{\prime}\right)-\left\langle x^{\prime}\left(x^{\prime \prime}, \xi^{\prime}, \theta^{\prime \prime}\right), \xi^{\prime}\right\rangle
$$

Conversely, if $u \in I_{\delta}^{m}(\Lambda), u$ can be written in the form $I(a, \varphi)$ in a neighborhood of any point $\left(x_{0}, \xi_{0}\right) \in \Lambda$.

Proof. We simply address what must be changed from Proposition 4.4.2. First, we still have (4.4.5), so we need only consider $U\left(x^{\prime \prime}, \xi^{\prime}\right)$. Now, $C_{\varphi}$ is locally a manifold of dimension $e+d$ and by Lemma 3.1.15 the map $C_{\varphi} \ni(x, \theta) \mapsto\left(x^{\prime \prime}, \varphi_{x^{\prime}}^{\prime}\right)$ has a fiber $C_{\left(x^{\prime \prime}, \eta^{\prime}\right)}$ over $\left(x^{\prime \prime}, \eta^{\prime}\right)$ where $x^{\prime}=H_{\xi^{\prime}}\left(x^{\prime \prime}, t \eta\right)$ and $\xi^{\prime \prime}=-H_{x^{\prime \prime}}^{\prime}\left(x^{\prime \prime}, t \eta\right)$. Now, since $d \varphi_{\theta}^{\prime}=d \varphi_{x^{\prime}}^{\prime}=d x^{\prime \prime}=0$ on the tangent space to $C_{\left(x^{\prime \prime}, \eta\right)}$, we can split the $\theta$ variables into $\left(\theta^{\prime}, \theta^{\prime \prime}\right)$ with the required properties. Then the Hessian of $\frac{\varphi(x, t \theta)+H\left(x^{\prime \prime}, t \eta\right)}{t}-\left\langle x^{\prime}, \eta^{\prime}\right\rangle$ with respect to $\left(x^{\prime}, \theta^{\prime}\right)$ is not zero and applying the principle of stationary phase in these variables together with the fact that

$$
\psi\left(x^{\prime \prime}, \xi^{\prime}, \theta^{\prime \prime}\right)=\varphi\left(x\left(x^{\prime \prime}, \xi^{\prime}, \theta^{\prime \prime}\right), \theta^{\prime}\left(x^{\prime \prime}, \xi^{\prime}, \theta^{\prime \prime}\right), \theta^{\prime \prime}\right)+H\left(x^{\prime \prime}, \xi^{\prime}\right)-\left\langle x^{\prime}\left(x^{\prime \prime}, \xi^{\prime}, \theta^{\prime \prime}\right), \xi^{\prime}\right\rangle
$$

has $\psi_{\xi^{\prime}}^{\prime}=\psi_{\theta^{\prime \prime}}^{\prime}=\psi_{x^{\prime \prime}}^{\prime}=0$ proves the proposition.
We now define the principal symbol of a distribution $u \in I_{\delta}^{m}(\Lambda)$. Let $\varphi(x, \theta)$ be a nondegenerate phase function. Define a $d$-form $d_{\varphi}$ on $C_{\varphi}$ by

$$
d_{\varphi} \wedge d\left(\partial_{\theta_{1}} \varphi\right) \wedge \cdots \wedge d\left(\partial_{\theta_{N}} \varphi\right)=d x_{1} \wedge \cdots \wedge d x_{d} \wedge d \theta_{1} \wedge \ldots d \theta_{d}
$$

Then if $\lambda_{1}, \ldots, \lambda_{d}$ are coordinates on $C_{\varphi}$ extended to smooth functions in an open neighborhood of a point in $C_{\varphi}, d_{\varphi}=f d \lambda_{1} \wedge \cdots \wedge d \lambda_{d}$ where

$$
f=\left(\operatorname{det}\left(\begin{array}{cc}
\partial_{x} \lambda & \partial_{\theta} \lambda \\
\varphi_{x \theta}^{\prime \prime} & \varphi_{\theta \theta}^{\prime \prime}
\end{array}\right)\right)^{-1}
$$

Notice that $d_{\varphi}$ does not depend on the choice of $\lambda$, but does depend on the choice of coordinates $\left(x_{i}\right)$. Now, we assume that

$$
\Lambda_{\varphi}=\left\{\left(\partial_{\xi^{\prime}} H\left(x^{\prime \prime}, \xi^{\prime}\right), x^{\prime \prime}, \xi^{\prime},-\partial_{x^{\prime \prime}} H\left(x^{\prime \prime}, \xi^{\prime}\right)\right\}\right.
$$

and use $\left(x^{\prime \prime}, \xi^{\prime}\right)$ as local coordinates on $C_{\varphi}$. Then, since $\xi_{i}=\partial_{x_{i}} \varphi$, we have that

$$
d_{\varphi}=(\operatorname{det} \Phi)^{-1} d x_{k+1} \wedge \cdots \wedge d x_{d} \wedge d \xi_{1} \wedge \ldots d \xi_{k}
$$

where $\Phi$ is as in Proposition 4.4.2. Now, define the density

$$
|d \varphi|=|\operatorname{det} \Phi|^{-1}\left|d x_{k+1} \wedge \ldots d x_{d} \wedge d \xi_{1} \wedge \ldots d \xi_{k}\right|
$$

If $\varphi_{1}$ and $\varphi_{2}$ parametrize $\Lambda_{\varphi}$ near $\left(x_{0}, \xi_{0}\right)$, $a_{1} \in S_{\delta}^{m+(d-2 N) / 4}\left(\mathbb{R}^{d} \times \mathbb{R}^{N}\right)$, and $I\left(a_{2}, \varphi_{2}\right) \equiv$ $I\left(a_{1}, \varphi_{1}\right)$, then (4.4.1) shows that

$$
e^{\frac{i}{h} A_{1}} a_{1}\left|d \varphi_{1}\right|^{1 / 2}=e^{\frac{i}{h} A_{2}} a_{2}\left|d \varphi_{2}\right|^{1 / 2} e^{i \frac{\pi}{4}\left(\operatorname{sgn} \Phi_{2}-\operatorname{sgn} \Phi_{1}\right)} \quad \bmod h^{1-2 \delta} S_{\delta}^{m+d / 4-1}\left(\Omega^{1 / 2}\right),
$$

where we have noted that $\left|d \xi_{i}\right|^{1 / 2}$ is homogeneous of degree $1 / 2$.
Now, in order to handle the fact that $d_{\varphi}$ depends on the coordinates $\left(x_{i}\right)$, we consider $u=$ $I(a, \varphi)$ as a $\frac{1}{2}$-density distribution. That is, in new coordinates $\left(\tilde{x}_{i}\right), \tilde{u}(\tilde{x})=|D x / D \tilde{x}|^{1 / 2} u(x)$, but then

$$
\begin{gathered}
\tilde{u}(\tilde{x})=(2 \pi h)^{-d+2 N) / 4} \int e^{\frac{i}{h} \tilde{\varphi}(\tilde{x}, \theta)} \tilde{a}(\tilde{x}, \theta) d \theta \\
\tilde{\varphi}(\tilde{x}, \theta)=\varphi(x, \theta), \quad \tilde{a}(\tilde{x}, \theta)=|D x / D \tilde{x}|^{1 / 2} a(x, \theta) .
\end{gathered}
$$

But, $\left|d_{\varphi}\right|=|D x / D \tilde{x}|\left|d_{\tilde{\varphi}}\right|$, so

$$
\begin{equation*}
e^{\frac{i}{h} \tilde{A}} \tilde{a}\left|d_{\tilde{\varphi}}\right|^{1 / 2}=e^{\frac{i}{h} A} a\left|d_{\varphi}\right|^{1 / 2} \tag{4.4.8}
\end{equation*}
$$

and hence is invariant under local coordinate changes. Finally, notice that an $(m \times m)$ nonsingular matrix has has signature congruent to $m \bmod 2$ Hence,

$$
\operatorname{sgn} \Phi_{2}-\operatorname{sgn} \Phi_{1} \equiv 0 \quad \bmod 2
$$

We now define the Maslov bundle (see also [41, Chapter XXI]) $\mathcal{L}$ over $\Lambda$ to be the complex line bundle with transition functions

$$
e^{i \frac{\pi}{4}\left(\operatorname{sgn} \Phi_{1}^{\prime \prime}-\operatorname{sgn} \Phi_{2}^{\prime \prime}\right)}
$$

in $\Lambda_{\varphi_{1}} \cap \Lambda_{\varphi_{1}}$ associated to a change of phase functions and

$$
e^{i \frac{\pi}{4}\left(\operatorname{sgn} \Phi_{x, \theta}-\operatorname{sgn} \Phi_{y, \theta}\right)}
$$

associated to a change of local coordinates. We have thus proved the following lemma
Lemma 4.4.8. If $\Lambda$ is exact, i.e. $\left.\sigma\right|_{\Lambda}=\left.d \psi\right|_{\Lambda}$, there exists a bijective map

$$
\sigma: I_{\delta}^{m}(\Lambda) / h^{1-2 \delta} I_{\delta}^{m}(\Lambda) \rightarrow S_{\delta}^{m+d / 4}\left(\Omega^{1 / 2} \otimes \mathcal{L}\right)
$$

called the symbol map.
Remark: The exactness of $\Lambda$ is used to fix a choice of $A$ in (??).

## Fourier Integral Operators

Let $\Lambda$ be a Lagrangian submanifold of $T^{*} M_{1} \times T^{*} M_{2}$ (with symplectic form $\sigma_{1}+\sigma_{2}$ ). Then we define the canonical relation

$$
C=\Lambda^{\prime}=\{(x, \xi, y,-\eta):(x, \xi, y, \eta) \in \Lambda\} .
$$

Notice that if $A: C_{c}^{\infty}\left(M_{1}\right) \rightarrow \mathcal{D}^{\prime}\left(M_{2}\right)$ is a linear operator with kernel $K \in I^{m}\left(M_{1} \times M_{2}, \Lambda\right)$ and suppose that $\Lambda \subset T^{*} Y \backslash\{0\} \times T^{*} X \backslash\{0\}$ then

$$
\mathrm{WF}_{\mathrm{h}}(A u) \subset C\left(\mathrm{WF}_{\mathrm{h}}(u)\right)
$$

by Corollary 4.3.17. With this in mind we define Fourier integral operators.
Definition 4.4.9. Suppose $M_{1}$ and $M_{2}$ are compact manifolds and $C^{\prime} \subset T^{*} M_{1} \times T^{*} M_{2}$ is an admissible Lagrangian submanifold. Then the set of operators with Schwartz kernels $K \in I_{\delta}^{m}\left(M_{1} \times M_{2} ; C\right)$ is the set of Fourier integral operators of order $m$ and class $\delta$ associated to $C$.

Notice that if $A \in I_{\delta}^{m}\left(M_{1} \times M_{2}, C\right)$, then its adjoint $A^{*} \in I_{\delta}^{m}\left(M_{2} \times M_{2}, C^{-1}\right)$ and the symbol of $A^{*}$ is the conjugate of that of $A$ in any local coordinates.

## Composition

We now study the composition of two Fourier integral operators under certain assumptions on the composition of their associated relations. Let $C$ be a canonical relation on $\bar{T}^{*} M_{1} \times \bar{T}^{*} M_{2}$. Then, define

$$
\begin{aligned}
& C_{M_{1}}:=\left\{(x, \xi) \in \partial \bar{T}^{*} M_{1}:(x, \xi, y, 0) \in C\right\} \\
& C_{M_{2}}:=\left\{(y, \eta) \in \partial \bar{T}^{*} M_{2}:(x, 0, y, \eta) \in C\right\}
\end{aligned}
$$

Then we have the following proposition
Proposition 4.4.10. Suppose that $A_{1} \in I_{\delta}^{m_{1}}\left(M_{1} \times M_{2}, C_{1}\right)$ and $A_{2} \in I_{\delta}^{m_{2}}\left(M_{2} \times M_{3}, C_{2}\right)$ such that $C_{1} \circ C_{2}$ is clean with excess $e$ and that $\left(C_{1}\right)_{M_{2}} \cap\left(C_{2}\right)_{M_{2}}=\emptyset$. Then $A_{1} A_{2} \in$ $h^{-e / 2} I_{\delta}^{m_{1}+m_{2}+e / 2}\left(M_{1} \times M_{3}, C_{1} \circ C_{2}\right)$ and its symbol at $\gamma \in C=C_{1} \circ C_{2}$ is given by

$$
\sigma\left(A_{1} A_{2}\right)=(2 \pi h)^{-e / 2} \int_{C_{\gamma}} \sigma\left(A_{1}\right) \times \sigma\left(A_{2}\right)
$$

$C_{\gamma}$ is the fiber over $\gamma$ of the intersection of $C_{1} \times C_{2}$ with $T^{*} M_{1} \times \Delta\left(T^{*} M_{2}\right) \times T^{*} M_{3}$.
Proof. We can reduce to the local situation by use of a partition of unity, so we assume $M_{i}=\mathbb{R}^{d_{i}}$ and

$$
\begin{aligned}
& A_{1}(x, y)=(2 \pi h)^{-\left(d_{1}+d_{2}+2 N_{1}\right) / 4} \int e^{\frac{i}{h} \varphi(x, y, \theta)} a_{1}(x, y, \theta) d \theta \\
& A_{2}(x, y)=(2 \pi h)^{-\left(d_{2}+d_{3}+2 N_{2}\right) / 4} \int e^{\frac{i}{h} \phi(y, z, \tau)} a_{2}(y, z, \tau) d \tau
\end{aligned}
$$

where $\phi$ and $\psi$ are nondegenerate phase functions in neighborhoods of $\left(x_{0}, y_{0}, \theta_{0}\right) \in M_{1} \times$ $M_{2} \times \overline{\mathbb{R}}^{N_{1}}$ and $\left(y_{0}, z_{0}, \tau_{0}\right) \in M_{2} \times M_{3} \times \overline{\mathbb{R}}^{N_{2}}$ respectively. Moreover, $\varphi$ parametrizes $C_{1}$ in a neighborhood of $\left(x_{0}, \xi_{0}, y_{0}, \eta_{0}\right) \in \bar{T}^{*} M_{1} \times \bar{T}^{*} M_{2}$ and $\phi$ parametrizes $C_{2}$ in a neighborhood of $\left(y_{0}, \eta_{0}, z_{0}, \zeta_{0}\right) \in \bar{T}^{*} M_{2} \times \bar{T}^{*} M_{3}$. Hence, we have

$$
\begin{aligned}
\varphi_{\theta}^{\prime} & =0, & \varphi_{x}^{\prime} & =\xi_{0},
\end{aligned} \quad \varphi_{y}^{\prime}=-\eta_{0} \quad \text { at }\left(x_{0}, y_{0}, \theta_{0}\right)
$$

Moreover, the amplitude $a_{1} \in S_{\delta}^{m_{1}+\left(d_{1}+d_{2}-2 N_{1}\right) / 4}$ has support in a small neighborhood of $\left(x_{0}, y_{0}, \theta_{0}\right)$ and $a_{2} \in S_{\delta}^{m_{2}+\left(d_{2}+d_{3}-2 N_{2}\right) / 4}$ has support in a small neighborhood of $\left(y_{0}, z_{0}, \tau_{0}\right)$. When $A_{j}$ are rapidly decreasing, we can write

$$
\begin{equation*}
A(x, z)=(2 \pi h)^{-\left(d_{1}+d_{3}+2\left(d_{2}+N_{1}+N_{2}\right)\right) / 4} \int e^{\frac{i}{h} \Phi(x, z, y, \theta, \tau)} a(x, z, y, \theta, \tau) d y d \theta d \tau \tag{4.4.9}
\end{equation*}
$$

where

$$
\begin{aligned}
\Phi(x, z, y, \theta, \tau) & =\varphi(x, y, \theta)+\phi(y, z, \tau) \\
a(x, z, y, \theta, \tau) & =a_{1}(x, y, \theta) a_{2}(y, z, \tau)
\end{aligned}
$$

By Lemma 3.1.23, we have that $\Phi$ is a clean phase with excess $e$ parametrizing $C=C_{1} \circ C_{2}$ in a neighborhood of $\left(x_{0}, z_{0}, y_{0}, \theta_{0}, \tau_{0}\right)$. So, once we reduce to a region where $a$ is a well behaved symbol, Proposition 4.4.7 will show that $A_{1} A_{2} \in h^{-e / 2} I_{\delta}^{m_{1}+m_{2}+e / 2}\left(M_{1} \times M_{3}, C\right)$.

Notice that $a_{1} a_{2}$ is not immediately a symbol since differentiating with respect to $\theta$ or $\tau$ only improves the symbol by $\langle\theta\rangle^{-1}$ or $\langle\tau\rangle^{-1}$ respectively rather than $\langle | \theta|+|\tau|\rangle^{-1}$. To remedy this, we show that the integral (4.4.8) over the region where $\theta$ and $\tau$ are not of the same magnitude or both bounded is residual. Notice that $\left(C_{1}\right)_{M_{2}} \cap\left(C_{2}\right)_{M_{2}}=\emptyset$ implies that either $\eta_{0} \neq 0$ or at least one of $\tau_{0}, \theta_{0} \notin S^{d-1}$. If either $\tau_{0}$ or $\theta_{0}$ is bounded, then clearly $a_{1} a_{2}$ has the required symbol property. Thus, we can assume $\eta_{0} \neq 0$. Since $\varphi \in S^{1}\left(\left(M_{1} \times M_{2}\right) \times \overline{\mathbb{R}}^{N_{1}}\right)$ and $\phi \in S^{1}\left(\left(M_{2} \times M_{3}\right) \times \overline{\mathbb{R}}^{N_{2}}\right)$,

$$
\begin{aligned}
& \varphi(x, y, \theta)=\varphi_{1}(x, y, \theta)+O_{S^{0}}(1) \\
& \phi(y, z, \tau)=\phi_{1}(y, z, \tau)+O_{S^{0}}(1)
\end{aligned}
$$

where $\varphi_{1}$ and $\phi_{1}$ are homogeneous of degree 1 in $\theta$ and $\tau$ respectively. Thus, since $\varphi$ and $\phi$ are non-degenerate phase functions, we may assume that $a_{1}$ and $a_{2}$ have small enough support in the base variables so that there exists $M$ with

$$
\begin{aligned}
& \left|\partial_{y} \varphi\right| \geq C|\theta| \text { on } \operatorname{supp} a_{1} \cap\{|\theta|>M\} \\
& \left|\partial_{y} \phi\right| \geq C|\tau| \text { on } \operatorname{supp} a_{2} \cap\{|\tau|>M\} .
\end{aligned}
$$

Hence, there exists $C>0$ such that if $\partial_{y} \varphi+\partial_{y} \phi=0$ and $(x, z, y, \theta, \tau) \in \operatorname{supp} a$, then

$$
(\theta, \tau) \in U=\{(\theta, \tau):|\theta|+|\tau|<2 M\} \quad \text { or } \quad\left\{C^{-1}|\tau|<|\theta|<C|\tau|\right\}
$$

Hence, integration by parts in $y$ shows that up to an $O_{C^{\infty}}\left(h^{\infty}\right)$ term, we can replace $a$ by $b=\chi(\theta, \tau) a$ where $\operatorname{supp} \chi \subset U$. Then the symbolic properties of $a_{i}$ imply that $b \in$ $S_{\delta}^{m_{1}+m_{2}+\left(d_{1}+d_{3}-2\left(N_{1}+N_{2}-d_{2}\right)\right) / 4}$. Now, take $\omega=\left(\left\langle\left(|\theta|^{2}+|\tau|^{2}\right)^{1 / 2}\right\rangle y, \theta, \tau\right) \in \mathbb{R}^{d_{2}+N_{1}+N_{2}}$ as new parameters. Then,

$$
|D \omega / D(y, \theta, \tau)|=\left\langle\left(|\theta|^{2}+|\tau|^{2}\right)^{1 / 2}\right\rangle^{d_{2}} .
$$

Hence,

$$
b(x, z, y, \theta, \tau) D(y, \theta, \tau) / D \omega \in S^{m_{1}+m_{2}+\left(d_{1}+d_{3}-2\left(N_{1}+N_{2}+d_{2}\right)\right) / 4}\left(M_{1} \times M_{2} \times \mathbb{R}^{d_{2}+N_{1}+N_{2}}\right)
$$

Now,

$$
\begin{equation*}
B(x, z)=(2 \pi h)^{-\left(d_{1}+d_{3}+2\left(d_{2}+N_{1}+N_{2}\right)\right) / 4} \int e^{\frac{i}{h} \Phi} b d y d \theta d \tau \tag{4.4.10}
\end{equation*}
$$

has $A_{1} A_{2}=B+O_{C}\left(h^{\infty}\right)$ and by Lemma 4.4.4 $B$ depends continuously on $a_{i}$ for $a_{i}$ in any symbol class and the equality remains true for $a_{j}$ rapidly decreasing. Thus Proposition 4.4.7 $A_{1} A_{2} \in h^{-e / 2} I_{\delta}^{m_{1}+m_{2}+e / 2}\left(M_{1} \times M_{3}, C\right)$ as desired. Note that the factor $h^{-e / 2}$ comes from the prefactor in 4.4.6.

To compute the principal symbol, we split the $(y, \theta, \tau)$ variables into $(\cdot)^{\prime}$ and $(\cdot)^{\prime \prime}$ variables so that $\Phi$ is nondegenerate in $\left(y^{\prime}, \theta^{\prime}, \tau^{\prime}\right)$ and the $e$ variables ( $y^{\prime \prime}, \theta^{\prime \prime}, \tau^{\prime \prime}$ ) parametrize the sets $C_{\gamma}$ where $\gamma=(x, \xi, z, \zeta) \in C$. More precisely, we should do this for the $\omega$ variables, but the invariance of the symbol under changes of coordinates shows that this is irrelevant. Let $B_{y^{\prime \prime}, \theta^{\prime \prime}, \tau^{\prime \prime}}$ denote the kernel obtained when we integrate only in ( $y^{\prime}, \theta^{\prime}, \tau^{\prime}$ ) in 4.4.9). The definition of the symbol of $A_{1}$ and $A_{2}$ then show that $\sigma\left(A_{1}\right) \times \sigma\left(A_{2}\right)$ is equal to $\left|d y^{\prime \prime} d \theta^{\prime \prime} d \tau^{\prime \prime}\right|$ times $B_{y^{\prime \prime}, \theta^{\prime \prime}, \tau^{\prime \prime}}$ and hence the formula 4.4.7) gives the result where we note that the $(2 \pi h)^{-e / 2}$ comes from the prefactor in (4.4.6).

Combining this theorem with Lemma 4.1.19 immediately gives us the following corollary
Corollary 4.4.11. Suppose that $A \in I_{\delta}^{0}\left(M_{1} \times M_{2}, C\right)$ where $C$ is locally a canonical graph. Then

$$
\|A\|_{L^{2} \rightarrow L^{2}} \leq \sup |\sigma(A)|+O\left(h^{1 / 4(1-2 \delta)}\right)
$$

Moreover, if $|\sigma(A)| \geq C$, then $A$ is invertible with inverse $A^{-1} \in I_{\delta}^{0}\left(M_{2} \times M_{1}, C^{-1}\right)$.
Proposition 4.4.10 gives us a good way of calculating with FIOs. However, in certain cases, the symbol of $A_{1}$ may vanish on $\pi_{2}\left(C_{2}\right)$ and hence this proposition does not give good information. Rather than studying the general case, we study the particular example which arises most often in applications. That is, the case of $A_{1}$ a pseudodifferential operator with symbol vanishing on the range of $C_{2}$.

Lemma 4.4.12. Let $P \in \Psi^{m}\left(M_{1}\right)$ with principal symbol $p=\sigma(P)$. Suppose that $C$ is $a$ canonical relation from $T^{*} M_{2}$ to $T^{*} M_{1}$ such that $\sigma(P)$ vanishes on the projection of $C$ to $T^{*} M_{1}$. Then if $A \in I_{\delta}^{m^{\prime}}\left(M_{1} \times M_{2}, C^{\prime}\right), P A \in h^{1-2 \delta} I_{\delta}^{m+m^{\prime}-1}\left(M_{1} \times M_{2}, C^{\prime}\right)$ with symbol

$$
i^{-1} h \mathcal{L}_{H_{p}} \sigma(A)+\sigma_{1}(P) \sigma(A)
$$

Here $H_{p}$ has been lifted to $T^{*} M_{1} \times T^{*} M_{2}$.
The Lie derivative of $a \in \Omega^{\kappa}(M)$ along a vector field $X$, denoted $\mathcal{L}_{X}(a)$ is given by

$$
\mathcal{L}_{X} a=\left.\frac{d}{d t}\left(\varphi^{t}\right)^{*} a\right|_{t=0}
$$

where $\varphi^{t}$ is the flow of $v$. Then, in local coordinates, write $a=u|d x|^{\kappa}$. Then $\left(\varphi^{t}\right)^{*} a=u_{t}|d x|^{\kappa}$ where

$$
u_{t}(X)=u\left(\varphi^{t}(x)\right)\left(D \varphi^{t}(x) / D x\right)^{\kappa}
$$

Hence,

$$
\begin{equation*}
\mathcal{L}_{X}\left(u|d x|^{\kappa}\right)=(\langle X, \partial u\rangle+\kappa(\operatorname{div} X) u)|d x|^{\kappa} . \tag{4.4.11}
\end{equation*}
$$

Proof. We need only argue locally, so assume that $\Lambda=C^{\prime}$ is given by

$$
\Lambda=\left\{\left(\partial_{\xi^{\prime}} H, x^{\prime \prime}, \xi^{\prime},-\partial_{x^{\prime \prime}} H, \partial_{\eta^{\prime}} H, y^{\prime \prime}, \eta^{\prime},-\partial_{y^{\prime \prime}} H\right)\right\}
$$

where $H=H\left(x^{\prime \prime}, \xi^{\prime}, y^{\prime \prime}, \eta^{\prime}\right) \in S^{1}\left(\mathbb{R}^{d_{1}-k_{1}} \times \overline{\mathbb{R}}^{k_{1}} \times \mathbb{R}^{d_{2}-k_{2}} \times \overline{\mathbb{R}}^{k_{2}}\right)$.
Then, letting

$$
\varphi\left(x, \xi^{\prime}, y, \eta^{\prime}\right)=\left\langle x^{\prime}, \xi^{\prime}\right\rangle+\left\langle y^{\prime}, \eta^{\prime}\right\rangle-H\left(x^{\prime \prime}, \xi^{\prime}, y^{\prime \prime}, \eta^{\prime}\right)
$$

generate $\Lambda$, the kernel of $P A$ is given by

$$
P A=(2 \pi h)^{-\mu-d_{1}} \int e^{\frac{i}{h}\left(\langle x-w, \zeta\rangle+\varphi\left(w, \xi^{\prime}, y, \eta^{\prime}\right)\right.} p((x+w) / 2, \zeta) a\left(w^{\prime \prime}, y^{\prime \prime}, \xi^{\prime}, \eta^{\prime}\right) d \xi^{\prime} d \eta^{\prime} d w d \zeta
$$

where $\mu=\left(d_{1}+d_{2}+2\left(k_{1}+k_{2}\right)\right) / 4$ and $p$ is a full symbol of $P$. Then, as in the proof of Proposition 4.4.10 we can restrict our attention to the region where the oscillatory variables $\left(\zeta, \xi^{\prime}, \eta^{\prime}\right)$ are all bounded or have comparable size. Thus, assuming without loss that $a$ has compact support in $w$, we may apply the principle of stationary phase in the $(w, \zeta)$ variables. The phase function $\Phi$ is given by

$$
\Phi\left(x, y, w, \xi^{\prime}, \eta^{\prime}, \zeta\right)=\langle x-w, \zeta\rangle+\varphi\left(w, \xi^{\prime}, y, \eta^{\prime}\right)
$$

Hence

$$
\partial_{w} \Phi=\left(\xi^{\prime}-\zeta^{\prime},-\zeta^{\prime \prime}-\partial_{w^{\prime \prime}} H\right), \quad \partial_{\zeta} \Phi=x-w
$$

and

$$
\partial^{2} \Phi=\begin{gathered}
\zeta^{\prime} \\
\zeta^{\prime \prime} \\
w^{\prime} \\
w^{\prime \prime}
\end{gathered}\left(\begin{array}{cccc}
\zeta^{\prime} & \zeta^{\prime \prime} & w^{\prime} & w^{\prime \prime} \\
0 & 0 & -I & 0 \\
0 & 0 & 0 & -I \\
-I & 0 & 0 & 0 \\
0 & -I & 0 & -\partial_{w^{\prime \prime}}^{2} H
\end{array}\right) \quad\left(\partial^{2} \Phi\right)^{-1}=\begin{array}{cccc}
\zeta^{\prime} & \zeta^{\prime \prime} & w^{\prime} & w^{\prime \prime} \\
\zeta^{\prime} \\
\zeta^{\prime \prime} \\
w^{\prime} \\
w^{\prime \prime}
\end{array}\left(\begin{array}{ccc}
0 & 0 & -I \\
0 & \partial_{w^{\prime \prime}}^{2} H & 0 \\
-I & 0 & 0 \\
0 & -I & 0 \\
0
\end{array}\right)
$$

Hence $\operatorname{sgn}\left(\partial^{2} \Phi\right)=2 d_{1}$ and

$$
\begin{aligned}
P A & =(2 \pi h)^{-\mu} \int e^{\frac{i}{h}\left(\left\langle x^{\prime}, \xi^{\prime}\right\rangle+\left\langle y^{\prime}, \eta^{\prime}\right\rangle-H\left(x^{\prime \prime}, \xi^{\prime}, y^{\prime \prime}, \eta^{\prime}\right)\right)} b\left(x^{\prime \prime}, y^{\prime \prime}, \xi^{\prime}, \eta^{\prime}\right) d \xi^{\prime} d \eta^{\prime} \\
b & \left.\sim \sum_{j} h^{j} A_{2 j} p((x+w) / 2, \zeta) a\left(w^{\prime \prime}, y^{\prime \prime}, \xi^{\prime}, \eta^{\prime}\right)\right|_{\substack{x=w, \zeta^{\prime}=\xi^{\prime}, \zeta^{\prime \prime}=-\partial_{x^{\prime \prime}} H}}
\end{aligned}
$$

where $A_{2 j}$ is a differential operator of order $2 j$. Write $p=\sigma(P)+\sigma_{1}(P)+O_{S^{m-2}}\left(h^{2}\right)$. Then $\sigma(P)$ vanishes on $C$, so we can write

$$
\begin{align*}
b= & \left(\sigma(P)+\sigma_{1}(P)\right)\left(x,\left(\xi^{\prime},-\partial_{x^{\prime \prime}} H\right)\right) a\left(x^{\prime \prime}, y^{\prime \prime}, \xi^{\prime}, \eta^{\prime}\right) \\
& +\left.h A_{2} \sigma(P)((x+w) / 2, \zeta) a\left(w^{\prime \prime}, y^{\prime \prime}, \xi^{\prime}, \eta^{\prime}\right)\right|_{(w, \zeta)=\gamma}+O_{S_{\delta}^{m+m^{\prime}+\left(d_{1}+d_{2}\right) / 4-2}}\left(h^{2-4 \delta}\right)  \tag{4.4.12}\\
= & h \tilde{b}_{1}+b_{2}+h b_{3}
\end{align*}
$$

where $\gamma=\left(x,\left(\xi^{\prime},-\partial_{x^{\prime \prime}} H\left(x^{\prime \prime}, \xi^{\prime}, y^{\prime \prime}, \eta^{\prime}\right)\right)\right)$. Now, because of the simple nature of the phase, we can see that (see for example 40, Theorem 7.7.5] )

$$
A_{2}=\frac{1}{2 i}\left\langle\left(\partial^{2} \Phi\right)^{-1} D, D\right\rangle=i\left(\left\langle D_{w}, D_{\zeta}\right\rangle-\frac{1}{2}\left\langle\partial_{w^{\prime \prime}}^{2} H D_{\zeta^{\prime \prime}}, D_{\zeta^{\prime \prime}}\right\rangle\right)
$$

So, letting $p_{0}:=\sigma(P)$, we have

$$
b_{3}=i\left(\frac{1}{2}\left(\left\langle D_{x}, D_{\xi}\right\rangle p_{0}(\gamma)-\left\langle\partial_{x^{\prime \prime}}^{2} H(\gamma) D_{\xi^{\prime \prime}}, D_{\xi^{\prime \prime}}\right\rangle p_{0}(\gamma)\right) a+\left\langle D_{x^{\prime \prime}} a(\gamma), D_{\xi^{\prime \prime}} p_{0}(\gamma)\right\rangle\right)
$$

Since $p_{0}$ vanishes on $C$,

$$
\begin{gathered}
p_{0}(x, \xi)=\left\langle q^{\prime}, x^{\prime}-\partial_{\xi^{\prime}} H\right\rangle+\left\langle q^{\prime \prime}, \xi^{\prime \prime}+\partial_{x^{\prime \prime}} H\right\rangle \\
D_{\xi^{\prime \prime}} p_{0}=-i q^{\prime \prime} \quad D_{x^{\prime}} p_{0}=-i q^{\prime} \quad \text { on } C \\
\left\langle D_{x^{\prime}}, D_{\xi^{\prime}}\right\rangle p_{0}=-i\left\langle D_{\xi^{\prime}}, q^{\prime}\right\rangle+i\left\langle\partial_{\xi^{\prime}}^{2} H D_{x^{\prime}}, q^{\prime}\right\rangle-i\left\langle\partial_{x^{\prime \prime} \xi^{\prime}}^{2} H D_{x^{\prime}}, q^{\prime \prime}\right\rangle \quad \text { on } C \\
\left\langle D_{x^{\prime \prime}}, D_{\xi^{\prime \prime}}\right\rangle p_{0}=i\left\langle\partial_{\xi^{\prime} x^{\prime \prime}}^{2} H D_{\xi^{\prime \prime}}, q^{\prime}\right\rangle-i\left\langle D_{x^{\prime \prime}}, q^{\prime \prime}\right\rangle-i\left\langle\partial_{x^{\prime \prime}}^{2} H D_{\xi^{\prime \prime}}, q^{\prime \prime}\right\rangle \quad \text { on } C \\
\left\langle A D_{\xi^{\prime \prime}}, D_{\xi^{\prime \prime}}\right\rangle p_{0}=-i\left\langle A D_{\xi^{\prime \prime}}, q^{\prime \prime}\right\rangle-i\left\langle A^{t} D_{\xi^{\prime \prime}}, q^{\prime \prime}\right\rangle \quad A \in M_{\left(d_{1}-k_{1}\right) \times\left(d_{1}-k_{1}\right)} \text { on } C .
\end{gathered}
$$

Using this for $b_{3}$ gives on $C$,

$$
\begin{aligned}
b_{3}= & \left\langle q^{\prime \prime}, D_{x^{\prime \prime}} a\right\rangle+\frac{1}{2}\left[\left\langle D_{\xi^{\prime}}-\partial_{\xi^{\prime}}^{2} H D_{x^{\prime}}-\partial_{\xi^{\prime} x^{\prime \prime}}^{2} H D_{\xi^{\prime}}, q^{\prime}\right\rangle\right] a \\
& +\frac{1}{2}\left[\left\langle\partial_{x^{\prime \prime} \xi^{\prime}}^{2} H D_{x^{\prime}}+D_{x^{\prime \prime}}-\partial_{x^{\prime \prime}}^{2} H D_{\xi^{\prime \prime}}, q^{\prime \prime}\right\rangle\right] a
\end{aligned}
$$

so, integrating by parts gives

$$
\begin{gathered}
\int e^{\frac{i}{h} \varphi} \tilde{b}_{1}=\int h^{-1}\left\langle a q^{\prime}, h D_{\xi^{\prime}} e^{\frac{i}{h} \varphi}\right\rangle=\int\left\langle-D_{\xi^{\prime}}, a q^{\prime}\left(x,\left(\xi^{\prime},-\partial_{x^{\prime \prime}} H\right)\right)\right\rangle e^{\frac{i}{h} \varphi}=\int e^{\frac{i}{h} \varphi} b_{1} \\
b_{1}:=-\left\langle q^{\prime}, D_{\xi^{\prime}} a\right\rangle-\left\langle D_{\xi^{\prime}}, q^{\prime}\right\rangle+\left\langle\partial_{\xi^{\prime} x^{\prime \prime}}^{2} H D_{\xi^{\prime \prime}}, q^{\prime}\right\rangle
\end{gathered}
$$

So, combining, we have on $C$

$$
\begin{align*}
b_{1}+b_{3}= & \left\langle q^{\prime \prime}, D_{x^{\prime \prime}} a\right\rangle-\left\langle q^{\prime}, D_{\xi^{\prime}} a\right\rangle  \tag{4.4.13}\\
& \left.+\frac{1}{2}\left[D_{x^{\prime \prime}}+\partial_{x^{\prime \prime} \xi^{\prime}}^{2} H D_{x^{\prime}}-\partial_{x^{\prime \prime}}^{2} H D_{\xi^{\prime \prime}}, q^{\prime \prime}\right\rangle-\left\langle D_{\xi^{\prime}}+\partial_{\xi^{\prime}}^{2} H D_{x^{\prime}}-\partial_{\xi^{\prime} x^{\prime \prime}}^{2} H D_{\xi^{\prime}}, q^{\prime}\right\rangle\right] a
\end{align*}
$$

The symbol of $A$ is $a\left(x^{\prime \prime}, \xi^{\prime}, y^{\prime \prime}, \eta^{\prime}\right)\left|d x^{\prime \prime} d \xi^{\prime} d y^{\prime \prime} d \eta^{\prime}\right|^{1 / 2}$ with $\left(x^{\prime \prime}, \xi^{\prime}, y^{\prime \prime}, \eta^{\prime}\right)$ parametrizing $C$. Now, since $p_{0}$ zero on $C, H_{p}$ is tangent to $C$ and hence in the ( $x^{\prime \prime}, \xi^{\prime}, y^{\prime \prime}, \eta^{\prime}$ ) coordinates

$$
\begin{align*}
H_{p_{0}} & \left.=\left\langle\partial_{\xi^{\prime \prime}} p_{0}, \partial_{x^{\prime \prime}}\right\rangle-\left\langle\partial_{x^{\prime}} p_{0}, \partial_{\xi^{\prime}}\right\rangle\right\rangle\left.\right|_{\substack{x^{\prime}=\partial_{\xi^{\prime}} H \\
\xi^{\prime \prime}=-x_{x^{\prime \prime}} H}}=\left\langle q^{\prime \prime}, \partial_{x^{\prime \prime}}\right\rangle-\left.\left\langle q^{\prime}, \partial_{\xi^{\prime}}\right\rangle\right|_{\substack{x^{\prime}=\partial_{\xi^{\prime}} H \\
\xi^{\prime \prime}=-\partial_{x^{\prime \prime}} H}} \\
\mathcal{L}_{H_{p}} \sigma(A) & =\left(\left\langle q^{\prime \prime}, \partial_{x^{\prime \prime}} a\right\rangle-\left\langle q^{\prime}, \partial_{\xi^{\prime}} a\right\rangle+\frac{1}{2} \operatorname{div}\left(H_{p}\right) a\right)\left|d x^{\prime \prime} d \xi^{\prime} d y^{\prime \prime} d \eta^{\prime}\right|^{1 / 2}  \tag{4.4.14}\\
\operatorname{div}\left(H_{p}\right) & =\left\langle\partial_{x^{\prime \prime}}+\partial_{x^{\prime \prime} \xi^{\prime}}^{2} H \partial_{x^{\prime}}-\partial_{x^{\prime \prime}}^{2} H \partial_{\xi^{\prime \prime}}, q^{\prime \prime}\right\rangle-\left\langle\partial_{\xi^{\prime}}+\partial_{\xi^{\prime}}^{2} H \partial_{x^{\prime}}-\partial_{\xi^{\prime} x^{\prime \prime}}^{2} H \partial_{\xi^{\prime \prime}}, q^{\prime}\right\rangle \tag{4.4.15}
\end{align*}
$$

So, recalling (4.4.11) and comparing (4.4.12) with (4.4.13) and (4.4.14), we have the result.

### 4.5 Shymbol

In Chapters 6 and 8 we will need to compute symbols of operators whose semiclassical order may vary from point to point in $T^{*} M$ One can often handle this type of behavior by using weights to compensate for the growth. However, this requires some a priori knowledge of how the order changes. In this section, we will develop a notion of a sheaf valued symbol, the shymbol, that can be used to work in this setting without such a priori knowledge.

Let $M$ be a compact manifold. Let $\mathcal{T}\left(T^{*} M\right)$ be the topology on $T^{*} M$. For $s \in \mathbb{R}$, denote the symbol map

$$
\sigma_{s}: h^{s} \Psi_{\delta}^{\text {comp }} \rightarrow h^{s} S_{\delta}^{\text {comp }} / h^{s+1-2 \delta} S_{\delta}^{\text {comp }}
$$

Suppose that for some $N>0$ and $\delta \in[0,1 / 2), A \in h^{-N} \Psi_{\delta}^{\text {comp }}(M)$. We define a finer notion of symbol for such a pseudodifferential operator. Fix $0<\epsilon \ll 1-2 \delta$. For each open set $U \in \mathcal{T}\left(T^{*} M\right)$, define the $\epsilon$-order of $A$ on $U$

$$
I_{A}^{\epsilon}(U):=\sup _{s \in \mathcal{S}_{\epsilon}} s+1-2 \delta
$$

where

$$
\mathcal{S}_{\epsilon}:=\left\{s \in \epsilon \mathbb{Z}: \text { there exists } \chi \in C_{c}^{\infty}\left(T^{*} M\right),\left.\chi\right|_{U}=1,\left.\sigma_{s}\left(\operatorname{Op}_{\mathrm{h}}(\chi) A \mathrm{Op}_{\mathrm{h}}(\chi)\right)\right|_{U} \equiv 0\right\} .
$$

Then it is clear that for any $V \Subset U$ there exists $\chi \in C_{c}^{\infty}(U)$ with $\chi \equiv 1$ on $V$ such that $\mathrm{Op}_{\mathrm{h}}(\chi) A \mathrm{Op}_{\mathrm{h}}(\chi) \in h^{I_{A}^{\epsilon}(U)} \Psi_{\delta}^{\text {comp }}(M)$.

Give $\mathcal{T}\left(T^{*} M\right)$ the ordering that $U \leq V$ if $V \subset U$ with morphisms $U \rightarrow V$ if $U \leq V$. Notice that $U \leq V$ implies $I_{A}^{\epsilon}(U) \leq I_{A}^{\epsilon}(V)$. Then define the functor $F_{A}^{\epsilon}: \mathcal{T}\left(T^{*} M\right) \rightarrow \mathbf{C o m m}$ (the category of commutative rings) by

$$
\begin{gathered}
F_{A}^{\epsilon}(U)= \begin{cases}\left.h^{I_{A}^{\epsilon}(U)} S_{\delta}^{\mathrm{comp}}(M)\right|_{U} /\left.h_{A}^{I_{A}^{\epsilon}(U)+1-2 \delta} S_{\delta}^{\mathrm{comp}}(M)\right|_{U} & I_{A}^{\epsilon}(U) \neq \infty \\
\{0\} & I_{A}^{\epsilon}(U)=\infty\end{cases} \\
F_{A}^{\epsilon}(U \rightarrow V)= \begin{cases}\left.h_{A}^{I_{A}^{\epsilon}(V)-I_{A}^{\epsilon}(U)}\right|_{V} & I_{A}^{\epsilon}(V) \neq \infty \\
0 & I_{A}^{\epsilon}(V)=\infty\end{cases}
\end{gathered}
$$

Then $F_{A}^{\epsilon}$ is a presheaf on $T^{*} M$. We sheafify $F_{A}^{\epsilon}$, still denoting the resulting sheaf by $F_{A}^{\epsilon}$, and say that $A$ is of $\epsilon$-class $F_{A}^{\epsilon}$. We define the stalk of the sheaf at $q$ by $F_{A}^{\epsilon}(q):=\lim _{\rightarrow q \in U} F_{A}^{\epsilon}(U)$.

Now, for every $U \subset \mathcal{T}\left(T^{*} M\right), I_{A}^{\epsilon}(U) \neq \infty$, there exists $\chi_{U} \in C_{c}^{\infty}\left(T^{*} M\right)$ with $\chi_{U} \equiv 1$ on $U$ such that $\left.\sigma_{I_{A}^{\epsilon}(U)}\left(\operatorname{Op}_{\mathrm{h}}\left(\chi_{U}\right) A \mathrm{Op}_{\mathrm{h}}\left(\chi_{U}\right)\right)\right|_{U} \neq 0$. Then we define the $\epsilon$-shymbol of $A$ to be the section of $F_{A}^{\epsilon}, \tilde{\sigma}_{(\cdot)}^{\epsilon}(A): \mathcal{T}\left(T^{*} M\right) \rightarrow F_{A}^{\epsilon}(\cdot)$, given by

$$
\tilde{\sigma}_{U}^{\epsilon}(A):=\left\{\begin{array}{ll}
\left.\sigma_{I_{A}^{\epsilon}(U)}\left(\mathrm{Op}_{\mathrm{h}}\left(\chi_{U}\right) A \mathrm{Op}_{\mathrm{h}}\left(\chi_{U}\right)\right)\right|_{U} & I_{A}^{\epsilon}(U) \neq \infty \\
0 & I_{A}^{\epsilon}(U)=\infty
\end{array} .\right.
$$

Define also the $\epsilon$-stalk shymbol, $\tilde{\sigma}^{\epsilon}(A)_{q}$ to be the germ of $\tilde{\sigma}^{\epsilon}(A)$ at $q$ as a section of $F_{A}^{\epsilon}$.

Now, define $I_{A}^{\epsilon}(q):=\sup _{q \in U} I_{A}^{\epsilon}(U)$. We then define the simpler compressed shymbol

$$
\tilde{\sigma}^{\epsilon}(A): T^{*} M \rightarrow \bigsqcup_{q} h^{I_{A}^{\epsilon}(q)} \mathbb{C} / h_{A}^{I_{A}^{\epsilon}(q)+1-2 \delta} \mathbb{C} \quad \text { by } \quad \tilde{\sigma}^{\epsilon}(A)(q):= \begin{cases}0 & I_{A}^{\epsilon}(q)=\infty  \tag{4.5.1}\\ \lim _{q \in U} \tilde{\sigma}_{U}^{\epsilon}(A)(q) & I_{A}^{\epsilon}(q)<\infty\end{cases}
$$

The limit in 4.5.1) exists since if $I_{A}^{\epsilon}(q)<\infty$, then there exists $U \ni q$ such that for all $V \subset U, I_{A}^{\epsilon}(V)=\overline{I_{A}^{\epsilon}(U)}$. This also shows that it is enough to take any sequence of $U_{n} \downarrow q$. It is easy to see from standard composition formulae that the compressed shymbol has

$$
\tilde{\sigma}^{\epsilon}(A B)(q)=\tilde{\sigma}^{\epsilon}(A)(q) \tilde{\sigma}(B)(q), \quad A \in h^{-N} \Psi_{\delta}^{\text {comp }} \text { and } B \in h^{-M} \Psi_{\delta}^{\text {comp }} .
$$

Moreover,

$$
\tilde{\sigma}^{\epsilon}([A, B])(q)=-i h\left\{\tilde{\sigma}^{\epsilon}(A)(q), \tilde{\sigma}^{\epsilon}(B)(q)\right\} .
$$

The following lemma follows from Proposition 4.4.10 combined with the definitions above:
Lemma 4.5.1. Suppose that $A \in \Psi_{\delta}^{\text {comp }}$ and let $T$ be a semiclassical FIO associated to the symplectomorphism $\kappa$ with elliptic symbol $t \in S_{\delta}$. Then for $0<N$ independent of $h$ $(A T)_{N}:=\left(T^{*} A^{*}\right)^{N}(A T)^{N}$ has

$$
\tilde{\sigma}^{\epsilon}\left((A T)_{N}\right)(q)=\prod_{i=1}^{N}\left(\left|\tilde{\sigma}^{\epsilon}(A) t\right|^{2} \circ \kappa^{i}(q)+O\left(h^{I_{A_{i}}^{\epsilon}\left(\beta^{k}(q)\right)+1-2 \delta}\right)\right) .
$$

Proof. Fix $q \in T^{*} M$. Let $\chi_{k} \in C_{c}^{\infty}$ have $\chi_{k} \equiv 1$ on $B_{q}\left(\frac{1}{k}\right)$ and $\operatorname{supp} \chi_{k} \subset B_{q}\left(\frac{2}{k}\right)$. Then let $D:=\operatorname{Op}_{\mathrm{h}}\left(\chi_{k}\right)(A T)_{N} \operatorname{Op}_{\mathrm{h}}\left(\chi_{k}\right)$. We have that

$$
D=\operatorname{Op}_{\mathrm{h}}\left(\chi_{k}\right)(B T)_{N} \mathrm{Op}_{\mathrm{h}}\left(\chi_{k}\right)+O_{\Psi_{\delta}^{\operatorname{comp}}}\left(h^{\infty}\right)
$$

where $B_{i}=\mathrm{Op}_{\mathrm{h}}\left(\psi_{k, i}\right) A_{i} \mathrm{Op}_{\mathrm{h}}\left(\psi_{k, i}\right)$ and $\psi_{k, i} \equiv 1$ in some neighborhood of $\beta^{i}(q)$ and is supported inside a neighborhood $U_{k, i}$ of $\beta^{i}(q)$ such that $U_{k, i} \downarrow q$. Then the result follows from standard composition formulae in Proposition 4.4.10.

Now, since $\epsilon>0$ is arbitrary, we define the semiclassical order of $A$ at $q$ by $I_{A}(q):=$ $\sup _{\epsilon>0} I_{A}^{\epsilon}(q)$ with the understanding that $f=O\left(h^{I_{A}(q)}\right)$ means that for any $\epsilon>0$,

$$
|f(q)| \leq C_{\epsilon} h^{I_{A}(q)-\epsilon}
$$

Furthermore, we suppress the $\epsilon$ in the notation $\tilde{\sigma}^{\epsilon}(A)(q)$ and denote the compressed shymbol, $\tilde{\sigma}(A)(q)$, again with the understanding that for any $\epsilon>0$,

$$
\tilde{\sigma}(A)(q) \in h^{I_{A}(q)-\epsilon} \mathbb{C} / h^{I_{A}(q)+1-2 \delta-\epsilon} \mathbb{C} .
$$

### 4.6 Semiclassical Intersecting Lagrangian Distributions

We follow [49] to construct intersecting Lagrangian distributions in the semiclassical regime.

## Definitions

A pair $\left(\Lambda_{0}, \Lambda_{1}\right)$ where $\Lambda_{0} \subset T^{*} X$ is a Lagrangian manifold and $\Lambda_{1} \subset T^{*} X$ is a Lagrangian manifold with boundary, is said to be an intersecting pair of Lagrangian manifolds if $\Lambda_{0} \cap \Lambda_{1}=$ $\partial \Lambda_{1}$ and the intersection is clean:

$$
T_{\lambda}\left(\Lambda_{0}\right) \cap T_{\lambda}\left(\Lambda_{1}\right)=T_{\lambda}\left(\partial \Lambda_{1}\right) \text { for all } \lambda \in \partial \Lambda_{1} .
$$

Two such pairs $\left(\Lambda_{0}, \Lambda_{1}\right)$ and $\left(\Lambda_{0}^{\prime}, \Lambda_{1}^{\prime}\right)$, with given base points $\lambda \in \partial \Lambda_{1}$ and $\lambda^{\prime} \in \partial \Lambda_{1}^{\prime}$ are said to be locally equivalent if there is a neighborhood $V$ of $\lambda$ and a symplectic transformation $\chi: V \rightarrow T^{*} X$ such that $\chi(\lambda)=\lambda^{\prime}, \chi\left(\Lambda_{0} \cap V\right) \subset \Lambda_{0}^{\prime}$ and $\chi\left(\Lambda_{1} \cap V\right) \subset \Lambda_{1}^{\prime}$. Then, we have the following lemma [41, Theorem 21.2.10 and remark thereafter].

Lemma 4.6.1. If $\Lambda_{1}, \Lambda_{2} \subset M$, and $\bar{\Lambda}_{1}, \bar{\Lambda}_{2} \subset \bar{M}$ are two pairs of intersecting Lagrangians with $\operatorname{dim} M=\operatorname{dim} \bar{M}$ and $\operatorname{dim} \Lambda_{1} \cap \Lambda_{2}=\operatorname{dim} \bar{\Lambda}_{1} \cap \bar{\Lambda}_{2}$ then $\left(\Lambda_{1}, \Lambda_{2}\right)$ is locally equivalent to $\left(\bar{\Lambda}_{1}, \bar{\Lambda}_{2}\right)$.

We associate spaces of distributions to the pair $\left(\tilde{\Lambda}_{0}, \tilde{\Lambda}_{1}\right)$ of intersecting Lagrangian manifolds, where $\tilde{\Lambda}_{0}=T_{0}^{*} \mathbb{R}^{d} \tilde{\Lambda}_{1}=\left\{\left(\left(x_{1}, x^{\prime}\right), \xi\right) \in T^{*} \mathbb{R}^{d}: x^{\prime}=0, \xi_{1}=0, x_{1} \geq 0\right\}$.
Remark: One can also associate distributions to intersecting Lagrangians with intersections of various dimensions as in [35], but we do not pursue that here.

Definition 4.6.2. For $\delta \in[0,1 / 2)$, denote by $I_{\delta}^{m}\left(\mathbb{R}^{d} ; \tilde{\Lambda}_{0}, \tilde{\Lambda}_{1}\right)$ the subspace of $C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ consisting of functions $u$ which can be written in the form $u=u_{1}+u_{2}$ with $u_{2} \in h^{1 / 2} I_{\delta}^{m-1 / 2}\left(\tilde{\Lambda}_{0}\right)$ and

$$
\begin{equation*}
u_{1}(x)=(2 \pi h)^{-(3 d+2) / 4} \int_{0}^{\infty} \int_{\mathbb{R}^{d}} e^{\frac{i}{h}\left(\left(x_{1}-s\right) \xi_{1}+\left\langle x^{\prime}, \xi^{\prime}\right\rangle\right)} a(s, x, \xi) d \xi d s=: J(a) \tag{4.6.1}
\end{equation*}
$$

where $a \in S_{\delta}^{m+\frac{1}{2}-\frac{d}{4}}$ has compact support in $x$.

## Remarks:

- 4.6.1 is well defined as an oscillatory integral and as such depends continuously on $a$ in the topology of $S_{\delta}^{m^{\prime}}$, for any $m^{\prime}>m+\frac{1}{2}-\frac{1}{4} d$.
- We show in Lemma 4.6.3 that functions of the form 4.6.1 are microlocalized on $\tilde{\Lambda}_{0} \cup \tilde{\Lambda}_{1}$.

Lemma 4.6.3. If $u \in I_{\delta}^{m}\left(\mathbb{R}^{d} ; \tilde{\Lambda}_{0}, \tilde{\Lambda}_{1}\right)$, then

$$
\begin{equation*}
\mathrm{WF}_{\mathrm{h}}(u) \subset \tilde{\Lambda}_{0} \cup \tilde{\Lambda}_{1} \tag{4.6.2}
\end{equation*}
$$

Suppose $\gamma \leq \delta$ and $B \in S_{\gamma}$ is a zeroth order pseudo-differential operator with $\operatorname{MS}_{\mathrm{h}}(B) \cap \tilde{\Lambda}_{0}=$ $\emptyset$ then $B u \in I_{\delta}^{m}\left(\mathbb{R}^{d} ; \tilde{\Lambda}_{1}\right)$. If $\operatorname{MS}_{\mathrm{h}}(B) \cap \tilde{\Lambda}_{1}=\emptyset$ then $\left.B u \in h^{1 / 2-\gamma} I_{\delta}^{m-1 / 2}\left(\mathbb{R}^{d} ; \tilde{\Lambda}_{0}\right)\right)$.

Proof. Let $\pi: \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ be the projection off the first factor, then $u=\pi_{*}(H(s) \tilde{u})$ where $H$ is the Heaviside function and

$$
\tilde{u}(s, x)=(2 \pi h)^{-(3 d+2) / 4} \int e^{\frac{i}{h}\left(\left(x_{1}-s\right) \xi_{1}+\left\langle x^{\prime}, \xi^{\prime}\right\rangle\right)} a(s, x, \xi) d \xi .
$$

We now use the standard bounds on wavefront sets for pullbacks, tensors, and pushforwards (see Lemmas 4.3.12, 4.3.16, and 4.3.15) to obtain 4.6.2).

Now, suppose $B \in \psi_{\gamma}^{0}$. Then,

$$
B\left(e^{\frac{i}{h}\langle x, \xi\rangle} a(x, \xi)\right)=e^{\frac{i}{h}\langle x, \xi\rangle}(\mathcal{B} a)
$$

defines a continuous linear map $\mathcal{B}: S_{\delta}^{m} \rightarrow S_{\delta}^{m}$. In particular, $B$ can be applied under the integral sign in 4.6.1). This shows that $B u_{1}$ is of the same form with $a$ replaced by $\mathcal{B} a$.

Observe that since $u_{2} \in h^{1 / 2} I_{\delta}^{m-1 / 2}\left(\tilde{\Lambda}_{0}\right), B u_{2}=O_{C_{c}^{\infty}}\left(h^{\infty}\right)$. Then, if $\operatorname{MS}_{\mathrm{h}}(B) \cap \tilde{\Lambda}_{0}=\emptyset$, we can assume, by disregarding an $O_{C_{c}^{\infty}}\left(h^{\infty}\right)$ term, that for some $\epsilon>0, \mathcal{B}(a)=0$ in $|x|<\epsilon h^{\gamma}$. Choose $\mu \in C^{\infty}(\mathbb{R})$ with $\mu(s)=1$ in $s \geq \frac{1}{2} \epsilon, \mu(s)=0$ in $s<\frac{1}{4} \epsilon$. From the definition of semiclassical Lagrangian distributions (see Section 2.3)

$$
v_{1}=(2 \pi h)^{-(3 d+2) / 4} \iint e^{\frac{i}{h}\left(\left(x_{1}-s\right) \xi_{1}+\left\langle x^{\prime}, \xi^{\prime}\right\rangle\right)} \mu\left(h^{-\delta} s\right)(\mathcal{B} a)(s, x, \xi) d \xi d s
$$

is an element of $I_{\delta}^{m}\left(\mathbb{R}^{d} ; \tilde{\Lambda}_{1}\right)$. To show that $B u$ is also in this space, we need to verify that

$$
\begin{equation*}
B u-v_{1}=(2 \pi h)^{-(3 d+2) / 4} \int_{0}^{\infty} \int e^{\frac{i}{h}\left(\left(x_{1}-s\right) \xi_{1}+\left\langle x^{\prime}, \xi^{\prime}\right\rangle\right.}\left(1-\mu\left(h^{-\delta} s\right)\right) \mathcal{B}(a) d \xi d s=\mathcal{O}_{C^{\infty}}\left(h^{\infty}\right) \tag{4.6.3}
\end{equation*}
$$

The operator

$$
L=\left(\left(x_{1}-s\right)^{2}+\left|x^{\prime}\right|^{2}\right)^{-1}\left[\left(x_{1}-s\right) h D_{\xi_{1}}+x^{\prime} h D_{\xi^{\prime}}\right]
$$

satisfies $L \exp \left(\frac{i}{h}\left(\left(x_{1}-s\right) \xi_{1}+\left\langle x^{\prime}, \xi^{\prime}\right\rangle\right)\right)=\exp \left(\frac{i}{h}\left(\left(x_{1}-s\right) \xi_{1}+\left\langle x^{\prime}, \xi^{\prime}\right\rangle\right)\right)$ and has coefficients in $h^{-\gamma} S_{\gamma}$ on the support of $(1-\mu) \mathcal{B} a$. Then, 4.6.3) follows from integration by parts. Thus, $B u \in I_{\delta}^{\text {comp }}\left(\mathbb{R}^{d} ; \tilde{\Lambda}_{1}\right)$.

Now, suppose that $\operatorname{MS}_{\mathrm{h}}(B) \cap \tilde{\Lambda}_{1}=\emptyset$. Then we can assume, with $a$ replaced by $\mathcal{B} a$ that $a=0$ if $\left|x^{\prime}\right|^{2}+\xi_{1}^{2}<\epsilon^{2} h^{2 \gamma}$ and $x_{1}>-\epsilon h^{\gamma}$. Thus, the operator

$$
M=\left(\left|x^{\prime}\right|^{2}+\xi_{1}^{2}\right)^{-1}\left(x^{\prime} h D_{\xi^{\prime}}-\xi_{1} h D_{s}\right)
$$

has coefficients in $h^{-\gamma} S_{\gamma}$ on supp $a$ provided $x_{1}>-\epsilon h^{\gamma}$. Since $\exp \left(\frac{i}{h}\left(\left(x_{1}-s\right) \xi_{1}+\left\langle x^{\prime}, \xi^{\prime}\right\rangle\right)\right)$ is an eigenfunction of $M$ with eigenvalue 1, integration by parts gives

$$
\begin{align*}
& B u=(2 \pi h)^{-(3 d+2) / 4} \int_{0}^{\infty} \int e^{\frac{i}{h}\left(\left(x_{1}-s\right) \xi_{1}+\left\langle x^{\prime}, \xi^{\prime}\right\rangle\right)} M^{t}(\mathcal{B} a) d \xi d s \\
&+(2 \pi h)^{1-(3 d+2) / 4} \int e^{\frac{i}{h}\langle x, \xi\rangle} \frac{-i \xi_{1} \mathcal{B} a(0, x, \xi)}{\left(\left|\xi_{1}\right|^{2}+\left|x^{\prime}\right|^{2}\right)} d \xi \tag{4.6.4}
\end{align*}
$$

The second term in 4.6.4) is a distribution in $h^{1 / 2-\gamma} I_{\delta}^{m-1 / 2}\left(\mathbb{R}^{d} ; \tilde{\Lambda}_{0}\right)$. Then, iterating this process, we have for any $k \in \mathbb{N}$,

$$
B u-(2 \pi h)^{-(3 d+2) / 4} \int_{0}^{\infty} \int e^{\frac{i}{h}\left(\left(x_{1}-s\right) \xi_{1}+\left\langle x^{\prime}, \xi^{\prime}\right\rangle\right)}\left(M^{t}\right)^{k} \mathcal{B} a d \xi d s
$$

lies in $h^{\frac{1}{2}-\gamma} I_{\delta}^{m-1 / 2}\left(\mathbb{R}^{d} ; \tilde{\Lambda}_{0}\right)$. Since $\left(M^{t}\right)^{k} \mathcal{B} a \in h^{k(1-\delta)} S_{\delta}^{m-k}$, we conclude that

$$
B u \in h^{\frac{1}{2}-\gamma} I_{\delta}^{m-1 / 2}\left(\mathbb{R}^{d} ; \tilde{\Lambda}_{0}\right)
$$

Next, we show that $a$ need not be allowed to depend on $s$.
Lemma 4.6.4. Suppose $u=J(a)$ for $a \in S_{\delta}^{m}$. Then there exists $b_{j}=b(x, \eta) \in S_{\delta}^{m-j}$ such that

$$
u-\sum_{j=0}^{N-1} J\left(b_{j}\right) \in h^{N(1-\delta)} I_{\delta}^{m-N}\left(\mathbb{R}^{d} ; \tilde{\Lambda}_{0}, \tilde{\Lambda_{1}}\right)
$$

Proof. By Taylor's theorem at $y_{1}=s$, there exists $b_{0}$ such that

$$
\left|a\left(y_{1}, y^{\prime}, s, \eta\right)-b_{0}(y, \eta)\right|=O\left(h^{-\delta}\left(y_{1}-s\right)\right) .
$$

Then, integrating by parts with respect to $\eta_{1}$ in the formula for $J\left(a-b_{0}\right)$ gives that

$$
J\left(a-b_{0}\right)=h^{1-\delta} J(c)
$$

with $c \in S_{\delta}^{m-1+1 / 2-d / 4}$. So, repeating this process gives the Lemma.
Finally, we show that an element of $I_{\delta}^{m}\left(\mathbb{R}^{d} ; \tilde{\Lambda}_{0}, \tilde{\Lambda}_{1}\right)$ can be written as a Lagrangian distribution with singular symbol.

Lemma 4.6.5. Suppose that $u=J(a)$ where $a=a(y, \eta) \in S^{m}$. Then,

$$
u=(2 \pi h)^{-(3 d-2) / 4} \int_{\mathbb{R}^{d}} e^{\frac{i}{h}\langle x, \xi\rangle} \frac{-i a(y, \eta)}{\eta-i 0} d \eta+O_{C_{c}^{\infty}}\left(h^{\infty}\right)
$$

Proof. Observe that by the Paley-Wiener theorem,

$$
f\left(\eta_{1}\right)=\int_{0}^{\infty} e^{-\frac{i}{h} s \eta_{1}} d s
$$

is holomorphic in $\operatorname{Im} \eta_{1}<0$. So, we can take limits from $\eta_{1}$ in the lower half plane to obtain

$$
f\left(\eta_{1}\right)=\frac{h}{i\left(\eta_{1}-i 0\right)}
$$

This gives the result.

## General Lagrangians

Suppose that $\left(\Lambda_{0}, \Lambda_{1}\right)$ is an intersecting pair of Lagrangian manifolds in a $C^{\infty}$ manifold $X$ with $\operatorname{dim} X=d$ and $\operatorname{dim} \Lambda_{0} \cap \Lambda_{1}=d-1$ and $\Lambda_{0} \cap \Lambda_{1} \Subset T^{*} X$. Given $\lambda \in \Lambda_{0} \cap \Lambda_{1}$, by Lemma 4.6.1 we can find a local parametrization of the the intersecting pair. Therefore, we define

Definition 4.6.6. $I_{\delta}^{m}\left(X ; \Lambda_{0}, \Lambda_{1}\right)$ consists of those $C^{\infty} \frac{1}{2}$ densities, $u$ on $X$ which are modelled microlocally on Definition 4.6.2. We say that $u \in I_{\delta}^{m}\left(X ; \Lambda_{0}, \Lambda_{1}\right)$ if there exist distributions $u_{0} \in h^{1 / 2} I_{\delta}^{m}\left(\Lambda_{0}\right), u_{1} \in I_{\delta}^{\text {comp }}\left(\Lambda_{1} \backslash \partial \Lambda_{1}\right)$, a finite set of parametrizations $\chi_{j}: V_{j} \rightarrow T^{*} \mathbb{R}^{d}$ reducing $\left(\Lambda_{0}, \Lambda_{1}\right)$ locally to normal form, zeroth order Fourier integral operators $F_{j}$ associated to $\chi_{j}^{-1}$ and distributions $v_{j} \in I_{\delta}^{\text {comp }}\left(\mathbb{R}^{d} ; \tilde{\Lambda}_{0}, \tilde{\Lambda}_{1}\right)$ such that

$$
u-u_{0}-u_{1}-\sum_{j} F_{j} v_{j}=\mathcal{O}_{\mathcal{S}}\left(h^{\infty}\right)
$$

Remark: Recall that for open $\Lambda$, all $u \in I_{\delta}^{\text {comp }}(\Lambda)$ are compactly microlocalized inside $\Lambda$. Thus, $I_{\delta}^{\text {comp }}\left(\Lambda_{1} \backslash \partial \Lambda_{1}\right)$ consists of distributions which are compactly microlocalized away from $\partial \Lambda_{1}$.

To show that these distributions are well defined, we need to show that if $\chi$ is a canonical transformation on $\mathbb{R}^{d}$ which leaves both $\tilde{\Lambda}_{0}$ and $\tilde{\Lambda}_{1}$ invariant and $F$ is a properly supported zeroth order Fourier integral operator associated to $\chi$, then $F u \in I_{\delta_{\tilde{N}}}^{\text {comp }}\left(\mathbb{R}^{d} ; \tilde{\Lambda}_{0}, \tilde{\Lambda}_{1}\right)$ provided $u$ is in this space. We will actually prove something stronger. Let $\tilde{\Lambda}_{i}^{d^{\prime}} \subset T^{*} \mathbb{R}^{d^{\prime}}, \tilde{\Lambda}_{i}^{d} \subset T^{*} \mathbb{R}^{d}$, $i=0,1$.
Lemma 4.6.7. Suppose that $d, d^{\prime} \geq 2$ and $\Gamma$ is a canonical relation such that $\Gamma \circ \tilde{\Lambda}_{0}^{d^{\prime}} \subset \tilde{\Lambda}_{0}^{d}$, $\Gamma \circ \tilde{\Lambda}_{1}^{d^{\prime}} \subset \tilde{\Lambda}_{1}^{d}$ and the compositions are transversal. Let $F \in I^{\text {comp }}(\Gamma)$. Then

$$
\begin{equation*}
F: I_{\delta}^{\text {comp }}\left(\mathbb{R}^{d^{\prime}} ; \tilde{\Lambda}_{0}^{d^{\prime}}, \tilde{\Lambda}_{1}^{d^{\prime}}\right) \rightarrow I_{\delta}^{\text {comp }}\left(\mathbb{R}^{d} ; \tilde{\Lambda}_{0}^{d}, \tilde{\Lambda}_{1}^{d}\right) \tag{4.6.5}
\end{equation*}
$$

Proof. We can always decompose $F$ by using a microlocal partition of unity and so assume that $\Gamma \circ \tilde{\Lambda}_{0}^{d^{\prime}}=\Lambda_{0}^{d}$ and $\Gamma \circ \tilde{\Lambda}_{1}^{d^{\prime}}=\tilde{\Lambda}_{1}^{d}$ in the region of interest. Suppose that

$$
\begin{gathered}
u=(2 \pi h)^{-\left(3 d^{\prime}+2\right) / 4} \int_{0}^{\infty} \int e^{\frac{i}{h}\left(\left(y_{1}-s\right) \eta_{1}+\left\langle y^{\prime}, \eta^{\prime}\right\rangle\right)} a(s, y, \eta) d \eta d s \\
F v=(2 \pi h)^{-\left(d+d^{\prime}+2 L\right) / 4} \int e^{\frac{i}{h} \phi(x, y, \theta)} b(x, y, \theta) v(y) d y d \theta
\end{gathered}
$$

where $\phi$ non-degenerate phase function defining $\Gamma$. Then,

$$
\begin{equation*}
F u=(2 \pi h)^{-\left(d+4 d^{\prime}+2 L+2\right) / 4} \int_{0}^{\infty} \int\left[\int e^{\frac{i}{h} \psi(x, y, s, \theta, \eta)} b a d y d \eta\right] d \theta d s \tag{4.6.6}
\end{equation*}
$$

with $\psi=\phi(x, y, \theta)+\left(y_{1}-s\right) \eta_{1}+\left\langle y^{\prime}, \eta^{\prime}\right\rangle$. Now, note that $d_{\eta} \psi=0$ if and only if $y_{1}=s, y_{2}=$ $\ldots=y_{d^{\prime}}=0, d_{y} \psi=0$ if and only if $\eta=-d_{y} \phi$ and

$$
\partial_{y \eta}^{2} \psi=\left(\begin{array}{cc}
\partial_{y}^{2} \phi & I \\
I & 0
\end{array}\right)
$$

which has determinant 1 .
Thus, by stationary phase,

$$
F u=(2 \pi h)^{-(d+2 L+2) / 4} \int_{0}^{\infty} \int e^{\frac{i}{h} \phi(x,(s, 0), \theta)} c(x, s, \theta) d \theta d s
$$

Notice that $\Gamma \circ \tilde{\Lambda}_{i}^{\left(d^{\prime}\right)}=\Lambda_{i}^{(d)}$ implies that

$$
\begin{aligned}
d_{\theta} \phi=0 \Rightarrow y & =0 \Leftrightarrow x=0 \\
d_{\theta} \phi=0, \Rightarrow \phi_{y_{1}}^{\prime} & =0 \Rightarrow \phi_{x_{1}}^{\prime}=0, x^{\prime}=0, x_{1} \geq 0 \\
d_{\theta} \phi=0, \phi_{x_{1}}^{\prime} & =0 \Rightarrow \phi_{y_{1}}^{\prime}=0, y^{\prime}=0 . y_{1} \geq 0
\end{aligned}
$$

Since we have assumed that the compositions $\Gamma \circ \tilde{\Lambda}_{1}^{d^{\prime}}$ is transversal $\varphi(x, s, \theta)=\phi(x,(s, 0), \theta)$ is non-degenerate and since $\Gamma \circ \tilde{\Lambda}_{i}^{d^{\prime}}=\tilde{\Lambda}_{1}^{d}$,

$$
\begin{gather*}
d_{\theta} \varphi=0, s=0 \Leftrightarrow x=0, d_{\theta} \varphi=0  \tag{4.6.7}\\
d_{\theta} \varphi=0, d_{s} \varphi=0 \Leftrightarrow x^{\prime}=0, d_{x_{1}} \varphi=0, x_{1} \geq 0, d_{\theta} \phi=0 \tag{4.6.8}
\end{gather*}
$$

Since away from $s=0, u \in I^{\text {comp }}\left(\Lambda_{1}\right)$, we may work in a small neighborhood of $s=0$. Suppose that there exists $\left\{\left(s_{i}, x_{i}, \theta_{i}\right)\right\}_{i=1}^{\infty}$ such that $s_{i} \rightarrow 0, s_{i}, x_{i} \neq 0, d_{\theta} \varphi\left(x_{i}, x_{i}, \theta_{i}\right)=$ 0 . Then, since $c$ has compact support, we may assume that $\left(x_{i}, \theta_{i}\right) \rightarrow(x, \theta)$. But, $\varphi \in$ $C^{\infty}$. Therefore, $d_{\theta} \varphi(x, 0, \theta)=0$ and hence $x=0$ by 4.6.7) and we may also work in a neighborhood of $x=0$.

Suppose that $\partial^{2} \varphi / \partial \theta \partial \theta(0,0, \theta) \neq 0$. Then there exist $i, j$ such that $\partial^{2} \varphi / \partial \theta_{i} \partial \theta_{j}(0,0, \theta) \neq$ 0 . Suppose $i=j$. Then $\partial^{2} \varphi / \partial \theta_{i}^{2}(0,0, \theta) \neq 0$ and we can use stationary phase to eliminate
the $\theta_{i}$ variable. Therefore, we may assume that $\partial^{2} \varphi / \partial \theta_{i}^{2}(0,0, \theta)=0$ for all $i$ and $\theta$ in $d_{\theta} \varphi=0$. Suppose that $i \neq j$. Then, since $\partial^{2} \varphi / \partial^{2} \theta_{i}(0,0, \theta)=0$ for all $i$, we may use stationary phase in the $\theta_{i}$ and $\theta_{j}$ variables. Now, observe that if $\partial^{2} \varphi / \partial \theta \partial \theta(0,0, \theta) \neq 0$ then the same is true in a neighborhood of $s=0, x=0$.

Hence, reducing the size of the neighborhood of $(0,0)$ if necessary and using stationary phase we can reduce the number of $\theta$ variables, $L$, until $\partial^{2} \varphi / \partial \theta \partial \theta=0$ at $(0,0, \tilde{\theta})$. Then, by (4.6.7) and the fact that $\Gamma \circ \tilde{\Lambda}_{0}^{d^{\prime}}$ is transverse $L=d$ and $\operatorname{det}\left(\partial^{2} \varphi / \partial x \partial \theta\right) \neq 0$. Therefore,

$$
\begin{gathered}
F u=(2 \pi h)^{-(3 d+2) / 4} \int_{0}^{\infty} \int e^{\frac{i}{h} \phi(x,(s, 0), \theta)} c(x, s, \theta) d \theta d s, \\
\frac{\partial \varphi}{\partial \theta_{j}}=\sum_{i} C_{j i}\left(x_{i}-s \alpha_{i}(x, s, \theta)\right)
\end{gathered}
$$

where $C$ is invertible. Now, we want to show that there is a change of variables $\theta=\Theta(x, s, \theta)$, $s=s T(x, s, \theta)$ where $T>0$ such that $F u$ is of the form 4.6.1.

First, replace $\theta_{i}$ by $\sum_{j} C_{j i} \theta_{j}$ to reduce $\varphi$ to

$$
\varphi(x, s, \theta)=\theta \cdot x-s \alpha(x, s, \theta)
$$

Now, write $\alpha=\alpha(0,0, \theta)+x \cdot \beta(x, s, \theta)+s \gamma(x, s, \theta)$ and let $\theta_{i}=\theta_{i}-s \beta_{i}$. Then,

$$
\varphi(x, s, \theta)=\theta \cdot x-s \alpha(\theta)+s^{2} \gamma(x, s, \theta)
$$

Now, 4.6.8 reads

$$
\begin{equation*}
x_{i}-s \frac{\partial \alpha}{\partial \theta_{i}}+s^{2} \frac{\partial \gamma}{\partial \theta_{i}}=0, \quad-\alpha(\theta)+\frac{\partial}{\partial s}\left(s^{2} \gamma(x, s, \theta)\right)=0 \tag{4.6.9}
\end{equation*}
$$

if and only if $x^{\prime}=0, \partial_{x_{1}} \varphi=0$, and $x_{1} \geq 0$. But, using (4.6.7), we have that $s \neq 0$ implies $x_{1} \neq 0$ and hence $x_{1}>0$. Thus, $\partial \alpha / \partial \theta_{1}>0$ and on the surface $S$, defined by (4.6.9). Hence, $s$ and $\theta^{\prime}=\left(\theta_{2}, \ldots, \theta_{d}\right)$ can be taken as coordinates. Moreover, for $i \geq 2, x_{i} \equiv 0$ on $S$ so that differentiating with respect to $s$ in the first equation of (4.6.9) and setting $s=0$ gives $\partial \alpha / \partial \theta^{\prime}=0$ on $\alpha=0$. But, $\partial \alpha / \partial \theta_{1} \neq 0$, so

$$
\alpha(\theta)=\left(\theta_{1}-\rho\left(\theta^{\prime}\right)\right) \beta(\theta)
$$

Now,

$$
0<\frac{\partial \alpha}{\partial \theta_{1}}=\beta(\theta)+\frac{\partial \beta}{\partial \theta_{1}}\left(\theta_{1}-\rho\left(\theta^{\prime}\right)\right) .
$$

But, on $\alpha=0, \theta_{1}-\rho\left(\theta^{\prime}\right)=0$ and hence $\beta>0$. Then, since $\beta=0$ implies $\alpha=0, \beta>0$. We also have that $\partial \rho / \partial \theta^{\prime}=0$ on $\alpha=0$ since

$$
0=\frac{\partial \alpha}{\partial \theta^{\prime}}=\frac{\partial \beta}{\partial \theta^{\prime}}\left(\theta_{1}-\rho\left(\theta^{\prime}\right)\right)+\beta(\theta) \frac{\partial \rho}{\partial \theta^{\prime}} .
$$

But, for every $\theta^{\prime}$ there is a $\theta_{1}$ such that $\theta_{1}-\rho\left(\theta^{\prime}\right)=0$ and hence $\alpha=0$. Therefore, $\partial \rho / \partial \theta^{\prime} \equiv 0$ and $\rho \equiv C_{1}$. Hence, by relabeling $\theta_{1}=\theta_{1}-C_{1}$, and $s=s \beta$, we have

$$
\varphi=\theta \cdot x+x_{1} C_{1}-s \theta_{1}+s^{2} \gamma(x, s, \theta)
$$

and, using (4.6.8), and setting $s=0, \theta_{1}=0$, we have $\partial \varphi / \partial x_{1}=\theta_{1}+C_{1}=0$. Therefore, $C_{1}=0$ and we have

$$
\varphi=\theta \cdot x-s \theta_{1}+s^{2} \gamma(x, s, \theta)
$$

Relabeling $\gamma=\alpha$ and repeating the argument gives for any fixed $k$ that

$$
\varphi(x, s, \theta)=x \cdot \theta-s \theta_{1}+s^{k} \gamma(x, s, \theta)
$$

Now, we apply the method used by Hörmander [39] to show that $\varphi$ is equivalent to $\phi(x, s, \theta)=x \cdot \theta-s \theta_{1}$ under a change of phase variables preserving $s=0$ and $s>0$.

The map

$$
\chi:(x, s, \theta) \mapsto\left(x, \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial \theta}, \frac{\partial \varphi}{\partial s}\right)
$$

is injective and, $\chi(x, s, \theta)=\left(x, \theta,\left(x_{1}-s, x^{\prime}\right),-\theta_{1}\right)+\mathcal{O}\left(s^{\infty}\right)$. Hence $\chi$ has a left inverse $\Psi(x, \xi, \eta, \sigma)$ such that, on the surface $\eta_{1}=x_{1}, \Psi\left(x, \theta,\left(x_{1}, \eta^{\prime}\right), \sigma\right)=(x, 0, \theta)$ to high order. Let $\kappa^{-1}(x, s, \theta)=\Psi\left(x, \theta,\left(x_{1}-s, x^{\prime}\right),-\theta_{1}\right)$ and put $\psi=\kappa^{*} \varphi$. Then $\kappa: \tilde{S} \rightarrow S$ and $\kappa$ is equal to the identity to high order at $s=0$.

Now, write

$$
\kappa(x, s, \theta)=(x, t(x, s, \theta), \eta(x, s, \theta)) .
$$

Then,

$$
\left.\frac{\partial \varphi}{\partial x}\right|_{(x, s, \theta)=(x, t, \eta)}=\theta,\left.\quad \frac{\partial \varphi}{\partial \theta}\right|_{(x, s, \theta)=(x, t, \eta)}=\left(x_{1}-s, x^{\prime}\right),\left.\quad \frac{\partial \varphi}{\partial s}\right|_{(x, s, \theta)=(x, t, \eta)}=-\theta_{1} .
$$

Hence, by (4.6.8) on $\tilde{S}$

$$
x_{1}-s=0, \quad x^{\prime}=0, \quad \theta_{1}=0 .
$$

Therefore, on $\theta_{1}=0, \partial_{s} \psi=0$ and hence $\partial_{x_{1}} \psi=0$. Thus, $(\psi-\phi)\left(x_{1}, x^{\prime}, x_{1}, 0, \theta^{\prime}\right)=0$. That is, $\psi-\phi$ vanishes on $\tilde{S}$. Note also that we have on $\tilde{S}$ that

$$
\partial \psi=\partial_{x^{\prime}} \varphi \frac{\partial x^{\prime}}{\partial x^{\prime}}=\partial_{x^{\prime}} \varphi=\theta^{\prime}=\partial \phi
$$

and we have $\partial(\psi-\phi)=0$. Hence $\psi-\phi$ vanishes to second order on $\tilde{S}$.
Thus,

$$
\psi(x, s, \theta)-\phi(x, s, \theta)=Z \cdot A \cdot Z
$$

where $Z=\left(x_{1}-s, x^{\prime},-\theta_{1}\right)=\left(\partial \phi / \partial \theta_{1}, \partial \phi / \partial \theta^{\prime}, \partial \phi / \partial s\right)$ and $A$ vanishes at $s=0$. We need to find a coordinate change $(\tilde{s}, \tilde{\theta})=(s, \theta)+B(x, s, \theta) \cdot Z$ such that $\varphi(x, s, \theta)=\phi(x, \tilde{s}, \tilde{\theta})$ and $B=0$ at $s=0$. Since

$$
\phi(x, \tilde{s}, \tilde{\theta})=\phi(x, s, \theta)+Z \cdot B \cdot Z+Z \cdot B \cdot G \cdot B \cdot Z
$$

where $G$ is a matrix depending smoothly on $x, \theta, s$ and $B$, it suffices to choose $B$ as the unique small solution of $B+B G B=A$. Then we have that $B=0$ at $s=0$ since $A=0$ there. Thus the phase functions $\phi$ and $\varphi$ are equivalent.

Now, the symbol calculus follows from [49]. We include the relevant results in the semiclassical setting.

First, suppose $\lambda_{0} \in \partial \Lambda_{1}$ and choose $h_{1}, \ldots, h_{d-1}$ functions whose differentials are linearly independent on $\partial \Lambda_{1}$ near $\lambda_{0}$. Choose also $f, g$ such that $f=0$ on $\Lambda_{0}, f>0$ on $\Lambda_{1} \backslash \partial \Lambda_{1}$, $d f\left(\lambda_{0}\right) \neq 0, g=0$ on $\Lambda_{1}, d g\left(\lambda_{0}\right) \neq 0$, and $\{f, g\}\left(\lambda_{0}\right)<0$. Let $a \in C^{\infty}\left(\Lambda_{0} \backslash \partial \Lambda_{1} ; \Omega^{1 / 2}\right)$ such that if $g \in C^{\infty}\left(\Lambda_{0}\right)$ vanishes on $\partial \Lambda_{1}$ then $g a \in C^{\infty}\left(\Lambda_{0}\right)$. Then write

$$
a=g^{-1} r\left|d h_{1} \wedge \cdots \wedge d h_{d-1} \wedge d g\right|^{1 / 2}
$$

and define

$$
R a:=r\left|d h_{1} \wedge \cdots \wedge d h_{d-1} \wedge d f\right|^{1 / 2}\{g, f\}^{-1 / 2}
$$

Then [49, Section 4] shows that $R$ is independent of the choice of $h_{i}, g$, and $f$ as above.
Definition 4.6.8. We define the symbol class

$$
S_{\delta}^{\text {comp }}\left(\Lambda_{0} \cup \Lambda_{1}\right) \subset h^{1 / 2} S_{\delta}^{\text {comp }}\left(\Lambda_{0} \backslash \partial \Lambda_{1} ; \Omega^{1 / 2}\right) \times S_{\delta}^{\text {comp }}\left(\Lambda_{1} ; \Omega^{1 / 2}\right)
$$

as the subspace consisting of those sections $(a, b)$ such that for all $g$ vanishing on $\partial \Lambda_{1}$, $g a \in C^{\infty}\left(\Lambda_{0}\right)$ and $\left.b\right|_{\partial \Lambda_{1}}=e^{\pi i / 4}(2 \pi)^{1 / 2} h^{-1 / 2} R(a)$.

Then we have the following [49, Theorem 4.13]
Lemma 4.6.9. The following sequence is exact:

$$
0 \hookrightarrow h^{1-2 \delta} I_{\delta}^{\text {comp }}\left(\Lambda_{0}, \Lambda_{1}\right) \hookrightarrow I_{\delta}^{\text {comp }}\left(\Lambda_{0}, \Lambda_{1}\right) \xrightarrow{\sigma} S_{\delta}^{\text {comp }}\left(\Lambda_{0} \cup \Lambda_{1}\right) \rightarrow 0 .
$$

Remark: Here $\sigma$ is the usual symbol map for Lagrangian distributions applied to each component $\Lambda_{0} \backslash \partial \Lambda_{1}$ and $\Lambda_{1}$ separately.

We need the analog of [49, Propositions 5.4 and 5.5] in the semiclassical setting. First, we characterize the appearance of transport equations. The following lemma follows from Proposition 4.4.12.

Lemma 4.6.10. Let $P \in \Psi_{\delta}^{m}(X)$ be a properly supported pseudodifferential operator such that $p:=\sigma(P)$ vanishes on the part $\Lambda_{1}$ of an intersecting pair $\left(\Lambda_{0}, \Lambda_{1}\right)$ of Lagrangians. Then for $u \in I_{\delta}^{m^{\prime}}\left(X ; \Lambda_{0}, \Lambda_{1}\right), P u=f+g, f \in h^{1 / 2} I_{\delta}^{m+m^{\prime}-1 / 2}\left(X, \Lambda_{0}\right), g \in h^{1-2 \delta} I_{\delta}^{m+m^{\prime}-1}\left(X ; \Lambda_{0}, \Lambda_{1}\right)$ and

$$
\left.\sigma(g)\right|_{\Lambda_{1}}=\left.\left(-i h \mathcal{L}_{H_{p}}+p_{1}\right) \sigma(u)\right|_{\Lambda_{1}}
$$

where $\mathcal{L}_{H_{p}}$ is the Lie action of the Hamilton vector field $H_{p}$ and $p_{1}$ is the subprincipal symbol of $P$.

Second, we need the asymptotic summability of the spaces $I_{\delta}^{m}\left(X ; \Lambda_{0}, \Lambda_{1}\right)$.
Lemma 4.6.11. Assume that $u_{j} \in h^{j(1-2 \delta)} I_{\delta}^{m-j}\left(X ; \Lambda_{0}, \Lambda_{1}\right)$ for $j=0,1, \ldots$ then there exists $u \in I_{\delta}^{m}\left(X ; \Lambda_{0}, \Lambda_{1}\right)$ such that for every $N$ there exists $N^{\prime}>0$ large enough such that

$$
u-\sum_{j=0}^{N^{\prime}} u_{j} \in h^{N} C^{N}(X)
$$

Finally, we need the following analog of [49, Proposition 6.6]. Define the characteristic set of $P$,

$$
\Sigma(P)=\left\{\nu \in T^{*} X: \sigma(P)(\nu)=0\right\}
$$

We say that $P \in \Psi_{\delta}^{m}$ is of real principal type if, letting $p:=\sigma(P)$ and $p_{1}:=\sigma_{1}(P)$, the subprincipal symbol, $p$ is real,

$$
\partial p(q) \neq 0 \text { for } q \in \Sigma(P)
$$

and $\operatorname{Im} p_{1} \geq 0$. We say that $P \in \Psi_{\delta}^{m}$ is elliptic if there exists $M>0$ such that for $|\xi| \geq M$, $|\sigma(P)| \geq C|\xi|^{m}$.

Lemma 4.6.12. Let $P \in \Psi_{\delta}^{m}(X)$ be elliptic and of real principal type. Then let

$$
\Lambda_{0}=\left\{(x, \xi, x,-\xi) \in T^{*} X \times T^{*} X\right\}
$$

and $\Lambda_{1}^{e}$ be the $H_{p}$ flow out of $\Lambda_{0} \cap \Sigma(P)$ with orientation $e$. Assume that $\exp \left(t H_{p}\right)$ is nontrapping on $\Sigma(P)$. Then there exists $u \in h^{-1 / 2} I_{\delta}^{-m}\left(X \times X ; \Lambda_{0}, \Lambda_{1}^{e}\right)$, such that for each $K \Subset M$,

$$
P(u+v)=\delta\left(y, y^{\prime}\right)+O_{\mathcal{D}^{\prime} \rightarrow C^{\infty}}\left(h^{\infty}\right) \text { for }\left(y, y^{\prime}\right) \in K \times K
$$

In particular, we have take

$$
\sigma(u)=\left(\sigma(P)^{-1}, r\right) \in h^{-1 / 2} S_{\delta}^{-m}\left(\Lambda_{0} \cup \Lambda_{1} ; \Omega^{1 / 2}\right)
$$

where $r$ solves

$$
\begin{equation*}
h \mathcal{L}_{H_{p}} r+i p_{1} r=0,\left.\quad r\right|_{\partial \Lambda_{1}}=e^{\pi i / 4}(2 \pi)^{1 / 2} h^{-1 / 2} R\left(\chi \sigma(P)^{-1}\right) \tag{4.6.10}
\end{equation*}
$$

and where $p_{1}$ is the subprincipal symbol of $P$.

Remark: Lemma 4.6.12 gives us the kernel of a right parametrix for $P u=f$.
Proof. First, let $\chi \in C_{c}^{\infty}\left(T^{*} M \times T^{*} M\right)$ have $\chi \equiv 1$ on $\Lambda_{1}^{e}$ and $\chi_{1} \in C_{c}^{\infty}(M)$ have $\chi_{1} \equiv 1$ on $K$. Then, since $\mathrm{WF}_{\mathrm{h}}\left(\mathrm{Op}_{\mathrm{h}}(1-\chi)\right) \cap \Lambda_{1}^{e}=\emptyset$, there exists $v \in I_{\delta}^{-m}\left(\Lambda_{0}\right)$ such that

$$
P v=\left(1-\operatorname{Op}_{\mathrm{h}}(\chi)\right) \delta\left(y, y^{\prime}\right) \chi_{1}(y) \chi_{1}\left(y^{\prime}\right)+O_{\mathcal{D}^{\prime} \rightarrow C^{\infty}}\left(h^{\infty}\right) .
$$

In particular, $v$ is the kernel of a pseudodifferential operator $V \in \Psi_{\delta}^{-m}(X)$.
We now solve $P u=\operatorname{Op}_{\mathrm{h}}(\chi) \delta\left(y, y^{\prime}\right) \chi_{1}(y) \chi_{1}\left(y^{\prime}\right)+O_{\mathcal{D}^{\prime} \rightarrow C^{\infty}}\left(h^{\infty}\right)$. To do so, we proceed symbolically. Suppose that $u_{0} \in h^{-1 / 2} I_{\delta}^{\text {comp }}\left(X \times X ; \Lambda_{0}, \Lambda_{1}^{e}\right)$. Then we have

$$
P u_{0}=f_{0}+g_{0}, \quad f_{0} \in I_{\delta}^{\text {comp }}\left(\Lambda_{0}\right), \quad g_{0} \in h^{1 / 2-2 \delta} I_{\delta}^{\text {comp }}\left(X \times X ; \Lambda_{0}, \Lambda_{1}^{e}\right) .
$$

Now,

$$
\sigma\left(f_{0}\right)=\left.\sigma(P) \sigma\left(u_{0}\right)\right|_{\Lambda_{0}}
$$

Thus, writing $p=\sigma(P)$, we have

$$
\left.\sigma\left(u_{0}\right)\right|_{\Lambda_{0} \backslash \partial \Lambda_{1}^{e}}=p^{-1} \sigma\left(\mathrm{Op}_{\mathrm{h}}(\chi) \delta\left(y, y^{\prime}\right) \chi_{1}(y) \chi_{1}\left(y^{\prime}\right)\right) \in S_{\delta}^{\text {comp }}\left(\Lambda_{0} \backslash \partial \Lambda_{1}^{e}\right)
$$

Thus, using the fact that $\chi \equiv 1$ on $\Lambda_{1}^{e}$,

$$
\left.\sigma\left(u_{0}\right)\right|_{\partial \Lambda_{1}}=e^{\pi i / 4}(2 \pi)^{1 / 2} h^{-1 / 2} R\left(p^{-1} \sigma\left(\delta\left(y, y^{\prime}\right) \chi_{1}(y) \chi_{1}\left(y^{\prime}\right)\right)\right)
$$

and hence

$$
\left.\sigma\left(g_{0}\right)\right|_{\Lambda_{1}}=\left(-i h \mathcal{L}_{H_{p}}+p_{1}\right) \sigma\left(u_{0}\right)
$$

where $p_{1}$ is the subprincipal symbol of $P$. Thus, $\sigma\left(g_{0}\right)=0$ on $\Lambda_{1}$ yields the transport equation

$$
h \mathcal{L}_{H_{p}} \sigma\left(u_{0}\right)+i p_{1} \sigma\left(u_{0}\right)=0 \text { on } \Lambda_{1} .
$$

Under our assumptions, [20, Section 6.4] gives that this equation has a unique solution. Then since $\operatorname{Im} p_{1} \geq 0$ and $\left.\sigma\left(u_{0}\right)\right|_{\Lambda_{0}} \in S_{\delta}^{\text {comp }}$, we have that for $q \in \partial \Lambda_{1}^{e},\left.u_{0}\right|_{\Lambda_{1}^{e} \cap T^{*} K} \in h^{-1 / 2} S_{\delta}^{\text {comp }}$. Thus, for $\left(y, y^{\prime}\right) \in K$,

$$
P u_{0}-\delta\left(y, y^{\prime}\right) \chi_{1}(y) \chi_{1}\left(y^{\prime}\right)=f_{1}+g_{1} \in h^{1-2 \delta} I_{\delta}^{\text {comp }}\left(\Lambda_{0}\right)+h^{3 / 2-4 \delta} I_{\delta}^{\text {comp }}\left(X ; \Lambda_{0}, \Lambda_{1}^{e}\right) .
$$

Finally, let $\chi_{2} \in C_{c}^{\infty}(M)$ have $\chi_{2} \equiv 1$ on supp $\chi_{1}$. Then relabel $u_{0}=\chi_{2}(y) \chi_{2}\left(y^{\prime}\right) u_{0}$. Now, we proceed iteratively to find $u_{j} \in h^{j(1-2 \delta)-1 / 2} I_{\delta}^{\text {comp }}\left(X ; \Lambda_{0}, \Lambda_{1}^{e}\right)$, given $f_{j} \in h^{j(1-2 \delta)} I_{\delta}^{\text {comp }}\left(\Lambda_{0}\right)$, and $g_{j} \in h^{j(1-2 \delta)+1 / 2-2 \delta} I_{\delta}^{\text {comp }}\left(X \times X ; \Lambda_{0}, \Lambda_{1}^{e}\right)$, such that $\left.\sigma\left(u_{j}\right)\right|_{\Lambda_{0}}=\left.p^{-1} \sigma\left(f_{j}\right)\right|_{\Lambda_{0}}$,

$$
h \mathcal{L}_{H_{p}} \sigma\left(u_{j}\right)+i p_{1} \sigma\left(u_{j}\right)=i \sigma\left(g_{j}\right) \text { on } \Lambda_{1}^{e} .
$$

As above, the transport equation has a unique solution satisfying the initial condition

$$
\left.\sigma\left(u_{j}\right)\right|_{\partial \Lambda_{1}}=e^{\pi i / 4}(2 \pi)^{1 / 2} h^{-1 / 2} R\left(p^{-1} \sigma\left(f_{j}\right)\right) .
$$

Then, letting

$$
u \sim \sum u_{j}
$$

gives that for $\left(y, y^{\prime}\right) \in K \times K$

$$
P(u+v)=\delta\left(y, y^{\prime}\right)+O_{\mathcal{D}^{\prime} \rightarrow C^{\infty}}\left(h^{\infty}\right)
$$

as desired. Then simply relabel $u=u+v$.

## Chapter 5

## The Semiclassical Melrose-Taylor Parametrix

Let $\Omega \subset \mathbb{R}^{d}$ be strictly convex with smooth boundary. We construct parametrices for

$$
\begin{equation*}
\left(-h^{2} \Delta-z\right) u=0 \text { in } \Omega_{i},\left.\quad u\right|_{\partial \Omega}=f \tag{5.0.1}
\end{equation*}
$$

where $\Omega_{1}=\Omega$ and $\Omega_{2}=\mathbb{R}^{d} \backslash \bar{\Omega}$, and $f$ is microlocalized near glancing.
We give a construction similar to that in [32, Appendix A.II.3] and [77, Chapter 11] in $\Omega_{2}$ and adapt the results there to the case of $\Omega_{1}$ using methods similar to those in 47, Chapter 7]. Throughout, we assume $z=1+i \operatorname{Im} z,-C h \log h^{-1} \leq \operatorname{Im} z \leq C h \log h^{-1}$.
Remark: To obtain $\operatorname{Re} z \neq 1$, we simply rescale $h$ in the resulting parametrices.
Define $\epsilon(h)$ and $\mu(h)$ by

$$
h \leq \epsilon(h):=\max (h,|\operatorname{Im} z|)=O\left(h \log h^{-1}\right) \quad \mu(h):=\operatorname{Im} z
$$

We construct parametrices in a neighborhood of glancing where the size of the neighborhood will depend on $\epsilon(h)$.

In particular, if $\chi \in C_{c}^{\infty}(\mathbb{R}), \chi \equiv 1$ in a neighborhood of 0 . Then, let $x_{0} \in \partial \Omega, \delta>0$ and define

$$
\chi_{h}^{\delta, \gamma}(x, \xi):=\chi\left(\frac{\left|\xi^{\prime}\right|_{g}-1}{\gamma\left[h(\epsilon(h))^{-1}\right]^{2}}\right) \chi\left(\delta^{-1}\left|x-x_{0}\right|\right)
$$

where $|\cdot|_{g}$ denotes the norm induced on the $T^{*} \partial \Omega$ by the euclidean metric restricted to the boundary. Then $\chi_{h}^{\delta, \gamma}$ localizes microlocally near a glancing point $\left(x_{0}, \xi_{0}\right)$. We construct an operator $H$ such that

$$
\left\{\begin{array}{l}
\left(-h^{2} \Delta-z\right) H f=\mathcal{O}_{C^{\infty}}\left(h^{\infty}\right) \quad \text { in } \Omega_{i}  \tag{5.0.2}\\
H f=\operatorname{Op}_{\mathrm{h}}\left(\chi_{h}^{\delta, \gamma}\right) f+\mathcal{O}_{C^{\infty}}\left(h^{\infty}\right) \quad \text { in a neighborhood of } x_{0} \in \partial \Omega \\
H f \text { is outgoing if } \Omega_{i}=\Omega_{2}
\end{array}\right.
$$

In fact, we need to construct two such operators $H_{g}$ for gliding points and $H_{d}$ for diffractive points corresponding to $\Omega_{1}$ and $\Omega_{2}$ respectively.

We are able to construct the operators $H_{d}$ in the entire region $|\operatorname{Im} z| \leq C h \log h^{-1}$. Then, using the arguments in [69, Appendix A.5], we show that $H_{d}$ is $O_{C^{\infty}}\left(h^{\infty}\right)$ close to the true solution operator.

On the other hand, while we are still able to construct the operator $H_{g}$ in the entire region $|\operatorname{Im} z| \leq C h \log h^{-1}$, it is not possible to show that $H_{g} f$ is close to the solution to (5.0.1) in $\Omega_{1}$ when $|\operatorname{Im} z|=O\left(h^{\infty}\right)$. This is due to the presence of Dirichlet eigenvalues on the real axis. When $|\operatorname{Im} z| \geq C h^{M}$ for some $M$, we can invert the Dirichlet problem to show that $H_{g} f$ is $O_{C^{\infty}}\left(h^{\infty}\right)$ close to the solution to (5.0.1). Despite the fact that $H_{g}$ may not be close to the solution operator near $\operatorname{Im} z=0$, we are able to use it to construct a microlocal model for boundary layer operators even when $|\operatorname{Im} z|=O\left(h^{\infty}\right)$.

### 5.1 Semiclassical Melrose-Taylor Parametrix for Complex Energies

Following [47, Chapter 7] and [52], the ansatze for our constructions will be Fourier-Airy integral operators [51] of the form:

$$
\begin{align*}
B_{1} F & :=(2 \pi h)^{-d+1} \int\left[g_{0} A_{-}\left(h^{-2 / 3} \rho\right)+i h^{1 / 3} g_{1} A_{-}^{\prime}\left(h^{-2 / 3} \rho\right)\right] A_{-}\left(h^{-2 / 3} \alpha\right)^{-1} e^{i \theta / h} \mathcal{F}_{h} F(\xi) d \xi  \tag{5.1.1}\\
B_{2} F & :=(2 \pi h)^{-d+1} \int\left[g_{0} A i\left(h^{-2 / 3} \rho\right)+i h^{1 / 3} g_{1} A i^{\prime}\left(h^{-2 / 3} \rho\right)\right] A i\left(h^{-2 / 3} \alpha\right)^{-1} e^{i \theta / h} \mathcal{F}_{h} F(\xi) d \xi \tag{5.1.2}
\end{align*}
$$

where $F$ and $f$ will be related below, $\left.\rho\right|_{\partial \Omega}=\alpha, \rho, \theta \in C^{\infty}$ solve certain eikonal equations, $g_{0}, g_{1}$ solve transport equations, and $A i$ is the solution to $-A^{\prime \prime}(s)+s A=0$ given by

$$
A i(s)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{i\left(s t+t^{3} / 3\right)} d t
$$

for $s$ real, and $A_{-}(z)=A i\left(e^{2 \pi i / 3} z\right)$. Finally, $\left.\theta\right|_{\partial \Omega}$ will parametrize the canonical transformation reducing the billiard ball map for glancing pair $\{x \in \partial \Omega\}$ and $\left\{|\xi|^{2}-1=0\right\}$ to that for the Friedlander normal form

$$
\begin{equation*}
Q_{\text {Fried }}:=\left\{x_{d}=0\right\} \subset T^{*} \mathbb{R}^{d} \quad \text { and } \quad P_{\text {Fried }}:=\left\{\xi_{d}^{2}-x_{d}+\xi_{1}=0\right\} \subset T^{*} \mathbb{R}^{d} \tag{5.1.3}
\end{equation*}
$$

The Hamiltonian flow for this system is shown in Figure 5.1.


Figure 5.1: The figure shows several trajectories of the Hamiltonian flow for the Friedlander model. When $\xi_{1}=0$, the trajectory is tangent to the boundary $x_{d}=0$ and hence is glancing. When, $\xi_{1}<0$, the billiard ball map takes the upper intersection with $x_{d}=0$ to the lower. This corresponds to the hyperbolic region. Finally, when $\xi_{1}>0$, the trajectory does not intersect the boundary and hence this corresponds to the elliptic region.

## The Friedlander Model

As a first, step, we consider the Friedlander model. This toy example guides us when we consider the general case. The Friedlander model is given by

$$
P=\left(h D_{x_{d}}\right)^{2}-x_{d}+h D_{x_{1}} \quad \partial \Omega=\left\{x_{d}=0\right\}
$$

Suppose that

$$
\begin{equation*}
(P-i \mu) u=\left.0 \quad u\right|_{\partial \Omega}=f \tag{5.1.4}
\end{equation*}
$$

Then, taking the semiclassical Fourier transform in the $x^{\prime}$ variables gives

$$
\left(-h^{2} \partial_{x_{d}}^{2}-x_{d}+\xi_{1}-i \mu\right) \mathcal{F}_{h, x^{\prime}} u\left(x_{d}, \xi^{\prime}\right)=0 \quad \mathcal{F}_{h, x^{\prime}} u\left(0, \xi^{\prime}\right)=\mathcal{F}_{h}(f)\left(\xi^{\prime}\right)
$$

The solution to this problem for $\mu=0$ is

$$
u=(2 \pi h)^{-d+1} \int \frac{A\left(h^{-2 / 3}\left(-x_{d}+\xi_{1}\right)\right)}{A\left(h^{-2 / 3} \xi_{1}\right)} e^{\frac{i}{h}\left\langle x^{\prime}, \xi^{\prime}\right\rangle} \mathcal{F}_{h}(f)\left(\xi^{\prime}\right) d \xi^{\prime}
$$

where $A$ is a solution to the Airy equation. Let $\rho_{0}:=-x_{d}+\xi_{1}$ and $\theta_{0}=\left\langle x^{\prime}, \xi^{\prime}\right\rangle$. Now, suppose that $\mu=O\left(h \log h^{-1}\right) \neq 0$. We could simply replace $\rho_{0}$ by $-x_{d}+\xi_{1}-i \mu$, however, because the function $A i$ has zeros on the real axis, it is more convenient when we consider the general case to make a perturbation of $\theta_{0}$ and $\rho_{0}$ so that uniformly in $\mu,\left.\rho_{0}\right|_{x_{d}=0}$ has
nonzero imaginary part. To do this, we compute

$$
\begin{aligned}
P\left(A\left(h^{-2 / 3} \rho\right) e^{\frac{i}{h} \theta}\right)= & {\left[\left(\partial_{x_{d}} \theta\right)^{2}-\rho\left(\partial_{x_{d}} \rho\right)^{2}+\partial_{x_{1}} \theta-x_{d}\right] A\left(h^{-2 / 3} \rho\right) e^{\frac{i}{h} \theta} } \\
& -i h^{1 / 3}\left[2 \partial_{x_{d}} \rho \partial_{x_{d}} \theta+\partial_{x_{1}} \rho\right] A^{\prime}\left(h^{-2 / 3} \rho\right) e^{\frac{i}{h} \theta}
\end{aligned}
$$

So, we seek to find $\theta$ and $\rho$ solving the model eikonal equations

$$
\left\{\begin{array}{l}
\left(\partial_{x_{d}} \theta\right)^{2}-\rho\left(\partial_{x_{d}} \rho\right)^{2}+\partial_{x_{1}} \theta-x_{d}=i \mu \\
2 \partial_{x_{d}} \rho \partial_{x_{d}} \theta+\partial_{x_{1}} \rho=0
\end{array}\right.
$$

We find $\rho \sim \sum_{n \geq 0} \rho_{n} \epsilon(h)^{n}$ and $\theta \sim \sum_{n \geq 0} \theta_{n} \epsilon(h)^{n}$ where $\rho_{0}$ and $\theta_{0}$ are as above, $\theta_{n}=$ $\theta_{n}\left(x, \xi^{\prime}, \mu\right)$, and $\rho_{n}=\rho_{n}\left(x, \xi^{\prime}, \mu\right)$. Then, we solve for $\rho_{n}, \theta_{n}$ successively by solving transport equations of the form

$$
\left\{\begin{array}{l}
2 \partial_{x_{d}} \theta_{0} \partial_{x_{d}} \theta_{n}-2 \rho_{0} \partial_{x_{d}} \rho_{0} \partial_{x_{d}} \rho_{n}-\rho_{n}\left(\partial_{x_{d}} \rho_{0}\right)^{2}+\partial_{x_{1}} \theta_{n}=F_{1} \\
2 \partial_{x_{d}} \rho_{0} \partial_{x_{d}} \theta_{n}+2 \partial_{x_{d}} \theta_{0} \partial_{x_{d}} \rho_{n}+\partial_{x_{1}} \rho_{n}=F_{2}
\end{array}\right.
$$

where $F_{1}$ and $F_{2}$ depend on $\theta_{n}$ and $\rho_{n}$ for $m<n$.
In the next section, we construct solutions to these equations with $\rho_{1}\left(x_{1}, 0, \xi^{\prime}\right)=i$. With these solutions in hand $u$ will solve (5.1.4) up to $O\left(h^{\infty}\right)$.

### 5.2 Eikonal and Transport Equations

First, we consider a general differential operator

$$
P(x, h D)=\sum a_{j k}(x) h D_{j} h D_{k}+\sum b_{j}(x) h D_{j}+c(x)
$$

with $a_{j k}=a_{k j}$ applied to (5.1.1) and (5.1.2). Then, for $A$ an Airy function, we have, letting $f_{j}$ denote $\partial_{j} f$, and $\rho_{h}=h^{-2 / 3} \rho$

$$
\begin{aligned}
h D_{j}\left(g A\left(\rho_{h}\right) e^{\frac{i}{h} \theta}\right)= & \theta_{j} g A\left(\rho_{h}\right) e^{\frac{i}{h} \theta}-i h g_{j} A\left(\rho_{h}\right) e^{\frac{i}{h} \theta}-i h^{1 / 3} \rho_{j} g A^{\prime}\left(\rho_{h}\right) e^{\frac{i}{h} \theta} \\
h D_{k} h D_{j}\left(g A\left(\rho_{h}\right) e^{\frac{i}{h} \theta}\right)= & {\left[\left(\theta_{k} \theta_{j}-\rho_{j} \rho_{k} \rho\right) g-i h\left(\theta_{k} g_{j}+\theta_{j} g_{k}+\theta_{j k} g\right)-h^{2} g_{j k}\right] A\left(\rho_{h}\right) e^{\frac{i}{h} \theta} } \\
& -i h^{1 / 3}\left[\left(\theta_{j} \rho_{k}+\rho_{j} \theta_{k}\right) g-i h\left(g_{j} \rho_{k}+\rho_{j} g_{k}+\rho_{j k} g\right)\right] A^{\prime}\left(\rho_{h}\right) e^{\frac{i}{h} \theta} \\
h D_{j}\left(g A^{\prime}\left(\rho_{h}\right) e^{\frac{i}{h} \theta}\right)= & \theta_{j} g A^{\prime}\left(\rho_{h}\right) e^{\frac{i}{h} \theta}-i h g_{j} A^{\prime}\left(\rho_{h}\right) e^{\frac{i}{h} \theta}-i h^{-1 / 3} \rho_{j} \rho g A\left(\rho_{h}\right) e^{\frac{i}{h} \theta} \\
h D_{k} h D_{j}\left(g A^{\prime}\left(\rho_{h}\right) e^{\frac{i}{h} \theta}\right)= & -i h^{-1 / 3}\left[\left(\theta_{j} \rho_{k}+\theta_{k} \rho_{j}\right) \rho g\right. \\
& \left.-i h\left(g_{j} \rho_{k} \rho+g_{k} \rho_{j} \rho+\rho_{j k} \rho g+\rho_{j} \rho_{k} g\right)\right] A\left(\rho_{h}\right) e^{\frac{i}{h} \theta} \\
& +\left[\left(\theta_{j} \theta_{k}-\rho_{j} \rho_{k} \rho\right) g-i h\left(\theta_{k j} g+\theta_{j} g_{k}+\theta_{k} g_{j}\right)-h^{2} g_{j k}\right] A^{\prime}\left(\rho_{h}\right) e^{\frac{i}{h} \theta}
\end{aligned}
$$

So,

$$
\begin{aligned}
P\left(g_{0} A\left(\rho_{h}\right) e^{\frac{i}{h} \theta}\right)= & {\left[\begin{array}{r}
(\langle a d \theta, d \theta\rangle-\rho\langle a d \rho, d \rho\rangle+\langle b, d \theta\rangle+c) g_{0} \\
-i h\left(2\left\langle a d \theta, d g_{0}\right\rangle-P_{2} \theta g_{0}+\left\langle b, d g_{0}\right\rangle\right)+h^{2} P_{2} g_{0}
\end{array}\right] A\left(\rho_{h}\right) e^{\frac{i}{h} \theta} } \\
& -i h^{1 / 3}\left[\begin{array}{r}
(2\langle a d \theta, d \rho\rangle+\langle b, d \rho\rangle) g_{0} \\
-i h\left(2\left\langle a d \rho, d g_{0}\right\rangle-\left(P_{2} \rho\right) g_{0}\right)
\end{array}\right] A^{\prime}\left(\rho_{h}\right) e^{\frac{i}{h} \theta} \\
P\left(i h^{1 / 3} g_{1} A^{\prime}\left(\rho_{h}\right) e^{\frac{i}{h} \theta}\right)= & {\left[\begin{array}{r}
\rho(2\langle a d \theta, d \rho\rangle+\langle b, d \rho\rangle) g_{1} \\
-i h\left(2 \rho\left\langle a d \rho, d g_{1}\right\rangle+\langle a d \rho, d \rho\rangle g_{1}-\rho\left(P_{2} \rho\right) g_{1}\right)
\end{array}\right] A\left(\rho_{h}\right) e^{\frac{i}{h} \theta} } \\
& +i h^{1 / 3}\left[\begin{array}{r}
(\langle a d \theta, d \theta\rangle-\rho\langle a d \rho, d \rho\rangle+\langle b, d \theta\rangle+c) g_{1} \\
-i h\left(2\left\langle a d \theta, d g_{1}\right\rangle-\left(P_{2} \theta\right) g_{1}+\left\langle b, d g_{1}\right\rangle\right)+h^{2} P_{2} g_{1}
\end{array}\right] A^{\prime}\left(\rho_{h}\right) e^{\frac{i}{h} \theta}
\end{aligned}
$$

where $a_{j k}=a_{j k}(x), P_{2}=h^{-2}(P-\langle b, h D\rangle-c(x))$ and $\langle\cdot, \cdot\rangle$ denotes the euclidean inner product.

Now, applying $P$ under the integral in (5.1.1) and (5.1.2) gives the eikonal equations

$$
\left\{\begin{array}{l}
\langle a d \theta, d \theta\rangle-\rho\langle a d \rho, d \rho\rangle+\langle b, d \theta\rangle+c=0  \tag{5.2.1}\\
2\langle a d \theta, d \rho\rangle+\langle b, d \rho\rangle=0
\end{array}\right.
$$

Writing

$$
\begin{equation*}
\phi^{ \pm}=\theta \pm \frac{2}{3}(-\rho)^{3 / 2}, \tag{5.2.2}
\end{equation*}
$$

the eikonal equations are equivalent to $p\left(x, d \phi^{ \pm}\right)=0$. Now, suppose that $\rho$ has the form $\sum_{n \geq 0} \rho_{n} \epsilon(h)^{n}$ and $\theta$ has the form $\sum_{n \geq 0} \theta_{n} \epsilon(h)^{n}$ and

$$
g_{i} \sim \sum_{n} g_{i}^{[n]}\left(x, \xi^{\prime}, \mu\right) h^{n}
$$

Then the transport equations have the form

$$
\left\{\begin{array}{r}
2\left\langle a d \theta_{0}, d g_{0}^{[n]}\right\rangle+2 \rho_{0}\left\langle a d \rho_{0}, d g_{1}^{[n]}\right\rangle+\left\langle b, d g_{0}^{[n]}\right\rangle+\left\langle a d \rho_{0}, d \rho_{0}\right\rangle g_{1}^{[n]}-P_{2} \theta_{0} g_{0}^{[n]}-\rho_{0}\left(P_{2} \rho_{0}\right) g_{1}^{[n]}  \tag{5.2.3}\\
=F_{1}^{[n]}\left(\theta, \rho, g_{i}^{[m]<[n]}, \mu\right) \\
2\left\langle a d \rho_{0}, d g_{0}^{[n]}\right\rangle-2\left\langle a d \theta_{0}, d g_{1}^{[n]}\right\rangle-\left\langle b, d g_{1}^{[n]}\right\rangle-\left(P_{2} \rho_{0}\right) g_{0}^{[n]}+\left(P_{2} \theta_{0}\right) g_{1}^{[n]}=F_{2}^{k, m}\left(\theta, \rho, g_{i}^{[m]<[n]}, \mu\right)
\end{array}\right.
$$

More generally, we consider transport equations of the form

$$
\left\{\begin{array}{l}
2\left\langle a d \theta_{0}, d g_{0}\right\rangle+2 \rho_{0}\left\langle a d \rho_{0}, d g_{1}\right\rangle+\left\langle b, d g_{0}\right\rangle+\left\langle a d \rho_{0}, d \rho_{0}\right\rangle g_{1}+B_{1} g_{0}+\rho_{0} B_{2} g_{1}=F_{1}  \tag{5.2.4}\\
2\left\langle a d \rho_{0}, d g_{0}\right\rangle-2\left\langle a d \theta_{0}, d g_{1}\right\rangle-\left\langle b, d g_{1}\right\rangle+B_{2} g_{0}-B_{1} g_{1}=F_{2}
\end{array}\right.
$$

Then, these equations are equivalent to

$$
\begin{equation*}
2\left\langle a d \phi^{ \pm}, g^{ \pm}\right\rangle+\left\langle b, d g^{ \pm}\right\rangle+G^{ \pm} g^{ \pm}=F^{ \pm} \tag{5.2.5}
\end{equation*}
$$

where

$$
g^{ \pm}=g_{0} \pm\left(-\rho_{0}\right)^{1 / 2} g_{1} \quad G^{ \pm}=B_{1} \mp\left(-\rho_{0}\right)^{1 / 2} B_{2} \quad F^{ \pm}=F_{1} \mp\left(-\rho_{0}\right)^{1 / 2} F_{2} .
$$

We use the equivalence of glancing hypersurfaces to construct solutions of the eikonal equations near a glancing point. In particular, let $p(x, \xi)$ be the symbol of $P(x, h D)$ and $B$ a hypersurface in $M$. Let $P=\{p(x, \xi)=0\}$ and $Q=\{(x, \xi): x \in B\}$ be a pair of glancing manifolds at $m=\left(x_{0}, \xi_{0}\right) \in P \cap Q$. That is, if $q(x, \xi)=q(x)$ is a defining function for $Q$

$$
\begin{gathered}
d p \text { and } d q \text { are linearly independent at } m \\
\{p, q\}=0 \quad\{p,\{p, q\}\} \neq 0 \quad\{q,\{q, p\}\} \neq 0
\end{gathered}
$$

Then the equivalence of glancing hypersurfaces (see for example [41, Theorem 21.4.8]) gives the existence of neighborhoods $V$ of $m$ and $U$ of 0 and a symplectomorphism $\kappa: U \rightarrow V$ reducing $P$ and $Q$ to the normal form (5.1.3). Since $Q$ is the lift of a hypersurface to $T^{*} \mathbb{R}^{d}$, this also induces a symplectomorphism

$$
\kappa_{\partial}: \gamma \rightarrow T^{*} B \quad \gamma:=\left\{\left(y^{\prime}, \eta^{\prime}\right) \in T^{*} \mathbb{R}^{d-1}:\left(y^{\prime}, y_{d}, \eta^{\prime}, \eta_{d}\right) \in U \text { for some } \eta_{d}\right\}
$$

such that $\kappa_{\partial}$ intertwines the billiard ball map on $T^{*} B$ with that on $T^{*} Q_{\text {Fried }}$.
We assume further that $H_{p}$ is not tangent to $T_{x} \mathbb{R}^{d}$ at $x=\pi(m)$. This allows us to conclude that

$$
\begin{equation*}
\kappa_{\partial}^{*}\left(d \eta_{j}\right), \quad j=1, \ldots d-1 \text { are linearly independent on } T_{x}^{*} B . \tag{5.2.6}
\end{equation*}
$$

To see this, observe that the projection of $H_{p}$ onto $T^{*} B$ is not tangent to $T_{x} B$. This image is the direction of the Hamilton vector field on the fold set and hence it follows that $\partial_{y_{1}}$ is not tangent to

$$
\mathcal{H}:=\kappa_{\partial}^{-1}\left(T_{x}^{*} B\right) .
$$

Observe that $\mathcal{H}$ is Lagrangian and hence $d \eta_{1} \neq 0$ on $\mathcal{H}$. Hence, there exists a symplectic change of coordinates on leaving $\left(y_{1}, \eta_{1}\right)$ fixed such that $d \eta_{j} j=1 \ldots d-1$ are independent on $\mathcal{H}$ and therefore that (5.2.6) holds. This transformation can clearly be extended to leave $Q_{\text {Fried }}$ and $P_{\text {Fried }}$ fixed.

Now, consider

$$
Y: P \rightarrow M \times \mathbb{R}^{d-1} \quad P \ni p \mapsto\left(\pi(p), \eta_{1}\left(\kappa^{-1}(p)\right), \ldots, \eta_{d-1}\left(\kappa^{-1}(p)\right)\right) .
$$

Lemma 5.2.1. The map $Y$ is a fold at $m$. Moreover, the fold set meets $Q_{\text {Fried }}$ transverally at $\xi_{d}=0$.

Proof. Let $q \in C^{\infty}(M)$ be a defining function for $B$. Then $d q \neq 0$ on $P$ near $m$. Thus, we need only consider the restriction of $Y$ to the intersection of $P$ and $Q$. That is,

$$
Y^{\prime}: P \cap Q \rightarrow B \times \mathbb{R}^{d-1}, \quad Y^{\prime}=\left.Y\right|_{P \cap Q} .
$$

But, the map from $P \cap Q$ to $T^{*} B$ is a fold and $Y^{\prime}$ is this projection composed with replacement of the fiber variables by $\eta_{j} j=1, \ldots d-1$ which has bijective differential. Hence, $Y^{\prime}$ and $Y$ are folds with the desired properties.

The construction of solutions to the eikonal equations near a glancing point now follows from [47, Proposition 4.3.1] which we include here.

Lemma 5.2.2. Let $p$ be the (real) principal symbol of a differential operator with $C^{\infty}$ coefficients in a neighborhood of $B \subset \mathbb{R}^{d}$ with defining function $x_{d}$. If $P$ and $Q$ form a glancing pair at $m$ then there exist real functions $\theta_{0}$ and $\rho_{0}$ smooth in a neighborhood, $\Sigma$, of $\pi(m) \times\{0\} \in \mathbb{R}^{d} \times \mathbb{R}^{d-1}$ such that

$$
\begin{gathered}
\rho_{0}=\eta_{1} \text { on } \Sigma \cap\left(B \times \mathbb{R}^{d-1}\right) \\
\left.\theta\right|_{B} \text { parametrizes } \kappa_{\partial}, \text { the reduction of } P \text { and } Q \text { to normal form } \\
d_{x} \partial_{\eta_{j}} \theta, \quad j=1 \ldots d-1 \text { are linearly independent on } \Sigma \\
\rho_{0} \text { is a defining function for the fold }
\end{gathered}
$$

and $\rho_{0}$ and $\theta_{0}$ solve the eikonal equations (5.2.1) in $\rho_{0} \leq 0$ and in Taylor series on $B$.
Proof. Let

$$
\Lambda_{\eta^{\prime}}=\left\{p \in P: Y(p)=\left(\cdot, \eta^{\prime}\right): \eta^{\prime} \in \mathbb{R}^{d-1}\right\}
$$

Then, $\Lambda_{\eta^{\prime}}$ are Lagrangian submanifolds foliating $P$ near $m$. To see that they are Lagrangian, observe that

$$
\kappa^{-1}\left(\Lambda_{\xi^{\prime}}\right)=\left\{\left(\left(y^{\prime}, \eta_{1}+\eta_{d}^{2}\right),\left(\eta^{\prime}, \eta_{d}\right)\right): y^{\prime} \in U \subset \mathbb{R}^{d-1}, \eta_{d} \in V \subset \mathbb{R}\right\}
$$

and hence is Lagrangian.
This implies that the canonical one form, $\omega=\left.\xi d x\right|_{\Lambda_{\eta^{\prime}}}$ is closed and hence there exists $\Phi$ a smooth function on $P$ such that

$$
d\left(\left.\Phi\right|_{\Lambda_{\eta^{\prime}}}\right)=\left.\omega\right|_{\Lambda_{\eta^{\prime}}} \text { for } \eta^{\prime} \text { near } \eta_{0}^{\prime}
$$

and hence $p(x, d \Phi)=0$.
In fact, since $\Phi$ is the integral of a one form, it is locally unique up to a normalization on each $\Lambda_{\eta^{\prime}}$. We fix this normalization by choosing $T \subset P$ a submanifold of dimension $d$ transverse to the fibration by $\Lambda_{\eta^{\prime}}$ and contained in the fold of $Y$. We then insist that $\left.\Phi\right|_{T}=0$. Now, since $Y$ is a fold

$$
\begin{equation*}
Y(P)=\left\{\eta_{1} \leq x_{d} f(x, \eta)\right\} \tag{5.2.7}
\end{equation*}
$$

with $f(m) \neq 0$. Then, since $Y$ is a fold and $\Lambda_{\eta^{\prime}}$ is Lagrangian, by [41, Theorem 21.4.1]

$$
\begin{equation*}
\Phi=Y^{*}\left(\theta_{0} \pm \frac{2}{3}\left(-\rho_{0}\right)^{3 / 2}\right) \tag{5.2.8}
\end{equation*}
$$

where $\theta_{0}, \rho_{0}: Y(P) \rightarrow \mathbb{R}$ are smooth and $\rho_{0}$ is a defining function for the fold. Moreover, the odd part of $\Phi$ vanishes to second order at the fold since $\Phi$ is the integral of a smooth 1-form.

Next, we show that $\rho_{0}=\eta_{1}$. Notice that this is independent of the choice of the reduction to normal form, $\kappa$ and the choice of $T$. Fix $\kappa$ and suppose that $\Phi_{1}$ and $\Phi_{2}$ are two smooth
solutions of $p(x, d \Phi)=0$ corresponding to different submanifolds $T_{1}$ and $T_{2}$. Then, let $w=\Phi_{1}-\Phi_{2} . w$ is constant on each leaf of $\Lambda_{\eta^{\prime}}$ and hence is a function of only $\eta^{\prime}$. Observe that the involution defined by $Y$ preserves $\Lambda_{\eta^{\prime}}$ and hence the $Y$ odd (and even) part of $w$ is a function of only $\eta$. But this implies that the $Y$ odd part vanishes identically. Hence, since $\theta$ is $Y$ even, $\rho_{\Phi_{1}}=\rho_{\Phi_{2}}$.

Observe that over $B$, the involution map of $Y$ is just the projection of $P \cap Q$ to $T^{*} B$ and the function $\Phi$ pulls back under $\kappa$ to a solution to the same problem for the model case. Together, these imply that the odd part of $\Phi$ restricted to the boundary is independent of the choice of $T$ and $\kappa$ and hence is the same as for the model case.

Next, observe that $\theta_{\Phi_{1}}-\theta_{\Phi_{2}}$ is a function of only $\eta^{\prime}$. Hence, $\partial_{x}\left(\theta_{\Phi_{1}}-\theta_{\Phi_{2}}\right)=0$. But, as in the previous paragraph, at the boundary $B, \Phi$ pulls back under $\kappa$ to a solution of $p(x, d \Phi)=0$ for the model problem and hence $\left.\partial_{x^{\prime} \eta^{\prime}}^{2} \kappa^{*} \theta\right|_{B}=I$. In particular,

$$
d_{x^{\prime}} \partial_{\eta_{j}} \theta \text { are linearly independent for } j=1 \ldots d-1
$$

Now, by construction

$$
\Lambda_{\eta^{\prime}}=\left\{\left(x, \partial_{x} \Phi\left(x, \eta^{\prime}\right)\right)\right\}
$$

and $\kappa^{-1}\left(x, \partial_{x} \Phi\left(x, \eta^{\prime}\right)\right)=\left(y\left(x, \partial_{x} \Phi\left(x, \eta^{\prime}\right)\right), \eta^{\prime},\left(y_{d}-\eta_{1}\right)^{1 / 2}\right)$ when $\left(y_{d}-\eta_{1}\right) \geq 0$. Now, on $B$, this holds for $\eta_{1} \leq 0$ and $\left.\partial_{x} \Phi\left(x, \eta^{\prime}\right)\right|_{B}=\partial_{x} Y^{*} \theta$, Now, if $\kappa(y, \eta)=(x, \xi)$, then, using that $\kappa$ is a symplectomorphism, we have $\left(\frac{\partial y}{\partial x}\right)^{t}=\frac{\partial \xi}{\partial \eta}$. Therefore,

$$
\left.\frac{\partial y}{\partial x}\left(x, \partial_{x} \Phi\left(x, \eta^{\prime}\right)\right)\right|_{B}=\left.\left(\partial_{\eta x}^{2} \theta\right)^{t}\right|_{B} .
$$

Thus,

$$
y=\partial_{\eta} \theta\left(x^{\prime}, \eta^{\prime}\right)+f\left(\eta^{\prime}\right)
$$

and hence, using that $\kappa_{\partial}$ is symplectic, we have that $f=\partial_{\eta}^{\prime} g$ and hence by adjusting the normalization $T$, we can arrange that $\left.\theta\right|_{B}$ generates $\kappa_{\partial}$.

At this point, we have solved the eikonal equations $p\left(x, d \phi^{ \pm}\right)=0$ with $\phi^{ \pm}$having the correct form in the region $\rho_{0} \leq 0$. This is a region of the form 5.2.7). Our last task is to extend these solutions so that the eikonal equations continue to hold in Taylor series at $B$ and $\rho_{0}=0$.

By the Malgrange preparation theorem, we can write

$$
p(x, \xi)=p^{\prime}\left[\left(\xi_{d}-a\left(x, \xi^{\prime}\right)\right)^{2}-b\left(x, \xi^{\prime}\right)\right]
$$

where $p^{\prime}$ is nonvanishing near $m, a, b$ are real, and $\xi^{\prime}=\left(\xi_{1}, \ldots \xi_{d}\right)$. Thus, we can drop $p^{\prime}$ when solving $p\left(x, d \phi^{ \pm}\right)=0$. Then, by the glancing hypothesis on $p$ and $q$ along with $H_{p}$ not tangent to the fiber at $x=\pi(m)$,

$$
\xi_{d}=a, \quad b=0, \quad d_{\xi^{\prime}} b \neq 0 \quad \text { at } m
$$

with $b=0, x_{d}=0$ the glancing surface. Then, $p\left(x, d \phi^{ \pm}\right)=0$ becomes

$$
\begin{equation*}
\partial_{x_{d}} \phi^{ \pm}-a\left(x, \partial_{x^{\prime}} \phi^{ \pm}\right)= \pm\left(b\left(x, \partial_{x^{\prime}} \phi^{ \pm}\right)^{1 / 2} \quad \text { in } \rho_{0} \leq 0\right. \tag{5.2.9}
\end{equation*}
$$

Then, extending $\rho_{0}$ and $\theta_{0}$ to smooth real valued functions across $\rho_{0}=0$ gives solutions to (5.2.9) in Taylor series at $\rho_{0}=0$. We write $\phi_{1}^{ \pm}$for the extended functions. Then,

$$
\begin{equation*}
\partial_{x_{d}} \phi_{1}^{ \pm}-a\left(x, \partial_{x^{\prime}} \phi_{1}^{ \pm}\right) \mp\left(b\left(x, \partial_{x^{\prime}} \phi_{1}^{ \pm}\right)^{1 / 2}=e_{ \pm}\right. \tag{5.2.10}
\end{equation*}
$$

with $e_{ \pm}=0$ in $\rho_{0} \leq 0$ and vanishing to all orders at $\rho_{0}=0$. Then to solve (5.2.9) to all orders at $x_{d}=0$, we add to $\phi_{1}$ function

$$
\phi_{2} \sim \sum_{k=1}^{\infty} x_{d}^{k} g_{k}\left(x^{\prime}, \xi\right)
$$

with $\phi_{2}$ vanishing in $\rho_{0} \leq 0$. Then, by (5.2.10), we can solve for the $g_{k}$ successively as functions vanishing in $\rho_{0} \leq 0$ and $\phi^{ \pm}=\phi_{1}^{ \pm}+\phi_{2}^{ \pm}$solves the required problem.

In addition, by two applications of [47, Section 4.4] (one for the real part and one for the imaginary), we have the following lemma.

Lemma 5.2.3. For $c, d \in S$ and $b_{0} \in \mathbb{C}, B_{1}, B_{2}, F_{1}, F_{2} \in C^{\infty}$ there exist $g_{0}, g_{1} \in S$ in $\rho \leq 0$ solving (5.2.4) Moreover, the equations (5.2.3) can be solved in Taylor series at $\rho_{0}=0$ and $y=0$ and we can arrange that $g_{1}\left(\left(0, x^{\prime}\right), \xi\right)=c g_{0}+d$ and $g_{0}((0,0), 0)=b_{0}$..

Proof. We saw in (5.2.5) that (5.2.4) is equivalent to

$$
2\left\langle a d_{x} \phi^{ \pm}, d_{x} g^{ \pm}\right\rangle+\left\langle b, d g^{ \pm}\right\rangle+G^{ \pm} g^{ \pm}=F^{ \pm}
$$

where $g^{ \pm}, G^{ \pm}$, and $F^{ \pm}$are smooth in $x, \xi$ and $\left(-\rho_{0}\right)^{1 / 2}$. Hence, pulling back by $Y$, this lifts to

$$
2\left\langle a d_{x} \Phi, d g\right\rangle+\langle b, d g\rangle+G a=F \quad \text { on } P .
$$

Then, reinterpreting this as an equation on each $\Lambda_{\eta^{\prime}}, x$ can be used as coordinates on $\Lambda_{\eta^{\prime}}$ and hence

$$
H_{p}=\partial_{\xi} p \partial_{x}=2\left\langle a d \Phi, \partial_{x}\right\rangle+\left\langle b, \partial_{x}\right\rangle
$$

and hence $2\langle a d \Phi, \cdot\rangle+\langle b, \cdot\rangle$ is the vector field $H_{p}$. That is, our equation becomes

$$
\begin{equation*}
H_{p} g+G g-F \tag{5.2.11}
\end{equation*}
$$

We can reduce our problem to solving $H_{p} u=0$ by first solving

$$
H_{p} a_{1}+G a_{1}=F \quad a_{1}(m)=0
$$

and

$$
H_{p} \alpha=G^{\prime}, \quad \alpha(m)=0
$$

and writing

$$
a-a_{1}=\exp (-\alpha) u .
$$

Then, the equation $H_{p} u=0$ just reduces to evenness of $u$ under the involution generated by projection from the manifold of bicharacteristics for $P$ to $P \cap Q$.

Our goal is to solve (5.2.11) with

$$
g_{1}=c g_{0}+d \quad g_{0}(m)=b_{0} .
$$

Let $\mathcal{I}_{Q}$ denote the involution on $P \cap Q$ coming from projection to $T^{*} B$. Then this amounts to

$$
[g]_{O}=c[g]_{e}+d, \text { where } v_{O}=\frac{1}{2}\left(\mathcal{I}_{Q}^{*} v-v\right) \rho_{0}^{-1} \quad v_{E}=\frac{1}{2}\left(\mathcal{I}_{Q}^{*} v+v\right) .
$$

After reducing to $H_{p} u=0$, we have changed the boundary condition to

$$
[\exp (-\alpha) u]_{O}=c[\exp (-\alpha) u]_{E}+e^{\prime} \quad u(m)=b_{0}
$$

where $e^{\prime}$ is some $\mathcal{I}_{Q}$ even function. Now, observe that

$$
[v w]_{E}=[v]_{E}[w]_{E}+\rho_{0}^{2}[v]_{O}[w]_{O}
$$

So, we can write our boundary condition as

$$
u_{O}=c^{\prime} u_{E}+f
$$

where $c^{\prime}$ and $f$ are $\mathcal{I}_{Q}$ even. Then, after applying $\kappa$ to reduce to normal form, we have by [47, Proposition 2.8.2] there exists such a function $u$.

This solves the transport equations in $\rho_{0} \leq 0$. The extension to $\rho_{0}>0$ follows as in the proof of Lemma 5.2.2.

## Full Phase and Amplitude Functions for the Dirichlet Parametrices

We now specialize to the case $P=-h^{2} \Delta-z$ and work in a neighborhood of the boundary $\partial \Omega$ of the for $O=[0, a) \times U$ with coordinates $\left(y, x^{\prime}\right)$ and $U$ an open set in $\partial \Omega$. Notice that in these coordinates,

$$
-h^{2} \Delta=\left\langle a\left(y, x^{\prime}\right) h D, h D\right\rangle+h\left\langle b\left(y, x^{\prime}\right), h D\right\rangle
$$

and hence that $h\left\langle b\left(y, x^{\prime}\right), h D\right\rangle$ term can be moved into the right hand side of the transport equations without difficulty.

By the results of Lemma 5.2 .2 (or [77, Chapter 11], and [32, Appendix A.II]), we have

Lemma 5.2.4. There exist $\theta_{0}, \rho_{0} \in C^{\infty}$ solving 5.2.1 for $P=-h^{2} \Delta-1$ for $\rho_{0} \leq 0$ and $\left(\left(y, x^{\prime}\right), \xi\right)$ near $\left(\left(0, x_{0}^{\prime}\right), \xi_{0}\right)$ and in Taylor series at $\rho_{0}=0$ and $y=0$. Moreover,

$$
\begin{gather*}
d_{x} \partial_{\xi_{j}} \theta \text { are linearly independent for } j=1 \ldots d-1  \tag{5.2.12}\\
\qquad \frac{\partial \rho_{0}}{\partial y}<0, \quad \rho_{0}=\alpha_{0} \text { on } y=0
\end{gather*}
$$

where $\alpha_{0}:=\left.\rho_{0}\right|_{y=0}=\xi_{1}$ and $\left.\theta_{0}\right|_{y=0}$ parametrizes the reduction of the billiard ball map to that for the normal form (5.1.3) (i.e. $\kappa_{\partial}$ ).

Now that we have constructed phase functions for $z=1$, we will correct them to obtain solutions of 5.2.1 to $O\left(h^{\infty}\right)$. To do this, let

$$
\begin{gathered}
z=1+i \mu, \quad \theta=\theta_{0}+\sum_{n>0} \theta_{n} \epsilon(h)^{n}=: \theta_{0}+\theta^{\prime} \\
\rho=\rho_{0}+\sum_{n>0} \rho_{n} \epsilon(h)^{n}=: \rho_{0}+\rho^{\prime}
\end{gathered}
$$

where $\theta_{0}$ and $\rho_{0}$ are the solutions found above. Then,

$$
\begin{aligned}
i \mu= & \left(2\left\langle a d \theta_{0}, d \theta^{\prime}\right\rangle-2 \rho_{0}\left\langle a d \rho_{0}, d \rho^{\prime}\right\rangle-\rho^{\prime}\left\langle a d \rho_{0}, d \rho_{0}\right\rangle+\left\langle b, d \theta^{\prime}\right\rangle\right) \\
& +\left(\left\langle a d \theta^{\prime}, d \theta^{\prime}\right\rangle-2 \rho^{\prime}\left\langle a d \rho_{0}, d \rho^{\prime}\right\rangle-\rho_{0}\left\langle a d \rho^{\prime}, d \rho^{\prime}\right\rangle\right)-\rho^{\prime}\left\langle a d \rho^{\prime}, d \rho^{\prime}\right\rangle \\
0= & \left(2\left\langle a d \theta_{0}, d \rho^{\prime}\right\rangle+2\left\langle a d \theta^{\prime}, d \rho_{0}\right\rangle+\left\langle b, d \rho^{\prime}\right\rangle\right)+\left(2\left\langle a d \theta^{\prime}, d \rho^{\prime}\right\rangle\right)
\end{aligned}
$$

where we have grouped terms according to homogeneity in $\epsilon(h)$. Note that if $\operatorname{Im} z=o(h)$, we have artificially introduced a perturbation of size $h$ to $\rho$ and $\theta$.

Then, equating powers of $\epsilon(h)$, and letting

$$
\theta_{<n}=\left\{\theta_{m}: m<n\right\}, \quad \rho_{<n}=\left\{\rho_{m}: m<n\right\}
$$

we have that

$$
\left\{\begin{array}{ll}
2\left\langle a d \theta_{0}, d \theta_{n}\right\rangle+2 \rho_{0}\left\langle a d \rho_{0}, d\left(-\rho_{n}\right)\right\rangle+\left(-\rho_{n}\right)\left\langle a d \rho_{0}, d \rho_{0}\right\rangle+\left\langle b, d \theta_{n}\right\rangle & =F_{n}\left(\theta_{<n}, \rho_{<n}, \mu\right)  \tag{5.2.13}\\
2\left\langle d \theta_{n}, d \rho_{0}\right\rangle-2\left\langle d \theta_{0}, d\left(-\rho_{n}\right)\right\rangle-\left\langle b, d\left(-\rho_{n}\right)\right\rangle & =G_{n}\left(\theta_{<n}, \rho_{<n}, \mu\right)
\end{array} .\right.
$$

These equations are of the form (5.2.4) with $-\rho_{n}$ playing the role of $g_{1}$. Thus, appealing to Lemma 5.2.3, we can take $\rho_{1}\left(\left(0, x^{\prime}\right), \xi\right)=i$. For $n>1$, Lemma 5.2.3, implies that 5.2.13) can be solved with $\rho_{n}\left(\left(0, x^{\prime}\right), \xi\right)=0$. Putting this together, we have

Lemma 5.2.5. Let $\theta_{0}$ and $\rho_{0}$ be the functions guaranteed by Lemma 5.2.4. Then there exist $\theta, \rho \in S$ solving

$$
\left\{\begin{array}{l}
\langle a d \theta, d \theta\rangle-\rho\langle a d \rho, d \rho\rangle+\langle b, d \theta\rangle=z+O\left(h^{\infty}\right) \\
2\langle d \theta, d \rho\rangle+\langle b, d \rho\rangle=0
\end{array}\right.
$$

in $\rho_{0} \leq 0$ and in Taylor series at $\rho_{0}=0$ and $y=0$. Moreover,

$$
\rho \sim \rho_{0}+\sum_{n>0} \rho_{n} \epsilon(h)^{n} \quad \theta \sim \theta_{0}+\sum_{n>0} \theta_{n} \epsilon(h)^{n}
$$

with $\rho_{n}, \theta_{n} \in S, \rho_{0}, \theta_{0}$ real valued, $\operatorname{Im} \theta_{1} \geq 0$ and $\left.\rho\right|_{y=0}=\alpha:=\xi_{1}+i \epsilon(h)$.
Remark: In this way, we arrange

$$
\alpha\left(\xi^{\prime}\right)=\xi_{1}+i \epsilon(h) .
$$

Now, to solve for the amplitudes $g_{0}$ and $g_{1}$, we expand them as formal power series in $h^{n}$. Then, the successive terms solve equations also of the form (5.2.4) (in particular, (5.2.3)). Since the inhomogeneities do not appear in the equations for $g_{i}^{[n]}$, there are solutions with boundary condition $g_{0}^{[0]}\left(\left(0, x^{\prime}\right), \xi\right)$ a real valued elliptic function and $g_{1}^{[0]}=0$. Then, for $n>0$, we have solutions with $g_{1}^{[n]}\left(\left(0, x^{\prime}\right), \xi\right)=0$.

## Semiclassical Fourier-Airy integral Operators

Before proceeding, we give the necessary results on semiclassical Fourier-Airy integral operators following [77, VIII. 6 and X.2] as well as 47, Chapter 6]. We denote $h^{-2 / 3} \alpha=\alpha_{h}$ and $h^{-2 / 3} \rho=\rho_{h}$.

We make the following basic assumptions throughout this section. Let $\Omega \subset \mathbb{R}^{d}$ be strictly convex and $U$ be a neighborhood of $x_{0} \in \partial \Omega$. Suppose that $c h^{M}<\epsilon(h)<C h \log h^{-1}$ and let $\rho, \theta, \in C^{\infty}(U)$ and $g_{0}, g_{1} \in S_{\delta}^{\text {comp }}(U)$. Suppose that $\theta_{0}, \rho_{0} \in C^{\infty}(U ; \mathbb{R})$. Suppose further that $\rho=\rho_{0}+\epsilon(h) \rho^{\prime}, \theta=\theta_{0}+\epsilon(h) \theta^{\prime}$ with $\left.\rho\right|_{\partial \Omega}=: \alpha$,

$$
\begin{equation*}
d_{x} \partial_{\xi} \theta_{0} \neq 0, \quad \partial_{\nu} \rho \leq a_{0}<0 \text { in case (5.1.1) }, \quad \partial_{\nu} \rho \geq a_{0}>0 \text { in case (5.1.2). } \tag{5.2.14}
\end{equation*}
$$

with $\left|\operatorname{Im} \rho^{\prime}\right|>c \epsilon(h)$ and $\theta^{\prime}, \rho^{\prime} \in C^{\infty}(U ; \mathbb{C})$,

$$
\theta^{\prime}=\theta_{1}+O(\epsilon(h)), \quad \operatorname{Im} \theta_{1}((0,0), 0)=0 .
$$

Next, assume $\alpha:=\alpha_{0}(\xi)+\epsilon(h) \alpha^{\prime}(\xi)$ with $\alpha_{0} \in C^{\infty}(\partial \Omega ; \mathbb{R})$ and $\alpha^{\prime}=i+O(\epsilon(h))$. Then, assume that $F \in \mathcal{E}^{\prime}$ with

$$
\begin{equation*}
\operatorname{MS}_{\mathrm{h}}(F) \subset T^{*} U \bigcap\left\{\left|\alpha_{0}\right| \leq \min \left[\gamma\left(\frac{h}{\epsilon(h)}\right)^{2}, \gamma\right]\right\} . \tag{5.2.15}
\end{equation*}
$$

The fact that (5.1.1) and (5.1.2) are well defined follows from the fact that $g_{0}$ and $g_{1}$ have compact support and that $\left|\operatorname{Im} \rho^{\prime}\right|>0$.
Remark: We could take $\alpha^{\prime}=-i+O(\epsilon(h))$, but this would change the wavefront relations in Lemma 5.3.1. In particular, for $\left(\mathcal{A}_{-} \mathcal{A} i\right)^{-1}$.

## Preliminary Estimates on Airy functions and multipliers

We start by recalling some preliminary estimates and asymptotics for Airy functions. We have

$$
\begin{equation*}
A_{-}(z)=\Xi_{-}(z) e^{i 2 / 3(-z)^{3 / 2}} \quad|\operatorname{Arg}(z)-\pi / 3|>\delta \tag{5.2.16}
\end{equation*}
$$

where, letting $\omega:=e^{i \pi / 3}, \Xi_{-}(z):=\Xi\left(z \omega^{2}\right) \in S^{-1 / 4}, \Xi$ has 77, Section X.1]

$$
\Xi(z)=z^{-1 / 4} \sum_{k=0}^{\infty}(-1)^{k} a_{k} z^{-3 k / 2}
$$

where $a_{k}>0$ and $a_{0}=(2 \sqrt{\pi})^{-1}$. and we take the branch of $z^{1 / 2}$ at $\operatorname{Arg}(z)=\pi$ with $(1)^{1 / 2}=1$. We also write $A_{+}(z)=\overline{A_{-}(\bar{z})}$ for another solution to the Airy equation. The asymptotics for $\Xi(z)$ can be differentiated a finite number of times to obtain asymptotic expansions for $A_{-}^{(k)}(z)$.

Next, recall

$$
\begin{equation*}
A i(z)=\Xi(z) e^{-2 / 3 z^{3 / 2}}, \quad|\operatorname{Arg}(z)-\pi|>\delta \tag{5.2.17}
\end{equation*}
$$

Moreover,

$$
\begin{equation*}
A i(z)=\omega A_{+}(z)+\bar{\omega} A_{-}(z) . \tag{5.2.18}
\end{equation*}
$$

So, using the asymptotics (5.2.16) and the analogous asymptotics for $A_{+}$, we have

$$
\begin{equation*}
A i(z)=\omega \Xi_{+}(z) e^{-2 / 3 i(-z)^{3 / 2}}+\bar{\omega} \Xi_{-}(z) e^{2 / 3 i(-z)^{3 / 2}}, \quad|\operatorname{Arg}(z)-\pi|<\delta \tag{5.2.19}
\end{equation*}
$$

where $\Xi_{+}(z)=\Xi\left(z \bar{\omega}^{2}\right)$.
Define

$$
\phi_{i}(z):=\frac{A i^{\prime}(z)}{A i(z)} \quad \phi_{-}(z):=\frac{A_{-}^{\prime}(z)}{A_{-}(z)} .
$$

We will need the following lemma (we follow the proof given in [80, Lemma 3.1]).
Lemma 5.2.6. Let $\phi_{i}$ be as above. Then there exists $\delta>0$ such that

$$
\left|\phi_{i}(z)\right| \leq C \begin{cases}\langle z\rangle^{1 / 2}+|\operatorname{Im} z|^{-1} & |z| \geq \delta, \operatorname{Re} z<0 \\ \langle z\rangle^{1 / 2} & \text { otherwise }\end{cases}
$$

and

$$
\left|\phi_{i}(z)\right|^{-1} \leq C \begin{cases}\langle z\rangle^{-1 / 2}+|\operatorname{Im} z|^{-1}\langle z\rangle^{-1} & |z| \geq \delta, \operatorname{Re} z<0 \\ \langle z\rangle^{-1 / 2} & \text { otherwise }\end{cases}
$$

Proof. Since $\phi_{i}$ is meromorphic and bounded above and below at $z=0$, there exists $\epsilon_{0}>0$ such that for $|z|<z_{0}, 0<c \leq\left|\phi_{i}\right| \leq C$. For $|\operatorname{Arg}(z)-\pi|>\delta$ and $|z| \gg 1$, the estimates follow from the asymptotics (5.2.17). Thus, we need to consider the regions $\epsilon_{0}<|z|<M$ and $|\operatorname{Arg}(z)-\pi| \leq \delta$.

First, we consider the region $\epsilon_{0}<|z|<M$.

Let $-\zeta_{j} \sim C_{1} j^{2 / 3}$ be the zeros of $A i(z)$ and $-\zeta_{j}^{\prime} \sim C_{2} j^{2 / 3}$ be the zeros of $A i^{\prime}(z)$. Recall that both $\zeta_{j}$ and $\zeta_{j}^{\prime}$ are positive and real for all $j$. Now, $A i$ and $A i^{\prime}$ are entire of order $\frac{3}{2}$. Therefore, we can use the Hadamard factorization theorem to write

$$
A i(z)=e^{C_{2} z+C_{1}} \prod_{j}\left(1+\frac{z}{\zeta_{j}}\right) e^{-\frac{z}{\zeta_{j}}}, \quad A i^{\prime}(z)=e^{C_{3} z+C_{4}} \prod_{j}\left(1+\frac{z}{\zeta_{j}^{\prime}}\right) e^{-\frac{z}{\zeta_{j}^{\prime}}}
$$

Hence taking the logarithmic derivative of $A i$ and $A i^{\prime}$ respectively,

$$
\phi_{i}(z)=C_{2}+\sum_{j} \frac{1}{z+\zeta_{j}}-\frac{1}{\zeta_{j}} \quad z \phi_{i}^{-1}(z)=C_{3}+\sum_{j} \frac{1}{z+\zeta_{j}^{\prime}}-\frac{1}{\zeta_{j}^{\prime}}
$$

Since $\zeta_{j}$ are real and positive,

$$
\left|z+\zeta_{j}\right|^{-1} \leq\left\{\begin{array}{ll}
|\operatorname{Im} z|^{-1} & \operatorname{Re} z<0 \\
C|z|^{-1} & \operatorname{Re} z \geq 0
\end{array}\right\}=: a(z)
$$

where and $\zeta_{j} \geq 2|z|,\left|z+\zeta_{j}\right|^{-1} \leq 2\left|\zeta_{j}\right|^{-1}$. Thus,

$$
\begin{aligned}
\left|\phi_{i}(z)\right| & \leq\left|C_{2}\right|+\sum_{j=1}^{2|z|}\left(\left|z+\zeta_{j}\right|^{-1}\right)+|z| \sum_{j=2|z|}^{\infty}\left|z+\zeta_{j}\right|^{-1}\left|\zeta_{j}\right|^{-1} \\
& \leq C\left(1+|z|\left(a(z)+1+\sum_{j}\left|\zeta_{j}\right|^{-2}\right)\right) \leq C a(z)
\end{aligned}
$$

since $\epsilon_{0}<|z|<M$. By an identical argument,

$$
\left|z \| \phi_{i}\right|^{-1} \leq C a(z)
$$

in this region.
Now, we consider the remaining region. Let $|z| \gg 1$ with $|\operatorname{Arg}(z)|<\delta$. First, using (5.2.18), we have that

$$
\begin{align*}
\phi_{i}(-z) & =\frac{A_{+}^{\prime}(-z)}{A_{+}(-z)}\left(1+\frac{A_{-}^{\prime}(-z)}{\omega^{2} A_{+}^{\prime}(-z)}\right)\left(1+\frac{A_{-}(-z)}{\omega^{2} A_{+}(-z)}\right)^{-1} \\
\phi_{i}^{-1}(-z) & =\frac{A_{+}(-z)}{A_{+}^{\prime}(-z)}\left(1+\frac{A_{-}(-z)}{\omega^{2} A_{+}(-z)}\right)\left(1+\frac{A_{-}^{\prime}(-z)}{\omega^{2} A_{+}^{\prime}(-z)}\right)^{-1} \tag{5.2.20}
\end{align*}
$$

Thus, to estimate $\phi_{i}$ and $\phi_{i}^{-1}$, we proceed by obtaining estimates on $A_{+}$and $A_{-}$. Defining $\zeta=\frac{2}{3} z^{3 / 2}$, we have

$$
\operatorname{Im} \zeta=\operatorname{Im} z(\operatorname{Re} z)^{1 / 2}(1+O(\delta)), \quad|\operatorname{Im} \zeta| \geq C_{\delta}|\operatorname{Im} z||z|^{1 / 2}
$$

Now, let

$$
\begin{gathered}
B_{ \pm}(z):=z^{1 / 4} e^{\mp i \pi / 12} \Xi\left(e^{ \pm i \pi / 3} z\right) \\
D_{ \pm}(z):= \pm i z^{-1 / 4} e^{\mp i / 12}\left(\mp i z^{1 / 2} \Xi\left(e^{ \pm \pi i / 3} z\right)-\Xi^{\prime}\left(e^{ \pm \pi i / 3}\right)\right)
\end{gathered}
$$

where $\Xi$ is as in (5.2.16) so that

$$
\begin{equation*}
A_{ \pm}(-z)=z^{-1 / 4} e^{ \pm i \pi / 12} B_{ \pm}(z) e^{ \pm i \zeta}, \quad A_{ \pm}^{\prime}(-z)=\mp i z^{1 / 4} e^{ \pm i \pi / 12} D_{ \pm}(z) e^{\mp \zeta} \tag{5.2.21}
\end{equation*}
$$

Then,

$$
\begin{array}{ll}
B_{ \pm}(z)=b_{0} \pm i b_{1} \zeta^{-1}+O\left(\zeta^{-2}\right), & -z B_{ \pm}^{\prime}(z)= \pm \frac{3 i b_{1}}{2} \zeta^{-1}+O\left(\zeta^{-2}\right) \\
D_{ \pm}(z)=d_{0} \pm i d_{1} \zeta^{-1}+O\left(\zeta^{-2}\right), & -z D_{ \pm}^{\prime}(z)= \pm \frac{3 i d_{1}}{2} \zeta^{-1}+O\left(\zeta^{-2}\right)
\end{array}
$$

where $b_{i}>0, d_{i}>0$ and

$$
\begin{aligned}
& \pm \operatorname{Im}\left(B_{ \pm}(z) \overline{B_{ \pm}^{\prime}(z)}\right)=\frac{3 b_{0} b_{1}}{2}|z|^{-5 / 2}\left(1+O(\delta)+O\left(|z|^{-3 / 2}\right)\right)>0 \\
& \pm \operatorname{Im}\left(D_{ \pm}(z) \overline{D_{ \pm}^{\prime}(z)}\right)=\frac{3 d_{0} d_{1}}{2}|z|^{-5 / 2}\left(1+O(\delta)+O\left(|z|^{-3 / 2}\right)\right)>0
\end{aligned}
$$

We first seek to show that $\pm\left|A_{-}(-z)\right| \leq \pm\left|A_{+}(-z)\right|$ in $\pm \operatorname{Im} z \geq 0$. To this end, define

$$
f_{a}(\tau)=\left|B_{+}(a+i \tau)\right|^{2}-\left|B_{-}(a+i \tau)\right|^{2}
$$

Then,

$$
f_{a}^{\prime}(\tau)=2 \operatorname{Im}\left(B_{+}(a+i \tau) \overline{B_{+}^{\prime}(a+i \tau)}-B_{-}(a+i \tau) \overline{B_{-}^{\prime}\left(a_{i} \tau\right)}\right)>0
$$

So taking $a=\operatorname{Re} z$ and using the fact that $\left|A_{+}(\operatorname{Re} z)\right|=\left|A_{-}(\operatorname{Re} z)\right|$, we have $f_{\operatorname{Re} z}(0)=0$ and $f_{\operatorname{Re} z}^{\prime}(\tau)>0$ for $0 \leq \tau<\delta \operatorname{Re} z$ and $\operatorname{Re} z \gg 1$. This implies

$$
\begin{equation*}
\pm\left|B_{+}(z)\right| \geq \pm\left|B_{-}(z)\right| \quad \pm \operatorname{Im} z \geq 0 \tag{5.2.22}
\end{equation*}
$$

An identical analysis with the function

$$
g_{a}(\tau)=\left|D_{+}(a+i \tau)\right|^{2}-\left|D_{-}(a+i \tau)\right|^{2}
$$

gives

$$
\begin{equation*}
\pm\left|D_{+}(z)\right| \geq \pm\left|D_{-}(z)\right| \quad \pm \operatorname{Im} z \geq 0 \tag{5.2.23}
\end{equation*}
$$

We now restrict our attention to $\operatorname{Im} z>0$ and hence $\operatorname{Im} \zeta>0$ since the other region is similar. By (5.2.21) and 5.2.22

$$
\begin{equation*}
\left|\frac{A_{-}(-z)}{A_{+}(-z)}\right|=e^{-2 \operatorname{Im} \zeta}\left|\frac{B_{-}(z)}{B_{+}(z)}\right| \leq e^{-2 \operatorname{Im} \zeta}, \quad\left|\frac{A_{-}^{\prime}(-z)}{A_{+}^{\prime}(-z)}\right|=e^{-2 \operatorname{Im} \zeta}\left|\frac{D_{-}(z)}{D_{+}(z)}\right| \leq e^{-2 \operatorname{Im} \zeta} . \tag{5.2.24}
\end{equation*}
$$

Now, the asymptotics (5.2.16) imply that

$$
\begin{equation*}
0<c\langle z\rangle^{1 / 2} \leq\left|\frac{A_{ \pm}^{\prime}(-z)}{A_{+}(-z)}\right| \leq C\langle z\rangle^{1 / 2} \quad 0<c \leq\left|\frac{A_{ \pm}(-z)}{A_{+}(-z)}\right| \leq C . \tag{5.2.25}
\end{equation*}
$$

So, using $|\operatorname{Im} \zeta| \geq C_{\delta}|\operatorname{Im} z||z|^{1 / 2}$ together with using (5.2.24) and (5.2.25) in (5.2.20) gives

$$
\begin{gathered}
\left|\phi_{i}(-z)\right| \leq \frac{C|z|^{1 / 2}}{1-e^{-2 \operatorname{Im} \zeta}} \leq \frac{C|z|^{1 / 2}}{\min (1,2 \operatorname{Im} \zeta)} \leq C|z|^{1 / 2}+C|\operatorname{Im} z|^{-1} \\
\left|\phi_{i}(-z)\right|^{-1} \leq C \frac{|z|^{-1 / 2}}{1-e^{-2 \operatorname{Im} \zeta}} \leq C \frac{|z|^{-1 / 2}}{\min (1,2 \operatorname{Im} \zeta)} \leq\langle z\rangle^{-1 / 2}\left(1+|\operatorname{Im} z|^{-1}\langle z\rangle^{-1 / 2}\right)
\end{gathered}
$$

The following bounds on products of Airy functions will be useful in our construction of $H_{g}$ and $H_{d}$

Lemma 5.2.7. Let $\alpha$ be as in 5.2.15 and $\alpha_{h}=h^{-2 / 3} \alpha$. Then for $\gamma$ small enough and

$$
|\alpha| \leq \gamma\left(h \epsilon(h)^{-1}\right)^{2}
$$

we have for $\operatorname{Re} \alpha_{h} \leq-\delta$

$$
\begin{gathered}
C h^{-2 / 3} \epsilon(h) \leq C\left|\operatorname{Im} \alpha_{h}\right| \leq\left|A i\left(\alpha_{h}\right) A_{-}\left(\alpha_{h}\right)\right| \leq C \\
c\left\langle\alpha_{h}\right\rangle^{1 / 2} \leq\left|A i^{\prime}\left(\alpha_{h}\right) A_{-}^{\prime}\left(\alpha_{h}\right)\right| \leq C\left\langle\alpha_{h}\right\rangle^{1 / 2} \\
c\left(\left|\operatorname{Im} \alpha_{h}\right|^{-1}\left\langle\alpha_{h}\right\rangle^{-1}+\left\langle\alpha_{h}\right\rangle^{-1 / 2}\right)^{-1} \leq\left|\phi_{i}\left(\alpha_{h}\right)\right| \leq\left|\operatorname{Im} \alpha_{h}\right|^{-1} \leq C h^{2 / 3} \epsilon(h)^{-1} \\
c\left\langle\alpha_{h}\right\rangle^{1 / 2} \leq\left|\phi_{-}\right| \leq C\left\langle\alpha_{h}\right\rangle^{1 / 2}
\end{gathered}
$$

and for $\operatorname{Re} \alpha_{h} \geq-\delta$

$$
\begin{gathered}
C h^{1 / 3} \leq\left|A i\left(\alpha_{h}\right) A_{-}\left(\alpha_{h}\right)\right| \leq C \\
c\left(\left|\operatorname{Im} \alpha_{h}\right|^{-1}\left\langle\alpha_{h}\right\rangle^{-1}+\left\langle\alpha_{h}\right\rangle^{-1 / 2}\right)^{-1} \leq\left|A i^{\prime}\left(\alpha_{h}\right) A_{-}^{\prime}\left(\alpha_{h}\right)\right| \leq C\left\langle\alpha_{h}\right\rangle^{1 / 2} \\
c\left\langle\alpha_{h}\right\rangle^{1 / 2} \leq\left|\phi_{i}\left(\alpha_{h}\right)\right|+\left|\phi_{-}\left(\alpha_{h}\right)\right| \leq C\left\langle\alpha_{h}\right\rangle^{1 / 2} .
\end{gathered}
$$

Proof. First observe that

$$
\begin{equation*}
c h^{-2 / 3} \epsilon(h)<\left|\operatorname{Im} \alpha_{h}\right|=O\left(h^{-2 / 3} \epsilon(h)\right) \ll \delta \tag{5.2.26}
\end{equation*}
$$

thus, either $\left|\alpha_{h}\right|<\delta$. or $\left|\operatorname{Im} \alpha_{h}\right| \ll\left|\alpha_{h}\right|$.
The upper bounds for $\operatorname{Ai}\left(\alpha_{h}\right) A_{-}\left(\alpha_{h}\right)$ and $A i^{\prime}\left(\alpha_{h}\right) A_{-}\left(\alpha_{h}\right)$ follow directly from the asymptotics (5.2.16), (5.2.17), and (5.2.19) together with the analyticity of these functions.

In order to estimate $\left(A_{-} A i\right)^{-1}$, we use the Wronskian to write

$$
\begin{equation*}
\frac{A_{-}^{\prime}(z)}{A_{-}(z)}-\frac{A i^{\prime}(z)}{A i(z)}=\frac{W\left(A i, A_{-}\right)(z)}{A i(z) A_{-}(z)}=\frac{e^{-\pi i / 6}}{2 \pi A i(z) A_{-}(z)} \tag{5.2.27}
\end{equation*}
$$

Thus, to estimate $\left|A_{-} A i\right|^{-1}$ it is enough to estimate $\phi_{i}$ and

$$
\phi_{-}:=\frac{A_{-}^{\prime}}{A_{-}} .
$$

Similarly, to estimate $\left(A_{-}^{\prime} A i^{\prime}\right)^{-1}$, we use the Wronskain to write

$$
\begin{equation*}
\frac{A_{-}(z)}{A_{-}^{\prime}(z)}-\frac{A i(z)}{A i^{\prime}(z)}=-\frac{W\left(A i, A_{-}\right)(z)}{A i(z) A_{-}(z)}=\frac{e^{5 \pi i / 6}}{2 \pi A i^{\prime}(z) A_{-}^{\prime}(z)} \tag{5.2.28}
\end{equation*}
$$

By Lemma 5.2.6, there exists $\delta>0$ such that

$$
\begin{gathered}
\left|\phi_{i}(z)\right| \leq C \begin{cases}|\operatorname{Im} z|^{-1}+\langle z\rangle^{1 / 2} & |z| \geq \delta, \operatorname{Re} z<0 \\
\langle z\rangle^{1 / 2} & \text { otherwise }\end{cases} \\
\left|\phi_{i}(z)\right|^{-1} \leq C\left\{\begin{array}{ll}
\langle z\rangle^{-1 / 2}+|\operatorname{Im} z|^{-1}\langle z\rangle^{-1} & |z| \geq \delta, \operatorname{Re} z<0 \\
\langle z\rangle^{-1 / 2} & \text { otherwise }
\end{array} .\right.
\end{gathered}
$$

and, since

$$
\phi_{-}(z)=e^{2 \pi i / 3} \phi_{i}\left(e^{2 \pi i / 3} z\right)
$$

we also have

$$
\begin{gathered}
\left|\phi_{-}(z)\right| \leq C \begin{cases}\left(\left|\operatorname{Im} e^{2 \pi i / 3} z\right|+\langle z\rangle^{1 / 2}\right) & |z| \geq \delta \\
1 & |z| \leq \delta\end{cases} \\
\left|\phi_{-}(z)\right|^{-1} \leq C\left\{\begin{array}{ll}
\langle z\rangle^{-1 / 2}+\left|\operatorname{Im} e^{2 \pi i / 3} z\right|^{-1}\langle z\rangle^{-1} & |z| \geq \delta \\
\langle z\rangle^{-1 / 2} & \text { otherwise }
\end{array} .\right.
\end{gathered}
$$

Now, by (5.2.26), either $\left|\alpha_{h}\right| \leq \delta$ or $\left|\operatorname{Im} e^{2 \pi i / 3} z\right| \geq \delta$, so we can estimate

$$
\begin{gathered}
c\left\langle\alpha_{h}\right\rangle^{1 / 2} \leq\left|\phi_{-}\left(\alpha_{h}\right)\right| \leq C\left\langle\alpha_{h}\right\rangle^{1 / 2} \\
\left|\phi_{i}\right|\left(\alpha_{h}\right) \leq \begin{cases}\left|\operatorname{Im} \alpha_{h}\right|^{-1}+\left\langle\alpha_{h}\right\rangle^{1 / 2} & \operatorname{Re} \alpha_{h}<0,\left|\alpha_{h}\right| \geq \delta \\
\left\langle\alpha_{h}\right\rangle^{1 / 2} & \text { otherwise }\end{cases} \\
\left|\phi_{i}^{-1}\right|\left(\alpha_{h}\right) \leq\left\{\begin{array}{ll}
\left|\operatorname{Im} \alpha_{h}\right|^{-1}\left\langle\alpha_{h}\right\rangle^{-1}+\left\langle\alpha_{h}\right\rangle^{-1 / 2} & \operatorname{Re} \alpha_{h}<0,\left|\alpha_{h}\right| \geq \delta \\
\left\langle\alpha_{h}\right\rangle^{-1 / 2} & \text { otherwise }
\end{array} .\right.
\end{gathered}
$$

Next, we have

$$
|\alpha| \leq \gamma\left(\frac{h}{\epsilon(h)}\right)^{2}
$$

and hence

$$
\left(1+\left|\alpha_{h}\right|\right)^{-1 / 2} \geq\left\langle h^{-2 / 3}\left(\gamma \frac{h^{2}}{\epsilon(h)^{2}}+C \epsilon(h)\right)\right\rangle^{-1 / 2} \geq \gamma^{-1 / 2} h^{-2 / 3} \epsilon(h) \geq\left|\operatorname{Im} \alpha_{h}\right|
$$

provided that $\gamma$ is small enough. This implies

$$
\left\langle\alpha_{h}\right\rangle^{1 / 2} \leq\left|\operatorname{Im} \alpha_{h}\right|^{-1}
$$

and hence gives the desired estimates
Define the Airy multipliers:

$$
\begin{aligned}
\left(\mathcal{A}_{-} \mathcal{A} i\right)^{-1} F & :=(2 \pi h)^{-d+1} \int\left[A i\left(\alpha_{h}\right) A_{-}\left(\alpha_{h}\right)\right]^{-1} e^{i\left\langle x, \xi^{\prime}\right\rangle / h} \mathcal{F}_{h} F(\xi) d \xi \\
\left(\mathcal{A}_{-} \mathcal{A} i\right) F & :=(2 \pi h)^{-d+1} \int A i\left(\alpha_{h}\right) A_{-}\left(\alpha_{h}\right) e^{i\left\langle x, \xi^{\prime}\right\rangle / h} \mathcal{F}_{h} F(\xi) d \xi \\
\left(\Phi_{i}\right) F & :=(2 \pi h)^{-d+1} \int \phi_{i}\left(\alpha_{h}\right) e^{i\left\langle x, \xi^{\prime}\right\rangle / h} \mathcal{F}_{h} F(\xi) d \xi \\
\left(\Phi_{i}^{-1}\right) F & :=(2 \pi h)^{-d+1} \int \phi_{i}^{-1}\left(\alpha_{h}\right) e^{i\left\langle x, \xi^{\prime}\right\rangle / h} \mathcal{F}_{h} F(\xi) d \xi \\
\left(\Phi_{-}\right) F & :=(2 \pi h)^{-d+1} \int \phi_{-}\left(\alpha_{h}\right) e^{i\left\langle x, \xi^{\prime}\right\rangle / h} \mathcal{F}_{h} F(\xi) d \xi \\
\left(\Phi_{-}^{-1}\right) F & :=(2 \pi h)^{-d+1} \int \phi_{-}^{-1}\left(\alpha_{h}\right) e^{i\left\langle x, \xi^{\prime}\right\rangle / h} \mathcal{F}_{h} F(\xi) d \xi .
\end{aligned}
$$

Then the following estimates follow from Lemma 5.2.7. (see also 47, Proposition 5.3.10])
Lemma 5.2.8.

$$
\begin{gathered}
\left(\mathcal{A}_{-} \mathcal{A} i\right)^{-1}=O_{H_{h}^{s} \rightarrow H_{h}^{s}}\left(h^{-1 / 3}\right), \quad \mathcal{A}_{-} \mathcal{A} i=O_{H_{h}^{s} \rightarrow H_{h}^{s}}(1) \\
\left(\mathcal{A}_{-}^{\prime} \mathcal{A} i^{\prime}\right)^{-1}=O_{H_{h}^{s} \rightarrow H_{h}^{s}}\left(h^{2 / 3} \epsilon(h)^{-1}\right), \quad \mathcal{A}_{-}^{\prime} \mathcal{A} i^{\prime}=O_{H_{h}^{s} \rightarrow H_{h}^{s}}\left(h^{-1 / 3}\right) \\
\Phi_{i}=O_{H_{h}^{s} \rightarrow H_{h}^{s}}\left(h^{-1 / 3}\right) \quad \Phi_{i}^{-1}=O_{H_{h}^{s} \rightarrow H_{h}^{s}}\left(h^{2 / 3} \epsilon(h)^{-1}\right) \\
\Phi_{-}=O_{H_{h}^{s} \rightarrow H_{h}^{s}}\left(h^{-1 / 3}\right) \quad \Phi_{-}^{-1}=O_{H_{h}^{s} \rightarrow H_{h}^{s}}(1) .
\end{gathered}
$$

Proof. This follows from the estimates in Lemma 5.2.7.

## Estimates for Fourier-Airy Integral Operators

Estimates for (5.1.2 type Fourier Airy Integral operators
To analyze the action of (5.1.2), we follow the analysis given in [47, Chapter 6]. We work in a neighborhood of the boundary $\partial \Omega$ of the form $O=[0, a) \times U$ with coordinates $\left(y, x^{\prime}\right)$ and define the symbol classes

Definition 5.2.9. We say $p\left(y, x^{\prime}, \xi ; h\right) \in a(h) S_{\rho, \delta, \nu}$ if

$$
\left|D_{y}^{k} D_{x^{\prime}}^{\beta}\left(h D_{\xi}\right)^{\alpha} p\left(y, x^{\prime}, \xi ; h\right)\right| \leq a(h) h^{\rho|\alpha|-\delta|\beta|-\nu k} .
$$

Write $B_{2} F:=B_{3} \circ\left(\mathcal{A}_{i} \mathcal{A}_{-}\right)^{-1} F$ where

$$
\begin{equation*}
B_{3} F:=(2 \pi h)^{-d+1} \int\left[g_{0} A i\left(\rho_{h}\right)+i h^{1 / 3} g_{1} A i^{\prime}\left(\rho_{h}\right)\right] A_{-}\left(\alpha_{h}\right) e^{i \theta / h} \mathcal{F}_{h} F(\xi) d \xi \tag{5.2.29}
\end{equation*}
$$

Then all that remains is to analyze $B_{3}$.
To analyze $B_{3}$, we break it into several pieces that can be handled using the theory of Fourier integral operators with singular phase. Let $p_{1}, p_{2}, p_{3}$ have $\operatorname{supp} p_{1} \subset[C, \infty)$, $\operatorname{supp} p_{2} \subset(-2 C, 2 C)$, $\operatorname{supp} p_{3} \subset(-\infty,-C]$ with $p_{1}+p_{2}+p_{3}=1$ and let $q_{1}=1-p_{3}$ where $C \gg 1$ will be chosen later.

We first examine the case where $\operatorname{Re} \alpha_{h}>-2 C$.

## Lemma 5.2.10.

$$
\begin{gathered}
y^{j} A i\left(\rho_{h}\right) A_{-}\left(\alpha_{h}\right) q_{1}\left(\operatorname{Re} \alpha_{h}\right) \in h^{2 / 3 j} e^{C \epsilon(h) / h} S_{1 / 3,2 / 3,1} \\
y^{j} A i^{\prime}\left(\rho_{h}\right) A_{-}\left(\alpha_{h}\right) q_{1}\left(\operatorname{Re} \alpha_{h}\right) \in e^{C \epsilon(h) / h} h^{-1 / 6+2 / 3 j} S_{1 / 3,2 / 3,1}
\end{gathered}
$$

for $j \geq 0$.
Proof. We first consider the term involving $p_{1}$. By 5.2.16 We have that

$$
A_{-}\left(\alpha_{h}\right)=\Xi_{-}\left(\alpha_{h}\right) e^{2 / 3 \alpha^{3 / 2} / h} \text { if } \operatorname{Re} \alpha>0, \quad A i\left(\rho_{h}\right)=\Xi\left(\rho_{h}\right) e^{-(2 / 3) \rho^{3 / 2} / h} \text { if } \operatorname{Re} \rho>0
$$

Thus, since $\rho_{0} \geq \alpha_{0}+c y$,

$$
A i\left(\rho_{h}\right) A_{-}\left(\alpha_{h}\right) p_{1}\left(\operatorname{Re} \alpha_{h}\right)=p_{1}\left(\operatorname{Re} \alpha_{h}\right) \Xi_{-}\left(\alpha_{h}\right) \Xi\left(\rho_{h}\right) e^{-(2 / 3)\left(\rho^{3 / 2}-\alpha^{3 / 2}\right) / h}
$$

Write

$$
p_{1}\left(\operatorname{Re} \alpha_{h}\right)=\chi_{1}^{2}\left(\alpha_{h}\right) \chi_{2}^{2}\left(\rho_{h}\right)
$$

where $\chi_{1}, \chi_{2}$ are supported in $\operatorname{Re} s \geq 1 / 4$ and equal to 1 for $\operatorname{Re} s \geq 2$. This is possible since $\alpha \leq \rho-C y+O\left(h^{-2 / 3} \epsilon(h)\right)$. It suffices to show that $\chi_{1}\left(\alpha_{h}\right) \Xi_{-}\left(\alpha_{h}\right) \in S_{1 / 3,0}, \chi_{2}\left(\rho_{h}\right) \Xi\left(\rho_{h}\right) \in$ $S_{1 / 3,2 / 3,2 / 3}$, and

$$
\begin{equation*}
\chi_{1}\left(\operatorname{Re} \alpha_{h}\right) \chi_{2}\left(\operatorname{Re} \rho_{h}\right) e^{-(2 / 3)\left(\rho^{3 / 2}-\alpha^{3 / 2}\right) / h} \in e^{C \epsilon(h) / h} S_{1 / 3,2 / 3,1} \tag{5.2.30}
\end{equation*}
$$

The first two estimates follow from elementary estimates on $\Xi$.
To prove (5.2.30), we apply the chain rule:

$$
\begin{aligned}
& D_{y}^{k} D_{x^{\prime}}^{\beta} D_{\xi}^{\gamma} e^{-(2 / 3)\left(\rho^{3 / 2}-\alpha^{3 / 2}\right) / h} \\
& =\sum C D_{y}^{k_{1}} D_{x^{\prime}}^{\beta_{1}} D_{\xi}^{\gamma_{1}}\left(h^{-1}\left(\rho^{3 / 2}-\alpha^{3 / 2}\right)\right) \ldots D_{y}^{k_{\mu}} D_{x^{\prime}}^{\beta_{\mu}} D_{\xi}^{\gamma_{\mu}}\left(h^{-1}\left(\rho^{3 / 2}-\alpha^{3 / 2}\right)\right) e^{-(2 / 3)\left(\rho^{3 / 2}-\alpha^{3 / 2}\right) / h}
\end{aligned}
$$

where the sum is over $\sum \gamma_{i}=\gamma, \sum \beta_{i}=\beta, \sum k_{i}=k$. Note that 5.2.14) implies that for $y$ small on $\operatorname{supp} \chi_{1}\left(\alpha_{h}\right) \chi_{2}\left(\rho_{h}\right)$

$$
\operatorname{Re}\left(\rho^{3 / 2}-\alpha^{3 / 2}\right) \geq C y^{3 / 2}, \quad \operatorname{Re} \alpha>0
$$

Hence,

$$
\begin{aligned}
\left|D_{y}^{k} D_{x^{\prime}}^{\beta} D_{\xi}^{\gamma}\left(\rho^{3 / 2}-\alpha^{3 / 2}\right) e^{-\frac{2}{3 h}\left(\rho^{3 / 2}-\alpha^{3 / 2}\right)}\right| & \leq e^{-c y^{3 / 2} / h} e^{C \epsilon(h) / h} C|\rho|^{3 / 2-k-|\beta|-|\gamma|}, \quad k>0, \\
\left|D_{x^{\prime}}^{\beta} D_{\xi}^{\gamma}\left(\rho^{3 / 2}-\alpha^{3 / 2}\right)\right| e^{-\frac{2}{3 h}\left(\rho^{3 / 2}-\alpha^{3 / 2}\right)} & \leq C e^{-c y^{3 / 2} / h} e^{C \epsilon(h) / h}|\rho|^{3 / 2-|\beta|-|\gamma|} \quad \beta>0, \\
\left|D_{\xi}^{\gamma}\left(\rho^{3 / 2}-\alpha^{3 / 2}\right)\right| e^{-\frac{2}{3 h}\left(\rho^{3 / 2}-\alpha^{3 / 2}\right)} & \leq e^{-c y^{3 / 2} / h} \max \left(|\rho|^{3 / 2-|\gamma|},|\alpha|^{3 / 2-|\gamma|}\right) .
\end{aligned}
$$

But, on supp $\chi_{1}\left(\operatorname{Re} \alpha_{h}\right) \chi_{2}\left(\operatorname{Re} \rho_{h}\right), C h^{2 / 3} \leq \alpha \leq \rho$. Thus,

$$
\left|D_{y}^{k} D_{x^{\prime}}^{\beta} D_{\xi}^{\gamma}\left(\rho^{3 / 2}-\alpha^{3 / 2}\right) e^{-(2 / 3)\left(\rho^{3 / 2}-\alpha^{3 / 2}\right)}\right| y^{j} \leq C e^{\epsilon(h) / h} h^{-2 / 3(k+|\beta|+|\gamma|-3 / 2-j)} .
$$

Now, for the term involving $p_{2}$, we have

$$
A i\left(\rho_{h}\right) p_{2}\left(\operatorname{Re} \alpha_{h}\right), \quad h^{1 / 3} A i\left(\rho_{h}\right) p_{2}\left(\operatorname{Re} \alpha_{h}\right) \in e^{C \epsilon(h) / h} S_{1 / 3,2 / 3,1}
$$

To see this observe that on $p_{2}\left(\operatorname{Re} \alpha_{h}\right) p_{2}\left(\operatorname{Re} \rho_{h}\right)$ we have $\left|\rho_{0}\right|,\left|\alpha_{0}\right| \leq C h^{2 / 3}$ and hence the main term in the exponential phase is bounded independently of $h$. Moreover, since $\operatorname{Re} \rho_{0} \geq$ $\alpha_{0}+C y,|y| \leq h^{2 / 3}$ so the second statement follows. On $p_{2}\left(\operatorname{Re} \alpha_{h}\right) p_{1}\left(\operatorname{Re} \rho_{h}\right)$, we estimate as above.

The estimate for terms involving $A i^{\prime}$ follows from the fact that

$$
A i^{\prime}(z)=\tilde{\Xi}(z) e^{-(2 / 3) z^{3 / 2}}
$$

where $\tilde{\Xi}=O\left(z^{1 / 4}\right)$. This completes the proof of the lemma.
Next, we analyze the case where $\alpha_{h}<-C$. Write

$$
B_{3}^{<} F=(2 \pi h)^{-d+1} \int\left[g_{0} A i\left(\rho_{h}\right)+i h^{1 / 3} g_{1} A i^{\prime}\left(\rho_{h}\right)\right] A_{-}\left(\alpha_{h}\right) p_{3}\left(\alpha_{h}\right) e^{i \theta / h} \mathcal{F}_{h} F d \xi
$$

We have similar to [47, Section 6.3]
Lemma 5.2.11. The operator defined by

$$
\mathcal{A}_{-}^{<}(F):=(2 \pi h)^{-d+1} \int A_{-}\left(\alpha_{h}\right) p_{3}\left(\alpha_{h}\right) e^{\frac{i\left\langle x, \xi^{\prime}\right\rangle}{h}} \mathcal{F}_{h} F d \xi^{\prime}
$$

is a Fourier integral operator with singular phase.

Remark: For a treatment of semiclassical Fourier integral operators with singular phase see Section 5.7 .

Let

$$
D G=\int\left[g_{0} A i\left(\rho_{h}\right)+i h^{1 / 3} g_{1} A i^{\prime}\left(\rho_{h}\right)\right] e^{i \theta / h} \mathcal{F}_{h} G(\xi) d \xi
$$

where $G \in \mathcal{E}^{\prime}$. Then $B_{3}^{<}=D \circ \mathcal{A}_{-}^{<}$.
Hence, we only need to analyze $D$. We decompose $D$ using $p_{3}\left(\operatorname{Re} \rho_{h}\right)$ and $q_{1}\left(\operatorname{Re}\left(\rho_{h}\right)\right)$ and write the resulting operators $D:=D_{1}+D_{2}$.

Then, using the same analysis as in Lemma 5.2.10 we have
Lemma 5.2.12. For $j \geq 0$,

$$
\begin{gathered}
\rho_{0}^{j} A i\left(\rho_{h}\right) q_{1}\left(\operatorname{Re} \rho_{h}\right) \in h^{2 / 3 j} e^{C \epsilon(h) / h} S_{1 / 3,2 / 3,1}, \\
\rho_{0}^{j} A i^{\prime}\left(\rho_{h}\right) q_{1}\left(\operatorname{Re} \rho_{h}\right) \in h^{-1 / 6+2 / 3 j} e^{C \epsilon(h) / h} S_{1 / 3,2 / 3,1} .
\end{gathered}
$$

Finally,

## Lemma 5.2.13.

$$
(2 \pi h)^{-d+1} \int\left[g_{0} A i\left(\rho_{h}\right)+i h^{1 / 3} g_{1} A i^{\prime}\left(\rho_{h}\right)\right] e^{i \theta / h} \mathcal{F}_{h} G(\xi) p_{3}\left(\rho_{h}\right) d \xi=B^{+}+B^{-}
$$

with

$$
B^{ \pm}=\omega^{\mp}(2 \pi h)^{-d+1} \int\left[g_{0} \Xi_{ \pm}\left(\rho_{h}\right)+i h^{1 / 3} g_{1} \tilde{\Xi}_{ \pm}\left(\rho_{h}\right)\right] e^{i\left[\theta \mp(2 / 3)(-\rho)^{3 / 2}\right] / h} p_{3}\left(\rho_{h}\right) \mathcal{F}_{h} G(\xi) d \xi
$$

where $\Xi_{ \pm}\left(\rho_{h}\right) \in S_{1 / 3,2 / 3,2 / 3}, \tilde{\Xi}_{ \pm} \in h^{-1 / 6} S_{1 / 3,2 / 3,2 / 3}$.
Proof. By (5.2.19) we have

$$
A i\left(\rho_{h}\right)=\omega \Xi_{+}\left(\rho_{h}\right) e^{-(2 i / 3)(-\rho)^{3 / 2} / h}+\bar{\omega} \Xi_{-}\left(\rho_{h}\right) e^{(2 i / 3)(-\rho)^{3 / 2} / h}, \quad \operatorname{Re} \rho<0
$$

Similarly for $A i^{\prime}$. Thus, the lemma follows from symbol estimates on $\Xi_{ \pm}$and $\tilde{\Xi}_{ \pm}$.

## Estimates for (5.1.1) type Fourier Airy Integral operators

The analysis of (5.1.1) is similar to that of (5.1.2). This time, we decompose $B_{1}$ into $\rho_{h}<-C$ and $\rho_{h}>-2 C$. We have

Lemma 5.2.14. For $j \geq 0$,

$$
\begin{aligned}
& y^{j} A_{-}\left(\rho_{h}\right) A_{-}\left(\alpha_{h}\right)^{-1} q_{1}\left(\operatorname{Re} \rho_{h}\right) \in e^{C \epsilon(h) / h} h^{-1 / 6} h^{2 / 3 j} S_{1 / 3,2 / 3,1}, \\
& y^{j} A_{-}^{\prime}\left(\rho_{h}\right) A_{-}\left(\alpha_{h}\right)^{-1} q_{1}\left(\operatorname{Re} \rho_{h}\right) \in e^{C \epsilon(h) / h} h^{-1 / 3} h^{2 / 3 j} S_{1 / 2,3 / 2,1}
\end{aligned}
$$

Proof. Since $\rho_{0} \leq \alpha_{0}-C y$ and $|\operatorname{Re} \rho| \leq c h^{2 / 3}$ on $\operatorname{supp} p_{2}\left(\operatorname{Re} \rho_{h}\right)$, we may analyze terms involving only $p_{1}$ instead of $q_{1}$.

By (5.2.16), we have

$$
\frac{A_{-}\left(\rho_{h}\right)}{A_{-}\left(\alpha_{h}\right)}=\frac{\Xi_{-}\left(\rho_{h}\right)}{\Xi_{-}\left(\alpha_{h}\right)} e^{2 / 3\left(\rho^{3 / 2}-\alpha^{3 / 2}\right) / h}=\frac{\Xi_{-}\left(\rho_{h}\right)}{\Xi_{-}\left(\alpha_{h}\right)} e^{2 / 3\left(\rho_{0}^{3 / 2}-\alpha_{0}^{3 / 2}\right) / h+O(\epsilon(h) / h)}
$$

We have that $\rho_{0} \leq \alpha_{0}-c y$. Therefore, the estimates follow as in Lemma 5.2.10
We have

## Lemma 5.2.15.

$$
\left(\mathcal{A}_{-}^{<}\right)^{-1} F:=(2 \pi h)^{-d+1} \int\left(A_{-}\left(\alpha_{h}\right)\right)^{-1} p_{3}\left(\operatorname{Re} \alpha_{h}\right) \mathcal{F}_{h} F d \xi
$$

is a Fourier integral operator with singular phase.
Moreover, $\left(A_{-}\left(\alpha_{h}\right)\right)^{-1}$ is bounded on $\operatorname{supp} q_{1}$. Then, similar to above, we have

## Lemma 5.2.16.

$$
(2 \pi h)^{-d+1} \int\left[g_{0} A_{-}\left(\rho_{h}\right)+i h^{1 / 3} g_{1} A_{-}^{\prime}\left(\rho_{h}\right)\right] e^{i \theta / h} \mathcal{F}_{h} G(\xi) p_{3}\left(\rho_{h}\right) d \xi=B^{-}
$$

with

$$
B^{-}=\omega(2 \pi h)^{-d+1} \int\left[g_{0} \Xi_{-}\left(\rho_{h}\right)+i h^{1 / 3} g_{1} \tilde{\Xi}_{-}\left(\rho_{h}\right)\right] e^{i\left[\theta+(2 / 3)(-\rho)^{3 / 2}\right] / h} p_{3}\left(\rho_{h}\right) \mathcal{F}_{h} G(\xi) d \xi
$$

where $\Xi_{-}\left(\rho_{h}\right) \in S_{1 / 3,2 / 3,2 / 3}, \tilde{\Xi}_{-} \in h^{-1 / 6} S_{1 / 3,2 / 3,2 / 3}$.
Together with Lemma 5.2.14 and the fact that $A_{-}\left(\alpha_{h}\right)^{-1}$ is bounded on $\operatorname{supp} q_{1}\left(\operatorname{Re} \alpha_{h}\right)$, this shows that on $\left.\operatorname{supp} p_{3}\left(\operatorname{Re} \rho_{h}\right), 5.1 .1\right)$ is a Fourier integral operator with singular phase.

## Verification of the properties (5.0.2)

We now prove that using the phase and amplitudes constructed in the previous section that (5.0.2) is satisfied. First, we construct $F$ so that the boundary conditions are satisfied. We have that $\left.g_{1}\right|_{\partial \Omega}=0$ and $\left.\rho\right|_{\partial \Omega}=\alpha$. Hence, restricting (5.1.1) or (5.1.2) to $\partial \Omega$ gives

$$
\left.B F\right|_{\partial \Omega}=(2 \pi h)^{-d+1} \int g e^{i \theta_{b} / h} \mathcal{F}_{h} F(\xi) d \xi
$$

where $\theta_{b}=\left.\theta\right|_{\partial \Omega}$ and $g=\left.g_{0}\right|_{\partial \Omega}$. Now, $d_{x} \partial_{\xi_{j}} \theta_{0}$ are linearly independent and hence $\theta_{0}$ is a phase function. Fix $\delta>\delta_{1}>0$. Then, since $\epsilon(h)=O\left(h \log h^{-1}\right), e^{\frac{i}{h} \epsilon(h) \theta^{\prime}} \in S_{\delta}$, and shrinking the neighborhood on which we work if necessary

$$
\frac{\sup \left|e^{\frac{i}{h} \epsilon(h) \theta^{\prime}}\right|}{\inf \left|e^{\frac{i}{h} \epsilon(h) \theta^{\prime}}\right|} \leq C h^{-\delta_{1}}
$$

Thus, $J:=\left.B\right|_{\partial \Omega}$ is a semiclassical Fourier integral operator that is invertible by the symbol calculus of FIOs. Hence, we just need to take $F=J^{-1} f$ to obtain the appropriate boundary conditions where $J^{-1}$ is a microlocal parametrix for $J$. Thus, we let $H_{d}=B_{1} J^{-1}$ and $H_{g}=B_{2} J^{-1}$. We need to verify that if

$$
\operatorname{MS}_{\mathrm{h}}(f) \subset\left\{\left.| | \xi^{\prime}\right|_{g}-1 \mid<\eta(h) \ll 1\right\}
$$

then

$$
\operatorname{MS}_{\mathrm{h}}\left(J^{-1} f\right) \subset\left\{\left|\xi_{1}\right|<C \eta(h)\right\}
$$

but this follows from the fact that $\theta$ parametrizes the reduction of $\partial \Omega$ and $|\xi|^{2}=1$ to the normal form (5.1.3) combined with the wavefront set bound 4.4.5).

After a change of variables near $x_{0}$, we may assume that locally $\Omega_{1}=\{y<0\}$ and $\Omega_{2}=\{y>0\}$. with $x=\left(y, x^{\prime}\right)$.

## Diffractive points

Now, we have that

$$
\left(-h^{2} \Delta-z^{2}\right) B_{1} F=(2 \pi h)^{-d+1} \int\left[a \frac{A\left(\rho_{h}\right)}{A\left(\alpha_{h}\right)}+b \frac{A^{\prime}\left(\rho_{h}\right)}{A\left(\alpha_{h}\right)}\right] e^{i \theta / h}
$$

where $a \sim \sum a_{j, m} h^{j} \epsilon(h)^{m}$ and $b \sim \sum b_{j, m} h^{j} \epsilon(h)^{m}$ such that

$$
\begin{gather*}
a_{j, m}, b_{j, m}=0 \quad \text { for } \rho_{0} \leq 0  \tag{5.2.31}\\
a_{j, m}, b_{j, m}=O\left(y^{n}\right), \quad \text { for any }(x, \xi) \text { and all } n>0 .
\end{gather*}
$$

Thus, for diffractive points, by Lemma 5.2.14

$$
\left(-h^{2} \Delta-z^{2}\right) B_{1} F=O_{C^{\infty}}\left(h^{\infty}\right)
$$

as desired.

## Gliding Points

For gliding points, the verification is more complicated because $\rho_{0}$ may become positive away from the boundary. The case when $\alpha_{0}>0$ are taken care of by Lemma 5.2.10 and the estimates 5.2.31. Suppose that $\alpha_{0} \leq 0$, but $\rho_{0}=0$ at $y_{1}$. Then, since the eikonal and transport equations can be solved in Taylor series at $\rho_{0}=0$ and $\rho_{0} \geq \alpha_{0}+C y$, we have that $a_{j, m}, b_{j, m}=c_{j, m, n} h^{j} \rho_{0}^{n}$, but by Lemma 5.2.12, for $\alpha_{0} \leq 0$ and $\rho_{0} \geq 0$, such an integrand is $O\left(h^{\infty}\right)$ as desired. Hence, we also have

$$
\left(-h^{2} \Delta-z^{2}\right) B_{2} F=O_{C^{\infty}}\left(h^{\infty}\right)
$$

in the gliding case.

### 5.3 Microlocal description of $H_{d}, H_{g}$ and the Airy multipliers

In 9.1, we need the following microlocal characterization of the operator $\mathcal{A}_{-} \mathcal{A} i$ similar to that in [47, Theorem 5.4.19]

Lemma 5.3.1. The Airy multipliers have wavefront set bounds as follows:

$$
\begin{gathered}
\left\{\begin{array}{c}
\mathrm{WF}_{\mathrm{h}}{ }^{\prime}\left(\mathcal{A}_{-} \mathcal{A} i\right) \cup \mathrm{WF}_{\mathrm{h}}{ }^{\prime}\left(\mathcal{A}_{-} \mathcal{A} i^{\prime}\right) \\
\mathrm{WF}_{\mathrm{h}}{ }^{\prime}\left(\mathcal{A}_{-}^{\prime} \mathcal{A} i\right) \cup \mathrm{WF}_{\mathrm{h}}{ }^{\prime}\left(\mathcal{A}_{-}^{\prime} \mathcal{A} i^{\prime}\right)
\end{array}\right\} \subset C_{\beta} \cup \operatorname{graph}(\mathrm{Id})=: C_{b} \\
\mathrm{WF}_{\mathrm{h}}{ }^{\prime}\left(\left(\mathcal{A}_{-} \mathcal{A} i\right)^{-1}\right) \subset \cup_{n=0}^{\infty} C_{\beta^{n}} \cap E_{+}=: C_{b}^{\infty} \\
E_{+}:=\left\{\xi_{1} \neq 0\right\} \cup\left\{x_{1} \geq y_{1}, x_{i}=y_{i}, 2 \leq i \leq d, \xi=\eta, \xi_{1}=0\right\}
\end{gathered}
$$

where $C_{\beta^{n}}$ is the relation generated by $\beta^{n}$ and graph(Id) denotes the graph of the identity map.

Remark: Note that $C_{b}^{\infty}=\overline{\bigcup_{n \geq 0} C_{\beta^{n}}}$
Proof. We have that $\alpha_{h}=h^{-2 / 3}\left(\xi_{1}+\epsilon(h) \alpha^{\prime}(\xi)\right)$ where $\xi_{1}$ is dual to $y$. First, fix $\delta>0$ and suppose that $\psi_{1} \in S^{0}\left(\mathbb{R}^{d}\right)$ is a cutoff function with $\psi(\xi)=0,\left|\xi_{1}\right| \leq \delta$, and $\psi(\xi)=1,\left|\xi_{1}\right| \geq$ $2 \delta$. Then, we show that $\mathrm{WF}_{\mathrm{h}}{ }^{\prime}\left(\psi(h D) \mathcal{A}_{-} \mathcal{A} i\right) \subset C_{b}, \mathrm{WF}_{\mathrm{h}}{ }^{\prime}\left(\left(\psi(h D) \mathcal{A}_{-} \mathcal{A} i\right)^{-1}\right) \subset C_{b}^{\infty}$. Write $\psi=: \psi_{+}+\psi_{-}$where $\operatorname{supp} \psi_{ \pm} \subset\left\{ \pm \xi_{1}>0\right\}$. Then, in $|\operatorname{Arg} z|<\epsilon$,

$$
A_{-}(z) A i(z)=\Xi_{-} \Xi
$$

with $\Xi_{-} \Xi$ an elliptic symbol. Hence,

$$
\begin{gathered}
\psi_{+} A_{-}\left(\alpha_{h}\right) A i\left(\alpha_{h}\right)=\psi_{+} \Xi_{-}\left(\alpha_{h}\right) \Xi\left(\alpha_{h}\right) \\
\psi_{+}\left(A_{-}\left(\alpha_{h}\right) A i\left(\alpha_{h}\right)\right)^{-1}=\psi_{+} \Xi_{-}^{-1}\left(\alpha_{h}\right) \Xi\left(\alpha_{h}\right)^{-1}
\end{gathered}
$$

and $\psi_{+}\left(\mathcal{A}_{-} \mathcal{A}_{i}\right), \psi_{+}\left(\mathcal{A}_{-} \mathcal{A} i\right)^{-1}$ are classical pseudodifferential operators. Thus, we have

$$
\mathrm{WF}_{\mathrm{h}}{ }^{\prime}\left(\psi_{+} \mathcal{A}_{-} \mathcal{A} i\right), \mathrm{WF}_{\mathrm{h}}{ }^{\prime}\left(\psi_{+}\left(\mathcal{A}_{-} \mathcal{A} i\right)^{-1}\right) \subset \text { graph } I d
$$

Now, for the term involving $\psi_{-}$, we use the asymptotic expansion of $A i$ and $A_{-}$to write in $|\operatorname{Arg} z-\pi|<\epsilon, A i A_{-}(z)=\omega \Xi_{+}(z) \Xi_{-}(z)+\bar{\omega} \Xi_{-}^{2}(z) e^{4 / 3 i(-z)^{3 / 2}}$. Thus,

$$
\begin{equation*}
\psi_{-}(\xi) A_{-} A i\left(\alpha_{h}\right)=a_{1} \exp \left(\frac{4}{3 h} i\left(-\xi_{1}-\epsilon(h) \alpha^{\prime}\right)^{3 / 2}\right)+a_{2} \tag{5.3.1}
\end{equation*}
$$

where $a_{i} \in h^{1 / 3} S^{-1 / 2}$. Therefore $\psi_{-} \mathcal{A} \mathcal{A} i \in h^{1 / 3} I^{0}\left(C_{b} \cap\left\{\left|\xi_{1}\right| \neq 0\right\}\right)$ since $\varphi=\langle x-y, \xi\rangle+$ $\frac{4}{3}\left(-\xi_{1}\right)^{3 / 2}$ parametrizes $\beta$ for the Friedlander model and the $\alpha^{\prime}$ term is a symbolic perturbation since $\epsilon(h)=O\left(h \log h^{-1}\right)$. Identical arguments give the wavefront set bound from $\mathcal{A}_{-}^{\prime} \mathcal{A} i^{\prime}$.

Similarly, using [47, Section 5] or simply expanding in power series,

$$
\left.\psi_{-}\left(\left(A_{-} A i\right)^{-1}\left(\alpha_{h}\right)\right)=\sum_{k \geq 0} a_{k} \exp \left(\frac{4 k}{3 h} i\left(-\xi_{1}-\epsilon(h) \alpha^{\prime}\right)\right)^{3 / 2}\right)
$$

where for any $S^{1 / 2}$ seminorm, $\|\cdot\|_{S^{1 / 2}}$,

$$
\sum_{k \geq 0}\left\|a_{k}\right\|_{S^{1 / 2}}<C h^{-1 / 3}
$$

Thus,

$$
\psi_{-}\left(\mathcal{A}_{-} \mathcal{A} i\right)^{-1} \in h^{-1 / 3} I^{1 / 3}\left(\mathbb{R}^{d} ; C_{b}^{\infty} \cap\left\{\left|\xi_{1}\right| \neq 0\right\}\right)
$$

Now, by Lemma 5.2.10,

$$
A i\left(\alpha_{h}\right) A_{-}\left(\alpha_{h}\right) q_{1}\left(\operatorname{Re} \alpha_{h}\right) \in e^{C \epsilon(h) / h} S_{1 / 2,2 / 3,1}
$$

Thus,

$$
h D_{\xi}^{\beta} A i A_{-}\left(\alpha_{h}\right) q_{1}\left(\operatorname{Re} \alpha_{h}(h D)\right)=O\left(h^{|\beta| / 3}\right) e^{C \epsilon(h) / h} .
$$

So, if $b$ is the kernel of $\mathcal{A} i \mathcal{A}_{i} q\left(\operatorname{Re}\left(\alpha_{h}(h D)\right)\right.$, then

$$
\left(x_{i}-y_{i}\right)^{k} b=O\left(h^{|\beta| / 3}\right) e^{C \epsilon(h) / h} .
$$

Hence, for any $N>0$, taking $|\beta|$ large enough and using that $\epsilon(h)=O\left(h \log h^{-1}\right)$.,

$$
\left(x_{i}-y_{i}\right)^{|\beta|} b=O\left(h^{N}\right)
$$

But, $x_{i}-y_{i}$ is elliptic away from $x_{i}-y_{i}=0$. Hence,

$$
\mathrm{WF}_{\mathrm{h}}{ }^{\prime}\left(\mathcal{A}_{-} \mathcal{A} i q_{1}\left(\operatorname{Re}\left(\alpha_{h}(h D)\right)\right) \subset \operatorname{graph}(\mathrm{Id}) .\right.
$$

But on $\operatorname{supp}\left(1-q_{1}\right)$, the asymptotics (5.3.1) hold and we have studied this wavefront set.
Next, observe that $\partial_{\xi_{j}}\left(A_{-} A_{i}\left(\alpha_{h}\right)\right)^{-1}=0$ for $2 \leq j \leq d$. Hence,

$$
\mathrm{WF}_{\mathrm{h}}{ }^{\prime}\left(\left(\mathcal{A}_{-} \mathcal{A} i\right)^{-1}\right) \subset\left\{x_{2}=y_{2}, \ldots x_{d}=y_{d}\right\} .
$$

The sign condition on $x_{1}$ follows from the fact that $\left(A_{-} A_{i}\left(h^{-2 / 3} \xi_{1}+i \epsilon(h)\right)\right)^{-1}$ is holomorphic in $\operatorname{Im} \xi_{1}>0$. Hence, by the Paley-Weiner theorem [40, Theorem 7.3.8]

$$
\operatorname{supp}\left[\left(\mathcal{A}_{-} \mathcal{A} i\right)^{-1} \delta(x)\right] \subset\left\{x_{1}>0\right\}
$$

Remark: This is where we use the assumption $\alpha^{\prime}=i+O(\epsilon(h))$ rather than $\alpha^{\prime}=-i+O(\epsilon(h))$.

We need the following characterization of $\mathrm{WF}_{\mathrm{h}}{ }^{\prime}\left(H_{d}\right)$ [69, Appendix A.3]

## Lemma 5.3.2.

$$
\mathrm{WF}_{\mathrm{h}}{ }^{\prime}\left(H_{d}\right) \subset\left\{\begin{array}{c}
(x, \xi, y, \eta) \in T^{*} \Omega_{2} \times T^{*} \partial \Omega: \\
|\xi|=1,(x, \xi) \text { in the outgoing ray from }(y, \eta)
\end{array}\right\}
$$

Proof. We decompose the operator into pieces where $\operatorname{Re} \rho \geq-2 C h^{2 / 3}$ and $\operatorname{Re} \rho \leq-C h^{2 / 3}$. When $\operatorname{Re} \rho \geq-2 C h^{2 / 3}$, Lemma 5.2.10 shows that in the interior of $\Omega_{1}, H_{d}=O\left(h^{\infty}\right)$. When $\operatorname{Re} \rho \leq-C h^{2 / 3}$, Lemmas 5.2.15 and 5.2.16 show that $H_{d} J$ is a Fourier integral operator with singular phase

$$
\psi=\theta-\frac{2}{3}\left[(-\rho)^{3 / 2}-(-\alpha)^{3 / 2}\right] .
$$

Thus it has $\left.\mathrm{WF}_{\mathrm{h}}{ }^{\prime}\left(H_{d} J\right)\right|_{\Omega_{2}} \subset C_{\psi}$ where $C_{\psi}=\left\{\left(x, \nabla_{x} \psi, \nabla_{\xi} \psi, \xi\right)\right\}$. But, this parametrizes the outgoing geodesics ( 77 , Section X.4], [47, Section 6.5]).

Now, at $\partial \Omega, H_{d}$ is a microlocally invertible Fourier integral operator with phase $\theta_{b}\left(x^{\prime}, \xi\right)-$ $\theta_{b}\left(y^{\prime}, \xi\right)$. Hence, on $\partial \Omega$

$$
\left.\mathrm{WF}_{\mathrm{h}}{ }^{\prime}\left(H_{d}\right)\right|_{\partial \Omega} \subset \text { graph Id. }
$$

Similar arguments together with the wavefront set bound on $\left(\mathcal{A}_{-} \mathcal{A} i\right)^{-1}$ show 47 , Section 6.5],

## Lemma 5.3.3.

$$
\begin{aligned}
\mathrm{WF}_{\mathrm{h}}{ }^{\prime}\left(B_{3} J^{-1}\right) \subset\left\{\begin{array}{c}
(x, \xi, y, \eta) \in T^{*} \Omega_{1} \times T^{*} \partial \Omega: \\
|\xi|=1,(x, \xi) \text { is in an outgoing ray from a point }(y, \eta)
\end{array}\right\} \\
\mathrm{WF}_{\mathrm{h}}{ }^{\prime}\left(H_{g}\right) \subset\left\{\begin{array}{c}
(x, \xi, y, \eta) \in T^{*} \Omega_{1} \times T^{*} \partial \Omega:|\xi|=1 \\
(x, \xi) \text { is in an outgoing ray from } \overline{\cup_{n \geq 0} \beta^{n}((y, \eta))}
\end{array}\right\}
\end{aligned}
$$

Proof. We first prove wave front set bounds on operators of type (5.2.29) decompose the operator into pieces where $\operatorname{Re} \rho \geq-2 C h^{2 / 3}$ and $\operatorname{Re} \rho \leq-C h^{2 / 3}$. When $\operatorname{Re} \rho \geq-2 C h^{2 / 3}$, Lemma 5.2 .10 shows that in the interior of $\Omega_{2}, B_{3}=O\left(h^{\infty}\right)$. When Re $\rho \leq-C h^{2 / 3}$, Lemmas 5.2 .11 and 5.2 .13 show that $B_{3}$ is a Fourier integral operator with singular phase

$$
\psi=\theta-\frac{2}{3}\left[(-\rho)^{3 / 2}-(-\alpha)^{3 / 2}\right] .
$$

Thus it has

$$
\left.\mathrm{WF}_{\mathrm{h}}^{\prime}\left(B_{3}\right)\right|_{\Omega_{1}} \subset C_{\psi}, \quad \text { where } \quad C_{\psi}:=\left\{\left(x, \nabla_{x} \psi, \nabla_{\xi} \psi, \xi\right\} .\right.
$$

But, this parametrizes the outgoing geodesics ( 77 , Section X.4], [47, Section 6.5]).
Now, at $\partial \Omega, B_{3}$ is a microlocally invertible Fourier integral operator with phase $\theta_{b}$. Hence, on $\partial \Omega$

$$
\left.\mathrm{WF}_{\mathrm{h}}{ }^{\prime}\left(H_{g}\right)\right|_{\partial \Omega} \subset \text { graph Id. }
$$

Combining this with the wavefront relation for $\left(\mathcal{A}_{-} \mathcal{A} i\right)^{-1}$ completes the proof of the lemma.

### 5.4 Parametrix for diffractive points

We follow [69] to show that the parametrices $H_{d}$ constructed above are $O_{C^{\infty}}\left(h^{\infty}\right)$ close to the exact solution near $\partial \Omega$. We have that for $f$ microsupported near a glancing point $\left(y_{0}, \eta_{0}\right)$

$$
\begin{equation*}
\left(-h^{2} \Delta-z^{2}\right) H_{d} f=K f \text { in } U,\left.\quad H_{d} f\right|_{\partial \Omega}=f+S f \tag{5.4.1}
\end{equation*}
$$

Here $K=O_{\mathcal{S}^{\prime} \rightarrow C^{\infty}}\left(h^{\infty}\right)$ and $S=O_{\mathcal{D}^{\prime} \rightarrow C^{\infty}}\left(h^{\infty}\right)$. Let $\chi \in C_{0}^{\infty}$ have supp $\chi \subset U$ and $\chi \equiv 1$ in a neighborhood of $\partial \Omega$.

Define

$$
\tilde{H}_{d}:=\chi H_{d}-R_{0}\left(\chi K-\left[h^{2} \Delta, \chi\right] H_{d}\right) .
$$

Then $\tilde{H}_{d}$ is $z$ outgoing and has $\left(-h^{2} \Delta-z\right) \tilde{H}_{d}=0$. Next,

$$
\left.\left(\tilde{H}_{d} f\right)\right|_{\partial \Omega}=f+S f-\gamma R_{0}\left(\chi K f+\left[-h^{2} \Delta, \chi\right] H_{d} f\right)
$$

The last term is the only potentially problematic term. However, since $\mathrm{WF}_{\mathrm{h}}\left(\left[-h^{2} \Delta, \chi\right]\right)$ is away from $\partial \Omega, H_{d}$ and $R_{0}$ are outgoing, and $\Omega$ is convex, this term is $O_{C^{\infty}}\left(h^{\infty}\right)$ when restricted to a neighborhood of $\partial \Omega$.

Thus, writing

$$
\tilde{R}=S-\gamma R_{0}\left(\chi K+\left[-h^{2} \Delta, \chi\right] H_{d}\right)
$$

we have that the exact solution operator is given by $\mathcal{H}_{d}=\tilde{H}_{d}(I+\tilde{R})^{-1}$ where $I+\tilde{R}$ is invertible for $h$ small since $\tilde{R}$ is $O_{C^{\infty}}\left(h^{\infty}\right)$. Hence, we have

Lemma 5.4.1. Then the solution operator for the exterior Dirichlet problems is given by

$$
\mathcal{H}_{d}=\chi H_{d}-R_{0}\left(\chi K-\left[h^{2} \Delta, \chi\right] H_{d}\right)+O_{C^{\infty}}\left(h^{\infty}\right)
$$

In a neighborhood, $U$ of $\partial \Omega$, this is

$$
\left.\mathcal{H}_{d}\right|_{U}=\left.\chi H_{d}\right|_{U}+O_{C^{\infty}(U)}\left(h^{\infty}\right) .
$$

## Dirichlet to Neumann Maps in the Diffractive Case

Using the parametrices constructed above, we construct a microlocal representation of the Dirichlet to Neumann map near glancing. In order to do this, we simply take the normal derivative of $H$ from the previous section. That is, let $\nu^{\prime}$ denote the inward unit normal to $\Omega$,

$$
\left.\partial_{\nu^{\prime}} H_{d}(f)\right|_{\partial \Omega}=(2 \pi h)^{-d+1} \int\left(g_{0}^{\prime}+i h^{1 / 3} g_{1}^{\prime} \frac{A_{-}^{\prime}\left(h^{-2 / 3} \alpha\right)}{A_{-}\left(h^{-2 / 3} \alpha\right)}\right) e^{i \theta_{b} / h} \mathcal{F}_{h} F d \xi
$$

The new symbols $g_{0}^{\prime}$ and $g_{1}^{\prime}$ have $g_{0}^{\prime}=\partial_{\nu^{\prime}} g_{0}+i h^{-1} g_{0} \partial_{\nu^{\prime}} \theta+i h^{-1} g_{1} \rho \partial_{\nu^{\prime}} \rho$ and $g_{1}^{\prime}=\partial_{\nu^{\prime}} g_{1}-$ $i h^{-1} g_{0} \partial_{\nu^{\prime}} \rho+h^{-1} g_{1} \partial_{\nu^{\prime}} \theta$. By construction $g_{1}$ vanishes at the boundary and, moreover $\partial_{\nu^{\prime}} \rho \neq 0$ with $\nabla \rho=\partial_{\nu^{\prime}} \rho \nu^{\prime}$. Hence, $\partial_{\nu^{\prime}} \theta=0$ by (5.2.1). So, we have

$$
\begin{equation*}
g_{0}^{\prime}=\partial_{\nu^{\prime}} g_{0} \quad g_{1}^{\prime}=-i h^{-1} g_{0} \partial_{\nu^{\prime}} \rho+\partial_{\nu^{\prime}} g_{1} \tag{5.4.2}
\end{equation*}
$$

Now, $g_{0}^{\prime} \in S$ and $g_{1}^{\prime} \in h^{-1} S$ with $g_{1}^{\prime}$ elliptic and hence we have

$$
\begin{gathered}
\frac{1}{(2 \pi h)^{d-1}} \int g_{0}^{\prime} e^{i \theta_{b} / h} \mathcal{F}_{h} F(\xi) d \xi=: J B(F) \\
\frac{i h^{1 / 3}}{(2 \pi h)^{d-1}} \int g_{1}^{\prime} \frac{A_{-}^{\prime}\left(h^{-2 / 3} \alpha\right)}{A_{-}\left(h^{-2 / 3} \alpha\right)} e^{i \theta_{b} / h} \mathcal{F}_{h} F(\xi) d \xi=: J h^{-2 / 3} C \Phi_{-}(F) .
\end{gathered}
$$

with $C \in \Psi$ elliptic, $B \in \Psi$, and $\Phi_{-}$the operator defined by

$$
\widehat{\Phi_{-}(F)}:=\frac{A_{-}^{\prime}\left(\alpha_{h}\right)}{A_{-}\left(\alpha_{h}\right)} \mathcal{F}_{h} F=: \phi_{-}\left(\alpha_{h}\right) \mathcal{F}_{h} F \text {. }
$$

Hence, microlocally,

$$
\begin{equation*}
N_{2}=J\left(h^{-2 / 3} C \Phi_{-}+B\right) J^{-1} . \tag{5.4.3}
\end{equation*}
$$

A simple nonstationary phase argument shows that $\mathrm{WF}_{\mathrm{h}}{ }^{\prime}\left(\Phi_{-}\right) \subset$ graph Id. This together with the microlocal model (5.4.3) implies the following bounds for the exterior Dirichlet to Neumann maps near glancing.

Theorem 5.1. Let $N_{2}$ denote the Dirichlet to Neumann map for the exterior of $\Omega$. Let $\chi \in$ $C_{c}^{\infty}(\mathbb{R})$. Fix $0<\epsilon<1 / 2$ and let $X_{\epsilon}=\mathrm{Op}_{\mathrm{h}}\left(\chi\left(\left.h^{-\epsilon}| | \xi^{\prime}\right|_{g}-1 \mid\right)\right)$. Then for $|\operatorname{Im} z| \leq C h \log h^{-1}$,

$$
\left\|N_{2} X_{\epsilon}\right\|_{L^{2} \rightarrow L^{2}} \leq h^{-1+\epsilon / 2} .
$$

Remark: Note that one can let $0<\epsilon \leq 2 / 3$ if we apply the second microlocal calculus of 65.

### 5.5 Relation with exact operators in gliding case

In the gliding case, we cannot make a simple wavefront set argument to show that $H_{g}$ is $h^{\infty}$ close to the exact solution operator. Instead, we focus on constructing functions that are used in section 6.7 to produce microlocal descriptions of boundary layer operators and potentials near glancing. In particular, we examine operators of the form $\tilde{A}_{g}:=B_{3} J^{-1}$ where

$$
B_{3} F:=\frac{1}{(2 \pi h)^{-d+1}} \int\left(g_{0} A i\left(\rho_{h}\right)+i h^{1 / 3} g_{1} A i^{\prime}\left(\rho_{h}\right)\right) A_{-}\left(\alpha_{h}\right) e^{\frac{i}{h} \theta(x, \xi)} \mathcal{F}_{h}(F)(\xi) d \xi
$$

Let $\left(y_{0}, \eta_{0}\right) \in S^{*} \partial \Omega$ be a glancing point. Then we have that there exists $U$ a neighborhood of $y_{0}$ in $\Omega$ such that for $\delta$ and $\gamma$ small enough and $\psi$ with

$$
\begin{gathered}
\psi \equiv 1 \text { on }\left\{\left|y-y_{0}\right|<\delta,\left|\eta-\eta_{0}\right|<\delta_{1},\left||\eta|_{g}-1\right| \leq \gamma h^{2} \epsilon(h)^{-2}\right\} \\
\operatorname{supp} \psi \subset\left\{\left|y-y_{0}\right|<2 \delta,\left|\eta-\eta_{0}\right|<2 \delta_{1},\left||\eta|_{g}-1\right| \leq 2 \gamma h^{2} \epsilon(h)^{-2}\right\} \\
\left\{\begin{array}{r}
\left(-h^{2} \Delta-z^{2}\right) \tilde{A}_{g} f=K f \\
\left.\tilde{A}_{g} f\right|_{\partial \Omega}=J \mathcal{A} i \mathcal{A}_{-} J^{-1} \operatorname{Op}_{\mathrm{h}}(\psi) f+S f
\end{array}\right.
\end{gathered}
$$

where $K=O_{\mathcal{D}^{\prime}(\partial \Omega) \rightarrow C^{\infty}(U)}\left(h^{\infty}\right)$ and $S=O_{\mathcal{D}^{\prime} \rightarrow C^{\infty}(\partial \Omega)}\left(h^{\infty}\right)$. Now, shrinking $\delta$ if necessary, we assume that $B\left(y_{0}, 3 \delta\right) \subset U$. Now, fix $\chi \in C^{\infty}(\Omega)$ supported in $U$ with $\chi \equiv 1$ on $B\left(y_{0}, 2 \delta\right)$. Then, using the wavefront set bound on $B_{3}$, we have that shrinking $\delta$ again if necessary, $\mathrm{WF}_{\mathrm{h}}\left(\tilde{A}_{g}\right) \cap \operatorname{supp} \partial \chi=\emptyset$. So, defining $A_{g}:=\chi H_{g}$, we have

$$
\left\{\begin{aligned}
\left(-h^{2} \Delta-z^{2}\right) A_{g} f & =\chi K f+\left[h^{2} \Delta, \chi\right] \tilde{A}_{g} f=K_{1} f \\
\left.A_{g} f\right|_{\partial \Omega} & =\chi J \mathcal{A} i \mathcal{A}_{-} J^{-1} \operatorname{Op}_{\mathrm{h}}(\psi) f+\chi S f \\
& =J \mathcal{A} i \mathcal{A}_{-} J^{-1} \operatorname{Op}_{\mathrm{h}}(\psi) f+S_{1} f
\end{aligned}\right.
$$

where $K_{1}=O_{\mathcal{D}^{\prime}(\partial \Omega) \rightarrow C^{\infty}(\Omega)}\left(h^{\infty}\right)$ and $S_{1}=O_{\mathcal{D}^{\prime}(\partial \Omega) \rightarrow C^{\infty}(\partial \Omega)}\left(h^{\infty}\right)$.
Similarly, there exists $B_{g}=\chi B_{4} J^{-1}$ with

$$
B_{4} F:=\frac{1}{(2 \pi h)^{-d+1}} \int\left(g_{0} A i\left(\rho_{h}\right)+i h^{1 / 3} g_{1} A i^{\prime}\left(\rho_{h}\right)\right) A_{-}^{\prime}\left(\alpha_{h}\right) e^{\frac{i}{h} \theta(x, \xi)} \mathcal{F}_{h}(F)(\xi) d \xi
$$

such that

$$
\left\{\begin{aligned}
\left(-h^{2} \Delta-z^{2}\right) B_{g} f & =K_{2} f \\
\left.B_{g} f\right|_{\partial \Omega} & =J \mathcal{A} i \mathcal{A}_{-}^{\prime} J^{-1} \mathrm{Op}_{\mathrm{h}}(\psi) f+S_{2} f
\end{aligned}\right.
$$

where $K_{2}=O_{\mathcal{D}^{\prime}(\partial \Omega) \rightarrow C^{\infty}(\Omega)}\left(h^{\infty}\right)$ and $S_{2}=O_{\mathcal{D}^{\prime}(\partial \Omega) \rightarrow C^{\infty}(\partial \Omega)}\left(h^{\infty}\right)$.
Note also that with $\nu$ the outward unit normal to $\Omega$,

$$
\begin{aligned}
\left.\partial_{\nu} A_{g} f\right|_{\partial \Omega} & =-J\left(h^{-2 / 3} C \mathcal{A} i^{\prime} \mathcal{A}_{-}+B \mathcal{A} i \mathcal{A}_{-}\right) J^{-1} \mathrm{Op}_{\mathrm{h}}(\psi) f+S_{3} f \\
\left.\partial_{\nu} B_{g} f\right|_{\partial \Omega} & =-J\left(h^{-2 / 3} C \mathcal{A} i^{\prime} \mathcal{A}_{-}^{\prime}+B \mathcal{A} i \mathcal{A}_{-}^{\prime}\right) J^{-1} \mathrm{Op}_{\mathrm{h}}(\psi) f+S_{4} f
\end{aligned}
$$

where $S_{i}=O_{\mathcal{D}^{\prime}(\partial \Omega) \rightarrow C^{\infty}(\partial \Omega)}\left(h^{\infty}\right)$ and $B, C \in \Psi$ are as in 5.4.3). Then we have,

Lemma 5.5.1. Near a gliding point, there exist operators $A_{i, g} i=1,2$ so that

$$
\left\{\begin{aligned}
\left(-h^{2} \Delta-z^{2}\right) A_{i, g} f & =K_{i} f \text { in } \Omega \\
\left.A_{1, g}\right|_{\partial \Omega} & =J \mathcal{A} i \mathcal{A}_{-} J^{-1} \mathrm{Op}_{\mathrm{h}}(\psi) f+S_{1, r} f \\
\left.A_{2, g}\right|_{\partial \Omega} & =J \mathcal{A} i \mathcal{A}_{-}^{\prime} J^{-1} \mathrm{Op}_{\mathrm{h}}(\psi) f+S_{2, r} f \\
\left.\partial_{\nu} A_{1, g}\right|_{\partial \Omega} & =-J\left(h^{-2 / 3} C \mathcal{A} i^{\prime} \mathcal{A}_{-}+B \mathcal{A} i \mathcal{A}_{-}\right) J^{-1} \mathrm{Op}_{\mathrm{h}}(\psi) f+S_{1, \nu} f \\
\left.\partial_{\nu} A_{2, g}\right|_{\partial \Omega} & =-J\left(h^{-2 / 3} C \mathcal{A} i^{\prime} \mathcal{A}_{-}^{\prime}+B \mathcal{A} i \mathcal{A}_{-}^{\prime}\right) J^{-1} \mathrm{Op}_{\mathrm{h}}(\psi) f+S_{2, \nu} f
\end{aligned}\right.
$$

where $K_{i}=O_{\mathcal{D}^{\prime}(\partial \Omega) \rightarrow C^{\infty}(\Omega)}\left(h^{\infty}\right)$ and $S_{i, \cdot}=O_{\mathcal{D}^{\prime}(\partial \Omega) \rightarrow C^{\infty}(\partial \Omega)}\left(h^{\infty}\right)$.

### 5.6 Wave equation parametrices

Let $\Omega \subset \mathbb{R}^{d}$ be a strictly convex domain with smooth boundary. Let $\Omega_{1}=\Omega$ and $\Omega_{2}=\mathbb{R}^{d} \backslash \bar{\Omega}$. In order to handle the glancing region, we construct microlocal parametrices for

$$
\begin{cases}\left(\partial_{t}^{2}-\Delta\right) u_{i}=0, & \text { in } \Omega_{i} \\ u_{1}=u_{2} & \text { on } \partial \Omega \\ \partial_{\nu_{1}} u_{1}+\partial_{\nu_{2}} u_{2}=f & \text { on } \partial \Omega\end{cases}
$$

That is, we construct $H$ such that if $f$ has wavefront set in a small conic neighborhood of $\left(t_{0}, x_{0}, \tau_{0}, \xi_{0}\right) \in T^{*}(\mathbb{R} \times \partial \Omega)$ over which a glancing ray passes, then

$$
\begin{cases}\left(\partial_{t}^{2}-\Delta\right) H f & \in C^{\infty}\left(\Omega_{i}\right)  \tag{5.6.1}\\ \left.\left(H_{1} f-H_{2} f\right)\right|_{\partial \Omega} & \in C^{\infty}(\partial \Omega) \\ \partial_{\nu_{1}} H_{1} f+\partial_{\nu_{2}} H_{2} f-f & \in C^{\infty}(\partial \Omega) \\ H f \in C^{\infty} & t \ll 0\end{cases}
$$

First, the wavefront set property for $f$ implies that $f$ is $C^{\infty}$ outside of a compact set in $t$. Hence, by [40, Theorem 6.24], the solution when $f$ is replaced by $\chi(t) f$ differs only by a $C^{\infty}$ function. Thus, without loss of generality, we assume that $f$ has compact support.

We will use the construction in Section 6.7. To pass from the parametrix for $-h^{2} \Delta-z^{2}$ to a (5.6.1), set $z=1, h=\tau^{-1}$, and rescale $\xi^{\prime} \rightarrow \xi^{\prime} \tau$.
That is, letting

$$
\begin{gathered}
H_{h} f(x, h):=(2 \pi h)^{-d+1} \int g\left(x, \xi^{\prime}, y, h\right) f(y, h) d y d \xi^{\prime} \\
\tilde{H} f(x, \tau):=\left(2 \pi \tau^{-1}\right)^{-d+1} \int g\left(x, \tau \xi^{\prime}, y, \tau^{-1}\right) f\left(y, \tau^{-1}\right) d y d \xi^{\prime}
\end{gathered}
$$

We then have that $\tilde{H}$ acts on functions $f$ with wavefront set in $\left|\left|\xi^{\prime} \tau^{-1}\right|-1\right| \leq \epsilon$ and is $O\left(\tau^{-\infty}\right)$ on functions with wavefront set away from this set. That is, $\tilde{H}$ acts on functions with wavefront set in a conic neighborhood of glancing. Then,

$$
H:=\mathcal{F}_{t \rightarrow \tau}^{-1} \tilde{H} \mathcal{F}_{t \rightarrow \tau}
$$

is the desired parametrix.

### 5.7 Semiclassical Fourier integral operators with singular phase

We now define the semiclassical analog of Fourier integral operators with singular phase. We follow the treatment in the homogeneous setting given in [77, Section VII.6] (For another treatment of Fourier integral operators with singular phase in the homogeneous setting, see [47, Appendix D].).

Throughout this section, we assume that $U \subset \mathbb{R}^{d}$ is open and $\varphi \in C^{\infty}(U)$ is a nondegenerate phase function with the caveat that, letting $\gamma$ be a boundary defining function for $\bar{U}$ and $0 \leq a<1$, it only has

$$
\begin{equation*}
\varphi \in C^{1}(\bar{U}) \quad\left|D_{x}^{\beta} D_{\xi}^{\alpha} \varphi\right| \leq C_{\alpha, \beta} \gamma^{(1+a)-|\alpha|-|\beta|}, \quad \text { if }|\alpha|+|\beta| \geq 2 \tag{5.7.1}
\end{equation*}
$$

Then, let $a \in S_{\delta}(U)$ have

$$
\begin{equation*}
\operatorname{supp} a \subset\left\{\gamma \geq c h^{b}\right\}, \quad \operatorname{supp} a \Subset \bar{U} \tag{5.7.2}
\end{equation*}
$$

where $c>0$ and $0<b<1$. Here, we allow $\delta \in[0, b)$.
A Fourier integral operator with singular phase $\varphi$ is an operator $A u$ defined by

$$
A u(x)=(2 \pi h)^{-d} \int a(x, \xi) e^{\frac{i}{h}(\varphi(x, \xi)-\langle y, \xi\rangle)} u(y) d y d \xi
$$

Since $a$ has compact support, this operator is well defined. We need to prove the following lemma.

Lemma 5.7.1. Let $\varphi$ have (5.7.1) and $a \in S_{\delta}$ have (5.7.2). Let $A$ be a Fourier integral operator with singular phase $\varphi$. Then

$$
\mathrm{WF}_{\mathrm{h}}{ }^{\prime}(A) \subset\left\{\left(x, \partial_{x} \varphi(x, \xi), \xi, \partial_{\xi} \varphi(x, \xi)\right\} .\right.
$$

Proof. To see this, consider

$$
\left\langle\chi(x) e^{-\frac{i}{h}\langle x, \theta\rangle}, A u\right\rangle=(2 \pi h)^{-d} \int u(y) \chi(x) a(x, \xi) e^{\frac{i}{h} \Phi(x, \xi, y, \theta)} d y d x d \xi
$$

where $\Phi(x, \xi, y, \theta)=\varphi(x, \xi)-\langle y, \xi\rangle-\langle x, \theta\rangle$. Then, away from $\partial_{x} \varphi=\partial_{\xi} \varphi=0$, there exists $L$ such that $L e^{\frac{i}{h} \Phi}=e^{\frac{i}{h} \Phi}$. By (5.7.1 we have

$$
\left(L^{t}\right)^{k}=\sum_{0 \leq|\sigma| \leq k} h^{k} A_{\sigma}^{k}(x, \xi) D_{x}^{\sigma}
$$

Here, $A_{\sigma}^{k}$ has

$$
\left|D_{x}^{\beta} D_{\xi}^{\alpha} h^{k} A_{\sigma}^{k}\right| \leq h^{k}\left(1+\gamma^{a-(k-|\sigma|)-|\alpha|-|\beta|}\right)
$$

Thus, on $\operatorname{supp} a$,

$$
\left|h^{k} A_{\sigma}^{k}\right| \leq C h^{k}\left(1+h^{(a-(k-|\sigma|)) b}\right) \leq C h^{k(1-b)+a b+|\sigma| b}
$$

Thus,

$$
\left|\left(L^{t}\right)^{k} a\right| \leq \sum_{0 \leq|\sigma| \leq k} C h^{k(1-b)+a b+|\sigma| b-\delta|\sigma|)} \leq h^{k \min (1-b, b-\delta)}
$$

Since $\delta<b<1$, this gives the result.

## Chapter 6

## Boundary Layer Operators

In Chapters 7 and 10, the existence of resonances for $-\Delta_{\Gamma, \delta}$ and $-\Delta_{\partial \Omega, \delta^{\prime}}$ will be related to a certain equation involving boundary layer operators of the Helmholtz equation. In this Chapter we prepare for the analysis of these equations by understanding the classical boundary layer potentials from a semiclassical point of view. We first review some of the classical theory of boundary layer potentials. We then proceed to prove (nearly) sharp high frequency estimates on layer potentials using $L^{2}$ estimates on restrictions of quasimodes and their derivatives to hypersurfaces. We then give a microlocal description of the single and derivative double layer operators for domains with smooth boundary away from glancing. In the process, we give a description of the free resolvent as a semiclassical intersecting Lagrangian distribution. Finally, in the case that the domain is strictly convex, we use the Melrose-Taylor parametrix from Chapter 5 to give a microlocal description of the single and derivative double layer operators near glancing. As a consequence of the microlocal models for the single layer potential and derivative double layer potential, we improve the nearly sharp estimates on these operators to sharp estimates in the case the the domain has smooth, strictly convex boundary.

### 6.1 Classical Layer Potential Theory

We review here some facts about boundary layer potentials in the context of the Helmholtz equation. We start by considering $\operatorname{Im} \lambda>0$. Then,

$$
\left(-\Delta_{x}-\lambda^{2}\right) R_{0}(\lambda)(x, y)=\delta_{y}(x)
$$

Moreover, the equality continues analytically through $\operatorname{Re} \lambda \geq 0$ to $\mathbb{C}$ in the case that $d$ is odd and to the logarithmic cover of $\mathbb{C} \backslash\{0\}$ if $d$ is even.

Let

$$
\mathcal{S} f(x):=\int_{\partial \Omega} R_{0}(\lambda, x, y) d S(y), \quad \mathcal{D} f(x):=\int_{\partial \Omega} \partial_{\nu_{y}} R_{0}(\lambda, x, y) f(y) d S(y) \quad x \notin \partial \Omega
$$

be respectively the single and double layer potential. We prove the following lemma similar to [76, Propositions 11.1, 11.2]

Lemma 6.1.1. Let $\Omega \Subset \mathbb{R}^{d}$ be open with smooth boundary. For $x \in \Omega$, let $v_{+}(x)$ and $v_{-}(x)$ denote limits respectively from $x \in \Omega$ and $x \in \mathbb{R}^{d} \backslash \bar{\Omega}$. Then for $x \in \Omega$,

$$
\begin{aligned}
&(\mathcal{S} f)_{ \pm}(x)= G f(x), \quad(\mathcal{D} f)_{ \pm}(x)=\mp \frac{1}{2} f(x)+\tilde{N} f(x) \\
&\left(\partial_{\nu_{x}} \mathcal{S} f\right)_{ \pm}(x)= \pm \frac{1}{2} f(x)+\tilde{N}^{\#} f(x)
\end{aligned}
$$

where for $x \in \partial \Omega$,

$$
\begin{gathered}
G f(x):=\int_{\partial \Omega} R_{0}(\lambda)(x, y) f(y) d S(y) \quad \tilde{N} f(x):=\int_{\partial \Omega} \partial_{\nu_{y}} R_{0}(\lambda)(x, y) f(y) d S(y) \\
\tilde{N}^{\#} f(x):=\int_{\partial \Omega} \partial_{\nu_{x}} R_{0}(\lambda)(x, y) f(y) d S(y)
\end{gathered}
$$

and $\partial_{\nu_{x}}$ denotes the outward unit normal derivative to $\partial \Omega$ at $x$.
We call $G$ the single layer operator and $\tilde{N}$ the double layer operator.
Proof. We start by considering a general pseudodifferential operator $P(x, D)$. Let $S \in \mathcal{E}^{\prime}\left(\mathbb{R}^{d}\right)$ denote the surface measure on $\partial \Omega$ and make a local change of coordinates so that $\partial \Omega=\left\{x_{1}=\right.$ $0\}$ with $\Omega \cap U=\left\{x_{1}<0\right\} \cap V$. Then, for $f \in \mathcal{D}^{\prime}(\partial \Omega)$, letting $x=\left(x_{1}, x^{\prime}\right)$ and $y=\left(y_{1}, y^{\prime}\right)$

$$
\begin{aligned}
P(x, D)(f S) & =(2 \pi)^{-d} \iint e^{i\left\langle x^{\prime}-y^{\prime}, \xi^{\prime}\right\rangle+i x_{1} \xi_{1}} p\left(x, \xi^{\prime}, \xi_{1}\right) f\left(y^{\prime}\right) d y^{\prime} d \xi^{\prime} d \xi_{1} \\
& =q\left(x_{1}, x^{\prime}, D^{\prime}\right) f
\end{aligned}
$$

where

$$
\begin{equation*}
q\left(x_{1}, x^{\prime}, \xi^{\prime}\right)=(2 \pi)^{-1} \int e^{i x_{1} \xi_{1}} p\left(x_{1}, x^{\prime}, \xi\right) d \xi_{1} \tag{6.1.1}
\end{equation*}
$$

Now, suppose that $p \in S_{0, c l}^{m}$. Then, for $m<-1$, 6.1.1) is absolutely integrable and hence continuous at $x_{1}=0$. On the other hand, if $m \geq-1$, we can write

$$
p \sim \sum_{j=-\infty}^{m} C_{ \pm}^{j}\left(x, \xi^{\prime}\right) \xi_{1}^{j} \quad \pm \xi_{1} \rightarrow \infty
$$

Then, by for example [76, Chapter 3] (or Lemma 6.6.2) $q$ is smooth away from $x_{1}=0$ and, if $C_{-}^{j}\left(x, \xi^{\prime}\right)=(-1)^{j} C_{+}^{j}\left(x, \xi^{\prime}\right)$ for $j \geq-1$, there is a jump discontinuity at $x_{1}=0$.

Now, we apply this to $\mathcal{S}$ and $\mathcal{D}$. Note that the (homogeneous) symbol of $R_{0}(\lambda)$ is $|\xi|^{-2}$ so we immediately obtain that there is no jump for $\mathcal{S}$.

On the other hand, let $L$ be a vector field equal to $\partial_{\nu}$ on $\partial \Omega$. Then,

$$
\mathcal{D} f=R_{0}(\lambda) L^{*}(f S), \quad \partial_{\nu} \mathcal{S} f=L R_{0}(\lambda)(f S)
$$

where $L^{*}=-L-(\operatorname{div} L)$. So, the symbol of $R_{0} L^{*}$ is $-|\xi|^{-2} i\langle\nu(x), \xi\rangle$ and that of $L R_{0}$ is $|\xi|^{-2} i\langle\nu(x), \xi\rangle$. Then, writing

$$
|\xi \pm \tau \nu(x)|^{-2} i\langle\nu(x), \xi \pm \tau \nu(x)\rangle
$$

we see that $\pm C_{ \pm}\left(x, \xi^{\prime}\right)^{-1} \equiv i$. Computing the integral (6.1.1) with $p=-|\xi|^{2} i \xi_{1}$ gives the constant $\mp \frac{1}{2}$ for $\mathcal{D}$ and, since the symbols are related by multiplication by $-1, \pm \frac{1}{2}$ for $\partial_{\nu} \mathcal{S}$.

Now, suppose that $\operatorname{Im} \lambda>0$ and that $u$ solves

$$
\begin{equation*}
\left(-\Delta-\lambda^{2}\right) u(x)=0 \quad x \in \Omega \tag{6.1.2}
\end{equation*}
$$

Then, using Green's formula and the fact that $R_{0}(\lambda)(x, y)=R_{0}(\lambda)(x, y)$,

$$
\left.\mathcal{S} \partial_{\nu_{i}} u\right|_{\partial \Omega}-\left.\mathcal{D} u\right|_{\partial \Omega}= \begin{cases}u(x) & x \in \Omega  \tag{6.1.3}\\ 0 & x \notin \bar{\Omega}\end{cases}
$$

So, taking limits from inside and outside $\Omega$ in (6.1.3), we have

$$
G \partial_{\nu_{i}} u+\frac{1}{2} u-\tilde{N} u=u \quad G \partial_{\nu_{i}} u-\frac{1}{2} u-\tilde{N} u=0 .
$$

That is,

$$
\begin{equation*}
G \partial_{\nu_{i}} u=\frac{1}{2} u+\tilde{N} u . \tag{6.1.4}
\end{equation*}
$$

Next, apply $\partial_{\nu_{i}}$ to 6.1.3) and take limits from inside and outside $\Omega$ to obtain

$$
\frac{1}{2} \partial_{\nu_{i}} u+\tilde{N}^{\#} \partial_{\nu_{i}} u-\left(\partial_{\nu_{i}} \mathcal{D} u\right)_{+}=\partial_{\nu_{i}} u \quad-\frac{1}{2} \partial_{\nu_{i}} u+\tilde{N}^{\#} \partial_{\nu_{i}} u-\left(\partial_{\nu_{i}} \mathcal{D} u\right)_{-}=0
$$

That is,

$$
\begin{equation*}
\left(\partial_{\nu_{i}} \mathcal{D} u\right)_{ \pm}=-\frac{1}{2} \partial_{\nu_{i}} u+\tilde{N}^{\#} \partial_{\nu_{i}} u \tag{6.1.5}
\end{equation*}
$$

On the other hand, suppose that $u$ solves

$$
\begin{equation*}
\left(-\Delta-\lambda^{2}\right) u(x)=0 \quad x \notin \bar{\Omega} \quad u \text { is } \lambda \text {-outgoing. } \tag{6.1.6}
\end{equation*}
$$

Then, using Green's formula and the fact that $R_{0}(\lambda)(x, y)=R_{0}(\lambda)(x, y)$,

$$
\left.\mathcal{S} \partial_{\nu_{e}} u\right|_{\partial \Omega}+\left.\mathcal{D} u\right|_{\partial \Omega}= \begin{cases}0 & x \in \Omega  \tag{6.1.7}\\ u(x) & x \notin \bar{\Omega}\end{cases}
$$

So, taking limits from inside and outside $\Omega$ in (6.1.7), we have

$$
G \partial_{\nu_{e}} u-\frac{1}{2} u+\tilde{N} u=0 \quad G \partial_{\nu_{e}} u+\frac{1}{2} u+\tilde{N} u=0
$$

That is,

$$
\begin{equation*}
G \partial_{\nu_{e}} u=\frac{1}{2} u-\tilde{N} u \tag{6.1.8}
\end{equation*}
$$

Next, apply $\partial_{\nu_{i}}$ to 6.1.7) and take limits from inside and outside $\Omega$ to obtain

$$
\frac{1}{2} \partial_{\nu_{e}} u+\tilde{N}^{\#} \partial_{\nu_{e}} u+\left(\partial_{\nu_{i}} \mathcal{D} u\right)_{+}=0 \quad-\frac{1}{2} \partial_{\nu_{e}} u+\tilde{N}^{\#} \partial_{\nu_{e}} u+\left(\partial_{\nu_{i}} \mathcal{D} u\right)_{-}=\partial_{\nu_{i}} u
$$

That is,

$$
\begin{equation*}
\left(\partial_{\nu_{i}} \mathcal{D} u\right)_{ \pm}=-\frac{1}{2} \partial_{\nu_{e}} u-\tilde{N}^{\#} \partial_{\nu_{e}} u \tag{6.1.9}
\end{equation*}
$$

Now, let $f \in C^{\infty}(\partial \Omega)$ and $u_{i}$ be the unique solution to (6.1.2) with $\left.u_{i}\right|_{\partial \Omega}=f$. Then the interior Dirichlet to Neumann Map is given by $\mathcal{N}_{i}: f \mapsto \partial_{\nu_{i}} u_{i}$. If $u_{e}$ solves 6.1.6) with $\left.u_{e}\right|_{\partial \Omega}=f$, then the exterior Dirichlet to Neumann Map is given by $\mathcal{N}_{e}: f \mapsto \partial_{\nu_{e}} u_{e}$.

Next, suppose that $v_{i}$ is the unique solution to (6.1.2 with $\partial_{\nu_{i}} v_{i}=f$. Then the interior Neumann to Dirichlet Map is given by $\mathcal{D}_{N_{i}}: f \mapsto v_{i} \mid \partial \Omega$. Finally, suppose that $v_{e}$ solves (6.1.6) with $\partial_{\nu_{e}} v_{e}=f$. Then, the exteriror Neumann to Dirichlet Map is given by $\mathcal{D}_{N_{e}}:\left.f \mapsto v_{e}\right|_{\partial \Omega}$.

Then (6.1.4) 6.1.5 (6.1.8) and (6.1.9) combined with density of $C^{\infty}$ in distributions give the following
Lemma 6.1.2. Let $G, \tilde{N}$, and $\tilde{N}^{\#}$ be as in Lemma 6.1.1. Then for $\operatorname{Im} \lambda>0$,

$$
G \mathcal{N}_{i}=\frac{1}{2} I+\tilde{N} \quad G \mathcal{N}_{e}=\frac{1}{2} I-\tilde{N}
$$

Moreover, $\partial_{\nu_{i}} \mathcal{D}$ has no jump across $\partial \Omega$ and

$$
\partial_{\nu} \mathcal{D} \ell=\left(\partial_{\nu_{i}} \mathcal{D}\right)_{ \pm}=\left(-\frac{1}{2} I+\tilde{N}^{\#}\right) \mathcal{N}_{i}=\left(-\frac{1}{2} I-\tilde{N}^{\#}\right) \mathcal{N}_{e}
$$

where

$$
\partial_{\nu} \mathcal{D} \ell(\lambda) f(x)=\int_{\partial \Omega} \partial_{\nu_{x}} \partial_{\nu_{y}} R_{0}(\lambda)(x, y) f(y) d S(y)
$$

Finally,

$$
\partial_{\nu} \mathcal{D} \ell \mathcal{D}_{N_{i}}=-\frac{1}{2} I+\tilde{N}^{\#}, \quad \partial_{\nu} \mathcal{D} \ell \mathcal{D}_{N_{e}}=-\frac{1}{2} I-\tilde{N}^{\#} .
$$

We call $\partial_{\nu} \mathcal{D} \ell$ the derivative double layer operator.
Now let $\operatorname{Im} \lambda>0$ and fix $h \in C^{\infty}(\partial \Omega)$ and suppose that $u(x)=\mathcal{S} h$. Then $\left.u\right|_{\partial \Omega}=G h$ and hence $\partial_{\nu_{i}} u=\mathcal{N}_{i} G h, \partial_{\nu_{e}} u=\mathcal{N}_{e} G h$. On the other hand, taking limits from inside and outside $\Omega$ and using Lemma 6.1.1, we have

$$
\partial_{\nu_{i}} u=\left(\frac{1}{2} I+\tilde{N}^{\#}\right) h \quad \partial_{\nu_{e}} u=\left(\frac{1}{2} I-\tilde{N}^{\#}\right) h .
$$

Similarly, if we let $u(x)=\mathcal{D} h$. Then, $\partial_{\nu_{i}} u=\partial_{\nu} \mathcal{D} \ell h$ and $(u)_{+}=\mathcal{D}_{N_{i}} \partial_{\nu} \mathcal{D} \ell h,(u)_{-}=$ $-\mathcal{D}_{N_{e}} \partial_{\nu} \mathcal{D} \ell h$. On the other hand, taking limits from inside and outside $\Omega$, and using Lemma 6.1.1, we have

$$
(u)_{+}=\left(-\frac{1}{2} I+\tilde{N}\right) h \quad(u)_{-}=\left(\frac{1}{2} I+\tilde{N}\right) h .
$$

Again, using the density of $C^{\infty}$ in $\mathcal{D}^{\prime}$, we have proven
Lemma 6.1.3. Let $G, \tilde{N}$, and $\tilde{N} \#$ and $\partial_{\nu} \mathcal{D} \ell$ be as in Lemma 6.1.2 and $\operatorname{Im} \lambda>0$. Then

$$
\mathcal{N}_{i} G=\frac{1}{2} I+\tilde{N}^{\#} \quad \mathcal{N}_{e} G=\frac{1}{2} I-\tilde{N}^{\#} .
$$

Moreover,

$$
\mathcal{D}_{N_{i}} \partial_{\nu} \mathcal{D} \ell=-\frac{1}{2} I+\tilde{N}, \quad \mathcal{D}_{N_{e}} \partial_{\nu} \mathcal{D} \ell=-\frac{1}{2} I-\tilde{N} .
$$

Now, to see that Lemmas 6.1.2 and 6.1.3 hold for $\lambda$ in the domain of $R_{0}(\lambda)$, observe that computing symbols as in Lemma 6.1.1 (see also Lemma 6.6.2) for $G$ and $\partial_{\nu} \mathcal{D} \ell$, we have that $G \in \Psi_{\text {hom }}^{-1}$ elliptic and $\partial_{\nu} \mathcal{D} \ell \in \Psi_{\text {hom }}^{1}$ elliptic. Thus, $G$ and $\partial_{\nu} \mathcal{D} \ell$ are meromorphic families of Fredholm operators on the domain of $R_{0}(\lambda)$. Now, Lemma 6.1.2 together with Lemma 6.1 .3 imply that $G$ and $\partial_{\nu} \mathcal{D} \ell$ are invertible for $\operatorname{Im} \lambda>0$. Thus, the meromorphic Fredholm theorem implies that they have meromorphic inverses. This implies that $\mathcal{N}_{i}, \mathcal{N}_{e}, \mathcal{D}_{N_{i}}$, and $\mathcal{D}_{N_{e}}$ are meromorphic families of operators. Hence, we have

Proposition 6.1.4. For $\lambda$ in the domain of meromorphy of $R_{0}(\lambda)$,

$$
\begin{array}{cc}
G \mathcal{N}_{i}=\frac{1}{2} I+\tilde{N} & G \mathcal{N}_{e}=\frac{1}{2} I-\tilde{N} \\
\mathcal{N}_{i} G=\frac{1}{2} I+\tilde{N}^{\#} & \mathcal{N}_{e} G=\frac{1}{2} I-\tilde{N}^{\#}
\end{array}
$$

Moreover, $\partial_{\nu_{i}} \mathcal{D}$ has no jump across $\partial \Omega$ and

$$
\partial_{\nu} \mathcal{D} \ell=\left(\partial_{\nu_{i}} \mathcal{D}\right)_{ \pm}=\left(-\frac{1}{2} I+\tilde{N}^{\#}\right) \mathcal{N}_{i}=\left(-\frac{1}{2} I-\tilde{N}^{\#}\right) \mathcal{N}_{e} .
$$

Furthermore

$$
\begin{aligned}
\partial_{\nu} \mathcal{D} \ell \mathcal{D}_{N_{i}} & =-\frac{1}{2} I+\tilde{N}^{\#}, & \partial_{\nu} \mathcal{D} \ell \mathcal{D}_{N_{e}} & =-\frac{1}{2} I-\tilde{N}^{\#} \\
\mathcal{D}_{N_{i}} \partial_{\nu} \mathcal{D} \ell & =-\frac{1}{2} I+\tilde{N}, & \mathcal{D}_{N_{e}} \partial_{\nu} \mathcal{D} \ell & =-\frac{1}{2} I-\tilde{N} .
\end{aligned}
$$

### 6.2 Quasimode Estimates

We next prove a restriction estimate for quasimodes for the Laplacian. In particular, we show

Lemma 6.2.1. Let $U \Subset \mathbb{R}^{d}$ be open with $\Gamma \Subset U$ a $C^{1,1}$ embedded hypersurface. Suppose that $\|u\|_{L^{2}(U)}=1$ and

$$
\left(-h^{2} \Delta-1\right) u=O_{L^{2}}(h) .
$$

Then for $0<h<h_{0}$,

$$
\|u\|_{L^{2}(\Gamma)} \leq\left\{\begin{array}{l}
C h^{-1 / 4}  \tag{6.2.1}\\
C h^{-1 / 6} \quad \Gamma \in C^{2,1}, \text { curved }
\end{array}\right.
$$

In the setting of smooth Riemannian manifolds with restriction to a submanifold, these estimates along with their $L^{p}$ generalizations appear in the work of Tataru [75] who also notes that the $L^{2}$ bounds are a corollary of an estimate of Greenleaf and Seeger [33]. Such $L^{p}$ generalizations were also studied by Burq, Gérard and Tzvetkov in [10]. Semiclassical analogues were proved by Tacy [73] and Hassell-Tacy [38]. These estimates were generalized to the setting of restriction to smooth submanifolds in Riemannian manifolds with metrics of $C^{1,1}$ regularity by Blair [9]. In making a change of coordinates to flatten a submanifold the resulting metric has one lower order of regularity, thus the estimates of [9] do not apply directly to $C^{1,1}$ submanifolds, and so we include here the proof of the $L^{2}$ estimate on $C^{1,1}$ hypersurfaces of Euclidean space. The estimate with $h^{-1 / 6}$ for curved $C^{2,1}$ hypersurfaces does follow from [9], so we consider here just the case of a general $C^{1,1}$ hypersurface.

We now prove Lemma 6.2.1.
Proof. We derive (6.2.1) from a square function estimate, Lemma 6.2.2. The estimate (6.2.2) is a characteristic trace estimate for solutions to the wave equation, but the proof more closely resembles that of dispersive estimates for the wave equation. Our proof of Lemma 6.2.2 is inspired by [9], although the analysis here is simpler since we work on Euclidean space, and seek only $L^{2}$ bounds on the restriction of eigenfunctions.

Let $\chi \in C_{c}^{\infty}(U)$ have $\chi \equiv 1$ on $\Gamma$. Then, we have

$$
\left(-h^{2} \Delta-1\right) \chi u=O_{L^{2}}(h) .
$$

Moreover, letting $\psi \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ have $\psi \equiv 1$ on $7 / 8<|\xi|<5 / 4$, $\operatorname{supp} \psi \subset\{3 / 4<|\xi|<3 / 2\}$, and using the fact that $\left(-h^{2} \Delta-1\right)$ is elliptic on $\operatorname{supp}(1-\psi)$, we have

$$
(1-\psi(h D)) \chi u=O_{H_{h}^{2}}(h) .
$$

Hence,

$$
\|(1-\psi(h D)) \chi u\|_{L^{2}(\Gamma)}=O\left(h^{1 / 2}\right)
$$

Thus, we need only consider $\psi(h D) \chi u$. Now, let $v=\psi(h D) \chi u$. Then

$$
\cos (t \sqrt{-\Delta}) v=\cos (t r) v+\int_{0}^{t} \cos ((t-s) \sqrt{-\Delta}) \sin (s r) E r^{-1} d s
$$

where

$$
\left(-\Delta-r^{2}\right) v=E=O_{L^{2}}(r)
$$

Then, Minkowski's integral inequality together with Jensen's inequality and (6.2.2) imply that

$$
\left\|\left\|\int_{0}^{t}(\cos ((t-s) \sqrt{-\Delta}) f)\left(x^{\prime}, F\left(x^{\prime}\right)\right)\right\|_{L^{2}\left(\mathbb{R}^{d-1}, d x^{\prime}\right)}\right\|_{L^{4}([0,1])} \leq C r^{1 / 4}\|f\|_{L^{2}\left(\mathbb{R}^{d}\right)}
$$

for $f$ with $\hat{f}(\xi)$ supported in $\frac{3}{4} r<|\xi|<\frac{3}{2} r$. Since $v=\psi(h D) \chi u, E$ has the required Fourier support property with $r=1 / h$, the proof of $(6.2 .1)$ is then completed by the following.
Lemma 6.2.2. Suppose that $r \geq 1$ and $f \in L^{2}\left(\mathbb{R}^{d}\right)$, and $\hat{f}(\xi)$ is supported in the region $\frac{3}{4} r \leq|\xi| \leq \frac{3}{2} r$. If $F \in C^{1,1}\left(\mathbb{R}^{d-1}\right)$ is real valued, with $\|\nabla F\|_{L^{\infty}} \leq \frac{1}{20}$, and $F(0)=0$, then

$$
\begin{equation*}
\left(\int_{0}^{1}\left\|(\cos (t \sqrt{-\Delta}) f)\left(x^{\prime}, F\left(x^{\prime}\right)\right)\right\|_{L^{2}\left(\mathbb{R}^{d-1}, d x^{\prime}\right)}^{4} d t\right)^{\frac{1}{4}} \leq C r^{\frac{1}{4}}\|f\|_{L^{2}\left(\mathbb{R}^{d}\right)} \tag{6.2.2}
\end{equation*}
$$

Proof. Given a function $F_{r}$ such that $\sup _{x^{\prime}}\left|F_{r}\left(x^{\prime}\right)-F\left(x^{\prime}\right)\right| \leq r^{-1}$, then 6.2.2 holds if we can show that

$$
\begin{equation*}
\left(\int_{0}^{1}\left\|(\cos (t \sqrt{-\Delta}) f)\left(x^{\prime}, F_{r}\left(x^{\prime}\right)\right)\right\|_{L^{2}\left(\mathbb{R}^{d-1}, d x^{\prime}\right)}^{4} d t\right)^{\frac{1}{4}} \leq C r^{\frac{1}{4}}\|f\|_{L^{2}\left(\mathbb{R}^{d}\right)} \tag{6.2.3}
\end{equation*}
$$

This follows from the fact that (6.2.3), together with the frequency localization of $f$ and translation invariance, implies the gradient bound, uniformly over $s \in \mathbb{R}$,

$$
\left(\int_{0}^{1}\left\|\partial_{s}(\cos (t \sqrt{-\Delta}) f)\left(x^{\prime}, F_{r}\left(x^{\prime}\right)+s\right)\right\|_{L^{2}\left(\mathbb{R}^{d-1}, d x^{\prime}\right)}^{4} d t\right)^{\frac{1}{4}} \leq C r^{\frac{5}{4}}\|f\|_{L^{2}\left(\mathbb{R}^{d}\right)}
$$

We will take $F_{r}$ to be a mollification of the $C^{1,1}$ function $F$ on the $r^{-\frac{1}{2}}$ spatial scale. Precisely, let $F_{r}=\phi_{r^{1 / 2}} * F$, where $\phi_{r^{1 / 2}}=r^{\frac{d-1}{2}} \phi\left(r^{\frac{1}{2}} x^{\prime}\right)$, with $\phi$ a Schwartz function on $\mathbb{R}^{d-1}$ of integral 1. Then

$$
\sup _{x^{\prime}}\left|F_{r}\left(x^{\prime}\right)-F\left(x^{\prime}\right)\right| \leq C r^{-1}, \quad \sup _{x^{\prime}}\left|\nabla F_{r}\left(x^{\prime}\right)-\nabla F\left(x^{\prime}\right)\right| \leq C r^{-\frac{1}{2}},
$$

and $F_{r}$ is a smooth function with derivative bounds

$$
\begin{equation*}
\sup _{x^{\prime}}\left|\partial_{x^{\prime}}^{\alpha} F_{r}\left(x^{\prime}\right)\right| \leq C r^{\frac{|\alpha|-2}{2}}, \quad|\alpha| \geq 2 \tag{6.2.4}
\end{equation*}
$$

In establishing (6.2.3) we may replace $\cos (t \sqrt{-\Delta})$ by $\exp (i t \sqrt{-\Delta})$, the bounds for $\exp (-i t \sqrt{-\Delta})$ being similar. Let

$$
T f\left(t, x^{\prime}\right)=(\exp (i t \sqrt{-\Delta}) f)\left(x^{\prime}, F_{r}\left(x^{\prime}\right)\right)
$$

We deduce bounds for $T: L^{2}\left(\mathbb{R}^{d}\right) \rightarrow L^{4}\left([0,1], L^{2}\left(\mathbb{R}^{d-1}\right)\right)$ from bounds for $T T^{*}$. Precisely, let $K_{r}(t-s, x-y)$ denote the kernel of the operator

$$
\rho\left(r^{-1} D\right) \exp (i(t-s) \sqrt{-\Delta}), \quad D:=-i \partial,
$$

where $\rho$ is a smooth function supported in the region $\frac{1}{2}<|\xi|<2$. It then suffices to show that

$$
\begin{align*}
\| \int_{0}^{1} \int K_{r}\left(t-s,\left(x^{\prime}-y^{\prime}, F_{r}\left(x^{\prime}\right)-F_{r}\left(y^{\prime}\right)\right)\right) f\left(s, y^{\prime}\right) d y^{\prime} d s & \|_{L^{4}\left([0,1], L^{2}\left(\mathbb{R}^{d-1}\right)\right)} \\
& \leq C r^{\frac{1}{2}}\|f\|_{L^{4 / 3}\left([0,1], L^{2}\left(\mathbb{R}^{d-1}\right)\right)} \tag{6.2.5}
\end{align*}
$$

since this implies $\left\|T T^{*} f\right\|_{L^{4}\left([0,1], L^{2}\left(\mathbb{R}^{d-1}\right)\right)} \leq C r^{\frac{1}{2}}\|f\|_{L^{4 / 3}\left([0,1], L^{2}\left(\mathbb{R}^{d-1}\right)\right)}$, and hence 6.2.3). We recall the Hardy-Littlewood-Sobolev inequality,

$$
\left\||t|^{-\frac{1}{2}} * f\right\|_{L^{4}(\mathbb{R})} \leq C\|f\|_{L^{4 / 3}(\mathbb{R})}
$$

Translation invariance in $t$ then shows that (6.2.5) is a consequence of the following fixed-time estimate, for $|t|<1$,

$$
\begin{equation*}
\left\|\int K_{r}\left(t,\left(x^{\prime}-y^{\prime}, F_{r}\left(x^{\prime}\right)-F_{r}\left(y^{\prime}\right)\right)\right) f\left(y^{\prime}\right) d y^{\prime}\right\|_{L^{2}\left(\mathbb{R}^{d-1}\right)} \leq C r^{\frac{1}{2}}|t|^{-\frac{1}{2}}\|f\|_{L^{2}\left(\mathbb{R}^{d-1}\right)} \tag{6.2.6}
\end{equation*}
$$

If $|t| \leq r^{-1}$, where we recall $r \geq 1$, then (6.2.6) follows by the Schur test, since if $|t| r \leq 1$ then for any $N \geq 0$

$$
\left|K_{r}(t, x-y)\right| \leq C_{N} r^{d}(1+r|x-y|)^{-N}
$$

We thus restrict attention to $|t|>r^{-1}$, where we establish 6.2.6 using wave packet techniques that were developed to prove dispersive estimates for wave equations with $C^{1,1}$ coefficients; see [68].

To prove (6.2.6) for a given $t$ with $|t|>r^{-1}$, we make an almost orthogonal decomposition $K_{r}=\sum_{j} K_{j}$ of the convolution kernel $K_{r}(t, \cdot)$. This decomposition is based on dividing the frequency space into essentially disjoint cubes of sidelength $\approx r^{\frac{1}{2}}|t|^{-\frac{1}{2}}$. On each of these cubes the phase of the wave operator is essentially linear in the frequency variable, and hence each term $K_{j}$ behaves as a normalized convolution operator in $x$.

We fix $t$ with $|t| \in\left[r^{-1}, 1\right]$, and let $\delta=r^{\frac{1}{2}}|t|^{-\frac{1}{2}}$. Let $\eta_{j}$ count the elements of the lattice of spacing $\delta$ for which $\left|\eta_{j}\right| \in\left[\frac{1}{2} r, 2 r\right]$, and write

$$
\rho\left(r^{-1} \xi\right)=\sum_{j} Q_{j}(\xi)
$$

where $Q_{j}$ is supported in the cube of sidelength $\delta$ centered on $\eta_{j}$, and the following bounds hold on the derivatives of $Q_{j}$, uniformly over $r, t$ and $j$,

$$
\begin{equation*}
\left|\partial_{\xi}^{\alpha} Q_{j}(\xi)\right| \leq C_{\alpha} \delta^{-|\alpha|} \tag{6.2.7}
\end{equation*}
$$

We then write $K_{r}(t, x)=\sum K_{j}(x)$, where we suppress the dependence on $r$ and $t$, and set

$$
K_{j}(x)=(2 \pi)^{-d} \int e^{i\langle x, \xi\rangle+i t|\xi|} Q_{j}(\xi) d \xi
$$

The multiplier $t|\xi|-t\left|\eta_{j}\right|^{-1}\left\langle\eta_{j}, \xi\right\rangle$ satisfies the derivative bounds (6.2.7) on the support of $Q_{j}$, hence we may write

$$
e^{i\langle x, \xi\rangle+i t|\xi|} Q_{j}(\xi)=e^{\left.\left.i\langle x+t| \eta_{j}\right|^{-1} \eta_{j}, \xi\right\rangle} \tilde{Q}_{j}(\xi),
$$

with $\tilde{Q}_{j}$ having the same support and derivative conditions as $Q_{j}$. Consequently, we may write

$$
K_{j}(x)=\delta^{d} e^{i\left\langle x, \eta_{j}\right\rangle+i t\left|\eta_{j}\right|} \chi_{j}\left(\delta\left(x+t\left|\eta_{j}\right|^{-1} \eta_{j}\right)\right)
$$

where $\chi_{j}$ is a Schwartz function, with seminorm bounds independent of $j$. We let

$$
\tilde{K}_{j}\left(x^{\prime}, y^{\prime}\right)=K_{j}\left(x^{\prime}-y^{\prime}, F_{r}\left(x^{\prime}\right)-F_{r}\left(y^{\prime}\right)\right)
$$

It follows from the Schur test that

$$
\left\|\tilde{K}_{j}\right\|_{L^{2}\left(\mathbb{R}^{d-1}\right) \rightarrow L^{2}\left(\mathbb{R}^{d-1}\right)} \leq C \delta
$$

To handle the sum over $j$ we establish the estimate

$$
\begin{equation*}
\left\|\tilde{K}_{j} \tilde{K}_{i}^{*}\right\|_{L^{2} \rightarrow L^{2}}+\left\|\tilde{K}_{j}^{*} \tilde{K}_{i}\right\|_{L^{2} \rightarrow L^{2}} \leq C_{N} \delta^{2}\left(1+\delta^{-1}\left|\eta_{i}-\eta_{j}\right|\right)^{-N} \tag{6.2.8}
\end{equation*}
$$

from which the bound 6.2.6 follows by the Cotlar-Stein lemma. Since $\tilde{K}_{j}$ and $\tilde{K}_{j}^{*}$ have similar form, we restrict attention to the first term in 6.2.8).

The kernel $\left(\tilde{K}_{j} \tilde{K}_{i}^{*}\right)\left(x^{\prime}, z^{\prime}\right)$ has absolute value dominated by

$$
\begin{equation*}
\left|\left(\tilde{K}_{j} \tilde{K}_{i}^{*}\right)\left(x^{\prime}, z^{\prime}\right)\right| \leq C \delta^{2 d} \int\left(1+\left.\delta|x+t| \eta_{j}\right|^{-1} \eta_{j}-y \mid\right)^{-N}\left(1+\left.\delta|z+t| \eta_{i}\right|^{-1} \eta_{i}-y \mid\right)^{-N} d y^{\prime} \tag{6.2.9}
\end{equation*}
$$

where we use the notation $y=\left(y^{\prime}, F_{r}\left(y^{\prime}\right)\right)$, and similarly for $x$ and $z$.
Suppose that $\left|\left(\eta_{j}\right)_{n}\right| \geq \frac{1}{4}\left|\eta_{j}\right|$. Then since $\left|F_{r}\left(x^{\prime}\right)-F_{r}\left(y^{\prime}\right)\right| \leq \frac{1}{10}\left|x^{\prime}-y^{\prime}\right|$,

$$
\left.\left|x^{\prime}+t\right| \eta_{j}\right|^{-1} \eta_{j}^{\prime}-y^{\prime}|+10| F_{r}\left(x^{\prime}\right)+t\left|\eta_{j}\right|^{-1}\left(\eta_{j}\right)_{n}-F_{r}\left(y^{\prime}\right) \mid \geq 5 t
$$

hence (6.2.9) and the Schur test leads to the bound

$$
\left\|\tilde{K}_{j} \tilde{K}_{i}^{*}\right\|_{L^{2} \rightarrow L^{2}} \leq C_{N} \delta^{2}(1+\delta t)^{-N}
$$

which is stronger than (6.2.8) since $\left|\eta_{i}-\eta_{j}\right| \leq 6 r$. The same estimate holds if $\left|\left(\eta_{i}\right)_{n}\right| \geq \frac{1}{4}\left|\eta_{i}\right|$.
We thus assume that $\left|\left(\eta_{j}\right)_{n}\right| \leq \frac{1}{4}\left|\eta_{j}\right|$, and similarly for $\eta_{i}$. Consider then the case where $\left|\left(\eta_{i}-\eta_{j}\right)_{n}\right| \geq\left|\left(\eta_{i}-\eta_{j}\right)^{\prime}\right|$. Then we have

$$
\left|\left(\left|\eta_{j}\right|^{-1} \eta_{j}-\left|\eta_{i}\right|^{-1} \eta_{i}\right)_{n}\right| \geq \frac{1}{2+2 \sqrt{2}}\left|\left(\left|\eta_{j}\right|^{-1} \eta_{j}-\left|\eta_{i}\right|^{-1} \eta_{i}\right)^{\prime}\right|
$$

and since $\frac{1}{2} r \leq\left|\eta_{i}\right|,\left|\eta_{j}\right| \leq 2 r$,

$$
\left|\left(\left|\eta_{j}\right|^{-1} \eta_{j}-\left|\eta_{i}\right|^{-1} \eta_{i}\right)_{n}\right| \geq \frac{1}{4 \sqrt{2}} r^{-1}\left|\eta_{i}-\eta_{j}\right|
$$

Then since $\left|\nabla F_{r}\right| \leq \frac{1}{10}$,
$\left|x^{\prime}-z^{\prime}+t\left(\left|\eta_{j}\right|^{-1} \eta_{j}-\left|\eta_{i}\right|^{-1} \eta_{i}\right)^{\prime}\right|+10\left|F_{r}\left(x^{\prime}\right)-F_{r}\left(z^{\prime}\right)+t\left(\left|\eta_{j}\right|^{-1} \eta_{j}-\left|\eta_{i}\right|^{-1} \eta_{i}\right)_{n}\right| \geq \frac{5}{4 \sqrt{2}} \delta^{-2}\left|\eta_{j}-\eta_{i}\right|$, hence (6.2.9) and the Schur test show that $\left\|\tilde{K}_{j} \tilde{K}_{i}^{*}\right\|_{L^{2} \rightarrow L^{2}} \leq C_{N} \delta^{2}\left(1+\delta^{-1}\left|\eta_{j}-\eta_{i}\right|\right)^{-N}$ as desired.

We thus consider the case that $\left|\left(\eta_{j}-\eta_{i}\right)_{n}\right| \leq\left|\left(\eta_{j}-\eta_{i}\right)^{\prime}\right|$. In this case we need use the oscillations of the kernels to bound $\left\|\tilde{K}_{j} \tilde{K}_{i}^{*}\right\|_{L^{2} \rightarrow L^{2}}$. Up to a factor of modulus 1 , the kernel $\left(K_{j} K_{i}^{*}\right)\left(x^{\prime}, z^{\prime}\right)$ can be written as

$$
\delta^{2 d} \int e^{-i\left\langle y^{\prime}, \eta_{j}^{\prime}-\eta_{i}^{\prime}\right\rangle-i F_{r}\left(y^{\prime}\right)\left(\eta_{j}-\eta_{i}\right)_{n}} \chi_{j}\left(\delta\left(x+t\left|\eta_{j}\right|^{-1} \eta_{j}-y\right)\right) \overline{\chi_{i}}\left(\delta\left(z+t\left|\eta_{i}\right|^{-1} \eta_{i}-y\right)\right) d y^{\prime}
$$

where again $y=\left(y^{\prime}, F_{r}\left(y^{\prime}\right)\right)$, and similarly for $x$ and $z$. Since $\left|\nabla F_{r}\left(y^{\prime}\right)\right| \leq \frac{1}{10}$, and $\left|\left(\eta_{j}-\eta_{i}\right)_{n}\right| \leq$ $\left|\eta_{j}^{\prime}-\eta_{i}^{\prime}\right|$, we have

$$
\left|\eta_{j}^{\prime}-\eta_{i}^{\prime}+\nabla F_{r}\left(y^{\prime}\right)\left(\eta_{j}-\eta_{i}\right)_{n}\right| \geq \frac{1}{2}\left|\eta_{j}-\eta_{i}\right|
$$

Using the estimates (6.2.4), and that $r^{\frac{1}{2}} \leq \delta$, an integration by parts argument dominates the kernel $\left(K_{j} K_{i}^{*}\right)\left(x^{\prime}, z^{\prime}\right)$ by

$$
\delta^{2 d}\left(1+\delta^{-1}\left|\eta_{j}-\eta_{i}\right|\right)^{-N} \int\left(1+\left.\delta|x+t| \eta_{j}\right|^{-1} \eta_{j}-y \mid\right)^{-N}\left(1+\left.\delta|z+t| \eta_{i}\right|^{-1} \eta_{i}-y \mid\right)^{-N} d y^{\prime}
$$

which leads to the desired norm bounds, concluding the proof of (6.2.8), and hence of Lemma 6.2.2.

Lemma 6.2.2 then implies Lemma 6.2.1.
We also want the corresponding restriction estimates for normal derivatives which we include without proof.

Lemma 6.2.3. Let $U \Subset \mathbb{R}^{d}$ be open with $\Gamma \Subset U$ a $C^{\infty}$ embedded hypersurface. Suppose that $\|u\|_{L^{2}(U)}=1$ and

$$
\left(-h^{2} \Delta-1\right) u=O_{L^{2}}(h)
$$

Then for $0<h<1$

$$
\begin{equation*}
\left\|\partial_{\nu} u\right\|_{\left.L^{2}(\Gamma)\right)} \leq C h^{-1} \tag{6.2.10}
\end{equation*}
$$

where $\partial_{\nu}$ is a choice of normal derivative to $\Gamma$.
Estimates of this type first appear in the work of Tataru [75] in the form of regularity estimates for restrictions of solutions to hyperbolic equations. Semiclassical analogs of this estimate were proved in Christianson-Hassell-Toth [17] and Tacy (72.

### 6.3 Estimates on the Single, Double and Derivative Double Layer Operators

Next we give semiclassical estimates for the double and single layer operators and derivative double layer operator. The estimates on single layer operators appear in [31, Theorem 1.2], and those for double layer operators appear in 37] but we repeat them below for the convenience of the reader.

Let $\gamma: H_{\mathrm{loc}}^{1 / 2+\epsilon} \rightarrow L^{2}(\Gamma)$ denote restriction to $\Gamma$ for a $C^{1,1}$ embedded hypersurface $\Gamma$ and $\gamma^{*}: L^{2}(\Gamma) \rightarrow H_{\text {comp }}^{-1 / 2-\epsilon}\left(\mathbb{R}^{d}\right)$ its dual. Then $\gamma^{*}$ is the inclusion map $f \mapsto f \delta_{\Gamma}$ where $\delta_{\Gamma}$ is $d-1$ dimensional Hausdorff measure on $\Gamma$. Then when $\Gamma=\partial \Omega, G$ above can be written

$$
\begin{equation*}
G=\gamma R_{0} \gamma^{*} \tag{6.3.1}
\end{equation*}
$$

Because of this, we redefine the single layer operator to be given by 6.3.1
Similarly, if we assume that $\Gamma=\partial \Omega$ and $L$ is a vectorfield equal to $\partial_{\nu}$ on $\Gamma$, then

$$
\begin{equation*}
\partial_{\nu} \mathcal{D} \ell(\lambda)=\gamma L R_{0}(\lambda) L^{*} \gamma^{*} \tag{6.3.2}
\end{equation*}
$$

and we redefine the derivative double layer operator to be given by (6.3.3). Here we interpret $\gamma$ as a limit from either inside or outside $\Omega$ as in Lemma 6.1.2. Note that we cannot quite define $\tilde{N}$ by

$$
\gamma R_{0}(\lambda) L^{*} \gamma^{*}
$$

since there is a jump across $\partial \Omega$. However, we can bound $\tilde{N}$ by obtaining bounds on

$$
\left\langle R_{0}(\lambda) L^{*} \gamma^{*} f, \gamma^{*} g\right\rangle
$$

If $d=1$ then $\delta_{\Gamma}$ is a finite sum of point measures, and from the formula $G_{0}(\lambda, x, y)=$ $-(2 i \lambda)^{-1} e^{i \lambda|x-y|}$ we see, using the notation of Theorem 6.1 below, that

$$
\begin{equation*}
\|G(\lambda)\|_{L^{2}(\Gamma) \rightarrow L^{2}(\Gamma)} \leq C|\lambda|^{-1} e^{D_{\Gamma}(\operatorname{Im} \lambda)_{-}}, \quad d=1 \tag{6.3.4}
\end{equation*}
$$

In higher dimensions, we establish the following theorem:
Theorem 6.1. Let $\Gamma \subset \mathbb{R}^{d}$ be a finite union of compact subsets of embedded $C^{1,1}$ hypersurfaces. Then $G(\lambda)$ is a compact operator on $L^{2}(\Gamma)$ for $\lambda$ in the domain of $R_{0}(\lambda)$, and there exists $C$ such that

$$
\|G(\lambda)\|_{L^{2}(\Gamma) \rightarrow L^{2}(\Gamma)} \leq \begin{cases}C\langle\lambda\rangle^{-\frac{1}{2}} \log \langle\lambda\rangle \log \left\langle\lambda^{-1}\right\rangle e^{D_{\Gamma}(\operatorname{Im} \lambda)_{-}}, & d=2  \tag{6.3.5}\\ C\langle\lambda\rangle^{-\frac{1}{2}} \log \langle\lambda\rangle e^{D_{\Gamma}(\operatorname{Im} \lambda)_{-}}, & d \geq 3\end{cases}
$$

where $D_{\Gamma}$ is the diameter of the set $\Gamma$, and we assume $-\pi \leq \arg \lambda \leq 2 \pi$ if $d$ is even.

If $\Gamma$ can be written as a finite union of compact subsets of strictly convex $C^{2,1}$ hypersurfaces, then for some $C$ and all $\lambda$ in the domain of $R_{0}(\lambda)$ the following stronger estimate holds

$$
\|G(\lambda)\|_{L^{2}(\Gamma) \rightarrow L^{2}(\Gamma)} \leq \begin{cases}C\langle\lambda\rangle^{-\frac{2}{3}} \log \langle\lambda\rangle \log \left\langle\lambda^{-1}\right\rangle e^{D_{\Gamma}(\operatorname{Im} \lambda)_{-}}, & d=2  \tag{6.3.6}\\ C\langle\lambda\rangle^{-\frac{2}{3}} \log \langle\lambda\rangle e^{D_{\Gamma}(\operatorname{Im} \lambda)_{-}}, & d \geq 3\end{cases}
$$

Here we set $\langle\lambda\rangle=\left(2+|\lambda|^{2}\right)^{\frac{1}{2}}$, and $(\operatorname{Im} \lambda)_{-}=\max (0,-\operatorname{Im} \lambda)$. Compactness follows easily by Rellich's embedding theorem, or the bounds on $G_{0}(\lambda, x, y)$ in Section 6.3. The powers $\frac{1}{2}$ and $\frac{2}{3}$ in 6.3.5 and 6.3.6), respectively, are in general optimal. This follows from the fact that the corresponding estimates for the restriction of eigenfunctions are the best possible. The logarithmic divergence at $\lambda=0$ for $d=2$ in both (6.3.5) and (6.3.6) arises from similar divergence for $R_{0}(\lambda)$. The factor of $\log \langle\lambda\rangle$ in the estimates, which arises from our method of proof via restriction estimates, is likely not needed. For $\Gamma$ contained in a hyperplane, the estimate (6.3.5) for $d \geq 3$ holds without it, and it does not arise in our direct proof of (6.3.5) for $d=2$. We also expect that estimate (6.3.6) holds for subsets of strictly convex $C^{1,1}$ hypersurfaces, but do not pursue that here.

In the case that $\operatorname{Im} \lambda \geq|\lambda|^{\frac{1}{2}}$, respectively $\operatorname{Im} \lambda \geq|\lambda|^{\frac{2}{3}}$, the above bounds can be improved upon.

Theorem 6.2. Let $\Gamma \subset \mathbb{R}^{d}$ be a finite union of compact subsets of embedded $C^{1,1}$ hypersurfaces. Then there esists $C$ such that for $0 \leq \arg \lambda \leq \pi$,

$$
\|G(\lambda)\|_{L^{2}(\Gamma) \rightarrow L^{2}(\Gamma)} \leq \begin{cases}C\langle\operatorname{Im} \lambda\rangle^{-1} \log \left\langle\lambda^{-1}\right\rangle, & d=2 \\ C\langle\operatorname{Im} \lambda\rangle^{-1}, & d \geq 3\end{cases}
$$

Next, we give estimates on the double layer operator
Theorem 6.3. Let $\Omega \Subset \mathbb{R}^{d}$ and $\partial \Omega$ be Lipschitz and piecewise smooth. Then there exists $\lambda_{0}$ such that for $|\lambda|>\lambda_{0}$,

$$
\begin{equation*}
\|\tilde{N}\|_{L^{2}(\partial \Omega) \rightarrow L^{2}(\partial \Omega)} \leq C\langle\lambda\rangle^{\frac{1}{4}} \log \langle\lambda\rangle e^{D_{\Omega}(\operatorname{Im} \lambda)-} \tag{6.3.7}
\end{equation*}
$$

Moreover, if $\partial \Omega$ is a finite union of compact subsets of curved $C^{\infty}$ hypersurfaces, then

$$
\begin{equation*}
\|\tilde{N}\|_{L^{2}(\partial \Omega) \rightarrow L^{2}(\partial \Omega)} \leq C\langle\lambda\rangle^{\frac{1}{6}} \log \langle\lambda\rangle e^{D_{\Omega}(\operatorname{Im} \lambda)_{-}} \tag{6.3.8}
\end{equation*}
$$

Finally, we give estimates for the derivative double layer operator
Theorem 6.4. Let $\Omega \Subset \mathbb{R}^{d}$ and $\partial \Omega$ be smooth. Then there exists $\lambda_{0}$ such that for $|\lambda|>\lambda_{0}$,

$$
\begin{equation*}
\left\|\partial_{\nu} \mathcal{D} \ell\right\|_{H^{1}(\partial \Omega) \rightarrow L^{2}(\partial \Omega)} \leq C\langle\lambda\rangle \log \langle\lambda\rangle e^{D_{\Omega}(\operatorname{Im} \lambda)_{-}} \tag{6.3.9}
\end{equation*}
$$

In sections 6.4 and 6.4 we show that the exponents on $\langle\lambda\rangle$ in Theorems $6.1,6.3$ and 6.4 are sharp. However, if we impose the condition that $\Omega$ is convex with piecewise smooth, $C^{1,1}$ boundary, then we expect that $\tilde{N}$ is uniformly bounded in $\lambda$.

## Bounds on Green's function and estimates on $G$ for $d=2$

We conclude this section by reviewing bounds on the convolution kernel $G_{0}(\lambda, x, y)$ associated to the operator $R_{0}(\lambda)$. It can be written in terms of the Hankel functions of the first kind,

$$
G_{0}(\lambda, x, y)=C_{d} \lambda^{d-2}(\lambda|x-y|)^{-\frac{d-2}{2}} H_{\frac{d}{2}-1}^{(1)}(\lambda|x-y|),
$$

for some constant $C_{d}$. If $d \geq 3$ is odd, this can be written as a finite expansion

$$
G_{0}(\lambda, x, y)=\lambda^{d-2} e^{i \lambda|x-y|} \sum_{j=\frac{d-1}{2}}^{d-2} \frac{c_{d, j}}{(\lambda|x-y|)^{j}} .
$$

For $x \neq y$ this form extends to $\lambda \in \mathbb{C}$, and defines the analytic extension of $R_{0}(\lambda)$. In particular, for $d \geq 3$ odd we have the upper bounds

$$
\left|G_{0}(\lambda, x, y)\right| \lesssim \begin{cases}|x-y|^{2-d}, & |x-y| \leq|\lambda|^{-1}  \tag{6.3.10}\\ e^{-\operatorname{Im} \lambda|x-y|}|\lambda|^{\frac{d-3}{2}}|x-y|^{\frac{1-d}{2}}, & |x-y| \geq|\lambda|^{-1}\end{cases}
$$

If $d \geq 4$ is even, the bounds 6.3.10 hold for $\operatorname{Im} \lambda>0$, as well as for the analytic extension to $-\pi \leq \arg \lambda \leq 2 \pi$. For $-\pi<\arg \lambda<2 \pi$ this follows by the asymptotics of $H_{n}^{(1)}(z)$; see for example [1, (9.2.3)]. To see that it extends to the closed sector, we use Stone's formula (see [21]),

$$
\begin{aligned}
& G_{0}\left(e^{i \pi} \lambda, x, y\right)-G_{0}(\lambda, x, y)=\frac{i}{2} \frac{\lambda^{d-2}}{(2 \pi)^{d-1}} \int_{\mathbb{S}^{d-1}} e^{i \lambda\langle x-y, \omega\rangle} d \omega \\
&=C_{d} \lambda^{d-2}(\lambda|x-y|)^{-\frac{d-2}{2}} J_{\frac{d}{2}-1}(\lambda|x-y|)
\end{aligned}
$$

where $e^{i \pi}$ indicates analytic continuation through positive angle $\pi$, and where $d \omega$ is surface measure on the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^{d}$. This holds in all dimensions for $\lambda>0$, and hence for the analytic continuation. The bounds 6.3.10) then follow from the asymptotics of $J_{n}(z)$ and the bounds for $\operatorname{Im} \lambda \geq 0$. We also note as a consequence of the above that, for $\lambda \in \mathbb{R} \backslash\{0\}$, and any sheet of the continuation in even dimensions,

$$
\begin{equation*}
G_{0}\left(e^{i \pi} \lambda, x, y\right)-G_{0}(\lambda, x, y)=\pi i(\operatorname{sgn} \lambda)^{d}|\lambda|^{-1}(2 \pi)^{-d} \widehat{\delta_{\mathbb{S}_{\lambda}^{d-1}}}(x-y), \tag{6.3.11}
\end{equation*}
$$

where $\delta_{\mathbb{S}_{\lambda}^{d-1}}$ denotes surface measure on the sphere $|\xi|=|\lambda|$ in $\mathbb{R}^{d}$, and

$$
\hat{g}(\xi)=\int e^{-i\langle x, \xi\rangle} g(x) d x
$$

If $d=2$, one has the bounds, see [1, (9.1.8)-(9.2.3)],

$$
\left|G_{0}(\lambda, x, y)\right| \lesssim \begin{cases}|\log (\lambda|x-y|)|, & |x-y| \leq \frac{1}{2}|\lambda|^{-1}  \tag{6.3.12}\\ e^{-\operatorname{Im} \lambda|x-y|}|\lambda|^{-\frac{1}{2}}|x-y|^{-\frac{1}{2}}, & |x-y| \geq \frac{1}{2}|\lambda|^{-1}\end{cases}
$$

Using the above asymptotics, we give an elementary proof of estimate (6.3.5) of Theorem 6.1 for $d=2$. Indeed, we can prove the following stronger result, which holds on subsets of Lipschitz graphs.

Theorem 6.5. Suppose that $d=2$, and that $\Gamma$ is a finite union $\Gamma=\bigcup_{j} \Gamma_{j}$ where each $\Gamma_{j}$ is a compact subset of a Lipschitz graph. Then for $-\pi \leq \arg \lambda \leq 2 \pi$, with 1-dimensional Hausdorff measure on $\Gamma$,

$$
\|G(\lambda) f\|_{L^{2}(\Gamma)} \leq \begin{cases}C\langle\lambda\rangle^{-\frac{1}{2}} \log \left\langle\lambda^{-1}\right\rangle\langle\operatorname{Im} \lambda\rangle^{-\frac{1}{2}}\|f\|_{L^{2}(\Gamma)}, & \operatorname{Im} \lambda \geq 0 \\ C\langle\lambda\rangle^{-\frac{1}{2}} \log \left\langle\lambda^{-1}\right\rangle e^{-D_{\Gamma} \operatorname{Im} \lambda}\|f\|_{L^{2}(\Gamma)}, & \operatorname{Im} \lambda \leq 0\end{cases}
$$

Proof. The following kernel bounds hold by (6.3.12), since $|x-y|$ is bounded above,

$$
\left|G_{0}(\lambda, x, y)\right| \leq C e^{-\operatorname{Im} \lambda|x-y|}\langle\lambda\rangle^{-\frac{1}{2}} \log \left\langle\lambda^{-1}\right\rangle|x-y|^{-\frac{1}{2}}
$$

By the Schur test and symmetry of the kernel, the operator norm of $G(\lambda)$ is bounded by the following

$$
\sup _{x} \int_{\Gamma}\left|G_{0}(\lambda, x, y)\right| d \sigma(y)
$$

where $\sigma$ is 1-dimensional Hausdorff measure, which equals arclength measure on each $\Gamma_{j}$.
First consider $\operatorname{Im} \lambda \leq 0$. Then $e^{-\operatorname{Im} \lambda|x-y|} \leq e^{-D_{\Gamma} \operatorname{Im} \lambda}$ for $x, y \in \Gamma$. After rotation, we can write $\Gamma_{j}$ as the graph $y_{2}=F_{j}\left(y_{1}\right)$ for $y_{1}$ in a compact set $K_{j}$, and with uniform Lipschitz bounds on $F_{j}$. Then on $\Gamma_{j}$ we have $d \sigma(y) \approx d y_{1}$, and

$$
\sup _{x} \int_{\Gamma_{j}}|x-y|^{-\frac{1}{2}} d \sigma(y) \leq C \sup _{x_{1}} \int_{K_{j}}\left|x_{1}-y_{1}\right|^{-\frac{1}{2}} d y_{1} \leq C D_{K_{j}}^{1 / 2}
$$

For $\operatorname{Im} \lambda \geq 0$, we use instead the bound

$$
\sup _{x_{1}} \int_{K_{j}} e^{-\operatorname{Im} \lambda\left|x_{1}-y_{1}\right|}\left|x_{1}-y_{1}\right|^{-\frac{1}{2}} d y_{1} \leq C_{j}\langle\operatorname{Im} \lambda\rangle^{-\frac{1}{2}}
$$

Summing over finitely many $j$ then yields the desired bounds over $\Gamma$.

## Proof of the Theorems

We start by proving a conditional result which assumes a certain estimate on restriction of the Fourier transform of surfrace measures to the sphere of radius $r$.

Lemma 6.3.1. Suppose that for $\Gamma \Subset \mathbb{R}^{d}$ any compact embedded $C^{\infty}$ hypersurface, and some $\alpha, \beta>0$,

$$
\begin{align*}
\int\left|\widehat{L^{*} f \delta_{\Gamma}}\right|^{2}(\xi) \delta(|\xi|-r) d \xi & \leq C_{\Gamma}\langle r\rangle^{2 \alpha}\|f\|_{L^{2}(\Gamma)}^{2}  \tag{6.3.13}\\
\int\left|\widehat{f \delta_{\Gamma}}\right|^{2}(\xi) \delta(|\xi|-r) d \xi & \leq C_{\Gamma}\langle r\rangle^{2 \beta}\|f\|_{L^{2}(\Gamma)}^{2} \tag{6.3.14}
\end{align*}
$$

with $2 \beta<1$. Let $\Gamma_{1}, \Gamma_{2} \Subset \mathbb{R}^{d}$ ben compact embedded $C^{\infty}$ hypersurfaces. Let $L$ be a vector field with $L=\partial_{\nu}$ on $\Gamma_{1}$ for some choice of normal $\nu$ on $\Gamma_{1}$ and $\psi \in C_{c}^{\infty}(\mathbb{R})$ with $\psi \equiv 1$ in neighborhood of 0 . Then define for $f \in L^{2}\left(\Gamma_{1}\right), g \in L^{2}\left(\Gamma_{2}\right)$

$$
\begin{gathered}
Q_{\lambda}^{G}(f, g):=\int R_{0}(\lambda)\left(\psi\left(\lambda^{-1} D\right) f \delta_{\Gamma_{1}}\right) \bar{g} \delta_{\Gamma_{2}}, \quad Q_{\lambda}^{D}(f, g):=\int R_{0}(\lambda)\left(\psi\left(\lambda^{-1} D\right) L_{1}^{*}\left(f \delta_{\Gamma_{1}}\right)\right) \bar{g} \delta_{\Gamma_{2}} \\
Q_{\lambda}^{d D}(f, g):=\int R_{0}(\lambda)\left(\psi\left(\lambda^{-1} D\right) L_{1}^{*}\left(f \delta_{\Gamma_{1}}\right) \overline{L_{2}^{*}\left(g \delta_{\Gamma_{2}}\right)}\right.
\end{gathered}
$$

Then for $\operatorname{Im} \lambda>0$,

$$
\begin{align*}
\left|Q_{\lambda}^{G}(f, g)\right| & \leq C_{\Gamma_{1}, \Gamma_{2}}\langle\lambda\rangle^{2 \beta-1} \log \langle\lambda\rangle\|f\|_{L^{2}\left(\Gamma_{1}\right)}\|g\|_{L^{2}\left(\Gamma_{2}\right)}  \tag{6.3.15}\\
\left|Q_{\lambda}^{D}(f, g)\right| & \leq C_{\Gamma_{1}, \Gamma_{2}, \psi}\langle\lambda\rangle^{\alpha+\beta-1} \log \langle\lambda\rangle\|f\|_{L^{2}\left(\Gamma_{1}\right)}\|g\|_{L^{2}\left(\Gamma_{2}\right)}  \tag{6.3.16}\\
\left|Q_{\lambda}^{d D}(f, g)\right| & \leq C_{\Gamma_{1}, \Gamma_{2}, \psi}^{d}\langle\lambda\rangle^{2 \alpha-1} \log \langle\lambda\rangle\|f\|_{L^{2}\left(\Gamma_{1}\right)}\|g\|_{L^{2}\left(\Gamma_{2}\right)} . \tag{6.3.17}
\end{align*}
$$

Proof. We follow [31] [37] to prove the lemma. First, observe that due to the compact support of $f \delta_{\Gamma_{i}}$, 6.3.13 and (6.3.14) imply that for $\Gamma \Subset \mathbb{R}^{d}$,

$$
\begin{gather*}
\int\left|\nabla_{\xi} \widehat{L^{*} f \delta_{\Gamma}}(\xi)\right|^{2} \delta(|\xi|-r) \leq C\langle r\rangle^{2 \alpha}\|f\|_{L^{2}(\Gamma)}^{2}  \tag{6.3.18}\\
\int\left|\nabla_{\xi} \widehat{f \delta_{\Gamma}}(\xi)\right|^{2} \delta(|\xi|-r) \leq C\langle r\rangle^{2 \beta}\|f\|_{L^{2}(\Gamma)}^{2} \tag{6.3.19}
\end{gather*}
$$

Now, $g \delta_{\Gamma_{2}} \in H^{-\frac{1}{2}-\epsilon}\left(\mathbb{R}^{d}\right), L_{2}^{*}\left(g \delta_{\Gamma_{2}}\right) \in H^{-3 / 2-\epsilon}\left(\mathbb{R}^{d}\right)$ and

$$
\begin{equation*}
\left.R_{0}(\lambda)\left(\psi\left(\lambda^{-1}|D|\right) L^{*}\left(f \delta_{\Gamma_{1}}\right)\right) \in C^{\infty}\left(\mathbb{R}^{d}\right), \quad R_{0}(\lambda)\left(\psi\left(\lambda^{-1}|D|\right)\right) f \delta_{\Gamma_{1}}\right) \in C^{\infty}\left(\mathbb{R}^{d}\right) \tag{6.3.20}
\end{equation*}
$$

For $|\lambda| \leq 2$, the bounds 6.3.15) and 6.3.16) follow from (6.3.20) and $g \delta_{\Gamma_{2}} \in H^{-\frac{1}{2}-\epsilon}\left(\mathbb{R}^{d}\right)$ and the analyticity of $R_{0}(\lambda)$ in the upper half plane. Therefore, we need only consider $|\lambda| \geq 2$.

By Plancherel's theorem,

$$
\begin{gathered}
Q_{\lambda}^{D}(f, g)=\int \psi\left(\lambda^{-1}|\xi|\right) \frac{\widehat{L^{*} f \delta_{\Gamma_{1}}}(\xi) \widehat{\widehat{g \delta_{\Gamma_{2}}}(\xi)}}{|\xi|^{2}-\lambda^{2}}, \quad Q_{\lambda}^{G}(f, g)=\int \psi\left(\lambda^{-1}|\xi|\right) \frac{\widehat{f \delta_{\Gamma_{1}}}(\xi) \widehat{\widehat{g \delta_{\Gamma_{2}}}}(\xi)}{|\xi|^{2}-\lambda^{2}} \\
Q_{\lambda}^{d D}(f, g)=\int \psi\left(\lambda^{-1}|\xi|\right) \frac{\widehat{L_{1}^{*} f \delta_{\Gamma_{1}}}(\xi) \widehat{\widehat{L_{2}^{*} g \delta_{\Gamma_{2}}}}(\xi)}{|\xi|^{2}-\lambda^{2}}
\end{gathered}
$$

Thus, to prove the lemma, we only need estimate

$$
\begin{equation*}
\int \psi\left(\lambda^{-1}|\xi|\right) \frac{F(\xi) G(\xi)}{|\xi|^{2}-\lambda^{2}} \tag{6.3.21}
\end{equation*}
$$

where by (6.3.13), 6.3.14), 6.3.18, and 6.3.19)

$$
\begin{aligned}
&\|F\|_{L^{2}\left(S_{r}^{d-1}\right)}+\left\|\nabla_{\xi} F\right\|_{L^{2}\left(S_{r}^{d-1}\right)} \leq C\langle r\rangle^{\delta_{1}}\|f\|_{L^{2}(\Gamma)} \\
&\|G\|_{L^{2}\left(S_{r}^{d-1}\right)}+\left\|\nabla_{\xi} G\right\|_{L^{2}\left(S_{r}^{d-1}\right)} \leq C\langle r\rangle^{\delta_{2}}\|g\|_{L^{2}(\Gamma)}
\end{aligned}
$$

Consider first the integral in (6.3.21) over $||\xi|-|\lambda|| \geq 1$. Since $\left||\xi|^{2}-\lambda^{2}\right| \geq\left||\xi|^{2}-|\lambda|^{2}\right|$, by the Schwartz inequality, 6.3.13), and (6.3.14) this piece of the integral is bounded by

$$
\begin{align*}
\int_{\| \xi|-|\lambda|| \geq 1}\left|\psi\left(\lambda^{-1}|\xi|\right) \frac{F(\xi) G(\xi)}{|\xi|^{2}-\lambda^{2}}\right| & \leq \int_{M \lambda \geq|r-|\lambda| \geq 1} \frac{1}{r^{2}-|\lambda|^{2}} \int_{S_{r}^{d-1}} F(r \theta) G(r \theta) d S(\theta) d r \\
& \leq C\|f\|_{L^{2}(\Gamma)}\|g\|_{L^{2}(\Gamma)} \int_{M|\lambda| \geq|r-|\lambda|| \geq 1}\langle r\rangle^{\delta_{1}+\delta_{1}}\left|r^{2}-|\lambda|^{2}\right|^{-1} d r \\
& \leq C\|f\|_{L^{2}(\Gamma)}\|g\|_{L^{2}(\Gamma)} \lambda^{\delta_{1}+\delta_{2}-1} \int_{M|\lambda| \geq|r-|\lambda|| \geq 1}|r-| \lambda \|^{-1} d r \\
& \leq C|\lambda|^{\delta_{1}+\delta_{2}-1} \log |\lambda|\|f\|_{L^{2}(\Gamma)}\|g\|_{L^{2}(\Gamma)} \tag{6.3.22}
\end{align*}
$$

Remark: The estimate 6.3.22 is the only term where the log appears.
Next, if $\operatorname{Im} \lambda \geq 1$, then $\left||\xi|^{2}-\lambda^{2}\right| \geq|\lambda|$, and by (6.3.13), (6.3.14)

$$
\left|\int_{\|\xi|-| \lambda\| \leq 1} \frac{F(\xi) G(\xi)}{|\xi|^{2}-\lambda^{2}} d \xi\right| \leq C|\lambda|^{\delta_{1}+\delta_{2}-1}\|f\|_{L^{2}(\Gamma)}\|g\|_{L^{2}(\Gamma)}
$$

Thus, we may restrict our attention to $0 \leq \operatorname{Im} \lambda \leq 1$ and $||\xi|-|\lambda|| \leq 1$.
We consider $\operatorname{Re} \lambda \geq 0$, the other case following similarly, and write

$$
\frac{1}{|\xi|^{2}-\lambda^{2}}=\frac{1}{|\xi|+\lambda} \frac{\xi}{|\xi|} \cdot \nabla_{\xi} \log (|\xi|-\lambda)
$$

where the logarithm is well defined since $\operatorname{Im}(|\xi|-\lambda)<0$. Let $\chi(r)=1$ for $|r| \leq 1$ and vanish for $|r| \geq \frac{3}{2}$. We then use integration by parts, together with (6.3.13), (6.3.14), (6.3.18), and (6.3.19) to bound

$$
\left|\int \chi(|\xi|-|\lambda|) \frac{1}{|\xi|+\lambda} F(\xi) G(\xi) \frac{\xi}{|\xi|} \cdot \nabla_{\xi} \log (|\xi|-\lambda) d \xi\right| \leq C|\lambda|^{\delta_{1}+\delta_{2}-1}\|f\|_{L^{2}(\Gamma)}\|g\|_{L^{2}(\Gamma)}
$$

Now, taking $\delta_{1}=\delta_{2}=\alpha$ gives 6.3.15, and taking $\delta_{1}=\alpha$ and $\delta_{2}=\beta$ gives 6.3.16) and taking $\delta_{1}=\delta_{2}=\beta$ gives 6.3.17).

Remark: Note that the estimate on $Q_{\lambda}^{G}$ holds uniformly in $\psi$ and so putting in the cutoff $\psi$ is unnecessary. However, so that the presentation of all of the estimates are similar, we include the cutoff here.

We now prove the estimates (6.3.13) and (6.3.14).
Lemma 6.3.2. Let $\Gamma \Subset \mathbb{R}^{d}$ be a compact $C^{1,1}$ embedded hypersurface. Then estimate (6.3.14) holds with $\beta=1 / 4$. Moreover, if $\Gamma$ is curved and $C^{2,1}$, then (6.3.14) holds with $\beta=1 / 6$. Finally, if $\Gamma$ is $C^{\infty}$ then for $L=\partial_{\nu}$ on $\Gamma$, estimate 6.3.13) holds with $\alpha=1$.

Proof. Let $A: H^{s}\left(\mathbb{R}^{d}\right) \rightarrow H^{s-1}\left(\mathbb{R}^{d}\right)$. To estimate

$$
\left.\int \mid \widehat{A^{*}\left(f \delta_{\Gamma}\right.}\right)\left.(\xi)\right|^{2} \delta(|\xi|-r)
$$

write

$$
\left\langle\widehat{A^{*}\left(f \delta_{\Gamma}\right)}(\xi) \delta(|\xi|-r), \phi(\xi)\right\rangle=\iint A^{*}\left(f(x) \delta_{\Gamma}\right) \delta(|\xi|-r) \overline{\phi(\xi) e^{i\langle x, \xi\rangle}} d x d \xi=\int_{\Gamma} f A T_{r} \phi d x
$$

where

$$
\begin{equation*}
T_{r} \phi=\int \delta(|\xi|-r) \phi(\xi) e^{i\langle x, \xi\rangle} d \xi \tag{6.3.23}
\end{equation*}
$$

For $\chi \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right), \chi T \phi$ is a quasimode of the Laplacian with eigenvalue $\lambda=r$ in the sense of Lemma 6.2.1 with $h=r^{-1}$. Thus, we can use the restriction bounds for eigenfunctions found their to obtain estimates on $T \phi$.

To prove 6.3.14), let $A=I$. Then, by Lemma 6.2.1

$$
\begin{equation*}
\left\|\chi T_{r} \phi\right\|_{L^{2}(\Gamma)} \leq r^{\frac{1}{4}}\|\chi T \phi\|_{L^{2}\left(\mathbb{R}^{d}\right)} \tag{6.3.24}
\end{equation*}
$$

and if $\Gamma$ is curved then

$$
\begin{equation*}
\left\|\chi T_{r} \phi\right\|_{L^{2}(\Gamma)} \leq r^{\frac{1}{6}}\|\chi T \phi\|_{L^{2}\left(\mathbb{R}^{d}\right)} \tag{6.3.25}
\end{equation*}
$$

Next, we take $A=L$ to obtain (6.3.13). Observe that

$$
\chi L T_{r} \phi=L \chi T_{r} \phi+[\chi, L] T_{r} \phi
$$

with $[\chi, L] \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$. Therefore, $[\chi, L] T_{r} \phi$ is a quasimode of the Laplacian with eigenvalue $r$.

Hence, using the fact that $L=\partial_{\nu}$ on $\Gamma$ together with Lemma 6.2.3, we can estimate $L T \phi$.

$$
\begin{equation*}
\left\|\chi L T_{r} \phi\right\|_{L^{2}(\Gamma)} \leq\left\|L \chi T_{r} \phi\right\|_{L^{2}(\Gamma)}+\left\|[L, \chi] T_{r} \phi\right\|_{L^{2}(\Gamma)} \leq C r\left\|\chi T_{r} \phi\right\|_{L^{2}\left(\mathbb{R}^{d}\right)} \tag{6.3.26}
\end{equation*}
$$

To complete the proof of the Lemma, we estimate $\|\chi T \phi\|_{L^{2}\left(\mathbb{R}^{d}\right)}$. We have that

$$
\left\|\chi T_{r} \phi\right\|_{L^{2}\left(\mathbb{R}^{d}\right)}=\|\hat{\chi} * g \delta(|\xi|-r)\|_{L^{2}\left(\mathbb{R}^{d}\right)}
$$

Therefore,

$$
\begin{aligned}
\|\hat{\chi} * g \delta(|\xi|-r)\|_{L^{2}\left(\mathbb{R}^{d}\right)}^{2} & =\int\left|\int_{S_{r}^{d-1}} \hat{\chi}(\xi-\eta) g(\eta) d \eta\right|^{2} d \xi \\
& \leq\|g\|_{L^{2}\left(S_{r}^{d-1}\right)}^{2} \iint_{S_{r}^{d-1}}|\hat{\chi}(\xi-\eta)|^{2} d \eta d \xi \\
& \leq\|g\|_{L^{2}\left(S_{r}^{d-1}\right)}^{2} \iint_{S_{r}^{d-1}} C_{N}\langle | \xi|-r\rangle^{-N} d \eta d \xi \leq C\|g\|_{L^{2}\left(S_{r}^{d-1}\right)}^{2}
\end{aligned}
$$

Combining this with (6.3.24), 6.3.25 and 6.3.26) completes the proof of the Lemma.

Next, we obtain an estimate on the high frequency component of $\tilde{N}$ and $\partial_{\nu} \mathcal{D} \ell$. We start by analyzing the high frequency components of the free resolvent.

Lemma 6.3.3. Suppose that $|z| \in[E-\delta, E+\delta]$ Let $\psi \in C_{c}^{\infty}(\mathbb{R})$ with $\psi \equiv 1$ on $\left[-2 E^{2}, 2 E^{2}\right]$. Then for $\chi \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$.

$$
\chi R_{0}(z / h) \chi(1-\psi(|h D|))=B
$$

where $B \in h^{2} \Psi^{-2}\left(\mathbb{R}^{d}\right)$ with

$$
\sigma(B)=\frac{\chi^{2} h^{2}(1-\psi(|\xi|))}{|\xi|^{2}-z^{2}}
$$

If $\operatorname{Im} z>0$, then $\chi$ can be removed from all of the above statements.
Proof. Let $\chi_{0}=\chi \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ and $\chi_{n} \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ have $\chi_{n} \equiv 1$ on $\operatorname{supp} \chi_{n-1}$ for $n \geq 1$. Let $\psi_{0}=\psi \in C_{c}^{\infty}(\mathbb{R})$ have $\psi \equiv 1$ on $\left[-2 E^{2}, 2 E^{2}\right]$, let $\psi_{n} \in C_{c}^{\infty}(\mathbb{R})$ have $\psi_{n} \equiv 1$ on $\left[-3 E^{2} / 2,3 E^{2} / 2\right]$ and $\operatorname{supp} \psi_{n} \subset\left\{\psi_{n-1} \equiv 1\right\}$ for $n \geq 1$. Finally, let $\varphi_{n}=\left(1-\psi_{n}\right)$. Then, $h^{-2} \chi R_{0} \chi \varphi_{1}(|h D|)\left(-h^{2} \Delta-z\right)=h^{-2} \chi R_{0} \chi_{1}$

$$
\begin{aligned}
& \left(\left(-h^{2} \Delta-z\right) \chi \varphi_{1}(|h D|)+\left[\chi \varphi_{1}(|h D|),-h^{2} \Delta-z\right]\right) \\
= & \left(\chi^{2} \varphi_{1}(|h D|)+h^{-2} \chi R_{0} \chi_{1}\left[\chi \varphi_{1}(|h D|),-h^{2} \Delta-z\right]\right) \\
= & \left(\chi^{2} \varphi_{1}(|h D|)\right. \\
& \left.+h^{-2} \chi \chi_{1} R_{0} \chi_{1} \varphi_{2}(|h D|)\left[\chi \varphi_{1}(|h D|),-h^{2} \Delta-z\right]+O_{C_{c}^{\infty}}\left(h^{\infty}\right)\right)
\end{aligned}
$$

Now, by Lemma 4.3.5 there exists $A_{0} \in h^{2} \Psi^{-2}\left(\mathbb{R}^{d}\right) \mathrm{WF}_{\mathrm{h}}\left(A_{0}\right) \subset\left\{\operatorname{supp} \varphi_{0}\right\}$, such that

$$
h^{-2} \varphi_{1}(|h D|)\left(-h^{2} \Delta-z\right) A=\varphi(|h D|)+O_{\Psi^{-\infty}}\left(h^{\infty}\right)
$$

and $A_{0}$ has

$$
\sigma\left(A_{0}\right)=\frac{h^{2} \varphi(|h D|)}{|\xi|^{2}-z^{2}}
$$

Composing $h^{-2} \chi R_{0} \chi \varphi_{1}(|h D|)$ on the right with $A_{0}$, we have

$$
\begin{aligned}
\chi R_{0} \chi \varphi(|h D|) & =\chi^{2} A_{0}+\chi \chi_{1} R_{0} \chi_{1} \varphi_{2}(|h D|)\left[\chi \varphi_{1}(|h D|),-h^{2} \Delta-z\right] h^{-2} A_{0}+O_{C_{c}^{\infty}}\left(h^{\infty}\right) \\
& =\chi^{2} A_{0}+\chi \chi_{1} R_{0} \chi_{1} \varphi_{2} O_{\Psi^{-1}}(h)+O_{C_{c}^{\infty}}\left(h^{\infty}\right)
\end{aligned}
$$

Now, applying the same arguments, there exists $A_{n} \in h^{2} \Psi^{-2}\left(\mathbb{R}^{d}\right)$ such that

$$
\chi_{n} R_{0} \chi_{n} \varphi_{n}(|h D|)=\chi_{n}^{2} A_{n}+\chi_{n+1} R_{0} \chi_{n+1} \varphi_{n+2}(|h D|) O_{\Psi^{-1}}(h)+O_{C_{c}^{\infty}}\left(h^{\infty}\right) .
$$

Hence, by induction

$$
\chi R_{0} \chi \varphi(|h D|)=B \in h^{2} \Psi^{-2}\left(\mathbb{R}^{d}\right)
$$

with

$$
\sigma(B)=\frac{h^{2} \chi^{2}(1-\psi(|\xi|)}{|\xi|^{2}-z^{2}}
$$

as desired.

Now, let $\gamma^{ \pm}: H^{s}\left(\Omega^{ \pm}\right) \rightarrow H^{s-1 / 2}(\partial \Omega), s>1 / 2$ denote the restriction map where $\Omega^{+}=\Omega$ and $\Omega^{-}=\mathbb{R}^{d} \backslash \bar{\Omega}$. Then we have

Lemma 6.3.4. Let $M>1$ and $\psi \in C_{c}^{\infty}(\mathbb{R})$ with $\psi \equiv 1$ for $|\xi|<M$. Suppose that $\partial \Omega$ is a compact embedded $C^{\infty}$ hypersurface. Then there exists $\lambda_{0}>0$ such that for $|\lambda|>\lambda_{0}$ and $\operatorname{Im} \lambda \geq 0$,

$$
\begin{align*}
\gamma R_{0}(\lambda)\left(1-\psi\left(|\lambda|^{-1}|D|\right)\right) \gamma^{*} & =O_{L^{2}(\partial \Omega) \rightarrow L^{2}(\partial \Omega)}\left(|\lambda|^{-1}\right) .  \tag{6.3.27}\\
\gamma^{ \pm} R_{0}(\lambda)\left(1-\psi\left(|\lambda|^{-1}|D|\right)\right) L^{*} \gamma^{*} & =O_{L^{2}(\partial \Omega) \rightarrow L^{2}(\partial \Omega)}(1) .  \tag{6.3.28}\\
\gamma^{ \pm} L R_{0}(\lambda)\left(1-\psi\left(|\lambda|^{-1}|D|\right)\right) L^{*} \gamma^{*} & =O_{H^{1}(\partial \Omega) \rightarrow L^{2}(\partial \Omega)}(|\lambda|) . \tag{6.3.29}
\end{align*}
$$

Moreover, for $|\lambda|>\lambda_{0}$, and $\chi \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$

$$
\begin{aligned}
\gamma R_{0}(\lambda) \chi\left(1-\psi\left(|\lambda|^{-1}|D|\right)\right) \gamma^{*} & =O_{L^{2}(\partial \Omega) \rightarrow L^{2}(\partial \Omega)}\left(|\lambda|^{-1}\right) . \\
\gamma^{ \pm} R_{0}(\lambda) \chi\left(1-\psi\left(|\lambda|^{-1}|D|\right)\right) L^{*} \gamma^{*} & =O_{L^{2}(\partial \Omega) \rightarrow L^{2}(\partial \Omega)}(1) . \\
\gamma^{ \pm} L R_{0}(\lambda) \chi\left(1-\psi\left(|\lambda|^{-1}|D|\right)\right) L^{*} \gamma^{*} & =O_{H^{1}(\partial \Omega) \rightarrow L^{2}(\partial \Omega)}(|\lambda|) .
\end{aligned}
$$

Proof. Let $h^{-1}=|\lambda|$ and $\chi \equiv 1$ on $\Omega$. For $\operatorname{Im} \lambda>0$, we take $\chi \equiv 1$ and for $\arg \lambda \in$ $[-\pi, 0] \cup[\pi, 2 \pi], \chi \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$. Then by Lemma 6.3.3

$$
\begin{equation*}
\chi R_{0}(\lambda) \chi(1-\psi(h D)) \in h^{2} \Psi^{-2} \tag{6.3.30}
\end{equation*}
$$

Note that for $s>1 / 2, \gamma$ is a semiclassical FIO and

$$
\begin{equation*}
\gamma=O_{H_{h}^{s}\left(\mathbb{R}^{d}\right) \rightarrow H_{h}^{s-1 / 2}(\partial \Omega)}\left(h^{-1 / 2}\right) . \tag{6.3.31}
\end{equation*}
$$

The bounds 6.3.27) and the corresponding bound in the lower half plane follow from (6.3.30) and composition with $\gamma$ and $\gamma^{*}$.

Remark: Note that we only apply $\gamma: H_{h}^{1 / 2+\epsilon} \rightarrow L^{2}=O\left(h^{1 / 2}\right)$ in the case of 6.3.27) and hence this bound is valid for $\Gamma$ only $C^{1,1}$.

The strategy for obtaining the bounds $(6.3 .28)$ and $(6.3 .29)$ is to compare $\tilde{N}$ and $\partial_{\nu} \mathcal{D} \ell$ at high frequency with the corresponding operators for $\lambda=i$. Note that $\chi R_{0}(i) \chi(1-\psi(|h D|)) \in$ $h^{2} \Psi^{-2}$. For $\operatorname{Im} \lambda>0$, we consider

$$
A_{h}:=\chi\left(R_{0}(\lambda)-R_{0}(i)\right) \chi(1-\psi(|h D|))=h^{-2} \chi R_{0}(\lambda) R_{0}(i) \chi(1-\psi(|h D|)) .
$$

Hence, $A_{h} \in h^{2} \Psi^{-4}$. Let

$$
B_{h}:=\gamma A_{h} L^{*} \gamma^{*}, \quad C_{h}:=\gamma L A_{h} L^{*} \gamma^{*}
$$

Then, using 6.3.31 and the fact that $L, L^{*}=O_{H_{h}^{s} \rightarrow H_{h}^{s-1}}\left(h^{-1}\right)$, we have that $B_{h}=O_{L^{2} \rightarrow L^{2}}(1)$ and $C_{h}=O_{L^{2} \rightarrow L^{2}}\left(h^{-1}\right)$.

Now, by [76, Section 7.11]

$$
\begin{aligned}
\gamma^{ \pm} R_{0}(i) \chi(1-\psi(|h D|)) L^{*} \gamma: H^{s}(\partial \Omega) & \rightarrow H^{s}(\partial \Omega) \\
\gamma^{ \pm} L R_{0}(i) \chi(1-\psi(|h D|)) L^{*} \gamma: H^{s}(\partial \Omega) & \rightarrow H^{s-1}(\partial \Omega)
\end{aligned}
$$

for $\partial \Omega$ a smooth hypersurface. Hence,

$$
\gamma^{ \pm} R_{0}(\lambda) \chi(1-\psi(|h D|)) L^{*} \gamma=\gamma^{ \pm} R_{0}(i) \chi(1-\psi(|h D|)) L^{*} \gamma+\gamma L B_{h} L^{*} \gamma^{*}=O_{L^{2} \rightarrow L^{2}}(1)
$$

and

$$
\gamma^{ \pm} L R_{0}(\lambda) \chi(1-\psi(|h D|)) L^{*} \gamma=\gamma^{ \pm} L R_{0}(i) \chi(1-\psi(|h D|)) L^{*} \gamma+\gamma^{ \pm} L C_{h} L^{*} \gamma^{*}=O_{H^{1} \rightarrow L^{2}}\left(h^{-1}\right)
$$

Taking $\partial \Omega=\bigcup_{i} \Gamma_{i}$ and applying Lemmas 6.3.1 and Lemma 6.3.4 finishes the proof of Theorems 6.1 6.3 and 6.4 for $\operatorname{Im} \lambda \geq 0$.

Our final task is to extend the estimates into the lower half plane.
Lemma 6.3.5. Suppose that for $|\lambda| \geq \lambda_{0}$ and for $\lambda$ in the upper half plane,

$$
\left|Q_{\lambda}(f, g)\right| \leq C\langle\lambda\rangle^{\alpha}(\log \langle\lambda\rangle)^{\beta}\|f\|_{\mathcal{A}}\|g\|_{\mathcal{B}}
$$

where $Q$ is one of $Q_{\lambda}^{G}, Q_{\lambda}^{D}, Q_{\lambda}^{d D}$. Then for $|\lambda| \geq \lambda_{0}$ and $\operatorname{Im} \lambda \leq 0$, if $d$ is odd and for $\arg \lambda \in[-\pi, 0] \cup[\pi, 2 \pi]$ if $d$ is even

$$
\left|Q_{\lambda}(f, g)\right| \leq C\langle\lambda\rangle^{\alpha}(\log \langle\lambda\rangle)^{\beta} e^{D_{\Omega} \operatorname{Im} \lambda}\|f\|_{\mathcal{A}}\|g\|_{\mathcal{B}}
$$

where $D_{\Omega}$ is the diameter of $\Omega$.
Proof. We first consider $d$ odd. Let $\|f\|_{\mathcal{A}}=1$ and $\|g\|_{\mathcal{B}}=1$. Let $\chi \equiv 1$ on $\Omega$. Then consider

$$
F(\lambda)=e^{-i D_{\Omega} \lambda} \lambda^{-\alpha}(\log \lambda)^{-\beta} Q_{\lambda}(f, g), \quad|\lambda| \geq \lambda_{0}, \operatorname{Im} \lambda \leq 0
$$

where $\log \lambda$ is defined for $\arg \lambda \in(\pi / 2,5 \pi / 2)$. Then, $|F(\lambda)| \leq C$ on $\mathbb{R} \backslash\left[-\lambda_{0}, \lambda_{0}\right]$.
Lemma 6.3.4 shows that

$$
\begin{gathered}
\gamma R_{0}(\lambda) \chi\left(1-\psi\left(|\lambda|^{-1}|D|\right)\right) \gamma^{*}=O_{L^{2} \rightarrow L^{2}}\left(|\lambda|^{-1}\right) \\
\gamma^{ \pm} R_{0}(\lambda) \chi\left(1-\psi\left(|\lambda|^{-1} D\right)\right) L^{*} \gamma^{*}=O_{L^{2} \rightarrow L^{2}}(1) \\
\gamma^{ \pm} L R_{0}(\lambda) \chi\left(1-\psi\left(|\lambda|^{-1}|D|\right)\right) L^{*} \gamma^{*}=O_{H^{1} \rightarrow L^{2}}(|\lambda|)
\end{gathered}
$$

For all $s$

$$
\left\|\chi R_{0}(\lambda) \chi\right\|_{H^{s} \rightarrow H^{s}} \leq C\langle\lambda\rangle^{-1} e^{D_{\chi}(\operatorname{Im} \lambda)_{-}}
$$

where $D_{\chi}=\operatorname{diam}(\operatorname{supp} \chi)$ is the diameter of $\operatorname{supp} \chi$. Moreover,

$$
\psi\left(\left|\lambda^{-1} D\right|\right): H^{s} \rightarrow H^{s+M}=\mathcal{O}\left(|\lambda|^{M}\right) .
$$

So, there exists $N>0$ such that

$$
\begin{aligned}
\left\|\gamma L R_{0}(\lambda) \chi \psi\left(|\lambda|^{-1} D\right) L^{*} \gamma^{*}\right\|_{H^{1} \rightarrow L^{2}}+ & \left\|\gamma^{ \pm} R_{0}(\lambda) \chi \psi\left(|\lambda|^{-1} D\right) L^{*} \gamma^{*}\right\|_{L^{2} \rightarrow L^{2}} \\
& +\left\|\gamma R_{0}(\lambda) \chi \psi\left(|\lambda|^{-1} D\right) \gamma^{*}\right\|_{L^{2} \rightarrow L^{2}} \leq C\langle\lambda\rangle^{N} e^{D_{\chi}(\operatorname{Im} \lambda)_{-}}
\end{aligned}
$$

Letting supp $\chi \rightarrow \Omega$, we see that $|F(\lambda)|$ has at most polynomical growth in the lower half plane. Thus, the Phragmén-Lindelöf theorem shows that $|F(\lambda)| \leq C$.

When $d$ is even, we note that the assumed bounds hold for $\arg \lambda=2 \pi$ and $|\lambda| \geq \lambda_{0}$. This follows since $R_{0}\left(\lambda e^{\pi i}\right)-R_{0}(\lambda)$ satisfies the same bounds as $R_{0}(\lambda)$ for $\arg \lambda=0$. Moreover, $R_{0}\left(\lambda e^{2 \pi i}\right)-R_{0}\left(\lambda e^{\pi i}\right)=R_{0}\left(\lambda e^{\pi i}\right)-R_{0}(\lambda)$. Thus, we apply the Phragmén-Lindelöf theorem on the sheet $\pi \leq \arg z \leq 2 \pi$. Using a similar argument, we can apply the Phragmén-Lindelöf theorem on $-\pi \leq \arg \lambda \leq 0$.

Applying Lemma 6.3.5 together with Lemmas 6.3.1, 6.3.2 and 6.3.4 implies Theorems 6.16 .3 and 6.4

We conclude this section with the proof of Theorem6.6. The estimates for $0<\operatorname{Im} \lambda<1$ follow from Theorem 6.1, so we consider $\operatorname{Im} \lambda \geq 1$. To do this, we establish bounds on $Q_{\lambda}(f, g)$ defined by

$$
Q_{\lambda}(f, g):=\int \frac{\widehat{\delta_{\Gamma} f} \widehat{\widehat{\delta_{\Gamma} g}}}{|\xi|^{2}-\lambda^{2}} d \xi
$$

First consider the case that $f=g$ and $\Gamma$ is a graph $x_{n}=F\left(x^{\prime}\right)$. We then have uniform bounds

$$
\left.\sup _{\xi_{n}} \int \widehat{\gamma^{*} f}\left(\xi^{\prime}, \xi_{n}\right)\right|^{2} d \xi^{\prime} \leq C\|f\|_{L^{2}(\Gamma)}^{2}
$$

We use the lower bound $\left||\xi|^{2}-\lambda^{2}\right| \geq|\lambda||\operatorname{Im} \lambda|$ to dominate

$$
\int_{\left|\xi_{n}\right| \leq 2|\lambda|} \frac{\left|\widehat{\gamma^{*} f}(\xi)\right|^{2}}{\left.| | \xi\right|^{2}-\lambda^{2} \mid} d \xi \leq C\langle\operatorname{Im} \lambda\rangle^{-1}\|f\|_{L^{2}(\Gamma)}^{2}
$$

For $\left|\xi_{n}\right| \geq 2|\lambda|$ we have $\left||\xi|^{2}-\lambda^{2}\right| \gtrsim\left|\xi_{n}\right|^{2}$, hence

$$
\int_{\left|\xi_{n}\right| \geq 2|\lambda|} \frac{\left|\widehat{\gamma^{*} f}(\xi)\right|^{2}}{\left.| | \xi\right|^{2}-\lambda^{2} \mid} d \xi \leq C\langle\lambda\rangle^{-1}\|f\|_{L^{2}(\Gamma)}^{2}
$$

The case $f \neq g$ and $\Gamma$ a finite union of graphs then follows by a partition of unity argument and the Schwarz inequality.

### 6.4 Sharpness of the Estimates for $\lambda \in \mathbb{R}$

## Sharpness for the single and derivative double layer operators

We now show that the estimates on $G$ and $\partial_{\nu} \mathcal{D} \ell$ in Theorems 6.1 and 6.4 are sharp modulo the $\log \operatorname{losses}$ when $\lambda \in \mathbb{R}$ with $|\lambda| \gg 1$.

First, observe that for $\lambda>0$, the spectral measure $d E_{\lambda}$ has

$$
\begin{equation*}
\pi i d E_{\lambda}=R_{0}\left(e^{i \pi} \lambda\right)(x, y)-R_{0}(-\lambda)(x, y)=\frac{i}{2} \frac{1}{\lambda(2 \pi)^{d-1}} \int_{S_{\lambda}^{d-1}} e^{i\langle x-y, \omega\rangle} d \omega \tag{6.4.1}
\end{equation*}
$$

Thus,

$$
A d E_{\lambda} A^{*}=C_{d} A T_{\lambda} T_{\lambda}^{*} A^{*}
$$

where $T_{\lambda}$ is the operator in (6.3.23). By [10], 38, , 72$]$ the estimates (6.3.24), (6.3.25), and (6.3.26) are sharp and hence

$$
\left\|\gamma d E_{\lambda} \gamma^{*}\right\|_{L^{2}(\Gamma) \rightarrow L^{2}(\Gamma)} \geq \begin{cases}C\langle\lambda\rangle^{-\frac{1}{2}} & \Gamma \text { general } \\ C\langle\lambda\rangle^{-\frac{2}{3}} & \Gamma \text { curved }\end{cases}
$$

and

$$
\left\|\gamma L d E_{\lambda} L^{*} \gamma^{*}\right\|_{H^{1}(\partial \Omega) \rightarrow L^{2}(\partial \Omega)} \geq C\langle\lambda\rangle .
$$

Putting this together with 6.4.1 gives that for $\lambda \in \mathbb{R}$,

$$
\|G\|_{L^{2}(\Gamma) \rightarrow L^{2}(\Gamma)} \geq\left\{\begin{array}{ll}
C\langle\lambda\rangle^{-\frac{1}{2}} & \Gamma \text { general } \\
C\langle\lambda\rangle^{-\frac{2}{3}} & \Gamma \text { curved }
\end{array}, \quad\left\|\partial_{\nu} \mathcal{D} \ell\right\|_{H^{1}(\partial \Omega) \rightarrow L^{2}(\partial \Omega)} \geq C\langle\lambda\rangle\right.
$$

as desired.

## Sharpness for the double layer operator

We next show that there exist smooth embedded hypersurfaces $\Gamma$ such that for $\lambda \in \mathbb{R}$ with $|\lambda| \gg 1$,

$$
\|\tilde{N}\|_{L^{2}(\Gamma) \rightarrow L^{2}(\Gamma)} \geq \begin{cases}C\langle\lambda\rangle^{1 / 4} & \Gamma \text { general }  \tag{6.4.2}\\ C\langle\lambda\rangle^{1 / 6} & \Gamma \text { curved }\end{cases}
$$

Similar estimates and examples in the flat case are given in [15, Theorems 4.6, 4.7] in dimension 2. In the curved case, they prove an estimate $\|\tilde{N}\|_{L^{2}(\Gamma)} \geq C \lambda^{1 / 8}$.

The idea will be to use a family functions which is microlocalized at a point $\left(\left(x^{\prime}, 0\right), \xi^{\prime}\right) \in$ $T^{*} \Gamma$ such that $\left|\xi^{\prime}\right|<1$ and the geodesic

$$
\left\{\left(x^{\prime}, 0\right)+t\left(\xi^{\prime}, \sqrt{1-\left|\xi^{\prime}\right|^{2}}\right): t \in \mathbb{R}\right\}
$$

is tangent to $\Gamma$ at some point away from $\left(x^{\prime}, 0\right)$.

## Flat case

Let

$$
\begin{aligned}
\Gamma_{1}:= & \left\{\left(x_{1}, x_{2}, x^{\prime}\right) \in \mathbb{R}^{d}: 1 / 2<x_{1}<3 / 2, x_{2}=0,\left|x^{\prime}\right|<1\right\} \\
& \Gamma_{2}:=\left\{\left(x_{1}, x_{2}, x^{\prime}\right) \in \mathbb{R}^{d}: x_{1}=0, x_{2}^{2}+\left|x^{\prime}\right|^{2}<1\right\} .
\end{aligned}
$$

Let $\chi \in C_{c}^{\infty}\left(\mathbb{R}^{d-1}\right)$ have $\chi \geq 0,\|\chi\|_{L^{2}}=1$, and $\hat{\chi}(0) \geq 1 / 2$. That is

$$
\int \chi\left(x_{2}, x^{\prime}\right) d x_{2} d x^{\prime} \geq 1 / 2
$$

Then, denote by $\chi_{\lambda}:=\chi\left(M \lambda^{\gamma}\left(x_{2}, x^{\prime}\right)\right)$ and observe that

$$
\left\|\chi_{\lambda}\right\|_{L^{2}}=C_{M} \lambda^{-(d-1) \gamma / 2}, \quad \int \chi_{\lambda} d x_{2} d x^{\prime} \geq C_{M} \lambda^{-(d-1) \gamma}
$$

where $M>0$ will be chosen later and $\gamma \geq 1 / 2$.
Now, let $\Gamma \Subset \mathbb{R}^{d}$ be a smooth embedded hypersurface such that $\Gamma_{1} \cup \Gamma_{2} \subset \Gamma$. Suppose also that $f \in L^{2}(\Gamma)$ is supported on $\Gamma_{2}$. Then,

$$
\left.\tilde{N} f\right|_{\Gamma_{1}}=\int_{\Gamma_{2}}\left(\partial_{\nu_{y}} R_{\lambda}^{+}(x-y)\right) f(y) d y
$$

Now, for $|x-y|>\epsilon$,

$$
\begin{equation*}
\partial_{\nu_{y}} R_{\lambda}^{+}(x-y)=C_{d} \lambda^{d-1} \frac{\left\langle x-y, \nu_{y}\right\rangle}{|x-y|} e^{i \lambda|x-y|}\left(\lambda^{-(d-1) / 2}|x-y|^{(d-1) / 2}+\mathcal{O}\left((\lambda|x-y|)^{-(d+1) / 2}\right)\right) . \tag{6.4.3}
\end{equation*}
$$

We will consider $\chi_{\lambda}$ as a function in $L^{2}\left(\Gamma_{2}\right)$. Thus, since for $x \in \Gamma_{1}$ and $y \in \Gamma_{2},|x-y| \geq \epsilon$, we consider

$$
\lambda^{(d-1) / 2} \int_{\Gamma_{2}} \frac{e^{i \lambda|x-y|}\left\langle x-y, \nu_{y}\right\rangle}{|x-y|^{(d+1) / 2}} \chi_{\lambda}(y) d y
$$

We are interested in obtaining lower bounds for the $L^{2}$ norm on $\Gamma_{1}$. In particular, let $\psi \in$ $C_{c}^{\infty}(\mathbb{R})$ with $\psi(z)=1$ for $|z| \leq 1$. Then, let $\psi_{\lambda, 1}(z)=\psi\left(M \lambda^{\gamma}|z|\right)$ and $\psi_{\lambda, 2}(z)=\psi\left(M \lambda^{\gamma_{2}}|z|\right)$.

We estimate

$$
u=\psi_{\lambda, 2}\left(x_{1}-1\right) \psi_{\lambda, 1}\left(x^{\prime}\right) \lambda^{(d-1) / 2} \int_{\Gamma_{2}} \frac{e^{i \lambda|x-y|}\left\langle x-y, \nu_{y}\right\rangle}{|x-y|^{(d+1) / 2}} \chi_{\lambda}(y) d y
$$

on $\Gamma_{1}$. For $x \in \Gamma_{1} \cap \operatorname{supp} \psi_{\lambda, 1}\left(x^{\prime}\right) \psi_{\lambda, 2}\left(x_{1}-1\right)$ and $y \in \operatorname{supp} \chi_{\lambda}$

$$
\begin{equation*}
\frac{\left\langle x-y, \nu_{y}\right\rangle}{|x-y|}=1+O\left(\lambda^{-2 \gamma}\right), \quad|x-y|=x_{1}\left(1+O\left(\lambda^{-2 \gamma}\right)\right) \tag{6.4.4}
\end{equation*}
$$

Hence, we have

$$
\left.u\right|_{\Gamma_{1}}=C_{d} \psi_{\lambda, 2}\left(x_{1}-1\right) \psi_{\lambda, 1}\left(x^{\prime}\right) \lambda^{(d-1) / 2} \frac{e^{i \lambda x_{1}}}{x_{1}^{(d-1) / 2}} \int\left(1+O\left(\left\langle\lambda^{1-2 \gamma}\right\rangle M^{-2}\right)+O\left(\left\langle M^{-2} \lambda^{-2 \gamma}\right\rangle\right) \chi_{\lambda} d y\right.
$$

and on $\left|x^{\prime}\right|<M^{-1} \lambda^{-\gamma}$,

$$
\left.u\right|_{\Gamma_{1}}\left(x_{1}, x^{\prime}\right) \geq C \psi_{\lambda, 2}\left(x_{1}-1\right) \lambda^{(d-1) / 2} \int \chi_{\lambda} d y \geq C \lambda^{(d-1) / 2-(d-1) \gamma}
$$

So,

$$
\|u\|_{L^{2}\left(\Gamma_{1}\right)}^{2} \geq C \int_{\Gamma_{1} \cap\left|x^{\prime}\right|<C \lambda^{-\gamma}} \psi_{\lambda, 2}^{2}\left(x_{1}-1\right) \lambda^{d-1-2(d-1) \gamma} \geq C \lambda^{d-1-(3 d-4) \gamma-\gamma_{2}}
$$

Thus, using elementary estimates on the remainder terms

$$
\left\|\tilde{N} \chi_{\lambda}\right\| \geq C\|u\| \geq C \lambda^{\frac{d-1-(3 d-4) \gamma-\gamma_{2}}{2}}
$$

Hence,

$$
\frac{\left\|\tilde{N} \chi_{\lambda}\right\|}{\left\|\chi_{\lambda}\right\|} \geq C \lambda^{\frac{(d-1)(1-2 \gamma)+\gamma-\gamma_{2}}{2}}
$$

Thus, choosing $\gamma=1 / 2, \gamma_{2}=0$ and $M$ large enough,

$$
\left\|\tilde{N} \chi_{\lambda}\right\| \geq C \lambda^{1 / 4}\left\|\chi_{\lambda}\right\|
$$

as desired.

## Curved case

In order to obtain the lower bound in the curved case, we will need to arrange to hypersurfaces, $\Gamma_{1}$ and $\Gamma_{2}$ parametrized respectively by $\gamma, \sigma: B(0, \epsilon) \subset \mathbb{R}^{d-1} \rightarrow \mathbb{R}^{d}$ such that

$$
|\gamma(x)-\sigma(y)|=|\gamma(0)-\sigma(0)|+O\left(\left|x_{1}-y_{1}\right|^{3}\right)+\mathcal{O}\left(\left|x^{\prime}-y^{\prime}\right|^{2}\right)
$$

where $x=\left(x_{1}, x^{\prime}\right) \in \mathbb{R}^{d-1}$. To do this, let $\tilde{\gamma}:(-\epsilon, \epsilon) \rightarrow \mathbb{R}^{2}$ be a smooth unit speed curve with curvature $\kappa(t)=\left\|\gamma^{\prime \prime}(t)\right\|$ and normal vector $n(t)=\gamma^{\prime \prime}(t) / \kappa(t)$. We assume $\kappa(0) \neq 0$ and $\kappa^{\prime}(0) \neq 0$. Then, let $\sigma(t)$ be the loci of the osculating circle for $\tilde{\gamma}(t)$. That is,

$$
\tilde{\sigma}(t)=\tilde{\gamma}(t)+\frac{n(t)}{\kappa(t)}
$$

Finally, define

$$
\gamma(x):=\left(\tilde{\gamma}\left(x_{1}\right)+n\left(x_{1}\right)\left|x^{\prime}\right|^{2}, x^{\prime}\right), \quad \sigma(x):=\left(\tilde{\sigma}\left(x_{1}\right)+\gamma^{\prime}\left(x_{1}\right)\left|x^{\prime}\right|^{2}, x^{\prime}\right)
$$

Then we have

$$
|\gamma(y)-\sigma(x)|^{2}=\left|\tilde{\gamma}\left(y_{1}\right)-\tilde{\sigma}\left(x_{1}\right)\right|^{2}+O\left(\left|x^{\prime}\right|^{2}+\left|y^{\prime}\right|^{2}\right)+\left|x^{\prime}-y^{\prime}\right|^{2}
$$

Let $d\left(x_{1}, y_{1}\right)=\left|\tilde{\gamma}\left(y_{1}\right)-\tilde{\sigma}\left(x_{1}\right)\right|$. Then,

$$
\partial_{x_{1}} d\left(x_{1}, x_{1}\right)=\partial_{x_{1}}^{2} d\left(x_{1}, x_{1}\right)=0 .
$$



Figure 6.1: We show an example of a curve $\tilde{\gamma}$ and its loci of osculating circles, $\tilde{\sigma}$.

Hence,

$$
d\left(x_{1}, y_{1}\right)=d\left(x_{1}, x_{1}\right)+O\left(\left|x_{1}-y_{1}\right|^{3}\right)
$$

and we have near $x=y=0$,

$$
|\gamma(y)-\sigma(x)|=|\gamma(0)-\sigma(0)|+O\left(\left|x_{1}-y_{1}\right|^{3}\right)+O\left(\left|x^{\prime}\right|^{2}+\left|y^{\prime}\right|^{2}\right)
$$

Moreover,

$$
\frac{\left\langle\sigma(x)-\gamma(y), \nu_{y}\right\rangle}{|\gamma(y)-\sigma(x)|}=1+O(|x-y|)
$$

Now, with $\chi \in C_{c}^{\infty}\left(\mathbb{R}^{d-1}\right)$, let

$$
\chi_{\lambda}=\chi\left(M\left(\lambda^{\gamma_{1}} x_{1}, \lambda^{\gamma_{2}} x^{\prime}\right)\right) .
$$

Then,

$$
\left\|\chi_{\lambda}\right\|_{L^{2}\left(\mathbb{R}^{d-1}\right)}=C_{M} \lambda^{-\frac{d-2}{2} \gamma_{2}-\frac{1}{2} \gamma_{1}}, \quad \int_{B(0, \epsilon)} \chi_{\lambda} d x_{1} d x^{\prime} \geq C_{M} \lambda^{-(d-2) \gamma_{2}-\gamma_{1}}
$$

Next, define $\chi_{\lambda, 1} \in L^{2}\left(\Gamma_{1}\right)$ by $\chi_{\lambda, 1}(\gamma(y)):=\chi_{\lambda}(y)$ and $\chi_{\lambda, 2} \in L^{2}\left(\Gamma_{2}\right)$ by $\chi_{\lambda, 2}(\sigma(x)):=\chi_{\lambda}(x)$. Then

$$
\begin{equation*}
\left\|\chi_{\lambda, 1}\right\|_{L^{2}\left(\Gamma_{1}\right)},\left\|\chi_{\lambda, 2}\right\|_{L^{2}\left(\Gamma_{2}\right)} \geq C\left\|\chi_{\lambda}\right\|_{L^{2}\left(\mathbb{R}^{d-1}\right)}, \int_{\Gamma_{1}} \chi_{\lambda, 1}, \int_{\Gamma_{2}} \chi_{\lambda, 2} \geq C \int_{B(0, \epsilon)} \chi_{\lambda} \tag{6.4.5}
\end{equation*}
$$

Moreover, for $x, y \in \operatorname{supp} \chi_{\lambda}$

$$
\begin{gather*}
|\gamma(y)-\sigma(x)|=|\gamma(0)-\sigma(0)|+O\left(M^{-1}\left(\lambda^{-3 \gamma_{1}}+\lambda^{-2 \gamma_{2}}\right)\right), \\
\frac{\left\langle\sigma(x)-\gamma(y), \nu_{y}\right\rangle}{|\gamma(y)-\sigma(x)|}=1+O\left(\lambda^{-\gamma_{1}}+\lambda^{-\gamma_{2}}\right) . \tag{6.4.6}
\end{gather*}
$$

Hence, choosing $\gamma_{1}=1 / 3$ and $\gamma_{2}=1 / 2$ and $M$ large enough, using (6.4.5), (6.4.6) in (6.4.3) we have

$$
\left\|\chi_{\lambda, 2}(x) \tilde{N} \chi_{\lambda, 1}\right\|_{L^{2}\left(\Gamma_{2}\right)} \geq C \lambda^{\frac{d-1}{2}-\frac{d-2}{2}-\frac{1}{3}-\frac{d-2}{4}-\frac{1}{6}}
$$

which implies

$$
\left\|\tilde{N} \chi_{\lambda, 1}\right\|_{L^{2}\left(\Gamma_{2}\right)} \geq C \lambda^{\frac{1}{6}}\left\|\chi_{\lambda, 1}\right\|_{L^{2}\left(\Gamma_{1}\right)}
$$

All that remains to show is that $\Gamma_{2}$ and $\Gamma_{1}$ can be chosen so that they are curved. To see this, let $\tilde{\gamma}$ be a unit speed reparametrization of $t \mapsto\left(t+1,(t+1)^{2}\right)$. (This example is shown in Figure 6.1.) Then, a parametrization of $\Gamma_{1}$ is given by

$$
\left(t, x^{\prime}\right) \mapsto\left(\left(t+1,(t+1)^{2}\right)+\frac{(-2(t+1), 1)}{\sqrt{1+4(t+1)^{2}}}\left|x^{\prime}\right|^{2}, x^{\prime}\right)
$$

and a parametrization of $\Gamma_{2}$ is given by

$$
\left(t, x^{\prime}\right) \mapsto\left(\left(-4(t+1)^{3}, 3(t+1)^{2}+\frac{1}{2}\right)+\frac{(1,2(t+1))}{\sqrt{1+4(t+1)^{2}}}\left|x^{\prime}\right|^{2}, x^{\prime}\right)
$$

Then, a simple calculation verifies that near $(0,0)$ these surfaces are curved. Hence, letting $\Gamma$ be a curved hypersurface containing $\Gamma_{1}$ and $\Gamma_{2}$ completes the proof of the estimate 6.4 .2

### 6.5 Microlocal Description of the Free Resolvent

We have already analyzed the high frequency components of the free resolvent in Lemma 6.3.3. In this section, we analyze the remaining kernel of the free resolvent as a semiclassical intersecting Lagrangian distribution (see Section 4.6). In particular, we prove

Theorem 6.6. Suppose that $a, b>0, M>0$, and $\gamma<1 / 2$. and

$$
z \in[a, b] \times i\left[-C h \log h^{-1}, M h^{1-\gamma}\right]
$$

with $\operatorname{Re} z=E+O\left(h^{1-\gamma}\right)$. Then for $\chi \in C_{c}^{\infty}$, the cut-off free resolvent, $\chi R_{0}(z / h) \chi$, is given by

$$
\chi R_{0}(z / h) \chi=K_{R}+K_{\Delta}+O_{\mathcal{S}^{\prime} \rightarrow C_{c}^{\infty}}\left(h^{\infty}\right),
$$

where $K_{R}$ has kernel $K(x, y) \in h^{3 / 2} e^{\frac{1}{h}(\operatorname{Im} z)-D_{\chi}} I_{\gamma}^{\text {comp }}\left(\mathbb{R}^{d} ; \Lambda_{0}, \Lambda_{1}\right)$ with $D_{\chi}=\operatorname{diam}(\operatorname{supp} \chi)$,

$$
\Lambda_{0}=\left\{(x, \xi, x,-\xi) \in T^{*} \mathbb{R}^{d} \times T^{*} \mathbb{R}^{d}\right\} \text { and } \Lambda_{1}=\left\{\exp _{t} \Lambda_{0} \cap\{|\xi|=E\}: t \geq 0\right\}
$$

and $K_{\Delta} \in h^{2} \Psi_{\gamma}^{-2}$. Moreover, for any $\chi_{1} \in C_{c}^{\infty}$ with $\chi_{1}(\xi) \equiv 1$ on $|\xi| \leq 1$ we can take

$$
\begin{aligned}
& \sigma\left(e^{\frac{\operatorname{Im} z|x-y|}{h}} \chi_{1}(h D) K\right)=\left(\chi_{1}(\xi) \chi^{2}(x) h^{2}\left(|\xi|^{2}-E^{2}\right)^{-1}|d x \wedge d \xi|^{1 / 2},\right. \\
& \left.\quad h^{3 / 2} e^{\frac{i}{h}(\operatorname{Re} z-E)|x-y|} E^{(d-3) / 2} e^{(-d+3) \pi i / 4} 2^{-1 / 2} \pi^{1 / 2}|x-y|^{-(d-1) / 2} \chi(x) \chi(y)|d y \wedge d x|^{1 / 2}\right)
\end{aligned}
$$

and

$$
\sigma\left(K_{\Delta}\right)=\left(1-\chi_{1}(\xi)\right) \chi^{2}(x) h^{2}\left(|\xi|^{2}-E^{2}\right)^{-1}
$$

Proof. We now prove Theorem 6.6. Recall that in the context of Fourier integral operator relations we denote a point in $\bar{T}^{*} M \times \bar{T}^{*} M^{\prime}$ by $(x, \xi, y, \eta)$. By Lemma 4.6.12, for $C h^{1-\gamma} \geq$ $\operatorname{Im} z \geq 0,|\operatorname{Re} z-E| \leq C h^{1-\gamma}$, and each $M>0$ there exists an operator $U$ that is $z / h$ outgoing with kernel $K(x, y, z / h) \in h^{\frac{3}{2}} I_{\gamma}^{\text {comp }}\left(\mathbb{R}^{d} ; \Lambda_{0}, \Lambda_{1}\right)+h^{2} \Psi_{\gamma}^{-2}$ where

$$
\begin{gathered}
\Lambda_{0}:=\left\{(x, \xi, x,-\xi): x \in \mathbb{R}^{d}, \xi \in T_{x}^{*}\left(\mathbb{R}^{d}\right)\right\} \\
\Lambda_{1}:=\left\{\left(\exp _{t}(x, \xi), x,-\xi\right): x \in \mathbb{R}^{d},|\xi|=E, t \geq 0\right\}
\end{gathered}
$$

such that for all $\chi, \chi_{2} \in C_{c}^{\infty}(B(0, M))$ with $\chi_{2} \equiv 1$ on supp $\chi$,

$$
\begin{gathered}
\chi_{2}\left(-\Delta-(z / h)^{2}\right) U \chi=\chi+O_{\mathcal{D}^{\prime} \rightarrow C_{c}^{\infty}}\left(h^{\infty}\right) \\
\left(-\Delta-(z / h)^{2}\right) \chi_{2} U \chi=\chi+\left[-\Delta, \chi_{2}\right] U \chi+O_{\mathcal{D}^{\prime} \rightarrow C_{c}^{\infty}}\left(h^{\infty}\right) .
\end{gathered}
$$

Hence, since for $\operatorname{Im} z>0, R_{0}(z / h)=O_{H^{s} \rightarrow H^{s+2}}\left(h^{-1}\right)$ and $\mathrm{WF}_{\mathrm{h}}\left(R_{0}\right) \subset \Lambda_{0} \cup \Lambda_{1}$, we have for $\operatorname{Im} z \geq 0$ and $\chi \in C_{c}^{\infty}$ with $\operatorname{supp} \chi \subset B(0, M)$,

$$
\chi U(z / h) \chi=\chi R_{0}(z / h) \chi+O_{\mathcal{D}^{\prime} \rightarrow C_{c}^{\infty}}\left(h^{\infty}\right)
$$

In order to prove Theorem 6.6 and analyze $G$, we need to compute the symbol of $\chi U \chi$. First, define $P:=-\Delta-z^{2} / h^{2}=\mathrm{Op}_{\mathrm{h}, 1 / 2}\left(h^{-2}\left(|\xi|^{2}-z^{2}\right)\right)$. Then, for any $\delta>0, P$ has principal symbol

$$
p:=\sigma(P)=h^{-2}\left(|\xi|^{2}-E^{2}\right)
$$

and sub-principal symbol

$$
\sigma_{1}(P):=h^{-2} 2(E-z):-h^{-2} 2 \omega_{0}
$$

as an operator in $\Psi_{\delta}^{2}$.
Then, by Lemma 4.6 .12 we have that

$$
r_{0}(x, \xi, x,-\xi):=\left.\sigma(U)\right|_{\Lambda_{0} \cap T^{*} B(0, R)}=p^{-1} \sigma(\delta)|d x \wedge d \xi|^{1 / 2}=h^{2}\left(|\xi|^{2}-E^{2}\right)^{-1}|d x \wedge d \xi|^{1 / 2}
$$

Remark: Moreover, we see that in any coordinates each term in the full symbol of $\left.U\right|_{\Lambda_{0}}$ has the form

$$
a(x, \xi)=\frac{\sum_{|\alpha|=0}^{m} a_{\alpha}(x) \xi^{\alpha}}{|\xi|^{2}-E^{2}}
$$

where $a(x) \in C^{\infty}$
Next, we compute $r_{1}=\left.\sigma\left(R_{0}\right)\right|_{\Lambda_{1} \cap T^{*} B(0, R) \times T^{*} B(0, R)}$. Again, by Lemma 4.6.12, we need to solve

$$
\left\{\begin{array}{l}
h H_{p} r_{1}+i p_{1} r_{1}=0 \\
\left.r_{1}\right|_{\partial \Lambda_{1}}=e^{\pi i / 4}(2 \pi)^{1 / 2} h^{-1 / 2} R\left(r_{0}\right)
\end{array}\right.
$$

where $H_{p}$ is the Hamiltonian flow of $p$. Using that $\exp \left(t H_{p}\right)(x, \xi)=\exp _{2 t h^{-2}}(x, \xi)$, we have

$$
r_{1}\left(\exp _{t}(x, \xi), x,-\xi\right)=e^{i \omega_{0} t E / h} e^{\pi i / 4}(2 \pi)^{1 / 2} h^{-1 / 2} R\left(r_{0}\right)(x, \xi, x,-\xi)
$$

So, all that remains is to determine $R\left(r_{0}\right)(x, \xi, x,-\xi)$. But, taking $g=|\xi|^{2}-E^{2}$ and $f=$ $\langle x-y, \xi\rangle$, gives

$$
r_{0}=h^{2}\left(\left|2 \xi_{d}\right|^{1 / 2} g\right)^{-1}\left|d x \wedge d \xi^{\prime} \wedge d g\right|^{1 / 2}
$$

Hence,
$R r_{0}=\frac{h^{2}}{\left|2 \xi_{d}\right|^{1 / 2}}\{g, f\}^{-1 / 2}\left|d x \wedge d \xi^{\prime} \wedge d f\right|^{1 / 2}=\frac{h^{2}}{2\left|\xi_{d}\right|^{1 / 2}|\xi|}\left|d x \wedge d \xi^{\prime} \wedge(\xi d x-\xi d y+(x-y) d \xi)\right|^{1 / 2}$.
Now, parametrizing of $\Lambda_{1}$ near $\xi=(0, \ldots, 0, E)$ by $\left(y, \xi^{\prime}, t\right)$ using the map

$$
\Gamma\left(y, \xi^{\prime}, t\right)=\left(y+t\left(\xi^{\prime}, \sqrt{E^{2}-\left|\xi^{\prime}\right|^{2}}\right),\left(\xi^{\prime}, \sqrt{E^{2}-\left|\xi^{\prime}\right|^{2}}\right), y,-\left(\xi^{\prime}, \sqrt{E^{2}-\left|\xi^{\prime}\right|^{2}}\right)\right)
$$

gives

$$
d x \wedge d \xi^{\prime} \wedge d y_{i}=\xi_{i} d y \wedge d \xi^{\prime} \wedge d t
$$

Hence, using $\frac{E}{\sqrt{E^{2}-\left|\xi^{\prime}\right|^{2}}} d \xi^{\prime} \wedge d t=d \mu_{S_{E}^{d-1}}(\xi) \wedge d t$,

$$
R\left(r_{0}\right)\left(y, \xi^{\prime}, t\right)=\frac{h^{2}}{2\left(E^{2}-\left|\xi^{\prime}\right|^{2}\right)^{1 / 4}}\left|d y \wedge d \xi^{\prime} \wedge d t\right|^{1 / 2}=\frac{h^{2}}{2}\left|E^{-1} d y \wedge d \mu_{S_{E}^{d-1}}(\xi) \wedge d t\right|^{1 / 2}
$$

Thus,

$$
r_{1}(y, \theta, t)=\frac{1}{2} e^{\frac{i}{h} \omega_{0} t E} e^{\pi i / 4}(2 \pi)^{1 / 2} h^{3 / 2}\left|E^{-1} d y \wedge d \mu_{S_{E}^{d-1}}(\xi) \wedge d t\right|^{1 / 2}
$$

and parametrizing $\Lambda_{1}$ by $(y, x)\left(\right.$ instead of $\left.\left(y, \xi^{\prime}, t\right)\right)$ for $y \neq x$ gives

$$
r_{1}(x, y)=\frac{E^{(d-3) / 2}}{2|x-y|^{(d-1) / 2}} e^{\frac{i}{h} \omega_{0}|x-y|} e^{(-d+3) \pi i / 4}(2 \pi)^{1 / 2} h^{3 / 2}|d y \wedge d x|^{1 / 2}
$$

Here, the extra $e^{-\pi(d-2) i / 4}$ results from reparametrizing by $x$ instead of $\xi^{\prime}, t$.
Now, taking $\chi \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$, we have that $R_{\chi}:=\chi R_{0} \chi: L^{2} \rightarrow L^{2}$ continues meromorphically to $\mathbb{C}$ for $d$ odd and to the logarithmic covering space of $\mathbb{C} \backslash\{0\}$ for $d$ even. We show that for $-C h \log h^{-1} \leq \operatorname{Im} z$,

$$
R_{\chi} \in h^{3 / 2} e^{(\operatorname{Im} z)-D_{\chi} / h} I_{\gamma}^{\text {comp }}\left(\mathbb{R}^{d} ; \Lambda_{0}, \Lambda_{1}\right)+h^{2} \Psi^{-2}+O_{\mathcal{D}^{\prime} \rightarrow C_{c}^{\infty}}\left(h^{\infty}\right)
$$

Moreover, we show that the principal symbol of $R_{\chi}$ is the analytic continuation of that for $\operatorname{Im} z \geq 0$.

To do this, we need the following analog of the three line lemma and semiclassical maximum principle (74, Lemma 4.2], 70, Lemma 5.1]).
Lemma 6.5.1. Suppose that $f(z, h)$ is analytic in

$$
D(h):=[E-5 w(h), E+5 w(h)]+i[-\alpha(h), \alpha(h)] .
$$

Let $S(h)=w(h) \alpha^{-1}$. Assume that $S(h) \rightarrow \infty$ and suppose that $|f(z, h)| \leq M_{0}(h)$ on

$$
[E-5 w(h), E+5 w(h)]+i \alpha(h)
$$

and $|f(z, h)| \leq M_{1}(h)$ on $D(h)$ with $\log \max \left(M_{0}(h), M_{1}(h)\right)=o\left(S(h)^{2}\right)$. Then, $|f(z, h)| \leq$ $C M_{0}^{\frac{\operatorname{If} z+\alpha}{2 \alpha}} M_{1}^{\frac{\alpha-\operatorname{II} z}{2 \alpha}}$ for

$$
z \in[E-2 w(h), E+2 w(h)]+i[-\alpha(h), \alpha(h)]=: \tilde{D}(h) .
$$

Proof. We follow the proof of [74, Lemma 4.2]. First, define

$$
g(z, h)=\left(\pi \alpha^{2}\right)^{-1 / 2} \int e^{-\frac{(x-z)^{2}}{\alpha^{2}}} \psi_{h}(x) d x \quad \text { where } \quad \psi_{h}(x)= \begin{cases}0 & |x-E| \geq 3 w(h) \\ 1 & |x-E| \leq 2 w(h)\end{cases}
$$

Then $|g(z, h)|$ is holomorphic in $D(h),|g(z, h)| \geq C$ in $\tilde{D}(h),|g(z, h)| \leq C$ in $D(h)$, and

$$
|g(z, h)| \leq C e^{-C S(h)^{2}} \text { on } D(h) \cap|\operatorname{Re} z-E| \geq 4 w(h) .
$$

Let

$$
F(z, h):=g(z, h) f(z, h) M_{0}^{-\frac{i(z-i \alpha)}{2 \alpha}} M_{1}^{\frac{i(z+i \alpha)}{2 \alpha}} .
$$

Then $|F(z, h)| \leq 1$ on $\partial D$ by our assumptions. By the maximum principle $|F(z, h)| \leq 1$ on $D$. Together with the properties of $g(z, h)$, this gives the result.

Since for $\operatorname{Im} z \geq 0$, we have that $\chi\left(R_{0}-U\right) \chi=O_{\mathcal{D}^{\prime} \rightarrow C_{c}^{\infty}}\left(h^{\infty}\right)$, in order to apply Lemma 6.5.1 to our situation, we need to bound $\chi\left(R_{0}-U\right) \chi$ for $\operatorname{Im} z \leq 0$. In particular, we show that for $\operatorname{Im} z \leq 0$, there exists $N>0$ such that

$$
U=O_{H^{s} \rightarrow H^{s+2}}\left(h^{-N} e^{\frac{1}{h} D_{\chi}(\operatorname{Im} z)_{-}}\right) .
$$

Let $\psi \in C_{c}^{\infty}(\mathbb{R})$ have $\psi \equiv 1$ on $|s|<2$. Then by Lemmas 4.6.3 and 6.3.3.

$$
\chi\left(R_{0}-U\right) \chi(1-\psi(|h D|)) \in h^{\infty} \Psi_{\gamma}^{-2}=O\left(h^{\infty}\right)_{\mathcal{D}^{\prime} \rightarrow C_{c}^{\infty}} .
$$

Now, $\chi \psi(|h D|) U \chi \in h^{3 / 2} e^{1 / h D_{\chi}(\operatorname{Im} z)-} I_{\gamma}^{\text {comp }}$. Thus, we see from the definition of an intersecting Lagrangian distribution (Definition 4.6.6) that there exists $N>0$ such that

$$
\chi \psi(|h D|) U \chi=O_{\mathcal{D}^{\prime} \rightarrow C_{c}^{\infty}}\left(h^{-N} e^{\frac{1}{h} D_{\chi}(\operatorname{Im} z)_{-}}\right) .
$$

This together with standard bounds on the free resolvent (see for example [11, Theorem 1.2], [21, Chapter 3]) gives that

$$
\chi\left(R_{0}-U\right) \chi=O_{\mathcal{D}^{\prime} \rightarrow C_{c}^{\infty}}\left(h^{-N} e^{\frac{1}{h} D_{\chi}(\operatorname{Im} z)_{-}}\right)
$$

By Lemma 6.5.1 with

$$
\alpha(h)=C h \log h^{-1}, \quad w(h)=h^{1-\gamma}, \quad M_{0}=O\left(h^{\infty}\right), \quad M_{1}=h^{-N} e^{D_{\chi}(\operatorname{Im} z)-/ h}
$$

we have that for $|\operatorname{Im} z| \leq C h \log h^{-1}$

$$
\begin{equation*}
\chi\left(R_{0}-U\right) \chi=O_{\mathcal{D}^{\prime} \rightarrow C_{c}^{\infty}}\left(h^{\infty} e^{D_{\chi}(\operatorname{Im} z)-/ h}\right)=O_{\mathcal{D}^{\prime} \rightarrow C_{c}^{\infty}}\left(h^{\infty}\right) \tag{6.5.1}
\end{equation*}
$$

### 6.6 Microlocal Decomposition of $G$ and $\partial_{\nu} \mathcal{D} \ell$ Away from Glancing

## Decomposition of $G$

Recall that $G(z)=\gamma R_{0}(z / h) \gamma^{*}$ where $\gamma$ denotes restriction to $\partial \Omega$ and $R_{0}(\lambda)$ denotes the free outgoing resolvent of $-\Delta-\lambda^{2}$. We have that $\gamma$ is a semiclassical Fourier integral operator of order 0 associated to the relation $C \subset T^{*} \partial \Omega \times T^{*} \mathbb{R}^{d}$ given by

$$
C=\left\{\left(x, \xi^{\prime}, x, \xi\right): x \in \partial \Omega, \xi \in T_{x}^{*} \mathbb{R}^{d}, \xi=\xi_{\nu}+\xi^{\prime} \text { with } \xi_{\nu} \in N_{x}^{*}(\partial \Omega), \xi^{\prime} \in T_{x}^{*} \partial \Omega\right\} .
$$

Thus, $\gamma^{*}$ is a Fourier integral operator of order $1 / 4$ associated to the relation $C^{-1} \subset T^{*} \mathbb{R}^{d} \times$ $T^{*} \partial \Omega$ given by

$$
C^{-1}=\left\{\left(x, \xi, x, \xi^{\prime}\right): x \in \partial \Omega, \xi \in T_{x}^{*} \mathbb{R}^{d}, \xi=\xi_{\nu}+\xi^{\prime} \text { with } \xi_{\nu} \in N_{x}^{*}(\partial \Omega), \xi^{\prime} \in T_{x}^{*} \partial \Omega\right\} .
$$

Then $\gamma \in h^{-1 / 4}\left(I^{0}(C)\right)$ and $\gamma^{*} \in h^{-1 / 4}\left(I^{0}\left(C^{-1}\right)\right)$ have symbols given by

$$
\left.\sigma\left(\gamma^{*}\right)\right|_{C^{-1}}=(2 \pi h)^{-1 / 4}|d y \wedge d \xi|^{1 / 2}, \quad \text { and }\left.\quad \sigma(\gamma)\right|_{C}=(2 \pi h)^{-1 / 4}|d x \wedge d \eta|^{1 / 2}
$$

When $\Omega$ is convex, we decompose $G(z)$ into three parts: $G_{\Delta}, G_{B}$, and $G_{g} . G_{\Delta}$ is a mildly exotic pseudodifferential operator of order $-1 . G_{B}$ is a semiclassical FIO of order -1 associated to the billiard ball map. $G_{g}$ is an operator $G_{g}$ microsupported in an $h^{\epsilon}$ neighborhood of $S^{*} \partial \Omega \times S^{*} \partial \Omega$ intersected with the diagonal of $T^{*} \partial \Omega \times T^{*} \partial \Omega$.

We now decompose $G$ as claimed above. We begin by showing that the compositions $C \circ\left(\Lambda_{i}\right)^{\prime} \circ C^{-1}$ are clean away from the diagonal or away from $S^{*} \partial \Omega$. First, consider $C \circ\left(\Lambda_{0}\right)^{\prime} \circ C^{-1}$. We need only work locally, so we assume that $\partial \Omega=\left\{\left(x^{\prime}, \Gamma\left(x^{\prime}\right): x^{\prime} \in U\right\}\right.$. Then,

$$
\begin{aligned}
& C=\left\{\left(x^{\prime}, \xi,\left(x^{\prime}, \Gamma\left(x^{\prime}\right)\right),\left(\xi, \nabla \Gamma\left(x^{\prime}\right) \cdot \xi\right)+\tau\left(-\nabla \Gamma\left(x^{\prime}\right), 1\right)\right): \tau \in \mathbb{R}, x^{\prime} \in U\right\} \\
& T C=\left\{\left(\delta_{x^{\prime}}, \delta_{\xi},\left(\delta_{x^{\prime}}, \nabla \Gamma\left(x^{\prime}\right) \cdot \delta_{x^{\prime}}\right)\right.\right. \\
&\left.\left.\left(\delta_{\xi}, \nabla \Gamma\left(x^{\prime}\right) \cdot \delta_{\xi}\right)+\left(0, \delta_{x^{\prime}} \cdot \partial^{2} \Gamma \xi^{\prime}\right)+\tau\left(-\partial^{2} \Gamma\left(x^{\prime}\right) \delta_{x^{\prime}}, 0\right)+\delta_{\tau}\left(-\nabla \Gamma\left(x^{\prime}\right), 1\right)\right)\right\}
\end{aligned}
$$

and $C^{-1}$ and $T C^{-1}$ are obtained by reversing the roles of $T^{*} \partial \Omega$ and $T^{*} \mathbb{R}^{d}$. Then, it is easy to check that $C \circ \Lambda_{0}$ is clean (indeed, even transverse) and given by $C \circ \Lambda_{0}=C$. Now, without loss of generality, we can assume that $\nabla \Gamma\left(y^{\prime}\right)=\Gamma\left(y^{\prime}\right) 0$. so

$$
\begin{gathered}
A:=\left(C \times C^{-1}\right) \cap\left(T^{*} \partial \Omega \times \Delta\left(T^{*} \mathbb{R}^{d}\right) \times T^{*} \partial \Omega\right) \\
=\left\{\left(y^{\prime}, \eta,\left(y^{\prime}, 0\right),(\eta, 0)+\tau(0,1), y^{\prime}, \eta\right)\right\} \\
T A=\left\{\left(\delta_{y^{\prime}}, \delta_{\eta},\left(\delta_{y^{\prime}}, 0\right),\left(\left(\delta_{\eta}, 0\right)+\left(0, \delta_{y^{\prime}} \partial^{2} \Gamma \eta\right)+\delta_{\tau}(0,1)+\tau\left(-\partial^{2} \Gamma\left(x^{\prime}\right) \delta_{x^{\prime}}, 0\right)\right)\right\}\right.
\end{gathered}
$$

Remark: Since we intersect with the diagonal in these formulae, we have suppresed one of the pairs in $T^{*} \mathbb{R}^{d}$.
On the other hand $B=T C \times T C^{-1} \cap\left(T\left(T^{*} \partial \Omega \times \Delta\left(T^{*} \mathbb{R}^{d}\right) \times T^{*} \partial \Omega\right)\right)$ at $\left(y^{\prime}, \eta,\left(y^{\prime}, 0\right),(\eta, 0)+\right.$ $\left.\sigma(0,1), y^{\prime}, \eta\right)$ is given by

$$
B=\left\{\left(\delta_{x^{\prime}}, \delta_{\xi},\left(\delta_{y^{\prime}}, 0\right),\left(\delta_{\eta}, 0\right)+\left(0, \delta_{y} \partial^{2} \Gamma \eta\right)+\sigma\left(-\partial^{2} \Gamma\left(y^{\prime}\right) \delta_{y^{\prime}}, 0\right)+\delta_{\sigma}(0,1) \cdot \delta_{y^{\prime}}, \delta_{\eta}\right)\right\}
$$

where

$$
\begin{aligned}
\left(\delta_{\eta}, 0\right)+\left(0, \delta_{y^{\prime}} \partial^{2} \Gamma \eta\right)+\sigma\left(-\partial^{2} \Gamma\left(y^{\prime}\right) \delta_{y^{\prime}}\right. & , 0)+\delta_{\sigma}(0,1) \\
& =\left(\delta_{\xi}, 0\right)+\left(0, \delta_{x^{\prime}} \partial^{2} \Gamma \eta\right)+\sigma\left(-\partial^{2} \Gamma\left(y^{\prime}\right) \delta_{x^{\prime}}, 0\right)+\delta_{\tau}(0,1)
\end{aligned}
$$

and $\delta_{y^{\prime}}=\delta_{x^{\prime}}$. But, since $(0,1)$ is linearly independent from $T_{y} \partial \Omega$, this implies that $\delta_{\eta}=\delta_{\xi}$, $\delta_{\tau}=\delta_{\sigma}$ and hence the composition is clean.

Now, recall that

$$
\Lambda_{1}=\left\{(x+t \xi, \xi, x, \xi): \xi \in S^{d-1}, t \geq 0\right\}
$$

we consider or $\xi \notin T^{*} \partial \Omega$ Thus,

$$
T \Lambda_{1}=\left\{\left(\delta_{x}+t \delta_{\xi}+\delta_{t} \xi, \delta_{\xi}, \delta_{x}, \delta_{\xi}\right): \delta_{\xi} \in T_{\xi} S^{d-1}\right\}
$$

To see that $T\left(\Lambda_{1} \times C^{-1}\right) \cap T\left(T^{*} \mathbb{R}^{d} \times \Delta\left(T^{*} \mathbb{R}^{d}\right) \times T^{*} \partial \Omega\right)$ is transverse at $\left(\left(y^{\prime}, \Gamma\left(y^{\prime}\right)\right)+\right.$ $\left.t \xi, \xi,\left(y^{\prime}, \Gamma\left(y^{\prime}\right)\right), \xi, y^{\prime}, \eta\right)$ where $\xi-\eta \in N_{y^{\prime}}^{*} \partial \Omega$, we choose $\delta_{z}=\alpha\left(-\nabla \Gamma\left(y^{\prime}\right), 1\right)$ and $\delta_{\zeta}=\beta \xi$. Then for any $v \in \mathbb{R}^{d}, v=\beta \xi+\delta_{\xi}$ for some $\beta \in \mathbb{R}$ and $\delta_{\xi} \in T_{\xi} S^{d-1}$. Moreover, any $w \in \mathbb{R}^{d}$ can be written $w=\delta_{y^{\prime}}+\alpha\left(-\nabla \Gamma\left(y^{\prime}\right), 1\right)$ for some $\delta_{y^{\prime}} \in T_{y^{\prime}} \partial \Omega$ and $\alpha \in \mathbb{R}$ Thus,

$$
T\left(\Lambda_{1} \times C^{-1}\right)+T\left(T^{*} \mathbb{R}^{d} \times \Delta\left(T^{*} \mathbb{R}^{d}\right) \times T * \partial \Omega\right)=T\left(T^{*} \mathbb{R}^{d} \times T^{*} \mathbb{R}^{d} \times T^{*} \mathbb{R}^{d} \times T^{*} \partial \Omega\right.
$$

and the composition is transverse. Now

$$
\begin{gathered}
\Lambda_{1} \circ C^{-1}=\left\{\left(\left(y^{\prime}, \Gamma\left(y^{\prime}\right)\right)+t \xi, \xi, y^{\prime}, \eta\right): t \geq 0 \xi \in S^{d-1}, \xi-\eta \in N_{y^{\prime}}^{*} \partial \Omega\right\} \\
T\left(\Lambda_{1} \circ C^{-1}\right)=\left\{\left(\left(\delta_{y^{\prime}}, \nabla \Gamma\left(y^{\prime}\right)\right)+\delta_{t} \xi+t \delta_{\xi}, \delta_{\xi}, \delta_{y^{\prime}}, \delta_{\eta} \delta_{\eta}=d \pi \delta_{\xi}, \delta_{\xi} \in T_{\xi} S^{d-1}\right\}\right.
\end{gathered}
$$

Now, if $t>0$, it is clear that any vector $w \in \mathbb{R}^{d}$ can be written $w=\delta_{t} \xi+t \delta_{\xi}$. On the other hand, if $t=0$, but $\xi \notin T_{y}^{*} \partial \Omega$, then we have that $w$ can be written as

$$
w=\left(\delta_{y^{\prime}}, \nabla \Gamma\left(y^{\prime}\right) \cdot \delta_{y^{\prime}}\right)+\delta_{t} \xi
$$

Moreover, parametrizing $\partial \Omega$ near a point $x$ in the intersection with $C$ by $\left(x^{\prime}, \Gamma_{1}\left(x^{\prime}\right)\right), w$ can be written

$$
w=\delta_{\zeta}+\tau\left(-\partial^{2} \Gamma_{1} \delta_{x^{\prime}}, 0\right)+\delta_{\tau}\left(-\nabla \Gamma_{1}\left(x^{\prime}\right), 1\right)
$$

for $\delta_{\zeta} \in T T_{y}^{*} \partial \Omega$. So, an identical analysis to that for the composition on the right by $C^{-1}$ gives that $C \circ \Lambda_{1} \circ C^{-1}$ is transverse away from the diagonal as well as at the diagonal, but away from $T^{*} \partial \Omega$.

Since $\partial \Omega \Subset \mathbb{R}^{d}$, we may take $\chi \equiv 1$ on $\Omega$ in Theorem 6.6. Then by composing relations, using Lemma 4.6.7, and observing that the composition is transverse, we see that for $-3 / 2<$ $s$,

$$
\begin{equation*}
R_{\chi} \gamma^{*} \in h^{5 / 4} I_{\gamma}^{\text {comp }}\left(\mathbb{R}^{d} \times \partial \Omega ; \Lambda_{0} \circ C^{-1}, \Lambda_{1} \circ C^{-1}\right)+h^{7 / 4} I_{\gamma}^{-2}\left(\Lambda_{0} \circ C^{-1}\right)+O\left(h^{\infty}\right)_{H^{s}(\partial \Omega) \rightarrow C^{\infty}\left(\mathbb{R}^{d}\right)} \tag{6.6.1}
\end{equation*}
$$

Remark: This implies that the single layer potential has the above decomposition.
We have that $C$ composes on the left with $\Lambda_{1} \circ C^{-1}$ transversally. However, $C$ composes on the left with $\Lambda_{0} \circ C^{-1}$ only cleanly. Thus, we cannot apply Lemma 4.6.7 in this case to obtain $\gamma R_{\chi} \gamma^{*}=\gamma R_{0} \gamma^{*}$. Note also that Proposition4.4.10 does not apply directly to the composition forming $C \circ \Lambda_{0} \circ C^{-1}$ since $(C)_{\mathbb{R}^{d}} \cap\left(\Lambda_{0} \circ C^{-1}\right)_{\mathbb{R}^{d}}$ is nonempty. Instead we microlocalize away from the intersection of the two Lagrangians and use the following lemma combined with more detailed analysis near fiber infinity.

Lemma 6.6.1. Suppose that $\partial \Omega$ is smooth and $A \in I^{\text {comp }}\left(\mathbb{R}^{d} \times \partial \Omega ; \Lambda_{0} \circ C^{-1}, \Lambda_{1} \circ C^{-1}\right)$. Then

$$
\gamma A=A_{1}+A_{2}+R \in h^{-1 / 4-\delta / 2} I_{\delta}^{\text {comp }}\left(C \circ \Lambda_{1} \circ C^{-1}\right)+h^{-1 / 4-\delta / 2} I_{\delta}^{\text {comp }}\left(C \circ \Lambda_{0} \circ C^{-1}\right)+R
$$

where $R$ is microlocalized on an $h^{\delta}$ neighborhood of the intersection of $S_{E}^{*} \partial \Omega \times S_{E}^{*} \partial \Omega$ with the diagonal. Moreover, the symbol $A_{2}$ can be computed using Proposition 4.4.10 in the sense that

$$
\sigma\left(A_{2}\right)=\left.(2 \pi h)^{-3 / 4} \int \sigma\left(A \psi\left(|h D|^{\prime}\right)\right)\right|_{\Lambda_{0} \circ C^{-1}}\left|d\left\langle\nu_{x}, \xi\right\rangle\right|^{1 / 2}
$$

where $\psi$ is supported $h^{\delta}$ away from $\left|\xi^{\prime}\right|=E$ and the integral is interpreted as a distributional pairing.

Proof. By lemma 4.6.3, we need only consider an $h^{\delta}$ neighborhood of the diagonal intersected with $\left.S^{*} \mathbb{R}^{d}\right|_{\partial \Omega} \times\left. S^{*} \mathbb{R}^{d}\right|_{\partial \Omega}$. Let $\chi \in C_{c}^{\infty}(\mathbb{R})$ with $\chi \equiv 1$ near 0 . Let

$$
\begin{gathered}
A_{1}(x, y)=\chi\left(|x-y| / h^{\gamma}\right) \chi\left(|x-y| / h^{\delta}\right) A(x, y) \\
A_{2}(x, y)=\left(1-\chi\left(|x-y| / h^{\gamma}\right)\right) \chi\left(|x-y| / h^{\delta}\right) A(x, y)
\end{gathered}
$$

where $A(x, y)$ is the kernel of $A$. Then, we can write for $B \in \Psi_{\delta}(\partial \Omega)$

$$
B \gamma A_{2}(x, y)=C h^{-M} \int_{\mathbb{R}^{d-1} \times \mathbb{R}^{d}} e^{\langle x-w, \eta\rangle+\frac{i E}{h}|w-y|}\left(1-\chi\left(|w-y| / h^{\gamma}\right)\right) b(x, \eta) a(w, y) d w d \eta
$$

But, since $d_{w}|w-y| \rightarrow 1$ as $w \rightarrow y$, we have that the phase is nonstationary with gradient bounded below by $c h^{\delta}$ if $b$ is supported $h^{\delta}$ away from $|\eta|=E$. Hence, integrating by parts we
lose at most $h^{\gamma}$ and gain $h^{1-\delta}$, so when $\gamma<1-\delta$, we obtain a kernel in $O_{C^{\infty}}\left(h^{\infty}\right)$. Similarly, we have the same result for $A_{2} B$.

Next, consider $A_{1}$. Then, let $B$ be microlocalized $h^{\delta}$ away from $|\eta|=E$, the kernel of $B A_{1}$ can be written

$$
B \gamma A_{1}(x, y)=(2 \pi h)^{-d-1 / 4} \int_{0}^{\infty} \int e^{\frac{i}{h}\left(\left\langle x-y, \eta+\xi_{\nu} \nu_{x}\right\rangle-\frac{t}{2}\left(|\eta|^{2}+\xi_{\nu}^{2}-E\right)\right)} b(x, \eta) a(x, \xi) d \xi d \eta d t
$$

Then, using Lemma 4.6 .5 evaluating the $t$ integral as a distribution, we have
$\gamma A_{1}(x, y)=-2 i(2 \pi h)^{-d+3 / 4} \int e^{\frac{i}{h}\left(\left\langle x-y, \eta+\xi_{\nu} \nu_{x}\right\rangle\right)} b(x, \eta) a\left(x, \eta+\xi_{\nu} \nu_{x}\right)\left(|\eta|^{2}+\xi_{\nu}^{2}-E-i 0\right)^{-1} d \xi_{\nu} d \eta$
$\eta \in T_{x} \partial \Omega$ and $\nu_{x}$ is the unit normal to $\partial \Omega$ at $x$. Note that since $||\eta|-E| \geq c h^{\delta}\left(\xi_{\nu}^{2}+|\eta|^{2}-E-\right.$ $i 0)^{-1} \in h^{-\delta / 2} \mathcal{S}^{\prime}$ as a distribution in $\xi_{\nu}$. We are working in a small neighborhood of $|\xi|=E$, so we can assume that the integrand is compactly supported in $\xi_{\nu}$. Now, $\left\langle x-y, \nu_{y}\right\rangle=O\left(|x-y|^{2}\right)$ and $|x-y|=O\left(h^{\gamma}\right)$ with $\gamma>1 / 2$. So, we obtain an accurate representation using the Taylor expansion of $e^{\frac{i}{h}\left\langle x-y, \nu_{x} \xi_{\nu}\right\rangle}$. Then, a typical term is of the form

$$
-2 i(2 \pi h)^{-d+3 / 4} \int e^{\frac{i}{h}(\langle x-y, \eta\rangle)} \frac{\left(\left\langle x-y, \nu_{x}\right\rangle \xi_{\nu}\right)^{j}}{h^{j} j!} b(x, \eta) a\left(x, \eta+\xi_{\nu} \nu_{x}\right)\left(|\eta|^{2}+\xi_{\nu}^{2}-E-i 0\right) d \xi_{\nu} d \eta
$$

So, integrating by parts $2 j$ times in $\eta$, we gain $h^{2 j}$. Integrating in $\xi_{\nu}$ gives the result.
Now, let $\psi \in C_{c}^{\infty}(\mathbb{R})$ with $\psi \equiv 1$ near 0 and let $0 \leq \epsilon<1 / 2$. Then, writing $R_{\chi}(x, y)$ for the kernel of $R_{\chi}$, define

$$
R_{\chi}(x, y)=R_{\chi}(x, y)\left(1-\psi\left(h^{-\epsilon}|x-y|\right)\right)+R_{\chi}(x, y) \psi\left(h^{-\epsilon}(|x-y|)\right)=: R_{1}(x, y)+R_{2}(x, y) .
$$

Then, recalling that $G:=\gamma R_{0} \gamma^{*}$,
$G=\gamma R_{1} \gamma^{*}+\gamma R_{2} \gamma^{*}\left(1-\psi\left(h^{-\epsilon}\left(\left|h D^{\prime}\right|_{g}-E\right)\right)\right)+\gamma R_{2} \gamma^{*} \psi\left(h^{-\epsilon}\left(\left|h D^{\prime}\right|_{g}-E\right)\right)=: G_{B}+G_{\Delta}+G_{g}$
We will see that in spite of the difficulty at fiber infinity, $G_{\Delta}$ is still a pseudodifferential operator. As in Section 6.1, to interpret $G_{\Delta}$ appropriately, we must view $\gamma$ as one of two objects, $\gamma^{+}$for the limit from inside $\Omega$ and $\gamma^{-}$for that from outside $\Omega$. In Lemma 6.1.1 we saw that $G$ is independent of the choice of $\gamma^{ \pm}$, so we choose $\gamma^{+}$.

Lemma 6.6.2. Suppose $A \in \Psi_{\delta}^{m}\left(\mathbb{R}^{d}\right)$. Choose coordinates so that $\partial \Omega=\left\{x_{d}=0\right\}$ and let a have $A=\mathrm{Op}_{\mathrm{h}, 0}(a)$ for $a \in S_{\delta, c l}^{m}\left(T^{*} \mathbb{R}^{d}\right)$. Suppose further that

$$
a(y, \xi) \sim \sum_{j=-\infty}^{m} C_{j, \pm}\left(y, \xi^{\prime}\right)\left|\xi_{d}\right|^{j} \quad \xi_{d} \rightarrow \pm \infty
$$

and for $j>-2, C_{j,+}=(-1)^{j} C_{j,-}$. Then, $\gamma^{ \pm} A \gamma^{*} \in h^{-1} \Psi_{\delta}^{m+1}(\partial \Omega)$.

Moreover, for such operators A, the symbol calculus contained in Proposition 4.4.10 applies in the sense that

$$
\sigma\left(\gamma^{ \pm} A \gamma^{*}\right)=(2 \pi h)^{-1} \int \sigma(A)\left(x, \xi^{\prime}-\nu_{x} \xi_{d}\right) d \xi_{d}
$$

where $\nu_{x}$ is the outward unit normal to $\partial \Omega$ and the integral is interpreted as the sum of residues in $\pm \operatorname{Im} \xi_{d}>0$ if $\sigma(A)$ is not integrable.

Proof. Let

$$
q(y, \xi)=\frac{\sum_{j=-1}^{m} C_{j,+}\left(y, \xi^{\prime}\right) \xi_{d}^{j+2}}{|\xi|^{2}+1}
$$

and $r=a-q$. Then, $r \in S_{\delta}^{\min (-2, m)}$. Now, $A \gamma^{*}$ has kernel

$$
(2 \pi h)^{-d} \iint e^{\frac{i}{h}\left(x_{d} \xi_{d}+\left\langle x^{\prime}-y^{\prime}, \xi^{\prime}\right\rangle\right)} q\left(y^{\prime}, \xi\right)+r\left(y^{\prime}, \xi\right) d \xi_{d} d \xi^{\prime}
$$

The integral in involving $r$ in $\xi_{d}$ is well defined at $x_{d}=0$ since $|r| \leq C\left|\xi_{d}\right|^{-2}$ for $\left|\xi_{d}\right|$ large. Moreover, since for $m<-1$,

$$
\int\langle\xi\rangle^{m} d \xi_{d} \leq\left\langle\xi^{\prime}\right\rangle^{m} \int\left\langle\xi_{d}\left\langle\xi^{\prime}\right\rangle^{-1}\right\rangle^{m} d \xi_{d} \leq\left\langle\xi^{\prime}\right\rangle^{m+1}
$$

$\gamma^{*} \mathrm{Op}_{\mathrm{h}, 0}(r) \gamma \in h^{-1} \Psi_{\delta}^{\min (-1, m+1)}(\partial \Omega)$.
Now, consider $q$. In this case, we must take a limit as $x_{d} \rightarrow 0$ from above or below since the integral is not apriori well defined. Consider

$$
u=(2 \pi h)^{-1} \int e^{i x_{d} \xi_{d}} q\left(y^{\prime}, \xi^{\prime}\right) d \xi_{d} \in \mathcal{S}^{\prime}(\mathbb{R})
$$

Let $f_{ \pm} \in C_{c}^{\infty}\left(\mathbb{R}_{ \pm}\right)$and write

$$
\begin{align*}
u(f) & =(2 \pi h)^{-1} \iint e^{\frac{i}{h} x_{d} \xi_{d}} q\left(y^{\prime}, \xi\right) f\left(x_{d}\right) d x_{d} d \xi_{d} \\
& =(2 \pi h)^{-1} \iint e^{\frac{i}{h} x_{d} \xi_{d}}\left\langle\xi_{d}\right\rangle^{-2 k}\left(1-\left(h \partial_{x_{d}}\right)^{2}\right)^{k}\left(q\left(y^{\prime}, \xi\right) f\left(x_{d}\right)\right) d x_{d} d \xi_{d}  \tag{6.6.3}\\
& =(2 \pi h)^{-1} \iint e^{\frac{i}{h} x_{d} \xi_{d}}\left\langle\xi_{d}\right\rangle^{-2 k} q\left(y^{\prime}, \xi\right)\left(1-\left(h \partial_{x_{d}}\right)^{2}\right)^{k} f d \xi_{d} d x_{d}  \tag{6.6.4}\\
& =\frac{i}{h} \int e^{\mp \frac{x_{d}}{h} \sqrt{\left|\xi^{\prime}\right|^{2}+1}}\left(\sqrt{\left|\xi^{\prime}\right|^{2}+1}\right\rangle^{-2 k} \frac{\sum_{j=-1}^{m} C_{j,+}\left(y^{\prime}, \xi^{\prime}\right)\left( \pm i \sqrt{\left|\xi^{\prime}\right|^{2}+1}\right)^{j}}{ \pm 2 i \sqrt{\left|\xi^{\prime}\right|^{2}+1}}\left(1-\left(h \partial_{x_{d}}^{2}\right)\right)^{k} f d x_{d}
\end{align*}
$$

$$
\begin{equation*}
= \pm \frac{i}{h} \int e^{\frac{x_{d}}{h}} \sqrt{\left|\xi^{\prime}\right|^{2}+1} \frac{\sum_{j=-1}^{m} C_{j,+}\left(y^{\prime}, \xi^{\prime}\right)\left( \pm i \sqrt{\left|\xi^{\prime}\right|^{2}+1}\right)^{j}}{ \pm 2 i \sqrt{\left|\xi^{\prime}\right|^{2}+1}} f\left(x_{d}\right) d x_{d} \tag{6.6.5}
\end{equation*}
$$

Since $q(y, \xi)$ grows polynomially in $\xi_{d}$, we use Jordan's lemma to obtain 6.6.5). Now, let $f_{n} \rightarrow \delta_{0} f_{n} \in C_{c}^{\infty}\left(\mathbb{R}_{ \pm}\right)$. Then, we have

$$
\lim _{ \pm x_{d} \downarrow 0} u\left(x_{d}\right)= \pm \frac{i}{h} \frac{\sum_{j=-1}^{m} C_{j,+}\left(y^{\prime}, \xi^{\prime}\right)\left( \pm i \sqrt{\left|\xi^{\prime}\right|^{2}+1}\right)^{j}}{ \pm 2 i \sqrt{\left|\xi^{\prime}\right|^{2}+1}} \in h^{-1} S_{\delta}^{m+1}
$$

So, we have that $\gamma^{ \pm} \mathrm{Op}_{\mathrm{h}, 0}(q) \gamma^{*} \in h^{-1} \Psi_{\delta}^{m+1}(\partial \Omega)$ as desired.
Now, by Lemma 6.6.1 for the purposes of understanding the compositiong $\gamma R_{0} \gamma^{*}$ near the diagonal and away from $S^{*} \partial \Omega$, we can view $R_{2} \gamma^{*}\left(1-\psi\left(h^{-\epsilon}\left(\left|h D^{\prime}\right|_{g}-E\right)\right)\right)$ as a pseudodifferential operator.

Applying Lemmas 6.6.1 and 6.6.2, we have that away from glancing or the diagonal, $\gamma R_{0} \gamma^{*}$ is composed of a Fourier integral operator, $G_{B}$, associated with the relation

$$
C_{b}:=\left\{\begin{array}{c}
\left(\pi\left(\exp _{t}(x, \xi)\right), x, \xi_{2}^{\prime}\right) \\
\left.:\left(x, \xi_{2}^{\prime}\right) \in B^{*} \partial \Omega,(x, \xi) \in \pi^{-1}\left(x, \xi_{2}\right), t \geq 0,\left.\exp _{t}(x, \xi) \in S_{E}^{*} \mathbb{R}^{d}\right|_{\partial \Omega}\right\} .
\end{array}\right.
$$

and a pseudodifferential operator $G_{\Delta}$. Here $\pi$ is orthogonal projection $\left.S_{E}^{*} \mathbb{R}^{d}\right|_{\partial \Omega} \rightarrow \overline{B_{E}^{*} \partial \Omega}$ and $S_{E}^{*} \mathbb{R}^{d}$ and $B_{E}^{*} \partial \Omega$ are respectively the cosphere and coball bundles of radius $E$.
Remark: Note that in the case $\Omega$ is strictly convex, $C_{b}$ is parametrized by $\beta_{E}$ where

$$
\beta_{E}\left(x, \xi^{\prime}\right)=\left(\pi_{x} \circ \beta\left(x, \xi^{\prime} / E\right), E \pi_{\xi} \circ \beta\left(x, \xi^{\prime} / E\right)\right) .
$$

Next, observe that by (6.6.1), when we compose $\gamma R_{\chi}$ on the right by $\gamma^{*}$, the remainder term is $O_{L^{2} \rightarrow C^{\infty}}\left(h^{\infty}\right)$ as desired.

Putting this together, we have

$$
G(z):=G_{\Delta}(z)+G_{B}(z)+G_{g}(z)+O_{L^{2} \rightarrow C^{\infty}}\left(h^{\infty}\right)
$$

where $G_{\Delta}$ is pseudodifferential, $G_{B}$ is a Fourier integral operator associated with the relation $C_{b}$, and $G_{g}$ has $\mathrm{MS}_{\mathrm{h}}\left(G_{g}\right) \subset U_{h} \times U_{h} \cap V_{h}$ where $U_{h}$ is an $h^{\epsilon}$ neighborhood of $S_{E}^{*} \partial \Omega$ and $V_{h}$ is an $h^{\epsilon}$ neighborhood of the diagonal of $\partial \Omega \times \partial \Omega$ lifted to $T^{*} \partial \Omega$. Moreover, if $\Omega$ is strictly convex, $G_{B}$ is associated to the billiard ball map.

Next, we compute the symbols of $G_{\Delta}$ and $G_{B}$. Using Lemmas 6.6.1 and 6.6.2 we have

$$
\begin{aligned}
\sigma\left(G_{\Delta}\right)= & (2 \pi h)^{-1 / 2} \sigma(\gamma) \int r_{0} \sigma\left(\gamma^{*}\right) d \xi_{d}=(2 \pi h)^{-1} \int h^{2}\left(|\xi|^{2}-(E+i 0)^{2}\right)^{-1} d \xi_{d} \\
& \int\left(\xi_{d}^{2}-{\sqrt{E^{2}-\left|\xi^{\prime}\right|_{g}^{2}+i 0}}^{2}\right)^{-1} d \xi_{d}=\pi i\left(E^{2}-\left|\xi^{\prime}\right|_{g}^{2}\right)^{-1 / 2}
\end{aligned}
$$

Here we take the branch of the square root such that $\sqrt{a}$ is positive on $a>0$. This choice is unambiguous since $\operatorname{Arg}\left(E^{2}-\left|\xi^{\prime}\right|_{g}^{2}\right) \in\{0, \pi\}$. Thus,

$$
\sigma\left(G_{\Delta}\right)=\left.\sigma(G)\right|_{C \circ \Lambda_{0} \circ C^{-1}}=i h\left(2 \sqrt{E^{2}-\left|\xi^{\prime}\right|_{g}^{2}}\right)^{-1}
$$

Remark: Note that the symbol of $G_{\Delta}$ is the same as that if we had naively applied Proposition 4.4.10.

Note also that using the transversality of the intersection $C \circ \Lambda_{1} \circ C^{-1}$, Proposition 4.4.10 gives that

$$
\begin{equation*}
\left.\sigma\left(e^{\frac{\operatorname{Im} z}{h}|x-y|} G_{B}\right)\right|_{C_{b}^{\prime}}=\frac{h E^{(d-3) / 2} e^{(-d+3) \pi i / 4} e^{\frac{i}{h} \operatorname{Re} \omega_{0}|x-y|}}{2|x-y|^{(d-1) / 2}}|d y \wedge d x|^{1 / 2}, \quad(x, y) \in \partial \Omega \tag{6.6.6}
\end{equation*}
$$

Then, assuming that $\Omega$ is strictly convex so that $C_{b}$ is parametrized by $\beta_{E}$, we have by composing symbols (see also [34, Proposition 6.1]) that

Lemma 6.6.3. Let $q=(y, \eta) \in B_{y}^{*} \partial \Omega$. Then $G_{B}$ has symbol

$$
\begin{align*}
& \sigma\left(\left.G_{B} e^{\frac{\operatorname{Im} z}{h} \mathrm{Oph}_{\mathrm{h}}\left(l\left(q, \beta_{E}(q)\right)\right.} \varphi\left(h^{-\epsilon}\left(\left|h D^{\prime}\right|_{g}-E\right)\right)\right|_{C_{b}^{\prime}}\right. \\
&=\frac{h e^{\frac{i}{h} \operatorname{Re} \omega_{0} l\left(q, \beta_{E}(q)\right)} \varphi\left(h^{-\epsilon}\left(|\eta(q)|_{g}-E\right)\right)}{2\left(E^{2}-\left|\eta\left(\beta_{E}(q)\right)\right|_{g}^{2}\right)^{1 / 4}\left(E^{2}-|\eta(q)|_{g}^{2}\right)^{1 / 4}} d q^{1 / 2} . \tag{6.6.7}
\end{align*}
$$

$\chi \in C^{\infty}(\mathbb{R})$ has $\chi \equiv 0$ near 0 and $\varphi=1-\chi$.
Proof. To convert from (6.6.6) to (6.6.7), we reparametrize by $(y, \eta)$. That is, we write $\left|d \xi^{\prime}\right|$ in terms of $|d y|$. Observe that by (3.2.5 $\eta=E d_{y}|y-x|$ on $C_{\beta}$. Thus, we compute

$$
|d \eta|=E^{d-1} \operatorname{det}\left(\left.\frac{\partial^{2}}{\partial s_{i} \partial t_{j}}\right|_{\substack{\mathbf{t}=0 \\ \mathbf{s}=0}}\left|y+\sum_{i} s_{i} e_{i}-\left(x+\sum_{i} t_{i} e_{i}^{\prime}\right)\right|\right)|d x|
$$

where $e_{i}$ and $e_{i}^{\prime}(i=2, \ldots, d)$ are respectively orthonormal bases for $T_{y}^{*} \partial \Omega$ and $T_{x}^{*} \partial \Omega$. Without loss of generality, we assume that

$$
\begin{aligned}
& \nu_{y}=(1,0,0 \ldots, 0), \quad \nu_{x}=(\cos \beta, \sin \beta, 0, \ldots 0) \\
& y=(0,0,0, \ldots, 0), \quad x=\left(r_{1}, r_{2}, r_{3}, 0, \ldots 0\right)=\mathbf{r}
\end{aligned}
$$

Then we choose as our orthonormal bases $e_{i}=\mathbf{e}_{i} i=2, \ldots d$ where $\mathbf{e}_{j}$ is the standard basis and

$$
e_{2}^{\prime}=(-\sin \beta, \cos \beta, 0, \ldots, 0), \quad e_{i}^{\prime}=e_{i} \text { for } i=3, \ldots d
$$

Next we compute derivatives of $w=w\left(s_{2}, \ldots, s_{d}, t_{2}, \ldots, t_{d}\right)$

$$
w=\left|\left(t_{2} \sin \beta-r_{1}, s_{2}-r_{2}-t_{2} \cos \beta, s_{3}-r_{3}-t_{3}, s_{4}-t_{4}, \ldots, s_{d}-t_{d}\right)\right|
$$

A long but straightforward computation gives

$$
\left.\frac{\partial^{2} w}{\partial s_{i} \partial t_{j}}\right|_{\substack{\mathbf{t}=0 \\
\mathbf{s}=0}}=|\mathbf{r}|^{-3}\left(\begin{array}{ccc}
-|\mathbf{r}|^{2} \cos \beta+r_{2}\left(r_{2} \cos \beta-\sin \beta r_{1}\right) & r_{3}\left(r_{2} \cos \beta-\sin \beta r_{1}\right) & 0 \\
r_{2} r_{3} & -|\mathbf{r}|^{2}+r_{3}^{2} & 0 \\
0 & 0 & -|\mathbf{r}|^{2} I
\end{array}\right) .
$$

This matrix has

$$
\begin{aligned}
\left|\operatorname{det}\left(\left.\frac{\partial^{2} w}{\partial s_{i} \partial t_{j}}\right|_{\substack{\mathbf{t}=0 \\
\mathbf{s}=0}}\right)\right| & =|\mathbf{r}|^{-d+1}\left|\frac{r_{1}^{2} \cos \beta+r_{1} r_{2} \sin \beta}{|\mathbf{r}|^{2}}\right|=|\mathbf{r}|^{-d+1}\left|\partial_{\nu_{x}}\right| y-x| |\left|\partial_{\nu_{y}}\right| x-y| | \\
& =|x-y|^{-d+1} E^{-2} \sqrt{E^{2}-|\eta|_{g}^{2}} \sqrt{E^{2}-\left|\xi^{\prime}\left(\beta_{E}(q)\right)\right|_{g}^{2}}
\end{aligned}
$$

Remark: If $\Omega$ is strictly convex, then the cutoff away from the diagaonal in $R_{1}$ causes $G_{B}$ to be microlocalized $h^{\epsilon}$ away from $\left|\xi^{\prime}\right|_{g}=E$ and hence $G_{B} \varphi\left(h^{-\epsilon}\left(\left|h D^{\prime}\right|_{g}-E\right)\right)=G_{B}+O\left(h^{\infty}\right)$.

Now, to understand $\tilde{N}=\gamma^{+} R_{0} L^{*} \gamma^{*}+\frac{1}{2}$ Id and $\partial_{\nu} \mathcal{D} \ell=\gamma L R_{0} L^{*} \gamma^{*}$ microlocally away from glancing, we only need to compute the symbols of the various pieces since the geometry of the situation is identical to that for $G$. For $\partial_{\nu} \mathcal{D} \ell$, it is irrelevant whether we choose $\gamma^{+}$or $\gamma^{-}$ since we have verified that there is no jump at $\partial \Omega$ in Lemma 6.1.1. Write $\bar{N}=\gamma^{+} R_{0} L^{*} \gamma^{*}$. Then for $\bar{N}$, we write

$$
\begin{align*}
\overline{\tilde{N}}= & \gamma R_{1} L^{*} \gamma^{*}+\gamma^{+} R_{2} L^{*} \gamma^{*}\left(1-\psi\left(h^{-\epsilon}\left(\left|h D^{\prime}\right|_{g}-E\right)\right)\right. \\
& \quad+\gamma^{+} R_{2} L^{*} \gamma^{*} \psi\left(h^{-\epsilon}\left|h D^{\prime}\right|_{g}-E\right) \\
= & \tilde{N}_{B}+\tilde{N}_{\Delta}+\tilde{N}_{g}+\frac{1}{2} \mathrm{Id} \tag{6.6.8}
\end{align*}
$$

Also, write

$$
\begin{aligned}
\partial_{\nu} \mathcal{D} \ell= & \gamma L R_{1} L^{*} \gamma^{*}+\gamma^{+} L R_{2} L^{*} \gamma^{*}\left(1-\psi\left(h^{-\epsilon}\left(\left|h D^{\prime}\right|_{g}-E\right)\right)\right. \\
& \quad+\gamma^{+} L R_{2} L^{*} \gamma^{*} \psi\left(h^{-\epsilon}\left|h D^{\prime}\right|_{g}-E\right) \\
= & \partial_{\nu} \mathcal{D} \ell_{B}+\partial_{\nu} \mathcal{D} \ell_{\Delta}+\partial_{\nu} \mathcal{D} \ell_{g}
\end{aligned}
$$

The symbol of $\partial_{\nu} \mathcal{D} \ell_{B}$ is given by

$$
\begin{aligned}
& \sigma\left(\tilde{N}_{B} e^{\frac{\operatorname{Im} z}{h}|x-y|}\right)= \\
& \frac{i E^{(d+1) / 2} e^{(-d+3) \pi i / 4} e^{\frac{i}{h} \operatorname{Re} \omega_{0}|x-y|}}{2|x-y|^{(d-1) / 2}} d_{\nu_{y}}|x-y||d y \wedge d x|^{1 / 2}
\end{aligned}
$$

and using the computations from Lemma 6.6.3

$$
\sigma\left(\tilde{N}_{B} e^{\frac{\operatorname{Im} z}{h} \operatorname{OPp}_{\mathrm{h}}\left(l\left(q, \beta_{E}(q)\right)\right)}\right)=\frac{-i e^{\frac{i}{h} \operatorname{Re} \omega_{0} l\left(q, \beta_{E}(q)\right)}\left(E^{2}-\left|\xi^{\prime}(q)\right|_{g}^{2}\right)^{1 / 4}}{2\left(E^{2}-\left|\xi^{\prime}\left(\beta_{E}(q)\right)\right|_{g}^{2}\right)^{-1 / 4}} d q^{1 / 2}
$$

Then, the symbol of $\partial_{\nu} \mathcal{D} \ell_{B}$ is given by

$$
\begin{aligned}
& \sigma\left(\partial_{\nu} \mathcal{D} \ell_{B} e^{\left.\frac{\operatorname{Im} z|x-y|}{h} \right\rvert\, x-}\right)= \\
& \frac{-h^{-1} E^{(d+1) / 2} e^{(-d+3) \pi i / 4} e^{\frac{i}{h} \operatorname{Re} \omega_{0}|x-y|}}{2|x-y|^{(d-1) / 2}} d_{\nu_{x}}|x-y| d_{\nu_{y}}|x-y||d y \wedge d x|^{1 / 2}
\end{aligned}
$$

and using the computations from Lemma 6.6.3

$$
\begin{aligned}
\sigma\left(\partial_{\nu} \mathcal{D} \ell_{B} e^{\frac{\operatorname{Im} z}{h} \mathrm{Op}_{\mathrm{h}}\left(l\left(q, \beta_{E}(q)\right)\right)}\right) & = \\
& \frac{h^{-1} e^{\frac{i}{h} \operatorname{Re} \omega_{0} l\left(q, \beta_{E}(q)\right)}\left(E^{2}-\left|\xi^{\prime}\left(\beta_{E}(q)\right)\right|_{g}^{2}\right)^{1 / 4}\left(E^{2}-\left|\xi^{\prime}(q)\right|_{g}^{2}\right)^{1 / 4}}{2} d q^{1 / 2}
\end{aligned}
$$

To analyze $\tilde{N}_{\Delta}$ and $\partial_{\nu} \mathcal{D} \ell_{\Delta}$, write

$$
\begin{aligned}
& R_{2} L^{*} \gamma^{*}\left(1-\psi\left(h^{-\epsilon}\left(\left|h D^{\prime}\right|-1\right)\right)\right)(x, y) \\
& \quad=(2 \pi h)^{-d} \int e^{\frac{i}{h}\langle x-y, \xi\rangle} \frac{-i\left\langle\xi, \nu_{y}\right\rangle+p_{1}(x, y, \xi)}{|\xi|^{2}-E^{2}-i 0}\left(1-\psi\left(h^{-\epsilon}\left(\left|\xi^{\prime}\right|_{g}-E\right)\right)\right) d \xi
\end{aligned}
$$

and

$$
\begin{aligned}
& L R_{2} L^{*} \gamma^{*}\left(1-\psi\left(h^{-\epsilon}\left(\left|h D^{\prime}\right|-1\right)\right)\right)(x, y) \\
& \quad=(2 \pi h)^{-d} \int e^{\frac{i}{h}\langle x-y, \xi\rangle} \frac{\left\langle\xi, \nu_{x}\right\rangle\left\langle\xi, \nu_{y}\right\rangle+p_{2}(x, y, \xi)}{|\xi|^{2}-E^{2}-i 0}\left(1-\psi\left(h^{-\epsilon}\left(\left|\xi^{\prime}\right|_{g}-E\right)\right)\right) d \xi
\end{aligned}
$$

where the $p_{i}$ are polynomial in $\xi$. Then, in appropriate coordinates

$$
\begin{aligned}
R_{2} L^{*} \gamma^{*}\left(1-\psi\left(h ^ { - \epsilon } \left(\left|h D^{\prime}\right|\right.\right.\right. & -1)))(x, y) \\
& =(2 \pi h)^{-d} \int e^{\frac{i}{h}\langle x-y, \xi\rangle} \frac{-i \xi_{d}+p(x, y, \xi)}{|\xi|^{2}-E^{2}-i 0}\left(1-\psi\left(h^{-\epsilon}\left(\left|\xi^{\prime}\right|_{g}-E\right)\right)\right) d \xi
\end{aligned}
$$

and

$$
\begin{aligned}
L R_{2} L^{*} \gamma^{*}\left(1-\psi\left(h ^ { - \epsilon } \left(\left|h D^{\prime}\right|\right.\right.\right. & -1)))(x, y) \\
& =(2 \pi h)^{-d} \int e^{\frac{i}{h}\langle x-y, \xi\rangle} \frac{\xi_{d}^{2}+p(x, y, \xi)}{|\xi|^{2}-E^{2}-i 0}\left(1-\psi\left(h^{-\epsilon}\left(\left|\xi^{\prime}\right|_{g}-E\right)\right)\right) d \xi
\end{aligned}
$$

Hence, the relevant parts of $R_{2} L^{*} \gamma^{*}$ and $L R_{2} L^{*} \gamma^{*}$ satisfy the requirements of Lemma 6.6.2. When we compute the symbol of $\gamma^{+} R_{2} L^{*} \gamma^{*}$, we obtain $\frac{1}{2}$ which is exactly the $\frac{1}{2}$ Id appearing in (6.6.8). Hence, we can compute symbols to obtain:

For the case that $\Omega$ is strictly convex, we summarize the result of this decomposition in the following Lemma

Lemma 6.6.4. Let $\Omega \subset \mathbb{R}^{d}$ be strictly convex with $\partial \Omega \in C^{\infty}$. Then for all $1 / 2>\epsilon, \gamma>0$, and $z=E+O\left(h^{1-\gamma}\right)$ with $\operatorname{Im} z \geq-C h \log h^{-1}$. Then

$$
\begin{gathered}
G(z / h):=G_{\Delta}(z)+G_{B}(z)+G_{g}(z)+O_{\mathcal{D}^{\prime} \rightarrow C^{\infty}}\left(h^{\infty}\right) \\
\tilde{N}(z / h):=\tilde{N}_{\Delta}(z)+\tilde{N}_{B}(z)+\tilde{N}_{g}(z)+O_{\mathcal{D}^{\prime} \rightarrow C^{\infty}}\left(h^{\infty}\right) \\
\partial_{\nu} \mathcal{D} \ell(z / h):=\partial_{\nu} \mathcal{D} \ell_{\Delta}(z)+\partial_{\nu} \mathcal{D} \ell_{B}(z)+\partial_{\nu} \mathcal{D} \ell_{g}(z)+O_{\mathcal{D}^{\prime} \rightarrow C^{\infty}}\left(h^{\infty}\right)
\end{gathered}
$$

$\tilde{N}_{B}$ here $G_{\Delta} \in h^{1-\frac{\epsilon}{2}} \Psi_{\epsilon}^{-1}, \tilde{N}_{\Delta} \in h^{1-2 \epsilon} \Psi_{\epsilon}^{-1}, \partial_{\nu} \mathcal{D} \ell_{\Delta} \in h^{-1} \Psi_{\epsilon}^{1}$, and $G_{B} \in h^{1-\frac{\epsilon}{2}} e^{(\operatorname{Im} z)-d_{\Omega} / h} I_{\delta}^{\text {comp }}\left(C_{b}\right)$, $\tilde{N}_{B} \in e^{(\operatorname{Im} z)-d_{\Omega} / h} I_{\delta}^{\text {comp }}\left(C_{b}\right)$, and $\partial_{\nu} \mathcal{D} \ell_{B} \in h^{-1} e^{(\operatorname{Im} z)-d_{\Omega} / h} I_{\delta}^{\text {comp }}\left(C_{b}\right)$ are FIOs associated to $\beta_{E}$ where $\delta=\max (\epsilon, \gamma)$. Moreover,

$$
\begin{gathered}
\mathrm{MS}_{\mathrm{h}}{ }^{\prime}\left((\cdot)_{B}\right) \subset\left\{\begin{array}{c}
(q, p) \in B_{E}^{*} \partial \Omega \times B_{E}^{*} \partial \Omega: \\
\min \left(E-\left|\xi^{\prime}(q)\right|_{g}, E-\left|\xi^{\prime}(q)\right|_{g}, l(q, p)\right)>c h^{\epsilon}
\end{array}\right\} \\
\mathrm{MS}_{\mathrm{h}}{ }^{\prime}\left((\cdot)_{g}\right) \subset\left\{\begin{array}{c}
(q, p) \in T^{*} \partial \Omega \times T^{*} \partial \Omega: \\
\max \left(\left|E-\left|\xi^{\prime}(q)\right|_{g}\right|,\left|E-\left|\xi^{\prime}(p)\right|_{g}\right|, l(q, p)\right)<c h^{\epsilon}
\end{array}\right\} \\
\sigma\left(G_{\Delta}\right)=\frac{i h}{2 \sqrt{E^{2}-\left|\xi^{\prime}\right|_{g}^{2}}}, \quad \sigma\left(\partial_{\nu} \mathcal{D} \ell_{\Delta}\right)=\frac{i h^{-1} \sqrt{E^{2}-\left|\xi^{\prime}\right|_{g}^{2}}}{2}, \\
\sigma\left(G_{B} e^{\frac{\operatorname{Im} z}{h} \operatorname{Opp}_{\mathrm{h}}\left(l\left(q, \beta_{E}(q)\right)\right)}\right)=\frac{h e^{\frac{i}{h} \operatorname{Re} \omega_{0} l\left(q, \beta_{E}(q)\right)}}{2\left(E^{2}-\left|\xi^{\prime}\left(\beta_{E}(q)\right)\right|_{g}^{2}\right)^{1 / 4}\left(E^{2}-\left|\xi^{\prime}(q)\right|_{g}^{2}\right)^{1 / 4}} d q^{1 / 2}, \\
\sigma\left(\tilde{N}_{B} e^{\frac{\operatorname{Im} z}{h} \operatorname{Op}_{\mathrm{h}}\left(l\left(q, \beta_{E}(q)\right)\right)}\right)=\frac{-i e^{\frac{i}{h} \operatorname{Re} \omega_{0} l\left(q, \beta_{E}(q)\right)}\left(E^{2}-\left|\xi^{\prime}(q)\right|_{g}^{2}\right)^{1 / 4}}{2\left(E^{2}-\left|\xi^{\prime}\left(\beta_{E}(q)\right)\right|_{g}^{2}\right)^{1 / 4}} d q^{1 / 2}, \\
\sigma\left(\partial_{\nu} \mathcal{D} \ell_{B} e^{\frac{\operatorname{Im} z}{h} \operatorname{Op}_{\mathrm{h}}\left(l\left(q, \beta_{E}(q)\right)\right)}\right)= \\
\frac{h^{-1} e^{\frac{i}{h} \operatorname{Re} \omega_{0} l\left(q, \beta_{E}(q)\right)}\left(E^{2}-\left|\xi^{\prime}\left(\beta_{E}(q)\right)\right|_{g}^{2}\right)^{1 / 4}\left(E^{2}-\left|\xi^{\prime}(q)\right|_{g}^{2}\right)^{1 / 4}}{2} d q^{1 / 2} .
\end{gathered}
$$

where we take $\sqrt{z}=\sqrt{|z|} e^{\frac{1}{2} \operatorname{Arg}(z)}$ for $-\pi / 2<\operatorname{Arg}(z)<3 \pi / 2$.

## Remarks:

- The change in this Lemma when $\Omega$ is only assumed to be convex is that we lose restriction on $\left|\xi^{\prime}\right|_{g}$ and $\left|\eta^{\prime}\right|_{g}$ in $\mathrm{MS}_{\mathrm{h}}{ }^{\prime}\left(G_{B}\right)$ and thus must use (6.6.6) for the symbol of $G_{B}$ near glancing points and away from the diagonal.
- The microsupports of the various components of $G$ are shown graphically in Figure 6.2


### 6.7 Boundary layer operators and potentials near glancing

In this section, we complete the microlocal descriptions of the boundary layer operators using the Melrose-Taylor parametrix constructed in Appendx 5.


Figure 6.2: We show the wavefront relation for each of the pieces in the decomposition of $G$ (or $\partial_{\nu} \mathcal{D} \ell$ ). The formulae for these wavefront sets are contained in Lemmas 6.6.4. We label the elliptic, glancing, and hyperbolic regions by $\mathcal{E}, \mathcal{G}$, and $\mathcal{H}$ respectively. The top, middle, and bottom pictures correspond to $G_{B}, G_{g}$ and $G_{\Delta}$ respectively. In the left copy of $T^{*} \partial \Omega$, we show the wavefront set of each operator in the fiber over $y \in \partial \Omega$. The right copy of $T^{*} \partial \Omega$ shows how each operator maps the wavefront set in the fiber over $y$. Note that the curve shown in the right copy of $T^{*} \partial \Omega$ for $\mathrm{WF}_{\mathrm{h}}{ }^{\prime}\left(G_{B}\right)$ continues outside of the portion of $T^{*} \partial \Omega$ shown.

## Estimates for a simple transmission problem

We start by proving estimates for the following transmission problem. Let $\Omega_{1}=\Omega, \Omega_{2}=$ $\mathbb{R}^{d} \backslash \bar{\Omega}$, and $u=u_{1} 1_{\Omega_{1}} \oplus u_{2} 1_{\Omega_{2}}$. Suppose that $\chi \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ with $\chi \equiv 1$ on $\Omega_{1}$ and

$$
\begin{align*}
\left(-h^{2} \Delta-z^{2}\right) u_{i} & =h^{2} \chi f_{i} & & \text { in } \Omega_{i} \\
u_{1}-u_{2} & =g_{1} & & \text { on } \partial \Omega \\
\partial_{\nu_{1}} u_{1}+\partial_{\nu_{2}} u_{2} & =g_{2} & & \text { on } \partial \Omega  \tag{6.7.1}\\
u_{2} & \text { is } z / h \text { outgoing } & &
\end{align*}
$$

Then, it is easy to check that as a distribution,

$$
\left(-h^{2} \Delta-z^{2}\right) u=h^{2}\left(f+L^{*} \delta_{\partial \Omega} \otimes g_{1}+\delta_{\partial \Omega} \otimes g_{2}\right)
$$

where $f=1_{\Omega_{1}} f_{1} \oplus 1_{\Omega_{2}} f_{2}$ and $L$ is a vector field with $\left.L\right|_{\partial \Omega}=\partial_{\nu_{1}}$. Thus, applying $h^{-2} R_{0}(z / h)$ to this equation shows that for $z / h$ in the domain of $R_{0}(z / h)$, 6.7.1) has a unique solution given by

$$
u=R_{0} \chi f+\mathcal{S} \ell g_{2}+\mathcal{D} \ell g_{1}
$$

Hence

$$
\begin{align*}
\left.u_{1}\right|_{\partial \Omega} & =\gamma R_{0} f+G g_{2}-\frac{1}{2} g_{1}+\tilde{N} g_{1} \\
\left.u_{2}\right|_{\partial \Omega} & =\gamma R_{0} f+G g_{2}+\frac{1}{2} g_{1}+\tilde{N} g_{1}  \tag{6.7.2}\\
\left.\partial_{\nu_{1}} u_{1}\right|_{\partial \Omega} & =\gamma \partial_{\nu_{1}} R_{0} f+\frac{1}{2} g_{2}+\tilde{N}^{\#} g_{2}+\partial_{\nu} \mathcal{D} \ell g_{1} \\
\left.\partial_{\nu_{2}} u_{2}\right|_{\partial \Omega} & =\gamma \partial_{\nu_{2}} R_{0} f+\frac{1}{2} g_{2}-\tilde{N}^{\#} g_{2}-\partial_{\nu} \mathcal{D} \ell g_{1}
\end{align*}
$$

To obtain an $L^{2}$ estimate on $u$, we simply apply standard resolvent estimates (see for example [21, Chapter 3]),

$$
\begin{equation*}
\left\|\chi R_{0}(z / h) \chi\right\|_{H_{h}^{s} \rightarrow H_{h}^{s+2}} \leq C h e^{D_{\chi}(\operatorname{Im} z)-/ h} . \tag{6.7.3}
\end{equation*}
$$

So

$$
\|\chi u\|_{L^{2}\left(\mathbb{R}^{d}\right)} \leq C e^{D_{\chi}(\operatorname{Im} z)-}\left(h\|\chi f\|_{L^{2}\left(\mathbb{R}^{d}\right)}+h^{1 / 2}\left\|g_{1}\right\|_{L^{2}(\partial \Omega)}+h^{1 / 2}\left\|g_{2}\right\|_{L^{2}(\partial \Omega)}\right)
$$

To upgrade this to estimates on $u_{i}$ in $H^{k}\left(\Omega_{i}\right)$, we observe that for $\chi_{1} \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ with $\chi_{1} \equiv 1$ on $\chi$, and $\chi_{2} \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ with $\chi_{2} \equiv 1$ on $\operatorname{supp} \chi_{1}$,

$$
\begin{aligned}
\left(-h^{2} \Delta-z^{2}\right) \chi_{1} u= & {\left[\chi_{1}, h^{2} \Delta\right] u+h^{2}\left(\chi f+L^{*} \delta_{\partial \Omega} \otimes g_{1}+\delta_{\partial \Omega} \otimes g_{2}\right) } \\
\chi_{1} u= & \chi_{2} R_{0}(0)\left(h^{-2}\left(\left[\chi_{1}, h^{2} \Delta\right]+z^{2} \chi_{1}\right) u+\chi f\right) \\
& +\chi_{2} \mathcal{D} \ell(0) g_{1}+\chi_{2} \mathcal{S} \ell(0) g_{2}
\end{aligned}
$$

and for $k \geq-1, \mathcal{D} \ell(0): H^{k+3 / 2}(\partial \Omega) \rightarrow H^{k+2}\left(\Omega_{1}\right) \oplus H^{k+2}\left(\Omega_{2}\right)$ and $\mathcal{S} \ell(0): H^{k+1 / 2}(\partial \Omega) \rightarrow$ $H^{k+2}\left(\Omega_{1}\right) \oplus H^{k+2}\left(\Omega_{2}\right), \chi R_{0}(0) \chi: H^{k}\left(\Omega_{1}\right) \oplus H^{k}\left(\Omega_{2}\right) \rightarrow H^{k+2}\left(\Omega_{1}\right) \oplus H^{k+2}(\Omega)_{2}$. (See 25 , Theorems 9, 10]) So,

$$
\begin{aligned}
& \left\|u_{1}\right\|_{H^{k+2}\left(\Omega_{1}\right)}+\left\|\chi_{1} u_{2}\right\|_{H^{k+2}\left(\Omega_{2}\right)} \\
& \leq h^{-2}\left(\left(\left\|u_{1}\right\|_{H^{k}\left(\Omega_{1}\right)}+h\left\|u_{1}\right\|_{H^{k+1}\left(\Omega_{1}\right)}\right)+\left(\left\|\chi_{1} u_{2}\right\|_{H^{k}(\Omega)}+h\left\|\chi_{2} u_{2}\right\|_{H^{k+1}\left(\Omega_{2}\right)}\right)\right) \\
& \quad+\|\chi f\|_{H^{k}\left(\mathbb{R}^{d}\right)}+\left\|g_{1}\right\|_{H^{k+1 / 2}(\partial \Omega)}+\left\|g_{2}\right\|_{H^{k+3 / 2}(\partial \Omega)}
\end{aligned}
$$

Using the description $G, \tilde{N}$, and $\partial_{\nu} \mathcal{D} \ell$ at high energy in Lemmas 6.6.4 as psuedodifferential operators, we have for $\psi \in C_{c}^{\infty}(\mathbb{R})$ with $\psi \equiv 1$ on $[-2 E, 2 E]$,

$$
\begin{aligned}
\|G u\|_{H_{h}^{k}} & \leq\|(1-\psi(|h D|)) G u\|_{H_{h}^{k}}+\|\psi(|h D|) G u\|_{H_{h}^{k}} \\
& \leq h\|u\|_{H_{h}^{k-1}}+\|G u\|_{L^{2}} \\
\|\tilde{N} u\|_{H_{h}^{k}} & \leq\|(1-\psi(|h D|)) \tilde{N} u\|_{H_{h}^{k}}+\|\psi(|h D|) \tilde{N} u\|_{H_{h}^{k}} \\
& \leq\|u\|_{H_{h}^{k}}+\|\tilde{N} u\|_{L^{2}} \\
\left\|\tilde{N}^{\#} u\right\|_{H_{h}^{k}} & \leq\left\|(1-\psi(|h D|)) \partial_{\nu} \mathcal{D} \ell^{\#} u\right\|_{H_{h}^{k}}+\left\|\psi(|h D|) \partial_{\nu} \mathcal{D} \ell^{\#} u\right\|_{H_{h}^{k}} \\
& \leq\|u\|_{H_{h}^{k}}+\|\tilde{N} \# u\|_{L^{2}} \\
\left\|\partial_{\nu} \mathcal{D} \ell u\right\|_{H_{h}^{k}} & \leq\left\|(1-\psi(|h D|)) \partial_{\nu} \mathcal{D} \ell u\right\|_{H_{h}^{k}}+\left\|\psi(|h D|) \partial_{\nu} \mathcal{D} \ell u\right\|_{H_{h}^{k}} \\
& \leq h^{-1}\|u\|_{H_{h}^{k+1}}+\left\|\partial_{\nu} \mathcal{D} \ell u\right\|_{L^{2}} .
\end{aligned}
$$

Together with 6.7.3 and Theorem 6.1, this implies the estimates
Lemma 6.7.1. Suppose that $z / h$ is in the domain of $R_{0}, \chi \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ with $\chi \equiv 1$ on $\Omega_{1}$ and $u \in L_{\mathrm{loc}}^{2}\left(\mathbb{R}^{d}\right)$ is the solution to 6.7.1). Then

$$
u=R_{0} \chi f+\mathcal{S} \ell g_{2}+\mathcal{D} \ell g_{1},
$$

(6.7.2) holds and for any $\epsilon>0, k \geq-1 / 2, m \geq 0$, there exists $h_{0}>0, C, N_{k}>0$ such that for $0<h<h_{0}$,

$$
\left(\begin{array}{c}
\left\|u_{1}\right\|_{H_{h}^{k+2}\left(\Omega_{1}\right)}+\left\|\chi u_{2}\right\|_{H_{h}^{k+2}\left(\Omega_{2}\right)}+\left\|u_{1}\right\|_{H_{h}^{k+\frac{3}{2}}(\partial \Omega)}+\left\|u_{2}\right\|_{H_{h}^{k+\frac{3}{2}}(\partial \Omega)} \\
+\left\|\partial_{\nu_{1}} u_{1}\right\|_{H^{k+\frac{1}{2}}(\partial \Omega)}+\left\|\partial_{\nu_{2}} u_{2}\right\|_{H^{k+\frac{1}{2}}(\partial \Omega)} \\
\leq C h^{-N_{k}} e^{\frac{D_{\chi}(\operatorname{Im} z)-}{h}}\left(\|\chi f\|_{H_{h}^{k}\left(\mathbb{R}^{d}\right)}+\left\|g_{2}\right\|_{H_{h}^{k+\frac{1}{2}}(\partial \Omega)}+\left\|g_{1}\right\|_{H_{h}^{k+\frac{3}{2}}(\partial \Omega)}\right)
\end{array}\right.
$$

## Microlocal Description of $G$ and $\mathcal{S} \ell$ near glancing

Now, let $u$ solve 6.7.1 with $f_{i} \equiv 0$ and $g_{1}=0$ and $g_{2}=g$ microlocalized sufficiently close to a glancing point $\left(x^{\prime}, \xi^{\prime}\right)$ so that the parametrices from Appendix 5 can be constructed.

In particular, let $\left(y_{0}, \eta_{0}\right) \in S^{*} \partial \Omega$ and

$$
\begin{gather*}
\psi \equiv 1 \text { on }\left\{\left|y-y_{0}\right|<\delta,\left|\eta-\eta_{0}\right|<\delta_{1},\left||\eta|_{g}-1\right|<\gamma h^{2} \epsilon(h)^{-2} .\right.  \tag{6.7.4}\\
\operatorname{supp} \psi \subset\left\{\left|y-y_{0}\right|<2 \delta,\left|\eta-\eta_{0}\right|<2 \delta_{1},\left||\eta|_{g}-1\right|<2 \gamma h^{2} \epsilon(h)^{-2}\right. \tag{6.7.5}
\end{gather*}
$$

and suppose that $g=\mathrm{Op}_{\mathrm{h}}(\psi) g+O_{\Psi^{-\infty}}\left(h^{\infty}\right) g$.
Recall that by Lemmas 5.4.1 and (5.4.3) a microlocal description of the exterior Dirichlet to Neumann map, $\mathcal{N}_{2}$, is given by

$$
\begin{equation*}
\mathcal{N}_{2} g=J\left(h^{-2 / 3} C \Phi_{-}+B\right) J^{-1} g+O_{\Psi^{-\infty}}\left(h^{\infty}\right) g \tag{6.7.6}
\end{equation*}
$$

where $C \in \Psi$ is elliptic, $B \in \Psi, \Phi_{-}$is the Fourier multiplier

$$
\begin{equation*}
\Phi_{-}(u)=(2 \pi h)^{-d+1} \int \frac{A_{-}^{\prime}\left(h^{-2 / 3} \alpha\right)}{A_{-}\left(h^{-2 / 3} \alpha\right)} e^{\frac{i}{h}\left\langle x-y, \xi^{\prime}\right\rangle} u d \xi^{\prime} \tag{6.7.7}
\end{equation*}
$$

where,

$$
\alpha\left(\xi^{\prime}\right)=\xi_{1}+i \epsilon(h) \quad \text { with } c h \leq \epsilon(h)=O\left(h \log h^{-1}\right) .
$$

Let $\mathcal{A} i \mathcal{A}_{-}, \mathcal{A} i^{\prime} \mathcal{A}_{-}, \mathcal{A} i \mathcal{A}_{-}^{\prime}$, and $\mathcal{A} i^{\prime} \mathcal{A}_{-}^{\prime}$ be the Fourier multiplies obtained by replacing $\frac{A_{-}^{\prime}}{A_{-}}$in (6.7.7) by $A i A_{-}, A i^{\prime} A_{-}, A i A_{-}^{\prime}$, and $A i^{\prime} A_{-}^{\prime}$ respectively.

Let $q_{1}=h^{2 / 3} \beta^{-1} J C^{-1} J^{-1} g$ and $q_{2}=h^{2 / 3} \beta^{-1} J \mathcal{A} i \mathcal{A}_{-} C^{-1} J^{-1} g$ where $\beta=\frac{e^{-\pi i / 6}}{2 \pi}$. Then, let $w_{1}=A_{1, g} q_{1}$ where $A_{1}$ is as in Lemma 5.5.1 and $w_{2}=\mathcal{H}_{d} q_{2}$ where $\mathcal{H}_{d}$ is the solution operator to

$$
\left\{\begin{array}{l}
\left(-h^{2} \Delta-z^{2}\right) \mathcal{H}_{d} q_{2}=0 \quad \text { in } \mathbb{R}^{d} \backslash \bar{\Omega} \\
\left.\mathcal{H}_{d} q_{2}\right|_{\partial \Omega}=q_{2} \\
\mathcal{H}_{d} q_{2} z / h \text { outgoing }
\end{array}\right.
$$

Then, by Lemma 5.5.1 and 6.7.6,

$$
\begin{equation*}
\left.w_{1}\right|_{\partial \Omega}=\left.w_{2}\right|_{\partial \Omega}=h^{2 / 3} \beta^{-1} J \mathcal{A} i \mathcal{A}_{-} C^{-1} J^{-1} g+O_{\Psi^{-\infty}}\left(h^{\infty}\right) \tag{6.7.8}
\end{equation*}
$$

and

$$
\begin{align*}
& \partial_{\nu} w_{1}+\partial_{\nu_{2}} w_{2}  \tag{6.7.9}\\
& =\beta^{-1} J\left(C \mathcal{A} i \mathcal{A}_{-}^{\prime} C^{-1}+B \mathcal{A} i \mathcal{A}_{-} C^{-1}-C \mathcal{A} i^{\prime} \mathcal{A}_{-} C^{-1}-B \mathcal{A} i \mathcal{A}_{-} C^{-1}\right) J^{-1} g \\
& \quad+O_{\Psi^{-\infty}}\left(h^{\infty}\right) g \\
& \quad=\beta^{-1} J\left(C\left(\mathcal{A} i^{\prime} \mathcal{A}_{-}-\mathcal{A} i \mathcal{A}_{-}^{\prime}\right) C^{-1}\right) J^{-1} g+O_{\Psi}-\infty\left(h^{\infty}\right) g  \tag{6.7.10}\\
& \quad= \\
& \quad J C C^{-1} J^{-1} g+O_{\Psi^{-\infty}}\left(h^{\infty}\right) g=g+O_{\Psi^{-\infty}}\left(h^{\infty}\right) g
\end{align*}
$$

## CHAPTER 6. BOUNDARY LAYER OPERATORS

where we have use the Wronskian for the Airy equation to reduce (6.7.10).
Thus, $\left(u_{1}-w_{1}, u_{2}-w_{2}\right)$ solves 6.7.1) with

$$
\|\chi f\|_{H_{h}^{N}\left(\mathbb{R}^{d}\right)}+\left\|g_{1}\right\|_{H_{h}^{N}(\partial \Omega)}+\left\|g_{2}\right\|_{H_{h}^{N}(\partial \Omega)}=O\left(h^{\infty}\right)\|g\|_{H^{-N}}
$$

for any $N$. Hence, $u=w+O_{C^{\infty} \text { loc }}\left(h^{\infty}\right)$ and we have that

$$
\begin{equation*}
G g=J \beta^{-1} h^{2 / 3} \mathcal{A} i \mathcal{A}_{-} C^{-1} J^{-1} g+O_{\Psi^{-\infty}}\left(h^{\infty}\right) g \tag{6.7.11}
\end{equation*}
$$

for any $\operatorname{Im} z=O\left(h \log h^{-1}\right)$. Moreover,

$$
\begin{equation*}
\left.\mathcal{S} \ell g\right|_{\Omega}=h^{2 / 3} \beta^{-1} A_{1, g} J C^{-1} J^{-1} g+O_{\mathcal{D}^{\prime}(\partial \Omega) \rightarrow C^{\infty}(\Omega)}\left(h^{\infty}\right) g \tag{6.7.12}
\end{equation*}
$$

Lemma 6.7.2. Suppose that $\varphi \in L^{2}(\partial \Omega)$ and there exists $\epsilon>0$ such that $\operatorname{MS}_{\mathrm{h}}(\varphi) \subset$ $\left\{\left|1-\left|\xi^{\prime}\right|_{g}\right| \leq h^{\epsilon}\right\}$, and $\operatorname{Im} z \geq-M h \log h^{-1}$. Then,

$$
\|G \varphi\|_{L^{2}} \leq C h^{2 / 3}\|\varphi\|_{L^{2}}
$$

Proof. Let $\chi_{\epsilon} \in S_{\epsilon}\left(T^{*} \partial \Omega\right)$ have $\chi_{\epsilon} \equiv 1$ on $\left\{\left|1-\left|\xi^{\prime}\right|_{g}\right| \leq h^{\epsilon}\right\}$ with $\operatorname{supp} \chi_{\epsilon} \subset\left\{\left|1-\left|\xi^{\prime}\right|_{g}\right| \leq 2 h^{\epsilon}\right\}$ and $X=\operatorname{Op}_{\mathrm{h}}\left(\chi_{\epsilon}\right)$. Then $X \varphi=\varphi+O\left(h^{\infty}\right) \varphi$.

By Lemma 6.6.4, there exists $1 / 2>\delta>0$ such that for any $x_{0} \in \partial \Omega$ and $\beta>0$, if $\zeta_{1} \in S_{\delta} \cap C_{c}^{\infty}(\partial \Omega)$ has $\operatorname{supp} \zeta_{1} \subset\left\{\left|x-x_{0}\right| \leq \beta h^{\delta}\right\}$ and $\zeta_{2} \in S_{\delta} \cap C_{c}^{\infty}(\partial \Omega)$ has $\zeta_{2} \equiv 1$ on $\left\{\left|x-x_{0}\right| \leq 2 \beta h^{\delta}\right\}$ then

$$
\begin{aligned}
& \zeta_{2} G \zeta_{1} X \varphi=G \zeta_{1} X \varphi+O\left(h^{\infty}\right) \varphi \\
& \zeta_{1} G \zeta_{2} X \varphi=\zeta_{1} G X \varphi+O\left(h^{\infty}\right) \varphi
\end{aligned}
$$

Now, by 6.7.11,

$$
\zeta_{i} G \zeta_{j} X \varphi=\zeta_{i} J \beta^{-1} h^{2 / 3} \mathcal{A} i \mathcal{A}_{-} C^{-1} J^{-1} \zeta_{j} X \varphi+O\left(h^{\infty}\right) \varphi
$$

So, since $\mathcal{A} i \mathcal{A}_{-}=O_{L^{2} \rightarrow L^{2}}(1)$, and the $\zeta J$ terms are elliptic semiclassical FIO's, with symbols in $h^{-\alpha} S_{\delta}$ for some $\alpha>0$, we have

$$
\left\|\zeta_{i} G \zeta_{j} X \varphi\right\|_{L^{2}} \leq C_{0} h^{2 / 3}\left\|\zeta_{j} X \varphi\right\|_{L^{2}}
$$

where $C_{0}$ is a constant depending only on $\Omega$.
Let $x_{i=1}^{R(\epsilon)}$ have $\partial \Omega \subset \bigcup_{i=1}^{R(\epsilon)} B\left(x_{i}, \epsilon\right)$ be such that for all $0<\epsilon<1$,

$$
\sup _{x \in \partial \Omega} \#\left\{i: x \in B\left(x_{i}, 10 \epsilon\right)\right\} \leq M_{\Omega} .
$$

To see that this is possible, see for example [Minicozzi]. Then, $R(\epsilon) \leq c \epsilon^{-d+1}$. Now, let $\left\{\zeta_{i, \beta}\right\}_{i=1}^{R\left(\beta h^{\delta}\right)}$ be a partition of unity with $\operatorname{supp} \zeta_{i, \beta} \subset B\left(x_{i}, 2 \beta h^{\delta}\right)$ and $\zeta_{i, \beta} \equiv 1$ on $B\left(x_{i}, \beta h^{\delta}\right)$.

$$
\begin{gathered}
\sum_{j=1}^{R\left(\beta h^{\delta}\right)}\left\|G \zeta_{i, \beta} X X^{*} \zeta_{j, \beta} G^{*}\right\|^{1 / 2} \leq C M_{\Omega} h^{2 / 3}+O\left(h^{\infty}\right) \leq C_{\Omega} h^{2 / 3} \\
\sum_{j=1}^{R\left(\beta h^{\delta}\right)}\left\|X^{*} \zeta_{i, \beta} G^{*} G \zeta_{j, \beta} X\right\|^{1 / 2}=\sum_{i=1}^{C h^{-\epsilon}}\left\|X^{*} \zeta_{i, \beta} G^{*} \zeta_{i, 4 \beta} \zeta_{j, 4 \beta} G \zeta_{j} X\right\|^{1 / 2}+O\left(h^{\infty}\right) \\
\leq C_{\Omega} h^{2 / 3}
\end{gathered}
$$

Hence, by the Cotlar-Stein Lemma (see for example [87, Theorem C.5]),

$$
\|G X\|_{L^{2} \rightarrow L^{2}}=\left\|\sum_{j} G \zeta_{j, \beta} X\right\|_{L^{2} \rightarrow L^{2}} \leq C_{\Omega} h^{2 / 3}
$$

Combining Lemma 6.7.2 with Lemma 6.6.4, the $L^{2}$ boundedness of semiclassical FIOs associated to canonical graphs gives, and Lemma 6.3.5 gives the following improvement of Theorem 6.1 in the case that $\Omega$ is strictly convex with smooth boundary

Theorem 6.7. Let $\Omega \subset \mathbb{R}^{d}$ be strictly convex with smooth boundary. Then there exists $\lambda_{0}>0$ such that for some $C$ and all $|\lambda|>\lambda_{0}$ the following estimate holds

$$
\|G(\lambda)\|_{L^{2}(\partial \Omega) \rightarrow L^{2}(\partial \Omega)} \leq C\langle\lambda\rangle^{-\frac{2}{3}} e^{D_{\partial \Omega}(\operatorname{Im} \lambda)_{-}}
$$

Remark: The improvement from Theorem 6.1 is that we have removed the $\log \lambda$ from the right hand side of 6.3.6)

## Microlocal description of $\tilde{N}$ near glancing

To obtain a microlocal description of $\tilde{N}$ near glancing, we combine Proposition 6.1.4 with the microlocal decomposition of $G$ and the microlocal parametrix for $\mathcal{N}_{e}$ constructed in Appendix 5. In particular, for $g$ microlocalized near glancing point $\left(y_{0}, \eta_{0}\right)$,

$$
\begin{gathered}
G g=J \beta^{-1} h^{2 / 3} \mathcal{A} i \mathcal{A}_{-} C^{-1} J^{-1}+O\left(h^{\infty}\right) g \\
\mathcal{N}_{2} g=J\left(h^{-2 / 3} C \Phi_{-}+B\right) J^{-1} g+O\left(h^{\infty}\right) g
\end{gathered}
$$

where $\mathcal{N}_{2}$ as denotes the exterior Dirichlet to Neumann map. Now, $\mathcal{N}_{2}$ has microsupport contained in an $h^{\epsilon}$ neighborhood of the diagonal and hence $\mathcal{N}_{2} g$ remains microlocalized near
glancing and we can use the microlocal model (6.7) in the composition $G \mathcal{N}_{2}$. Proposition 6.1.4 implies that

$$
\begin{aligned}
\tilde{N} g= & \frac{1}{2} g-G \mathcal{N}_{2} g \\
= & \frac{1}{2} g-\beta^{-1} J\left(\mathcal{A}^{2} \mathcal{A}_{-}^{\prime}+h^{2 / 3} \mathcal{A} i \mathcal{A}_{-} C^{-1} B\right) J^{-1} g+O_{\Psi^{-\infty}}\left(h^{\infty}\right) g \\
= & -\beta^{-1} J\left(\frac{1}{2}\left(\mathcal{A} i^{\prime} \mathcal{A}_{-}+\mathcal{A} i \mathcal{A}_{-}^{\prime}\right)\right. \\
& \quad+h^{2 / 3} \mathcal{A}_{-}{\left.\mathcal{A} i C^{-1} B\right) J^{-1} g+\mathcal{O}_{\Psi^{-\infty}}\left(h^{\infty}\right) g}=-\beta^{-1} J\left(\mathcal{A}^{\prime} \mathcal{A}_{-}+h^{2 / 3} \mathcal{A}_{\mathcal{A}} \mathcal{A}_{-} C^{-1} B\right) J^{-1} g-\frac{1}{2} g+O_{\Psi^{-\infty}}\left(h^{\infty}\right) g
\end{aligned}
$$

Hence, for $g$ microlocalized near glancing

$$
\begin{equation*}
\tilde{N} g=\frac{1}{2} g-\beta^{-1} J\left(\mathcal{A} i \mathcal{A}_{-}^{\prime}+h^{2 / 3} \mathcal{A} i \mathcal{A}_{-} C^{-1} B\right) J^{-1} g+O_{\Psi^{-\infty}}\left(h^{\infty}\right) g \tag{6.7.13}
\end{equation*}
$$

So, by analogous arguments to those in Lemma 6.7.2, we have
Lemma 6.7.3. Suppose that $\varphi \in L^{2}(\partial \Omega)$, and there exists $\epsilon>0$ such that $\operatorname{MS}_{\mathrm{h}}(\varphi) \subset$ $\left\{\left|1-\left|\xi^{\prime}\right|_{g}\right| \leq h^{\epsilon}\right\}$, and $\operatorname{Im} z \geq-M h \log h^{-1}$. Then,

$$
\|\tilde{N} \varphi\|_{L^{2}} \leq C\|\varphi\|_{L^{2}}
$$

Combining Lemma 6.7.3 with Lemma 6.6.4, the $L^{2}$ boundedness of semiclassical FIOs associated to canonical graphs, and Lemma 6.3 .5 gives the following improvement of Theorem 6.1 in the case that $\Omega$ is strictly convex with smooth boundary

Theorem 6.8. Let $\Omega \subset \mathbb{R}^{d}$ be strictly convex with smooth boundary. Then there exists $\lambda_{0}>0$ such that for some $C$ and all $|\lambda|>\lambda_{0}$ the following estimate holds

$$
\|\tilde{N}\|_{L^{2}(\partial \Omega) \rightarrow L^{2}(\partial \Omega)} \leq C e^{D_{\partial \Omega}(\operatorname{Im} \lambda)-}
$$

Remark: This theorem improves the estimate for $\tilde{N}$ in Theorem 6.1 by removing the factor $\langle\lambda\rangle^{\frac{1}{6}} \log \langle\lambda\rangle$. The improved estimate is sharp in the case of a strictly convex domain as can be seen by taking Neumann eigenfunctions on the ball.

## Microlocal description of $\partial_{\nu} \mathcal{D} \ell$ and $\mathcal{D} \ell$ near glancing

Now, let $u$ solve 6.7.1 with $f_{i} \equiv 0$ and $g_{2}=0$ and $g_{1}=g$ microlocalized sufficiently close to a glancing point $\left(y_{0}, \eta_{0}\right)$ so that the parametrices from Appendix 5 can be constructed. In particular, let $\psi$ be as in (6.7.4) and assume the $\mathrm{Op}_{\mathrm{h}}(\psi) g=g+O_{\Psi^{-\infty}}\left(h^{\infty}\right) g$.

We know that $u=\mathcal{D} \ell g$ and so by Lemma 6.1.1 $\left.u_{1}\right|_{\partial \Omega}=-\frac{1}{2} g+\tilde{N} g,\left.u_{2}\right|_{\partial \Omega}=\frac{1}{2} g+\tilde{N} g$. Motivated by this and 6.7.13), let

$$
\begin{aligned}
& w_{1}=-\beta^{-1} A_{2, g} g-\beta^{-1} h^{2 / 3} A_{1, g} J C^{-1} B J^{-1} g \\
& w_{2}=-\beta^{-1} \mathcal{H}_{d} J\left(\mathcal{A} i^{\prime} \mathcal{A}_{-}+h^{2 / 3} \mathcal{A}_{-} \mathcal{A} i C^{-1} B J^{-1} g\right.
\end{aligned}
$$

where $A_{i, g}$ are as in Lemma 5.5.1. Then,

$$
\begin{align*}
&\left.w_{1}\right|_{\partial \Omega}=-\beta^{-1} J\left(\mathcal{A} i \mathcal{A}_{-}^{\prime}+h^{2 / 3} \mathcal{A} i \mathcal{A}_{-} C^{-1} B\right) J^{-1} g+O_{\Psi^{-\infty}}\left(h^{\infty}\right) g \\
&\left.w_{2}\right|_{\partial \Omega}=- \beta^{-1} J\left(\mathcal{A} i^{\prime} \mathcal{A}_{-}+h^{2 / 3} \mathcal{A} i \mathcal{A}_{-} C^{-1} B\right) J^{-1} g+O_{\Psi^{-\infty}}\left(h^{\infty}\right) g \\
&\left.\partial_{\nu_{1}} w_{1}\right|_{\partial \Omega}=\beta^{-1} J\left(h^{-2 / 3} C \mathcal{A} i^{\prime} \mathcal{A}_{-}^{\prime}+B \mathcal{A} i \mathcal{A}_{-}^{\prime}+C \mathcal{A} i^{\prime} \mathcal{A}_{-} C^{-1} B J^{-1} g\right. \\
&\left.+h^{2 / 3} B \mathcal{A} i \mathcal{A}_{-} C^{-1} B\right) J^{-1} g+O_{\Psi^{-\infty}}\left(h^{\infty}\right) g  \tag{6.7.14}\\
&\left.\partial_{\nu_{2}} w_{2}\right|_{\partial \Omega}=- \beta^{-1} J\left(h^{-2 / 3} C \mathcal{A} i^{\prime} \mathcal{A}_{-}^{\prime}+C \mathcal{A} i \mathcal{A}_{-}^{\prime} C^{-1} B+B \mathcal{A} i^{\prime} \mathcal{A}_{-}\right) J^{-1} g \\
&\left.-h^{2 / 3} \beta^{-1} J B \mathcal{A} i \mathcal{A}_{-} C^{-1} B\right) J^{-1} g+O_{\Psi^{-\infty}}\left(h^{\infty}\right) g
\end{align*}
$$

Thus,

$$
\begin{aligned}
\left.\partial_{\nu_{1}} w_{1}\right|_{\partial \Omega}+\left.\partial_{\nu_{2}} w_{2}\right|_{\partial \Omega} & =O_{\Psi^{-\infty}}\left(h^{\infty}\right) g \\
w_{1}-w_{2} & =g+O_{\Psi^{-\infty}}\left(h^{\infty}\right) g
\end{aligned}
$$

where we have used the Wronksian for the Airy equation in simplifying the expressions in (6.7.14).

Thus, $\left(u_{1}-w_{1}, u_{2}-w_{2}\right)$ solves (6.7.1) with

$$
\|\chi f\|_{H_{h}^{N}\left(\mathbb{R}^{d}\right)}+\left\|g_{1}\right\|_{H_{h}^{N}(\partial \Omega)}+\left\|g_{2}\right\|_{H_{h}^{N}(\partial \Omega)}=O\left(h^{\infty}\right)\|g\|_{H^{-N}}
$$

for any $N>0$. This gives

$$
u_{i}=w_{i}+O_{\Psi^{-\infty}}\left(h^{\infty}\right) g
$$

So we have that

$$
\begin{align*}
\partial_{\nu} \mathcal{D} \ell g= & \beta^{-1} J\left(h^{-2 / 3} C \mathcal{A} i^{\prime} \mathcal{A}_{-}^{\prime}+B \mathcal{A} i \mathcal{A}_{-}^{\prime}+C \mathcal{A} i^{\prime} \mathcal{A}_{-} C^{-1} B\right) J g \\
& \left.+h^{2 / 3} \beta^{-1} J B \mathcal{A} i \mathcal{A}_{-} C^{-1} B\right) J^{-1} g+O_{\Psi^{-\infty}}\left(h^{\infty}\right) g  \tag{6.7.15}\\
= & \beta^{-1} h^{-2 / 3} J C \mathcal{A} i^{\prime} \mathcal{A}_{-}^{\prime} J^{-1} g+J O_{H_{h}^{s} \rightarrow H_{h}^{s}}(1) J^{-1} g
\end{align*}
$$

and

$$
\begin{equation*}
\left.\mathcal{D} \ell g\right|_{\Omega_{1}}=-\beta^{-1} A_{2, g} g-\beta^{-1} h^{2 / 3} A_{1, g} J C^{-1} B J^{-1} g+O_{\mathcal{D}^{\prime}(\partial \Omega) \rightarrow C^{\infty}\left(\Omega_{1}\right)}\left(h^{\infty}\right) \tag{6.7.16}
\end{equation*}
$$

for any $\operatorname{Im} z=O\left(h \log h^{-1}\right)$.
Lemma 6.7.4. Suppose that $\varphi \in L^{2}(\partial \Omega)$ and there exists $\epsilon>0$ such that $\operatorname{MS}_{\mathrm{h}}(\varphi) \subset$ $\left\{\left|1-\left|\xi^{\prime}\right|_{g}\right| \leq h^{\epsilon}\right\}$, and $\operatorname{Im} z \geq-M h \log h^{-1}$. Then,

$$
\left\|\partial_{\nu} \mathcal{D} \ell \varphi\right\|_{L^{2}} \leq C h^{1-\epsilon / 2}\|\varphi\|_{L^{2}}
$$

Proof. Let $\chi_{\epsilon} \in S_{\epsilon}\left(T^{*} \partial \Omega\right)$ have $\chi_{\epsilon} \equiv 1$ on $\left\{\left|1-\left|\xi^{\prime}\right|_{g}\right| \leq h^{\epsilon}\right\}$ with supp $\chi_{\epsilon} \subset\left\{\left|1-\left|\xi^{\prime}\right|_{g}\right| \leq 2 h^{\epsilon}\right\}$ and $X=\operatorname{Op}_{\mathrm{h}}\left(\chi_{\epsilon}\right)$. Then $X \varphi=\varphi+O\left(h^{\infty}\right) \varphi$. Fix $0<\epsilon_{2}<\epsilon_{1}=\epsilon$. Then let $x_{0} \in \partial \Omega$ and $\zeta_{1}, \zeta_{2} \in C_{c}^{\infty}(\partial \Omega) \cap S_{\epsilon}$ such that $\zeta_{i} \equiv 1$ on $\left\{\left|x-x_{0}\right|<C h^{\epsilon_{i}}\right\}$ and $\operatorname{supp} \zeta_{1} \subset\left\{\left|x-x_{0}\right|<2 C h^{\epsilon_{i}}\right\}$. Then by Lemma 6.6.4,

$$
\zeta_{2} \partial_{\nu} \mathcal{D} \ell \zeta_{1} X \varphi=\partial_{\nu} \mathcal{D} \ell \zeta_{1} X \varphi+O\left(h^{\infty}\right) \varphi
$$

Now, by 6.7.11),

$$
\begin{aligned}
\zeta_{2} \partial_{\nu} \mathcal{D} \ell \zeta_{1} X \varphi=-\zeta_{2} J \beta^{-1} h^{-2 / 3} \mathcal{A}_{-}^{\prime} \mathcal{A} i^{\prime} C J^{-1} J\left(1+O_{L^{2} \rightarrow L^{2}}\left(h^{2 / 3}\right)\right) J^{-1} \zeta_{1} X \varphi & \\
& +O_{L^{2} \rightarrow L^{2}}\left(h^{\infty}\right) \varphi .
\end{aligned}
$$

Next, observe that on for $|\operatorname{Im} z| \leq|\operatorname{Re} z|^{-1 / 2}$,

$$
\left|A i^{\prime}(z) A_{-}^{\prime}(z)\right| \leq C\langle z\rangle^{1 / 2}
$$

and $\zeta_{i} J$ are elliptic semiclassical FIO's, with symbol in $h^{-\alpha} S_{\delta}$ for some $\alpha>0$. Therefore,

$$
\left\|\zeta_{2} J h^{-2 / 3} \mathcal{A}_{-}^{\prime} \mathcal{A} i^{\prime} C J^{-1} \zeta_{1} X \varphi\right\|_{L^{2}} \leq C_{0} h^{-1+\epsilon / 2}\left\|\zeta_{1} X \varphi\right\|
$$

where $C_{0}$ is a constant depending only on $\Omega$. Taking a partitions of unity as in Lemma 6.7.2 completes the proof.

Combining Lemma 6.7.4 with Lemma 6.6.4, the $L^{2}$ boundedness of semiclassical FIOs associated to canonical graphs gives, and Lemma 6.3.5 gives the following improvement of Theorem 6.1

Theorem 6.9. Let $\Omega \subset \mathbb{R}^{d}$ be strictly convex with smooth boundary. Then there exists $\lambda_{0}>0$ such that for some $C$ and all $|\lambda|>\lambda_{0}$ the following estimate holds

$$
\left\|\partial_{\nu} \mathcal{D} \ell(\lambda)\right\|_{H^{1}(\partial \Omega) \rightarrow L^{2}(\partial \Omega)} \leq C\langle\lambda\rangle e^{D_{\partial \Omega}(\operatorname{Im} \lambda)_{-}} .
$$

## Chapter 7

## Harmonic Analysis of $-\Delta_{\Gamma, \delta}$

We assume that $\Gamma \subset \mathbb{R}^{d}$ is a finite union $\Gamma=\bigcup_{j=1}^{m} \Gamma_{j}$, where each $\Gamma_{j}$ is a compact subset of an embedded $C^{1,1}$ hypersurface; equivalently, by subdividing we may take $\Gamma_{j}$ to be a compact subset of the graph of a $C^{1,1}$ function with respect to some coordinate. Here, $C^{1,1}$ is the space of functions whose first derivatives are Lipschitz continuous. The Bunimovich stadium is an example of a domain in two dimensions with boundary that is $C^{1,1}$, but not $C^{2}$. Let $\delta_{\Gamma}$ denote $(d-1)$-dimensional Hausdorff measure on $\Gamma$, which on each $\Gamma_{j}$ agrees with the Lebesgue induced surface measure on $\Gamma_{j}$, and let $L^{2}(\Gamma)$ be the associated space of square-integrable functions on $\Gamma$. Although the compact sets $\Gamma_{j}$ and $\Gamma$ may be irregular, the estimates we need in $L^{2}(\Gamma)$ will follow from $L^{2}$ estimates over the hypersurfaces containing the $\Gamma_{j}$, hence the detailed analysis in this paper will take place on $C^{1,1}$ hypersurfaces. Let $\gamma u$ denote restriction of $u$ to $\Gamma$. We take $V$ to be a bounded, self-adjoint operator on $L^{2}(\Gamma)$, and for $u \in H_{\text {loc }}^{1}\left(\mathbb{R}^{d}\right)$ define $\left(V \otimes \delta_{\Gamma}\right) u:=(V \gamma u) \delta_{\Gamma}$. Let $-\Delta_{\Gamma, \delta}$ be the unbounded self-adjoint operator

$$
-\Delta_{\Gamma, \delta}:=-\Delta+V \otimes \delta_{\Gamma}
$$

(See Section 7.1 for the formal definition of $-\Delta_{\Gamma, \delta}$.) We will show that $\sigma_{\text {ess }}\left(-\Delta_{\Gamma, \delta}\right)=[0, \infty)$ (the essential spectrum of $-\Delta_{\Gamma, \delta}$ ), and that there are at most a finite number of eigenvalues, each of finite rank, in the interval $(-\infty, 0]$. In contrast to the case of potentials $V \in L_{\text {comp }}^{\infty}\left(\mathbb{R}^{d}\right)$ (see [60, Section XIII.13] or [21, Section 3.2]), there may be embedded eigenvalues in [0, $\infty$ ), which can arise from the allowed non-local nature of $V \otimes \delta_{\Gamma}$.

Resonances are defined as poles of the meromorphic continuation from $\operatorname{Im} \lambda \gg 1$ of the resolvent

$$
R_{V}(\lambda)=\left(-\Delta_{\Gamma, \delta}-\lambda^{2}\right)^{-1}
$$

If the dimension $d$ is odd, $R_{V}(\lambda)$ admits a meromorphic continuation to the entire complex plane, and to the logarithmic covering space of $\mathbb{C} \backslash\{0\}$ if $d$ is even (see Section 7.2). In even dimensions we will restrict attention to $-\pi \leq \arg \lambda \leq 2 \pi$, so $\operatorname{Im} \lambda>0$ implies $0<\arg \lambda<\pi$.

The imaginary part of a resonance gives the decay rate of the corresponding term in the resonance expansion of solutions to the wave equation. Thus, resonances close to the real axis give information about long term behavior of waves. In particular, since the seminal


Figure 7.1: Examples of a finite union of compact subsets of strictly convex hypersurfaces, and of the boundary of a domain of $C^{1,1}$ regularity.
work of Lax-Phillips [46] and Vainberg [78], resonance free regions near the real axis have been used to understand decay of waves.

In this chapter, we demonstrate the existence of a resonance free region for $-\Delta_{\Gamma, \delta}$ on a general class of $\Gamma$.

Theorem 7.1. Let $\Gamma \subset \mathbb{R}^{d}$ be a finite union of compact subsets of embedded $C^{1,1}$ hypersurfaces, and suppose $V$ is a bounded self-adjoint operator on $L^{2}(\Gamma)$. Then for all $\epsilon>0$ there exists $R_{\epsilon}<\infty$ such that, if $\lambda$ is a resonance for $-\Delta_{\Gamma, \delta}$, then

$$
\begin{equation*}
\operatorname{Im} \lambda \leq-\left(\frac{1}{2} D_{\Gamma}^{-1}-\epsilon\right) \log (|\operatorname{Re} \lambda|) \quad \text { if } \quad|\operatorname{Re} \lambda| \geq R_{\epsilon} \tag{7.0.1}
\end{equation*}
$$

where $D_{\Gamma}$ is the diameter of the set $\Gamma$. If $d=1$ then we can replace $\frac{1}{2}$ by 1 in 7.0.1). For $d \geq 2$, if $\Gamma$ can be written as a finite union of compact subsets of strictly convex embedded $C^{2,1}$ hypersurfaces, then we can replace $\frac{1}{2}$ by $\frac{2}{3}$ in (7.0.1).

By a strictly convex hypersurface we understand that, with proper choice of normal direction, the second fundamental form of the hypersurface is strictly positive definite, as in the example on the left in Figure 7.1.
Remark: Note that in this Chapter we take $V$ bounded on $L^{2}$ independent of $\lambda$ for simplicity. However, with appropriate changes in the constants in 7.0.1, our results generalize to the case that $V=V(\lambda)$ is an analytic family of operators with

$$
\|V\|_{L^{2}(\Gamma) \rightarrow L^{2}(\Gamma)} \leq\langle\lambda\rangle^{\alpha}
$$

for some $\alpha<2 / 3$.

## Remarks:

- In case of potential functions $V \in L_{\text {comp }}^{\infty}\left(\mathbb{R}^{d}\right)$ or $V \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$, the resonance free region can be improved. For any $M<\infty$ and $\epsilon>0$, the inequality in (7.0.1) can be replaced by, see [45], [50], and [21, Section 3.2]

$$
\operatorname{Im} \lambda \leq\left\{\begin{array}{l}
-\left(D_{\mathrm{supp} V}^{-1}-\epsilon\right) \log (|\operatorname{Re} \lambda|) \quad \text { if } \quad|\operatorname{Re} \lambda| \geq R_{\epsilon}, \quad V \in L_{\mathrm{comp}}^{\infty}\left(\mathbb{R}^{d}\right), \\
-M \log (|\operatorname{Re} \lambda|) \quad \text { if } \quad|\operatorname{Re} \lambda| \geq R_{M}, \quad V \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)
\end{array}\right.
$$

- The bounds on the size of the resonance free region for $-\Delta_{\Gamma, \delta}$ are not generally optimal, for example in the case that $\Gamma=\partial B(0,1) \subset \mathbb{R}^{2}$. In Chapter 8 , we use a microlocal analysis of the transmission problem (7.0.5) to obtain sharp bounds in the case that $\Gamma=\partial \Omega$ is $C^{\infty}$ with $\Omega$ strictly convex. In this case, one can replace the constant $\frac{1}{2}$ in (1) by 1 and, under certain nontrapping conditions, one obtains an arbitrarily large logarithmic resonance free region as in the case of $V \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$. In particular, if all billiards trajectories eventually leave the support of $V$, then one obtains an arbitrarily large logarithmic resonance free region.
- Cardoso, Popov, and Vodev [13, 14, 58] studied resonances and local energy decay for the transmission problem with differing wave speeds on $\Omega$ and $\mathbb{R}^{d} \backslash \Omega$, and with prescribed matching conditions at $\partial \Omega$.

We next show that the operator $-\Delta_{\Gamma, \delta}$ and its resonances are a good model for the resonances of $-\Delta+V$ when $V \in L_{\text {comp }}^{\infty}$ is supported in a narrow interaction region. Let $\left(x^{\prime}, x_{d}\right)$ be coordinates in a collar neighborhood of $\Gamma$ with $\Gamma=\left\{x_{d}=0\right\}$ and $\partial_{\nu_{x}^{\prime}}=\partial_{x_{d}}$. Then for $V$ supported in the collar neighborhood, define

$$
V_{\epsilon}\left(x^{\prime}, x_{d}\right):=\epsilon^{-1} V\left(x^{\prime}, \epsilon^{-1} x_{d}\right) \quad \text { and } \quad V_{\bmod }\left(x^{\prime}\right):=\int V\left(x^{\prime}, x_{d}\right) d x_{d}
$$

Then
Theorem 7.2. Suppose that $\Gamma=\cup_{i} \Gamma_{i}$ where $\Gamma_{i}$ are non-intersecting embedded $C^{1,1}$ hypersurfaces. Then for $V \in L_{\text {comp }}^{\infty}$ supported in a small enough collar neighborhood of $\Gamma,-\Delta+V_{\epsilon}$ converges to $-\Delta+V_{\bmod } \otimes \delta_{\Gamma}$ in the norm resolvent sense and, moreover, the resonances of $-\Delta+V_{\epsilon}$ converge to those of $-\Delta_{\Gamma, \delta}$ uniformly in compact sets.

We next use the results on resonance free regions and the estimates in Theorem 6.1 to analyze the long term behavior of waves scattered by the potential $V \otimes \delta_{\Gamma}$. Theorem 7.1 implies that there are only a finite number of resonances in the set $\operatorname{Im} \lambda>-A$, for any $A<\infty$. We give a resonance expansion in odd dimensions for the wave equation

$$
\begin{equation*}
\left(\partial_{t}^{2}-\Delta+V \otimes \delta_{\Gamma}\right) u=0, \quad u(0, x)=0, \quad \partial_{t} u(0, x)=g \in L_{\mathrm{comp}}^{2} \tag{7.0.2}
\end{equation*}
$$

with solution given by $U(t) g$, where $U(t) g$ can be expressed as an integral (7.4.1) of the resolvent $R_{V}(\lambda) g$. This is also equivalent to the more standard functional calculus expression $\sqrt{-\Delta_{\Gamma, \delta}}-1 \sin \left(t \sqrt{-\Delta_{\Gamma, \delta}}\right) g$.

Let $m_{R}(\lambda)$ be the order of the pole of $R_{V}(\lambda)$ at $\lambda$. We let $\mathcal{D}_{N}$ be the domain of $\left(-\Delta_{\Gamma, \delta}\right)^{N}$, and define

$$
\mathcal{D}_{\text {loc }}=\left\{u: \chi u \in \mathcal{D}_{1} \text { whenever } \chi \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right) \text { and } \chi=1 \text { on a neighborhood of } \Gamma\right\} .
$$

Theorem 7.3. Let $d \geq 1$ be odd, and assume that $\Gamma \subset \mathbb{R}^{d}$ is a finite union of compact subsets of embedded $C^{1,1}$ hypersurfaces, and that $V$ is a self-adjoint operator on $L^{2}(\Gamma)$.

Let $0>-\mu_{1}^{2}>\cdots>-\mu_{K}^{2}$ and $0<\nu_{1}^{2}<\cdots<\nu_{M}^{2}$ be the nonzero eigenvalues of $-\Delta_{\Gamma, \delta}$, with $\mu_{j}, \nu_{j}>0$, and $\left\{\lambda_{k}\right\}$ the resonances with $\operatorname{Im} \lambda<0$. Then for any $A>0$ and $g \in L_{\mathrm{comp}}^{2}$, the solution $U(t) g$ to (7.0.2 admits an expansion

$$
\begin{align*}
U(t) g=\sum_{j=1}^{K}\left(2 \mu_{j}\right)^{-1} e^{t \mu_{j}} \Pi_{\mu_{j}} g+t \Pi_{0} g & +\mathcal{P}_{0} g+\sum_{k=1}^{M} \nu_{j}^{-1} \sin \left(t \nu_{j}\right) \Pi_{\nu_{j}} g \\
& +\sum_{\operatorname{Im} \lambda_{k}>-A} \sum_{\ell=0}^{m_{R}\left(\lambda_{k}\right)-1} e^{-i t \lambda_{k}} t^{\ell} \mathcal{P}_{\lambda_{k}, \ell} g+E_{A}(t) g \tag{7.0.3}
\end{align*}
$$

where $\Pi_{\mu_{j}}$ and $\Pi_{\nu_{j}}$ respectively denote the projections onto the $-\mu_{j}^{2}$ and $\nu_{j}^{2}$ eigenspaces, and $\Pi_{0}$ the projection onto the 0-eigenspace. The maps $\mathcal{P}_{\lambda_{k}, \ell}$ and $\mathcal{P}_{0}$ are bounded from $L_{\text {comp }}^{2} \rightarrow \mathcal{D}_{\text {loc }}$.

The operator $E_{A}(t): L_{\text {comp }}^{2} \rightarrow L_{\text {loc }}^{2}$ has the following property: for any $\chi \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ equal to 1 on a neighborhood of $\Gamma$, and $N \geq 0$, there exists $T_{A, \chi, N}<\infty$ and $C_{A, \chi, N}<\infty$ so that

$$
\left\|\chi E_{A}(t) \chi\right\|_{L^{2} \rightarrow \mathcal{D}_{N}} \leq C_{A, \chi, N} e^{-A t}, \quad t>T_{A, \chi, N}
$$

We refer to Section 7.4 for more details on the operators $\mathcal{P}_{\lambda_{k}, \ell}$ and $\mathcal{P}_{0}$. The restriction that $t$ be larger than a constant depending on the diameter of $\chi$ is necessary to ensure that $\chi E_{A}(t) \chi g$ has no $H^{2 N}$ singularities away from $\Gamma$ in $\operatorname{supp}(\chi)$, although our argument does not give an optimal value for $T_{A, \chi, N}$.

Under the assumption that $\Gamma=\partial \Omega$ for a bounded open domain $\Omega \subset \mathbb{R}^{d}$, and that $V$ and $\partial \Omega$ satisfy higher regularity assumptions, for $g \in L^{2}$ we obtain estimates on $\chi E_{A}(t) \chi g$ in the spaces

$$
\mathcal{E}_{N}:=H^{1}\left(\mathbb{R}^{d}\right) \cap\left(H^{N}(\Omega) \oplus H^{N}\left(\mathbb{R}^{d} \backslash \bar{\Omega}\right)\right), \quad N \geq 1
$$

If $\partial \Omega$ is of $C^{1,1}$ regularity, and $V$ is bounded $H^{\frac{1}{2}}(\partial \Omega) \rightarrow H^{\frac{1}{2}}(\partial \Omega)$, then we show in Section 2.3 that $\mathcal{D}_{1} \subset \mathcal{E}_{2}$, and convergence in $\mathcal{E}_{2}$ follows from Theorem 7.3. If $\partial \Omega$ is of $C^{\infty}$ regularity, and $V$ is bounded $H^{s}(\partial \Omega) \rightarrow H^{s}(\partial \Omega)$ for every $s$, then $\mathcal{D}_{N}$ is a closed subspace of $\mathcal{E}_{2 N}$ (see (7.1.3) below), and we have the following.

Theorem 7.4. Suppose that $\Gamma=\partial \Omega$ is $C^{\infty}$, and that $V$ is a self-adjoint map on $L^{2}(\partial \Omega)$ which is bounded from $H^{s}(\partial \Omega) \rightarrow H^{s}(\partial \Omega)$ for all $s$. Then the operator $E_{A}(t)$ defined in (7.0.3) has the following property: for any $\chi \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ equal to 1 on a neighborhood of $\bar{\Omega}$, and $N \geq 1$, there exists $T_{A, \chi, N}<\infty$ and $C_{A, \chi, N}<\infty$ so that

$$
\left\|\chi E_{A}(t) \chi\right\|_{L^{2} \rightarrow \mathcal{E}_{N}} \leq C_{A, \chi, N} e^{-A t}, \quad t>T_{A, \chi, N}
$$

In addition to describing resonances as poles of the meromorphic continuation of the resolvent, we will give a more concrete description of resonances in Sections 7.2 and 7.4 . We
show in Proposition 7.2.2 and the comments following it, that if $\Gamma$ is as in Theorem 7.1 then $\lambda$ is a resonance if and only if there is a nontrivial $\lambda$-outgoing solution $u \in H_{\mathrm{loc}}^{1}\left(\mathbb{R}^{d}\right)$ to the equation

$$
\begin{equation*}
\left(-\Delta-\lambda^{2}+V \otimes \delta_{\Gamma}\right) u=0 \tag{7.0.4}
\end{equation*}
$$

Here we say that $u$ is $\lambda$-outgoing if for some $R<\infty$, and some compactly supported distribution $g$, we can write

$$
u(x)=\left(R_{0}(\lambda) g\right)(x) \quad \text { for } \quad|x| \geq R
$$

In case $d=1$ this definition needs to be modified for $\lambda=0$. Noting that for $d=1$ and $\lambda \neq 0$, a $\lambda$-outgoing solution equals $c_{\operatorname{sgn}(x)} e^{i \lambda|x|}$ for $|x| \geq R$, we say $u$ is 0 -outgoing when $d=1$ if $u$ is separately constant on $x \geq R$ and $x \leq-R$, for some $R<\infty$.

In case $\Gamma=\partial \Omega$ for a bounded domain $\Omega$, and $V: H^{\frac{1}{2}}(\partial \Omega) \rightarrow H^{\frac{1}{2}}(\partial \Omega)$, we show that $\lambda$ outgoing solvability of $(7.0 .4$ is equivalent to solving the following transmission problem. We remark that the Sobolev spaces $H^{s}(\partial \Omega)$ are well defined for $|s| \leq 2$ if $\partial \Omega$ is $C^{1,1}$, since these spaces are preserved under $C^{1,1}$ changes of coordinates. Also, since $\mathcal{E}_{2, \text { loc }}\left(\mathbb{R}^{d}\right) \subset H_{\text {loc }}^{1}\left(\mathbb{R}^{d}\right)$, if $u \in \mathcal{E}_{2, \text { loc }}$ then its trace $\gamma u$ belongs to $H^{\frac{1}{2}}(\partial \Omega)$.

Proposition 7.0.1. Assume $\Omega$ is a bounded domain in $\mathbb{R}^{d}$ with $\partial \Omega$ a $C^{1,1}$ hypersurface, and $V$ is a self-adjoint map on $L^{2}(\partial \Omega)$ which maps $H^{\frac{1}{2}}(\partial \Omega) \rightarrow H^{\frac{1}{2}}(\partial \Omega)$. Then $\lambda$ is a resonance of $-\Delta_{\partial \Omega, \delta}$ if and only if the following system has a nontrivial solution $u \in \mathcal{E}_{2, \mathrm{loc}}\left(\mathbb{R}^{d}\right)$ such that, with $\left.u\right|_{\Omega}=u_{1},\left.u\right|_{\mathbb{R}^{d} \backslash \bar{\Omega}}=u_{2}$,

$$
\begin{cases}\left(-\Delta-\lambda^{2}\right) u_{1}=0 & \text { in } \Omega  \tag{7.0.5}\\ \left(-\Delta-\lambda^{2}\right) u_{2}=0 & \text { in } \mathbb{R}^{d} \backslash \bar{\Omega} \\ \partial_{\nu} u_{1}+\partial_{\nu^{\prime}} u_{2}+V \gamma u=0 & \text { on } \partial \Omega \\ u_{2} \text { is } \lambda \text {-outgoing } & \end{cases}
$$

Here, $\partial_{\nu}$ and $\partial_{\nu^{\prime}}$ are respectively the interior and exterior normal derivatives of $u$ at $\partial \Omega$.
The outline of this chapter is as follows. In Section 7.1 we present the definition of $-\Delta_{V, \Omega}$ and its domain. In Section 7.2 we demonstrate the meromorphic continuation of $R_{V}(\lambda)$, give the proof of Theorem 7.1, relate resonances to solvability of 7.0.4 by reduction to an equation on $\Gamma$, and prove Proposition 7.0.1. In Section 7.3, we prove Theorem 7.2 , In Section 7.4 we give more details on the structure of the meromorphic continuation of $R_{V}(\lambda)$. We establish mapping bounds for compact cutoffs of $R_{V}(\lambda)$, and use these to prove Theorems 7.3 and 7.4 by a contour integration argument. In Section 7.5 we prove a needed transmission property estimate for boundaries of regularity $C^{1,1}$.

### 7.1 Formal Definition of the Operator

We define the operator $-\Delta_{\Gamma, \delta}$ using the symmetric quadratic form, with dense domain $H^{1}\left(\mathbb{R}^{d}\right) \subset L^{2}\left(\mathbb{R}^{d}\right)$,

$$
Q_{V, \Gamma}(u, w):=\langle\nabla u, \nabla w\rangle_{L^{2}\left(\mathbb{R}^{d}\right)}+\langle V \gamma u, \gamma w\rangle_{L^{2}(\Gamma)} .
$$

For $\Gamma$ a finite union of compact subsets of $C^{1,1}$ hypersurfaces (indeed for $\Gamma$ a bounded subset of a Lipschitz graph), as a special case of (7.4.7) we can bound

$$
\|\gamma u\|_{L^{2}(\Gamma)} \leq C\|u\|_{L^{2}}^{\frac{1}{2}}\|u\|_{H^{1}}^{\frac{1}{2}} \leq C \epsilon\|u\|_{H^{1}}+C \epsilon^{-1}\|u\|_{L^{2}} .
$$

It follows that there exist $c, C>0$ such that

$$
\left|Q_{V, \Gamma}(u, w)\right| \leq C\|u\|_{H^{1}}\|w\|_{H^{1}} \quad \text { and } \quad c\|u\|_{H^{1}}^{2} \leq Q_{V, \Gamma}(u, u)+C\|u\|_{L^{2}}^{2} .
$$

By Reed-Simon [59, Theorem VIII.15], $Q_{V, \Gamma}(u, w)$ is determined by a unique self-adjoint operator $-\Delta_{\Gamma, \delta}$, with domain $\mathcal{D}$ consisting of $u \in H^{1}$ such that $Q_{V, \Gamma}(u, w) \leq C\|w\|_{L^{2}}$ for all $w \in H^{1}\left(\mathbb{R}^{d}\right)$. By Rellich's embedding lemma, the potential term is compact relative to $H^{1}$. It follows by Weyl's essential spectrum theorem, see [60, Theorem XIII.14], that $\sigma_{\text {ess }}\left(-\Delta_{\Gamma, \delta}\right)=[0, \infty)$. Additonally, there are at most a finite number of eigenvalues in $(-\infty, 0]$, each of finite multiplicity.

If $u \in \mathcal{D}$, by the Riesz representation theorem we then have $Q_{V, \Gamma}(u, w)=\langle g, w\rangle$ for some $g \in L^{2}\left(\mathbb{R}^{d}\right)$, and taking $w \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ shows that in the sense of distributions

$$
\begin{equation*}
-\Delta u+(V \gamma u) \delta_{\Gamma}=g \tag{7.1.1}
\end{equation*}
$$

Conversely, if $u \in H^{1}\left(\mathbb{R}^{d}\right)$ and (7.1.1) holds for some $g \in L^{2}\left(\mathbb{R}^{d}\right)$, then by density of $C_{c}^{\infty} \subset H^{1}$ we have $Q_{V, \Gamma}(u, w)=\langle g, w\rangle$ for $w \in H^{1}\left(\mathbb{R}^{d}\right)$, hence $u \in \mathcal{D}$, and $-\Delta_{\Gamma, \delta} u$ is given by the left hand side of (7.1.1). We thus can define

$$
\|u\|_{\mathcal{D}}=\|u\|_{H^{1}}+\left\|\Delta_{\Gamma, \delta} u\right\|_{L^{2}},
$$

where finiteness of the second term carries the assumption that $\Delta_{\Gamma, \delta} u \in L^{2}$.
We set $\mathcal{D}_{1}=\mathcal{D}$, and recursively define $\mathcal{D}_{N} \subset \mathcal{D}_{1}$ for $N \geq 2$ by the condition $\Delta_{\Gamma, \delta} u \in$ $\mathcal{D}_{N-1}$. We also recursively define

$$
\|u\|_{\mathcal{D}_{N}}=\|u\|_{H^{1}}+\left\|\Delta_{\Gamma, \delta} u\right\|_{\mathcal{D}_{N-1}}, \quad N \geq 2
$$

Suppose that $\chi \in C_{c}^{\infty}\left(\mathbb{R}^{d} \backslash \Gamma\right)$ and that $u \in H^{1}\left(\mathbb{R}^{d}\right)$ solves 7.1.1). Then,

$$
\Delta(\chi u)=\chi g+2 \nabla \chi \cdot \nabla u+(\Delta \chi) u \in L^{2}\left(\mathbb{R}^{d}\right)
$$

Hence,

$$
\|\chi u\|_{H^{2}} \leq C_{\chi}\|u\|_{\mathcal{D}} .
$$

That is, $\mathcal{D} \subset H^{1}\left(\mathbb{R}^{d}\right) \cap H_{\text {loc }}^{2}\left(\mathbb{R}^{d} \backslash \Gamma\right)$, with continuous inclusion. Similar arguments show that

$$
\mathcal{D}_{N} \subset H^{1}\left(\mathbb{R}^{d}\right) \cap H_{\mathrm{loc}}^{2 N}\left(\mathbb{R}^{d} \backslash \Gamma\right)
$$

The behavior of $u$ near $\Gamma$ may be more singular. For $V$ and $\Gamma$ as in Theorem 7.1, from (7.1.1) and the fact that $(V \gamma u) \delta_{\Gamma} \in H^{-\frac{1}{2}-\epsilon}\left(\mathbb{R}^{d}\right)$ for all $\epsilon>0$, we conclude that $u \in H^{\frac{3}{2}-\epsilon}\left(\mathbb{R}^{d}\right)$. However, under additional assumptions on $V$ and $\Gamma$ we can give a full description of $\mathcal{D}$ near $\Gamma$.

For the purposes of the remainder of this section we assume that $\Gamma=\partial \Omega$ for some bounded open domain $\Omega \subset \mathbb{R}^{d}$, and that $\partial \Omega$ is a $C^{1,1}$ hypersurface; that is, locally $\partial \Omega$ can be written as the graph of a $C^{1,1}$ function. We assume also that $V: H^{\frac{1}{2}}(\partial \Omega) \rightarrow H^{\frac{1}{2}}(\partial \Omega)$. Then since $u \in H^{1}\left(\mathbb{R}^{d}\right)$, and $\gamma: H^{s}\left(\mathbb{R}^{d}\right) \rightarrow H^{s-\frac{1}{2}}(\partial \Omega)$ for $s \in\left(\frac{1}{2}, 2\right]$, we have $V \gamma u \in H^{\frac{1}{2}}(\partial \Omega)$. By (7.1.1) we can write $u$ as $(-\Delta)^{-1} g$ plus the single layer potential of a $H^{\frac{1}{2}}(\partial \Omega)$ function, hence Proposition 7.5.2 shows that

$$
\mathcal{D} \subset \mathcal{E}_{2}=H^{1}\left(\mathbb{R}^{d}\right) \cap\left(H^{2}(\Omega) \oplus H^{2}\left(\mathbb{R}^{d} \backslash \bar{\Omega}\right)\right)
$$

with continuous inclusion. We remark that $H^{2}(\Omega)$ and $H^{2}\left(\mathbb{R}^{d} \backslash \bar{\Omega}\right)$ can be identified as restrictions of $H^{2}\left(\mathbb{R}^{d}\right)$ functions; see [12] and [71, Theorem VI.5]. Thus, if $u \in \mathcal{D}$ then $u$ has a well defined trace on $\partial \Omega$ of regularity $H^{\frac{3}{2}}(\partial \Omega)$, and the first derivatives of $u$ have one-sided traces from the interior and exterior, of regularity $H^{\frac{1}{2}}(\partial \Omega)$.

For $w \in H^{1}\left(\mathbb{R}^{d}\right)$ and $u \in \mathcal{E}_{2}$, it follows from Green's identities that

$$
Q_{V, \partial \Omega}(u, w)=\langle-\Delta u, w\rangle_{L^{2}(\Omega)}+\langle-\Delta u, w\rangle_{L^{2}\left(\mathbb{R}^{d} \backslash \bar{\Omega}\right)}+\left\langle\partial_{\nu} u+\partial_{\nu^{\prime}} u+V \gamma u, \gamma w\right\rangle_{L^{2}(\partial \Omega)}
$$

where $\partial_{\nu}$ and $\partial_{\nu^{\prime}}$ denote the exterior normal derivatives from $\Omega$ and $\mathbb{R}^{d} \backslash \bar{\Omega}$. Thus, in the case that $V$ is bounded from $H^{\frac{1}{2}}(\partial \Omega) \rightarrow H^{\frac{1}{2}}(\partial \Omega)$, we can completely characterize the domain $\mathcal{D}$ of the self-adjoint operator $-\Delta_{\partial \Omega, \delta}$ as

$$
\begin{equation*}
\mathcal{D}=\left\{u \in \mathcal{E}_{2} \quad \text { such that } \quad \partial_{\nu} u+\partial_{\nu^{\prime}} u+V \gamma u=0\right\} \tag{7.1.2}
\end{equation*}
$$

in which case $\Delta_{\partial \Omega, \delta} u=\left.\left.\Delta u\right|_{\Omega} \oplus \Delta u\right|_{\mathbb{R}^{d} \backslash \bar{\Omega}}$.
If $\partial \Omega$ is a $C^{\infty}$ hypersurface, and $V: H^{s}(\partial \Omega) \rightarrow H^{s}(\partial \Omega)$ is bounded for all $s$, then Proposition 7.5.1 and induction, as in the proof of Lemma 7.4.2, show that $\mathcal{D}_{N} \subset \mathcal{E}_{2 N}$. Induction also shows that $\mathcal{D}_{N}$ can be characterized as the subspace of $\mathcal{E}_{2 N}$ consisting of $u$ that satisfy the following matching conditions:

$$
\begin{gather*}
\gamma\left(\left.\Delta^{j} u\right|_{\Omega}\right)=\gamma\left(\left.\Delta^{j} u\right|_{\mathbb{R}^{d} \backslash \bar{\Omega}}\right),  \tag{7.1.3}\\
\partial_{\nu}\left(\left.\Delta^{j} u\right|_{\Omega}\right)+\partial_{\nu^{\prime}}\left(\left.\Delta^{j} u\right|_{\mathbb{R}^{d} \backslash \bar{\Omega}}\right)+V \gamma\left(\left.\Delta^{j} u\right|_{\Omega}\right)=0, \quad \text { for } \quad 0 \leq j \leq N-1
\end{gather*}
$$

### 7.2 Meromorphy of the resolvent and Relation with Outgoing Solutions

We demonstrate the meromorphic continuation of $R_{V}(\lambda)$ from $\operatorname{Im} \lambda \gg 1$ to $\lambda \in \mathbb{C}$ (to the logarithmic cover in even dimensions) following arguments similar to those in the case where $V \in L_{\text {comp }}^{\infty}\left(\mathbb{R}^{d}\right)$. We assume $\Gamma$ is a finite union of compact subsets of $C^{1,1}$ hypersurfaces. We use $\rho$ to denote a function in $C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ with $\rho=1$ on a neighborhood of $\Gamma$; the following results hold for any such choice of $\rho$. Observe that when the right hand side is defined

$$
R_{V}(\lambda)=R_{0}(\lambda)(I+K(\lambda) \rho)^{-1}(I-K(\lambda)(1-\rho))
$$

where, for $\lambda$ in the domain of $R_{0}(\lambda)$,

$$
K(\lambda)=\left(V \otimes \delta_{\Gamma}\right) R_{0}(\lambda)=\gamma^{*} V \gamma R_{0}(\lambda)
$$

The operator $K(\lambda) \rho: H^{-1}\left(\mathbb{R}^{d}\right) \rightarrow H_{\text {comp }}^{-\frac{1}{2}-\epsilon}$ is compact on $H^{-1}\left(\mathbb{R}^{d}\right)$ by Rellich's embedding theorem. Furthermore, $I+K(\lambda) \rho$ is invertible if $\operatorname{Im} \lambda \gg 1$. To see this, note that $g+$ $K(\lambda) \rho g=0$ and $g \in H^{-1}\left(\mathbb{R}^{d}\right)$ implies that $g=\gamma^{*} f$ where $f \in L^{2}(\Gamma)$. It follows that $f+V G(\lambda) f=0$, which implies $f=0$ for $\operatorname{Im} \lambda \gg 1$ by Theorem 6.2. This also shows that $I+K(\lambda) \rho$ is invertible on $H^{-1}\left(\mathbb{R}^{d}\right)$ if and only if $I+V G(\lambda)$ is invertible on $L^{2}(\Gamma)$.

Then $(I+K(\lambda) \rho)^{-1}$ is a meromorphic family of Fredholm operators on $H^{-1}\left(\mathbb{R}^{d}\right)$ for $\lambda$ in the domain of $R_{0}(\lambda)$. This follows by analytic Fredholm theory, see e.g. Proposition 7.4 of [76, Chapter 9]. Note that for $d=1$ the domain is $\mathbb{C} \backslash\{0\}$. We prove meromorphicity at 0 for $d=1$ following Proposition 7.2 .2 below; for now if $d=1$ we assume $\lambda \in \mathbb{C} \backslash\{0\}$.

Since $K=\gamma^{*} V \gamma R_{0}$, we have that

$$
\begin{equation*}
(I+K(\lambda) \rho)^{-1} \gamma^{*}=\gamma^{*}(I+V G(\lambda))^{-1} \tag{7.2.1}
\end{equation*}
$$

where $(I+V G(\lambda))^{-1}$ acts on $L^{2}(\Gamma)$. Hence,

$$
\begin{aligned}
(I+K(\lambda) \rho)^{-1} & =I-(I+K(\lambda) \rho)^{-1} K(\lambda) \rho \\
& =I-\gamma^{*}(I+V G(\lambda))^{-1} V \gamma R_{0}(\lambda) \rho .
\end{aligned}
$$

The meromorphic extension of the resolvent $R_{V}(\lambda)$ for $-\Delta_{\Gamma, \delta}$ then equals, for any $\rho$ as above,

$$
\begin{align*}
R_{V}(\lambda) & =R_{0}(\lambda)(I+K(\lambda) \rho)^{-1}(I-K(\lambda)(1-\rho))  \tag{7.2.2}\\
& =\left(R_{0}(\lambda)-R_{0}(\lambda) \gamma^{*}(I+V G(\lambda))^{-1} V \gamma R_{0}(\lambda) \rho\right)(I-K(\lambda)(1-\rho))
\end{align*}
$$

In particular, given $g \in H_{\text {comp }}^{-1}$ we can take $\rho g=g$ to obtain

$$
\begin{equation*}
R_{V}(\lambda) g=R_{0}(\lambda) g-R_{0}(\lambda) \gamma^{*}(I+V G(\lambda))^{-1} V \gamma R_{0}(\lambda) g \tag{7.2.3}
\end{equation*}
$$

Consequently, $R_{V}(\lambda): H_{\text {comp }}^{-1} \rightarrow H_{\text {loc }}^{1}$, and its image is $\lambda$-outgoing.

The resolvent set $\Lambda$ is defined as the set of poles of $R_{V}(\lambda)$. Since

$$
(I-K(\lambda)(1-\rho))(I+K(\lambda)(1-\rho))=I,
$$

the preceding arguments show that $\Lambda$ agrees with the poles of $(I+V G(\lambda))^{-1}$, except possibly $\lambda=0$ when $d=1$. If $\|G(\lambda)\|_{L^{2} \rightarrow L^{2}}<\|V\|_{L^{2} \rightarrow L^{2}}^{-1}$, then $I+V G(\lambda)$ is invertible by Neumann series. By Theorem 6.1 and (6.3.4), when $\operatorname{Im} \lambda<0$ this is the case provided that $|\lambda|>2$ and

$$
|\operatorname{Im} \lambda| \leq D_{\Gamma}^{-1}(a \log |\lambda|-\log C-\log (\log |\lambda|))
$$

for some $C$, where $a=\frac{1}{2}$ or $\frac{2}{3}$ or 1 accordingly. This completes the proof of Theorem 7.1.
Remark: For $\lambda$ in the domain of $R_{0}(\lambda)$, the $L^{2}(\Gamma)$ kernel of $I+G(\lambda) V$ is in one-to-one correspondence with the kernel of $I+V G(\lambda)$ by the map $h \rightarrow V h$. That $(I+G(\lambda) V) h=0$ implies $(I+V G(\lambda)) V h=0$ is immediate. Conversely, $(I+V G(\lambda)) f=0$ expresses $f=$ $-V G(\lambda) f:=V h$, and $(I+G(\lambda) V) h=-G(\lambda)(I+V G(\lambda)) f=0$.

To equate resonances to $\lambda$-outgoing solutions of (7.0.4), we use the following extension of the Rellich uniqueness theorem.

Proposition 7.2.1 (Rellich uniqueness). If $\lambda$ belongs to the domain of $R_{0}(\lambda)$, then a global $\lambda$-outgoing solution to $\left(-\Delta-\lambda^{2}\right) u=0$ must vanish identically.
Proof. For $0<\arg \lambda<\pi$ and $g$ a compactly supported distribution, $R_{0}(\lambda) g$ is exponentially decreasing in $|x|$, so Green's identities yield, for $u=R_{0}(\lambda) g$ and for $R \gg 1$, that

$$
u(x)=\int_{|y|=R}\left(G_{0}(\lambda, x, y) \partial_{\nu^{\prime}} u(y)-\partial_{\nu_{y}^{\prime}} G_{0}(\lambda, x, y) u(y)\right) d \sigma(y), \quad|x|>R
$$

By analytic continuation this holds for all $\lambda$ in the domain of $R_{0}(\lambda)$. If $u$ is an entire solution then the right hand side is real-analytic in $R$, and we may let $R \rightarrow 0$ to deduce that $u \equiv 0$.

Proposition 7.2.2. For $\lambda$ in the domain of $R_{0}(\lambda)$, there is a one-to-one correspondence of $\lambda$-outgoing solutions $u \in H_{\mathrm{loc}}^{1}$ to (7.0.4) and solutions $f \in L^{2}(\Gamma)$ to $(I+V G(\lambda)) f=0$, given by $u=R_{0}(\lambda)\left(\gamma^{*} f\right)$, and $f=-V \gamma u$.
Proof. If $(I+V G(\lambda)) f=0, f \in L^{2}(\Gamma)$, then $u=R_{0}(\lambda)\left(\gamma^{*} f\right)$ is a $\lambda$-outgoing solution to $-\Delta_{\Gamma, \delta} u=\lambda^{2} u$. Indeed $u \in H_{\mathrm{loc}}^{1}$ and is $\lambda$-outgoing by definition, $\left(-\Delta-\lambda^{2}\right) u=\gamma^{*} f$, and $\left(V \otimes \delta_{\Gamma}\right) u=\gamma^{*} V G(\lambda) f=-\gamma^{*} f$.

Conversely, if $u \in H_{\text {loc }}^{1}$ is a $\lambda$-outgoing solution to $-\Delta u-\lambda^{2} u=-\left(V \otimes \delta_{\Gamma}\right) u$, then by Proposition 7.2.1

$$
\begin{equation*}
u=-R_{0}(\lambda)\left(V \otimes \delta_{\Gamma}\right) u=-\int_{\Gamma} G_{0}(\lambda, x, y)(V \gamma u)(y) \tag{7.2.4}
\end{equation*}
$$

Hence if $f=-V \gamma u$, then $f+V G(\lambda) f=0$. By (7.2.4 the correspondence between $u$ and $V \gamma u$ is one-to-one. As a result, the space of solutions $u$ for given $\lambda$ is finite dimensional, since it is in one-to-one correspondence with the kernel of a Fredholm operator.

The case $d=1$ and $\lambda=0$. For $d=1$, we need to prove that $R_{V}(\lambda)$ is meromorphic at $\lambda=0$, and equate existence of 0 -outgoing (i.e. separately constant near $\pm \infty$ ) solutions of $-\Delta_{\Gamma, \delta} u=0$ to 0 being a pole. When $d=1, I+V G(\lambda)$ is a matrix valued meromorphic function for $\lambda \in \mathbb{C}$, invertible on $L^{2}(\Gamma) \equiv \mathbb{C}^{m}$ for $\operatorname{Im} \lambda \gg 1$, so $\operatorname{det}(I+V G(\lambda))$, hence $(I+V G(\lambda))^{-1}$, is meromorphic on $\mathbb{C}$. Equation 7.2.3), which holds for $\lambda \in \mathbb{C} \backslash\{0\}$, and meromorphicity of $R_{0}(\lambda)$ on $\mathbb{C}$, then establishes meromorphicity of $R_{V}(\lambda)$ on $\mathbb{C}$, in particular that 0 is either a regular point or a pole. Furthermore, since $\gamma^{*}$ has finite dimensional range, so do the singular terms of $R_{V}(\lambda)$ at $\lambda=0$. It remains to show that $R_{V}(\lambda)$ is singular at 0 if and only if there is a nontrivial solution $u \in H_{\mathrm{loc}}^{1}(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$ to $\Delta u=\gamma^{*} V \gamma u$, since $u \in L^{\infty}(\mathbb{R})$ is equivalent to 0 -outgoing for such $u$. By the discussion preceding (7.4.4) below, a pole at 0 implies existence of a 0 -outgoing solution to (7.0.4). Conversely, if $R_{V}(\lambda)$ is holomorphic at $\lambda=0$, then (7.2.1) and the identity (see [21, Section 2.2])

$$
(I+K(\lambda) \rho)^{-1}=I-\gamma^{*} V \gamma R_{V}(\lambda) \rho
$$

shows that the matrix $(I+V G(\lambda))^{-1}$ is then holomorphic at $\lambda=0$. Considering adjoints, we must then have

$$
\begin{equation*}
\|f\|_{L^{2}(\Gamma)} \leq C\|(I+G(\lambda) V) f\|_{L^{2}(\Gamma)}, \quad f \in L^{2}(\Gamma), \quad|\lambda| \ll 1 \tag{7.2.5}
\end{equation*}
$$

Suppose $u \in H_{\mathrm{loc}}^{1} \cap L^{\infty}$ satisfies $\Delta u=\gamma^{*} V \gamma u$. Let $\Gamma=\left\{x_{1}, \ldots, x_{m}\right\} \subset \mathbb{R}$, and $V \gamma u=$ $\left(c_{1}, \ldots, c_{m}\right) \in \mathbb{C}^{m}$. Then

$$
u(x)=\frac{1}{2} \sum_{x_{j} \in \Gamma} c_{j}\left|x-x_{j}\right|+a x+b, \quad \text { for some } a, b \in \mathbb{C} .
$$

Since $u \in L^{\infty}$ we must have $\sum_{j} c_{j}=0$ and $a=0$. Hence, with $E_{i j}=-\frac{1}{2}\left|x_{i}-x_{j}\right|$, we have

$$
\langle\gamma \mathbf{1}, V \gamma u\rangle=0, \quad(I+E V) \gamma u=\gamma \boldsymbol{b}
$$

where $\mathbf{1}$ and $\boldsymbol{b}$ are constant functions on $\mathbb{R}$. Since $G(\lambda)_{j k}=-(2 i \lambda)^{-1} \exp \left(i \lambda\left|x_{j}-x_{k}\right|\right)$, then for $f \in L^{2}(\Gamma)$

$$
(I+G(\lambda) V) f=-(2 i \lambda)^{-1}\langle\gamma \mathbf{1}, V f\rangle \gamma \mathbf{1}+(I+E V) f+\mathcal{O}(\lambda) f
$$

Assume first that $V \gamma \mathbf{1} \neq 0$, and take $f=\gamma u+2 i \lambda\|V \gamma \mathbf{1}\|^{-2} V \gamma \boldsymbol{b}$. Then $(I+G(\lambda) V) f=\mathcal{O}(\lambda)$, contradicting (7.2.5) unless $\gamma u=0$, hence $u \equiv 0$.

We conclude by showing that $R_{V}(\lambda)$ regular at $\lambda=0$ implies $V \gamma \mathbf{1} \neq 0$. To see this, note that if $V \gamma \mathbf{1}=0$ (in which case $-\Delta_{\Gamma, \delta} \mathbf{1}=0$ would give a 0 -outgoing solution) then $K(\lambda)$ is regular at $\lambda=0$, since for $g \in H_{\text {comp }}^{-1}$

$$
V \gamma R_{0}(\lambda) g=V \gamma\left(R_{0}(\lambda) g+(2 i \lambda)^{-1}\langle\mathbf{1}, g\rangle \mathbf{1}\right)=-V \gamma \int\left(\frac{e^{i \lambda|x-y|}-1}{2 i \lambda}\right) g(y) d y
$$

Then (7.2.2 shows that for $g \in H_{\text {comp }}^{-1}$, by taking $\rho=1$ on a neighborhood of $\operatorname{supp}(g) \cup \Gamma$, we can write $R_{0}(\lambda) g=R_{V}(\lambda)(I+K(\lambda)) g$, hence $R_{V}(\lambda)$ must be singular at 0 since $R_{0}(\lambda)$ is.

We now relate the existence of resonances to the solution of the transmission problem (7.0.5).

Proof of Proposition 7.0.1. Suppose now that $\Gamma=\partial \Omega$ for a compact domain $\Omega \subset \mathbb{R}^{d}$ with $C^{1,1}$ boundary. Assume also that $V: H^{\frac{1}{2}}(\partial \Omega) \rightarrow H^{\frac{1}{2}}(\partial \Omega)$. Then the analysis leading to (7.1.2) shows that a $\lambda$-outgoing solution of 7.0 .4 with $u \in H_{\text {loc }}^{1}$ belongs to $\mathcal{E}_{2, \text { loc }}$ and satisfies the transmission problem (7.0.5). Conversely, suppose $u \in \mathcal{E}_{2, \text { loc }}$ satisfies (7.0.5). For $w \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$, Green's identities yield

$$
\int_{\mathbb{R}^{d}} u\left(-\Delta-\lambda^{2}\right) w=\int_{\partial \Omega}\left(\partial_{\nu} u+\partial_{\nu^{\prime}} u\right) \gamma w=-\int_{\partial \Omega}(V \gamma u) \gamma w .
$$

Hence $u$ is a $\lambda$-outgoing $H_{\text {loc }}^{1}$ distributional solution to $\left(-\Delta-\lambda^{2}\right) u+\left(V \otimes \delta_{\partial \Omega}\right) u=0$, and by the above $\lambda$ is a resonance.

### 7.3 Approximation by Regular Potentials

In this section, we prove Theorem 7.2 which shows that the poles of the resolvent $(-\Delta+$ $\left.V-\lambda^{2}\right)^{-1}$ converge to those of $\left(-\Delta_{V_{\bmod , \Gamma}}-\lambda^{2}\right)^{-1}$ as the interaction region narrows.

Let $U$ be an open set on $\Gamma$, a compact $C^{1,1}$ hypersurface without self-intersection and $W$ be a collar neighborhood of $U$. Let $\left(x^{\prime}, x_{d}\right)$ be coordinates on $W$ where $x_{d}$ is the signed normal distance to $\Gamma$ and $x^{\prime}$ is a coordinate on $\Gamma$. Suppose that $V \in L_{\text {comp }}^{\infty}\left(\mathbb{R}^{d}\right)$ has $\operatorname{supp} V \subset W$ and define

$$
V_{\mathrm{mod}}\left(x^{\prime}\right):=\int V\left(x^{\prime}, x_{d}\right) d x_{d}
$$

Finally, let $V_{\epsilon}=\epsilon^{-1} V\left(x^{\prime}, \epsilon^{-1} x_{d}\right)$. When it will not cause confusion, we also use the notation $V_{\epsilon}$ to denote the operator given by multiplication by $V_{\epsilon}$.

Lemma 7.3.1. Denote by $V_{\epsilon}$ the operator $u \mapsto V_{\epsilon} u$. For $s, t>1 / 2$,

$$
\left\|V_{\epsilon}-\gamma^{*} V_{\bmod } \gamma\right\|_{H^{s}\left(\mathbb{R}^{d}\right) \rightarrow H^{-t}\left(\mathbb{R}^{d}\right)} \underset{\epsilon \rightarrow 0}{ } 0 .
$$

Proof. Let $u \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$. Then,

$$
\begin{aligned}
\mathcal{F}_{\xi_{d}}\left(\gamma^{*} V_{\bmod } \gamma u-V_{\epsilon} u\right) & \left.=V_{\bmod }\left(x^{\prime}\right) u\left(x^{\prime}, 0\right)-\widehat{V_{\epsilon}\left(x^{\prime}, \cdot\right.}\right) * \widehat{u\left(x^{\prime}, \cdot\right)}\left(\xi_{d}\right) \\
& \left.=\int V_{\bmod }\left(x^{\prime}\right) \widehat{u\left(x^{\prime}, \cdot\right)}\left(\xi_{d}-\eta\right)-\widehat{V\left(x^{\prime}, \cdot\right.}\right)(\epsilon \eta) \widehat{u\left(x^{\prime}, \cdot\right)}\left(\xi_{d}-\eta\right) d \eta \\
& \left.=\int\left[V_{\bmod }\left(x^{\prime}\right)-\widehat{V\left(x^{\prime}, \cdot\right.}\right)(\epsilon \eta)\right] \widehat{u\left(x^{\prime}, \cdot\right)}\left(\xi_{d}-\eta\right) d \eta
\end{aligned}
$$

So,

$$
\begin{aligned}
\left\|\left(\gamma^{*} V_{\bmod } \gamma-V_{\epsilon}\right) u\right\|_{L_{x^{\prime}}^{2} H_{x_{d}}^{-t}} & \left.=\| \| \int\left[V_{\bmod }\left(x^{\prime}\right)-\widehat{V\left(x^{\prime}, \cdot\right.}\right)(\epsilon \eta)\right] \widehat{u\left(x^{\prime}, \cdot\right)}\left(\xi_{d}-\eta\right) d \eta\left\langle\xi_{d}\right\rangle^{-t}\left\|_{L_{\xi_{d}}^{2}}\right\|_{L_{x^{\prime}}^{2}} \\
& \left.\left.\leq C\| \| \int\left[V_{\bmod }\left(x^{\prime}\right)-\widehat{V\left(x^{\prime}, \cdot\right.}\right)(\epsilon \eta)\right] \widehat{u\left(x^{\prime}, \cdot\right.}\right)\left(\xi_{d}-\eta\right) d \eta\left\|_{L_{\xi_{d}}^{\infty}}\right\|_{L_{x^{\prime}}^{2}} \\
& \left.\leq C\| \|\left[V_{\bmod }\left(x^{\prime}\right)-\widehat{V\left(x^{\prime}, \cdot\right.}\right)(\epsilon \eta)\right]\langle\eta\rangle^{-s}\left\|_{L_{\eta}^{2}}\right\| u\left(x^{\prime}, \cdot\right)\left\|_{H_{x_{d}}^{s}}\right\|_{L_{x^{\prime}}^{2}} \\
& \leq C\| \|\left[V_{\bmod }\left(x^{\prime}\right)-\widehat{V\left(x^{\prime}, \cdot\right)}(\epsilon \eta)\right]\langle\eta\rangle^{-s}\left\|_{L_{\eta}^{2}}\right\|_{L_{x^{\prime}}^{\infty}}\|u\|_{L_{x^{\prime}}^{2} H_{x_{d}}^{s}} \\
& =o(1)\|u\|_{L_{x^{\prime}}^{2} H_{x_{d}}^{s}}=o(1)\|u\|_{H^{s}}
\end{aligned}
$$

where in the last step we use the that $V \in L_{x^{\prime}}^{\infty} L_{x_{d}}^{1}$. The fact that $\|\cdot\|_{H^{-t}\left(\mathbb{R}^{d}\right)} \leq C\|\cdot\|_{L_{x^{\prime}}^{2} H_{x_{d}}^{-t}}$ completes the proof.

A partition of unity then gives the following corollary
Corollary 7.3.2. Let $\Gamma \Subset \mathbb{R}^{d}$ be a finite union of compact $C^{1,1}$ hypersurfaces, $\Gamma_{i}$ such that each $\Gamma_{i}$ does not self intersect and $\Gamma_{i} \cap \Gamma_{j}=\emptyset$ for $i \neq j$. Then for $V \in L_{\text {comp }}^{\infty}\left(\mathbb{R}^{d}\right)$ supported in a small enough neighborhood of $\Gamma, V_{\epsilon}$ as above, and $s, t>1 / 2$, there exists $V_{\bmod } \in L^{\infty}(\Gamma)$ such that

$$
\left\|V_{\epsilon}-\gamma^{*} V_{\bmod } \gamma\right\|_{H^{s}\left(\mathbb{R}^{d}\right) \rightarrow H^{-t}\left(\mathbb{R}^{d}\right)} \xrightarrow[\epsilon \rightarrow 0]{ } 0
$$

We now show that $-\Delta+V_{\epsilon}$ converges to $-\Delta_{V_{\text {mod }}}$ in the norm resolvent sense. Let $R_{V_{\epsilon}}(\lambda):=\left(-\Delta-\lambda^{2}\right)^{-1}$ and $R_{V_{\text {mod }}}(\lambda):=\left(-\Delta_{V_{\text {mod }}, \Gamma}-\lambda^{2}\right)^{-1}$.

Lemma 7.3.3. Let $\Gamma, V, V_{\epsilon}$, and $V_{\bmod }$ be as in Corollary 7.3.2. Then for $\lambda$ not a pole of $R_{V_{\text {mod }}}(\lambda)$,

$$
R_{V_{\epsilon}}(\lambda) \xrightarrow[\epsilon \rightarrow 0]{\longrightarrow} R_{V_{\text {mod }}}(\lambda): H_{\text {comp }}^{-1}\left(\mathbb{R}^{d}\right) \rightarrow H_{\mathrm{loc}}^{1}\left(\mathbb{R}^{d}\right)
$$

Moreover, the poles of $R_{V_{\epsilon}}(\lambda)$ converge to those of $R_{V_{\bmod }}(\lambda)$ uniformly on compact sets.
Proof. Fix $s, t>1 / 2$ and $\rho \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ with $\rho \equiv 1$ on supp $V$. Let $K_{\bmod }(\lambda):=\gamma^{*} V_{\bmod } \gamma R_{0}(\lambda)$ and $K_{\epsilon}(\lambda)=V_{\epsilon} R_{0}(\lambda)$. Notice that by Corollary 7.3.2

$$
K_{\epsilon}(\lambda) \rho \xrightarrow[\epsilon \rightarrow 0]{\longrightarrow} K_{\bmod }(\lambda) \rho: H_{\mathrm{loc}}^{-1}\left(\mathbb{R}^{d}\right) \rightarrow H_{\mathrm{comp}}^{-t}
$$

Hence, when $I+K_{\bmod }(\lambda) \rho$ is invertible, so is $I+K_{\epsilon}(\lambda) \rho$ for $\epsilon$ small enough. Moreover,

$$
\left(I+K_{\epsilon}(\lambda) \rho\right)^{-1} \xrightarrow[\epsilon \rightarrow 0]{ }\left(I+K_{\bmod }(\lambda) \rho\right)^{-1}: H_{\text {comp }}^{-t} \rightarrow H_{\mathrm{loc}}^{-1} .
$$

Then by (7.2.2) and 21, Chapter 2,3]

$$
\begin{aligned}
R_{V_{\epsilon}}(\lambda) & =R_{0}(\lambda)\left(I+K_{\epsilon}(\lambda) \rho\right)^{-1}\left(I-K_{\epsilon}(\lambda)(1-\rho)\right) \\
R_{V_{\bmod }}(\lambda) & =R_{0}(\lambda)\left(I+K_{\bmod }(\lambda) \rho\right)^{-1}\left(I-K_{\bmod }(\lambda)(1-\rho)\right)
\end{aligned}
$$

Finally, noting that for $\operatorname{Im} \lambda>0$,

$$
K_{\epsilon}(\lambda)(1-\rho) \xrightarrow[\epsilon \rightarrow 0]{ } K_{\bmod }(\lambda)(1-\rho): H^{-1}\left(\mathbb{R}^{d}\right) \rightarrow H_{\text {comp }}^{-t}
$$

and for all $\lambda$ in the domain of $R_{0}(\lambda)$,

$$
K_{\epsilon}(\lambda)(1-\rho) \xrightarrow[\epsilon \rightarrow 0]{ } K_{\bmod }(\lambda)(1-\rho): H_{\text {comp }}^{-1}\left(\mathbb{R}^{d}\right) \rightarrow H_{\text {comp }}^{-t}
$$

we have for $\operatorname{Im} \lambda \gg 1$,

$$
R_{V_{\epsilon}}(\lambda) \xrightarrow[\epsilon \rightarrow 0]{\longrightarrow} R_{V_{\bmod }}: H^{-1}\left(\mathbb{R}^{d}\right) \rightarrow H^{1}\left(\mathbb{R}^{d}\right)
$$

and for $\lambda$ not a pole of $\left(I+K_{\bmod }(\lambda) \rho\right)^{-1}$,

$$
R_{V_{\epsilon}}(\lambda) \xrightarrow[\epsilon \rightarrow 0]{\longrightarrow} R_{V_{\text {mod }}}: H_{\text {comp }}^{-1}\left(\mathbb{R}^{d}\right) \rightarrow H_{\mathrm{loc}}^{1}\left(\mathbb{R}^{d}\right)
$$

Hence, since as operators $H_{\text {comp }}^{-1}\left(\mathbb{R}^{d}\right) \rightarrow H_{\mathrm{loc}}^{1}\left(\mathbb{R}^{d}\right), R_{V_{\epsilon}}(\lambda)$ and $R_{V_{\text {mod }}}$ are meromorphic, the poles of $R_{V_{\epsilon}}(\lambda)$ converge to those of $R_{V_{\bmod }}$.

To see that the convergence of poles is uniform on compact sets, we use the regularized determinant. First, the operators $K_{\epsilon} \rho$ and $K_{\bmod } \rho$ can be thought of as acting : $H^{-1}\left(\mathbb{T}_{R}^{d}\right) \rightarrow$ $H^{-t}\left(\mathbb{T}_{R}^{d}\right)$ for $R$ large enough. Hence,

$$
s_{j}(K \rho) \leq s_{j}\left(\left(-\Delta_{\mathbb{T}}+1\right)^{-l}\right)\left\|\left(-\Delta_{\mathbb{T}}+1\right)^{l} K \rho\right\|_{H^{-1} \rightarrow H^{-1}} \leq j^{-2 l / d}\|K \rho\|_{H^{-1} \rightarrow H^{-1+2 l}} .
$$

So, taking $l=(-t+1) / 2$, gives that $K \rho$ is in the $p-$ Schatten class (see for example 21, Appendix B]) for $p>2 d$. Thus, (away from $\lambda=0$ ), the poles of $I+K_{\epsilon} \rho$ and $I+K_{\bmod } \rho$ respectively agree with the zeros of

$$
f_{\epsilon}(\lambda)=\lambda \operatorname{det}_{p}\left(I+K_{\epsilon} \rho\right) \quad \text { and } \quad f_{\bmod }(\lambda)=\lambda \operatorname{det}_{p}\left(I+K_{\bmod } \rho\right)
$$

where $\operatorname{det}_{p}$ is the $p$-regularized determinant (see [21, Section 3.4]). Note that $f_{\epsilon}$ and $f_{\text {mod }}$ are analytic and hence by Hurwitz's theorem it suffices to check that $f_{\epsilon}(\lambda)$ converges locally uniformly to $f_{\bmod }(\lambda)$. Now,

$$
s_{j}\left(K_{\epsilon} \rho-K_{\bmod } \rho\right) \leq j^{-2 l / d}\left\|K_{\epsilon} \rho-K_{\bmod } \rho\right\|_{H^{-1} \rightarrow H^{-t}}
$$

so, $\lambda K_{\epsilon} \rho \rightarrow \lambda K_{\bmod } \rho$ in the $p$-Schatten locally uniformly in $\lambda$. But this implies the local uniform convergence of $f_{\epsilon}$ and $f_{\text {mod }}$.

### 7.4 Resonance Expansion for the Wave Equation

In this section we prove Theorems 7.3 and 7.4 . We will use the following representation of the wave group $U(t)$ acting on $g \in L_{\text {comp }}^{2}\left(\mathbb{R}^{d}\right)$,

$$
\begin{equation*}
U(t) g=\frac{1}{2 \pi} \int_{-\infty+i \alpha}^{\infty+i \alpha} e^{-i t \lambda} R_{V}(\lambda) g d \lambda \tag{7.4.1}
\end{equation*}
$$

where $\alpha \geq 1$ is chosen so that $\mu_{j}<\alpha$ for all $j$, where $-\mu_{j}^{2}$ are the negative eigenvalues of $-\Delta_{\Gamma, \delta}$ with $\mu_{j}>0$. This representation follows by the spectral theorem and the resolvent estimates we establish in this section; see (7.4.11). The expansion (7.0.3) is proven by a contour integration argument applied to (7.4.1). We start this section by studying the structure of the resolvent $R_{V}(\lambda)$ near its poles, and then prove norm estimates on $R_{V}(\lambda)$ that justify the change of contour used to prove Theorem 7.3. We then establish higher order estimates on $R_{V}(\lambda)$, which are used to prove Theorem 7.4 .

Let $\Lambda$ denote the set of resonances; since we work in odd dimensions $\Lambda$ is a discrete subset of $\mathbb{C}$. The elements of $\Lambda$ such that $\operatorname{Im} \lambda>0$ consist of $i \mu_{j}$ where $-\mu_{j}^{2}$ are the eigenvalues of $-\Delta_{\Gamma, \delta}$ in $(-\infty, 0)$ with $\mu_{j}>0$. That there are only a finite number of such eigenvalues follows by relative compactness of $V \otimes \delta_{\Gamma}$ with respect to $-\Delta$. The resolvent near $i \mu_{j}$ takes the form

$$
R_{V}(\lambda)=\frac{-\Pi_{\mu_{j}}}{\lambda^{2}+\mu_{j}^{2}}+\text { holomorphic }=\frac{i \Pi_{\mu_{j}}}{2 \mu_{j}\left(\lambda-i \mu_{j}\right)}+\text { holomorphic }
$$

where $\Pi_{\mu_{j}}$ is projection onto the $-\mu_{j}^{2}$-eigenspace of $-\Delta_{\Gamma, \delta}$. In particular we note that

$$
\begin{equation*}
\operatorname{Res}\left(e^{-i t \lambda} R_{V}(\lambda), i \mu_{j}\right)=i\left(2 \mu_{j}\right)^{-1} e^{t \mu_{j}} \Pi_{\mu_{j}} \tag{7.4.2}
\end{equation*}
$$

We note that if there is a compactly supported eigenfunction $u$ for $-\mu_{j}^{2}$, then $-i \mu_{j}$ must also be a resonance. To see this, by compact support of $u$ we can write

$$
u(x)=\int G_{0}\left(-i \mu_{j}, x, y\right)\left(-\Delta+\mu_{j}^{2}\right) u(y) d y=-R_{0}\left(-i \mu_{j}\right)\left(V \otimes \delta_{\Gamma}\right) u
$$

hence $u$ is also $-i \mu_{j}$ outgoing, and $-i \mu_{j}$ is a resonance by the results of Section 7.2 .
In contrast to the case of $V \in L_{\text {comp }}^{\infty}\left(\mathbb{R}^{d}\right)$, there may be resonances $\lambda \in \mathbb{R} \backslash\{0\}$. For an example in one dimension of $V$ and $\Gamma$ with a positive (hence embedded) eigenvalue, consider $\Gamma=\left\{-\frac{\pi}{2}, 0, \frac{\pi}{2}\right\}$, and $V$ given by

$$
(V \gamma u)(x)= \begin{cases}u(0), & x= \pm \frac{\pi}{2} \\ u\left(\frac{\pi}{2}\right)+u\left(-\frac{\pi}{2}\right), & x=0\end{cases}
$$

Then the function

$$
u(x)= \begin{cases}\cos (x), & |x| \leq \frac{\pi}{2} \\ 0, & |x| \geq \frac{\pi}{2}\end{cases}
$$

is compactly supported, and satisfies $-\Delta_{\Gamma, \delta} u-u=0$. It is $\lambda$-outgoing for both $\lambda= \pm 1$ by the argument above, hence yields resonances at $\lambda= \pm 1$. Using piecewise linear functions one can also produce an example of a compactly supported eigenfunction with eigenvalue 0 , and using piecewise combinations of $\left\{e^{x}, e^{-x}\right\}$ produce a compactly supported eigenfunction with eigenvalue -1 , for appropriate choices of $V$ and $\Gamma$.

For $\lambda \in \mathbb{R} \backslash\{0\}$ and any dimension $d$, a $\lambda$-outgoing solution $u \in H_{\text {loc }}^{1}$ to $-\Delta_{\Gamma, \delta} u=\lambda^{2} u$ must in fact be a compactly supported eigenfunction. To see this, observe that for $R \gg 1$
$0=\int_{|x| \leq R} \bar{u}\left(-\Delta u+\left(V \otimes \delta_{\Gamma}\right) u-\lambda^{2} u\right)=\int_{|x| \leq R}\left(|\nabla u|^{2}-\lambda^{2}|u|^{2}\right)+\int_{|x|=R} \bar{u} \partial_{\nu} u+\int_{\Gamma} \overline{\gamma u} V \gamma u$
shows that $\operatorname{Im} \int_{|x|=R} \bar{u} \partial_{\nu} u=0$. The proof of Proposition 1.1 and Lemma 1.2 of [76, Chapter 9] then show that $u \equiv 0$ on $|x| \geq R_{0}$, hence by analytic continuation $u$ vanishes on the unbounded component of $\mathbb{R}^{d} \backslash \Gamma$. For $V \in L_{\text {comp }}^{\infty}\left(\mathbb{R}^{d}\right)$, unique continuation (see $\sqrt[60]{ }$, Theorem XIII.63]) would yield $u \equiv 0$. For singular potentials and non-local $V$ unique continuation can fail by the example above, but we note that if $\Gamma$ coincides with the boundary of the unbounded component of $\mathbb{R}^{d} \backslash \Gamma$ then there are no resonances $\lambda \in \mathbb{R} \backslash\{0\}$, since in that case $\gamma u=0$, hence $\left(V \otimes \delta_{\Gamma}\right) u=0$. Thus $u$ is a compactly supported eigenfunction of $-\Delta$ on $\mathbb{R}^{d}$, and must vanish identically.

The resonances in $\mathbb{R} \backslash\{0\}$ form a finite set by Theorem 7.1. By Proposition 7.2.2 and the preceding, $\lambda \in \mathbb{R} \backslash\{0\}$ is a resonance if and only if $\lambda^{2}$ is an eigenvalue of $-\Delta_{\Gamma, \delta}$, and the real resonances are thus symmetric about 0 . We indicate them by $\pm \nu_{j}$, with $\nu_{j}>0$. The spectral bound $\left\|R_{V}(\lambda)\right\|_{L^{2} \rightarrow L^{2}} \leq C \epsilon^{-1}|\operatorname{Im} \lambda|^{-1}$, for $|\operatorname{Re} \lambda| \geq \epsilon$ and $\operatorname{Im} \lambda>0$, shows that the pole at $\nu_{j}$ is simple. By inspection, for $\operatorname{Im} \lambda>0$ near $\pm \nu_{j}$ we have

$$
R_{V}(\lambda)=\frac{-\Pi_{\nu_{j}}}{\lambda^{2}-\nu_{j}^{2}}+\text { holomorphic }=\frac{\mp \Pi_{\nu_{j}}}{2 \nu_{j}\left(\lambda \mp \nu_{j}\right)}+\text { holomorphic }
$$

where $\Pi_{\nu_{j}}$ is projection onto the $\nu_{j}^{2}$ eigenspace, hence

$$
\begin{equation*}
\operatorname{Res}\left(e^{-i t \lambda} R_{V}(\lambda), \pm \nu_{j}\right)=\mp\left(2 \nu_{j}\right)^{-1} e^{\mp i t \nu_{j}} \Pi_{\nu_{j}} . \tag{7.4.3}
\end{equation*}
$$

The nature of the residue at 0 depends on the dimension $d$. For $d \geq 5, \lambda$-outgoing solutions to (7.0.4) for $\lambda=0$ must be square-integrable, hence if $0 \in \Lambda$ there is a corresponding eigenfunction. For $d=1$, a square-integrable solution to 7.0.4 must be compactly supported; there may also be 0 -outgoing solutions (i.e. constant near $\pm \infty$ ) that are not eigenfunctions for 0 . For $d=3$, if $0 \in \Lambda$ there may be square-integrable and/or non squareintegrable solutions to (7.0.4) since, depending on whether the integral over $\Gamma$ of $f=V \gamma u$ vanishes or not, $u=R_{0}(0) \gamma^{*} f$ satisfies $|u| \lesssim|x|^{-2}$ or $|u| \approx|x|^{-1}$ for $|x| \gg 1$.

For $|\lambda| \ll 1$ and $\operatorname{Im} \lambda>0$, the spectral bound $\left\|R_{V}(\lambda)\right\|_{L^{2} \rightarrow L^{2}} \leq C(|\lambda| \operatorname{Im} \lambda)^{-1}$ shows that

$$
R_{V}(\lambda)=-\frac{\Pi_{0}}{\lambda^{2}}+\frac{i \mathcal{P}_{0}}{\lambda}+\text { holomorphic } .
$$

Since $R_{V}^{*}(-\bar{\lambda})=R_{V}(\lambda)$ for $\operatorname{Im} \lambda>0$, it follows that $\Pi_{0}$ and $\mathcal{P}_{0}$ are symmetric maps of $L_{\text {comp }}^{2}$ to $\mathcal{D}_{\text {loc }}$, in that $\left\langle\mathcal{P}_{0} g, h\right\rangle=\left\langle g, \mathcal{P}_{0} h\right\rangle$ for $g, h \in L_{\text {comp }}^{2}$, similarly for $\Pi_{0}$, and their images are solutions of 7.0 .4 with $\lambda=0$. Since $\Pi_{0}$ is bounded on $L^{2}\left(\mathbb{R}^{d}\right)$ it is then self-adjoint, and since it is the identity on the 0 -eigenspace we see that $\Pi_{0}$ is projection onto the 0 -eigenspace of $-\Delta_{\Gamma, \delta}$. For $d \geq 3$ the range of $\Pi_{0}$ is 0 -outgoing, since $u=-R_{0}(0)\left(V \otimes \delta_{\Gamma}\right) u$ when $u \in \mathcal{D}$ solves $-\Delta_{\Gamma, \delta} u=0$. We remark that the arguments of [21, Section 3.3] show that $\mathcal{P}_{0}=0$ for $d \geq 5$, and that the range of $\mathcal{P}_{0}$ is 0 -outgoing if $d=3$, although we do not use that here. To see that the range of $\Pi_{0}$ and $\mathcal{P}_{0}$ are 0 -outgoing when $d=1$, we note that $\left(\partial_{x}-i \operatorname{sgn}(x) \lambda\right)\left(R_{V}(\lambda) g\right)(x)=0$ for $|x| \gg 1$ and $g \in L_{\text {comp }}^{2}$. The range of $\Pi_{0}$ is supported in the convex hull of $\Gamma$ when $d=1$ (hence is 0 -outoing), and by letting $\lambda \rightarrow 0$ this implies $\partial_{x}\left(\mathcal{P}_{0} g\right)(x)=0$ for $|x| \gg 1$, hence the range of $\mathcal{P}_{0}$ is 0-outgoing.

We can then write

$$
\begin{equation*}
\operatorname{Res}\left(e^{-i t \lambda} R_{V}(\lambda), 0\right)=i t \Pi_{0}+i \mathcal{P}_{0} \tag{7.4.4}
\end{equation*}
$$

The remaining resonances form a discrete set $\left\{\lambda_{k}\right\} \subset\{\operatorname{Im} \lambda<0\}$, with respective order $m_{R}\left(\lambda_{k}\right)$. Since $\lambda_{k} \neq 0$, the Laurent expansion of $R_{V}(\lambda)$ about $\lambda_{k}$ can be written in the following form

$$
R_{V}(\lambda)=i \sum_{\ell=1}^{m_{R}\left(\lambda_{k}\right)} \frac{\left(-\Delta_{\Gamma, \delta}-\lambda_{k}^{2}\right)^{\ell-1} \mathcal{P}_{\lambda_{k}}}{\left(\lambda^{2}-\lambda_{k}^{2}\right)^{\ell}}+\text { holomorphic } .
$$

Here $\mathcal{P}_{\lambda_{k}}: L_{\text {comp }}^{2} \rightarrow \mathcal{D}_{\text {loc }}$ is given by

$$
\mathcal{P}_{\lambda_{k}}=-\frac{1}{2 \pi} \oint_{\lambda_{k}} R_{V}(\lambda) 2 \lambda d \lambda,
$$

and $\left(-\Delta_{\Gamma, \delta}-\lambda_{k}^{2}\right)^{m_{R}\left(\lambda_{k}\right)} \mathcal{P}_{\lambda_{k}}=0$. We can thus write

$$
\begin{equation*}
\operatorname{Res}\left(e^{-i t \lambda} R_{V}(\lambda), \lambda_{k}\right)=i \sum_{\ell=0}^{m_{R}\left(\lambda_{k}\right)-1} t^{\ell} e^{-i t \lambda_{k}} \mathcal{P}_{\lambda_{k}, \ell} \tag{7.4.5}
\end{equation*}
$$

where $\mathcal{P}_{\lambda_{k}, \ell}: L_{\text {comp }}^{2} \rightarrow \mathcal{D}_{\text {loc }}$. When $\ell=m_{R}\left(\lambda_{k}\right)-1, \mathcal{P}_{\lambda_{k}, \ell} g$ is $\lambda_{k}$-outgoing, as seen by writing the Laurent expansion of $R_{V}(\lambda)$ in terms of that for $(I+K(\lambda) \rho)^{-1}$. In particular, if $m_{R}\left(\lambda_{k}\right)=1$, then $\operatorname{Res}\left(e^{-i t \lambda} R_{V}(\lambda), \lambda_{k}\right)=i\left(2 \lambda_{k}\right)^{-1} e^{-i t \lambda_{k}} \mathcal{P}_{\lambda_{k}}$, where $\mathcal{P}_{\lambda_{k}}$ maps $L_{\text {comp }}^{2}$ to $\lambda_{k}$-outgoing solutions of $\left(-\Delta_{\Gamma, \delta}-\lambda_{k}^{2}\right) u=0$.

## Resolvent Estimates

We first establish bounds on the cutoff of $R_{V}(\lambda)$, for $\lambda$ in the resonance free region established in Section 7.2 .

Lemma 7.4.1. Suppose that $\Gamma$ is a finite union of compact subsets of $C^{1,1}$ hypersurfaces. Then for all $\epsilon>0$ there exists $R<\infty$, so that if $\chi \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ equals 1 on a neighborhood of
$\Gamma,|\operatorname{Re} \lambda|>R$, and $\operatorname{Im} \lambda \geq-\left(\frac{1}{2} D_{\Gamma}^{-1}-\epsilon\right) \log (|\operatorname{Re} \lambda|)$, then

$$
\begin{aligned}
&\left\|\chi R_{V}(\lambda) \chi g\right\|_{L^{2}} \leq C\langle\lambda\rangle^{-1} e^{2 D_{\chi}(\operatorname{Im} \lambda)-}\|g\|_{L^{2}} \\
&\left\|\chi R_{V}(\lambda) \chi g\right\|_{H^{1}} \leq C e^{2 D_{\chi}(\operatorname{Im} \lambda)-}\|g\|_{L^{2}} \\
&\left\|\chi R_{V}(\lambda) \chi g\right\|_{\mathcal{D}} \leq C\langle\lambda\rangle e^{2 D_{\chi}(\operatorname{Im} \lambda)-}\|g\|_{L^{2}}
\end{aligned}
$$

where $R_{V}(\lambda)$ is the meromorphic continuation of $\left(-\Delta_{\Gamma, \delta}-\lambda^{2}\right)^{-1}, D_{\chi}=\operatorname{diam}(\operatorname{supp} \chi)$, and $(\operatorname{Im} \lambda)_{-}=\max (0,-\operatorname{Im} \lambda)$. If $\operatorname{Im} \lambda \geq 1,|\operatorname{Re} \lambda|>R$, then the estimates hold with $\chi \equiv 1$, setting $D_{\chi}(\operatorname{Im} \lambda)_{-}=0$.

Remark: The region in which this estimate is valid can be improved by replacing $\frac{1}{2}$ by $\frac{2}{3}$ if the components of $\Gamma$ are subsets of strictly convex $C^{2,1}$ hypersurfaces.

Proof. We recall the Sobolev estimates for the cutoff of the free resolvent if $|\lambda| \geq 1$, see e.g. [21, Chapter 3]

$$
\left\|\chi R_{0}(\lambda) \chi\right\|_{H^{s} \rightarrow H^{t}} \leq C\langle\lambda\rangle^{t-s-1} e^{D_{\chi}(\operatorname{Im} \lambda)_{-}}, \quad s \leq t \leq s+2
$$

In addition, when $\operatorname{Im} \lambda \geq 1$ these estimates hold globally, that is with $\chi \equiv 1$ and taking $D_{\chi}(\operatorname{Im} \lambda)_{-}=0$.

This in turn leads to the following restriction estimates

$$
\begin{align*}
& \left\|\gamma R_{0}(\lambda) \chi g\right\|_{L^{2}(\Gamma)} \leq C\langle\lambda\rangle^{-s-\frac{1}{2}} e^{D_{\chi}(\operatorname{Im} \lambda)-}\|g\|_{H^{s}}, \quad-\frac{3}{2}<s<\frac{1}{2},  \tag{7.4.6}\\
& \left\|\gamma \nabla R_{0}(\lambda) \chi g\right\|_{L^{2}(\Gamma)} \leq C\langle\lambda\rangle^{-s+\frac{1}{2}} e^{D_{\chi}(\operatorname{Im} \lambda)}-\|g\|_{H^{s}}, \quad-\frac{1}{2}<s<\frac{3}{2} .
\end{align*}
$$

To prove (7.4.6 we apply the following trace bound separately on each component of $\Gamma$,

$$
\begin{equation*}
\|\gamma g\|_{L^{2}(\Gamma)} \leq C_{t, t^{\prime}}\|g\|_{H^{t}}^{\theta}\|g\|_{H^{t^{\prime}}}^{1-\theta}, \quad 0 \leq t<\frac{1}{2}<t^{\prime}, \quad \theta\left(t-\frac{1}{2}\right)+(1-\theta)\left(t^{\prime}-\frac{1}{2}\right)=0 . \tag{7.4.7}
\end{equation*}
$$

The estimate 7.4.7) follows by considering the case of a graph $x_{n}=F\left(x^{\prime}\right)$, and applying Hölder's inequality in $x^{\prime}$ and the following scale-invariant one dimensional estimate in $x_{n}$

$$
\|g\|_{L^{\infty}(\mathbb{R})} \leq C_{t, t^{\prime}}\left\||D|^{t} g\right\|_{L^{2}(\mathbb{R})}^{\theta}\left\||D|^{t^{\prime}} g\right\|_{L^{2}(\mathbb{R})}^{1-\theta}
$$

with $t, t^{\prime}, \theta$ as in 7.4.7). This estimate follows by fixing $r$ so that

$$
\left\||D|^{t} g_{r}\right\|_{L^{2}(\mathbb{R})}=\left\||D|^{t^{\prime}} g_{r}\right\|_{L^{2}(\mathbb{R})}
$$

where $g_{r}(x)=g(r x)$, and noting $\left\|\widehat{g_{r}}\right\|_{L^{1}(\mathbb{R})} \leq \frac{1}{2} C_{t, t^{\prime}}\left(\left\||\xi|^{t} \widehat{g_{r}}\right\|_{L^{2}(\mathbb{R})}+\left\||\xi|^{t^{\prime}} \widehat{g}_{r}\right\|_{L^{2}(\mathbb{R})}\right)$ if $t<\frac{1}{2}<$ $t^{\prime}$, for some $C_{t, t^{\prime}}<\infty$.

By duality (7.4.6) implies the following extension estimate,

$$
\begin{equation*}
\left\|\chi R_{0}(\lambda) \gamma^{*} f\right\|_{H^{s}} \leq C\langle\lambda\rangle^{s-\frac{1}{2}} e^{D_{\chi}(\operatorname{Im} \lambda)-}\|f\|_{L^{2}(\Gamma)}, \quad-\frac{1}{2}<s<\frac{3}{2} \tag{7.4.8}
\end{equation*}
$$

Now fix $g \in L^{2}\left(\mathbb{R}^{d}\right)$, and set $u=R_{V}(\lambda) \chi g$. Then by (7.2.3) we have $u=R_{0}(\lambda) \chi g-w$, where

$$
w=R_{0}(\lambda) \gamma^{*}(I+V G(\lambda))^{-1} V \gamma R_{0}(\lambda) \chi g
$$

By Theorem 6.1, for $|\operatorname{Re} \lambda|$ large enough and $\operatorname{Im} \lambda \geq-\left(\frac{1}{2} D_{\Omega}^{-1}-\epsilon\right) \log (|\operatorname{Re} \lambda|)$, the operator $I+V G(\lambda)$ is invertible on $L^{2}(\Gamma)$, and we have

$$
\left\|(I+V G(\lambda))^{-1}\right\|_{L^{2}(\Gamma) \rightarrow L^{2}(\Gamma)} \leq C, \quad\|V G(\lambda)\|_{L^{2}(\Gamma) \rightarrow L^{2}(\Gamma)}<1
$$

It follows from (7.4.6 that, for $-\frac{3}{2}<s<\frac{1}{2}$,

$$
\left\|(I+V G(\lambda))^{-1} V \gamma R_{0}(\lambda) \chi g\right\|_{L^{2}(\Gamma)} \leq C\langle\lambda\rangle^{-s-\frac{1}{2}} e^{D_{\chi}(\operatorname{Im} \lambda)-}\|g\|_{H^{s}}
$$

Then 7.4.8 gives the following, for $-\frac{3}{2}<s<\frac{1}{2}$, and with global bounds if $\operatorname{Im} \lambda \geq 1$,

$$
\begin{align*}
\|\chi w\|_{L^{2}} & \leq C\langle\lambda\rangle^{-s-1} e^{2 D_{\chi}(\operatorname{Im} \lambda)-}\|g\|_{H^{s}}  \tag{7.4.9}\\
\|\chi w\|_{H^{1}} & \leq C\langle\lambda\rangle^{-s} e^{2 D_{\chi}(\operatorname{Im} \lambda)-}\|g\|_{H^{s}} \tag{7.4.10}
\end{align*}
$$

By the $L^{2} \rightarrow H^{t}$ bounds for $\chi R_{0}(\lambda) \chi$ the same holds for $s=0$ with $w$ replaced by $u$, which yields the bounds of Lemma 7.4.1 except for the ones on $\|\chi u\|_{\mathcal{D}}$.

To obtain bounds on $\|\chi u\|_{\mathcal{D}}$, we write

$$
\Delta(\chi u)=-\chi^{2} g+2(\nabla \chi) \cdot \nabla u+(\Delta \chi) u-\lambda^{2} \chi u+\left(V \otimes \delta_{\Gamma}\right) u
$$

and note by (7.4.9) and 7.4.10 that

$$
\|(\nabla \chi) \cdot \nabla u\|_{L^{2}}+\|(\Delta \chi) u\|_{L^{2}}+\langle\lambda\rangle^{2}\|\chi u\|_{L^{2}} \leq C\langle\lambda\rangle e^{2 D_{\chi}(\operatorname{Im} \lambda)-}\|g\|_{L^{2}}
$$

Consequently,

$$
\left\|\Delta_{\Gamma, \delta}(\chi u)\right\|_{L^{2}} \leq C\langle\lambda\rangle e^{2 D_{\chi}(\operatorname{Im} \lambda)-}\|g\|_{L^{2}}
$$

yielding the desired bound on $\|\chi u\|_{\mathcal{D}}$.

## Proof of Theorem 7.3

We prove here the case $N=1$ of Theorem 7.3; that is, that the expansion holds with bounds on $\left\|\chi E_{A}(t) \chi\right\|_{L^{2} \rightarrow \mathcal{D}}$. The case $N \geq 2$ will be handled following the proof of Theorem 7.4 . We follow the treatment in [74] and suppose that $g \in H^{s}$ for some $0<s<\frac{1}{2}$, then proceed by density of $H^{s}$ in $L^{2}$. As above write

$$
R_{V}(\lambda) \chi g=w(\lambda)+R_{0}(\lambda) \chi g
$$



Figure 7.2: The various contours used in Section 7.4 to obtain the resonance expansion in odd dimensions.

Choose $\alpha \geq 1$ so that $\mu_{j}<\alpha$ for all $j$, where $-\mu_{j}^{2}$ are the negative eigenvalues of $-\Delta_{\Gamma, \delta}$. By the spectral theorem we can write

$$
\begin{align*}
U(t) \chi g & =\frac{1}{2 \pi} \int_{-\infty+i \alpha}^{\infty+i \alpha} e^{-i t \lambda} R_{V}(\lambda) \chi g d \lambda \\
& =\frac{1}{2 \pi} \int_{-\infty+i \alpha}^{\infty+i \alpha} e^{-i t \lambda}\left(w(\lambda)+R_{0}(\lambda) \chi g\right) d \lambda \tag{7.4.11}
\end{align*}
$$

The integral is norm convergent in $L^{2}\left(\mathbb{R}^{d}\right)$, by $(7.4 .9)$ and the norm convergence of the free resolvent integral. After localizing by $\chi$ on the left, for $t$ sufficiently large we seek to deform the contour $\mathbb{R}+i \alpha$ to

$$
\Sigma_{A}=\{\lambda \in \mathbb{C}: \operatorname{Im} \lambda=-A-c \log (2+|\operatorname{Re} \lambda|)\}
$$

where we choose $c<\frac{1}{2} D_{\Gamma}^{-1}$, and assume $A$ is such that there are no resonances on $\Sigma_{A}$. We will show that the integral over $\Sigma_{A}$ is norm convergent for $g \in H^{s}$ if $s>0$, so to justify the contour change we need to show that for $t$ sufficiently large the integrals over

$$
\gamma_{ \pm R}(v)=\{ \pm R+i v:-(A+c \log (2+R)) \leq v \leq \alpha\}, \quad \text { and } \quad \gamma_{R, \infty}=\{x+i \alpha:|x| \geq R\}
$$

tend to 0 as $R \rightarrow \infty$. Note that for $R$ large enough, Theorem 7.1 shows that there are no resonances between $\mathbb{R}+i \alpha$ and $\Sigma_{A}$ with $|\operatorname{Re} \lambda| \geq R$, and hence none on $\gamma_{ \pm R}$.

We introduce the following notation,

$$
E_{\gamma}(t) g=\frac{1}{2 \pi} \int_{\gamma} e^{-i t \lambda} R_{V}(\lambda) g d \lambda
$$

Then for $t>2 D_{\chi}$, and $R$ large enough,

$$
\left\|\chi E_{\gamma_{ \pm R}}(t) \chi g\right\|_{L^{2}} \leq C e^{\alpha t}\langle R\rangle^{-1}(\alpha+A+c \log (2+R))\|g\|_{L^{2}} \rightarrow 0 \quad \text { as } \quad R \rightarrow \infty .
$$

The norm convergence of (7.4.11) shows that $\left\|\chi E_{\gamma_{R, \infty}} \chi g\right\|_{L^{2}} \rightarrow 0$ as $R \rightarrow \infty$. We then assume $c\left(t-2 D_{\chi}\right) \geq 3$ and calculate

$$
\left\|\chi E_{\Sigma_{A}}(t) \chi g\right\|_{\mathcal{D}} \leq C_{A, \chi} e^{-A\left(t-2 D_{\chi}\right)} \int_{-\infty}^{\infty} e^{-3 \log (2+|R|)}\langle A+| R| \rangle d R \leq C_{A, \chi} e^{-A t}\|g\|_{L^{2}}
$$

In particular the integral is norm convergent, and the contour deformation is allowed.
Thus, if we let $\Omega_{A}$ denote the collection of poles of $R_{V}(\lambda)$ in the set $\operatorname{Im} \lambda>-A-c \log (2+$ $|\operatorname{Re} \lambda|)$, then

$$
\chi U(t) \chi g=\chi E_{\Sigma_{A}}(t) \chi g-i \chi \sum_{z \in \Omega_{A}} \operatorname{Res}\left(e^{-i t \lambda} R_{V}(\lambda), z\right) \chi g
$$

and by density this holds for $g \in L^{2}\left(\mathbb{R}^{d}\right)$. Observe that if $g \in L_{\text {comp }}^{2}$ then we can take $\chi=1$ on the support of $g$, and drop the cutoff $\chi$ to write a global equality in $L_{\mathrm{loc}}^{2}$. To have estimates on the remainder in $\mathcal{D}$, though, requires cutting off by $\chi$ and taking $t>2 D_{\chi}+C$, which is required for $\chi U(t) \chi$ to map $L^{2}$ into $\mathcal{D} \subset H^{1}$. The expressions 7.4.2, (7.4.4), 7.4.3), and (7.4.5) now complete the proof of Theorem 7.3 for $N=1$, where we observe that the terms from poles in $\Omega_{A}$ with $\operatorname{Im} \lambda \leq-A$ can be absorbed into $E_{A}(t)$.

## Higher Order Estimates for Smooth Domains

We start with the following lemma, where we now assume that $\Gamma=\partial \Omega$ is $C^{\infty}$, and that $V: H^{s}(\partial \Omega) \rightarrow H^{s}(\partial \Omega)$ for all $s \geq 0$. Recall that we set $\mathcal{E}_{0}=L^{2}\left(\mathbb{R}^{d}\right)$, and for $N \geq 1$,

$$
\mathcal{E}_{N}=H^{1}\left(\mathbb{R}^{d}\right) \cap\left(H^{N}(\Omega) \oplus H^{N}\left(\mathbb{R}^{d} \backslash \bar{\Omega}\right)\right)
$$

In this setting $\mathcal{D}$ equals the subspace of $\mathcal{E}_{2}$ satisfying $\partial_{\nu} u+\partial_{\nu^{\prime}} u+V \gamma u=0$.
Lemma 7.4.2. Suppose that $\partial \Omega$ is of regularity $C^{\infty}$, and $N \geq 0$. Then for all $\epsilon>0$ there exists $R<\infty$, so that if $|\operatorname{Re} \lambda|>R,|\operatorname{Im} \lambda| \leq\left(\frac{1}{2} D_{\Omega}^{-1}-\epsilon\right) \log (|\operatorname{Re} \lambda|)$, and $\chi \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ equals 1 on a neighborhood of $\bar{\Omega}$, then

$$
\left\|\chi\left(R_{V}(\lambda)-R_{V}(-\lambda)\right) \chi g\right\|_{\mathcal{E}_{N}} \leq C_{N}\langle\lambda\rangle^{N-1} e^{2 D_{\chi}|\operatorname{Im} \lambda|}\|g\|_{L^{2}} .
$$

Proof. We proceed by induction on $N$. By Lemma 7.4.1, the result holds for $N=0,1,2$. We assume then that the result is true for integers less than or equal to $N$.

Letting $u=\left(R_{V}(\lambda)-R_{V}(-\lambda)\right) \chi g$, we write

$$
\Delta(\chi u)=2(\nabla \chi) \cdot \nabla u+(\Delta \chi) u-\lambda^{2} \chi u+\left(V \otimes \delta_{\partial \Omega}\right) u
$$

By the induction hypothesis, we have the following estimates:

$$
\begin{aligned}
\|(\Delta \chi) u\|_{H^{N-1}(\Omega) \oplus H^{N-1}\left(\mathbb{R}^{d} \backslash \bar{\Omega}\right)}+\|\chi u\|_{H^{N-1}(\Omega) \oplus H^{N-1}\left(\mathbb{R}^{d} \backslash \bar{\Omega}\right)} \leq C\langle\lambda\rangle^{N-2} e^{2 D_{\chi}|\operatorname{Im} \lambda|}\|g\|_{L^{2}} \\
\|(\nabla \chi) \cdot \nabla u\|_{H^{N-1}(\Omega) \oplus H^{N-1}\left(\mathbb{R}^{d} \backslash \bar{\Omega}\right)}+\|V \gamma u\|_{H^{N-\frac{1}{2}}(\partial \Omega)} \leq C\langle\lambda\rangle^{N-1} e^{2 D_{\chi}|\operatorname{Im} \lambda|}\|g\|_{L^{2}}
\end{aligned}
$$

Proposition 7.5.1 then gives the desired result for $\mathcal{E}_{N+1}$.
We now present the proof of Theorem 7.4. We use the notation from the proof of Theorem 7.3 above. We first note that

$$
\frac{1}{2 \pi} \int_{\Sigma_{A}} e^{-i t \lambda} R_{V}(-\lambda) d \lambda=-\sum_{\mu_{j}>A+\log 2}\left(2 \mu_{j}\right)^{-1} e^{-t \mu_{j}} \Pi_{\mu_{j}}
$$

where the completion of the contour to the lower half plane is justified by Lemma 7.4.1 and the rapid decrease of $e^{-i t \lambda}$ for $t>0$. We thus can write

$$
\chi E_{\Sigma_{A}}(t) \chi g=\frac{1}{2 \pi} \int_{\Sigma_{A}} e^{-i t \lambda} \chi\left(R_{V}(\lambda)-R_{V}(-\lambda)\right) \chi g d \lambda-\sum_{\mu_{j}>A+\log 2}\left(2 \mu_{j}\right)^{-1} e^{-t \mu_{j}} \chi \Pi_{\mu_{j}} \chi g
$$

Assume $c\left(t-2 D_{\chi}\right) \geq N+1$, by Lemma 7.4 .2 the norm in $\mathcal{E}_{N}$ of the integral term is dominated by

$$
C_{A, \chi} e^{-A\left(t-2 D_{\chi}\right)} \int_{-\infty}^{\infty} e^{-(N+1) \log (2+|R|)}\langle A+| R| \rangle^{N-1} d R \leq C_{A, \chi, N} e^{-A t}\|g\|_{L^{2}}
$$

It remains to show that for the eigenvalues $-\mu_{j}^{2}$ with $\mu_{j}>A$, and the resonances $\lambda_{k}$ with $\operatorname{Im} \lambda_{k}<-A$, then

$$
e^{-t \mu_{j}}\left\|\chi \Pi_{\mu_{j}} \chi g\right\|_{\mathcal{E}_{N}}+\left\|\chi \operatorname{Res}\left(e^{-i t \lambda} R_{V}(\lambda), \lambda_{k}\right) \chi g\right\|_{\mathcal{E}_{N}} \leq C_{A, \chi, N} e^{-t A}\|g\|_{L^{2}}
$$

since the difference of $\chi E_{A}(t) \chi$ and $\chi E_{\Sigma_{A}}(t) \chi$ is a sum of such terms.
A similar argument to the proof of Lemma 7.4 .2 gives the bound

$$
\left\|\Pi_{\mu_{j}} g\right\|_{\mathcal{E}_{N}} \leq C_{N}\left\langle\mu_{j}\right\rangle^{N}\|g\|_{L^{2}}
$$

which handles the eigenvalues. To handle the resonances in the lower half plane, consider first the case that $-\lambda_{k}$ is not a pole (that is, $\lambda_{k} \neq-i \mu_{j}$ for any $j$ ). We can then write

$$
\operatorname{Res}\left(e^{-i t \lambda} R_{V}(\lambda), \lambda_{k}\right)=\frac{1}{2 \pi i} \oint_{\lambda_{k}} e^{-i t \lambda}\left(R_{V}(\lambda)-R_{V}(-\lambda)\right) d \lambda
$$

and the estimate follows from Lemma 7.4.2, by choosing a small contour about $\lambda_{k}$ which is contained in $\operatorname{Im} \lambda<-A$. In the case that $-\lambda_{k}$ is a pole, hence an eigenvalue, then the term $R_{V}(-\lambda)$ contributes an eigenvalue projection, which is handled as above.

We now complete the proof of Theorem7.3 by considering the case $N \geq 2$. Eigenfunctions clearly belong to $\mathcal{D}_{N}$, and by an induction argument we have $\left\|\chi \Pi_{\mu_{j}} \chi g\right\|_{\mathcal{D}_{N}} \leq C_{N}\left\langle\mu_{j}\right\rangle^{2 N}\|g\|_{L^{2}}$. The proof then follows from that of Theorem 7.4 , using the following lemma.

Lemma 7.4.3. Suppose that $\Gamma$ is a finite union of compact subsets of $C^{1,1}$ hypersurfaces, and $N \geq 1$. Then for all $\epsilon>0$ there exists $R<\infty$ so that if $|\operatorname{Re} \lambda|>R,|\operatorname{Im} \lambda| \leq$ $\left(\frac{1}{2} D_{\Gamma}^{-1}-\epsilon\right) \log (|\operatorname{Re} \lambda|)$, and $\chi \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ equals 1 on a neighborhood of $\Gamma$, then

$$
\left\|\chi\left(R_{V}(\lambda)-R_{V}(-\lambda)\right) \chi g\right\|_{\mathcal{D}_{N}} \leq C\langle\lambda\rangle^{2 N-1} e^{2 D_{\chi}|\operatorname{Im} \lambda|}\|g\|_{L^{2}}
$$

Proof. The result was proven in Lemma 7.4 .1 for $N=1$. We then proceed by induction, writing

$$
\begin{aligned}
\Delta_{\Gamma, \delta \chi} \chi\left(R_{V}(\lambda)-R_{V}(-\lambda)\right) \chi g & =\left([\Delta, \chi]-\lambda^{2} \chi\right)\left(R_{V}(\lambda)-R_{V}(-\lambda)\right) \chi g \\
& =\left(2 \nabla \chi \cdot \nabla+(\Delta \chi)-\lambda^{2} \chi\right)\left(R_{V}(\lambda)-R_{V}(-\lambda)\right) \chi g
\end{aligned}
$$

By induction, and $\operatorname{since} \operatorname{supp}(\Delta \chi) \subset \operatorname{supp}(\chi)$,

$$
\begin{equation*}
\left\|\left((\Delta \chi)-\lambda^{2} \chi\right)\left(R_{V}(\lambda)-R_{V}(-\lambda)\right) \chi g\right\|_{\mathcal{D}_{N-1}} \leq C\langle\lambda\rangle^{2 N-1} e^{2 D_{\chi}|\operatorname{Im} \lambda|}\|g\|_{L^{2}} \tag{7.4.12}
\end{equation*}
$$

By Lemma 7.4.1. if $\chi_{1} \in C_{c}^{\infty}$ with $\operatorname{supp}\left(\chi_{1}\right) \subset \operatorname{supp}(\chi)$, and $u=\left(R_{V}(\lambda)-R_{V}(-\lambda)\right) \chi g$,

$$
\langle\lambda\rangle\left\|\chi_{1} u\right\|_{L^{2}}+\left\|\chi_{1} u\right\|_{H^{1}} \leq C e^{2 D_{\chi}|\operatorname{Im} \lambda|}\|g\|_{L^{2}} .
$$

On the complement of $\Gamma$, the function $u=\left(R_{V}(\lambda)-R_{V}(-\lambda)\right) \chi g$ satisfies $-\Delta u=\lambda^{2} u$. Since $\nabla \chi$ vanishes on a neighborhood of $\Gamma$, and $\operatorname{supp}(\nabla \chi) \subset \operatorname{supp}(\chi)$, an induction argument and elliptic regularity yields

$$
\left\|\nabla \chi \cdot \nabla\left(R_{V}(\lambda)-R_{V}(-\lambda)\right) \chi g\right\|_{H^{2 N-1}} \leq C\langle\lambda\rangle^{2 N-1} e^{2 D_{\chi}|\operatorname{Im} \lambda|}\|g\|_{L^{2}}, \quad N \geq 1
$$

Since $H_{\text {comp }}^{2 N-1}\left(\mathbb{R}^{d} \backslash \Gamma\right) \subset \mathcal{D}_{N-1}$ with continuous inclusion, this term also satisfies the bound of (7.4.12), and the result follows.

### 7.5 The Transmission Property for $C^{1,1}$ Domains

We provide here a proof of the transmission estimate, Proposition 7.5.2, that we used in Section 7.1 to establish $H^{2}$ regularity of solutions on the complement of $\partial \Omega$ in case $\partial \Omega$ is of $C^{1,1}$ regularity. In case $\partial \Omega$ is smooth, the following estimate, which we used in the proof of Lemma 7.4.2, is well known; see [53], and in particular Theorems 9 and 10 of [25]. We record it here for reference.

Proposition 7.5.1. Suppose that $\Omega \subset \mathbb{R}^{d}$ is a bounded open set, and $\partial \Omega$ is locally the graph of a $C^{\infty}$ function. Let $G_{0}(x, y)$ be Green's kernel for $\Delta^{-1}$, and define the single layer potential map by

$$
\mathcal{S} f(x)=\int_{\partial \Omega} G_{0}(x, y) f(y) d S(y)
$$

Then for $\chi \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$, and $N \geq-1$, $\chi S \ell$ is a continuous map from $H^{N+\frac{1}{2}}(\partial \Omega) \rightarrow$ $H^{N+2}(\Omega) \oplus H^{N+2}\left(\mathbb{R}^{d} \backslash \bar{\Omega}\right)$.

Additionally, for $N \geq 0$ the map

$$
\left(\chi G_{0} \chi g\right)(x)=\chi(x) \int G_{0}(x, y) \chi(y) g(y) d y
$$

is a continuous map from $H^{N}(\Omega) \oplus H^{N}\left(\mathbb{R}^{d} \backslash \bar{\Omega}\right)$ to $H^{N+2}(\Omega) \oplus H^{N+2}\left(\mathbb{R}^{d} \backslash \bar{\Omega}\right)$.
We need the same result for $N=0$ and $\partial \Omega$ of $C^{1,1}$ regularity. The second statement in Proposition 7.5.1 does not depend on the regularity of $\partial \Omega$, so we just need to prove the single layer potential result.

Proposition 7.5.2. Suppose that $\Omega \subset \mathbb{R}^{d}$ is a bounded open set, and $\partial \Omega$ is locally the graph of a $C^{1,1}$ function. Let $G_{0}(x, y)$ be Green's kernel for $\Delta^{-1}$, and let

$$
\mathcal{S} f(x)=\int_{\partial \Omega} G_{0}(x, y) f(y) d \sigma(y)
$$

Then for $\chi \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$, $\chi S \ell$ is a continuous map from $H^{\frac{1}{2}}(\partial \Omega) \rightarrow H^{2}(\Omega) \oplus H^{2}\left(\mathbb{R}^{d} \backslash \bar{\Omega}\right)$.
Proof. Since the kernel is smooth away from the diagonal we may work locally, and assume that $\partial \Omega$ is given as a graph $x_{n}=F\left(x^{\prime}\right)$, with $F \in C^{1,1}\left(\mathbb{R}^{d-1}\right)$. Since surface measure $d \sigma(y)=m\left(y^{\prime}\right) d y^{\prime}$ where $m$ is Lipschitz, we can absorb the $m$ into $f$. Assuming then that $f \in C_{c}^{1}\left(\mathbb{R}^{d-1}\right)$, consider the maps

$$
\begin{aligned}
& T^{\prime} f(x)=\left(\nabla_{x^{\prime}} S \ell f\right)\left(x^{\prime}, F\left(x^{\prime}\right)+x_{d}\right)=c_{d} \int \frac{\left(x^{\prime}-y^{\prime}\right) f\left(y^{\prime}\right) d y^{\prime}}{\left(\left|x^{\prime}-y^{\prime}\right|^{2}+\left|x_{d}+F\left(x^{\prime}\right)-F\left(y^{\prime}\right)\right|^{2}\right)^{\frac{d}{2}}} \\
& T_{d} f(x)=\left(\partial_{x_{d}} S \ell f\right)\left(x^{\prime}, F\left(x^{\prime}\right)+x_{d}\right)=c_{d} \int \frac{\left(x_{d}+F\left(x^{\prime}\right)-F\left(y^{\prime}\right)\right) f\left(y^{\prime}\right) d y^{\prime}}{\left(\left|x^{\prime}-y^{\prime}\right|^{2}+\left|x_{d}+F\left(x^{\prime}\right)-F\left(y^{\prime}\right)\right|^{2}\right)^{\frac{d}{2}}}
\end{aligned}
$$

We seek $H^{\frac{1}{2}} \rightarrow H^{1}\left(x_{d} \neq 0\right)$ bounds for both terms. We have $\partial_{x_{d}} T^{\prime}=\nabla_{x^{\prime}} T_{d}-\left(\nabla_{x^{\prime}} F\right) \partial_{x_{d}} T_{d}$, and since $\Delta S \ell f=0$, for $x_{d} \neq 0$ we can write

$$
\left(1+\left|\nabla_{x^{\prime}} F\right|^{2}\right) \partial_{x_{d}} T_{d} f=\nabla_{x^{\prime}} T^{\prime} f-\left(\nabla_{x^{\prime}} F\right) \nabla_{x^{\prime}} T_{d} f
$$

Thus it suffices to prove $H^{\frac{1}{2}} \rightarrow L^{2}$ bounds for $\chi \nabla_{x}^{\prime} T^{\prime}$ and $\chi \nabla_{x^{\prime}} T_{d}$.
By the dual of the trace estimate we have

$$
\|\chi S \ell f\|_{H^{1}} \leq C\|f\|_{H^{-1 / 2}(\partial \Omega)}
$$

and hence we can bound

$$
\left\|\chi T^{\prime}\left(\nabla_{y^{\prime}} f\right)\right\|_{L^{2}}+\left\|\chi T_{d}\left(\nabla_{y^{\prime}} f\right)\right\|_{L^{2}} \leq C\|f\|_{H^{1 / 2}(\partial \Omega)}
$$

The desired bound will thus follow from showing that

$$
\begin{equation*}
\left\|\chi\left[\nabla_{x^{\prime}}, T^{\prime}\right] f\right\|_{L^{2}}+\left\|\chi\left[\nabla_{x^{\prime}}, T_{d}\right] f\right\|_{L^{2}} \leq C\|f\|_{L^{2}(\partial \Omega)} . \tag{7.5.1}
\end{equation*}
$$

One can write $\left(\chi\left[\nabla_{x^{\prime}}, T_{d}\right] f\right)(x)=\int K\left(x^{\prime}, x_{d}, y^{\prime}\right) f\left(y^{\prime}\right) d y^{\prime}$, where

$$
K\left(x^{\prime}, x_{d}, y^{\prime}\right)=\left(\nabla_{x^{\prime}}+\nabla_{y^{\prime}}\right) \frac{\left(x_{d}+F\left(x^{\prime}\right)-F\left(y^{\prime}\right)\right)}{\left(\left|x^{\prime}-y^{\prime}\right|^{2}+\left|x_{d}+F\left(x^{\prime}\right)-F\left(y^{\prime}\right)\right|^{2}\right)^{\frac{d}{2}}}
$$

and one verifies that $\left|K\left(x^{\prime}, x_{d}, y^{\prime}\right)\right| \lesssim\left(x_{d}^{2}+\left|x^{\prime}-y^{\prime}\right|^{2}\right)^{(1-d) / 2}$ since $\nabla F$ is Lipschitz. Consequently,

$$
\sup _{x^{\prime}} \int_{\left|y^{\prime}\right| \leq L}\left|K\left(x^{\prime}, x_{d}, y^{\prime}\right)\right| d y^{\prime}+\sup _{y^{\prime}} \int_{\left|x^{\prime}\right| \leq L}\left|K\left(x^{\prime}, x_{d}, y^{\prime}\right)\right| d x^{\prime} \leq C_{L} \log \left\langle x_{d}^{-1}\right\rangle .
$$

The bound 7.5.1) for this term is obtained by applying the Schur test in $x^{\prime}$ for each $x_{d}$, followed by integration over $x_{d}$, where we fix $L$ so $f$ and $\chi$ are supported in $\left|x^{\prime}\right| \leq L$. The corresponding kernel of $\chi\left[\nabla_{x^{\prime}}, T^{\prime}\right]$ satisfies the same bounds, which completes the proof of Proposition 7.5.2.

## Chapter 8

## Microlocal Analysis of $-\Delta_{\partial \Omega, \delta}$

In Chapter 7, we demonstrated the existence of a logarithmic resonance free region for a very general class of $\Omega$ and for $V: L^{2}(\partial \Omega) \rightarrow L^{2}(\partial \Omega)$ (see Figure 8.1). However, the result does not exploit the fine properties of $\Omega$ or $V$. In the early 1900s, Sabine [61] postulated that the decay rate of acoustic waves in a region with leaky walls is determined by the average decay over billiards trajectories. Such a Sabine type law incorporates the detailed properties of both the potential and the domain and has been suggested as a way to study resonances in quantum corrals [6] and to study propagation of cellular signals in indoor environments [27]. Our main theorem will give a Sabine type law for the size of the resonance free region when $\partial \Omega \in C^{\infty}$ is strictly convex and $V$ is a pseudodifferential operator.

Denote the set of rescaled resonances and the set of rescaled resonances that are logarithmically close to the real axis by

$$
\begin{equation*}
\Lambda(h):=\left\{z \in \mathbb{C}: z / h \text { is a resonance of }-\Delta_{V, \partial \Omega}\right\} \tag{8.0.1}
\end{equation*}
$$

and

$$
\Lambda_{\log }(h):=\left\{z \in \Lambda(h): z \in[1-C h, 1+C h]+i\left[-M h \log h^{-1}, 0\right]\right\}
$$

respectively.
Remark: All of our proofs go through when

$$
z \in E+[-C h, C h]+i\left[-C h \log h^{-1}, C h^{1-\gamma}\right]
$$

and $\gamma<1 / 2$, but for simplicity we use $\Lambda_{\log }$.
The following theorem is a consequence of the much finer Theorem 8.2
Theorem 8.1. Let $\Omega \subset \mathbb{R}^{d}$ be a strictly convex domain with $C^{\infty}$ boundary, $V \in \Psi(\partial \Omega)$ with $|\sigma(V)|>c>0$. Suppose that $z \in \Lambda_{\log }$. Then for every $\epsilon>0$ there is an $h_{0}>0$ such that for $0<h<h_{0}$

$$
-\frac{\operatorname{Im} z}{h} \geq \frac{1}{d_{\Omega}}\left[\log h^{-1}-\frac{1}{2} \sup _{(a, b) \in \mathcal{A}} \log \left(\frac{|\sigma(V)(a, 0) \sigma(V)(b, 0)|}{4}\right)\right]-\epsilon
$$

where $d_{\Omega}$ is the diameter of $\Omega$ and

$$
\mathcal{A}=\left\{(x, y) \in \partial \Omega \times \partial \Omega:|x-y|=d_{\Omega}\right\} .
$$

We now introduce the dynamical and microlocal objects for the finer version of Theorem 8.1. Let $\pi: T^{*} \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ denote projection to the base, $B^{*} \partial \Omega$ be the coball bundle of the boundary, and $g$ be the induced metric on $\partial \Omega$. Define also

$$
B_{r}^{*} \partial \Omega:=\left\{q \in T^{*} \partial \Omega:\left|\xi^{\prime}(q)\right|_{g}<r\right\}
$$

so that $B^{*} \partial \Omega=B_{1}^{*} \partial \Omega$. Then we denote the billiard ball map (see Section 3.2) by $\beta$ : $B^{*} \partial \Omega \rightarrow \overline{B^{*} \partial \Omega}$. We also denote for $A \subset B^{*} \partial \Omega, \beta_{-N}(A)=\bigcap_{i=1}^{N} \beta^{-i}(A)$.

Let $l: T^{*} \partial \Omega \times T^{*} \partial \Omega \rightarrow \mathbb{R}$ be given by $l\left(q, q^{\prime}\right):=\left|\pi(q)-\pi\left(q^{\prime}\right)\right|$ and write $l_{N}: B^{*} \partial \Omega \rightarrow \mathbb{R}$ where

$$
\begin{equation*}
l_{N}(q):=\frac{1}{N} \sum_{n=0}^{N-1} l\left(\beta^{n}(q), \beta^{n+1}(q)\right) \tag{8.0.2}
\end{equation*}
$$

is the average length between the first $N$ iterates of the billiard ball map originating at $q$.
Let $\gamma: H^{s}\left(\mathbb{R}^{d}\right) \rightarrow H^{s-\frac{1}{2}}(\partial \Omega)$ for $s>1 / 2$ be the restriction map. Then we write

$$
\begin{equation*}
G(z ; h):=\gamma R_{0}(z / h) \gamma^{*}=G_{\Delta}(z ; h)+G_{B}(z ; h)+G_{g}(z ; h)+O_{L^{2} \rightarrow H^{1-}}\left(h^{\infty}\right) \tag{8.0.3}
\end{equation*}
$$

where $G_{\Delta}$ is a pseudodifferential operator and $G_{B}$ is a semiclassical Fourier integral operator associated to $\beta$ and $G_{g}$ is microlocalized near $\left|\xi^{\prime}\right|_{g}=1$ and the diagonal (See Section 6.5 for the proof of this decomposition of $G$. In particular, see Lemma 6.6.4 and 6.6.2). Here, boundedness in $H^{1-}$ is boundedness in $H^{1-\epsilon}$ for all $\epsilon>0$. In the sequel, we suppress the dependence of these operators on $h$ to simplify notation.

Next, let $\chi \in C^{\infty}(\mathbb{R})$ with $\chi \equiv 1$ for $x>2 C$ and $\chi \equiv 0$ for $x<C$. Then fix $\epsilon>0$ and let

$$
\begin{equation*}
R_{\delta}(z):=-\left(I+G_{\Delta}^{1 / 2}(z) V G_{\Delta}^{1 / 2}(z)\right)^{-1} G_{\Delta}^{1 / 2}(z) V G_{\Delta}^{1 / 2}(z) \chi\left(\frac{1-\left|h D^{\prime}\right|_{g}}{h^{\epsilon}}\right) \in h^{1-\frac{\epsilon}{2}} \Psi_{\epsilon}^{-1} \tag{8.0.4}
\end{equation*}
$$

where $\Psi_{\epsilon}^{k}$ denotes a set of slightly exotic semiclassical pseudodifferential operators (see Section 2.3). The order of $R_{\delta}(z)$ in $h$ may vary from point to point in $B^{*} \Omega$. In Section 4.5, we develop the notion of the shymbol of a pseudodifferential operator, a notion of symbol which is sensitive to local changes of order. Using this idea, we have that the compressed shymbol of $R_{\delta}$ (see 4.5.1) ),

$$
\tilde{\sigma}\left(R_{\delta}\right)=\frac{h \tilde{\sigma}(V)}{2 i \sqrt{1-\left|\xi^{\prime}\right|_{g}^{2}}-h \tilde{\sigma}(V)} \chi\left(\frac{1-\left|\xi^{\prime}\right|_{g}}{h^{\epsilon}}\right)
$$

is the reflection coefficient at the point $\left(x^{\prime}, \xi^{\prime}\right) \in B^{*}(\partial \Omega)$. We call $R_{\delta}$ the reflection operator. Since the compressed shymbol of $R_{\delta}$ is independent of $z$, we suppress the dependence of $R_{\delta}(z)$ on $z$ to simplify our notation.

Remark: The compressed shymbol of the reflection operator agrees, up to lower order terms, with the reflection coefficient found when a plane wave with tangential frequency $\xi^{\prime}$ interacts with a delta function potential of constant amplitude $V$ on a hyperplane.

Let $T(z):=G_{\Delta}^{-1 / 2}(z) G_{B}(z) G_{\Delta}^{-1 / 2}(z)$ where $G_{B}$ is the Fourier integral operator component of $G(z)$. Then define $r_{N}(z): B^{*}(\partial \Omega) \rightarrow \mathbb{R}$, the logarithmic average of the reflectivity at successive iterates of the billiard map, by

$$
\begin{equation*}
r_{N}(z, q):=\frac{\operatorname{Im} z}{h} l_{N}(q)+\frac{1}{2 N} \log \tilde{\sigma}\left(h^{-2 N}\left(\left(R_{\delta} T(z)\right)^{*}\right)^{N}\left(R_{\delta} T(z)\right)^{N}\right)(q) \tag{8.0.5}
\end{equation*}
$$

The term $\frac{\operatorname{Im} z}{h} l_{N}$ in 8.0.5 serves to cancel the growth of $T(z)$ in the right hand term. In fact, for $0<N$ independent of $h$ we have

$$
\begin{equation*}
r_{N}(z, q)=\frac{1}{2 N} \sum_{n=1}^{N} \log \left|h^{-1}\left(\tilde{\sigma}\left(R_{\delta}\right) \circ \beta^{n}(q)+O\left(h^{I_{R_{\delta}}(q)+1-2 \epsilon}\right)\right)\right|^{2} \tag{8.0.6}
\end{equation*}
$$

where $I_{R_{\delta}}(q)$ is the local order of $R_{\delta}$ at $q$ (see Section 4.5). The expression (8.0.6) illustrates that $r_{N}$ is the logarithmic average reflectivity over $N$ iterations of the billiard ball map. Moreover, it shows that $r_{N}$ is independent of $z$ up to lower order terms. Because of this, we suppress the dependence on $z$ throughout the rest of this paper.
Note that, if for any $1 \leq i \leq N, \beta^{i}(q) \notin \mathrm{WF}_{\mathrm{h}}(V)$, where $\mathrm{WF}_{\mathrm{h}}(V)$ denotes the semiclassical wavefront set of $V$ (see Section 2.3), then for all $M>0$ there exists $h_{0}$ such that for $0<h<h_{0}$,

$$
\begin{equation*}
r_{N}(q) \leq-M \log h^{-1} \tag{8.0.7}
\end{equation*}
$$

Using Lemma 3.2.1 and 3.2.2, we have for $h$ small enough and $\epsilon<1 / 2$ with $V \in$ $\left.h^{-2 / 3}\right) \Psi(\partial \Omega)$,

$$
\begin{aligned}
\inf _{1-\delta \leq\left|\xi^{\prime}\right| g \leq 1-h^{\epsilon}}-\frac{\log \left|\tilde{\sigma}\left(R_{\delta}\right)(\beta(q))\right|^{2}}{2 l(q, \beta(q))} & =\inf _{C h^{\epsilon} 2 \leq r \leq \delta^{1 / 2}}-\frac{1+O(r)}{2 C r} \log \left(\frac{O\left(h^{2 / 3}\right)}{4 r^{2}+O\left(r^{3}\right)+O\left(h^{2 / 3}\right)}\right) \\
& \geq C \delta^{-1 / 2} \log h^{-1}
\end{aligned}
$$

where $\tilde{\sigma}\left(R_{\delta}\right)$ denotes the shymbol of $R_{\delta}$ (see Section 4.5).
Thus, we see that for all strictly convex domains $\Omega, 0<\epsilon<1 / 2, N_{1}>0$, and $V=O\left(h^{-2 / 3}\right)$

$$
\begin{equation*}
\sup _{N<N_{1} B_{1-c h t}^{*} \partial \Omega} \inf _{N} l_{N}^{-1}\left[\log h^{-1}-r_{N}\right]=\sup _{N<N_{1}} \inf _{1-\delta_{1}} \operatorname{ing}_{N} l_{N}^{-1}\left[\log h^{-1}-r_{N}\right] \tag{8.0.8}
\end{equation*}
$$

for some $\delta_{1}>0$ small enough. That is, the slowest decay rates are those at least a fixed distance away from the glancing region.

With these definitions in hand, we state our main result.
Theorem 8.2. Let $\Omega \subset \mathbb{R}^{d}$ be a strictly convex domain with $C^{\infty}$ boundary. Then there exists $\epsilon_{\Omega}>0$ such that for all $V \in h^{-2 / 3} \Psi(\partial \Omega)$ with $\|\sigma(V)\|_{L^{\infty}}<\epsilon_{\Omega} h^{-2 / 3}$ the following holds. For
$z \in \Lambda_{\log }$ there exists $\delta_{1}>0$ such that for every $\epsilon>0$ and $N_{1}>0$, there is an $h_{0}>0$ such that for $0<h<h_{0}$

$$
\begin{equation*}
-\frac{\operatorname{Im} z}{h} \geq \sup _{N<N_{1}} \inf \left\{l_{N}^{-1}(q)\left[\log h^{-1}-r_{N}(q)\right]-\epsilon: q=\left(x^{\prime}, \xi^{\prime}\right) \in B_{1-\delta_{1}}^{*} \partial \Omega\right\} \tag{8.0.9}
\end{equation*}
$$

## Remarks:

- The proof of Theorem 8.2 also shows that for each $0<\epsilon$ small enough, and $\epsilon_{1}>0$,

$$
-\frac{\operatorname{Im} z}{h} \leq \inf _{N<N_{1}} \sup \left\{l_{N}^{-1}(q)\left[\log h^{-1}-r_{N}(q)\right]+\epsilon_{1}: q=\left(x^{\prime}, \xi^{\prime}\right) \in B_{1-c h^{\epsilon}}^{*} \partial \Omega\right\}
$$

However, in strictly convex domains with $V \in h^{-2 / 3} \Psi(\partial \Omega)$, the quantity on the left goes to infinity for $\left|\xi^{\prime}\right|_{g} \sim 1-c h^{\epsilon}$ for $\epsilon>0$ small enough.

- Theorem 8.2 is sharp in the case of the unit disk in two dimensions with potential $V \equiv h^{-\gamma}$ (See Chapter 2).
- In typical physical systems, the strength of the interaction between a wave and a potential is a function of the frequency of the waves. This corresponds to considering $h$-dependent $V$. The requirement $\|\sigma(V)\|_{L^{\infty}}<\epsilon_{\Omega} h^{-2 / 3}$ comes from the construction of a parametrix for 8.2.2 near glancing in Section 8.3.
However, this is not the natural bound for there to be glancing effects. In fact, the scaling of the problem near glancing dictates that the closest particles can concentrate to glancing is $h^{2 / 3}$ (i.e. $\left|\xi^{\prime}\right|_{g}-1 \sim h^{2 / 3}$ ). Under this restriction and naively intepreting $r_{N}$ as the expression 8.0.6), $|\sigma(V)|=C h^{-5 / 6}$ coincides with the first time that $\mid(\log h+$ $\left.r_{N}\right) / l_{N} \mid \geq c$. Hence, when $|\sigma(V)| \geq c h^{-5 / 6}$, we expect nontrivial effects from glancing points. In Chapter 2, we verified this for $\Omega$ the unit disk in $\mathbb{R}^{2}$ and $V \equiv h^{-\gamma}$.

To see that the characterization of the resonance free region (8.0.9) can be thought of as a time-averaged Sabine type law, observe that if a wave packet intersecting the boundary for the first time at $q \in B^{*} \partial \Omega$ starts with energy $E$, then the energy remaining in $\Omega$ after $N$ reflections is given by

$$
\prod_{n=1}^{N}\left|\sigma\left(R_{\delta}(z)\right)\left(\beta^{n}(q)\right)\right|^{2} E=\exp \left(-2 N\left(\log h^{-1}-r_{N}(q)\right)\right) E
$$

Thus $\log h^{-1}-r_{N}$ is the average exponential rate of decay of the $L^{2}$ norm over $N$ reflections. Moreover, during the $N$ reflections it takes an average of time $l_{N}(q)$ to undergo each reflection. Hence, the $L^{2}$ time rate of decay is $l_{N}^{-1}\left[\log h^{-1}-r_{N}\right]$. Together with the resonance expansions from Chapter 7, this characterization is a step towards mathematically justifying the use of Sabine laws in the analysis of quantum corrals [6], as well as propagation of cellular signals in indoor environments (27].

When there is no potential at a point in $B^{*} \partial \Omega$, the Sabine Law also predicts that wave packets will leave without reflection. Hence, there will be an arbitrarily large exponential rate of decay if every trajectory eventually intersects a point outside of the potential's support. Theorem 8.2 combined with (8.0.7) shows that any trajectory which leaves $\mathrm{WF}_{\mathrm{h}}(V)$ has

$$
l_{N}^{-1}(q)\left[\log h^{-1}-r_{N}(q)\right] \geq M \log h^{-1}
$$

for all $M>0$. Hence, the infimum in Theorem 8.2 excludes trajectories that leave $\mathrm{WF}_{\mathrm{h}}(V)$. Moreover, if every trajectory at least $\delta_{1}$ from glancing eventually leaves $\mathrm{WF}_{\mathrm{h}}(V)$, then there is an arbitrarily large logarithmic resonance free strip, verifying the predictions of the Sabine Law.

Theorem 8.2 immediately gives us the following corollary:
Corollary 8.0.1. Let $\Omega \subset \mathbb{R}^{d}$ be a strictly convex domain with $C^{\infty}$ boundary, $V \in h^{-\gamma} \Psi(\partial \Omega)$ with $\gamma<2 / 3$. Suppose that $z \in \Lambda_{\log }$. Then, for every $\epsilon>0$, there is an $h_{0}>0$ such that for $0<h<h_{0}$

$$
\begin{equation*}
-\frac{\operatorname{Im} z}{h \log h^{-1}} \geq \sup _{N>0} \inf _{\left|\xi^{\prime}\right| g<1}\left\{l_{N}^{-1}(q)(1-\gamma)-\epsilon: q \in \beta_{-N}\left(\mathrm{WF}_{\mathrm{h}}(V)\right)\right\} \tag{8.0.10}
\end{equation*}
$$

Remark: Unlike Theorem 8.2, Corollary 8.0.1 does not provide information about $C_{2}$ in $-\operatorname{Im} z / h \geq-C_{1} \log h^{-1}+C_{2}$. However, the dynamical quantities are easier to compute than those in Theorem 8.2.

### 8.1 Conjectures and Numerical Computation of Resonances

We conjecture that the conclusions of Theorem 8.2 hold for much more general domains $\Omega$. In particular, we conjecture that the results hold for convex domains $\Omega$ with piecewise smooth, $C^{1,1}$ boundary.

Moreover, we conjecture that
Conjecture 8.1.1. Let $\Omega \subset \mathbb{R}^{d}$ be a convex domain with piecewise smooth boundary, $V \in$ $h^{-\gamma} \Psi(\partial \Omega)$ for $\gamma<2 / 3$. Then, for every $\epsilon>0$,

$$
\begin{aligned}
& \#\left\{z \in \Lambda_{\log }:-\frac{\operatorname{Im} z}{h \log h^{-1}} \leq \underset{\left|\xi^{\prime}\right| g<1}{\operatorname{ess} \inf } \limsup _{N>0}\left\{l_{N}^{-1}(q)(1-\gamma): q \in \beta_{-N}\left(\mathrm{WF}_{\mathrm{h}}(V)\right)\right\}-\epsilon\right\} \\
&=o\left(h^{-d+1}\right)
\end{aligned}
$$

If, moreover $|\sigma(V)|>c h^{-\gamma}$, then

$$
\#\left\{z \in \Lambda_{\log }:-\frac{\operatorname{Im} z}{h \log h^{-1}} \geq \underset{\left|\xi^{\prime}\right|_{g}<1}{\operatorname{ess} \sup } \liminf _{N>0} l_{N}^{-1}(1-\gamma)+\epsilon\right\}=o\left(h^{-d+1}\right)
$$



Figure 8.1: We show the various resonance free regions for $V \in \Psi(\partial \Omega)$. The top two lines show the bounds from Chapter 7. The lowest is the resonance free region bound from Theorem 8.2. Since $l_{N}^{-1} \geq d_{\Omega}^{-1}$, the gap between the bounds from Chapter 7 and Theorem 8.2 is at least $\frac{1}{3} d_{\Omega}^{-1} h \log h^{-1}$. If $\sigma(V)=0$ at some points in $B^{*} \partial \Omega$, then the resonance free region given by Theorem 8.2 can be much larger, while those from Chapter 7 will not change.

This conjecture is a mathematical statement of the space-averaged Sabine law. This states that, for ergodic billiards, the exponential decay rate of waves is given by the reciprocal of the average chord length of billiards trajectories. In Figure 8.2, we can see that the resonances cluster around the line given by the space-averaged Sabine law. The authors of [6] numerically compute resonances for scattering by quantum corrals on various domains with ergodic billiard flow. They observe that the resulting resonances cluster around the logarithmic line given by the space-averaged Sabine law.

In order to compute the resonances of 8.2 .2 in some example domains $\Omega \subset \mathbb{R}^{2}$, we consider the boundary problem (8.2.3). We discretize $\partial \Omega$ in steps of equal length. After this process, 8.2.3 reduces to a matrix equation. We then use a maximum searching algorithm to maximize the condition number of the resulting matrix.


Figure 8.2: The figure shows resonances for $\Omega$ the Bunimovich and ellipse with $V=I$ on the top and bottom respectively. The solid lines show the bound of Theorem8.2. On the left, the dashed line is that of our Conjecture. As predicted by the Conjecture, the resonances appear to cluster around the dashed line for the Bunimovich stadium. On the right, observe that there are many resonances close to the solid line. This gives evidence that the conclusions of Theorem 9.1 are valid for the ellipse.

### 8.2 Outline of the Proof and Organization of the Chapter

Let $\mathcal{D}$ be the domain of $-\Delta_{V, \partial \Omega}$ (see Section 7.1). Then, as discussed in Chapter 7, $\lambda$ is a resonance of the system if and only if there is a nontrivial $\lambda$-outgoing solution $u \in \mathcal{D}_{\text {loc }}$ to the equation

$$
\begin{equation*}
\left(-\Delta-\lambda^{2}+V \otimes \delta_{\partial \Omega}\right) u=0 \tag{8.2.1}
\end{equation*}
$$

If $V: H^{1 / 2}(\partial \Omega) \rightarrow H^{1 / 2}(\partial \Omega)$, we saw in Section 7.2 that this is equivalent to solving the following transmission problem

$$
\begin{cases}\left(-\Delta-\lambda^{2}\right) u_{1}=0 & \text { in } \Omega  \tag{8.2.2}\\ \left(-\Delta-\lambda^{2}\right) u_{2}=0 & \text { in } \mathbb{R}^{d} \backslash \bar{\Omega} \\ u_{1}=u_{2}, \partial_{\nu} u_{1}+\partial_{\nu^{\prime}} u_{2}+V u_{1}=0 & \text { on } \partial \Omega \\ u_{2} \lambda \text {-outgoing } & \end{cases}
$$

where we set $\left.u\right|_{\Omega}=u_{1}$ and $\left.u\right|_{\mathbb{R}^{d} \backslash \bar{\Omega}}=u_{2}$. Here, we say that $u_{2}$ is $\lambda$-outgoing if there exists $R<\infty$ and $\varphi \in L_{\text {comp }}^{2}\left(\mathbb{R}^{d}\right)$ such that $u_{2}(x)=\left(R_{0}(\lambda) \varphi\right)(x)$ for $|x| \geq R$, where $R_{0}(\lambda)$ is the analytic continuation of the free resolvent $\left(-\Delta-\lambda^{2}\right)^{-1}$, defined initially for $\operatorname{Im} \lambda>0$. In odd dimensions, we take $\lambda \in \mathbb{C}$ for the above meromorphic continuation, but for even dimensions, we need to consider $\lambda$ as an element of the logarithmic covering of $\mathbb{C} \backslash\{0\}$.

The starting point for the proofs of Theorem 8.2 is the reduction of the solution of (8.2.2) to the solution of the boundary problems

$$
\begin{equation*}
\left(N_{1}+N_{2}+V\right) \psi=0 \quad \Leftrightarrow \quad G\left(N_{1}+N_{2}+V\right) \psi=(I+G V) \psi=0 \tag{8.2.3}
\end{equation*}
$$

where, $G$ is as in 8.0.3) and $N_{1}$ and $N_{2}$ are the Dirichlet to Neumann maps on $\Omega$ and $\mathbb{R}^{d} \backslash \bar{\Omega}$ respectively (see Chapter 7). The second equality above follows from Section 6.1 or 76 , Section 7.11].

The strategy for proving Theorem 8.2 is to microlocally decompose the boundary and treat each region separately. The hyperbolic, glancing, and elliptic regions $(\mathcal{H}, \mathcal{G}$, and $\mathcal{E}$ respectively) have the property that, letting $U^{\prime}$ denote a slightly enlarged version of $U$,

$$
\left(I-\chi_{\mathcal{H}^{\prime}}\right)(I+G V) \chi_{\mathcal{H}}=\left(I-\chi_{\mathcal{G}^{\prime}}\right)(I+G V) \chi_{\mathcal{G}}=\left(I-\chi_{\mathcal{E}^{\prime}}\right)(I+G V) \chi_{\mathcal{E}} \equiv 0
$$

microlocally. Thus, the invertibility of $I+G V$ can be treated separately on each region.
The first step (see Section 6.5) in the proof is to use Theorem 6.6 to microlocally decompose $G$ into a Fourier integral operator associated with the billiard ball map, a pseudodifferential operator, and an operator microsupported in an $h^{\epsilon}$ small neighborhood of the diagonal of $S^{*} \partial \Omega \times S^{*} \partial \Omega$ (that is, in a small neighborhood of glancing). We denote these operators by $G_{B}, G_{\Delta}$, and $G_{g}$ respectively.

Section 8.3 examines the hyperbolic region, $\mathcal{H}$. Let $\psi=\left.u\right|_{\partial \Omega}$ where $u$ is a solution to (8.2.2). After some algebraic manipulation of (8.2.3), we arrive at the equation

$$
\begin{equation*}
\left(I-\left(R_{\delta} T\right)^{N}\right) G_{\Delta}^{1 / 2} V \psi=0, \quad \text { microlocally in } \mathcal{H} \tag{8.2.4}
\end{equation*}
$$

where $T=G_{\Delta}^{-1 / 2} G_{B} G_{\Delta}^{-1 / 2}$ and $R_{\delta}$ is the reflection operator described in 8.0.4). The restrictions on resonances in Theorem 8.2 appear as a consequence of 8.2.4). The crucial fact that leads to a logarithmic resonance free region is that $R_{\delta}$ has semiclassical order $<0$.

In Section 8.3, we use the fact that $G_{\Delta}=O_{L^{2} \rightarrow L^{2}}\left(h^{1-\epsilon / 2}\right)$ to show that $\operatorname{MS}_{\mathrm{h}}(\psi) \cap \mathcal{E}=\emptyset$. Finally, we show that $\operatorname{MS}_{\mathrm{h}}(\psi) \cap \mathcal{G}=\emptyset$ and hence that $\operatorname{MS}_{\mathrm{h}}(\psi)=\emptyset$. To do this, we use a Melrose Taylor parametrix [47] [77] adapted to the semiclassical setting (see Chapter 5 for the construction in the semiclassical setting) to give a microlocal description of $G$ near glancing. We then use this to construct a microlocal parametrix for 8.2.3 near glancing (see Section 8.3).

Remark: If one assumes that $V \in h^{-\gamma} \Psi(\partial \Omega)$ for $\gamma<2 / 3$, then one can avoid the use of the Melrose-Taylor parametrix. We outline this proof in Section 8.3.

### 8.3 Resonance Free Regions - Analysis of the Boundary Equation

We let $z=1+i \omega_{0}$ with and $\omega_{0} \in\left[-C h \log h^{-1}, C h \log h^{-1}\right]$.

## Hyperbolic Region: Appearance of the Dynamics

Recall from Lemma 6.6.4 that

$$
G=G_{\Delta}+G_{B}+G_{g}+O_{L^{2} \rightarrow C^{\infty}}\left(h^{\infty}\right) .
$$

In order to obtain the dynamical restriction on $\operatorname{Im} z$, we localize away from an $h^{\epsilon}$ neighborhood of $S^{*} \partial \Omega$. For $k=1,2$, let $\chi_{k} \in S_{\epsilon}$ with $\chi_{k} \equiv 1$ on $\left\{\left|\xi^{\prime}\right|_{g} \leq 1-(2 k+1) C h^{\epsilon}\right\}$ and $\operatorname{supp} \chi_{k} \subset\left\{\left|\xi^{\prime}\right|_{g} \leq 1-2 k C h^{\epsilon}\right\}$. Let $X_{k}=\operatorname{Op}_{\mathrm{h}}\left(\chi_{k}\right)$. Then, suppose that

$$
(I+G V) X_{1} \psi=f
$$

and let $G_{\Delta}^{-1 / 2}$ be a microlocal inverse for $G_{\Delta}^{1 / 2}$ on

$$
\mathcal{H}:=\left\{\left|\xi^{\prime}\right|_{g} \leq 1-r_{\mathcal{H}} h^{\epsilon}\right\} .
$$

Then

$$
\begin{aligned}
(I+G V) X_{1} \psi & =\left(I+\left(G_{\Delta}+G_{B}\right) V\right) X_{1} \psi+O\left(h^{\infty}\right) \psi \\
& =\left(I+G_{\Delta}^{1 / 2}\left(I+G_{\Delta}^{-1 / 2} G_{B} G_{\Delta}^{-1 / 2}\right) G_{\Delta}^{1 / 2} V\right) X_{1} \psi+O\left(h^{\infty}\right) \psi=f
\end{aligned}
$$

Thus, $f$ is microlocalized on $\mathcal{H}$ and, following the formal algebra in [85, Section 2] multiplying by $G_{\Delta}^{1 / 2} V$ and writing $\varphi=G_{\Delta}^{1 / 2} V X_{1} \psi, T=G_{\Delta}^{-1 / 2} G_{B} G_{\Delta}^{-1 / 2}$, we have

$$
\varphi=-G_{\Delta}^{1 / 2} V G_{\Delta}^{1 / 2}(I+T) \varphi+O\left(h^{\infty}\right) \psi+G_{\Delta}^{1 / 2} V f
$$

Remark: By Lemma 3.2.2, a microlocal inverse on $\mathcal{H}$ will be a microlocal inverse on $\operatorname{MS}_{\mathrm{h}}\left(G_{B} X_{1}\right)$.

Hence, letting

$$
R_{\delta}:=-\left(I+G_{\Delta}^{1 / 2} V G_{\Delta}^{1 / 2}\right)^{-1} G_{\Delta}^{1 / 2} V G_{\Delta}^{1 / 2}
$$

we have

$$
\varphi=R_{\delta} T \varphi+O\left(h^{\infty}\right) \psi-R_{\delta} G_{\Delta}^{-1 / 2} f
$$

Here, $T$ is an FIO associated to the billiard map such that

$$
\sigma\left(\exp \left(\frac{\operatorname{Im} z}{h} \mathrm{Op}_{\mathrm{h}}\left(l\left(q, \beta_{E}(q)\right)\right)\right) T\right)(\beta(q), q)=e^{-i \pi / 4} d q^{1 / 2} \in S
$$

and $R_{\delta} \in h^{1-\frac{\epsilon}{2}} \Psi_{\epsilon}$ is as in (8.0.4).
Thus by standard composition formulae for FIOs, we have for $0<N$ independent of $h$,

$$
\begin{equation*}
\left(I-\left(R_{\delta} T\right)^{N}\right) \varphi=O\left(h^{\infty}\right) \psi-\sum_{m=0}^{N-1}\left(R_{\delta} T\right)^{m} R_{\delta} G_{\Delta}^{-1 / 2} f \tag{8.3.1}
\end{equation*}
$$

We have that

$$
\begin{equation*}
\left(R_{\delta} T\right)_{N}:=\left(\left(R_{\delta} T\right)^{*}\right)^{N}\left(R_{\delta} T\right)^{N}=\mathrm{Op}_{\mathrm{h}}\left(a_{N}\right)+O_{\Psi^{-\infty}}\left(h^{\infty}\right) \tag{8.3.2}
\end{equation*}
$$

where $a_{N} \in S_{\epsilon}$ and, moreover, for $u$ with $\operatorname{MS}_{\mathrm{h}}(u) \subset \mathcal{H}$, by the Sharp Gårding inequality and [87. Theorem 13.13],

$$
\begin{aligned}
& \inf _{\mathcal{H}}\left(\left|\tilde{\sigma}\left(\left(R_{\delta} T\right)_{N}\right)(q)\right|+O\left(h^{I_{\left(R_{\delta} T\right)_{N}}(q)+1-2 \epsilon}\right)\right)\|u\|_{L^{2}} \leq\left\|\left(R_{\delta} T\right)^{N} u\right\|_{L^{2}}^{2} \\
& \left\|\left(R_{\delta} T\right)^{N} u\right\|^{2} \leq \sup _{\mathcal{H}}\left(\left|\tilde{\sigma}\left(\left(R_{\delta} T\right)_{N}\right)(q)\right|+O\left(h^{I_{\left(R_{\delta} T\right)_{N}}(q)+1-2 \epsilon}\right)\right)\|u\|_{L^{2}} .
\end{aligned}
$$

Let

$$
\beta_{1}:=1-\sqrt{\sup _{\mathcal{H}} \tilde{\sigma}\left(\left(R_{\delta} T\right)_{N}\right)} \quad \beta_{2}:=\sqrt{\inf _{\mathcal{H}} \tilde{\sigma}\left(\left(R_{\delta} T\right)_{N}\right)}-1 .
$$

Finally, let $\beta=\max \left(\beta_{1}, \beta_{2}\right)$. Then, we have
Lemma 8.3.1. Suppose that $\beta>h^{\gamma_{1}}$ where $\gamma_{1}<\min (\epsilon / 2,1 / 2-\epsilon)$. Let $c>r_{\mathcal{H}}$ and $g \in L^{2}$ have $\operatorname{MS}_{\mathrm{h}}(g) \subset\left\{1-C h^{\epsilon} \leq\left|\xi^{\prime}\right|_{g} \leq 1-c h^{\epsilon}\right\}$. If

$$
\left(I-\left(R_{\delta} T\right)^{N}\right) u=g
$$

then for any $\delta>0$,

$$
\operatorname{MS}_{\mathrm{h}}(u) \subset\left\{1-(C+\delta) h^{\epsilon} \leq\left|\xi^{\prime}\right|_{g} \leq 1-(c-\delta) h^{\epsilon}\right\} .
$$

In particular, there exists an operator $A$ with $\|A\|_{L^{2} \rightarrow L^{2}} \leq 2 \beta^{-1}$,

$$
A\left(I-\left(R_{\delta} T\right)^{N}\right)=I \text { microlocally on } \mathcal{H}
$$

and if $\mathrm{MS}_{\mathrm{h}}(g) \subset\left\{1-C h^{\epsilon} \leq\left|\xi^{\prime}\right|_{g} \leq 1-c h^{\epsilon}\right\}$, then

$$
\operatorname{MS}_{\mathrm{h}}(A g) \subset\left\{1-(C+\delta) h^{\epsilon} \leq\left|\xi^{\prime}\right|_{g} \leq 1-(c-\delta) h^{\epsilon}\right\} .
$$

Proof. In the case that $\beta_{2}>h^{\gamma_{1}}$, we write

$$
\left(I-\left(R_{\delta} T\right)^{N}\right)=-\left(R_{\delta} T\right)^{N}\left(I-\left(R_{\delta} T\right)^{-N}\right)
$$

microlocally on $\mathcal{H}$ and invert by Neumann series to see that for any $g,\left(I-(R T)^{N}\right) u=g$ has a unique solution modulo $h^{\infty}$ with $\|u\| \leq \beta^{-1}\|g\|$. On the other hand, if $\beta_{1}>h^{\gamma_{1}}$, $\left\|\left(R_{\delta} T\right)^{N}\right\| \leq 1-\beta_{1}$, and we have that for any $g,\left(I-\left(R_{\delta} T\right)^{N}\right) u=g$ has a unique solution with $\|u\| \leq \beta_{1}^{-1}\|g\|$.

We consider the case of $\beta_{1}>h^{\gamma_{1}}$, the case of $\beta_{2}<h^{\gamma_{1}}$ being similar with $\left(R_{\delta} T\right)^{N}$ replace by $\left(R_{\delta} T\right)^{-N}$. Suppose $\left(I-\left(R_{\delta} T\right)^{N}\right) u_{1}=g$. Then, $\left\|u_{1}\right\| \leq \beta^{-1}\|g\|$.

For $k \geq 1$, let $\chi_{k}=\chi_{k}\left(\left|\xi^{\prime}\right|_{g}\right)$ with $\chi_{k+1} \equiv 1$ on $\operatorname{supp} \chi_{k}$ and $\chi_{1} \equiv 1$ on $\operatorname{MS}_{\mathrm{h}}(g)$ so that

$$
\operatorname{supp} \chi_{k} \subset\left\{1-(C+\delta) h^{\epsilon} \leq\left|\xi^{\prime}\right|_{g} \leq 1-(c-\delta) h^{\epsilon}\right\}
$$

Let $X_{k}=\mathrm{Op}_{\mathrm{h}}\left(\chi_{k}\right)$. Finally, let $\chi_{\infty} \in S_{\epsilon}$ with $\chi_{\infty} \equiv 1$ on $\bigcup_{k} \operatorname{supp} \chi_{k}$ and

$$
\operatorname{supp} \chi_{\infty} \subset\left\{1-(C+2 \delta) h^{\epsilon} \leq\left|\xi^{\prime}\right|_{g} \leq 1-(c-2 \delta) h^{\epsilon}\right\} .
$$

Then,

$$
\left(I-\left(R_{\delta} T\right)^{N}\right) X_{1} u_{1}=g+O\left(h^{\infty}\right) g+\left[X_{1},\left(R_{\delta} T\right)^{N}\right] X_{\infty} u_{1}=: g+g_{1}
$$

Then by Lemma 3.2 .2 together with the fact that $\chi_{1}$ depends only on $\left|\xi^{\prime}\right|_{g}$,

$$
\left[X_{1}, T\right]=T\left(T^{-1} X_{1} T-X_{1}\right)=T\left(h^{\epsilon} A+h^{1-2 \epsilon} B\right)
$$

with $A, B \in \Psi_{\epsilon}$. In fact,

$$
\begin{equation*}
T^{-1} X_{1} T=\mathrm{Op}_{\mathrm{h}}\left(\chi_{1}(\beta(q))+O_{\Psi_{\delta}}\left(h^{1-2 \epsilon}\right)\right. \tag{8.3.3}
\end{equation*}
$$

and

$$
\begin{aligned}
& \chi_{1}(\beta(q))-\chi_{1}(q) \\
&=\int_{0}^{1} \chi_{1}^{\prime}\left((1-t)\left|\xi^{\prime}(q)\right|_{g}+t\left|\xi^{\prime}\right|_{g}(\beta(q))\right)\left(\left|\xi^{\prime}(\beta(q))\right|_{g}-\left|\xi^{\prime}(q)\right|_{g}\right) d t \in h^{\epsilon} S_{\epsilon} .
\end{aligned}
$$

Hence, since $X_{\infty} u$ is microlocalized $h^{\epsilon}$ close to glancing,

$$
\operatorname{MS}_{\mathrm{h}}\left(\left[X_{1},\left(R_{\delta} T\right)^{N}\right] X_{\infty} u_{1}\right) \subset\left\{\chi_{2} \equiv 1\right\}
$$

and $g_{1}:=\left[X_{1},\left(R_{\delta} T\right)^{N}\right] X_{\infty} u_{1}$ has

$$
\left\|g_{1}\right\| \leq C\left(h^{\epsilon}+h^{1-2 \epsilon}\right) \beta^{-1}\|g\|_{L^{2}}
$$

Then, there exists $u_{2}$ such that

$$
\begin{gathered}
\left(I-\left(R_{\delta} T\right)^{N}\right) u_{2}=-g_{1} \\
\left\|u_{2}\right\| \leq \beta^{-1}\|g\|_{1} \leq C\left(h^{\epsilon}+h^{1-2 \epsilon}\right) \beta^{-2}\|g\|
\end{gathered}
$$

So,

$$
\left(I-\left(R_{\delta} T\right)^{N}\right)\left(X_{1} u+u_{2}\right)=g+O\left(h^{\infty}\right) g .
$$

Continuing in this way, let

$$
\left(I-\left(R_{\delta} T\right)^{N}\right) u_{k}=-g_{k-1}, \quad g_{k-1}=\left[X_{k-1},(R T)^{N}\right] X_{\infty} u_{k-1} .
$$

Then,

$$
\left\|u_{k}\right\| \leq \beta^{-2 k}\left(h^{k \epsilon}+h^{k(1-2 \epsilon)}\right)\|g\|_{L^{2}}
$$

Moreover, letting $\tilde{u} \sim \sum_{k} X_{k} u_{k}$, we have $X_{\infty} \tilde{u}=\tilde{u}+O\left(h^{\infty}\right) \tilde{u}$ and

$$
\left(I-\left(R_{\delta} T\right)^{N}\right) \tilde{u}=g+O\left(h^{\infty}\right) g
$$

which implies $\tilde{u}-u=O\left(h^{\infty}\right)$.
Now, assume that $\psi$ solves 8.2.3). Then

$$
(I+G V) X_{1} \psi=-\left[X_{1}, G V\right] \psi=: f
$$

and by Lemmas 3.2 .2 and 6.6 .4

$$
\operatorname{MS}_{\mathrm{h}}(f) \subset \mathcal{H} \cap\left\{\left|\xi^{\prime}\right|_{g} \geq 1-\frac{3}{2} C h^{\epsilon}\right\} .
$$

Hence, using (8.3.1) and Lemmas 3.2.2, and 8.3.1, provided that $\beta>h^{\gamma_{1}}$ for some $\gamma_{1}<$ $\min (\epsilon / 2,1 / 2-\epsilon)$,

$$
\begin{equation*}
X_{2} \varphi=O\left(h^{\infty}\right) \psi \tag{8.3.4}
\end{equation*}
$$

We now examine when $\beta \ll h^{\gamma_{1}}$. For this to occur,

$$
\liminf _{h \rightarrow 0} \frac{\inf \| \tilde{\sigma}\left(\left(R_{\delta} T\right)_{N}\right)(q)|-1|}{h^{\gamma_{1}}}=0 .
$$

So, let

$$
\left|\tilde{\sigma}\left(R_{\delta} T\right)_{N}(q)\right|=e^{e(q)}
$$

Taking logs and renormalizing we have

$$
\frac{2 \operatorname{Im} z}{h} N l_{N}(q)-\frac{2 \operatorname{Im} z}{h} N l_{N}(q)+\log \left|\tilde{\sigma}\left(\left(R_{\delta} T\right)_{N}\right)(q)\right|=e(q) .
$$

So,

$$
\begin{aligned}
-\frac{\operatorname{Im} z}{h} & =-l_{N}^{-1}(q)\left[\frac{\operatorname{Im} z}{h} l_{N}(q)+\frac{1}{2 N} \log \left|\tilde{\sigma}\left(\left(R_{\delta} T\right)_{N}\right)(q)\right|+e(q)\right] \\
& =-l_{N}^{-1}(q)\left(r_{N}^{\delta}(q)+e(q)\right) .
\end{aligned}
$$

where $r_{N}^{\delta}$ as in 8.0.5. Thus, if $X_{2} \varphi \neq O_{L^{2}}\left(h^{\infty}\right)$, for any $c>0$,

$$
\inf _{\mathcal{H}}-l_{N}^{-1}\left(r_{N}^{\delta}+c h^{\gamma_{1}}\right) \leq-\frac{\operatorname{Im} z}{h} \leq \sup _{\mathcal{H}}-l_{N}^{-1}\left(r_{N}^{\delta}-c h^{\gamma_{1}}\right) .
$$

Now, writing

$$
R_{\delta} T=\left[R_{\delta} \exp \left(-\frac{\operatorname{Im} z}{h} \mathrm{Op}_{\mathrm{h}}(l(q), \beta(q))\right)\right]\left[\exp \left(\frac{\operatorname{Im} z}{h} \mathrm{Op}_{\mathrm{h}}(l(q), \beta(q))\right) T\right]
$$

and applying Lemma 4.5.1 shows that

$$
\begin{aligned}
& \tilde{\sigma}\left(\left(R_{\delta} T\right)_{N}\right)(q)=\exp \left(-\frac{2 \operatorname{Im} z}{h} \sum_{n=0}^{N-1} l\left(\beta^{n}(q), \beta^{n+1}(q)\right)\right) \\
& \prod_{i=1}^{N}\left(\left|\tilde{\sigma}\left(R_{\delta}\right)\left(\beta^{i}(q)\right)\right|^{2}+O\left(h^{I_{R_{\delta}}\left(\beta^{i}(q)\right)+1-2 \epsilon}\right)\right) .
\end{aligned}
$$

Since we have assumed $z \in \Lambda_{\log }$, this implies that if $\beta^{i}(q) \notin \mathrm{WF}_{\mathrm{h}}(V)$ for some $0<i \leq N$ then $r_{N}^{\delta}(q) \leq-M \log h^{-1}$ for all $M$. Hence,

$$
\inf _{\mathcal{H} \cap \beta_{-N}\left(\mathrm{WF}_{\mathrm{h}} V\right)}-l_{N}^{-1}\left(r_{N}^{\delta}+c h^{\gamma_{1}}\right) \leq-\frac{\operatorname{Im} z}{h} \leq \sup _{\mathcal{H}}-l_{N}^{-1}\left(r_{N}^{\delta}-c h^{\gamma_{1}}\right)
$$

Now, suppose that $X_{2} \varphi=O_{L^{2}}\left(h^{\infty}\right)$. We have by Theorem 6.1 that

$$
\|G\|_{L^{2} \rightarrow L^{2}} \leq C h^{2 / 3} \log h^{-1} e^{C \frac{(\operatorname{Im} z)-}{h}} .
$$

Remark: We can also use Theorem 6.7 to remove the $\log h^{-1}$ from the above expression.
Therefore,

$$
X_{3}(I+G V) X_{1} \psi=X_{3} \psi+O\left(h^{\infty}\right) \psi=O\left(h^{\infty}\right) \psi
$$

We summarize the discussion above in the following lemma.
Lemma 8.3.2. Let $0<\epsilon<1 / 2$. If for some $\gamma_{1}<\min (\epsilon / 2,1 / 2-\epsilon)$, and $c>0$

$$
\left.-\frac{\operatorname{Im} z}{h}<\inf _{\mathcal{H} \cap \beta_{-N}\left(\mathrm{WF}_{\mathrm{h}}(V)\right)}-l_{N}^{-1}\left(r_{N}+c h^{\gamma_{1}}\right) \text { or }-\frac{\operatorname{Im} z}{h}>\sup _{\mathcal{H}}-l_{N}^{-1}\left(r_{N}-c h^{\gamma_{1}}\right)\right),
$$

where $l_{N}$ and $r_{N}$ are as in 8.0.2 and 8.0.5 respectively, then

$$
\operatorname{MS}_{\mathrm{h}}(\psi) \subset\left\{\left|\xi^{\prime}\right|_{g} \geq 1-c h^{\epsilon}\right\}
$$

## Elliptic Region

Next, we show that solutions to 8.2.3) cannot concentrate in the elliptic region $\mathcal{E}:=\left\{\left|\xi^{\prime}\right|_{g} \geq\right.$ $\left.1+c h^{\epsilon}\right\}$ for some $\epsilon>0$.

Fix $\epsilon<1 / 2$. Let $\chi_{1} \in S_{\epsilon}$ have $\chi_{1} \equiv 1$ on $\left|\xi^{\prime}\right|_{g} \geq 1+2 C h^{\epsilon}$ and supp $\chi_{1} \subset\left|\xi^{\prime}\right|_{g} \geq 1+C h^{\epsilon}$. Also, let $\chi_{2} \in S_{\epsilon}$ have $\operatorname{supp} \chi_{2} \subset\left|\xi^{\prime}\right|_{g} \geq 1+3 C h^{\epsilon}$ and $\chi_{2} \equiv 1$ on $\left|\xi^{\prime}\right|_{g} \geq 1+4 C h^{\epsilon}$. Let $X_{1}=\operatorname{Oph}_{\mathrm{h}}\left(\chi_{1}\right)$ and $X_{2}=\operatorname{Op}_{\mathrm{h}}\left(\chi_{2}\right)$.

Let $\psi$ solve 8.2.3). Then, we have

$$
X_{2}(I+G V) X_{1} \psi=-X_{2}(I+G V)\left(1-X_{1}\right) \psi
$$

Now, by Lemma 6.6.4

$$
G V\left(1-X_{1}\right)=\left(G_{B}+G_{\Delta}+G_{g}\right) V\left(1-X_{1}\right)+O_{L^{2} \rightarrow L^{2}}\left(h^{\infty}\right) .
$$

But, $X_{2}\left(G_{B}+G_{\Delta}+G_{g}\right) V\left(1-X_{1}\right)=O_{L^{2} \rightarrow L^{2}}\left(h^{\infty}\right)$ since

$$
\begin{gathered}
\operatorname{MS}_{\mathrm{h}}^{\prime}\left(G_{g}\right) \circ \operatorname{supp}\left(1-\chi_{1}\right) \subset\left\{\left.| | \xi^{\prime}\right|_{g}-1 \mid \leq c h^{\epsilon}\right\} \\
\operatorname{MS}_{\mathrm{h}}\left(G_{B}\right)^{\prime} \circ \operatorname{supp}\left(1-\chi_{1}\right) \subset\left\{\left|\xi^{\prime}\right|_{g}<1\right\}
\end{gathered}
$$

By similar arguments

$$
X_{2} G V X_{1}=X_{2} G_{\Delta} V X_{1}+O_{L^{2} \rightarrow L^{2}}\left(h^{\infty}\right)
$$

Thus,

$$
X_{2}\left(I+G_{\Delta} V\right) X_{1} \psi=O\left(h^{\infty}\right) \psi
$$

Since $\sigma\left(X_{2} G_{\Delta}\right)=O_{S_{\epsilon}^{-1}}\left(h^{1-\frac{\epsilon}{2}}\right)$ and $V \in h^{-2 / 3} \Psi(\partial \Omega),\left|\sigma\left(I+G_{\Delta} V\right)\right|>c>0$, this implies

$$
\operatorname{MS}_{\mathrm{h}}(\psi) \cap\left\{\left|\xi^{\prime}\right|_{g} \geq 1+2 h^{\epsilon}\right\}=\emptyset
$$

We also need an elliptic estimate. Let supp $\chi_{3} \subset\left\{\left|\xi^{\prime}\right|_{g} \geq E+c\right\}$ and $X_{3}=\mathrm{Op}_{\mathrm{h}}\left(\chi_{3}\right)$. Using the fact

$$
\mathrm{Op}_{\mathrm{h}}(q):=C\left(1-X_{3}\right)+X_{2}\left(I+G_{\Delta} V\right) X_{1}
$$

is elliptic and has

$$
\mathrm{Op}_{\mathrm{h}}(q) \psi=C\left(1-X_{3}\right) \psi+O\left(h^{\infty}\right) \psi,
$$

we have

$$
\left\|X_{1} \psi\right\|_{L^{2}} \leq C\left\|\left(1-X_{3}\right) \psi\right\|_{L^{2}}+O\left(h^{\infty}\right)
$$

Summarizing,
Lemma 8.3.3. Fix $1 / 2>\epsilon>0$. Suppose that $|\operatorname{Im} z| \leq C h \log h^{-1}$ and $\psi=\left.u\right|_{\partial \Omega}$ where $u$ solves (8.2.2). Then

$$
\operatorname{MS}_{\mathrm{h}}(\psi) \cap\left\{\left|\xi^{\prime}\right|_{g} \geq 1+h^{\epsilon}\right\}=\emptyset
$$

Moreover, for $\chi \in S$ with $\operatorname{supp} \chi \subset\left\{\left|\xi^{\prime}\right|_{g} \geq 1+c\right\}$,

$$
\begin{equation*}
\left\|X_{1} \psi\right\|_{L^{2}} \leq C\left\|\left(1-\operatorname{Op}_{\mathrm{h}}(\chi)\right) \psi\right\|_{L^{2}}+O\left(h^{\infty}\right) \tag{8.3.5}
\end{equation*}
$$

If, in addition, the hypotheses of Lemma 8.3.2 hold, then

$$
\operatorname{MS}_{\mathrm{h}}(\psi) \subset\left\{\left(x^{\prime}, \xi^{\prime}\right) \in T^{*} \partial \Omega:\left|\left|\xi^{\prime}\right|_{g}-1\right| \leq c h^{\epsilon}\right\}
$$

## Glancing Points

Now, we consider $I+G V$ microlocally near a glancing point. We use the estimate from Lemma 6.7.2.

Fix $\epsilon<1 / 2$. Let $\chi \in S_{\epsilon}$ have $\chi \equiv 1$ on $\left\{\left|1-\left|\xi^{\prime}\right|_{g}\right| \leq h^{\epsilon}\right\}$ and $\operatorname{supp} \chi \subset\left\{\left|1-\left|\xi^{\prime}\right|_{g}\right| \leq 2 h^{\epsilon}\right\}$ with $X=\operatorname{Op}_{\mathrm{h}}(\chi)$. Then suppose that $\psi$ solves (8.2.3) and the hypotheses of Lemma 8.3.2 hold. Then by Lemma 8.3.3 and 8.3.5), $X \psi=\psi+O\left(h^{\infty}\right) \psi$. Therefore,

$$
(I+G V) X \psi=O\left(h^{\infty}\right) \psi
$$

Now, by Lemma 6.7.2,

$$
\begin{aligned}
\|G V X \psi\|_{L^{2}} & \leq C_{\Omega} h^{2 / 3}\|V X \psi\|_{L^{2}} \\
& \leq C_{\Omega} h^{2 / 3}\left(\sup _{1-C h^{\epsilon} \leq\left|\xi^{\prime}\right| g \leq 1+C h^{\epsilon}}|\sigma(V)|+O\left(h^{1-\alpha-2 \epsilon}\right)\right)\|X \psi\|_{L^{2}} .
\end{aligned}
$$

So, provided that $|\sigma(V)| \leq \frac{1}{2 C_{\Omega}} h^{-2 / 3}, X \psi=O_{L^{2}}\left(h^{\infty}\right)$ and hence $\psi=O_{L^{2}}\left(h^{\infty}\right)$, a contradiction.

## Sketch of an Alternate Proof Near Glancing

For $V \in h^{-\alpha} \Psi(\partial \Omega)$ with $\alpha<2 / 3$ one can give an alternate proof avoiding the use of the Melrose-Taylor parametrix and instead use the estimates from Theorem 6.1 on $G$ and that, by Lemma 6.6.4, $G_{g}$ is microlocalized near the diagonal. In particular, note that for $\psi$ microlocalized to a $\delta$ neighborhood of glancing, and $\chi \in C_{c}^{\infty}(\partial \Omega)$ with $\chi_{1} \equiv 1$ near $x_{0}$, $\operatorname{supp} \chi_{1} \subset B\left(x_{0}, \delta\right)$, and $\chi \in C_{c}^{\infty}(\partial \Omega)$ with $\chi \equiv 1$ on $\operatorname{supp} \chi_{1}, \chi_{1}(I+G V) \chi \psi=O\left(h^{\infty}\right) \psi$ and hence $\chi_{1} \psi=o(1) \psi$ by Theorem 6.1 together with the fact that supp $\chi \subset B\left(x_{0}, \delta\right)$.

The improvement given by the Melrose-Taylor parametrix comes from the fact that the microlocal model for $G$ gives estimates $\|G\| \leq C h^{-2 / 3}$ in $h^{\epsilon}$ neighborhoods of the diagonal, while those in Theorem 6.1 are of the form $\|G\| \leq C h^{-2 / 3} \log h^{-1}$. We also expect that a more detailed analysis of the microlocal model for $G$ near glancing will allow the analysis of potentials $V \in h^{-\alpha} \Psi(\partial \Omega)$ for $\alpha>2 / 3$.

## Proof of Corollary 8.0.1

We deduce Corollary 8.0.1 from Theorem 8.2,
Proof. Suppose $V \in h^{-\alpha} \Psi(\partial \Omega)$ and fix $\epsilon>0$. Let

$$
\mathcal{H}_{\delta_{1}}:=\left\{\left|\xi^{\prime}\right|_{g}<1-\delta_{1}\right\} \cap \beta_{-N}\left(\mathrm{WF}_{\mathrm{h}} V\right) .
$$

First observe that if for some $0<i \leq N, \beta^{i}(q) \notin \mathrm{WF}_{\mathrm{h}}(V)$, then for all $M>0$, there exists $h_{0}$ such that for $0<h<h_{0}, r_{N} \leq-M \log h^{-1}$. Together with the fact that we assume
$\operatorname{Im} z=O\left(h \log h^{-1}\right)$, this shows that the infimum in

$$
-\frac{\operatorname{Im} z}{h} \geq \inf _{\left|\xi^{\prime}\right|_{g}<1-\delta_{1}}-l_{N}^{-1}(q) r_{N}(q)
$$

excludes trajectories leaving $\beta_{-N}\left(\mathrm{WF}_{\mathrm{h}} V\right)$ (see also (8.0.9). Hence, we may reduce to taking an infimum over $\mathcal{H}_{\delta_{1}}$. Observe that

$$
\begin{aligned}
\inf _{\mathcal{H}_{\delta_{1}}}-\frac{r_{N}}{l_{N}} & \geq \inf _{\mathcal{H}_{\delta_{1}}} l_{N}^{-1}\left(\log h^{\alpha-1}+\inf _{\mathcal{H}_{\delta_{1}}}-\left(r_{N}+\log h^{\alpha-1}\right)\right) \\
& \geq \inf _{\mathcal{H}_{\delta_{1}}} l_{N}^{-1}(1-\alpha) \log h^{-1}-C \geq \inf _{\mathcal{H}_{0}} l_{N}^{-1}(1-\alpha) \log h^{-1}-C
\end{aligned}
$$

since $h^{2 \alpha}\left|\tilde{\sigma}\left(h^{-1} R_{\delta}\right)\right|^{2} \leq C$ on $\left|\xi^{\prime}\right|_{g} \leq 1-\delta_{1}$.
Now, fix $N_{1}>0$ such that

$$
\sup _{N>0} \inf _{\mathcal{H}_{0}} l_{N}^{-1}-\frac{\epsilon}{2} \leq \inf _{\mathcal{H}_{0}} l_{N_{1}}^{-1} .
$$

Then, apply Theorem 8.2 and observe that for $h$ small enough, we can absorb $C$ into the first term increasing the factor of $\log h^{-1}$ by at most $\epsilon / 2$.

## Chapter 9

## Existence Resonances for the Delta Potential

In this chapter we show that the resonance free region given by Corollary 8.0.1 is generically optimal for $V \in C^{\infty}(\partial \Omega)$. In particular, for every periodic billiards trajectory with $M$ reflections whose intersection with $T^{*} \partial \Omega$ does not leave $\{V \neq 0\}$, there are infinitely many resonances with

$$
-\operatorname{Im} z \leq\left(l_{M}^{-1}(q)+\epsilon\right) h \log h^{-1}
$$

where $q$ is a point in the billiards trajectory.
Theorem 9.1. There exists an open dense collection

$$
\mathcal{A} \subset\left\{\Omega \subset \mathbb{R}^{d}: \partial \Omega \in C^{\infty} \text { and } \Omega \text { is strictly convex }\right\}
$$

such that for all $\Omega \in \mathcal{A}$ the following statement holds. Suppose that there exists $q \in B^{*} \partial \Omega$, $M \in \mathbb{Z}^{+}$such that $\beta^{M}(q)=q$. Then for $V \in C^{\infty}(\partial \Omega)$, if $V\left(\pi \circ \beta^{i}(q)\right) \neq 0$ for $0 \leq i<M$, we have that for all $\delta>0$ and $\rho>l_{M}^{-1}(q)$ there exists $h_{0}>0$ such that for $0<h<h_{0}$,

$$
\#\left\{z \in \Lambda:|z| \leq 1, \operatorname{Im} z>-\rho h \log \left(\operatorname{Re} z h^{-1}\right)\right\} \geq c h^{-1+\delta}
$$

Remark: The bound on $\rho$ in Theorem 9.1 matches that in Corollary 8.0.1. Hence, the bounds from Corollary 8.0.1 are generically sharp up to $o\left(h \log h^{-1}\right)$ corrections. However, one must note that in Theorem 9.1, $V \in C^{\infty}(\partial \Omega)$ is a multiplication operator.

## Outline of the Proofs

Theorem 9.1 is proved in Section 9.1. The main component of the proof is to describe the singularities of the wave trace, $\sigma(t)$, at times $t>0$ for the problem

$$
\begin{equation*}
\left(\partial_{t}^{2}-\Delta+V(x) \otimes \delta_{\partial \Omega}\right) u=0 \tag{9.0.1}
\end{equation*}
$$

As in [16] and [19], we first show that the singularities occur at times $T$ such that $T$ is the length of a closed billiard trajectory. To examine contributions from non-glancing trajectories, we follow [19], using the parametrix for (9.0.1) constructed in [63] (see also [43]) along with a finer analysis near the boundary. In particular, we show that

$$
\left|\widehat{\psi_{\epsilon, T} \sigma_{\beta}}(\tau)\right| \geq c \tau^{-N}
$$

where $\sigma_{\beta}(t)$ denotes the wave trace microlocalized near a periodic trajectory of length $T$, $\psi_{\epsilon, T} \in C_{c}^{\infty}(\mathbb{R})$ is a cutoff function near $T$, and $N$ is the number of times the trajectory intersects the boundary. Finally, we use the Melrose Taylor parametrix 47) (see Appendix 5.6) to show that contributions from trajectories sufficiently close to glancing can be neglected. Moreover, we show that, generically, the wave trace is smooth at accumulation points of the length spectrum. In particular, we have the following consequence of 9.1.14

Proposition 9.0.1. For a generic strictly convex domain $\Omega$ we have that for any closed geodesic $\gamma \in \partial \Omega$, there exists a neighborhood, $U_{M} \ni T_{\gamma}$, such that $\sigma(t)$ is $C^{M}$ on $U_{M}$. In particular, $\sigma(t)$ is smooth at $T_{\gamma}$.

Remark: There has been interest in the singularities of wave traces near accumulation points in the length spectrum. In [79], the authors show that the wave trace is smooth at such points for the Dirichlet Laplacian inside the unit disk in $\mathbb{R}^{2}$. Proposition 9.0.1 gives an analog of such a result in our setting. Generically, the only accumulation points in $L_{\Omega}$ are the lengths of closed geodesics, $\gamma \in \partial \Omega$, [57, Section 7.4].

Next, the Poisson formula of [86] shows that the wave trace $\sigma$ is a distribution of the form

$$
\sigma(t)=\sum_{\lambda \in \Lambda} e^{-i t \lambda}
$$

Hence, we are able to use the estimate on the singularities of the wave trace along with [67, Theorem 1] to obtain Theorem 9.1.

### 9.1 Existence for generic domains and potentials

We will use 67 to establish a lower bound on the number of resonances in a logarithmic region. In particular, letting $\Lambda(h)$ be as in (8.0.1), we prove

Lemma 9.1.1. Let $\Omega \subset \mathbb{R}^{d}$ be strictly convex with smooth boundary and $V \in C^{\infty}(\partial \Omega)$. Suppose that the length spectrum of the billiard trajectories, $L_{\Omega}$, is simple, that the length of all periodic billiards trajectories are isolated inside $L_{\Omega}$, and that all periodic billiards trajectories are clean. Finally, suppose that there exists $q \in B^{*} \partial \Omega$ and $M \in \mathbb{Z}^{+}$such that $\beta^{M}(q)=q$ and $V\left(\pi \circ \beta^{i}(q)\right) \neq 0$ for all $0 \leq i<M$. Then, for all $\rho>l_{M}^{-1}(q)$ and $\delta>0$, there exist $h_{0}>0$ and $c>0$ such that for $0<h<h_{0}$,

$$
\#\left\{z \in \Lambda(h):|z| \leq 1, \operatorname{Im} z \geq-\rho h \log \left(\operatorname{Re} z h^{-1}\right)\right\} \geq c h^{-1+\delta}
$$

Remark: Our convention is not to include closed geodesics in the boundary in the set of periodic billiards trajectories. We do, however, include the length of such geodesics in $L_{\Omega}$.

To do this, we describe the singularities of the wave trace for our problem. We then apply the Poisson formula of $[\mathbf{S j Z w J F A}, 86]$ to see that for $t>0$ and andy $k>0$, the wave trace is of the form

$$
\sum_{\lambda \in \Lambda_{\gamma}} m(\lambda) e^{-i \lambda|t|}+O_{C^{\infty}}(1)
$$

where

$$
\Lambda_{\gamma}=\left\{\lambda \text { a resonance for }-\Delta_{V, \partial \Omega}: \operatorname{Im} \lambda \geq-\gamma|\lambda|\right\}
$$

Last, the results of 67] can be applied to yield the lemma.
Let $U_{0}$ denote the forward free wave propagator. Then for all $T>0$, there exist $M>0$ such that, for $t \leq T$ and $|x|>M, U(t, x, x)=U_{0}(t, x, x)$. Hence, letting $\chi \in C_{c}^{\infty}, \chi \equiv 1$ on $B(0, M)$, we have

$$
\begin{aligned}
\sigma(t) & =\int U(t, x, x)-U_{0}(t, x, x) d x \\
& =\int \chi U(t, x, x)-\chi U_{0}(t, x, x) d x=: \sigma_{1}(t)+\sigma_{2}(t)
\end{aligned}
$$

But, the singularities of $\sigma_{2}(t)$ occur only at times $t$ for which there exist a periodic geodesic on $\mathbb{R}^{d}$ with period $t[16]$. Thus, $\sigma_{2} \in C^{\infty}((0, T])$ and we only need to consider singularities of $\sigma_{1}(t)$ as a distribution in $t$. We denote

$$
\sigma_{1, \alpha}(t)=\int \chi(x)(U \circ \alpha)(t, x, x) d x
$$

where $\alpha$ is a microlocal cutoff.

## A non-glancing parametrix

Safarov [63, Section 3] (see also [43, Appendix B]) constructs a local parametrix for the wave transmission problem associated to (9.0.1). We recall the results of the construction in Lemmas 9.1.2 and 9.1.3.

Let $x=\left(x_{1}, x^{\prime}\right)$ be coordinates near $\partial \Omega$ where $x_{1}$ is the signed distance from the point to $\partial \Omega$ and $x^{\prime}$ are coordinates on $\partial \Omega$. (Here $x_{1}>0$ in $\Omega$ and $x_{1}<0$ in $\mathbb{R}^{d} \backslash \bar{\Omega}$ ). Then, let $\left\{g^{i j}\right\}$ be the inverse metric tensor and $a(x, \xi)$ the Riemannian quadratic form. Finally, let $g^{\prime}$ and $a^{\prime}$ be the restrictions of $g$ and $a$ to $T^{*} \partial \Omega$. In $\left(x_{1}, x^{\prime}\right)$ coordinates,

$$
\begin{gathered}
g^{\prime}\left(x^{\prime}\right)=g\left(0, x^{\prime}\right), \quad a\left(x_{1}, x^{\prime}, \xi_{1}, \xi^{\prime}\right)=\xi_{1}^{2}+\tilde{a}\left(x_{1}, x^{\prime}, \xi^{\prime}\right), \\
a^{\prime}\left(x^{\prime}, \xi^{\prime}\right)=\tilde{a}\left(0, x^{\prime}, \xi^{\prime}\right)
\end{gathered}
$$

Let $\alpha$ be a pseudodifferential operator. We seek operators $U_{\alpha}(t)$ with kernel $U_{\alpha}(t, x, y)$ such that, writing $\left.\right|_{x_{1}^{+}=0}$ for restriction from $\Omega$ and $\left.\right|_{x_{1}^{-}=0}$ for restriction from $\mathbb{R}^{d} \backslash \bar{\Omega}$, we have

$$
\begin{cases}\left(\partial_{t}^{2}-\Delta_{x}\right) U_{\alpha}=0, & \text { for } x \in \mathbb{R}^{d} \backslash \partial \Omega  \tag{9.1.1}\\ \left.U_{\alpha}\right|_{x_{1}^{+}=0}=\left.U_{\alpha}\right|_{x_{1}^{-}=0}, & \\ \left.\partial_{x_{1}} U_{\alpha}\right|_{x_{1}^{-}=0}-\left.\partial_{x_{1}} U_{\alpha}\right|_{x_{1}^{+}=0}+V(x) U_{\alpha}=0 & \text { on } \partial \Omega \\ \left.U_{\alpha}\right|_{t=0}=\alpha(x, y), & \\ U_{\alpha}(t) \in C^{\infty}, & \text { for } t \ll 0\end{cases}
$$

Recall that $G_{k}^{t}$, the billiards flow of type $k$, is defined as in (3.2.1), (3.2.2), and $\mathcal{O}_{T}$; the glancing set is defined as in (3.2.3).

When $t$ is small enough so that no geodesics starting in $\operatorname{supp} \alpha$ hit the boundary, then $U_{\alpha}$ is, modulo $C^{\infty}$, the solution to the free wave equation on $\mathbb{R}^{d}$. Hence it is a homogeneous FIO associated to $G_{0}^{t}\left[63\right.$, Section 3]. Also, if $\alpha_{0}$ is a pseudodifferential operator such that $\alpha_{0}=1$ in a neighborhood of $G_{0}^{t_{1}}(\operatorname{supp} \alpha)$, then

$$
U_{\alpha_{0}}(t) U_{\alpha}\left(t_{1}\right)=U_{\alpha}\left(t+t_{1}\right)
$$

modulo a smoothing operator.
Thus, to construct $U_{\alpha}$ we only need to consider the case when geodesics from $\operatorname{supp} \alpha$ hit the boundary in short times. We do this using the ansatz

$$
\begin{equation*}
U_{\alpha}(x, y, t)=\sum_{j=1}^{3} \int e^{i \varphi_{j}(x, y, t, \theta)} b_{j}(x, y, t, \theta) d \theta \tag{9.1.2}
\end{equation*}
$$

with terms in the sum corresponding to the incident $(j=1)$, reflected $(j=2)$ and transmitted $(j=3)$ components. (Here $x_{1} \leq 0$ for $j=1,2$ and $x_{1} \geq 0$ for $j=3$.) The phase functions $\varphi_{j}$ coincide when $x \in \partial \Omega$ and satisfy the eikonal equations

$$
\left\{\begin{array}{l}
\partial_{x_{1}} \varphi_{1}+\left[\left(\partial_{t} \varphi_{1}\right)^{2}-\tilde{a}\left(x_{1}, x^{\prime}, \nabla_{x^{\prime}} \varphi_{1}\right)\right]^{1 / 2}=0  \tag{9.1.3}\\
\partial_{x_{1}} \varphi_{3}+\left[\left(\partial_{t} \varphi_{3}\right)^{2}-\tilde{a}\left(x_{1}, x^{\prime}, \nabla_{x^{\prime}} \varphi_{3}\right)\right]^{1 / 2}=0 \\
\partial_{x_{1}} \varphi_{2}-\left[\left(\partial_{t} \varphi_{2}\right)^{2}-\tilde{a}\left(x_{1}, x^{\prime}, \nabla_{x^{\prime}} \varphi_{2}\right)\right]^{1 / 2}=0 \\
\left.\varphi_{1}\right|_{x_{1}^{+}=0}=\left.\varphi_{2}\right|_{x_{1}^{+}=0}=\left.\varphi_{3}\right|_{x_{1}^{-}=0}=0
\end{array}\right.
$$

Let the amplitudes $b_{j} \sim \sum_{n=0}^{\infty} b_{j}^{n}$ where $b_{j}^{n}$ is homogeneous in $\theta$ of degree $-n$. The functions $b_{j}^{n}$ can be found using the transport equations

$$
\begin{equation*}
2 i\left(\partial_{t} b_{k}^{j} \partial_{t} \varphi_{k}-\nabla b_{k}^{j} \cdot \nabla \varphi_{k}\right)+i\left(\partial_{t}^{2} \varphi_{k}-\Delta \varphi_{k}\right) b_{k}^{j}=\left(\partial_{t}^{2}-\Delta\right) b_{k}^{j-1} \tag{9.1.4}
\end{equation*}
$$

once boundary conditions are imposed. These boundary conditions follow from (9.1.1) and are given by the equations

$$
\begin{equation*}
i b_{1}^{j} \partial_{x_{1}} \varphi_{1}+i b_{2}^{j} \partial_{x_{1}} \varphi_{2}-i b_{3}^{j} \partial_{x_{1}} \varphi_{3}+\partial_{x_{1}}\left(b_{1}^{j-1}+b_{2}^{j-1}-b_{3}^{j-1}\right)+V\left(x^{\prime}\right) b_{3}^{j-1}=0 \tag{9.1.5}
\end{equation*}
$$

Remark: Equation (9.1.5) is the only place where we require that $V \in C^{\infty}$ rather than $V \in \Psi_{\text {hom }}(\partial \Omega)$. This is due to the fact that the zero frequency would otherwise appear in the symbol of $V$.

$$
\begin{equation*}
b_{1}^{j}+b_{2}^{j}=b_{3}^{j} \tag{9.1.6}
\end{equation*}
$$

at $x_{1}=0$. We use the convention that $b_{k}^{-1} \equiv 0$.
Remark: Notice that $\varphi_{1}$ and $\varphi_{3}$ solve the same eikonal equation and hence there is a $C^{\infty}$ function $\varphi^{\prime}$ that has $\left.\varphi^{\prime}\right|_{x_{1} \leq 0}=\varphi_{3}$ and $\left.\varphi^{\prime}\right|_{x_{1}>0}=\varphi_{1}$.

Now, combining [62, Lemma 1.3.17] with [63, Propositions 3.2 and 3.3] gives that in the case that $\Omega$ is strictly convex

Lemma 9.1.2. For $t_{0}<T, \nu \in S^{*} \mathbb{R}^{d} \backslash\left(\left.S^{*} \mathbb{R}^{d}\right|_{\partial \Omega} \cup \mathcal{O}_{T}\right)$ there is a conical neighborhood, $W$ of $\nu$ such that for every $\alpha$ with $\mathrm{WF}(\alpha) \subset W$, and $\left.G_{k}^{t_{0}} \nu \notin S^{*} \mathbb{R}^{d}\right|_{\partial \Omega}$ for all $k \in K$

$$
U_{\alpha}\left(t_{0}\right)=\sum_{k} U_{\alpha, k}\left(t_{0}\right)
$$

where $U_{\alpha, k}\left(t_{0}\right)$ is an FIO associated to the canonical relation $G_{k}^{t_{0}}$.
In order to describe the singularities of the wave trace, we need to determine the symbols of the $U_{\alpha, k}$. We again follow [63] to do this in our special case.

Remark: We note that although our case falls into the framework of [63], unlike in the cases explicitly considered there, the FIO associated to a reflected geodesic will decrease in order and hence become smoother with increasing numbers of reflections.

By conjugating by $\left(\operatorname{det} g^{i j}\right)^{1 / 2}$, we can associate operators $\alpha^{\prime}=: \gamma, U_{\alpha}^{\prime}=V_{\gamma}$, and $U_{\alpha, k}=$ : $V_{\gamma, k}$, to $\alpha, U_{\alpha}$, and $U_{\alpha, k}$ that act on half densities instead of functions. Let $C_{k} \subset \mathbb{R}_{+} \times S^{*} \mathbb{R}^{d} \times$ $S^{*} \mathbb{R}^{d}$ be the graph of $G_{k}^{t}$ with the points $\left(t+0, G_{k}^{t+0} \nu, \nu\right)$ and $\left(t-0, G_{k}^{t-0} \nu, \nu\right)$ sewn together when $\left.G_{k}^{t} \nu \in S^{*} \mathbb{R}^{d}\right|_{\partial \Omega}$. Then, one can use $(t, y, \eta)$ as coordinates on $C_{k}$.

Now, we compute the half density component of the symbol of $V_{\gamma, k}(t)$, as in 63]. Define the section $E_{k}$, by

- $E_{k}\left(t, y_{0}, \eta_{0}\right)$ is right continuous for fixed $\left(y_{0}, \eta_{0}\right)$
- $\left.E_{k}\right|_{t=0}=\exp (i \pi d / 4)$
- $E_{k}\left(t, y_{0}, \eta_{0}\right)$ is locally constant for $\left(y_{0}, \eta_{0}\right)$ fixed and has discontinuities at the points $t_{n}$ where the geodesic of type $k$ starting at $\left(y_{0}, \eta_{0}\right)$ hits the boundary.
- At discontinuity points, $E_{k}\left(t_{n}+0, y_{0}, \eta_{0}\right)=F_{n} E_{k}\left(t_{n}-0, y_{0}, \eta_{0}\right)$.

Lemma 9.1.3. Let $\gamma, \gamma_{0} \in \Psi_{\text {hom }}^{0}\left(\partial \Omega ; \Omega^{1 / 2}\right)$ be pseudodifferential operators with

$$
\left(\mathrm{WF}(\gamma) \cup \mathrm{WF}\left(\gamma_{0}\right)\right) \cap T^{*} \partial \Omega=\emptyset, \quad \text { and } \quad \mathrm{WF}(\gamma) \cap \mathcal{O}_{T}=\emptyset
$$

acting on half densities. Then, for any $t_{0}<T, \gamma_{0}^{0} E_{k}\left(t_{0}\right) \gamma^{0}$ is the symbol of $\gamma_{0}^{0} V_{\gamma, k}$. Furthermore, the order of the FIO $V_{\gamma, k}$ decreases by 1 after each reflection.

This follows from [63, Proposition 3.3] and [43, Appendix B]. We compute the $F_{n}$ for use below. In particular, assume that $b^{j}=0$ for $j<N$. Then we have $F_{N}=1$ if the geodesic is transmitted through $\partial \Omega$ and

$$
F_{N}=0, \quad F_{N+1}=-\frac{V+\partial_{x_{1}}\left(b_{1}^{N}+b_{2}^{N}-b_{3}^{N}\right)}{2 i \xi_{1}}=-\frac{V}{2 i \xi_{1}}
$$

if it is reflected at $\partial \Omega$. For the second equality, we use the transport equations at $x_{1}=0$. To determine $F_{n}$ when the geodesic is reflected, we notice that the principal term is 0 so the computation above follows from (9.1.5) and 9.1.6) with $j=N+1$.

## Analysis for Non-Glancing Trajectories Away from the Boundary

Let $\partial \Omega \Subset U$ and $\chi \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ have $\chi \equiv 1$ on $U$. Let $\chi_{1}=1-\chi$. We now consider $\sigma_{1, \chi_{1}}(t)$, that is, the contributions from points away from the boundary. Fix $T>0$ and let $\alpha$ be a pseudodifferential operator with $\mathrm{WF}(\alpha) \cap \mathcal{O}_{T}=\emptyset$.

Using Lemma 9.1.2 and an analysis similar to that in [16] and [19], singularities of $\sigma_{1, \chi_{1} \alpha}$ can only occur at $T_{j}$ where $T_{j}$ is the period of some billiard flow. That is, $G_{k}^{T_{j}}(x, \xi)=(x, \xi)$. We assume that $\Omega$ is strictly convex. Thus, the only periodic trajectories are of type $k=0$ and are trapped inside $S^{*} \Omega$.

Since we have assumed that the length spectrum of $\Omega$ is simple and discrete, the fixed point set for the billiards trajectory at a time $T$ is always a submanifold of $S^{*} M$ of dimension 1 with boundary. Moreover, we have assumed that it is a clean submanifold in the sense of [19] away from the boundary of $\Omega$.

We now follow [19] to compute the symbol $\sigma_{1, \chi_{1} \alpha}$ as a Lagrangian distribution. Let $\rho: \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R} \times \mathbb{R}^{d} \times \mathbb{R}^{d}$ be the diagonal map. Then, given a half density, $u$ on $\mathbb{R} \times \mathbb{R}^{d} \times \mathbb{R}^{d}$, we can pull it back to the diagonal and multiply the two half density factors in $\mathbb{R}^{d}$ to get, $\rho^{*} u$, a density in $\mathbb{R}^{d}$ times a half density in $\mathbb{R}$. Then, we integrate over $\mathbb{R}^{d}$ to get a half density on $\mathbb{R}$. We denote this by $\pi_{*} \rho^{*} u$ where $\pi: \mathbb{R} \times X \rightarrow \mathbb{R}$ is the projection map.

Then, letting $\alpha_{i}$ be a partition of unity for $S^{*} B(0, M)$, up to smooth terms, $\pi_{*} \rho^{*} \chi U_{\chi_{1} \alpha}=$ $\sum_{\alpha_{i}} \chi \sigma_{1, \chi_{1} \alpha \alpha_{i}}$. Fix $T$, the length of a periodic billiard trajectory. For $\alpha_{i}$ supported away from the periodic trajectory, Lemma 9.1.3 together with analysis similar to that in [16] shows that $\sigma_{1, \chi_{1} \alpha \alpha_{i}} \in C^{\infty}$. Therefore, we assume without loss that $\alpha_{i}$ is supported near a periodic trajectory with $N$ reflections and period $T$.

Then,

$$
\sigma_{1, \chi_{1} \alpha \alpha_{i}} \in I^{\frac{1}{2}-\frac{1}{4}-N}\left(\Lambda_{T}\right)
$$

with $\Lambda_{T}:=\{(T, \tau): \tau \in \mathbb{R} \backslash\{0\}\}$. Thus, $\sigma_{1, \chi_{1} \alpha \alpha_{i}}(t)$ is its symbol times

$$
\frac{1}{2 \pi} \int s^{-N} e^{-i s(t-T)} d s
$$

plus lower order terms. The symbol of $\sigma_{1, \chi_{1} \alpha \alpha_{i}}(t)$ is given, modulo Maslov factors, by

$$
\sqrt{d s} \int_{Z} \chi_{1} \alpha E_{0}(T, \cdot, \cdot) \alpha_{i} d \mu_{Z}
$$

where $Z$ is the fixed point set of the billiard trajectories of length $T$.
Now, $Z$ consists of a single billiard trajectory and hence $E_{0}$ is constant on the fixed point set. Thus the symbol is nonzero as long as $V \neq 0$ on $\left.\pi\left(Z \cap S^{*} B(0, M)\right)\right|_{\partial \Omega}$. Hence, we have that, summing over $\alpha_{i}$ supported near the periodic trajectory,

$$
\begin{equation*}
\left|\hat{\sigma}_{1, \chi_{1} \alpha}(\tau)\right|=\left|\sum_{i} \hat{\sigma}_{1, \chi_{1} \alpha \alpha_{i}}(\tau)\right| \geq c \tau^{-N} \tag{9.1.7}
\end{equation*}
$$

## Analysis for Non-Glancing Trajectories Near the Boundary

We now analyze $\sigma_{1, \chi \alpha}$. That is, we analyze the wave propagator near boundaries. To do this, we assume without loss of generality that the geodesic starting at $\nu \in S^{*} \mathbb{R}^{d}$ intersects $\partial \Omega$ for the first time at $0 \leq t_{1}$ and that the geodesic is traveling in the $-x_{1}$ direction. Let $\alpha$ have support in a small conic neighborhood of $\nu$. By Lemma 9.1.3, and formula (9.1.2), we see that for $t$ sufficiently close to $t_{1}$,

$$
\begin{equation*}
U_{\alpha}(t)=A_{\alpha}+H\left(x_{1}\right) R_{\alpha}+H\left(-x_{1}\right) T_{\alpha} \tag{9.1.8}
\end{equation*}
$$

where $H$ is the Heaviside function, $A_{\alpha}$, and $T_{\alpha}$ are classical FIOs associated to $G_{1 / 3}^{t}$ and $R$ is an FIO associated with $\tilde{G}_{0}^{t}$ where

$$
\tilde{G}_{0}^{t}=\exp _{t-t_{1}} \circ G_{0}^{t_{1}}
$$

That is, $A$ and $T$ are associated to trajectories that are transmitted through the boundary while $R$ is associated to a reflected trajectory. Also, $R$ and $T$ are one order lower than $A$.

We now check that the multiplication of $R_{\alpha}$ and $T_{\alpha}$ by the Heaviside function is well defined as a distribution and compute its wavefront set.

We have that

$$
\begin{aligned}
& \begin{aligned}
& \mathrm{WF}\left(H\left(x_{1}\right)\right)=\mathrm{WF}\left(H\left(-x_{1}\right)\right) \\
&=\left\{\left(t,\left(0, x^{\prime}\right), y, 0,\left(\xi_{1}, 0\right), 0\right), \xi_{1} \neq 0\right\}=N^{*}\left(\left\{x_{1}=0\right\}\right) \\
& \mathrm{WF}(A)=\mathrm{WF}(T) \\
& \quad=\left\{(t, x, y, \tau,-\xi, \eta): \tau^{2}=|\eta|^{2}, G_{1 / 3}^{t}((y, \eta))=(x, \xi),(\tau,-\xi, \eta) \neq 0\right\} . \\
& \mathrm{WF}(R)=\left\{(t, x, y, \tau,-\xi, \eta): \tau^{2}=|\eta|^{2}, \tilde{G}_{0}^{t}((y, \eta))=(x, \xi),(\tau, \xi, \eta) \neq 0\right\} .
\end{aligned}
\end{aligned}
$$

Therefore, it is easy to check that

$$
\mathrm{WF}(R) \cap-\mathrm{WF}(H)=\mathrm{WF}(T) \cap-\mathrm{WF}(H)=\emptyset
$$

where

$$
-B=\{(t, x, y,-\tau,-\xi,-\eta):(t, x, y, \tau, \xi, \eta) \in B)\}
$$

Thus, the multiplication is a well defined distribution. Moreover, $\mathrm{WF}(H T) \subset \mathrm{WF}(H) \cup$ $\mathrm{WF}(T) \cup A$ where

$$
A:=\left\{\begin{array}{c}
\left(t,\left(0, x^{\prime}\right), y, \tau,\left(\xi_{1}, \xi^{\prime}\right), \eta\right): \\
\exists \xi_{\nu},\left(t,\left(0, x^{\prime}\right), y, \tau,\left(\xi_{\nu}+\xi_{1}, \xi^{\prime}\right), \eta\right) \in \mathrm{WF}(T)
\end{array}\right\}
$$

The computation for $\mathrm{WF}(H R)$ follows similarly.
Putting this together with Lemma 9.1.2, we have the following description of $U_{\alpha}$ away from glancing

Lemma 9.1.4. For $t_{0}<T, \nu \in S^{*} \mathbb{R}^{d} \backslash \mathcal{O}_{T}$ there is a conical neighborhood, $U$, of $\nu$ such that for every $\alpha$ with support in $U$,

$$
U_{\alpha}\left(t_{0}\right)=\sum_{k} U_{\alpha, k}\left(t_{0}\right)
$$

where, if $\left.G_{k}^{t_{0}} \nu \notin S^{*} \mathbb{R}^{d}\right|_{\partial \Omega}, U_{\alpha, k}\left(t_{0}\right)$ is an FIO associated to the canonical relation $G_{k}^{t_{0}}$ and if $\left.G_{k}^{t_{0}} \nu \in S^{*} \mathbb{R}^{d}\right|_{\partial \Omega}, U_{\alpha, k}\left(t_{0}\right)$ is of the form (9.1.8).

At this point, we can prove a result analogous to that of Chazarain [16].
Lemma 9.1.5. Fix $T<\infty$ and $\nu \in S^{*} \mathbb{R}^{d} \backslash \mathcal{O}_{T}$. Then there exists $W$, a neighborhood of $\nu$, such that for $\alpha$ with $\mathrm{WF}(\alpha) \subset W$

$$
\begin{aligned}
& \mathrm{WF}\left(\sigma_{\alpha}\right) \subset\left\{(t, \tau): \exists \nu \in S^{*} \mathbb{R}^{d} \cap \operatorname{supp} \alpha, k,\right. \text { such that } \\
& \left.\qquad G_{k}^{t} \nu=\nu \text { or }\left.(x, \xi) \in S^{*} \mathbb{R}^{d}\right|_{\partial \Omega} \text { and } G_{k}^{t}(x, \xi)=\left(x,\left(-\xi_{1}, \xi^{\prime}\right)\right)\right\}
\end{aligned}
$$

Proof. First, we compute the wave front set of $\pi_{*} \rho^{*} H T$. The computation for $H R$ will follow similarly.

We need to check that the pull back $\rho^{*}(H T)$ is well defined, that is, that $\mathrm{WF}(H T) \cap$ $N^{*}(\{x=y\})=\emptyset$. Here

$$
N^{*}(\{x=y\})=\{(t, x, x, 0, \xi,-\xi): \xi \neq 0\}
$$

Thus, $\operatorname{WF}(H) \cap N^{*}(\{x=y\})=\emptyset$. To see that $\operatorname{WF}(T) \cap N^{*}(\{x=y\})$ is empty, observe that if the intersection were nonempty, then there would be a point with $\tau=0$ in $\mathrm{WF}(T)$. But this implies that $\xi=\eta=0$ and thus $(\tau, \xi, \eta)=0$. Last, we need to check that the third
piece of $\mathrm{WF}(H T)$ does not intersect $N^{*}(\{x=y\})$. This follows from the same arguments as those for the intersection with $\mathrm{WF}(T)$.

Finally, we compute

$$
\mathrm{WF}\left(\pi_{*} \rho^{*}(H T)\right) \subset\{(t, \tau): \exists x, \eta \text { such that }(t, x, x, \tau,-\eta, \eta) \in \mathrm{WF}(H T)\}
$$

Thus, since $\xi_{1} \neq 0$ and $\eta_{1}=0$ in $\mathrm{WF}(H)$, the contribution from $\mathrm{WF}(H)$ is empty. As usual, the contribution from $\mathrm{WF}(T)$ is

$$
\left\{(t, \tau): \exists(x, \eta), G_{1 / 3}^{t}((x, \eta))=(x, \eta)\right\}
$$

Finally, the contribution from

$$
\left\{\left(t,\left(0, x^{\prime}\right), y, \tau,\left(\xi_{1}, \xi^{\prime}\right), \eta\right): \exists \xi_{\nu},\left(t,\left(0, x^{\prime}\right), y, \tau,\left(\xi_{\nu}, \xi^{\prime}\right), \eta\right) \in \mathrm{WF}(T)\right\}
$$

is given by

$$
\left\{(t, \tau): G_{1 / 3}^{t}\left(\left(0, x^{\prime}\right),\left(\xi_{1}, \xi^{\prime}\right)\right) \in\left\{\left(\left(0, x^{\prime}\right),\left(\xi_{1}, \xi^{\prime}\right)\right),\left(\left(0, x^{\prime}\right),\left(-\xi_{1}, \xi^{\prime}\right)\right)\right\}\right\}
$$

Putting this together with Lemma 9.1.4, we obtain the result.
Now, we need to estimate the size of the singularities of $\operatorname{tr}(A), \operatorname{tr}(H R)$ and $\operatorname{tr}(H T)$ after a given number of reflections. Suppose that there is a periodic trajectory of length $T$ containing $N$ reflections starting at $\left.\nu \in S^{*} \mathbb{R}^{d}\right|_{\partial \Omega} \backslash \mathcal{O}_{2 T}$.

The terms in 9.1.8) of the form $A$ are classical FIO's and can be analyzed using the methods from the previous section. However, we must determine the size of the singularities. We have that $G_{0}^{T} \nu=\nu$. There are two cases. First, suppose that $\nu$ is inward pointing. Then the relevant term of the form $A$ has no singularities in its trace since it is associated to $G_{0 \ldots 01}^{T} \nu \neq \nu$ where there are $N-10$ 's. Now, if $\nu$ is outward pointing, then at $t=0, U_{\alpha}(t)$ is of the form (9.1.8) and the term associated to $G_{0}^{t}$ is of the form $H R$ and hence has order -1. Therefore, since at time $T$ the term of the form $A$ has undergone $N-1$ additional reflections, it can be treated as in section 9.1 and cut off in an arbitrarily small neighborhood of the boundary to obtain

$$
\begin{equation*}
\hat{\sigma}_{A}(\tau)=o\left(\tau^{-N}\right) \tag{9.1.9}
\end{equation*}
$$

To handle $\operatorname{tr}(H R)$ and $\operatorname{tr}(H T)$, observe that, for any $\chi \in C_{c}^{\infty}(\mathbb{R})$ with $\chi(0)=1$,

$$
\chi\left(x_{1}\right) H\left(x_{1}\right)=(2 \pi)^{-1} \int\left(1-\chi_{1}\left(\xi_{\nu}\right)\right)\left(\xi_{\nu}^{-1}+\mathcal{O}\left(\xi_{\nu}^{-\infty}\right)\right) e^{i x_{1} \xi_{\nu}} d \xi_{\nu}
$$

for some $\chi_{1} \in C_{c}^{\infty}$ with $\chi_{1}(0)=1$. By Lemma 9.1 .5 (and the fact that $\Omega$ is convex) we only need to consider times $t$ for which there are periodic billiards trajectories. Suppose that there is a billiard trajectory with period $T$ undergoing $N$ reflections. Then, for $t$ near $T$,

$$
\chi H\left(x_{1}\right) R=\sum_{j} \int e^{i x_{1} \xi_{\nu}} e^{i \varphi(t, x, y, \theta)} a_{j}\left(t, x, y, \theta, \xi_{\nu}\right) d \xi_{\nu} d \theta
$$

where $a_{j}$ is homogeneous of degree $-j$ in $\theta$ and -1 in $\xi_{\nu}$. Also, since there are $N$ reflections, $a_{j} \equiv 0$ for $j<N$.

Now, let $\psi \in C_{c}^{\infty}, \psi(0)=1$, and $\psi_{\epsilon, T}(t)=\psi\left(\epsilon^{-1}(t-T)\right)$. Let $\chi \in C_{c}^{\infty}$ with $\chi \equiv 1$ in a neighborhood of $x_{0} \in \partial \Omega$ and be supported in a neighborhood of the boundary with volume $\delta^{d}$, we are interested in the decay rate of

$$
\begin{aligned}
\hat{\sigma}_{H R}(\tau):= & \mathcal{F}_{t \rightarrow \tau}\left(\operatorname{tr}\left(\psi_{\epsilon, T}(t) \chi(x) H R\right)\right)(\tau)= \\
& \sum_{j} \int \psi_{\epsilon, T}(t) e^{i x_{1} \xi_{\nu}} e^{i \varphi(t, x, x, \theta)} e^{-i t \tau}\left(\chi(x) a_{j}\left(t, x, x, \theta, \xi_{\nu}\right)\right) d \xi_{\nu} d x d \theta d t+\mathcal{O}\left(\tau^{-\infty}\right)
\end{aligned}
$$

for sufficiently small $\epsilon$. We rescale $\xi_{\nu}$ and $\theta$ to $\xi_{\nu} / \tau$ and $\theta / \tau$ to obtain

$$
\begin{aligned}
\hat{\sigma}_{H R}(\tau)= & \\
& \sum_{j} \tau^{d-j} \int \psi_{\epsilon, T}(t) \chi(x) a_{j}\left(t, x, x, \theta, \xi_{\nu}\right) e^{i \tau\left(x_{1} \xi_{\nu}+\varphi(t, x, x, \theta)-t\right)} d \xi_{\nu} d x d \theta d t+\mathcal{O}\left(\tau^{-\infty}\right) .
\end{aligned}
$$

Next, observe that the phase is stationary in $x, t$, and $\theta$ on

$$
d_{\theta} \varphi=d_{t} \varphi-1=d_{x^{\prime}} \varphi+d_{y^{\prime}} \varphi=\xi_{\nu}+d_{x_{1}} \varphi+d_{y_{1}} \varphi=0 .
$$

Now, since $R$ is associated to $\tilde{G}_{0}^{t}$, these equations are equivalent to

$$
\left|\left(\xi_{1}, \xi^{\prime}\right)\right|=1, \quad x=\pi \tilde{G}_{0}^{t}(x, \xi), \quad \xi^{\prime}=\pi_{\xi^{\prime}} \tilde{G}_{0}^{t}(x, \xi), \quad \xi_{1}=\pi_{\xi_{1}} \tilde{G}_{0}^{t}(x, \xi)+\xi_{\nu}
$$

Hence, since the length spectrum is simple and discrete, the critical points form a submanifold of dimension 1 with volume less than $c \delta$. Now, by assumption the fixed point set of the billiards trajectory is clean and hence $\varphi$ is clean with excess 1 . Let $\psi=x_{1} \xi_{\nu}+\varphi-t$ then since as functions of $(x, \theta, t), \partial^{2} \psi=\partial^{2} \varphi$ these stationary points are clean with excess 1 .

Thus, after cutting off to a compact set in $\xi_{\nu}$, we may apply the clean intersection theory of Duistermaat and Guillemin [19]. To handle $\left|\xi_{\nu}\right| \geq C$, observe that there are no critical points for $\left|\xi_{\nu}\right| \geq 3$ and hence that this piece of the integral can be handled using the principle of nonstationary phase.

Now, letting $\delta \rightarrow 0$, and observing that we are integrating in $2 d+1$ variables and have a 1 dimensional submanifold of critical points, we obtain

$$
\begin{equation*}
\hat{\sigma}_{H R}(\tau)=o\left(\tau^{-N+d-\frac{2 d+1}{2}+\frac{1}{2}}\right)=o\left(\tau^{-N}\right) \tag{9.1.10}
\end{equation*}
$$

Remark: Observe that this computation relies on the fact that, after reflection, the order of the FIO decreases by 1 .

Now, putting (9.1.10) together with 9.1.7), and 9.1.9),

$$
\begin{equation*}
\left|\hat{\sigma}_{1, \alpha}(\tau)\right| \geq c \tau^{-N} \tag{9.1.11}
\end{equation*}
$$

## Glancing Trajectories

We are interested in $\operatorname{tr}\left(U(t)-U_{0}(t)\right)$ as a distribution. Let

$$
\left(u_{1}, u_{2}\right)(t)=\left(U(t)-U_{0}(t)\right) u_{0}
$$

Then

$$
\begin{cases}\left(\partial_{t}^{2}-\Delta\right) u_{i}=0 & \text { in } \Omega_{i}  \tag{9.1.12}\\ u_{1}-u_{2}=0, \text { and } \partial_{\nu} u_{1}+\partial_{\nu^{\prime}} u_{2}+V u_{1}=-V U_{0}(t) u_{0} & \text { on } \partial \Omega\end{cases}
$$

By the arguments in Sections 9.1 and 9.1, we can assume that $u_{0}$ has wave front set in a neighborhood of $\nu \in \mathcal{O}_{T}$. Then, by arguments identical to those in Section 8.3, we can use the operators constructed in Appendix 5.6 (see also [47, Section 11.3]) together with the ideas in Section 6.7 to find a microlocal description of the solution to this problem restricted to the boundary of the form

$$
\left.\left(I+J \beta^{-1} \mathcal{A} i \mathcal{A}_{-} J^{-1} B\right) u\right|_{\partial \Omega}=-\left.V U_{0}(t) u_{0}\right|_{\partial \Omega}
$$

where $B \in \Psi_{\text {hom }}^{-2 / 3}$. Because $B \in \Psi_{\text {hom }}^{-2 / 3}$, we can invert the operator on the left hand side by Neumann series

$$
\begin{equation*}
\left(I+\beta^{-1} J \mathcal{A} i \mathcal{A}_{-} J^{-1} B\right)^{-1}=\sum_{k=0}^{\infty}\left(-\beta^{-1} J \mathcal{A}_{-} \mathcal{A} i J^{-1} B\right)^{k} \tag{9.1.13}
\end{equation*}
$$

Hence, truncating the sum (9.1.13) at a sufficiently high $K>0$ gives a $C^{M}$ parametrix and the remainder contributes a term of size $O\left(\tau^{-M}\right)$ to the trace.

To analyze the singularities near glancing, we need the following lemma (see 47, Sections 5.4] and Lemma 5.3.1).

Lemma 9.1.6. $\mathrm{WF}^{\prime}(\mathcal{A}-\mathcal{A} i) \subset \operatorname{graph}(I d) \cup \operatorname{graph}(\beta)=: C_{b}$
Thus, up to glancing, the wavefront set of the solution $u$ is contained in the billiard trajectories through $\nu$.

Proof. The proof follows that of Lemma 5.3 .1 by letting $h=\tau^{-1}, \epsilon(h)=h$, and rescaling $\xi \rightarrow \tau \xi$. Using this transformation and the fact that in the construction of the Melrose Taylor parametrix, we have that $\alpha=\alpha_{0}+i$, we replace $\alpha_{h}$ with $\zeta=\tau^{-1 / 3}\left(\xi_{1}+i\right)$. Then in $\xi_{1}<0$, up to a lower order term $O\left(\zeta^{\prime}\right)$, we obtain the phase function

$$
\tau=\langle t-s, \tau\rangle+\langle x-y, \xi\rangle+\frac{4}{3}\left(-\xi_{1}\right)^{3 / 2} \tau^{-1 / 2}
$$

which parametrizes $\beta$. The $i$ term is a symbolic perturbation hence the wavefront set is given by the billiard relation.

The only thing that remains is to show that

$$
\mathrm{WF}^{\prime}\left(\mathcal{A}_{-} \mathcal{A} i\right) \cap\left\{\xi_{1}=0\right\} \subset\{t=s\} .
$$

To do this, let

$$
V_{1}=\partial_{\tau}+\frac{1}{3} \tau^{-1}\left(\xi_{1}+i\right) \partial_{\xi_{1}}
$$

Then, $V_{1} \zeta=0$ and hence

$$
V_{1}\left(A_{-} A i(\zeta)\right)=0
$$

The symbol of $V_{1}$ is given by $i t+\frac{1}{3} \tau^{-1}\left(\xi_{1}+i\right) x_{1}$. Thus, it is elliptic on $\xi_{1}=0$ away from $t=0$, and we have

$$
\mathrm{WF}^{\prime}\left(\mathcal{A}_{-} \mathcal{A} i\right) \cap\left\{\xi_{1}=0\right\} \subset\{t=0\}
$$

Combining this with the results of Lemma 5.3.1 completes the proof
By Lemma 9.1.6 together with the arguments used to prove Lemma 9.1.5, we have that the singularities of $\operatorname{tr} \sigma_{\alpha}(t)$ for $\alpha$ supported near a glancing trajectory are contributed by closed billiards trajectories. Observe that since $\Omega$ is strictly convex, as a trajectory approaches glancing, the number of reflections in a closed billiard trajectory increases without bound. Hence we see that in a small enough neighborhood of glancing the first $K$ terms in 9.1.13) contribute no singularities to the trace. Thus for all $M>0$, by choosing a small enough neighborhood of glancing, we have

$$
\begin{equation*}
\left|\hat{\sigma}_{1,1-\alpha}(\tau)\right|=\mathcal{O}\left(\tau^{-M}\right) \tag{9.1.14}
\end{equation*}
$$

One does not need the precise estimate (9.1.14) to prove Theorem 9.1. One only needs that the singularities up to glancing are contained in the length of periodic billiards trajectories. That is, Lemma 9.1 .6 or another propagation of singularities result is enough. However, the precise estimate 9.1.14 proves Proposition 9.0.1.

## Completion of the proof of Lemma 9.1.1

Let $T$ be the primitive length of the periodic billiard trajectory with $M$ reflection points contained in $\sigma(V) \neq 0$. Then, let $\varphi \in C_{c}^{\infty}, \varphi(0) \equiv 1$ and $\operatorname{supp} \varphi \subset(-1,1)$. Define $\varphi_{T, \epsilon}(t)=$ $\varphi\left(\epsilon^{-1}(t-T)\right)$. Then, Lemma 9.1.6, the fact that $T$ is isolated in $L_{\Omega}$, and 9.1.11) show that for $\epsilon>0$ small enough and $\tau$ large enough,

$$
\left|\widehat{\varphi_{T, \epsilon} \sigma_{1}}(\tau)\right| \geq c \tau^{-M} .
$$

Now, using [31, Lemma 7.1] we see that

$$
\mu_{j}\left(\chi R_{V}(1+i M)\right) \leq C j^{-1 / d}
$$

and hence by 66], 81, [82], and 83],

$$
\#\left\{z_{j}:\left|z_{j}\right| \leq 1, \operatorname{Im} z_{j} \geq-\gamma / h\right\}=\mathcal{O}\left(h^{-d}\right)
$$

Thus, by 67. Theorem 1] for all $\delta>0$ and $\rho>\frac{d+M}{T}$, there exists $c>0$ such that

$$
\begin{equation*}
\#\left\{z_{j}:\left|z_{j}\right| \leq 1, \operatorname{Im} z_{j} \geq-\rho h \log \left(\operatorname{Re} z_{j} h^{-1}\right)\right\} \geq c h^{-1+\delta} \tag{9.1.15}
\end{equation*}
$$

for $h$ small enough. But, if there is a periodic geodesic with $M$ reflections of length $T$, then there is also one with $n M$ reflections of length $n T$. Therefore, taking $n$ large enough, we have 9.1.15) for all $\rho>\frac{M}{T}$. This completes the proof of Lemma 9.1.1.

Theorem 9.1 follows from Lemma 9.1.1, together with the fact that strictly convex domains in $\mathbb{R}^{d}$ generically have simple length spectrum such that the only accumulation points are the lengths of closed geodesics in $\partial \Omega$ and have periodic billiards trajectory which are clean submanifolds ([57, Chapter 3, Section 7.4]).

## Chapter 10

## Analysis of $-\Delta_{\partial \Omega, \delta^{\prime}}$

Here we consider resonances for the operator $-\Delta_{\partial \Omega, \delta^{\prime}}$. Recall that resonances are defined as poles of the meromorphic continuation of the resolvent

$$
R_{V}(\lambda)=\left(-\Delta_{\partial \Omega, \delta^{\prime}}-\lambda^{2}\right)^{-1}, \quad \operatorname{Im} \lambda \gg 1
$$

and $-\Delta_{\partial \Omega, \delta^{\prime}}$ is the unbounded operator

$$
-\Delta_{\partial \Omega, \delta^{\prime}}:=-\Delta+\delta_{\partial \Omega}^{\prime}\left(\left.V(\lambda) \partial_{\nu}\right|_{\partial \Omega}\right)
$$

Remark: Note the sign change in the definition of $-\Delta_{\partial \Omega, \delta^{\prime}}$ comes from the fact that $\delta_{\partial \Omega}^{\prime}(u)=$ $-\int_{\partial \Omega} \partial_{\nu} u$.
(See Section 10.1 for the formal definition of $-\Delta_{\partial \Omega, \delta^{\prime} .}$.) We first show
Theorem 10.1. Let $\Omega \Subset \mathbb{R}^{d}$ have smooth boundary. Suppose that $V: L^{2}(\partial \Omega) \rightarrow L^{2}(\partial \Omega)$ is self adjoint and satisfies

$$
\|u\|_{L^{2}(\partial \Omega)} \leq C_{N}\left(\|V u\|_{L^{2}(\partial \Omega)}+\|u\|_{H^{-N}(\partial \Omega)}\right)
$$

Then

$$
R_{V}(\lambda):=\left(-\Delta_{\partial \Omega, \delta^{\prime}}-\lambda^{2}\right)^{-1}
$$

has a meromorphic continuation from $\operatorname{Im} \lambda \gg 1$ to $\mathbb{C}$ if $d$ is odd and to the logarithmic cover of $\mathbb{C} \backslash\{0\}$ if $d$ is even.

Moreover, the poles of $R_{V}(\lambda)$ are in 1-1 correspondence with solutions $u \in H_{\Delta, \text { loc }}^{3 / 2}\left(\mathbb{R}^{d} \backslash \partial \Omega\right)$ to

$$
\left\{\begin{array}{l}
\left(-\Delta_{\partial \Omega, \delta^{\prime}}-\lambda^{2}\right) u=0  \tag{10.0.1}\\
u \text { is } \lambda \text {-outgoing. }
\end{array}\right.
$$

Here,

$$
H_{\Delta}^{3 / 2}(U):=\left\{u \in H^{3 / 2}(U): \Delta u \in L^{2}(U)\right\}
$$

We also show that $u$ solving (10.0.1) is equivalent to $u=u_{1} \oplus u_{2} \in H_{\Delta}^{3 / 2}(\Omega) \oplus H_{\Delta, \text { loc }}^{3 / 2}\left(\mathbb{R}^{d} \backslash \bar{\Omega}\right)$ solving

$$
\begin{cases}\left(-\Delta-\lambda^{2}\right) u_{1}=0 & \left(-\Delta-\lambda^{2}\right) u_{2}=0  \tag{10.0.2}\\ \left.\partial_{\nu} u_{1}\right|_{\partial \Omega}=\left.\partial_{\nu} u_{2}\right|_{\partial \Omega} & u_{1}-u_{2}+V \partial_{\nu} u_{1}=0 \\ u_{2} \text { is } \lambda \text {-outgoing. } & \end{cases}
$$

In order to say something about the location of the resonances near the real axis, we assume that $\Omega$ is strictly convex and $\partial \Omega \subset \mathbb{R}^{d}$ is a smooth hypersurface and take $V=$ $V(h) \in h^{\alpha} \Psi(\partial \Omega)$ an elliptic (semiclassical) pseudodifferential operator with semiclassical parameter $h=\operatorname{Re} \lambda^{-1}$ and $\alpha>5 / 6$.

Denote the set of rescaled resonances and the set of rescaled resonances that are $h \log h^{-1}$ close to the real axis by

$$
\begin{equation*}
\Lambda(h):=\left\{z \in \mathbb{C}: z / h \text { is a resonance of }-\Delta_{\partial \Omega, \delta^{\prime}}\right\} \tag{10.0.3}
\end{equation*}
$$

and

$$
\Lambda_{\log }(h):=\left\{z \in \Lambda(h): z \in[1-C h, 1+C h]+i\left[-M h \log h^{-1}, 0\right]\right\}
$$

respectively.
Remark: When $V \in h^{\alpha} \Psi(\partial \Omega)$, all of our proofs go through when

$$
z \in E+[-C h, C h]+i\left[-M h \log h^{-1}, C h^{1-\gamma}\right]
$$

and $\gamma<\min (\alpha-1 / 2,1 / 2)$, but for simplicity we use $\Lambda_{\mathrm{log}}$.
Then the following theorem is a consequence of the much finer Theorem 10.3
Theorem 10.2. Let $\Omega \subset \mathbb{R}^{d}$ be a strictly convex domain with $C^{\infty}$ boundary, $V \in C^{\infty}(\partial \Omega)$ with $V>c>0$, and $\alpha>5 / 6$. Then there exists a constant $C_{V, \Omega, \alpha}$ such that for every $\epsilon>0$ there exists $h_{0}>0$ such that for $0<h<h_{0}$ and $z \in \Lambda_{\log }(h)$

$$
-\operatorname{Im} z \geq\left(C_{V, \Omega, \alpha}-\epsilon\right) \begin{cases}h^{3-2 \alpha} & 5 / 6<\alpha \leq 1 \\ h \log h^{-1} & \alpha>1\end{cases}
$$

We now introduce the dynamical and microlocal objects for the finer version of Theorem 8.1. Let $\pi, B^{*} \partial \Omega, \beta, l, l_{N}$ be as in Chapter 8 .

As before denote $\gamma: H^{s}\left(\mathbb{R}^{d}\right) \rightarrow H^{s-\frac{1}{2}}(\partial \Omega)$ for $s>1 / 2$ be the restriction map and let $L$ be a vector field with $\left.L\right|_{\partial \Omega}=\partial_{\nu}$ and $\gamma_{1}:=L \gamma$. We write

$$
\begin{equation*}
\partial_{\nu} \mathcal{D} \ell(z ; h):=\gamma_{1} R_{0}(z / h) \gamma_{1}^{*}=\partial_{\nu} \mathcal{D} \ell_{\Delta}(z ; h)+\partial_{\nu} \mathcal{D} \ell_{B}(z ; h)+\partial_{\nu} \mathcal{D} \ell_{g}(z ; h)+O_{L^{2} \rightarrow C^{\infty}}\left(h^{\infty}\right) \tag{10.0.4}
\end{equation*}
$$

where $\partial_{\nu} \mathcal{D} \ell_{\Delta}$ is a pseudodifferential operator, $\partial_{\nu} \mathcal{D} \ell_{B}$ is a semiclassical Fourier integral operator associated to $\beta$ and $D_{g}$ is microlocalized near $\left|\xi^{\prime}\right|_{g}=1$ and the diagonal (See Section 6.5 for the proof of this decomposition of $\partial_{\nu} \mathcal{D} \ell$ ). In the sequel, we suppress the dependence of these operators on $h$ to simplify notation.

We now suppose $V \in h^{\alpha} \Psi(\partial \Omega)$ for $\alpha>5 / 6$ and is self-adjoint with $\sigma(V)>c h^{\alpha}>0$. Let $\chi \in C^{\infty}(\mathbb{R})$ with $\chi \equiv 1$ for $x>2 C$ and $\chi \equiv 0$ for $x<C$. Then fix $\epsilon>0$ and let

$$
\begin{equation*}
R_{\delta^{\prime}}(z):=\left(I-\partial_{\nu} \mathcal{D} \ell_{\Delta}^{1 / 2}(z) V \partial_{\nu} \mathcal{D} \ell_{\Delta}^{1 / 2}(z)\right)^{-1} \partial_{\nu} \mathcal{D} \ell_{\Delta}^{1 / 2}(z) V \partial_{\nu} \mathcal{D} \ell_{\Delta}^{1 / 2}(z) \chi\left(\frac{1-\left|h D^{\prime}\right|_{g}}{h^{\epsilon}}\right) \in \Psi_{\epsilon}^{1} \tag{10.0.5}
\end{equation*}
$$

where $\Psi_{\epsilon}^{k}$ denotes a set of slightly exotic semiclassical pseudodifferential operators (see Section 2.3).

$$
\sigma\left(R_{\delta^{\prime}}\right)=\frac{i \sigma(V) \sqrt{1-\left|\xi^{\prime}\right|_{g}^{2}}}{i \sigma(V) \sqrt{1-\left|\xi^{\prime}\right|_{g}^{2}}-2 h} \chi\left(\frac{1-\left|\xi^{\prime}\right|_{g}}{h^{\epsilon}}\right)
$$

is the reflection coefficient at the point $\left(x^{\prime}, \xi^{\prime}\right) \in B^{*}(\partial \Omega)$. We call $R_{\delta^{\prime}}$ the reflection operator. Since the symbol of $R_{\delta^{\prime}}$ is independent of $z$, we suppress the dependence of $R_{\delta^{\prime}}(z)$ on $z$ to simplify our notation.
Remark: The symbol of the reflection operator agrees up to lower order terms, with the reflection coefficient found when a plane wave with tangential frequency $\xi^{\prime}$ interacts with a derivative delta function potential of constant amplitude $V$ on a hyperplane.

Let $T(z):=\partial_{\nu} \mathcal{D} \ell_{\Delta}^{-1 / 2}(z) \partial_{\nu} \mathcal{D} \ell_{B}(z) \partial_{\nu} \mathcal{D} \ell_{\Delta}^{-1 / 2}(z)$ where $\partial_{\nu} \mathcal{D} \ell_{B}$ is the Fourier integral operator component of $\partial_{\nu} \mathcal{D} \ell(z)$. Then, using the notion of shymbol defined in Chapter 4, we define $r_{N}(z): B^{*}(\partial \Omega) \rightarrow \mathbb{R}$, the logarithmic average of the reflectivity at successive iterates of the billiard map, by

$$
\begin{equation*}
r_{N}(z, q):=\frac{\operatorname{Im} z}{h} l_{N}(q)+\frac{1}{2 N} \log \tilde{\sigma}\left(h^{-2 N}\left(\left(R_{\delta^{\prime}} T(z)\right)^{*}\right)^{N}\left(R_{\delta^{\prime}} T(z)\right)^{N}\right)(q) \tag{10.0.6}
\end{equation*}
$$

The term $\frac{\operatorname{Im} z}{h} l_{N}$ in 10.0.6) serves to cancel the growth of $T(z)$ in the right hand term. In fact, for $0<N$ independent of $h$ we have

$$
\begin{equation*}
\left.\left.r_{N}(z, q)=\frac{1}{2 N} \sum_{n=1}^{N} \log \right\rvert\, h^{-1}\left(\sigma\left(R_{\delta^{\prime}}\right) \circ \beta^{n}(q)+O\left(h^{I_{R_{\delta^{\prime}}}(q)+1-2 \epsilon}\right)\right)\right)\left.\right|^{2} \tag{10.0.7}
\end{equation*}
$$

The expression 10.0.7 illustrates that $r_{N}$ is the logarithmic average reflectivity over $N$ iterations of the billiard ball map. Moreover, it shows that $r_{N}$ is independent of $z \in \Lambda_{\log }$ up to lower order terms. Because of this, we suppress the dependence on $z$ throughout the rest of this paper.

Using Lemma 3.2.1 and 3.2.2, we have for $h$ small enough and $\epsilon<1 / 2$ with $V \in h^{\alpha} \Psi(\partial \Omega)$,

$$
\begin{aligned}
\inf _{1-\delta \leq\left|\xi^{\prime}\right| g \leq 1-h^{\epsilon}}-\frac{\left.\overline{\log \mid \sigma( }\left(R_{\delta^{\prime}}\right)(\beta(q))\right|^{2}}{2 l(q, \beta(q))} & \left.=\inf _{C h^{\epsilon / 2} \leq r \leq \delta^{1 / 2}}-\frac{1}{2 C r+\mathcal{O}\left(r^{2}\right)} \log \left(\frac{O\left(h^{-2(\alpha-1)} r^{2}\right)}{4+O\left(h^{-2(\alpha-1)} r^{2}\right.}\right)\right) \\
& \geq C \delta^{-1 / 2} \min \left(\log h^{-1}, h^{3-2 \alpha}\right)
\end{aligned}
$$

Thus, we see that for all strictly convex domains $\Omega, 0<\epsilon<1 / 2, N_{1}>0$, and $V=O\left(h^{\alpha}\right)$

$$
\begin{equation*}
\sup _{N<N_{1}} \inf _{\left|\xi^{\prime}\right|_{g} \leq 1-c h^{\epsilon}} l_{N}^{-1}\left[\log h^{-1}-r_{N}\right]=\sup _{N<N_{1}} \inf _{\xi^{\prime} \mid g \leq 1-\delta_{1}} l_{N}^{-1}\left[\log h^{-1}-r_{N}\right] \tag{10.0.8}
\end{equation*}
$$

for some $\delta_{1}>0$ small enough. That is, the slowest decay rates are those at least a fixed distance away from the glancing region.

With these definitions in hand, we state our main result.
Theorem 10.3. Let $\Omega \subset \mathbb{R}^{d}$ be a strictly convex domain with $C^{\infty}$ boundary, $\alpha>5 / 6$. Then for all $V \in h^{\alpha} \Psi(\partial \Omega)$ self-adjoint and elliptic the following holds. There exists $\delta_{1}>0$ such that for every $\epsilon>0$ and $N_{1}>0$, there is an $h_{0}>0$ such that for $z \in \Lambda_{\log }(h)$ and $0<h<h_{0}$

$$
\begin{gather*}
-\frac{\operatorname{Im} z}{h} \geq \sup _{N<N_{1}} \inf _{q \in B_{1-\delta_{1}}^{*} \partial \Omega} l_{N}^{-1}(q)\left[\log h^{-1}-r_{N}(q)\right](1-\epsilon)  \tag{10.0.9}\\
B_{1-\delta_{1}}^{*} \partial \Omega:=\left\{q \in T^{*} \partial \Omega:\left|\xi^{\prime}(q)\right|^{g} \leq 1-\delta_{1}\right\} . \tag{10.0.10}
\end{gather*}
$$

## Outline of the Proofs

We begin in Section 10.1 by using the quasiboundary triple theory of 88 to define $-\Delta_{\partial \Omega, \delta^{\prime}}$ when $V$ is as in Theorem 10.1. When $V$ is invertible this can be done using quadratic forms similar to the way $-\Delta_{\Gamma, \delta}$ was defined, but in the more general situation of Theorem 10.1 , this does not seem to be possible. In the process of giving the definition of $-\Delta_{\partial \Omega, \delta^{\prime}}$ and identifying its domain, we see that a $\lambda$-outgoing function $u$ with $-\Delta_{\partial \Omega, \delta^{\prime}} u=0$ has

$$
\begin{gather*}
\left.\left(I-\partial_{\nu} \mathcal{D} \ell V\right) \partial_{\nu} u\right|_{\partial \Omega}=0  \tag{10.0.11}\\
u=R_{0}(\lambda) \gamma_{1}^{*} V \partial_{\nu} u .
\end{gather*}
$$

Because of this, we will see that the analysis reduces to an analysis of $I-\partial_{\nu} \mathcal{D} \ell V$
Once we have defined the operator, we give the proof of Theorem 10.1. This is done similarly to the analysis for $-\Delta_{\Gamma, \delta}$, however, the process is complicated by the lower regularity of $\delta_{\partial \Omega}^{\prime}$. We start by showing that $I-V \partial_{\nu} \mathcal{D} \ell$ has a meromorphic inverse and by writing a formula for $R_{V}$ in terms of $I-V \partial_{\nu} \mathcal{D} \ell$ we obtain Theorem 10.1. We also show that (except in $d=1$ with $\lambda=0$ when there is always a pole) $R_{V}$ has a pole at $\lambda$ if and only if there is a nontrivial solution to $\left(I-\partial_{\nu} \mathcal{D} \ell V\right) \varphi=0$.

Finally, as in Chapter 8 we decompose $\left(I-\partial_{\nu} \mathcal{D} \ell V\right)$ microlocally into the hyperbolic, elliptic, and glancing regions to prove Theorem 10.3. The hyperbolic, glancing, and elliptic regions ( $\mathcal{H}, \mathcal{G}$, and $\mathcal{E}$ respectively) have the property that, letting $U^{\prime}$ denote a slightly enlarged version of $U$,

$$
\left(I-\chi_{\mathcal{H}^{\prime}}\right)\left(I-\partial_{\nu} \mathcal{D} \ell V\right) \chi_{\mathcal{H}}=\left(I-\chi_{\mathcal{G}^{\prime}}\right)\left(I-\partial_{\nu} \mathcal{D} \ell V\right) \chi_{\mathcal{G}}=\left(I-\chi_{\mathcal{E}^{\prime}}\right)\left(I-\partial_{\nu} \mathcal{D} \ell V\right) \chi_{\mathcal{E}} \equiv 0
$$

microlocally. Thus, the invertibility of $I-\partial_{\nu} \mathcal{D} \ell V$ can be treated separately on each region. The analysis of $\mathcal{H}$ and $\mathcal{E}$ is nearly identical to that in Chapter 8 , however unlike $G$, the
operator $\partial_{\nu} \mathcal{D} \ell$ has a smaller operator norm when microlocally restricted near glancing. This, combined with a slightly more sophisticated microlocal decomposition allows us to complete the proof of Theorem 10.3.

### 10.1 Definition of the Operator and Identification of its Domain

The operator we consider is

$$
-\Delta_{\partial \Omega, \delta^{\prime}}:=\left(-\Delta+\delta_{\partial \Omega}^{\prime} V \gamma \partial_{\nu}\right)
$$

By $\delta_{\partial \Omega}^{\prime}$ we mean $L^{*} \gamma^{*}$ where $L$ is a vector field with $L=\partial_{\nu}$ on $\partial \Omega$ and $L^{*}$ is its dual operator. In order to understand what domain we should use for the operator, we let $v \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ and $u=u_{1} \oplus u_{2} \in L^{2} \cap H^{2}\left(\mathbb{R}^{d} \backslash \partial \Omega\right)$. compute

$$
\langle-\Delta u, v\rangle=\langle u,-\Delta v\rangle-\left\langle\partial_{\nu_{1}} u_{1}+\partial_{\nu_{2}} u_{2}, v\right\rangle_{\partial \Omega}+\left\langle u_{1}-u_{2}, \partial_{\nu_{1}} v\right\rangle_{\partial \Omega}
$$

Thus, if $-\Delta u=\delta_{\partial \Omega}^{\prime} V \partial_{\nu} u$ as a distribution then

$$
\begin{aligned}
\partial_{\nu_{1}} u_{1}+\partial_{\nu_{2}} u_{1} & =0 \\
u_{1}-u_{2} & =-V \partial_{\nu_{1}} u=V \partial_{\nu_{2}} u
\end{aligned}
$$

With this in mind, we show that $-\Delta_{\partial \Omega, \delta^{\prime}}$ is the operator $-\Delta$ with domain

$$
\mathcal{D}^{\prime}=\left\{u \in H_{\Delta}^{3 / 2}\left(\mathbb{R}^{d} \backslash \partial \Omega\right): \partial_{\nu_{1}} u_{1}=-\partial_{\nu_{2}} u_{2},-V \partial_{\nu_{1}} u_{1}=u_{1}-u_{2}\right\} .
$$

where

$$
H_{\Delta}^{3 / 2}(U)=\left\{u \in H^{3 / 2}(U): \Delta u \in L^{2}(U)\right\} .
$$

That is, for $u \in \mathcal{D}^{\prime},-\Delta_{\partial \Omega, \delta^{\prime}} u=-\left.\Delta\right|_{\Omega} \oplus-\left.\Delta\right|_{\mathbb{R}^{d} \backslash \bar{\Omega}}$. Note that for $U$ with $\partial U \in C^{\infty}$ the following Green's formulae hold on $H_{\Delta}^{3 / 2}(U)$ : for $f, g \in H_{\Delta}^{3 / 2}(U), h \in H^{1}(U)$,

$$
\begin{align*}
\langle-\Delta f, h\rangle_{U} & =\langle\nabla f, \nabla h\rangle_{U}-\left\langle\partial_{\nu} f, h\right\rangle_{\partial U}  \tag{10.1.1}\\
\langle-\Delta f, g\rangle_{U} & =\langle f,-\Delta g\rangle_{\Omega}+\left\langle f, \partial_{\nu} g\right\rangle_{\partial U}-\left\langle\partial_{\nu} f, g\right\rangle_{\partial U} \tag{10.1.2}
\end{align*}
$$

To see that $-\Delta_{\partial \Omega, \delta^{\prime}}$ is self-adjoint we proceed similar to [8, Section 3] Let $\Omega \subset \mathbb{R}^{d}$ be a bounded domain with $\partial \Omega \in C^{\infty}$. Denote by $\Omega_{1}=\Omega$ and $\Omega_{2}=\mathbb{R}^{d} \backslash \bar{\Omega}$. Let $P=-\Delta$ and $P_{i}=\left.P\right|_{\Omega_{i}}$. We write for a function $f \in L^{2}\left(\mathbb{R}^{d}\right) f=f_{1} \oplus f_{2}$ where $f_{i}=\left.f\right|_{\Omega_{i}}$. We denote by $\langle\cdot, \cdot\rangle,\langle\cdot, \cdot\rangle_{i}$, and $\langle\cdot, \cdot\rangle_{\partial \Omega}$ the inner products in $L^{2}\left(\mathbb{R}^{d}\right), L^{2}\left(\Omega_{i}\right)$, and $L^{2}(\partial \Omega)$ respectively. Let

$$
A_{i} f_{i}=P_{i} f_{i}, \quad \operatorname{dom}\left(A_{i}\right)=H_{0}^{2}\left(\Omega_{i}\right)
$$

be the minimal operators associated with $P_{i}$. Let $A=A_{1} \oplus A_{2}$. Then, $A$ is densely defined, closed as an operator on $L^{2}$. Next, let

$$
T_{i} f_{i}=P_{i} f_{i}, \quad \operatorname{dom}\left(T_{i}\right)=H_{\Delta}^{3 / 2}\left(\Omega_{i}\right)
$$

and $T=T_{1} \oplus T_{2}$ where
Finally, define the Dirichlet and Neumann realizations of $P_{i}$ by

$$
\begin{array}{ll}
A_{D, i}=P_{i} f_{i}, & \operatorname{dom}\left(A_{D, i}\right)=\left\{f_{i} \in H^{2}\left(\Omega_{i}\right):\left.f_{i}\right|_{\partial \Omega}=0\right\} \\
A_{N, i}=P_{i} f_{i}, & \operatorname{dom}\left(A_{D, i}\right)=\left\{f_{i} \in H^{2}\left(\Omega_{i}\right):\left.\partial_{\nu_{i}} f_{i}\right|_{\partial \Omega}=0\right\}
\end{array}
$$

where $\nu_{i}$ is the outward unit normal to $\partial \Omega$ from $\Omega_{i}$. Then, let $A_{D}=A_{D, 1} \oplus A_{D, 2}$ and $A_{N}=A_{N, 1} \oplus A_{N, 2}$. Let

$$
A_{0} f=P f, \quad \operatorname{dom}\left(A_{0}\right)=H^{2}\left(\mathbb{R}^{d}\right)
$$

be the free Laplacian. Finally, let

$$
\begin{aligned}
& \hat{A} f=P f, \quad \operatorname{dom}(\hat{A})=\left\{f \in H^{2}\left(\mathbb{R}^{d}\right): \partial_{\nu_{2}} f_{2}+\partial_{\nu_{1}} f_{1}=0\right\} \\
& \hat{T} f=P f, \quad \operatorname{dom}(\hat{T})=\left\{f \in H_{\Delta}^{3 / 2}\left(\mathbb{R}^{d} \backslash \partial \Omega\right): \partial_{\nu_{2}} f_{2}+\partial_{\nu_{1}} f_{1}=0\right\}
\end{aligned}
$$

The we have [8, Proposition 3.8]
Lemma 10.1.1. The triple $\Pi=\left\{L^{2}(\partial \Omega), B_{0}, B_{1}\right\}$ where

$$
B_{0} f=\partial_{\nu_{2}} f_{2}, \quad B_{1} f=\left.f_{2}\right|_{\partial \Omega}-\left.f_{1}\right|_{\partial \Omega}, \quad f \in \operatorname{dom}(\hat{T})
$$

is a quasi boundary triple for $\hat{A}^{*}$ in the sense of [8]. We have

$$
\left.\hat{T}\right|_{\text {ker } B_{0}}=A_{N},\left.\quad \hat{T}\right|_{\text {ker } B_{1}}=A_{0}
$$

and

$$
\operatorname{ran} B_{0}=L^{2}(\partial \Omega) \quad \operatorname{ran} B_{1}=H^{1}(\partial \Omega)
$$

Using this, we have
Lemma 10.1.2. If $V: L^{2}(\partial \Omega) \rightarrow L^{2}(\partial \Omega)$ is self-adjoint, then

$$
-\Delta_{\partial \Omega, \delta^{\prime}}:=T f, \quad \operatorname{dom}\left(-\Delta_{\partial \Omega, \delta^{\prime}}\right)=\left\{f \in \operatorname{dom}(T): f \in \operatorname{ker}\left(B_{1}+V B_{0}\right)\right.
$$

is a self-adjoint operator. If, moreover, $V$ has the property that $V u \in H^{1}(\partial \Omega)$ implies $V u \in H^{1 / 2}(\partial \Omega)$, then

$$
\operatorname{dom}\left(-\Delta_{\partial \Omega, \delta^{\prime}}\right) \subset L^{2}\left(\mathbb{R}^{d}\right) \cap\left(H^{2}\left(\Omega_{1}\right) \oplus H^{2}\left(\Omega_{2}\right)\right)
$$

Proof. Since $\sigma_{\text {ess }}(V)=\emptyset$, and $\operatorname{dom}(V)=L^{2}(\partial \Omega), V$ satisfies the hypotheses of [7, Theorem 1.17].

To see the second claim, suppose $f \in \operatorname{dom}\left(-\Delta_{\partial \Omega, \delta^{\prime}}\right)$. Then, $f \in \operatorname{dom}(T) \subset H_{\Delta}^{3 / 2}\left(\mathbb{R}^{d} \backslash \partial \Omega\right)$ and hence

$$
V B_{0} f=-B_{1} f \in H^{1}(\partial \Omega)
$$

Therefore, $B_{0} f \in H^{1 / 2}(\partial \Omega)$ by assumption. Next, fix $\lambda \notin \mathbb{R}$. Then

$$
\operatorname{dom}(\hat{T})=\operatorname{dom}\left(A_{N}\right)+\operatorname{ker}(\hat{T}-\lambda) .
$$

Hence, $f=f_{N}+f_{\lambda}$ where $f_{N} \in \operatorname{dom}\left(A_{N}\right)$ and $f_{\lambda} \in \operatorname{ker}(\hat{T}-\lambda)$. Then, since $A_{N}=\left.\hat{T}\right|_{\text {ker } B_{0}}$,

$$
B_{0} f_{\lambda}=B_{0} f \in H^{1 / 2}(\partial \Omega)
$$

Now, $B_{0}$ maps $\operatorname{dom}(\hat{T}) \cap H^{2}\left(\mathbb{R}^{d} \backslash \partial \Omega\right)$ surjectively onto $H^{1 / 2}(\partial \Omega)$. But,

$$
\operatorname{ker} B_{0} \cap H^{2}\left(\mathbb{R}^{d} \backslash \partial \Omega\right) \cap \operatorname{dom}\left(\hat{T}=\operatorname{dom}\left(A_{N}\right)\right.
$$

and $\operatorname{ker}(\hat{T}-\lambda) \cap H^{2}\left(\mathbb{R}^{d} \backslash \partial \Omega\right) \cap \operatorname{dom}\left(A_{N}\right)=\emptyset$. Hence, $B_{0}$ maps $\operatorname{ker}(\hat{T}-\lambda) \cap H^{2}\left(\mathbb{R}^{d} \backslash \partial \Omega\right)$ bijectively onto $H^{1 / 2}(\partial \Omega)$. This together with $B_{0} f_{\lambda} \in H^{1 / 2}(\partial \Omega), f_{N} \in H^{2}\left(\mathbb{R}^{d} \backslash \partial \Omega\right)$ imply $f \in H^{2}\left(\mathbb{R}^{d} \backslash \partial \Omega\right)$.

Letting $h \in C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$ and $u \in \mathcal{D}^{\prime}$ with $-\Delta_{\partial \Omega, \delta^{\prime}} u=f$, 10.1.1 (10.1.2 imply that as a distribution,

$$
\left(-\Delta-\delta_{\partial \Omega}^{\prime} V \partial_{\nu}\right) u=f
$$

for some $f \in L^{2}\left(\mathbb{R}^{d}\right)$. Thus if $\left(-\Delta_{\partial \Omega, \delta^{\prime}}-\lambda^{2}\right) u=f$,

$$
\begin{align*}
-\Delta u & =-\delta_{\partial \Omega}^{\prime} V \partial_{\nu} u+f+\lambda^{2} u \\
\left(-\Delta-\lambda^{2}\right) u & =-\delta_{\partial \Omega}^{\prime} V \partial_{\nu} u+f \\
u & =R_{0}(\lambda)\left(f-\delta_{\partial \Omega}^{\prime} V \partial_{\nu} u\right)  \tag{10.1.3}\\
\partial_{\nu} u & =\partial_{\nu} R_{0}(\lambda) f-\partial_{\nu} R_{0}(\lambda) \delta_{\partial \Omega}^{\prime} V \partial_{\nu} u \\
\partial_{\nu} R_{0}(\lambda) f & =\left(I+\partial_{\nu} R_{0}(\lambda) \delta_{\partial_{\Omega}}^{\prime} V\right) \partial_{\nu} u \\
& =\left(I-\partial_{\nu} \mathcal{D} \ell(\lambda) V\right) \partial_{\nu} u \tag{10.1.4}
\end{align*}
$$

Lemma 10.1.3. Suppose that $V: L^{2}(\partial \Omega) \rightarrow H^{3 / 2}(\partial \Omega)$ Then

$$
\mathcal{D}^{\prime} \subset L^{2}\left(\mathbb{R}^{d}\right) \cap\left(H^{2}\left(\Omega_{1}\right) \oplus H^{2}\left(\Omega_{2}\right)\right)
$$

Proof. Suppose $u \in \mathcal{D}^{\prime}$. Then $u \in H_{\Delta}^{3 / 2}$ and hence $\partial_{\nu} u \in L^{2}(\partial \Omega)$. Thus, $V: L^{2}(\partial \Omega) \rightarrow$ $H^{3 / 2}(\partial \Omega)$, together with $(10.1 .3)$ and the fact that the double layer potential, $\mathcal{D}$, has $\mathcal{D}$ : $H^{3 / 2}\left(\partial \Omega_{i}\right) \rightarrow H_{\text {loc }}^{2}\left(\Omega_{i}\right)$, imply that $u \in H^{2}\left(\mathbb{R}^{d} \backslash \partial \Omega\right)$.

### 10.2 Dynamical Resonance Free Regions for the Delta Prime Potential

Let $V \in h^{\alpha} \Psi(\partial \Omega), \alpha>5 / 6$ be a positive definite self-adjoint operator that has $\sigma(V) \geq$ $C h^{\alpha}>0$. In this case, $V$ is invertible for $h$ small enough and $-\Delta_{\partial \Omega, \delta^{\prime}}$ can be defined as in Section 10.1. Next, let $z=1+i \omega_{0}$ with

$$
\omega_{0} \in\left[-M h \log h^{-1}, C h\right] .
$$

Recall also that $z / h(z \neq 0$ or $d \neq 1)$ is a resonance if and only if there is a nontrivial solution $\psi$ to

$$
\begin{equation*}
\left(I-\partial_{\nu} \mathcal{D} \ell V\right) \psi=0 \tag{10.2.1}
\end{equation*}
$$

## Hyperbolic Region: Appearance of the Dynamics

Recall from Lemma 6.6.4 that

$$
\partial_{\nu} \mathcal{D} \ell(z)=\partial_{\nu} \mathcal{D} \ell_{\Delta}+\partial_{\nu} \mathcal{D} \ell_{B}+\partial_{\nu} \mathcal{D} \ell_{g}+O_{L^{2} \rightarrow C^{\infty}}\left(h^{\infty}\right)
$$

Let $0<\epsilon<1 / 2$. Then, suppose that $\chi \in S_{\epsilon}$ has $\operatorname{supp} \chi \subset\left\{\left|\xi^{\prime}\right| \leq 1-2 h^{\epsilon}\right\}$ and let $X=\mathrm{Op}_{\mathrm{h}}(\chi)$. Finally, suppose that

$$
\left(I-\partial_{\nu} \mathcal{D} \ell V\right) X \psi=f
$$

and let $\partial_{\nu} \mathcal{D} \ell_{\Delta}^{-1 / 2}$ be a microlocal inverse for $\partial_{\nu} \mathcal{D} \ell_{\Delta}^{1 / 2}$ on

$$
\mathcal{H}:=\left\{\left|\xi^{\prime}\right|_{g} \leq 1-r_{\mathcal{H}} h^{\epsilon}\right\}
$$

Then, following the same process used in section 8.3 for $-\Delta_{\partial \Omega, \delta}$, we have, writing $\varphi=$ $\partial_{\nu} \mathcal{D} \ell_{\Delta}^{1 / 2} V X \psi$ and $T=\partial_{\nu} \mathcal{D} \ell_{\Delta}^{-1 / 2} \partial_{\nu} \mathcal{D} \ell_{B} \partial_{\nu} \mathcal{D} \ell_{\Delta}^{-1 / 2}$,

$$
\left(I-\partial_{\nu} \mathcal{D} \ell_{\Delta}^{1 / 2} V \partial_{\nu} \mathcal{D} \ell_{\Delta}^{1 / 2}\right) \varphi=\partial_{\nu} \mathcal{D} \ell_{\Delta}^{1 / 2} V \partial_{\nu} \mathcal{D} \ell_{\Delta}^{1 / 2} T \varphi+O\left(h^{\infty}\right) \psi+\partial_{\nu} \mathcal{D} \ell_{\Delta}^{1 / 2} V f
$$

Lemma 10.2.1. The operator $I-\partial_{\nu} \mathcal{D} \ell_{\Delta}^{1 / 2} V \partial_{\nu} \mathcal{D} \ell_{\Delta}^{1 / 2}$ has a microlocal inverse,

$$
\left(I-\partial_{\nu} \mathcal{D} \ell_{\Delta}^{1 / 2} V \partial_{\nu} \mathcal{D} \ell_{\Delta}^{1 / 2}\right)^{-1} \in \min \left(h^{1-\alpha-\epsilon / 2}, 1\right) \Psi_{\epsilon}(\partial \Omega)
$$

on $\mathcal{H}$.
Proof. Let $B:=\partial_{\nu} \mathcal{D} \ell_{\Delta}^{1 / 2} V \partial_{\nu} \mathcal{D} \ell_{\Delta}^{1 / 2}$. We show that $I-B$ is microlocally invertible. To see this, let $\epsilon_{1}=2-2 \alpha<\frac{1}{2}$. Then for $\tilde{\chi}$ supported on $\left|\xi^{\prime}\right|_{g} \leq 1-C h^{\epsilon_{1}}, B^{-1} \tilde{\chi} \in \Psi_{\epsilon}$ and on $|\tilde{\chi}|>c>0,\left|\sigma\left(\left(I-B^{-1}\right) \mathrm{Op}_{\mathrm{h}}(\tilde{\chi})\right)\right| \geq c>0$. Therefore, we can write, microlocally on $\left|\xi^{\prime}\right|_{g} \leq 1-C h^{\epsilon_{1}}$,

$$
(I-B)^{-1}=-\left(B^{-1}\left(I-B^{-1}\right)\right)^{-1}=-\left(I-B^{-1}\right)^{-1} B
$$

On the other hand, for $\tilde{\chi}$ supported on $1-2 C h^{\epsilon_{1}} \leq\left|\xi^{\prime}\right|_{g} \leq 1-r_{\mathcal{H}} h^{\epsilon}, B \tilde{\chi} \in \Psi_{\epsilon}$ and on $|\tilde{\chi}|>c>0,\left|\sigma\left((I-B) \mathrm{Op}_{\mathrm{h}}(\tilde{\chi})\right)\right|>c>0$. Therefore, $(I-B)^{-1}$ exists microlocally on $1-2 C h^{\epsilon_{1}} \leq\left|\xi^{\prime}\right|_{g} \leq 1-r_{\mathcal{H}} h^{\epsilon}$.

Combining these two statements, we see that $\left(I-\partial_{\nu} \mathcal{D} \ell_{\Delta}^{1 / 2} V \partial_{\nu} \mathcal{D} \ell_{\Delta}^{1 / 2}\right)^{-1}$ exists microlocally on $\mathcal{H}$ and has the required property.

Letting

$$
\begin{aligned}
R_{\delta^{\prime}} & :=\left(I-\partial_{\nu} \mathcal{D} \ell_{\Delta}^{1 / 2} V \partial_{\nu} \mathcal{D} \ell_{\Delta}^{1 / 2}\right)^{-1} \partial_{\nu} \mathcal{D} \ell_{\Delta}^{1 / 2} V \partial_{\nu} \mathcal{D} \ell_{\Delta}^{1 / 2} \\
& =-I+\left(I-\partial_{\nu} \mathcal{D} \ell_{\Delta}^{1 / 2} V \partial_{\nu} \mathcal{D} \ell_{\Delta}^{1 / 2}\right)^{-1} \in \min \left(1, h^{\alpha-1+\epsilon / 2}\right) \Psi_{\epsilon}(\partial \Omega)
\end{aligned}
$$

we have

$$
\varphi=R_{\delta^{\prime}} T \varphi+O\left(h^{\infty}\right) \psi-R_{\delta^{\prime}} \partial_{\nu} \mathcal{D} \ell_{\Delta}^{-1 / 2} f
$$

Here, $T$ is an FIO associated to the billiard map such that

$$
\sigma\left(e^{\left.\frac{\operatorname{Im} z}{h} \mathrm{Op}_{\mathrm{h}} l\left(q, \beta_{E}(q)\right)\right)} T\right)\left(\beta_{E}(q), q\right)=e^{-i \pi / 4} d q^{1 / 2} \in S
$$

and $R_{\delta^{\prime}}$ is as in 10.0.5).
Thus by standard composition formulae for FIOs, we have for $0<N$ independent of $h$,

$$
\begin{equation*}
\left(I-\left(R_{\delta^{\prime}} T\right)^{N}\right) \varphi=O\left(h^{\infty}\right) \psi-\sum_{m=0}^{N-1}\left(R_{\delta^{\prime}} T\right)^{m} R_{\delta^{\prime}} \partial_{\nu} \mathcal{D} \ell_{\Delta}^{-1 / 2} f \tag{10.2.2}
\end{equation*}
$$

We also have that

$$
\begin{equation*}
\left(R_{\delta^{\prime}} T\right)_{N}:=\left(\left(R_{\delta^{\prime}} T\right)^{*}\right)^{N}\left(R_{\delta^{\prime}} T\right)^{N}=\mathrm{Op}_{\mathrm{h}}\left(a_{N}\right)+O_{\Psi^{-\infty}}\left(h^{\infty}\right) \tag{10.2.3}
\end{equation*}
$$

where $a_{N} \in \min \left(1, h^{N(\alpha-1+\epsilon)}\right) S_{\epsilon}\left(T^{*} \partial \Omega\right)$.
We now analyze the case that $\psi$ solves (10.2.1). We start by showing that under a dynamical condition on $\operatorname{Im} z$, there is an $1 / 2>\epsilon>0$ so that if $\chi_{0}=\chi_{0}\left(\left|\xi^{\prime}\right|_{g}\right) \in S_{\epsilon}$ with $\operatorname{supp} \chi_{0} \subset\left\{1-C h^{\epsilon}<\left|\xi^{\prime}\right|_{g}<1-2 h^{\epsilon}\right\}$

$$
\begin{equation*}
\left\|\mathrm{Op}_{\mathrm{h}}\left(\chi_{0}\right) \psi\right\|=O\left(h^{\infty}\right) \psi \tag{10.2.4}
\end{equation*}
$$

We then let $\chi_{1}=\chi_{1}\left(\left|\xi^{\prime}\right|_{g}\right) \in S_{\epsilon}$ with $\chi_{1} \equiv 1$ on $\left\{\left|\xi^{\prime}\right|_{g} \leq 1-2 h^{\epsilon}\right\}$ and $\operatorname{supp} \chi_{1} \subset\left\{\left|\xi^{\prime}\right|_{g} \leq\right.$ $\left.E-h^{\epsilon}\right\}$ and show that there exists $\epsilon>0$ such that both

$$
\left\|\mathrm{Op}_{\mathrm{h}}\left(\chi_{1}\right) \psi\right\| \leq\left(\left\|\mathrm{Op}_{\mathrm{h}}\left(\chi_{0}\right) \psi\right\|\right)+O\left(h^{\infty}\right)\|\psi\|
$$

and (10.2.4) hold.
To simplify notation, let $X_{1}=\operatorname{Op}_{\mathrm{h}}\left(\chi_{1}\right)$. For $i=1,2$, let $\chi_{0}^{(i)}=\chi_{0}^{(i)}\left(\left|\xi^{\prime}\right|_{g}\right) \in S_{\epsilon}$ with $\chi_{0}^{(i)} \equiv 1$ on supp $\chi_{0}^{(i-1)}$ and $\operatorname{supp} \chi_{0}^{(i)} \subset\left\{1-(i+1) C h^{\epsilon}<\left|\xi^{\prime}\right|_{g}<1-\left(2-\frac{i}{2}\right) h^{\epsilon}\right\}$. Here, $\chi_{0}^{(0)}=\chi_{0}$. Finally, let $X_{0}^{(i)}=\operatorname{Op}_{\mathrm{h}}\left(\chi_{0}^{(i)}\right)$. We have that

$$
\left(I-\partial_{\nu} \mathcal{D} \ell V\right) X_{0}^{(1)} \psi=\left[X_{0}^{(1)}, \partial_{\nu} \mathcal{D} \ell V\right] \psi
$$

So, by 10.2.2

$$
\left(I-\left(R_{\delta^{\prime}} T\right)^{N}\right) \varphi=O\left(h^{\infty}\right) \psi-\sum_{m=0}^{N-1}\left(R_{\delta^{\prime}} T\right)^{m} R_{\delta^{\prime}} \partial_{\nu} \mathcal{D} \ell_{\Delta}^{-1 / 2}\left[X_{0}, \partial_{\nu} \mathcal{D} \ell V\right] \psi
$$

with $\varphi=\partial_{\nu} \mathcal{D} \ell_{\Delta}^{1 / 2} V X_{0}^{(1)} \psi$. Moreover, since $\chi_{0}^{(2)} \equiv 1$ on $\operatorname{supp} \chi_{0}^{(1)}$,

$$
\begin{equation*}
\left(I-\left(R_{\delta^{\prime}} T\right)^{N} X_{0}^{(2)}\right) \varphi=O\left(h^{\infty}\right) \psi-\sum_{m=0}^{N-1}\left(R_{\delta^{\prime}} T\right)^{m} R_{\delta^{\prime}} \partial_{\nu} \mathcal{D} \ell_{\Delta}^{-1 / 2}\left[X_{0}, \partial_{\nu} \mathcal{D} \ell V\right] \psi \tag{10.2.5}
\end{equation*}
$$

Now, let

$$
\mathcal{N G}:=\left\{1-2 C h^{\epsilon} \leq\left|\xi^{\prime}\right|_{g} \leq 1-\frac{1}{100} h^{\epsilon}\right\}
$$

Then

$$
\left\|\left(R_{\delta^{\prime}} T\right)^{N} X_{0}^{(2)} u\right\|^{2} \leq \quad \sup _{\mathcal{N} \mathcal{G}}\left(\left|\tilde{\sigma}\left(\left(R_{\delta^{\prime}} T\right)_{N}\right)(q)\right|^{2}+O\left(\min \left(1, h^{N(2 \alpha-2+\epsilon)}\right) h^{1-2 \epsilon}\right)\|u\|_{L^{2}}^{2}\right.
$$

Let

$$
\beta_{0}:=1-\sqrt{\sup _{\mathcal{N G}} \sigma\left(\left(R_{\delta^{\prime}} T\right)_{N}\right)}
$$

Then the proof of the following lemma is nearly identical to that of Lemma 8.3.1.
Lemma 10.2.2. Suppose that $\beta_{0}>h^{\gamma_{1}}$ where $\gamma_{1}<\min (\epsilon / 2,1 / 2-\epsilon)$. Let $c>r_{\mathcal{H}}$ and $g \in L^{2}$ have $\mathrm{MS}_{\mathrm{h}}(g) \subset\left\{1-C h^{\epsilon} \leq\left|\xi^{\prime}\right|_{g} \leq 1-c h^{\epsilon}\right\}$. Then if

$$
\left(I-\left(R_{\delta^{\prime}} T\right)^{N} X_{0}^{(2)}\right) u=g,
$$

for any $\delta>0$,

$$
\operatorname{MS}_{\mathrm{h}}(u) \subset\left\{1-(C+\delta) h^{\epsilon} \leq\left|\xi^{\prime}\right|_{g} \leq 1-(c-\delta) h^{\epsilon}\right\}
$$

In particular, there exists an operator $A$ with $\|A\|_{L^{2} \rightarrow L^{2}} \leq 2 \beta_{0}^{-1}$,

$$
A\left(I-\left(R_{\delta} T\right)^{N}\right)=I \text { microlocally on } \mathcal{N G}
$$

and if $\mathrm{MS}_{\mathrm{h}}(g) \subset\left\{1-C h^{\epsilon} \leq\left|\xi^{\prime}\right|_{g} \leq 1-c h^{\epsilon}\right\}$, then

$$
\mathrm{MS}_{\mathrm{h}}(A g) \subset\left\{1-(C+\delta) h^{\epsilon} \leq\left|\xi^{\prime}\right|_{g} \leq 1-(c-\delta) h^{\epsilon}\right\}
$$

Writing

$$
R_{\delta^{\prime}} T=\left(R_{\delta^{\prime}} e^{-\frac{\operatorname{Im} z}{h} \operatorname{Oph}_{\mathrm{h}}\left(l(q), \beta_{E}(q)\right)}\right)\left(e^{\frac{\operatorname{Im} z}{h} \operatorname{Op}_{\mathrm{h}}\left(l(q), \beta_{E}(q)\right)} T\right)
$$

and applying Lemma 4.5.1 shows that

$$
\begin{aligned}
& \tilde{\sigma}\left(\left(R_{\delta^{\prime}} T\right)_{N}\right)(q)=\exp \left(-\frac{2 \operatorname{Im} z}{h} \sum_{n=0}^{N-1} l\left(\beta^{n}(q), \beta^{n+1}(q)\right)\right) \\
& \prod_{i=1}^{N}\left(\left|\tilde{\sigma}\left(R_{\delta^{\prime}}\right)\left(\beta^{i}(q)\right)\right|^{2}+O\left(h^{I_{R_{\delta^{\prime}}}\left(\beta^{i}(q)\right)+1-2 \epsilon}\right)\right) .
\end{aligned}
$$

Now, on $\mathcal{N G}$,

$$
\left|\sigma\left(R_{\delta^{\prime}}\right)\right|^{2} \leq 1-C h^{2-2 \alpha-\epsilon}
$$

so, using Lemma 3.2.2, we have that on $\mathcal{N G}$

$$
\left|\sigma\left(R_{\delta^{\prime}} T\right)_{N}\right| \leq 1-c h^{2-2 \alpha-\epsilon}+C \frac{\operatorname{Im} z}{h} h^{\frac{\epsilon}{2}}
$$

Hence, for $\operatorname{Im} z \geq-M h^{3-2 \alpha-\frac{\epsilon}{4}}$,

$$
\left|\sigma\left(R_{\delta^{\prime}} T\right)_{N}\right| \leq 1-c h^{2-2 \alpha-\epsilon} \quad \Rightarrow \quad \beta>h^{2-2 \alpha-\epsilon} .
$$

So, using that $V$ is invertible and applying $X_{0} V^{-1} \partial_{\nu} \mathcal{D} \ell^{-1 / 2} A$ to (10.2.5) gives
Lemma 10.2.3. Fix $M>0$ and suppose that

$$
\operatorname{Im} z \geq-M \min \left(h^{3-2 \alpha-\frac{\epsilon}{4}}, h \log h^{-1}\right)
$$

and $2-2 \alpha-\epsilon<\min \left(\frac{\epsilon}{2}, \frac{1}{2}-\epsilon\right)$. Then

$$
\left\|X_{0} \psi\right\|=O\left(h^{\infty}\right)\|\psi\|
$$

In particular, the estimate holds when $\frac{2}{3}(2-2 \alpha)<\epsilon<\frac{1}{2}$.
Now, we obtain an estimates on $X_{1} \psi$. Following the same argument used to get 10.2.5), we have

$$
\left(I-\left(R_{\delta^{\prime}} T\right)^{N}\right) \varphi_{1}=O\left(h^{\infty}\right) \psi-\sum_{m=0}^{N-1}\left(R_{\delta^{\prime}} T\right)^{m} R_{\delta^{\prime}} \partial_{\nu} \mathcal{D} \ell_{\Delta}^{-1 / 2}\left[X_{1}, \partial_{\nu} \mathcal{D} \ell V\right] \psi
$$

where $\varphi_{1}=\partial_{\nu} \mathcal{D} \ell_{\Delta}^{1 / 2} V X_{1} \psi$.
Next, by 87, Theorem 13.13]

$$
\left\|\left(R_{\delta^{\prime}} T\right)^{N} \varphi\right\|^{2} \leq \sup _{\mathcal{H}}\left(\left|\tilde{\sigma}\left(\left(R_{\delta^{\prime}} T\right)_{N}\right)(q)\right|^{2}+O\left(h^{I_{\left(R_{\delta^{\prime}}, T\right)_{N}}(q)+1-2 \epsilon}\right)\right)\|\varphi\|_{L^{2}}^{2} .
$$

Define

$$
\beta_{1}:=\min \left(\frac{1}{2}, 1-\sqrt{\sup _{\mathcal{H}} \sigma\left(\left(R_{\delta^{\prime}} T\right)_{N}\right)}\right) .
$$

Now, if $\beta_{1} \geq 0$, then $I_{\left(R_{\delta^{\prime}} T\right)_{N}} \geq 0$ on $\mathcal{H}$ and hence

$$
\begin{align*}
\left(\beta_{1}-C h^{1-2 \epsilon}\right)\left\|\varphi_{1}\right\|_{L^{2}} \leq & \left\|\left(I-\left(R_{\delta^{\prime}} T\right)^{N}\right) \varphi_{1}\right\|_{L^{2}} \\
= & \left\|\sum_{m=0}^{N-1}\left(R_{\delta^{\prime}} T\right)^{m} R_{\delta^{\prime}} \partial_{\nu} \mathcal{D} \ell_{\Delta}^{-1 / 2}\left[X_{1}, \partial_{\nu} \mathcal{D} \ell V\right] \psi\right\|  \tag{10.2.6}\\
& +O\left(h^{\infty}\right)\|\psi\|
\end{align*}
$$

But, by Lemma 10.2 .3 , if $\operatorname{Im} z \geq-\min \left(h^{3 \alpha-2-\frac{\epsilon}{4}}, h \log h^{-1}\right)$, then $\left[X_{1}, \partial_{\nu} \mathcal{D} \ell V\right] \psi=$ $O\left(h^{\infty}\right) \psi$. So, provided that $\beta_{1} \gg h^{1-2 \epsilon}$,

$$
\left\|\varphi_{1}\right\|=O\left(h^{\infty}\right)\|\psi\|
$$

and hence, since $X_{1} \psi=V^{-1} \partial_{\nu} \mathcal{D} \ell^{-1 / 2} \varphi$,

$$
\left\|X_{1} \psi\right\|=O\left(h^{\infty}\right)\|\psi\|
$$

Thus, in order for 10.2 .2 to hold with $\operatorname{MS}_{\mathrm{h}}(\psi) \cap \mathcal{H} \neq \emptyset$, and $z \in \Lambda_{\mathrm{log}}$, for any $\gamma_{1}<1-2 \epsilon$,

$$
\begin{equation*}
\limsup _{h \rightarrow 0} \frac{\sup \tilde{\sigma}\left(\left(R_{\delta^{\prime}} T\right)_{N}\right)(q)-1}{h^{\gamma_{1}}} \geq 0 \tag{10.2.7}
\end{equation*}
$$

Let

$$
\left|\sigma\left(\left(R_{\delta^{\prime}} \tilde{T}\right)_{N}\right)(q)\right|=e^{e(q)}
$$

Taking logs and renormalizing in 10.2.7), we have

$$
\frac{2 \operatorname{Im} z}{h} N l_{N}(q)-\frac{2 \operatorname{Im} z}{h} N l_{N}(q)+\log \left|\tilde{\sigma}\left(\left(R_{\delta^{\prime}} T\right)_{N}\right)(q)\right|=e(q)
$$

and hence

$$
\begin{aligned}
-\frac{\operatorname{Im} z}{h} & =l_{N}^{-1}(q)\left[-\left(\frac{\operatorname{Im} z}{h} l_{N}(q)+\frac{1}{2 N} \log \left|\tilde{\sigma}\left(\left(R_{\delta^{\prime}} T\right)_{N}\right)(q)\right|\right)+e(q)\right] \\
& =l_{N}^{-1}(q)\left[-r_{N}(q)+e(q)\right] .
\end{aligned}
$$

where $r_{N}$ as in 10.0.6. Thus, if $\operatorname{MS}_{\mathrm{h}}(\psi) \cap \mathcal{H} \neq \emptyset$, for any $c>0$,

$$
\begin{equation*}
\inf _{\mathcal{H}}-l_{N}^{-1}\left[r_{N}+c h^{\gamma_{1}}\right] \leq-\frac{\operatorname{Im} z}{h} . \tag{10.2.8}
\end{equation*}
$$

Notice that when $\operatorname{Im} z=0$ and $\left|\xi^{\prime}\right|_{g}<1-c$ for some $c>0$,

$$
-r_{N} \sim \min \left(h^{2-2 \alpha}, h \log h^{-1}\right)
$$

This implies that 10.2 .8 provides information about $\operatorname{Im} z$ when $2-2 \alpha<1-2 \epsilon$. However, by Lemma 10.2 .3 , we also need that $\frac{2}{3}(2-2 \alpha)<\epsilon$. Since we have assumed that $\gamma_{1}<$ $\min \left(2 \alpha-\frac{3}{2}, \frac{1}{2}\right)$, we can choose such an $\epsilon$ when $\alpha>11 / 14$.

Summarizing, we have the following lemma

CHAPTER 10. ANALYSIS OF $-\Delta_{\partial \Omega, \delta^{\prime}}$

Lemma 10.2.4. Fix $c>0$ and $\frac{2}{3}(2-2 \alpha)<\epsilon<\min \left(\frac{1}{2}, \alpha-\frac{1}{2}\right)$. Let $\gamma_{1}<1-2 \epsilon$. If

$$
\begin{equation*}
-\frac{\operatorname{Im} z}{h}<\inf _{\left\{\left|\xi^{\prime}\right| g<1-C h \in\right\} \cap \beta_{-N}\left(\mathrm{WF}_{\mathrm{h}}(V)\right)}-l_{N}^{-1}\left[r_{N}+c h^{\gamma_{1}}\right] \tag{10.2.9}
\end{equation*}
$$

where $l_{N}$ and $r_{N}$ are as in (8.0.2) and (10.0.6) respectively, and $\psi$ solves (10.2.1) then

$$
\begin{equation*}
\operatorname{MS}_{\mathrm{h}}(\psi) \subset\left\{\left|\xi^{\prime}\right|_{g} \geq 1-C h^{\epsilon}\right\} \tag{10.2.10}
\end{equation*}
$$

## Elliptic Region

Next, we show that solutions to (10.2.1) cannot concentrate in the elliptic region $\mathcal{E}:=\left\{\left|\xi^{\prime}\right|_{g} \geq\right.$ $\left.1+c h^{\epsilon}\right\}$ for any $\epsilon<\frac{1}{2}$.

Fix $\epsilon<\frac{1}{2}$. Let $\chi_{1} \in S_{\epsilon}$ have $\chi_{1} \equiv 1$ on $\left|\xi^{\prime}\right|_{g} \geq 1+2 C h^{\epsilon}$ and supp $\chi_{1} \subset\left|\xi^{\prime}\right|_{g} \geq E+C h^{\epsilon}$. Also, let $\chi_{2} \in S_{\epsilon}$ have supp $\chi_{2} \subset\left|\xi^{\prime}\right|_{g} \geq 1+3 C h^{\epsilon}$ and $\chi_{2} \equiv 1$ on $\left|\xi^{\prime}\right|_{g} \geq 1+4 C h^{\epsilon}$. Finally, define $X_{i}:=\operatorname{Op}_{\mathrm{h}}\left(\chi_{i}\right) i=1,2$.

Let $\psi$ solve 10.2.1). Then, we have

$$
\left(I-\partial_{\nu} \mathcal{D} \ell V\right) X_{1} \psi=\left[X_{1}, \partial_{\nu} \mathcal{D} \ell V\right] \psi
$$

and by Lemma 6.6.4

$$
\begin{aligned}
\partial_{\nu} \mathcal{D} \ell V X_{1} & =\partial_{\nu} \mathcal{D} \ell_{\Delta} V X_{1}+O_{L^{2} \rightarrow L^{2}}\left(h^{\infty}\right) \\
X_{1} \partial_{\nu} \mathcal{D} \ell V & =X_{1} \partial_{\nu} \mathcal{D} \ell_{\Delta} V+O_{L^{2} \rightarrow L^{2}}\left(h^{\infty}\right)
\end{aligned}
$$

Observe that the ellipticity of $V, \sigma(V) \geq 0, \sigma\left(\partial_{\nu} \mathcal{D} \ell_{\Delta}\right) \leq 0$ and arguments similar to those giving Lemma 10.2 .1 show that microlocally on $\left|\xi^{\prime}\right|_{g} \geq 1+C h^{\epsilon}$,

$$
\left(I-\partial_{\nu} \mathcal{D} \ell_{\Delta} V\right)^{-1} \in \min \left(h^{1-\alpha-\epsilon / 2}, 1\right) \Psi_{\epsilon}^{-1}(\partial \Omega)
$$

Hence

$$
X_{2} \psi=X_{2}\left(I-\partial_{\nu} \mathcal{D} \ell_{\Delta} V\right)^{-1}\left[X_{1}, \partial_{\nu} \mathcal{D} \ell V\right] \psi+O\left(h^{\infty}\right) \psi=O\left(h^{\infty}\right) \psi
$$

which implies

$$
\operatorname{MS}_{\mathrm{h}}(\psi) \cap\left\{\left|\xi^{\prime}\right|_{g} \geq 1+2 h^{\epsilon}\right\}=\emptyset
$$

We also need an elliptic estimate. Let supp $\chi_{3} \subset\left\{\left|\xi^{\prime}\right|_{g} \geq E+c\right\}$ and $X_{3}=\operatorname{Op}_{\mathrm{h}}\left(\chi_{3}\right)$. Using the fact

$$
\mathrm{Op}_{\mathrm{h}}(q):=C\left(1-X_{3}\right)+X_{2}\left(I-\partial_{\nu} \mathcal{D} \ell_{\Delta} V\right) X_{1}
$$

is elliptic and has

$$
\mathrm{Op}_{\mathrm{h}}(q) \psi=C\left(1-X_{3}\right) \psi+O\left(h^{\infty}\right) \psi
$$

we have

$$
\left\|X_{1} \psi\right\|_{L^{2}} \leq C\left\|\left(1-X_{3}\right) \psi\right\|_{L^{2}}+O\left(h^{\infty}\right)
$$

Summarizing,

Lemma 10.2.5. Suppose that $|\operatorname{Im} z| \leq C h \log h^{-1}$ and $\psi=\left.u\right|_{\partial \Omega}$ where $u$ solves (10.0.1) and

$$
\frac{2}{3}(2-2 \alpha)<\epsilon<\min \left(\frac{1}{2}, \alpha-\frac{1}{2}\right) .
$$

Then

$$
\operatorname{MS}_{\mathrm{h}}(\psi) \cap\left\{\left|\xi^{\prime}\right|_{g} \geq 1+h^{\epsilon}\right\}=\emptyset
$$

Moreover, for $\chi \in S$ with $\operatorname{supp} \chi \subset\left\{\left|\xi^{\prime}\right|_{g} \geq 1+c\right\}$,

$$
\begin{equation*}
\left\|X_{1} \psi\right\|_{L^{2}} \leq C\left\|\left(1-\mathrm{Op}_{\mathrm{h}}(\chi)\right) \psi\right\|_{L^{2}}+O\left(h^{\infty}\right) \tag{10.2.11}
\end{equation*}
$$

If, in addition, the hypotheses of Lemma 8.3.2 hold, then for $\epsilon<\min (2 \alpha-1,1 / 2)$,

$$
\operatorname{MS}_{\mathrm{h}}(\psi) \subset\left\{\left(x^{\prime}, \xi^{\prime}\right) \in T^{*} \partial \Omega: \|\left.\xi^{\prime}\right|_{g}-E \mid \leq c h^{\epsilon}\right\}
$$

## Glancing Points

Now, we consider $I-\partial_{\nu} \mathcal{D} \ell V$ microlocally near a glancing point. We use the estimate from Lemma 6.7.4.

Suppose that $\varphi$ solves (10.2.1), then by Lemma 8.3.3, if $\operatorname{Im} z$ satisfies 10.2.9),

$$
\begin{equation*}
\mathrm{MS}_{\mathrm{h}} \varphi \subset\left\{\left|1-\left|\xi^{\prime}\right|_{g}\right| \leq \delta h^{\epsilon}\right\}, \quad \frac{2}{3}(2-2 \alpha)<\epsilon<\min \left(\frac{1}{2}, \alpha-\frac{1}{2}\right) \tag{10.2.12}
\end{equation*}
$$

So, let $\chi \in S_{\epsilon}$ have $\chi \equiv 1$ on $\left\{\left|1-\left|\xi^{\prime}\right|_{g}\right| \leq h^{\epsilon}\right\}$ and $\operatorname{supp} \chi \subset\left\{\left|1-\left|\xi^{\prime}\right|_{g}\right| \leq 2 h^{\epsilon}\right\}$ with $X=\operatorname{Op}_{\mathrm{h}}(\chi)$. Then $X \varphi=\varphi+O\left(h^{\infty}\right) \varphi$. Therefore,

$$
\left(I-\partial_{\nu} \mathcal{D} \ell V\right) X \varphi=O\left(h^{\infty}\right) \varphi
$$

Then, by Lemma 6.7.4

$$
\begin{aligned}
\left\|\partial_{\nu} \mathcal{D} \ell V X \varphi\right\|_{L^{2}} & \leq C_{\Omega} h^{-1+\epsilon / 2}\|V X \varphi\|_{L^{2}} \\
& \leq C_{\Omega, V} h^{-1+\epsilon / 2+\alpha}\|X \varphi\|_{L^{2}} .
\end{aligned}
$$

Since $\alpha>5 / 6$, we can take $2-2 \alpha<\epsilon<\min \left(\alpha-\frac{1}{2}, \frac{1}{2}\right)$, and we obtain $X \varphi=O\left(h^{\infty}\right) \varphi$ and hence $\varphi=O_{L^{2}}\left(h^{\infty}\right)$, a contradiction.

## Bibliography

[1] M. Abramowitz and I. A. Stegun, eds. Handbook of mathematical functions with formulas, graphs, and mathematical tables. Reprint of the 1972 edition. Dover Publications, Inc., New York, 1992, pp. xiv -1046.
[2] S. Albeverio et al. Solvable models in quantum mechanics. Second. With an appendix by Pavel Exner. AMS Chelsea Publishing, Providence, RI, 2005, pp. xiv -488.
[3] S. Albeverio, S. Fassari, and F. Rinaldi. "A remarkable spectral feature of the Schrödinger Hamiltonian of the harmonic oscillator perturbed by an attractive $\delta^{\prime}$ interaction centred at the origin: double degeneracy and level crossing". In: J. Phys. A 46.38 (2013), pp. 385305, 16.
[4] I. Alexandrova. "Semi-classical wavefront set and Fourier integral operators". In: Canad. J. Math. 60.2 (2008), pp. 241-263.
[5] A. A. Aligia and A. M. Lobos. "Mirages and many-body effects in quantum corrals". In: J. Phys.: Condens. Matter 17 (2005).
[6] M. Barr, M. Zaletel, and E. Heller. "Quantum Corral Resonance Widths: Lossy Scattering as Acoustics". In: Nano Letters 10 (2010), pp. 3253-3260.
[7] J. Behrndt and M. Langer. "Elliptic operators, Dirichlet-to-Neumann maps and quasi boundary triples". In: Operator methods for boundary value problems. Vol. 404. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2012, pp. 121-160.
[8] J. Behrndt, M. Langer, and V. Lotoreichik. "Schrödinger operators with $\delta$ and $\delta^{\prime}$ potentials supported on hypersurfaces". In: Ann. Henri Poincaré 14.2 (2013), pp. 385423.
[9] M. Blair. " $L^{q}$ bounds on restrictions of spectral clusters to submanifolds for low regularity metrics". In: Analysis and PDE (To appear).
[10] N. Burq, P. Gérard, and N. Tzvetkov. "Restrictions of the Laplace-Beltrami eigenfunctions to submanifolds". In: Duke Math. J. 138.3 (2007), pp. 445-486.
[11] N. Burq. "Semi-classical estimates for the resolvent in nontrapping geometries". In: Int. Math. Res. Not. 5 (2002), pp. 221-241.
[12] A.-P. Calderón. "Lebesgue spaces of differentiable functions and distributions". In: Proc. Sympos. Pure Math., Vol. IV. American Mathematical Society, Providence, R.I., 1961, pp. 33-49.
[13] F. Cardoso, G. Popov, and G. Vodev. "Asymptotics of the number of resonances in the transmission problem". In: Comm. Partial Differential Equations 26.9-10 (2001), pp. 1811-1859.
[14] F. Cardoso, G. Popov, and G. Vodev. "Distribution of resonances and local energy decay in the transmission problem. II". In: Math. Res. Lett. 6.3-4 (1999), pp. 377-396.
[15] S. N. Chandler-Wilde et al. "Condition number estimates for combined potential boundary integral operators in acoustic scattering". In: J. Integral Equations Appl. 21.2 (2009), pp. 229-279.
[16] J. Chazarain. "Formule de Poisson pour les variétés riemanniennes". In: Invent. Math. 24 (1974), pp. 65-82.
[17] H. Christianson, A. Hassell, and J. A. Toth. "Exterior Mass Estimates and L2Restriction Bounds for Neumann Data Along Hypersurfaces". In: International Mathematics Research Notices (2014), rnt342.
[18] M. Crommie et al. "Quantum corrals". In: Physica D: Nonlinear Phenomena 83.1-3 (1995), pp. 98-108.
[19] J. J. Duistermaat and V. W. Guillemin. "The spectrum of positive elliptic operators and periodic bicharacteristics". In: Invent. Math. 29.1 (1975), pp. 39-79.
[20] J. J. Duistermaat and L. Hörmander. "Fourier integral operators. II". In: Acta Math. 128.3-4 (1972), pp. 183-269.
[21] S. Dyatlov and M. Zworski. Mathematical theory of scattering resonances.
[22] S. Dyatlov. "Asymptotic distribution of quasi-normal modes for Kerr-de Sitter black holes". In: Ann. Henri Poincaré 13.5 (2012), pp. 1101-1166.
[23] S. Dyatlov and C. Guillarmou. "Microlocal limits of plane waves and Eisenstein functions". In: Ann. Sci. Éc. Norm. Supér. (4) 47.2 (2014), pp. 371-448.
[24] S. Dyatlov and M. Zworski. "Quantum ergodicity for restrictions to hypersurfaces". In: Nonlinearity 26.1 (2013), pp. 35-52.
[25] C. L. Epstein. "Pseudodifferential methods for boundary value problems". In: Pseudodifferential operators: partial differential equations and time-frequency analysis. Vol. 52. Fields Inst. Commun. Amer. Math. Soc., Providence, RI, 2007, pp. 171-200.
[26] P. Exner. "Leaky quantum graphs: a review". In: Analysis on graphs and its applications. Vol. 77. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 2008, pp. 523-564.
[27] O. Franek, J. B. Andersen, and G. F. Pedersen. "Diffuse scattering model of indoor wideband propagation". In: IEEE Trans. Antennas Propag. 59.8 (2011), pp. 3006-3012.
[28] M. Gadella, J. Negro, and L. M. Nieto. "Bound states and scattering coefficients of the $-a \delta(x)+b \delta^{\prime}(x)$ potential". In: Phys. Lett. A 373.15 (2009), pp. 1310-1313.
[29] J. Galkowski. "Distribution of Resonances in Scattering by Thin Barriers". In: arXiv preprint, arxiv : 1404.3709 (2014).
[30] J. Galkowski. "Resonances for then barriers on the circle". In: arXiv preprint, arxiv : 1410.0340 (2014).
[31] J. Galkowski and H. Smith. "Restriction bounds for the free resolvent and resonances in lossy scattering". In: Int. Math. Res. Not (2014).
[32] C. Gérard. "Asymptotique des pôles de la matrice de scattering pour deux obstacles strictement convexes". In: Mém. Soc. Math. France (N.S.) 31 (1988), p. 146.
[33] A. Greenleaf and A. Seeger. "Fourier integral operators with fold singularities". In: J. Reine Angew. Math. 455 (1994), pp. 35-56.
[34] V. Guillemin and G. Uhlmann. "Oscillatory integrals with singular symbols". In: Duke Math. J. 48.1 (1981), pp. 251-267.
[35] V. Guillemin and G. Uhlmann. "Oscillatory integrals with singular symbols". In: Duke Math. J. 48.1 (1981), pp. 251-267.
[36] V. Guillemin and S. Sternberg. Geometric asymptotics. Mathematical Surveys, No. 14. American Mathematical Society, Providence, R.I., 1977, xviii -474 pp. (one plate).
[37] X. Han and M. Tacy. "Semiclassical single and double layer potentials: boundedness and sharpness with an Appendix by J. Galkowski". In: arXiv preprint, arxiv : 1403.6576 (2014).
[38] A. Hassell and M. Tacy. "Semiclassical $L^{p}$ estimates of quasimodes on curved hypersurfaces". In: J. Geom. Anal. 22.1 (2012), pp. 74-89.
[39] L. Hörmander. "Fourier integral operators. I". In: Acta Math. 127.1-2 (1971), pp. 79183.
[40] L. Hörmander. The analysis of linear partial differential operators. I. Classics in Mathematics. Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)]. Springer-Verlag, Berlin, 2003, pp. x-440.
[41] L. Hörmander. The analysis of linear partial differential operators. III. Classics in Mathematics. Pseudo-differential operators, Reprint of the 1994 edition. Springer, Berlin, 2007, pp. viii -525 .
[42] L. Hörmander. The analysis of linear partial differential operators. IV. Classics in Mathematics. Fourier integral operators, Reprint of the 1994 edition. Springer-Verlag, Berlin, 2009, pp. viii -352 .
[43] D. Jakobson, Y. Safarov, and A. Strohmaier. "The semiclassical theory of discontinuous systems and ray-splitting billiards". In: arXiv preprint, arxiv : 1301.6783 (2013).
[44] P. Kurasov. "Distribution theory for discontinuous test functions and differential operators with generalized coefficients". In: J. Math. Anal. Appl. 201.1 (1996), pp. 297323.
[45] P. D. Lax and R. S. Phillips. "A logarithmic bound on the location of the poles of the scattering matrix". In: Arch. Rational Mech. Anal. 40 (1971), pp. 268-280.
[46] P. D. Lax and R. S. Phillips. Scattering theory. Second. Vol. 26. Pure and Applied Mathematics. With appendices by Cathleen S. Morawetz and Georg Schmidt. Academic Press, Inc., Boston, MA, 1989, pp. xii -309.
[47] R. Melrose and M. Taylor. Boundary problems for wave equations with grazing and gliding rays.
[48] R. B. Melrose. "Equivalence of glancing hypersurfaces". In: Invent. Math. 37.3 (1976), pp. 165-191.
[49] R. B. Melrose and G. A. Uhlmann. "Lagrangian intersection and the Cauchy problem". In: Comm. Pure Appl. Math. 32.4 (1979), pp. 483-519.
[50] R. Melrose. "Scattering theory and the trace of the wave group". In: J. Funct. Anal. 45.1 (1982), pp. 29-40.
[51] R. B. Melrose. "Local Fourier-Airy integral operators". In: Duke Math. J. 42.4 (1975), pp. 583-604.
[52] R. B. Melrose. "Microlocal parametrices for diffractive boundary value problems". In: Duke Math. J. 42.4 (1975), pp. 605-635.
[53] L. Boutet de Monvel. "Comportement d'un opérateur pseudo-différentiel sur une variété à bord. I. La propriété de transmission". In: J. Analyse Math. 17 (1966), pp. 241-253.
[54] J. M. Munoz-Castaneda and J. M. Guilarte. " $\delta--\delta^{\prime}$ generalized Robin boundary conditions and quantum vacuum fluctuations". In: arXiv preprint, arxiv : 1407.4212 (2014).
[55] S. Nonnenmacher and M. Zworski. "Decay of correlations for normally hyperbolic trapping". In: to appear in Invent. Math. (2013).
[56] F. W. J. Olver et al., eds. NIST handbook of mathematical functions. With 1 CD-ROM (Windows, Macintosh and UNIX). U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010, pp. xvi -951.
[57] V. M. Petkov and L. N. Stoyanov. Geometry of reflecting rays and inverse spectral problems. Pure and Applied Mathematics (New York). John Wiley \& Sons, Ltd., Chichester, 1992, pp. vi -313.
[58] G. Popov and G. Vodev. "Distribution of the resonances and local energy decay in the transmission problem". In: Asymptot. Anal. 19.3-4 (1999), pp. 253-265.
[59] M. Reed and B. Simon. Methods of modern mathematical physics. I. Second. Functional analysis. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1980, pp. xv -400.
[60] M. Reed and B. Simon. Methods of modern mathematical physics. IV. Analysis of operators. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978, pp. xv -396.
[61] W. C. Sabine. Collected papers on acoustics. New York, NY: Dover Publications, 1964.
[62] Y. Safarov and D. Vassiliev. The asymptotic distribution of eigenvalues of partial differential operators. Vol. 155. Translations of Mathematical Monographs. Translated from the Russian manuscript by the authors. American Mathematical Society, Providence, RI, 1997, pp. xiv -354 .
[63] Y. G. Safarov. "On the second term of the spectral asymptotics of the transmission problem". In: Acta Appl. Math. 10.2 (1987), pp. 101-130.
[64] P. Šeba. "Some remarks on the $\delta^{\prime}$-interaction in one dimension". In: Rep. Math. Phys. 24.1 (1986), pp. 111-120.
[65] J. Sjöstrand and M. Zworski. "Asymptotic distribution of resonances for convex obstacles". In: Acta Math. 183.2 (1999), pp. 191-253.
[66] J. Sjöstrand and M. Zworski. "Complex scaling and the distribution of scattering poles". In: J. Amer. Math. Soc. 4.4 (1991), pp. 729-769.
[67] J. Sjöstrand and M. Zworski. "Lower bounds on the number of scattering poles". In: Comm. Partial Differential Equations 18.5-6 (1993), pp. 847-857.
[68] H. F. Smith. "Spectral cluster estimates for $C^{1,1}$ metrics". In: Amer. J. Math. 128.5 (2006), pp. 1069-1103.
[69] P. Stefanov and G. Vodev. "Distribution of resonances for the Neumann problem in linear elasticity outside a strictly convex body". In: Duke Math. J. 78.3 (1995), pp. 677714.
[70] P. Stefanov. "Approximating resonances with the complex absorbing potential method". In: Comm. Partial Differential Equations 30.10-12 (2005), pp. 1843-1862.
[71] E. M. Stein. Singular integrals and differentiability properties of functions. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970, pp. xiv -290 .
[72] M Tacy. "Semiclassical $L^{2}$ estimates for restrictions of the quantisation of normal velocity to interior hypersurfaces". In: arXiv preprint, arxiv : 1403.6575 (2014).
[73] M. Tacy. "Semiclassical $L^{p}$ estimates of quasimodes on submanifolds". In: Comm. Partial Differential Equations 35.8 (2010), pp. 1538-1562.
[74] S.-H. Tang and M. Zworski. "Resonance expansions of scattered waves". In: Comm. Pure Appl. Math. 53.10 (2000), pp. 1305-1334.
[75] D. Tataru. "On the regularity of boundary traces for the wave equation". In: Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 26.1 (1998), pp. 185-206.
[76] M. E. Taylor. Partial differential equations II. Qualitative studies of linear equations. Second. Vol. 116. Applied Mathematical Sciences. Springer, New York, 2011, pp. xxii $-614$.
[77] M. E. Taylor. Pseudodifferential operators. Vol. 34. Princeton Mathematical Series. Princeton University Press, Princeton, N.J., 1981, pp. xi -452.
[78] B. R. Vaĭnberg. Asymptotic methods in equations of mathematical physics. Translated from the Russian by E. Primrose. Gordon \& Breach Science Publishers, New York, 1989, pp. viii -498.
[79] Y. C. de Verdière, V. Guillemin, and D. Jerison. In: Singularities of the wave trace near cluster points of the length spectrum (2010).
[80] G. Vodev. "Transmission eigenvalues for strictly concave domains". In: arXiv preprint, arxiv : 1501.00797 (2015).
[81] G. Vodev. "Sharp bounds on the number of scattering poles for perturbations of the Laplacian". In: Comm. Math. Phys. 146.1 (1992), pp. 205-216.
[82] G. Vodev. "Sharp bounds on the number of scattering poles in even-dimensional spaces". In: Duke Math. J. 74.1 (1994), pp. 1-17.
[83] G. Vodev. "Sharp bounds on the number of scattering poles in the two-dimensional case". In: Math. Nachr. 170 (1994), pp. 287-297.
[84] G. N. Watson. A treatise on the theory of Bessel functions. Cambridge Mathematical Library. Reprint of the second (1944) edition. Cambridge University Press, Cambridge, 1995, pp. viii -804.
[85] M. Zaletel. "The Sabine law and a trace formula for lossy billiards". In: Unpublished note (2010).
[86] M. Zworski. "Poisson formula for resonances in even dimensions". In: Asian J. Math. 2.3 (1998), pp. 609-617.
[87] M. Zworski. Semiclassical analysis. Vol. 138. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2012, pp. xii -431.

## Appendix A

## Notation

## A. 1 Basic Notation

- Basic Definitions
$\mathbb{Z}:=$ integers $\quad \mathbb{N}:=$ nonnegative integers.
$\mathbb{R}:=$ the real numbers
$\mathbb{R}^{d}:=d$-dimensional euclidean space
$\mathbb{T}^{d}:=d$-dimensional flat torus $\mathbb{R}^{d} / \mathbb{Z}^{d}$.
$\mathbb{C}:=$ the complex plane and $\mathbb{C}^{d}:=d$-dimensional complex space.
$x, y$ denote typical points in $\mathbb{R}^{d}$ with $x=\left(x_{1}, \ldots x_{d}\right)$ and $y=\left(y_{1}, \ldots y_{1}\right)$
$S^{d-1}$ is the $d-1$ dimensional sphere
$\overline{\mathbb{R}}^{d}=\mathbb{R}^{d} \sqcup S^{d-1}$ is the radial compactification of $\mathbb{R}^{d}$ by the map $x \mapsto t x, t \neq 0$.
$\langle x, y\rangle:=\sum_{i=1}^{d} x_{i} \bar{y}_{i}$ is the inner product on $\mathbb{C}^{d}$
$\langle x\rangle:=\left(2+|x|^{2}\right)^{1 / 2}$
$\sigma(z, w):=\langle J z, w\rangle=$ the symplectic inner product on $\mathbb{R}^{2 d}$. (See below for the definition of $J$.)
$\kappa: \mathbb{R}^{2 d} \rightarrow \mathbb{R}^{2 d}$ usually denotes a smooth symplectomorphism.
- Sets
$\# S=$ cardinality of the set $S$
$U, V$ usually denote open subsets
$U \Subset V$ means that $\bar{U}$ is a compact subset of $V$.
- Matrices
$A^{t}=$ the transpose of a matrix $A$
$\operatorname{sgn} Q=$ the signature of the symmetric matrix $Q$
$I$ denotes the identity matrix and the identity mapping det $=$ determinant and $\operatorname{tr}=$ the trace
$J=\left(\begin{array}{cc}0 & I \\ -I & 0\end{array}\right)$ is the standard symplectic matrix.


## A. 2 Calculus Notations

- supp denotes the support of a function
- Partial derivatives

$$
\partial_{x_{j}}:=\frac{\partial}{\partial_{x_{j}}}, \quad D_{x_{j}}:=\frac{1}{i} \partial_{x_{j}} .
$$

- Multiindeces: A multiindex $\alpha=\left(\alpha_{1}, \ldots \alpha_{d}\right) \in \mathbb{N}^{d}$ has

$$
|\alpha|:=\alpha_{1}+\cdots+\alpha_{d} .
$$

For $x \in \mathbb{R}^{d}$, we define

$$
x^{\alpha}:=x_{1}^{\alpha_{1}} \ldots x_{d}^{\alpha_{d}}
$$

We also define

$$
\partial^{\alpha}:=\partial_{x_{1}}^{\alpha_{1}} \ldots \partial_{x_{d}}^{\alpha_{d}}
$$

and

$$
D^{\alpha}=D_{x_{1}}^{\alpha_{1}} \ldots D_{x_{d}}^{\alpha_{d}} .
$$

- If $\varphi: \mathbb{R}^{d} \rightarrow \mathbb{R}$, we write

$$
\nabla \varphi:=\partial \varphi:=\left(\varphi_{x_{1}}, \ldots, \varphi_{x_{d}}\right)=\text { the gradient }
$$

where $\varphi_{x_{i}}=\partial_{x_{i}} \varphi$. We write

$$
\partial^{2} \varphi:=\left(\begin{array}{lll}
\varphi_{x_{1} x_{1}} & \ldots & \varphi_{x_{1} x_{d}} \\
& \ddots & \\
\varphi_{x_{d} x_{1}} & \ldots & \varphi_{x_{d} x_{d}}
\end{array}\right)=\text { the Hessian. }
$$

Also, $D \varphi:=\frac{1}{i} \partial \varphi$.

- If $\varphi$ depends on $(x, y) \in \mathbb{R}^{d_{1}} \times \mathbb{R}^{d_{2}}$, then

$$
\varphi_{x}^{\prime \prime}:=\partial_{x}^{2} \varphi:=\left(\begin{array}{ccc}
\varphi_{x_{1} x_{1}} & \ldots & \varphi_{x_{1} x_{d_{1}}} \\
& \ddots & \\
\varphi_{x_{d_{1}} x_{1}} & \ldots & \varphi_{x_{d_{1}} x_{d_{1}}}
\end{array}\right), \quad \varphi_{y}^{\prime \prime}:=\partial_{y}^{2} \varphi:=\left(\begin{array}{ccc}
\varphi_{y_{1} y_{1}} & \ldots & \varphi_{y_{1} y_{d_{2}}} \\
& \ddots & \\
\varphi_{y_{d_{2}} y_{1}} & \ldots & \varphi_{y_{d_{2}} y_{d_{2}}}
\end{array}\right)
$$

and

$$
\varphi_{x y}^{\prime \prime}:=\partial_{x y}^{2} \varphi:=\left(\begin{array}{lll}
\varphi_{x_{1} y_{1}} & \cdots & \varphi_{x_{1} y_{d_{2}}} \\
& \ddots & \\
\varphi_{x_{d_{1}} y_{1}} & \cdots & \varphi_{x_{d_{1}} y_{d_{2}}}
\end{array}\right) .
$$

- The Poisson bracket: If $f, g: \mathbb{R}^{d} \rightarrow \mathbb{R}$ are continuously differentiable functions

$$
\{f, g\}:=\left\langle\partial_{\xi} f, \partial_{x} g\right\rangle-\left\langle\partial_{x} f, \partial_{\xi} g\right\rangle
$$

## A. 3 Function Spaces

- A subscript ' $c$ ' or comp on a space of functions means those with compact support
- $C^{k}(U)$ denotes the space of $k$ times continuously differentiable functions on $U$ with $C^{\infty}(U)$ denoting smooth function on $U$. The norm on $C^{k}(U)$ is given by

$$
\|u\|_{C^{k}(U)}:=\sum_{m=0}^{k} \sup _{\substack{x \in U \\|\alpha| \leq k}}\left|\partial^{\alpha} u(x)\right|
$$

- The Schwartz space is

$$
\mathcal{S}=\mathcal{S}\left(\mathbb{R}^{d}\right):=\left\{\varphi \in C^{\infty}\left(\mathbb{R}^{d}\right): \sup _{\mathbb{R}^{d}}\left|x^{\alpha} \partial^{\beta} \varphi\right|<\infty \text { for all } \alpha, \beta \in \mathbb{N}^{d}\right\}
$$

We say

$$
\varphi_{j} \rightarrow \varphi \quad \text { in } \mathcal{S}
$$

if

$$
\sup _{\mathbb{R}^{d}}\left|x^{\alpha} D^{\beta}\left(\varphi_{j}-\varphi\right)\right| \rightarrow 0 \quad \text { for all } \alpha, \beta \in \mathbb{N}
$$

- The space of tempered distributions $\mathcal{S}^{\prime}=\mathcal{S}^{\prime}\left(\mathbb{R}^{d}\right)$ is the dual of $\mathcal{S}\left(\mathbb{R}^{d}\right)$.
- The space of distributions $\mathcal{D}^{\prime}\left(\mathbb{R}^{d}\right)$ is the dual of $C_{c}^{\infty}\left(\mathbb{R}^{d}\right)$.


## A. 4 Operators

$A^{*}:=$ the adjoint of the operator $A$
$[A, B]:=A B-B A$ is the commutator of $A$ and $B$
$\sigma(A):=$ the symbol of the pseudodifferential operator $A$
$\sigma_{1}(A):=$ the sub-principal symbol of the pseudodifferential operator $A$
if $A: X \rightarrow Y$ is a bounded linear operator, we define the operator norm

$$
\|A\|_{X \rightarrow Y}:=\sup \left\{\|A u\|_{Y}:\|u\|_{X} \leq 1\right\} .
$$

## A. 5 Estimates

- We use ' $C$ ' and ' $c$ ' to denote a general positive constant appearing in inequalities. The constants $C$ and $c$ will generally vary from line to line.
- Order Estimates

We write

$$
f=O_{X}\left(h^{\infty}\right) \quad \text { as } h \rightarrow 0
$$

if there exists $h_{0}>0$ and for each positive integer $N$ a constant $C_{N}$ such that

$$
\|f\|_{X} \leq C_{N} h^{N} \quad \text { for all } 0<h<h_{0} .
$$

If we do not include a space $X$, then pointwise bounds are implied.
We write

$$
f=O_{X}(r(h)) \quad \text { as } h \rightarrow 0
$$

if there exists $h_{0}>0$ such that

$$
\|f\|_{X} \leq C r(h) \quad \text { for } 0<h<h_{0} .
$$

Again, if no space $X$ is specified, them pointwise bounds are to be understood. We write

$$
f=o_{X}(r(h)) \quad \text { as } h \rightarrow 0
$$

if

$$
\lim _{h \rightarrow 0^{+}} \frac{\|f\|_{X}}{r(h)}=0
$$

If no space $X$ is specified, then this limit is taken pointwise.

## A. 6 Symbol Classes

$$
\begin{gathered}
S_{\delta}(m):=\left\{a \in C^{\infty}:\left|\partial^{\alpha} a\right| \leq C_{\alpha} h^{-\delta|\alpha|} m \text { for all multiindeces } \alpha\right\} \\
S_{\delta}^{k}:=\left\{a \in C^{\infty}:\left|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} a\right| \leq C_{\alpha \beta} h^{-\delta(|\alpha|+|\beta|)}\langle\xi\rangle^{k-|\beta|} \text { for all multiindeces } \alpha\right\}
\end{gathered}
$$

We also have the classical symbol classes $S_{\delta, c l}^{m}\left(\mathbb{R}^{d} \times \mathbb{R}^{N}\right) \subset S_{\delta}^{m}\left(\mathbb{R}^{d} \times \mathbb{R}^{N}\right)$ such that $a \in$ $S_{\delta, c l}^{m}\left(\mathbb{R}^{d} \times \mathbb{R}^{N}\right)$ if there exists $M>0$ and $a_{j}(x, \theta) \in S_{\delta}^{m}\left(\mathbb{R}^{d} \times \mathbb{R}^{N}\right)$ homogeneous of degree $j$ for $|\theta| \geq M$ such that for all positive integers $N$,

$$
\left|\partial_{x}^{\alpha} \partial_{\theta}^{\beta}\left(a-\sum_{j=-N+1}^{m} a_{j}\right)\right| \leq C_{\alpha \beta N} h^{-\delta(|\alpha|+|\beta|)}\langle\theta\rangle^{m-N} .
$$

## A. 7 Semiclassical and Microlocal Operators

For $a \in S_{\delta}^{m}\left(\mathbb{R}^{d} \times \mathbb{R}^{d}\right)$, the operator $\mathrm{Op}_{\mathrm{h}} t(a)$ is the operator acting $C_{c}^{\infty} \rightarrow \mathcal{D}^{\prime}$ whose Schwartz kernel is given by

$$
(2 \pi h)^{-d} \int a(t x+(1-t) y, \xi) e^{\frac{i}{h}\langle x-y, \xi\rangle} d \xi
$$

The set of semiclassical pseudodifferential operators of order $m$ and class $\delta$ is

$$
\Psi_{\delta}^{m}:=\left\{\mathrm{Op}_{\mathrm{h}, 1 / 2}(a): a \in S_{\delta}^{m}\right\}
$$

The set of Kohn-Nirenberg pseudodifferential operators is

$$
\Psi_{\mathrm{hom}}^{m}:=\left\{\mathrm{Op}_{1,1 / 2}(a): a \in S_{\}, c l}^{m} .\right.
$$

$I_{\delta}^{m}(M ; \Lambda)$ is the set of semiclassical Lagrangian distributions of order $m$ and class $\delta$ associated to the Lagrangian $\Lambda$
$I_{\delta}^{m}\left(M_{1} \times M_{2}, C\right)$ is the set of semiclassical Fourier integral operators of order $m$ and class $\delta$ associated to the canonical relation $C$.


[^0]:    ${ }^{1}$ For $d$ even we do not expect to have an expansion of the for 1.0 .4 because of the failure of the strong Huygen's principle.

