AN INTRODUCTION TO MICROLOCAL COMPLEX DEFORMATIONS

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ABSTRACT. In this expository article we relate the presentation of weighted estimates in [Ma02] to the Bergman kernel approach of [Sj96]. It is meant as an introduction to the Helffer–Sjöstrand theory [HeSj86] in the simplest setting and to its adaptations to compact manifolds [Sj96], [GaZw].

1. INTRODUCTION

Suppose that P is a semiclassical differential operator (or a pseudodifferential operator, see (2.5)), for instance,

$$P = -h^2\Delta + V. \tag{1.1}$$

Conjugation by exponential weights has a very long tradition going back to the origins of Carleman and Agmon–Lithner estimates:

$$P_\varphi := e^{\varphi(x)/h} P e^{-\varphi(x)/h}, \quad \varphi \in C^\infty(\mathbb{R}^n; \mathbb{R}), \tag{1.2}$$

which in the case of (1.1) gives (with $D_x = -i\partial_x$, and $v^2 := v_1^2 + \cdots + v_n^2$, $v \in \mathbb{C}^n$)

$$\begin{aligned} P_\varphi &= (hD_x + i\nabla\varphi)^2 + V(x) \\ &= -h^2\Delta + \nabla\varphi \cdot h\nabla + V - |\nabla\varphi|^2 + h\Delta\varphi. \end{aligned}$$

Roughly speaking, exploiting the sign of $V - |\nabla\varphi|^2$ leads to exponential decay (tunneling) estimates for solutions of $Pu = 0$ – see for instance [Zw12, §7.1] and references given there. For exponential lower bounds (quanitative unique continuation, from the mathematical point of view), one exploits positivity properties of $[P_\varphi^*, P_\varphi]$ of suitably chosen φ using the identity (with L^2 norms and $u \in C_{\rm c}^\infty$)

$$\|P_\varphi u\|^2 = \|P_\varphi^* u\|^2 + \langle [P_\varphi^*, P_\varphi]u, u\rangle \geq \langle [P_\varphi^*, P_\varphi]u, u\rangle, \tag{1.3}$$

see for instance [Zw12, §7.2].

On the other hand conjugation (1.2) with $\varphi(x)$ replaced by $i\varphi(x)$ gives the simplest case of Egorov's Theorem (see for instance [Zw12, Theorem 11.1]):

$$P_{i\varphi} = (hD_x - \nabla\varphi(x))^2 + V(x),$$

which corresponds to the pull back of the symbol by the canonical transformation $(x, \xi) \mapsto (x, \xi - \nabla\varphi(x))$ associated to the operator $u(x) \mapsto e^{-\frac{i}{h}\varphi(x)}u(x)$.

When φ is real, we have implicitly used analyticity of $\xi \to \xi^2$ to obtain (1.3). If we formally conjugate (1.1) with $V(x) = x^2$ by $e^{\varphi(hD)/h}$ we obtain

$$e^{\varphi(hD)/h} P e^{-\varphi(hD)/h} = -h^2\Delta + V(x - i\nabla\varphi(hD)),$$

where we used the analyticity of $V(x) = x^2$. In general, we encounter problems akin to flowing the heat equation backwards which again requires analyticy.

In many problems it is advantageous to use $\varphi = G(x, hD_x)$ but, as the discussion above shows, the use of such weights requires analyticity assumptions (unless we use weights of moderate growth in h and ξ – see [Zw12, §8.2] for a textbook discussion and Faure–Sjöstrand [FaSj11], Dyatlov–Zworski [DyZw16] for recent applications and references).

The use of strong microlocal weights (φ in some sense equal to $G(x, hD)$) has been raised to the level of high art by Sjöstrand and his collaborators – see for instance Hitrik–Sjöstrand [HiSj15] and references given there. Here we would like to concentrate on the approach motivated by scattering resonances and introduced by Helffer–Sjöstrand [HeSj86].

The goal then is to justify the statement

$$\begin{aligned} e^{-G(x,hD)/h} P(x, hD) e^{G(x,hD)/h} &\sim P(x + i\nabla_\xi G(x, hD), hD_x - i\nabla_x G(x, hD)) \\ &= P(x, hD_x) - iH_P G(x, hD) + \mathcal{O}(\|G\|^2_{C^2}), \end{aligned} \tag{1.4}$$

and then to exploit the possible gain of ellipticity for the right hand side. In particular, the property $H_P G(x, \xi) > 0$ can be used to great advantage. Here

$$H_P := \sum_{j=1}^n \partial_{\xi_j} P(x, \xi)\partial_{x_j} - \partial_{x_j} P(x, \xi)\partial_{\xi_j},$$

is the Hamilton vector field of the symbol of P and $\mathcal{O}(\|G\|^2_{C^2})$ means a norm bound between suitable spaces, for instance $L^2 \to L^2$ if P is order 0 and G of order 1. The condition $H_P G > 0$ and its weaker forms are called the *escape function* property or the *positive commutator* property.

One tool for justifying (1.4) is the FBI transform[†] – see (2.1) and [HiSj15], [Ma02], [Zw12, Chapter 13] for three introductions. Roughly speaking it turns the action of the operator P to multiplication by its symbol (say, $\xi^2 + V(x)$, in the case of (1.1)). When weights are introduced, this action turns into multiplication by the "deformed symbol". That is, roughly speaking, the symbol of the operator on right hand side of (1.4).

[†]It is named after Fourier–Bros–Iagolnitzer and this name is used for its generalizations in microlocal analysis. In specific cases, and in other fields it is called Bargmann, Segal, Gabor, and wave packet transform.

Here we will present the simplest case of small (in C^2) compactly supported weights G. A very clear presentation (without the smallness assumption) following Nakamura [Na95] is provided in Martinez [Ma02, §3.5] but our goal is to make *simple things complicated* by explaining the theory in the way which adapts to the case of stronger (non-compactly supported) weights used in [HeSj86] and to the case of compactly supported weights on compact manifolds of [Sj96]. Our motivation comes from the study of viscosity limits for 0th order (analytic) pseudodifferential operators [GaZw]. It partly justifies claims made in the physics literature, see for instance [RGV01]‡.

A properly interpreted version of (1.4) is given in Theorem 2 which comes in this form from [Na95], [Ma02]. The proof however follows the strategy of [Sj96] and is based on the study of orthogonal projections onto weighted spaces of (essentially) holomorphic functions. Theorem 3 presents a more geometric version more directly in the spirit of [Sj96].

Our exposition of this material is structured as follows:

- In §2 we review the properties of the FBI (Bargmann/Segal/Gabor/wave packet) transform and the structure of pseudodifferential operator on the FBI transform side. No (non-quadratic) weights enter here but the simple geometric structure discussed in §2.2 provides a guide for more complicated constructions.
- §3 is devoted to the description of the projector onto the image of the FBI transform, orthogonal with respect to the norm on $L^2(T^*\mathbb{R}^n, e^{-\varphi(x,\xi)/h}dxd\xi)$. That follows the approach of [Sj96] which in turn is inspired by [BoSj76], [BoGu81] and [HeSj86]. The description of the action of *analytic* pseudodifferential operators on those spaces is then given in Theorem 2 in §4.
- §5 reviews some aspects of the analytic machinery of Melin–Sjöstrand [MeSj74] which is needed for the more geometric approach to the justification of (1.4) in §§6,7.
- A more geometric version, following the spirit of [HeSj86] and [Sj96], is presented in §6. Instead of putting in a weight, the phase space $T^*\mathbb{R}^n$ is deformed to $\Lambda := \{(x + iG_x(x,\xi), \xi - iG_\xi(x,\xi)) : (x,\xi) \in T^*\mathbb{R}^n\}$ (note the analogy with the right hand side (1.4); Λ is always Lagrangian with respect to $\operatorname{Im} d\zeta \wedge dz$ on $T^*\mathbb{C}^n$ and symplectic for $\operatorname{Re} d\zeta \wedge dz$ for G sufficiently small). That corresponds to continuing the FBI transform analytically and then restricting it to Λ. The action of an *analytic* pseudodifferential operator with symbol p on that space

‡Specifically, the claim that "The aim of this paper is to present what we believe to be the asymptotic limit of inertial modes in a spherical shell when viscosity tends to zero." These viscosity limits are essentially the *resonances* of zero order operators and hence it is natural to use the methods of [HeSj86]. Except in simplest cases the methods based on spacial deformations in the spirit of complex scaling – see [DyZw19, §4.5, §4.7] and references given there – are not sufficient.

is (in some sense) close to multiplication by $p|_\Lambda$ – see Theorem 3. That is again achieved by constructing an appropriate orthogonal projector.
- Finally, in §8 we discuss the equivalence of the two approaches by showing that each deformation Λ corresponds to putting in a weight without a deformation – see Theorem 4.

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2. The FBI transform

We define the usual FBI transform:

$$Tu(x,\xi) := ch^{-\frac{3n}{4}} \int_{\mathbb{R}^n} e^{\frac{i}{h}(\langle x-y,\xi\rangle + \frac{i}{2}(x-y)^2)} u(y)dy. \tag{2.1}$$

Then $T : L^2(\mathbb{R}^n) \to L^2(T^*\mathbb{R}^n)$ is an isometry as is easily checked using Plancherel's formula – see for instance [Zw12, Step 2 of the proof Theorem 13.7]. We then notice that

$$\begin{gathered} T(L^2(\mathbb{R}^n)) \subset \mathscr{H} := \{u \in L^2(T^*\mathbb{R}^n) : Z_j u = 0,\ j = 1, \cdots n\}, \\ \zeta_j(x,\xi,x^*,\xi^*) := x_j^* - \xi_j - i\xi_j^*, \\ Z_j := \zeta_j(x,\xi,hD_x,hD_\xi) = e^{-\xi^2/2h} 2hD_{\bar{z}_j} e^{\xi^2/2h}, \\ z_j = x_j - i\xi_j, \qquad D_{\bar{z}_j} = \tfrac{1}{2i}(\partial_{x_j} - i\partial_{\xi_j}). \end{gathered} \tag{2.2}$$

In fact, the range of T is exactly given by $\mathscr{H}$:

Proposition 1. *The orthogonal projector $\Pi_0 : L^2(T^*\mathbb{R}^n) \to \mathscr{H}$ is given by $\Pi_0 = TT^*$ and*

$$\begin{gathered} \Pi_0 u(\alpha) = h^{-n} c_0 \int_{T^*\mathbb{R}^n} e^{\frac{i}{h}\psi_0(\alpha,\beta)} u(\beta)d\beta, \quad \alpha = (x,\xi),\ \beta = (x',\xi'), \\ \psi_0(\alpha,\beta) = \tfrac{1}{2}(x\xi - x'\xi') + \tfrac{1}{2}(x\xi' - \xi x') + \tfrac{i}{4}(x-x')^2 + \tfrac{i}{4}(\xi-\xi')^2. \end{gathered} \tag{2.3}$$

In particular, $T(L^2(\mathbb{R}^n)) = \mathscr{H}$.

For the proof see [Zw12, Theorem 13.7] or [Ma02, Exercise 3.6.2].

Remark: Note that, using the holomorphic notation $z = x - i\xi$, $w = x' - i\xi'$,

$$\psi_0 = i\Big[\Phi_0(z) + \tfrac{1}{2}(z - \bar{w})^2 + \Phi_0(w)\Big], \qquad \Phi_0(z) := \tfrac{1}{2}|\operatorname{Im} z|^2. \tag{2.4}$$

2.1. **Pseudodifferential operators on the FBI side.** Suppose $P = p(x, hD)$,

$$Pu = p(x, hD)u := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} p(x, \xi)\, e^{\frac{i}{h}\langle x-y,\xi\rangle} u(y) dy d\xi, \qquad |\partial^\alpha_{x,\xi} p(x,\xi)| \le C_\alpha. \tag{2.5}$$

is a pseudodifferential operator (with the symbol, $p \in S(1)$ in the terminology of [Zw12, Chapter 4]). We want to consider

$$\mathscr{P} := TPT^* : \mathscr{H} \to \mathscr{H}. \tag{2.6}$$

There are many ways to think about this operator – see [Zw12, §13.4] for Sjöstrand's pseudodifferential approach. Here we look at it in the spirit of [Sj96].

Lemma 1. *The operator* (2.6) *is given by*

$$\mathscr{P}u = \int_{T^*\mathbb{R}^n} K_P(\alpha, \beta) u(\beta) d\beta$$

with

$$\begin{gathered} K_P(\alpha, \beta) = c_0 h^{-n} e^{\frac{i}{h}\psi_0(\alpha,\beta)} a(\alpha, \beta) + \mathcal{O}(h^\infty \langle \alpha - \beta\rangle^{-\infty}), \\ a(\alpha, \beta) \sim \sum_{j=0}^\infty h^j a_j(\alpha, \beta), \quad |\partial^\gamma_\alpha \partial^{\gamma'}_\beta a_j(\alpha,\beta)| \le C_{\gamma\gamma' j}, \quad a_0(\alpha,\alpha) = p(\alpha). \end{gathered} \tag{2.7}$$

Proof. We calculate the integral kernel using, again, the completion of squares and integration in y':

$$\begin{aligned} K_P &= \frac{e^{\frac{i}{h}(x\xi - x'\xi')}}{(2\pi h)^n} \int_{\mathbb{R}^{3n}} e^{\frac{i}{h}(y(\eta-\xi) - y'(\eta-\xi') + \frac{i}{2}(x-y)^2 + \frac{i}{2}(x'-y')^2)} p(y, \eta)\, dy' dy d\eta \\ &= \frac{e^{\frac{i}{h}x\xi}}{(2\pi h)^{\frac{n}{2}}} \int_{\mathbb{R}^{3n}} e^{\frac{i}{h}(y(\eta-\xi) - x'\eta + \frac{i}{2}(x-y)^2 + \frac{i}{2}(\xi'-\eta)^2)} p(y, \eta)\, dy d\eta. \end{aligned}$$

We note that the integral is now absolutely convergent and, denoting the phase by Φ, $\operatorname{Im}\Phi \ge 0$. The stationary points of Φ are given by solving (or completing squares)

$$\partial_\eta \Phi = y - x' + i(\eta - \xi') = 0, \quad \partial_y \Phi = \eta - \xi + i(y - x) = 0,$$

$$\Phi'' = \begin{bmatrix} iI_{\mathbb{R}^n} & I_{\mathbb{R}^n} \\ I_{\mathbb{R}^n} & iI_{\mathbb{R}^n} \end{bmatrix} \text{ is non-degenerate.}$$

The solutions are

$$y = y_c := \tfrac{1}{2}(x + x') + \tfrac{i}{2}(\xi' - \xi), \quad \eta = \eta_c := \tfrac{1}{2}(\xi + \xi') + \tfrac{i}{2}(x - x').$$

We note that

$$|\partial_\eta \Phi|^2 + |\partial_y \Phi|^2 \ge \tfrac{1}{2}|x - x'|^2 + \tfrac{1}{2}|\xi - \xi'|^2.$$

Hence non-stationary phase estimate shows that if we restrict the integration to $|y - y_c|^2 + |x - x_c|^2 < 1$ (using a smooth cut-off function), the remaining term is estimated by $\mathcal{O}(h^\infty\langle \alpha - \beta\rangle)$, $\alpha = (x, \xi)$, $\beta = (x', \xi')$.

For the integral over the set close to the critical points we apply the complex stationary phase method [MeSj74, Theorem 2.3, p.148] to obtain (2.7)

$$\begin{aligned} a &= a_0 + ha_1 + \cdots \\ &= \widetilde{p}\left(\tfrac{1}{2}(x+x') + \tfrac{i}{2}(\xi'-\xi), \tfrac{1}{2}(\xi+\xi') + \tfrac{i}{2}(x-x')\right) + \mathcal{O}(h), \end{aligned} \tag{2.8}$$

where $\widetilde{p}$ is an almost analytic extension of p. We note that $a_0(\alpha,\alpha) = p(\alpha)$. □

Also, just as for the kernel of Π_0, $K(\alpha,\beta)$ has to satisfy the equations

$$\zeta_j(\alpha, hD_\alpha)K(\alpha,\beta) = 0, \quad \widetilde{\zeta}_j(\beta, hD_\beta)K(\alpha,\beta) = 0, \quad \widetilde{\zeta}_j(\beta,\beta^*) := \overline{\zeta_j(\beta,-\beta^*)}. \tag{2.9}$$

The last condition follows from the fact that

$$\begin{aligned} 0 &= (\zeta_j TP^*T^*)^* u(\alpha) = (\zeta_j \mathscr{P}^*)^* u(\alpha) = \mathscr{P}\zeta_j^* u(\alpha) = \int K(\alpha,\beta)\bar{\zeta}_j(\beta, D_\beta)u(\beta)d\beta \\ &= \int \left[\bar{\zeta}_j(\beta, -D_\beta)K(\alpha,\beta)\right] u(\beta)d\beta. \end{aligned}$$

That means that

$$\begin{gathered} \zeta_j(\alpha, d_\alpha\psi_0(\alpha,\beta)) = 0, \quad \widetilde{\zeta}_j(\beta, d_\beta\psi_0(\alpha,\beta)) = 0, \quad \bar{\partial}_{z_j} a_k, \ \partial_{w_j} a_k = (|\alpha-\beta|^\infty), \\ z = x - i\xi, \quad w = x' - i\xi', \quad \alpha = (x,\xi), \quad \beta = (x',\xi'). \end{gathered} \tag{2.10}$$

Of course this is satisfied in our explicit construction. Note that (2.10) determines $a_k(\alpha,\beta)$ uniquely from $a_k(\alpha,\alpha)$ modulo $\mathcal{O}(|\alpha-\beta|^\infty)$. We can think about constructing a_k's as follows: define

$$\overline{\Delta} := \{(z,\bar{z}) : z \in \mathbb{C}^n\} \subset \mathbb{C}^n \times \mathbb{C}^n \simeq T^*\mathbb{R}^n \times T^*\mathbb{R}^n, \quad \mathbb{C}^n \ni z = x - i\xi \in \mathbb{C}^n, \tag{2.11}$$

which is a totally real subspace (see for instance [Zw12, 13.2]). With the variables of (2.10), write $a_k(\alpha,\beta) = b_k(z,\bar{w})$. Then $b_k(z,w)$ is the almost analytic extension of $b_k(z,\bar{z}) = b_k|_{\overline{\Delta}}$.

Concerning ψ_0, it is uniquely determined from (2.10) when we put

$$(d_\alpha\psi_0)(\alpha,\alpha) = \xi dx, \quad (d_\beta\psi_0)(\alpha,\alpha) = -\xi dx, \quad \psi_0(\alpha,\alpha) = 0, \quad \alpha = (x,\xi). \tag{2.12}$$

Note that we could just demand that $\psi_0(\alpha,\alpha) = 0$ as then the derivative conditions follow from the equations. Conversely, the derivative conditions determine ψ_0 up to an additive constant.

We can now compare $\mathscr{P}$ to the Toeplitz operator

$$\mathscr{T}_p := \Pi_0 M_p \Pi_0, \quad M_p u(\alpha) = p(\alpha)u(\alpha).$$

(We sometimes abuse notation and write p for M_p so that $\mathscr{T}_p := \Pi_0 p \Pi_0$.)

Using the stationary phase method we obtain

$$\mathscr{T}_p u(\alpha) = \int \widetilde{K}(\alpha,\beta)u(\beta)d\beta, \quad K(\alpha,\beta) = e^{\frac{i}{h}\psi_0(\alpha,\beta)}\tilde{a}(\alpha,\beta),$$
$$\tilde{a} = \tilde{a}_0 + h\tilde{a}_1 + \cdots, \quad \tilde{a}_0(\alpha,\alpha) = p(\alpha).$$

But the uniqueness statement means that

$$a_0(\alpha,\beta) = \tilde{a}_0(\alpha,\beta) + \mathcal{O}(|\alpha-\beta|^\infty).$$

This immediately gives

Theorem 1. *Suppose that P is a pseudodifferential operator (2.5) and that $\mathscr{P} := TPT^*$, $\mathscr{T}_p := \Pi_0 p \Pi_0$. Then*

$$\mathscr{P} = \mathscr{T}_p + \mathcal{O}(h)_{\mathscr{H}\to\mathscr{H}},$$

or

$$\langle TPu, Tv\rangle_{L^2(T^*\mathbb{R}^n)} = \langle pTu, Tv\rangle_{L^2(T^*\mathbb{R}^n)} + \mathcal{O}(h)\|u\|_{L^2(\mathbb{R}^n)}\|v\|_{L^2(\mathbb{R}^n)}. \tag{2.13}$$

Also,

$$TP = pT + \mathcal{O}(h^{\frac{1}{2}})_{L^2(\mathbb{R}^n)\to L^2(T^*\mathbb{R}^n)}. \tag{2.14}$$

That (2.13) holds was first observed by Cordoba–Fefferman [CoFe78] while (2.14) is an earlier result of Sjöstrand [Sj76]. (Both were formulated differently in the original versions and these are the versions from [Ma02] and [Sj96].)

In Theorem 2 we will see a stronger formulation which (when the weight is 0) applies here as well. We note however that when there is no weight we can use [Zw12, Theorem 13.10] to obtain an explicit q such that $\mathscr{P} = \Pi q \Pi + \mathcal{O}(h^\infty)_{L^2\to L^2}$.

2.2. **Geometry of Π_0.** We now revisit (2.10) and (2.12) in geometric terms. The (quadratic) phase ψ_0 generates a complex (linear) Lagrangian relation

$$\mathscr{C} := \{(\alpha, d_\alpha\psi_0(\alpha,\beta); \beta, -d_\beta\psi_0(\alpha,\beta)) : \alpha,\beta\in\mathbb{C}^{2n}\} \subset T^*\mathbb{C}^{2n}\times T^*\mathbb{C}^{2n}, \tag{2.15}$$

that is, a linear subspace of (complex) dimension $4n$ on which the (holomorphic) symplectic form

$$\sigma_2 := \pi_L^*\sigma - \pi_R^*\sigma, \quad \sigma := d(\alpha^* d\alpha), \quad \pi_L(\rho,\rho') = \rho, \quad \pi_R(\rho,\rho') = \rho',$$

vanishes. We note that $\mathscr{C}\subset S_1\times S_2$ where

$$S_1 = \{\rho\in T^*\mathbb{C}^{2n} : \zeta_j(\rho) = 0\}, \quad S_2 = \{\rho\in T^*\mathbb{C}^{2n} : \bar{\zeta}_j(\rho) = 0\}, \quad \bar{\zeta}_j(\rho) := \overline{\zeta_j(\bar{\rho})}. \tag{2.16}$$

That is a geometric version of (2.10). Since $\{\zeta_j,\zeta_k\} = 0\}$, S_j are involutive of (complex) dimension $3n$. (Note that we have $\bar{\zeta}_j$ rather than $\widetilde{\zeta}_j$ as we have the usual sign switch in the definition of $\mathscr{C}$.) We also identify the symplectic subspace

$$\{(\rho,\rho) : \rho\in S_1\cap S_2\}\subset\mathscr{C}. \tag{2.17}$$

Since

$$S_1 \cap S_2 = \{(x, \xi, \xi, 0) : (x, \xi) \in T^*\mathbb{C}^n\},$$

(2.17) is a geometric version of (2.12).

We can present this more abstractly without an explicit mention of ζ_j's. Thus we consider

$$V := T^*\mathbb{C}^m, \quad \sigma := \sum_{j=1}^m dz_j^* \wedge dz_j, \quad (z, z^*) \in T^*\mathbb{C}^m. \tag{2.18}$$

For a linear subspace of $W \subset V$ we define the symplectic annihilator of W by

$$W^\sigma := \{\rho \in V : \sigma(\rho, V) = 0\}.$$

We then consider *involutive* subspaces of V:

$$S \subset V, \quad S^\sigma \subset S, \quad \dim_{\mathbb{C}} S = 2m - k. \tag{2.19}$$

The Hamiltonian foliation of S is defined by the projection

$$p : S \longrightarrow S/S^\sigma. \tag{2.20}$$

Assume now S_1 and S_2 are two such subspaces and that

$$\dim_{\mathbb{C}}(S_1 \cap S_2) = 2m - 2k, \quad (S_1 \cap S_1)^\sigma \cap S_1 \cap S_2 = \{0\}. \tag{2.21}$$

This means that S_1 and S_2 intersect transversally at a symplectic subspace and that the (affine) leaves of the Hamiltonian foliations through points of $S_1 \cap S_2$ also intersect transversally and we have identifications

$$S_1 \cap S_2 \ni \rho \longmapsto \rho + S_j^\sigma \in S_j / S_j^\sigma.$$

Composing the inverse of this map with (2.20) we obtain complex linear maps

$$\begin{gathered} p_j : S_j \to S_1 \cap S_2, \\ p_1^{-1}(\rho) \cap p_2^{-1}(\rho) = \{\rho\}, \quad \rho \in S_1 \cap S_2, \quad \dim p_j^{-1}(\rho) = k. \end{gathered} \tag{2.22}$$

The abstract (linear) version of (2.15) is given in

Lemma 2. *Suppose that two involutive complex subspaces S_1 and S_2 satisfy (2.21) and that $\mathscr{C} \subset S_1 \times S_2$ is a complex Lagrangian subspace of $V \times V$. Then the following conditions are equivalent:*

(1) $\mathscr{C} \circ \mathscr{C} = \mathscr{C}$, $\mathscr{C} \cap ((S_1 \cap S_2) \times (S_1 \cap S_2)) = \Delta(S_1 \cap S_2)$;
(2) $\mathscr{C} \cap ((S_1 \cap S_2) \times (S_1 \cap S_2)) = \Delta(S_1 \cap S_2)$;
(3) $\mathscr{C} := \{(\rho_1, \rho_2) \in S_1 \times S_2 : p_1(\rho_1) = p_2(\rho_2)\}$,

where p_j are defined in (2.22) and, for $W \subset V$, $\Delta(W) := \{(\rho, \rho) : \rho \in W\} \subset V \times V$.

Proof. We can find defining functions of S_i's, ζ_j^i, $j = 1, \cdots, k$, $\{\zeta_j^i, \zeta_\ell^i\} = 0$, (chosen them globally here as we are in the linear case). Then $H_{\zeta_j^i}$ are tangent to S_j. $H_{\zeta_j^1} \oplus 0$ and $0 \oplus H_{\zeta_j^2}$ are tangent to $\mathscr{C}$. Defining $\Phi_i^t := \exp(t_1 H_{\zeta_1^i}) \cdots \exp(t_k H_{\zeta_k^i})$ we see that $\mathscr{C}$ is invariant under the action of $\Phi_t^1 \oplus \Phi_s^2$, $t, s \in \mathbb{C}^k$. It then follows that

$$(\rho_1, \rho_2) \in \mathscr{C} \iff (p_1(\rho_1), p_2(\rho_2)) \in \mathscr{C}. \tag{2.23}$$

With this in place the lemma is immediate: (1) $\Rightarrow$ (2) is obvious from the second condition in (1); (2) $\Rightarrow$ (3) follows from (2.23) and dimension counting; (3) $\Rightarrow$ (1) is clear: $(\rho_1, \rho) \in \mathscr{C}$ and $(\rho, \rho_2) \in \mathscr{C}$ implies that $\rho \in S_1 \cap S_2$ and hence $p_1(\rho_1) = p_2(\rho) = \rho = p_1(\rho) = p_2(\rho_2)$. $\square$

3. Projector with weights

We now want to prove an analogue of (2.13) in the case of weighted spaces. For that we assume that $P = p^w(x, hD)$ where $p \in S(1)$ has a bounded analytic continuation to a fixed neighbourhood of $T^*\mathbb{R}^n \subset \mathbb{C}^{2n}$. In that case, following Martinez [Ma02] and earlier works, we will show that for $\varphi \in C_c^\infty(T^*\mathbb{R}^n)$ with $\|\nabla\varphi\|_{L^\infty}$ sufficiently small (depending on the neighbourhood in which p is analytic) we have

$$\langle TPu, Tv\rangle_{L^2_\varphi} = \langle p_\varphi Tu, Tv\rangle_{L^2_\varphi} + \mathcal{O}(h)\|Tu\|_{L^2_\varphi}\|Tv\|_{L^2_\varphi}, \tag{3.1}$$

where

$$p_\varphi(x, \xi) := p(x + 2\partial_z\varphi, \xi - 2i\partial_z\varphi), \quad z = x - i\xi, \quad L^2_\varphi := L^2(T^*\mathbb{R}^n; e^{-2\varphi/h}d\alpha),$$

see Theorem 2 at the end of §4. To do this we follow the same strategy as in §2 and construct a self-adjoint projection

$$\begin{gathered}\Pi_\varphi : L^2(T^*\mathbb{R}^n) \to \mathscr{H}, \quad \Pi_\varphi^2 = \Pi_\varphi, \quad \Pi_\varphi|_{\mathscr{H}} = I_{\mathscr{H}}, \\ \langle \Pi_\varphi u, v\rangle_{L^2_\varphi} = \langle u, \Pi_\varphi v\rangle_{L^2_\varphi}.\end{gathered} \tag{3.2}$$

We write the last statement as $\Pi_\varphi^{*,\varphi} = \Pi_\varphi$. In what follows, for the sake of clarity we drop φ and, unless specifically stated, consider the adjoint in $L^2(e^{-2\varphi/h}d\alpha)$ only.

To describe Π_φ we make the assumption that $\|\varphi\|_{C^2}$ is sufficiently small.

The strategy for describing Π_φ as $h \to 0$ goes back to the works of Boutet de Monvel–Sjöstrand [BoSj76], Boutet de Monvel–Guillemin [BoGu81], Helffer–Sjöstrand and was outlined for compact manifolds and compactly supported weights in [Sj96]. The argument proceeds in the following steps:

- construction of a uniformly bounded operator (as $h \to 0$) $B : L^2_\varphi \to L^2_\varphi$ such that $Z_j B = \mathcal{O}(h^\infty)_{L^2_\varphi \to L^2_\varphi}$, $B^* = B$ and $B^2 = B + \mathcal{O}(h^\infty)_{L^2_\varphi \to L^2_\varphi}$;
- characterization of the *unique* properties of the Schwartz kernel of B: uniqueness of the phase and the determination of the amplitude from its restriction to the diagonal;

- finding a projector $P = \mathcal{O}(1)_{L^2_\varphi \to L^2_\varphi}$ onto the image of T.
- choosing $f \in S(1)$, $f \geq 1/C$ so that $A := PM_f P^*$ (in the notation of §2), satisfies $A = B + \mathcal{O}(h^\infty)_{L^2_\varphi \to L^2_\varphi}$; this relies on the uniqueness properties in the construction of B;
- expressing Π as a suitable contour integral of the resolvent of A and using it to show that $\Pi = B + \mathcal{O}(h^\infty)_{L^2_\varphi \to L^2_\varphi}$. (this, elementary and elegant part, can be copied verbatim from [Sj96]).

To construct B we postulate an ansatz

$$Bu(\alpha) = h^{-n} \int e^{i\psi(\alpha,\beta)/h - 2\varphi(\beta)/h} a(\alpha, \beta) u(\beta) d\beta,$$

and as in (2.9)

$$\begin{aligned} & e^{-i\psi(\alpha,\beta)/h} Z_j(\alpha, hD_\alpha) \left(e^{i\psi(\alpha,\beta)/h} a(\alpha,\beta)\right) = \mathcal{O}(|\alpha - \beta|^\infty), \\ & e^{-i\psi(\alpha,\beta)/h} \widetilde{Z}_j(\beta, hD_\beta) \left(e^{i\psi(\alpha,\beta)/h} a(\alpha,\beta)\right) = \mathcal{O}(|\alpha - \beta|^\infty), \\ & Z_j := hD_{x_j} - \xi_j - ihD_{\xi_j}, \quad \widetilde{Z}_j := -hD_{x_j} - \xi_j - ihD_{\xi_j}. \end{aligned} \tag{3.3}$$

We note that for $\bar{Z}_k(\alpha, hD_\alpha) := \widetilde{Z}_k(\alpha, -hD_\alpha)$ we have $(i/h)[Z_j, \bar{Z}_k] = -2i\delta_{jk}$.

Self adjointness of B implies that we should also have

$$\psi(\alpha, \beta) = -\overline{\psi(\beta, \alpha)}, \quad a(\alpha, \beta) = \overline{a(\beta, \alpha)},$$

which is consistent with (2.9).

The fact that the weights to do not appear in $\widetilde{Z}_j$ may seem surprising but is easily verified: put $K_B := e^{i\psi/h} a/h^n$ and note that

$$(Z_j)^* = e^{2\varphi/h} \bar{Z}_j e^{-2\varphi/h}, \quad \bar{Z}_j v := \overline{Z_j^t \bar{v}}.$$

Then

$$\begin{aligned} 0 \equiv (Z_j B)^* u(\alpha) &= B^* Z_j^* u(\alpha) = B Z_j^* u(\alpha) \\ &= \int K_B(\alpha, \beta) e^{-2\varphi(\beta)/h} (Z_j)^* u(\beta) d\beta \\ &= \int K_B(\alpha, \beta) \bar{Z}_j(\beta, hD_\beta) \left(e^{-2\varphi(\beta)/h} u(\beta)\right) d\beta \\ &= \int \widetilde{Z}_j(\beta, hD_\beta) K_B(\alpha, \beta) e^{-2\varphi(\beta)/h} d\beta, \quad \widetilde{Z}_j v := \overline{Z_j \, \bar{v}}. \end{aligned} \tag{3.4}$$

Going back to (3.3) we obtain simple eikonal and transport equations for ψ and a:

$$\begin{gathered} \psi_{x_j} - \xi_j - i\psi_{\xi_j} = \mathcal{O}(|\alpha - \beta|^\infty), \quad -\psi_{y_j} - \eta_j - i\psi_{\eta_j} = \mathcal{O}(|\alpha - \beta|^\infty), \\ \alpha = (x, \xi), \quad \beta = (y, \eta), \end{gathered} \tag{3.5}$$

and

$$\begin{gathered} a = a_0 + ha_1 + \cdots , \\ \bar\partial_z a_k = \mathcal{O}(|\alpha-\beta|^\infty), \quad \partial_w a_k = \mathcal{O}(|\alpha-\beta|^\infty), \\ z = x - i\xi, \ w = x' - i\xi'. \end{gathered} \tag{3.6}$$

To guarantee the boundedness on L^2_φ and decay away from the diagonal we also demand that

$$-\operatorname{Im}\psi(\alpha,\beta) - \varphi(\alpha) - \varphi(\beta) = c_0|\alpha-\beta|^2 + \mathcal{O}(|\alpha-\beta|^3), \quad c_0 > 0. \tag{3.7}$$

3.1. **Phase construction.** We now need to discuss the "initial conditions" for ψ: in the free case they were given in (2.12) and geometrically in Lemma 2. We start in an "ad hoc" way and then move to the geometric version. Thus we require that

$$\psi(\alpha,\alpha) = -2i\varphi(\alpha). \tag{3.8}$$

In the notation of §2 (specifically with ψ_0 as in (2.3)) we put

$$\psi(\alpha,\beta) = \psi_0(\alpha,\beta) + \widetilde\psi(z,\bar w),$$

so that the equations become

$$\partial_{\bar z}\widetilde\psi(z,\bar w), \quad \partial_w\widetilde\psi(z,\bar w) = \mathcal{O}(|z-w|^\infty), \quad \widetilde\psi(z,\bar z) = -2i\varphi(z).$$

(Recall that $\psi_0(\alpha,\alpha) = 0$ and that $2\partial_{\bar z}\psi_0 = \xi_j$, $2\partial_w\psi_0 = \xi_j'$, $z = x - i\xi$, $w = x' - i\xi'$, $\alpha = (x,\xi)$, $\beta = (x',\xi')$.)

We note that the analogue of (2.12) is

$$(\partial_z\widetilde\psi)(z,\bar z) = -2i\partial_z\varphi(z), \quad (\partial_w\widetilde\psi)(z,\bar z) = -2i\partial_{\bar z}\varphi(z). \tag{3.9}$$

This is solved by taking an almost analytic extension of $\widetilde\varphi = \widetilde\psi|_{\bar\Delta}$ from the totally real submanifold $\bar\Delta$ – see (2.11). We note here that $d((z,w),\bar\Delta) = |z-\bar w|$.

Remark. Note for any smooth function $f(z)$, if $g(z,w)$ is almost analytic near $\bar\Delta$ with $g(z,\bar z) = f(z)$, then since $\partial_{\bar z}g(z,w)$, $\partial_{\bar w}g(z,w) = O(|z-\bar w|^\infty)$ we have $\partial_z f(z) = \partial_z g|_{\bar\Delta} + \partial_{\bar w}g|_{\bar\Delta}$. Hence, $\partial_z g|_{\bar\Delta} = \partial_z f$. Similarly, $\partial_{\bar z}f = \partial_w g|_{\bar\Delta}$. □

We then get

$$\psi(\alpha,\beta) = \psi_0(\alpha,\beta) + \widetilde\psi(z,\bar w).$$

Near the diagonal we have

$$\begin{aligned} \operatorname{Im}\widetilde\psi(z,\bar w) &= -2\varphi(z) + \operatorname{Im}\Big((\partial_w\widetilde\psi)(z,\bar z)(\bar w - \bar z)\Big) + \mathcal{O}(\|\varphi\|_{C^2}|w-z|^2) \\ &= -2\varphi(z) + \operatorname{Im}(-2i\partial_{\bar z}\varphi(z)(\bar w - \bar z)) + \mathcal{O}(\|\varphi\|_{C^2}|w-z|^2). \end{aligned}$$

Similarly,

$$\begin{aligned}\operatorname{Im}\widetilde{\psi}(z,\bar{w}) &= -2\varphi(w) + \operatorname{Im}\Big((\partial_z\widetilde{\psi})(w,\bar{w})(z-w)\Big) + \mathcal{O}(\|\varphi\|_{C^2}|w-z|^2)\\ &= -2\varphi(w) + \operatorname{Im}\Big((\partial_z\widetilde{\psi})(z,\bar{z})(z-w)\Big) + \mathcal{O}(\|\varphi\|_{C^2}|w-z|^2)\\ &= -2\varphi(z) + \operatorname{Im}(-2i\partial_z\varphi(z)(z-w)) + \mathcal{O}(\|\varphi\|_{C^2}|w-z|^2).\end{aligned}$$

Adding up the two equalities we obtain

$$\operatorname{Im}\widetilde{\psi}(z,\bar{w}) = -\varphi(z) - \varphi(w) + \mathcal{O}(\|\varphi\|_{C^2}|w-z|^2).$$

Hence,

$$\begin{aligned}-\operatorname{Im}\psi(\alpha,\beta) - \varphi(\alpha) - \varphi(\beta) &= -\operatorname{Im}\psi_0(\alpha,\beta) + \mathcal{O}(\|\varphi\|_{C^2}|\alpha-\beta|^2)\\ &= -\tfrac{1}{4}|\alpha-\beta|^2 + \mathcal{O}(\|\varphi\|_{C^2}|\alpha-\beta|^2).\end{aligned}$$

Hence, if $\|\varphi\|_{C^2}$ is small enough, we obtain (3.7).

Remark. A more careful analysis of the quadratic terms would show that we only need $\sum_{i,j}\partial_{z_j\bar{z}_k}\varphi(z)\zeta_j\bar{\zeta}_k > -\frac{1}{4}|\zeta|^2$ which is a subharmonicity condition. We will not pursue this direction here.

We now discuss the property $B = B^2 + \mathcal{O}(h^\infty)_{L^2_\varphi\to L^2_\varphi}$ and that will lead naturally to the construction of a in (3.6). Denoting the kernel of B^2 by K_{B^2} (analogue of K_B in (3.4)) we have

$$\begin{aligned}K_{B^2}(\alpha,\beta) &= \int K_B(\alpha,\gamma)K_B(\gamma,\beta)e^{-2\varphi(\gamma)/h}d\gamma\\ &= h^{-2n}\int e^{\frac{i}{h}(\psi(\alpha,\gamma)+\psi(\gamma,\beta)+2i\varphi(\gamma)))}a(\alpha,\gamma)a(\gamma,\beta)d\gamma.\end{aligned}\tag{3.10}$$

In view of (3.7) we can assume that a is supported near the diagonal and that justifies an application of (complex) stationary phase. Let

$$\psi_1(\alpha,\beta) = \text{c.v.}_\gamma\left(\psi(\alpha,\gamma)+\psi(\gamma,\beta)+2i\varphi(\gamma)\right).$$

Since B^2 is self-adjoint on L^2_φ, $\psi_1(\alpha,\beta)$ satisfies the eikonal equations (3.5). If we show that (3.8) holds for ψ_1 then the uniqueness in the construction of ψ will show that $\psi_1 \equiv \psi$ to infinite order on the diagonal.

Remark. In our special case we can see that $\psi_1 = \psi$ quite immediately. Let us change to holomorphic coordinates and recall (2.4). Then, with

$$\Phi(z) := \tfrac{1}{2}|\operatorname{Im} z|^2 + \varphi(z), \qquad \Psi(z,w) = -\tfrac{1}{4}(z-w)^2 + i\widetilde{\psi}(z,w),\tag{3.11}$$

we have

$$\psi(\alpha,\beta) = i\left[\Phi_0(z)+\Phi_0(w)\right] - i\Psi(z,\bar{w}), \quad \Phi_0(z) := |\operatorname{Im} z|^2.$$

Therefore, with $\gamma \mapsto (v,\bar{v})$, the

$$i[\psi(\alpha,\gamma)+\psi(\gamma,\beta)+2i\varphi(\gamma)] = \Psi(z,\bar{v}) + \Psi(v,\bar{w}) - \Psi(v,\bar{v}) - \Phi_0(z) - \Phi_0(w).$$

Then immediately

$$\Psi(z,\bar{w}) = \text{c.v.}_{v,\bar{v}}(\Psi(z,\bar{v}) + \Psi(v,\bar{w}) - \Psi(v,\bar{v})). \tag{3.12}$$

In fact, treating v and $\bar{v}$ as independent variables (stationary phase is "real")

$$\begin{aligned} 0 &= \partial_v(\Psi(z,\bar{v}) + \Psi(v,\bar{w}) - \Psi(v,\bar{v})) = \partial_v\Psi(v,\bar{w}) - \partial_v\Psi(v,\bar{v}) \\ &= \partial_{w\bar{w}}\Phi(w)(\bar{w}-\bar{v}) + \mathcal{O}(|\bar{w}-\bar{v}|^2), \\ 0 &= \partial_{\bar{v}}(\Psi(z,\bar{v}) + \Psi(v,\bar{w}) - \Psi(v,\bar{v})) = \partial_{\bar{v}}\Psi(z,\bar{v}) - \partial_{\bar{v}}\Psi(v,\bar{v}) \\ &= \partial_{z\bar{z}}\Phi(z)(z-v) + \mathcal{O}(|z-v|^2). \end{aligned}$$

Since $\partial_{z\bar{z}}\Phi$ is non-degenerate we obtain $v = z$ and $\bar{v} = \bar{w}$. Inserting these critical values in on the right hand side of (3.12) yields the desired equality. In particular, (3.12) implies that $\psi(\alpha,\beta) = \text{c.v.}_\gamma\left(\psi(\alpha,\gamma) + \psi(\gamma,\beta) + 2i\varphi(\gamma)\right)$.

3.2. **Geometry of the phase.** We now proceed as in §2.2 but with complications due to the fact that the smooth weight φ makes the problem non-linear and non-holomorphic.

We start with a formal discussion assuming that φ has a holomorphic extension to a neighbourhood of $\mathbb{C}^{2n}$, U (if φ or even $\nabla\varphi$ are bounded, everything we have said so far remains valid).

Let

$$\mathscr{C} := \{(\alpha, d_\alpha\psi(\alpha,\beta) + id_\alpha\varphi(\alpha); \beta, -d_\beta\psi(\alpha,\beta) - id_\beta\varphi(\beta)) : (\alpha,\beta) \in U \times U\}.$$

We now define

$$\zeta_j^\varphi(\alpha,\alpha^*) := \zeta_j(\alpha, \alpha^* + i\partial_\alpha\varphi(\alpha)), \quad S_1 := \{\rho \in U : \zeta_j^\varphi(\rho) = 0\}.$$

We note that (since in our case so far $\zeta_j(\alpha,\alpha^*)$ are linear)

$$Z_j^\varphi(\alpha, hD_\alpha) = e^{\varphi(\alpha)/h} Z_j(\alpha, hD_\alpha) e^{-\varphi(\alpha)/h}.$$

Using $\widetilde{\zeta}_j$ we similarly define $\widetilde{\zeta}_j^\varphi$ and S_2.

Formally, we are in the situation described in Lemma 2 but for $\varphi \in C_c^\infty(T^*\mathbb{R}^n)$ we need an almost analytic version. In the setting here we already constructed the phase. However, the geometric point of view will be important in §6.1 where we consider a different approach.

3.3. **Amplitude construction.** To find the amplitude $a(\alpha,\beta)$ we once again use the fact that *it is enough to determine a on the diagonal.* Application of complex stationary phase to (3.10) yields

$$K_{B^2} = h^{-n} e^{\frac{i}{h}\psi(\alpha,\beta)} b(\alpha,\beta), \quad b(\alpha,\alpha) \sim \sum_j h^j L_{2j} a(\alpha,\gamma) a(\gamma,\alpha)|_{\gamma=\alpha}, \tag{3.13}$$

where L_{2j} are differential operators of order $2j$ in γ and $L_0|_\Delta = f(\alpha)$, $|f(\alpha)| > 0$. Since $\psi(\alpha,\beta) = -\overline{\psi(\beta,\alpha)}$, $f(\alpha) \in \mathbb{R}$. We note that if $a(\alpha,\beta) = \overline{a(\beta,\alpha)}$, then $b(\alpha,\beta) = \overline{b(\beta,\alpha)}$ as the operator B^2 is also self-adjoint. In particular, $b(\alpha,\alpha) \in \mathbb{R}$.

Writing $a \sim \sum_j h^j a_j$, we have

$$b(\alpha,\beta) \sim \sum_j h^j b_j(\alpha,\beta), \quad b_j(\alpha,\alpha) = \sum_{k+\ell+m=j} L_{2k} a_\ell(\alpha,\gamma) a_m(\gamma,\alpha)|_{\gamma=\alpha}.$$

We note that if $a_\ell(\alpha,\beta) = \overline{a_\ell(\beta,\alpha)}$ for $\ell \le M$ then $b_\ell|_\Delta \in \mathbb{R}$ for $\ell \le M$. Since

$$b_M(\alpha,\alpha) = 2f(\alpha)a_0(\alpha,\alpha)a_M(\alpha,\alpha) + \sum_{\substack{k+\ell+m=M\\ \ell,m<M}} L_{2k} a_\ell(\alpha,\gamma) a_m(\gamma,\alpha)|_{\gamma=\alpha},$$

it follows that

$$a_\ell(\alpha,\beta) = \overline{a_\ell(\beta,\alpha)},\ \ell < M \implies \sum_{\substack{k+\ell+m=M\\ \ell,m<M}} L_{2k} a_\ell(\alpha,\gamma) a_m(\gamma,\alpha)|_{\gamma=\alpha} \in \mathbb{R}. \tag{3.14}$$

We iteratively solve the following sequence of equations

$$\sum_{k+\ell+m=j} L_{2k} a_\ell(\alpha,\gamma) a_m(\gamma,\alpha)|_{\gamma=\alpha} = a_j(\alpha,\alpha) \tag{3.15}$$

with $a_j|_\Delta$ real. Since a is defined by its values on the diagonal taking almost analytic extensions from $\alpha = \beta$ will complete the proof. First, let

$$a_0(\alpha,\alpha) = \frac{1}{f(\alpha)} \in C^\infty(T^*\mathbb{R}^n;\mathbb{R})$$

so that $f(\alpha)a_0(\alpha,\alpha)^2 = a_0(\alpha,\alpha)$ (i.e. (3.15) is solved for $j = 0$). Next, take an almost analytic extension of $a_0|_{\bar\Delta}$ to define a_0 in a small neighbourhood of $\bar\Delta$ with $a_0(\alpha,\beta) = \overline{a_0(\beta,\alpha)}$.

Assume now that (3.15) is solved for $j \le M-1$. Then, (3.15) with $j = M$ reads

$$\begin{aligned} a_M(\alpha,\alpha) &= \sum_{k+\ell+m=M} L_{2k} a_\ell(\alpha,\gamma) a_m(\gamma,\alpha)|_{\gamma=\alpha} \\ &= 2a_M(\alpha,\alpha) + \sum_{\substack{k+\ell+m=M\\ \ell,m<M}} L_{2k} a_\ell(\alpha,\gamma) a_m(\gamma,\alpha)|_{\gamma=\alpha} \end{aligned}$$

Putting $a_M(\alpha,\alpha) = -\sum_{\substack{k+\ell+m=M\\ \ell,m<M}} L_{2k} a_\ell(\alpha,\gamma) a_m(\gamma,\alpha)|_{\gamma=\alpha}$ we solve (3.15) for $j = M$. From (3.14) we see that $a_M(\alpha,\alpha)$ is real. Taking an almost analytic continuation with $a_M(\alpha,\beta) = \overline{a_M(\beta,\alpha)}$ then completes the construction of a_M and hence by induction and the Borel summation lemma we have

$$b = a + O(h^\infty) + O(|\alpha-\beta|^\infty). \tag{3.16}$$

with $a(\alpha,\beta) = \overline{a(\beta,\alpha)}$.

Finally, it remains to check that an operator R with kernel

$$K_R(\alpha,\beta) = \chi(|\alpha-\beta|/C) r(\alpha,\beta) e^{\frac{i}{h}\psi(\alpha,\beta)-\frac{2\varphi(\beta)}{h}}, \qquad r = O(h^\infty + |\alpha-\beta|^\infty),\ \chi \in C_c^\infty(\mathbb{R})$$

has $R = O(h^\infty)_{L^2_\varphi \to L^2_\varphi}$. For that, consider the kernel of $e^{-\varphi/2h} R e^{\varphi/2h}$ given by

$$K_{R,\varphi}(\alpha,\beta) = r(\alpha,\beta) e^{\frac{i}{h}(\psi(\alpha,\beta)+i\varphi(\beta)+i\varphi(\alpha))}.$$

Now, by (3.7),

$$\left| e^{\frac{i}{h}(\psi(\alpha,\beta)+i\varphi(\beta)+i\varphi(\alpha))} \right| \le e^{-c|\alpha-\beta|^2/h}$$

and hence $K_{R,\varphi} = O(h^\infty)_{C_c^\infty}$ and is supported in $|\alpha-\beta| \le C$. Schur's lemma then implies $R = O(h^\infty)_{L^2_\varphi \to L^2_\varphi}$.

3.4. **Construction of the projector.** We first construct P with the following properties:

$$PTv = Tv, \quad v \in L^2(\mathbb{R}^n), \quad \|P\|_{L^2_\varphi \to L^2_\varphi} \le C, \tag{3.17}$$

with C independent of h.

The holomorphic structure will be used in the construction of P and we again write $z = x - i\xi$, $\Phi_0(z) := \frac{1}{2}|\operatorname{Im} z|^2$, and Φ as in (3.11). We then recall that

$$w = Tv,\ v \in L^2(\mathbb{R}^n) \ \Leftrightarrow\ u := e^{|\operatorname{Im} z|^2/2h} w \in L^2_{\Phi_0}(\mathbb{C}^n),$$

see for instance [Zw12, §13.3]. We construct

$$P_\Phi = \mathcal{O}(1) : L^2_\Phi \to H_\Phi, \quad P_\Phi u = u, \quad u \in H_\Phi. \tag{3.18}$$

We note that, since $|\Phi - \Phi_0| \le C$, as spaces $L^2_{\Phi_0} = L^2_\Phi$ and the issue is the uniform boundedness as $h \to 0$. The following P_Φ will satisfy (3.18):

$$P_\Phi u(z) := \frac{C^n}{(\pi h)^n} \int_{\mathbb{C}^n} e^{-C|z-w|^2/h + 2\langle z-w, \partial_z \Phi(z)\rangle/h} u(w) dm(w), \tag{3.19}$$

provided that C is suffiently large. To check uniform boundedness on L^2_Φ we note that

$$2\operatorname{Re}\langle z-w, \partial_z\Phi(z)\rangle = \Phi(z) - \Phi(w) + \mathcal{O}(\|\Phi''\|_{L^\infty}|z-w|^2). \tag{3.20}$$

Since Φ'' is uniformly bounded (in fact constant outside of a compact set) we see that for C sufficiently large,

$$-\Phi(z) + 2\operatorname{Re}\left(-C|z-w|^2 + \langle z-w, \partial_z\Phi(z)\rangle\right) + \Phi(w) \le -|w-z|^2,$$

which (using Schur's criterion) shows uniform boundedness of P_Φ on L^2_Φ. For $u \in H_\Phi$ we have $P_\Phi u = u$ – see for instance [Zw12, (13.3.16)] (the fact that $\Phi(z)$ is not quadratic plays no role in the argument). Returning to (3.17) we put

$$P := e^{-|\operatorname{Im} z|^2/2h} P_\Phi e^{|\operatorname{Im} z|^2/2h}.$$

We now construct the projector $\Pi = \Pi_\varphi$ and relate it to the parametrix B constructed above. That is done by repeating the argument presented in [Sj96].

We take $f \in S(T^*\mathbb{R}^n)$, $f \geq c > 0$, and consider

$$A := A_f = PM_f P^{*,\varphi}, \quad P^{*,\varphi} = e^{2\varphi/h} P^* e^{-2\varphi/h}.$$

We write the action of A as follows:

$$Au(\alpha) = \int K_A(\alpha,\beta)u(\beta)e^{-2\varphi(\beta)/h}d\beta, \quad K_A(\alpha,\beta) = h^{-n}e^{\frac{i}{h}\psi_A(\alpha,\beta)}a_f(\alpha,\beta),$$

where the phase and amplitude are obtained from the method of stationary phase in the composition defining A. We claim that

$$\psi_A(\alpha,\beta) = \psi(\alpha,\beta) + \mathcal{O}(|\alpha-\beta|^\infty).$$

To see that we note that (6.14) and hence (3.5),(3.6) hold with ψ and a replaced by ψ_A and a_f. Hence it is sufficient to check that (3.8) holds for ψ_A. We calculate the critical value on the diagonal in notation used in (3.19),(3.20):

$$\begin{aligned} i\psi_A(\alpha,\alpha) + \Phi_0(z) &= \text{c.v.}_w \left(-2C|z-w|^2 + 2\operatorname{Re}\langle z-w, \partial\Phi(z)\rangle + \Phi(w)\right) \\ &= \Phi(z) + \text{c.v.}_w\left(-C|z-w|^2 + \mathcal{O}(|z-w|^2)\right) \\ &= \Phi(z) = \Phi_0(z) + 2\varphi(z). \end{aligned}$$

Since the left hand side is equal to $i\psi_A(\alpha,\alpha)+\Phi_0(z)$, (3.8), and hence $\psi \equiv \psi_A$ follow.

Since equations (3.6) are satisified by a_f, a_f is determined up to $O(h^\infty + |\alpha-\beta|^\infty)$ by $a_f|_\Delta$. We want to choose $f \sim \sum_j h^j f_j$ so that

$$a_f(\alpha,\alpha) \sim \sum a_{f,j}(\alpha,\alpha)h^j, \quad \text{and} \quad a_{f,j}(\alpha,\alpha) = a_j(\alpha,\alpha),$$

where $a \sim \sum h^j a_j$ is the amplitude in the construction of B. As in §3.3 (but with different L_{2k}'s, $g := L_0|_\Delta \neq 0$) we have,

$$a_{f,j}(\alpha,\alpha) = \sum_{k+\ell=j} L_{2k} f_\ell(\alpha) = g(\alpha)f_j(\alpha) + \sum_{\substack{k+\ell=j\\ \ell<j}} L_{2k}f_\ell(\alpha).$$

(In our special case, the amplitude in P is constant which is not the case in generalizations – but the argument works easily just the same.) Using this, solving $a_{f,j}(\alpha) = a_j(\alpha)$ for f is immediate. As in the construction of the amplitude of B in §3.3 we see that f is real valued and f_0 is bounded from below.

To summarize, we constructed

$$Bu(\alpha) = \int e^{\frac{i}{h}\psi(\alpha,\beta)}a(\alpha,\beta)e^{-2\varphi(\beta)/h}d\beta$$

and found f such that

$$\begin{gathered} B = \mathcal{O}(1) : L^2_\varphi \to L^2_\varphi, \quad B = B^{*,\varphi}, \quad B = B^2 + \mathcal{O}(h^\infty)_{L^2_\varphi \to L^2_\varphi}, \\ B = A_f + \mathcal{O}(h^\infty)_{L^2_\varphi \to L^2_\varphi}, \quad A_f := PM_f P^{*,\varphi}, \\ f(\alpha) \sim \sum_j h^j f_j(\alpha) \in S(1), \quad f(\alpha) > 1/C. \end{gathered} \tag{3.21}$$

We can now quote [Sj96] verbatim to see that

$$\Pi_\varphi = B + \mathcal{O}(h^\infty)_{L^2_\varphi \to L^2_\varphi}. \tag{3.22}$$

For the sake completeness we recall the argument. To start we observe that for $u \in \mathscr{H} := T(L^2(\mathbb{R}^n))$, $\|u\|_{L^2_\varphi} > 0$,

$$\begin{aligned} \langle A_f u, u \rangle_{L^2_\varphi} &= \langle P f P^* u, u \rangle_{L^2_\varphi} = \langle f P^* u, P^* u \rangle_{L^2 \varphi} \geq \min_{\alpha \in T^*\mathbb{R}^n} f(\alpha) \|P^* u\|^2_{L^2_\varphi} \\ &\geq \frac{|\langle P^* u, u \rangle|^2}{C \|u\|^2_{L^2_\varphi}} = \|u\|^2_{L^2_\varphi}/C. \end{aligned}$$

Hence,

$$\begin{gathered} \|u\|_{L^2_\varphi}/C \leq \|A_f u\|_{L^2_\varphi} \leq C \|u\|_{L^2_\varphi}, \quad u \in \mathscr{H}, \\ A_f u = 0, \quad u \in \mathscr{H}^\perp, \quad A_f^* = A_f, \end{gathered} \tag{3.23}$$

and

$$\Pi_\varphi = \frac{1}{2\pi} \int_\gamma (\lambda - A_f)^{-1} d\lambda, \tag{3.24}$$

where γ is a positively oriented boundary of an open set in $\mathbb{C}$ containing $[1/C, C]$ and excluding 0. From (3.21) we know that

$$A_f = A_f^2 + \mathcal{O}(h^\infty)_{L^2_\varphi \to L^2_\varphi} \tag{3.25}$$

and we want to use this property to show that Π_φ is close to A_f. For that we note that if $A = A^2$ then, at first for $|\lambda| \gg 1$,

$$(\lambda - A)^{-1} = \sum_{j=0}^\infty \lambda^{-j-1} A^j = \lambda^{-1} + \lambda^{-1} \sum_{j=0}^\infty \lambda^{-j} A = \lambda^{-1} + A\lambda^{-1}(\lambda - 1)^{-1}.$$

Hence, it is natural to take the right hand side as the approximate inverse in the case when $A^2 - A$ is small:

$$(\lambda - A_f)(\lambda^{-1} + A_f \lambda^{-1}(\lambda - 1)^{-1}) = I - (A_f^2 - A_f)\lambda^{-2}(\lambda - 1)^{-1}.$$

In view of (3.25) and for h small enough, the right hand side is invertible for $\lambda \in \gamma$ with the inverse equal to $I + R$, $R = \mathcal{O}(h^\infty)_{L^2_\varphi \to L^2_\varphi}$. Hence for $\lambda \in \gamma$,

$$(\lambda - A_f)^{-1} = \lambda^{-1} + \lambda^{-1}(\lambda - 1)^{-1} A_f + \mathcal{O}(h^\infty)_{L^2_\varphi \to L^2_\varphi}.$$

Inserting this identity into (3.24) and using Cauchy's formula gives

$$\Pi_\varphi = A_f + \mathcal{O}(h^\infty)_{L^2_\varphi \to L^2_\varphi} = B + \mathcal{O}(h^\infty)_{L^2_\varphi \to L^2_\varphi},$$

which is (3.22).

4. Pseudodifferential operators on weighted spaces

We now want to present the action of pseudodifferential operators $P = p^{\mathrm{w}}(x, hD)$, $p \in S(1)$ on the FBI transform side.

We will use the notation of [Zw12, §13.4] and note that by [Zw12, Theorem 13.9]

$$\begin{gathered} TPT^* = e^{-\Phi_0(z)/h} q^w_{\Phi_0}(z, hD_z) e^{\Phi_0(z)/h}, \quad q(x - i\xi, \xi) := p(x, \xi), \\ q^{\mathrm{w}}_{\Phi_0}(z, hD_z)u := \frac{1}{(2\pi h)} \iint_{\Gamma_{\Phi_0}(z)} q\left(\frac{z+w}{2}, \zeta\right) e^{\frac{i}{h}\langle z-w, \zeta\rangle} u(w) d\zeta \wedge dw, \\ \Phi_0(z) := \tfrac{1}{2}|\operatorname{Im} z|^2, \quad \Gamma_{\Phi_0}(z) : w \mapsto \zeta = \frac{2}{i}\partial_z \Phi_0\left(\frac{z+w}{2}\right), \quad u \in H_{\Phi_0}. \end{gathered} \tag{4.1}$$

(See the remark after Lemma 3 concerning convergence of the integral.) We note here that the correspondence between q and p is formally valid for $(x, \xi) \in \mathbb{C}^{2n}$ and that $\kappa : (x, \xi) \mapsto (x - i\xi, \xi)$ defines a complex linear canonical transformation. The contour Γ_{Φ_0} corresponds to integrating $q|_{\Lambda_{\Phi_0}}$,

$$\Lambda_{\Phi_0} := \kappa(T^*\mathbb{R}^n) = \{(z, \zeta) : \zeta = -2i\partial_z \Phi_0(z)\}.$$

We have the following lemma (see [Sj02, §12.5] for a more general version and for applications to scattering resonances):

Lemma 3. *Suppose that p is holomorphic and bounded on $\mathbb{R}^{2n} + B_{\mathbb{C}^{2n}}(0, \rho_0) \subset \mathbb{C}^{2n}$ and that $\Phi(z) = \Phi_0(z) + 2\varphi(z)$ with $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{C}^n)$, $\|\varphi\|_{C^2}$ sufficiently small. Then on $H_\Phi = H_{\Phi_0}$,*

$$q^{\mathrm{w}}_{\Phi_0}(z, hD_z) = q^{\mathrm{w}}_{\Phi}(z, hD_z) = \mathcal{O}(1) : H_\Phi \to H_\Phi,$$

where for $u \in H_\Phi$,

$$\begin{gathered} q^{\mathrm{w}}_{\Phi}(z, hD_z)u = \frac{1}{(2\pi h)^n} \iint_{\Gamma_{\Phi,c}(z)} q\left(\frac{z+w}{2}, \zeta\right) e^{\frac{i}{h}\langle z-w, \zeta\rangle} u(w) d\zeta \wedge dw, \\ \Gamma_{\Phi,c}(z) : w \mapsto \zeta = \tfrac{2}{i}\partial_z \Phi\left(\frac{z+w}{2}\right) + ci\frac{\overline{z-w}}{\langle z-w\rangle}, \end{gathered} \tag{4.2}$$

where $c > 0$ is sufficiently small.

Remark. When $q|_{\Lambda_\Phi} \in \mathscr{S}(\Lambda_\Phi)$ then we can take $c = 0$ and have a convergent integral in (4.2). Since we assume analyticity the deformed contour provides a quick definition for q bounded near $\Lambda_\Phi + B_{\mathbb{C}^n}(0, \rho_0)$ which in the case of $q \in S(\Lambda_\Phi)$ requires the usual integration by parts and density (of $\mathscr{S} \subset S$ in the $\langle z\rangle^\epsilon S$ topology) arguments – see the proof of [Zw12, Theorem 13.8].

Proof. We can deform the integral in (4.1) to the contour given by $\Gamma_{\Phi,c}(z)$ (see the remark above concerning convergence): since φ is small and we take $c > 0$ small the deformation is allowed as q is holomorphic and bounded in $\Lambda_{\Phi_0} + B_{\mathbb{C}^n}(0, \rho_0)$. To see the boundedness on H_Φ we use (3.20). $\square$

We now discuss

$$\Pi_\varphi T P T^* \Pi_\varphi = \mathcal{O}(1) : H_\Phi \to H_\Phi,$$

where the uniform boundedness follows from Lemma 3. We can apply the method of stationary phase and for that it is useful to use the notation of (3.12). The phase then becomes

$$\psi(\alpha,\beta) + 2i\varphi(\beta) - i[\Psi(z,\bar w) + \Psi(z,\bar v) - \Phi(v) + i\langle v - v', \zeta\rangle + \Psi(v', \bar w)]$$
$$\zeta = \tfrac{2}{i}\partial_z \Phi\left(\frac{v+v'}{2}\right) + ci\frac{\overline{v - v'}}{\langle v - v'\rangle}.$$

We now let

$$\widetilde\Psi = \Psi(z,\bar v) - \Phi(v) + i\langle v - v', \zeta\rangle + \Psi(v', \bar w), \quad \zeta = \tfrac{2}{i}\partial_z \Phi\left(\frac{v+v'}{2}\right) + ci\frac{\overline{v - v'}}{\langle v - v'\rangle},$$

and show that

$$\mathrm{c.v}_{v,v',\bar v,\bar v'}\widetilde\Psi = \Psi(z,\bar w). \tag{4.3}$$

In fact, for simplicity we take $c = 0$ and first compute

$$\partial_v\widetilde\Psi = -\partial_v\Psi(v,\bar v) + \partial_v\Psi\left(\frac{v+v'}{2}, \frac{\bar v + \bar v'}{2}\right) + \tfrac12 \partial^2_{vv}\Psi\left(\frac{v+v'}{2}, \frac{\bar v + \bar v'}{2}\right)(v - v'),$$
$$\partial_{\bar v'}\widetilde\Psi = \tfrac12 \partial^2_{\bar v v}\Psi\left(\frac{v+v'}{2}, \frac{\bar v + \bar v'}{2}\right)(v - v').$$

Since $\partial_{\bar v v}\Psi$ is non-degenerate the second equation shows that $v = v'$. But then the first equation becomes $-\partial_v\Psi(v,\bar v) + \partial_v\Psi(v, (\bar v + \bar v')/2) = 0$ so that non-degeneracy of $\partial^2_{\bar v v}\Psi$ implies $\bar v = \bar v'$.

Computing the remaining two derivatives,

$$\partial_{\bar v}\widetilde\Psi|_{v=v'} = \partial_{\bar v}\Psi(z,\bar v) - \partial_{\bar v}\Psi(v,\bar v),$$
$$\partial_{v'}\widetilde\Psi|_{v=v'} = -\partial_v\Psi(v,\bar v) + \partial_v\Psi(v,\bar w),$$

we use the non-degeneracy of $\partial^2_{\bar v v}\Psi$ to see that $v = z$ and $\bar v = \bar w$. But then the critical value of $\widetilde\Psi$ is given by $\Psi(z,\bar w)$.

We conclude that we have an analogue of (2.8):

$$\begin{gathered}\Pi_\varphi T P T^* \Pi_\varphi u(\alpha) = c_\varphi h^{-n} \int K_{P,\varphi}(\alpha,\beta) u(\beta) e^{-2\varphi(\beta)/h} d\beta, \\ K_{P,\varphi}(\alpha,\beta) = e^{\frac{i}{h}\psi(\alpha,\beta)} a(\alpha,\beta), \quad a = a_0 + h a_1 + \cdots \\ a_0(\alpha,\beta) = q\left(\frac{z+w}{2}, \frac{2}{i}\partial_z \Phi\left(\frac{z+w}{2}\right)\right), \quad q(x - i\xi, \xi) = p(x,\xi), \\ \Phi(z) = \tfrac{1}{2}|\operatorname{Im} z|^2 + \varphi(z), \quad z = x - i\xi, \ w = y - i\eta, \quad \alpha = (x,\xi), \ \beta = (y,\eta).\end{gathered} \tag{4.4}$$

Since

$$Z_j(\alpha, hD_\alpha) K_P(\alpha,\beta) = 0, \quad \widetilde{Z}_j(\beta, hD_\beta) K_P(\alpha,\beta) = 0,$$

construction of B shows that $a(\alpha,\beta)$ is determined (modulo $\mathcal{O}(h^\infty)_{L^2_\varphi \to L^2_\varphi}$) by $a|_\Delta$. Hence,

$$\Pi_\varphi T P T^* \Pi_\varphi = \Pi_\varphi M_{p_\varphi} \Pi_\varphi + \mathcal{O}(h)_{L^2_\varphi \to L^2_\varphi}, \tag{4.5}$$

where

$$\begin{gathered}p_\varphi(x,\xi) = q(z, -i\partial_z \Phi(z)), \ z = x - i\xi, \quad (x,\xi) \in \mathbb{R}^{2n} \\ q(z,\zeta) = p(z + i\zeta, \zeta), \quad \Phi(z) = \tfrac{1}{2}|\operatorname{Im} z|^2 + \varphi(z).\end{gathered}$$

But this means that

$$\begin{aligned}p_\varphi(x,\xi) &= p(z + i(-2i\partial_z\Phi(z)), -2i\partial_z\Phi(z)) = p(x - i\xi + i(\xi - 2i\partial_z\varphi), \xi - 2i\partial_z\varphi) \\ &= p(x + 2\partial_z\varphi, \xi - 2i\partial_z\varphi),\end{aligned}$$

which agrees with (3.1). We also obtain the analogue of (2.14):

$$TP = p_\varphi T + \mathcal{O}(h^{\frac{1}{2}})_{L^2_\varphi(\mathbb{R}^n) \to L^2_\varphi(T^*\mathbb{R}^n)}. \tag{4.6}$$

We summarize this in the following version of [Ma02, Corollary 3.5.3]:

Theorem 2. *Suppose that P is given by (2.5) where the symbol p enjoys a holomorphic extension satisfying*

$$|p(z,\zeta)| \le M, \quad |\operatorname{Im} z| \le a, \quad |\operatorname{Im} \zeta| \le b.$$

Then for $\varphi \in C^\infty_{\rm c}(T^*\mathbb{R}^n)$ with $\|\varphi\|_{C^2}$ sufficiently small and $L^2_\varphi := L^2(T^*\mathbb{R}^n, e^{-2\varphi/h} dx d\xi)$,

$$\langle TPu, Tv\rangle_{L^2_\varphi} = \langle M_{P_\varphi} Tu, Tv\rangle_{L^2_\varphi} + \langle R_\varphi Tu, Tv\rangle_{L^2_\varphi}, \tag{4.7}$$

where $R_\varphi = \mathcal{O}(h^\infty)_{L^2_\varphi \to L^2_\varphi}$ and

$$\begin{gathered}P_\varphi(x,\xi,h) = p_\varphi(x,\xi) + h p^1_\varphi(x,\xi) + \cdots, \\ p_\varphi(x,\xi) = p(x + 2\partial_z\varphi(x,\xi), \xi - 2i\partial_z\varphi(x,\xi)), \quad z = x - i\xi.\end{gathered}$$

Proof. The leading term in (4.7) was already obtained in (4.5). Assume that we have obtained p_φ^j, $j = 1, \cdots, J-1$ so that

$$\Pi_\varphi T P T^* \Pi_\varphi = \Pi_\varphi \left(\sum_{j=0}^{J-1} h^j p_\varphi^j \right) \Pi_\varphi + R_\varphi^J, \tag{4.8}$$

where

$$R_\varphi^J u(\alpha) = h^{J-n} c_\varphi \int_{T^*\mathbb{R}^n} e^{\frac{i}{h}\psi(\alpha,\beta)} a^J(\alpha,\beta) e^{-2\varphi(\beta)/h} u(\beta) d\beta, \quad a^J \sim a_0^J + h a_1^J + \cdots ,$$

with a_k^J satisfying the transport equations (3.6). If we apply the method of stationary phase to the first term of the kernel of the first term on right hand side of (4.8) we obtain a kernel with the expansion

$$e^{\frac{i}{h}\psi(\alpha,\beta)}(a_0 + h a_1 + \cdots + h^{J-1} a_J + h^J r_0^J + h^{J+1} r_1^J + \cdots),$$

where a_j's are the same as in (4.4). Again all the terms satisfy (3.6) and hence are uniquely determined from their values on the diagonal. Hence, if we put

$$p_\varphi^J(\alpha) := r_0^J(\alpha,\alpha) + a_0^J(\alpha,\alpha),$$

we obtain (4.8) with J replaced by $J+1$. □

Remark. The equality (3.1) holds for more general weights, $\varphi \in C^{1,1}$, by more direct arguments – see [Sj90, Theorem 1.2]. Here we were interested in developing the approach of [HeSj86],[Sj96] based on Bergman-like projectors.

5. Review of some almost analytic constructions

In §6 we will follow [Sj96] and describe the orthogonal projector $L^2_\Lambda \to T_\Lambda(L^2(\mathbb{R}^n))$ (in the notation of Theorem 4). That will involve some more involved almost analytic machinery and hence we will first consider some simpler examples. They seem to be related to some (simpler) aspects of [Sj74].

5.1. General comments about almost analyticity.

We will be concerned with a neighbourhood of $\mathbb{R}^m$ in $\mathbb{C}^m$ and for $U \subset \mathbb{C}^m$ we define

$$f \in C^{\mathrm{aa}}(U) \iff \partial_{\bar z} f(z) = \mathcal{O}_K(|\operatorname{Im} z|^\infty), \quad z \in K \Subset U.$$

This definition is non-trivial only for $U \cap \mathbb{R}^m \neq \emptyset$. We write $f \sim 0$ in U if $f(z) = \mathcal{O}_K(|\operatorname{Im} z|^\infty)$, $z \in K \Subset U \subset \mathbb{C}^m$. We note that (see [Tr81, Lemma X.2.2]) that for $f \in C^\infty$ that implies $\partial^\alpha f \sim 0$ in U.

Suppose Λ is an almost analytic manifold and $\Lambda \cap \mathbb{R}^m = \Lambda_\mathbb{R}$. One way to define Λ is through almost analytic defining functions: near any point $z_0 \in \Lambda_\mathbb{R}$ there exist a neighbourhood U of z_0 in $\mathbb{C}^n$ and $f_1, \cdots, f_k \in C^\infty(\mathbb{C}^m)$ such that

$$\Lambda \cap U = \{z : f_j(z) = 0, 1 \le j \le n\}, \quad \partial_z f_j(z_0) \ \text{ are linearly independent,}$$
$$|\partial_{\bar z} f_j(z)| = \mathcal{O}(|\operatorname{Im} z|^\infty + |\sup_{1\le \ell \le k} f_\ell(z)|^\infty).$$

We now consider *almost analytic vector fields*:

$$V = \sum_{j=1}^m a_j(z)\partial_{z_j}, \quad a_j \in C^{\rm aa}(\mathbb{C}^n),$$

which we identify with *real vector fields* $\widehat V$ such that for u holomorphic $\widehat V f = V$:

$$\begin{aligned}\widehat V &:= V + \bar V = \operatorname{Re} V \\ &= \sum_{j=1}^m \operatorname{Re} a_j(z)(\partial_{z_j} + \partial_{\bar z_j}) + i \operatorname{Im} a_j(z)(\partial_{z_j} - \partial_{\bar z_j}) \\ &= \sum_{j=1}^m \operatorname{Re} a_j(z)\partial_{\operatorname{Re} z_j} + \operatorname{Im} a_j(z)\partial_{\operatorname{Im} z_j}.\end{aligned}$$

Example. Suppose $M \subset \mathbb{C}^m$, $\dim_{\mathbb{R}} M = 2k$ is almost analytic. Then vector fields tangent to M are spanned by almost analytic vector fields, $V_j = a_j(z)\cdot \partial_z$, $\partial_{\bar z} a_j(z) = \mathcal{O}(|\operatorname{Im} z|^\infty)$, $z \in M$, $j = 1, \cdots k$. In fact, using [MeSj74, Theorem 1. 4, 3°] we can write M locally near any $z \in M \cap \mathbb{R}^m$ as $\{(z', h(z')) : z' \in \mathbb{C}^k\}$, $h = (h_{k+1}, \cdots, h_m) : \mathbb{C}^k \to \mathbb{C}^{m-k}$, $\partial_{\bar z} h = \mathcal{O}(|\operatorname{Im} z'|^\infty + |\operatorname{Im} h(z')|^\infty)$. We then put

$$V_j = \partial_{z_j} + \sum_{\ell=k+1}^m \partial_{z_j} h_\ell(z')\partial_{z_\ell}. \tag{5.1}$$

The real vector fields $\widehat V_j$ then span vector fields tangent to M. □

Following [MeSj74] and [Sj74] we define the (small complex time) flow of V as follows for $s \in \mathbb{C}$, $|s| \le \delta$

$$\Phi_s(z) := \exp \widehat{sV}(z). \tag{5.2}$$

The right hand side is the flow out at time 1 of the real vector field $\widehat{sV}$. Unless the coefficients in V are holomorphic $[\widehat V, \widehat{iV}] \ne 0$ which means that $\exp(s+t)V \ne \exp sV \exp tV$ for $s, t \in \mathbb{C}$. However, we have $[\widehat{iV}, \widehat V] \sim 0$.

Lemma 4. *Suppose that $\Gamma \Subset \mathbb{C}^m$ is an embedded almost analytic submanifold and V is an almost analytic vector field. Assume that,*

$$\widehat V, \widehat{iV} \text{ are linearly independent and their span is transversal to } \Gamma, \tag{5.3}$$

and that, in the notation of (5.2),

$$|\operatorname{Im} \Phi_t(z)| \ge |t|/C_K, \quad z \in K \Subset \Gamma. \tag{5.4}$$

Then for any $U \Subset \mathbb{C}^m$*, there exists* δ *such that*

$$\Lambda := \left\{ \exp t\widehat{V}(\rho) : \rho \in \Gamma \cap U, \quad |t| < \delta, \ t \in \mathbb{C} \right\}$$

is an almost analytic manifold, $\Lambda_{\mathbb{R}} = \Gamma_{\mathbb{R}}$ *and* $\dim_{\mathrm{Re}} \Lambda = 2k + 2$.

We will use the following geometric lemma:

Lemma 5. *Suppose* $Z_j \in C^\infty(\mathbb{R}^m; T^*\mathbb{R}^m)$*,* $j = 1, \cdots, J$*, are smooth vector fields and, for* $s \in \mathbb{R}^J$,

$$\langle s, Z \rangle := \sum_{j=1}^{J} s_j Z_j \in C^\infty(\mathbb{R}^m; T^*\mathbb{R}^m).$$

Then for $f \in C^\infty(\mathbb{R}^m)$

$$f(e^{\langle s,Z\rangle}(\rho)) = \sum_{p=1}^{P} \frac{1}{p!}(\langle s, Z\rangle)^k f(\rho) + \mathcal{O}_K(|s|^{P+1}), \quad \rho \in K \Subset \mathbb{R}^m. \tag{5.5}$$

while for $Y \in C^\infty(\mathbb{R}^m; T^*\mathbb{R}^m)$,

$$e_*^{\langle s,Z\rangle} Y(\rho) = \sum_{p=1}^{P} \frac{1}{p!} \operatorname{ad}^k_{\langle s,Z\rangle} Y(\rho) + \mathcal{O}_K(|s|^{P+1}), \quad \rho \in K \Subset \mathbb{R}^m. \tag{5.6}$$

For a proof see for instance [Je14, Appendix A]. We recall that $F_* Y(F(\rho)) := dF(\rho)Y(\rho)$.

Proof of Lemma 4. Let $\iota : \Gamma \hookrightarrow \mathbb{C}^m$ the inclusion map. Then

$$\partial \exp(t_1 \widehat{V} + t_2 \widehat{iV}) \circ \iota(\rho) : T_{(0,\rho)}(\mathbb{R}^2_t \times \Gamma) \to T_\rho \mathbb{C}^m$$

is given by $(T, X) \mapsto T_1 \widehat{V} + T_2 \widehat{iV} + \iota_* X$, which, thanks to our assumptions, is surjective onto a $2k+2$ (real) dimensional subspace of $T^*\mathbb{C}^m$. Hence, by the implicit function theorem Λ is a $2k+2$ dimensional embedded submanifold of $\mathbb{C}^m$.

To fix ideas we start with the simplest case of $\Gamma = \{0\} \subset \mathbb{C}^n$. In that case $\{\Lambda = \{\Phi_t(0) : t \in \mathbb{C}, |t| < \delta\}$, and from our assumption $|\operatorname{Im} \Phi_t(0)| \sim |t_1 \widehat{V} + t_2 \widehat{iV}| \sim |t|$. The tangent space is given by

$$T_{\Phi_t(0)} \Lambda = \{\partial_t \Phi_t(0) T + \partial_{\bar{t}} \Phi_t(0) \bar{T} : T \in \mathbb{C}\} \subset \mathbb{C}^2.$$

If we show that

$$\partial_{\bar{t}} \Phi_t(0) = \mathcal{O}(|t|^\infty) \tag{5.7}$$

then $d(T_{\Phi_t(0)}\Lambda, iT_{\Phi_t(0)}\Lambda) = \mathcal{O}(t^\infty)$ and almost analyticity of Λ follows from [MeSj74, Theorem 1.4, 1°]. The estimate (5.7) will follow from showing that for any holomorphic function f, $\partial_{t_1}^{\alpha_1}\partial_{t_2}^{\alpha_2}\partial_{\bar t} f(\Phi_t(0))|_{t=0} = 0$. But this follows from (5.5) and the fact that $[\widehat V, \widehat{iV}] \sim 0$ at 0. Indeed,

$$\begin{aligned}\partial_{t_1}^{\alpha_1}\partial_{t_2}^{\alpha_2}\partial_{\bar t} f(\Phi_t(0))|_{t=0} &= \partial_{t_1}^{\alpha_1}\partial_{t_2}^{\alpha_2}\partial_{\bar t}\left(\sum_{k=0}^{\infty}\frac{1}{k!}\left(t_1\widehat V + t_2\widehat{iV}\right)^k f(0)\right)|_{t=0} \\ &= \partial_{t_1}^{\alpha_1}\partial_{t_2}^{\alpha_2}\left(\sum_{k=0}^{\infty}\frac{1}{k!}\left(t_1\widehat V + t_2\widehat{iV}\right)^k (\widehat V + i\widehat{iV}) f(0)\right)|_{t=0} \\ &= \widehat V^{\alpha_1}\widehat{iV}^{\alpha_2}(\widehat V + i\,\widehat{iV})f(0) = \widehat V^{\alpha_1}\widehat{iV}^{\alpha_2}(V - V)f(0) = 0.\end{aligned} \tag{5.8}$$

The fact that $\widehat V$ and $\widehat{iV}$ commute to infinite order at 0 was crucial in this calculation. Holomorphy of f was used to have $\widehat W f = Wf$.

We now move the general case. For $z \in \Gamma$, $T_{\Phi_t(z)}\Lambda$ is spanned by

$$\partial_t\Phi_t(z)T + \partial_{\bar t}\Phi_t(z)\bar T,\ T \in \mathbb{C},\ \ d\Phi_t(z)X,\ \ X \in T_z\Gamma. \tag{5.9}$$

We can repeat the calculation (5.8) with 0 replaced by z to see that, using the assumption (5.4) and the fact that $\operatorname{Im}\Phi_t(z) = \operatorname{Im} z + \mathcal{O}(t)$,

$$\partial_{\bar t}\Phi(z) = \mathcal{O}(|t|^\infty + |\operatorname{Im} z|^\infty) = \mathcal{O}(|\operatorname{Im}\Phi_t(z)|^\infty). \tag{5.10}$$

To consider $d\Phi_t(z)X = (\Phi_t)_*Y(\Phi_t(z))$ we choose a vector field tangent to Γ, Y, $Y_c(z) = X$. We choose

$$Y_c = \widehat{W}_c, \quad W_c = \sum_{j=1}^{k} c_j V_j, \quad c \in \mathbb{C}^k, \tag{5.11}$$

a constant coefficient linear combination of vector fields (5.1). Then $d\Phi_t(z)X = (\Phi_t)_*Y_c(\Phi_t(z))$ and we want to show that

$$c \mapsto (\Phi_t)_*Y_c(\Phi_t(z)) \ \text{ is complex linear modulo errors } \mathcal{O}(|\operatorname{Im}\Phi_t(z)|^\infty). \tag{5.12}$$

In view of (5.9) that shows that $d(T_{\Phi_t(z)}\Lambda, iT_{\Phi_t(z)}\Lambda) = \mathcal{O}(|\operatorname{Im}\Phi_t(z)|^\infty)$ and from [MeSj74, Theorem 1.4, 1°] we conclude that Λ is almost analytic.

To establish (5.12) we use (5.6) with $\langle s, X\rangle = s_1\widehat V + s_2\widehat{iV}$, $s_1 = \operatorname{Re} t$, $s_2 = \operatorname{Im} t$. Since $[\widehat V, \widehat{iV}] \sim 0$ and $\widehat V \sim \widehat{iV}/i$ at $\operatorname{Im} w = 0$, we see that

$$(\Phi_t)_*Y_c(w) = \sum_{p=0}^{\infty}\frac{t^p}{p!}\operatorname{ad}^p_{\widehat V} W_c(w) + \mathcal{O}(|t|^{K+1} + |\operatorname{Im} w|^\infty). \tag{5.13}$$

Because of the form of W_c (see (5.1) and (5.11))

$$\operatorname{ad}^p_{\widehat V} W_c(w) = \widehat{\operatorname{ad}^p_V W_c}(w) + \mathcal{O}(|\operatorname{Im} w'|^\infty + |\operatorname{Im} h(w')|^\infty),$$

and

$$c \mapsto \operatorname{ad}^p_V W_c(w) \ \text{ is complex linear.}$$

Since $w = \Phi_t(z)$, $z \in \Gamma$,

$$\begin{aligned}|\operatorname{Im} w'| + |\operatorname{Im} h(w')| &= \mathcal{O}(|\operatorname{Im} z'| + |\operatorname{Im} h(z')| + |t|) \\ &= \mathcal{O}(|\operatorname{Im} z| + |t|) = \mathcal{O}(|\operatorname{Im} w| + |t|) = \mathcal{O}(|\operatorname{Im} w|),\end{aligned}$$

since $|\operatorname{Im} w| = |\operatorname{Im} \Phi_t(z)| \geq |t|/C$. Combining this estimates with (5.13) gives (5.12). $\square$

5.2. **Quasimodes and a positivity condition.** We make the same assumptions on $p \in \mathscr{S}$ as above but assume in addition that at (x_0, ξ_0), $p(x_0, \xi_0) = 0$ and $\{\operatorname{Re} p, \operatorname{Im} p\}(x_0, \xi_0) < 0$. We want to show that there exists $u(h) \in C^\infty_c(\mathbb{R}^n)$ such that for $P = P(x, \xi, h) = p(x, \xi) + \mathcal{O}(h)_{\mathscr{S}}$,

$$P(x, hD, h)u = \mathcal{O}(h^\infty)_{L^2}, \quad \operatorname{WF}_h(u) = (x_0, \xi_0), \quad \|u\|_{L^2} = 1. \tag{5.14}$$

(See [Zw12, 12.5] for a different argument based on a semiclassical adaptation of the construction of Duistermaat–Sjöstrand.) The assumption that $p \in \mathscr{S}(\mathbb{R}^{2n})$ is made for convenience only: the construction is (micro)local in phase space.

5.2.1. *Eikonal equation.* Fix $\tilde{p}$ an almost analytic extension of p. We proceed as follows. Assume that $(x_0, \xi_0) = (0, 0)$ and write $p(x, \xi) = a(x, \xi) + ib(x, \xi) + \mathcal{O}(|x|^2 + |\xi|^2)$, a, b real valued and linear. Since $\{a, b\} = -c^2 < 0$, the linear version of Darboux's theorem [HöIII, Theorem 21.1.3] shows that there exists a *linear* symplectic change of variables $\kappa(y, \eta) = (x, \xi)$ (preserving $T^*\mathbb{R}^n$) such that

$$\kappa^* a = c\eta_1 + \mathcal{O}(|\eta|^2 + |y|^2), \qquad \kappa^* b = -cy_1 + \mathcal{O}(|\eta|^2 + |y|^2).$$

We now switch to coordinates (y, η) and we denote them again by (x, ξ). Writing p for $\kappa^* p$ and $\tilde{p}$ for $\kappa^* \tilde{p}$, we obtain,

$$p(0, 0) = 0, \quad p(x, \xi) = c(\xi_1 - ix_1) + \mathcal{O}(|x|^2 + |\xi|^2). \tag{5.15}$$

For $s \in \mathbb{C}^{n-1}$, small there exists $\zeta_1(s)$ such that

$$\tilde{p}((0, s), (\zeta_1(s), is)) = 0, \quad \zeta_1(0) = 0, \quad \partial_s \zeta_1(0) = 0, \qquad \partial_{\bar{s}} \zeta_1(s) = \mathcal{O}(|\operatorname{Im} s|^\infty), \quad \alpha > 0.$$

We put $\Lambda_0 := \{((0, s), (\zeta_1(s), is))\}$ and then, in the notation of (5.2) we define

$$\Lambda = \{\exp \widehat{tH_{\tilde{p}}}(\rho) : \rho \in \Lambda_0, \ t \in \mathbb{C}, \ |t| < \epsilon\} \subset T^*\mathbb{C}^n.$$

To check that Λ is an almost analytic Lagrangian submanifold of $T^*\mathbb{C}^n$ we use Lemma 4. The transversality condition (5.3) follows immediately form (5.15) and it remains to check (5.4). For that we note that with $t = t_1 + it_2$ (and recalling that $\zeta_1(s) = \mathcal{O}(|s|^2)$,

$$\operatorname{Im} \Phi_t((0, s), (\zeta_1(s), is))) = (t_2 c, \alpha \operatorname{Im} s, ct_1, -\operatorname{Re} s) + \mathcal{O}(|t|^2 + |s|^2).$$

Hence, we obtain (5.4):

$$\begin{aligned}|\operatorname{Im}\Phi_t((0,s),(\zeta_1(s),is)))| &\geq c(|t_1|+|t_2|)+|s|-\mathcal{O}(|t|^2+|s|^2)\\ &\geq |t|/C+|s|/C, \quad |s|\ll 1.\end{aligned}$$

We now claim that Λ is positive in the sense that for

$$\tfrac{1}{i}\sigma(X,\bar{X})\geq c|X|^2, \quad X\in T^*_{(0,0)}\Lambda\subset T^*_{(0,0)}\mathbb{C}^n. \tag{5.16}$$

(Here σ is the symplectic form (2.18).) In fact, vectors in $T^*_{(0,0)}\Lambda$ are given by

$$X=((T,S),(iT,iS)),\ S\in\mathbb{C}^{n-1},\ T\in\mathbb{C}, \tag{5.17}$$

from which (5.16) follows.

We now note that the (real) linear transformation κ extends to a complex linear transformation on $\mathbb{C}^n\times\mathbb{C}^n$ and we can go back to the original coordinates (x,ξ) by taking the almost analytic Lagrangian manifold $\kappa(\Lambda)$. We also note that the positivity condition (5.16) is invariant under linear symplectic transformations which are real when restricted to $\mathbb{R}^n\times\mathbb{R}^n$ (as then $\kappa(\bar{X})=\overline{\kappa}(X)$). Hence $\kappa(\Lambda)$ is an almost analytic positive Lagrangian and we now denote it by Λ.

From (5.17) we see that $\pi_*: T_{(0,0)}\Lambda\to T_0\mathbb{C}^n$ is onto and hence we have an almost analytic generating function, that is $\Psi(z)$,

$$\partial_{\bar{z}}\Psi=\mathcal{O}(|\operatorname{Im}z|^\infty+|\operatorname{Im}\Psi(z)|^\infty)$$

such that, as almost analytic manifolds,

$$\Lambda\sim\{(z,\Psi_z(z)):|z|<\epsilon\}, \quad \Psi_z(0)=0. \tag{5.18}$$

Proof of (5.18). Since Λ is a.a. Lagrangian, we have $\sigma|_\Lambda\sim 0$ (vanishes to infinite order at $\Lambda_\mathbb{R}$) while the projection property shows that, near $z=0$, $\Lambda=\{(z,\zeta(z)):z\in\mathbb{C}^n\}$, $\zeta(0)=0$. Hence $d(\zeta(z)dz)\sim 0$ and (see [MeSj74, Theorem 1.4, 3°])

$$\partial_{\bar{z}}\zeta(z)=\mathcal{O}(|\operatorname{Im}z|^\infty+|\operatorname{Im}\zeta(z)|^\infty).$$

We note that for $z=x\in\mathbb{R}^n$, the strict positivity at $\Lambda_\mathbb{R}=\{(0,0)\}$ shows that

$$|x|/C\leq|\operatorname{Im}\zeta(x)|\leq C|x|, \quad x\in\mathbb{R}^n,\ |x|<\epsilon. \tag{5.19}$$

We now see that

$$0\sim\sigma|_\Lambda=\sum_{j=1}^n\partial_z\zeta_j(z)\wedge dz_j+\mathcal{O}(|\operatorname{Im}z|^\infty+|\operatorname{Im}\zeta(z)|^\infty)_{C^\infty(\mathbb{C}^n;\wedge^{2n}\mathbb{C}^n)},$$

and in view of (5.19)

$$\partial_{z_k}\zeta_j(x)-\partial_{z_j}\zeta_k(x)=\mathcal{O}(|x|^\infty), \quad x\in\mathbb{R}^n,\ |x|<\epsilon.$$

For $x \in \mathbb{R}^n$ define Ψ by the simplest version of the Poincaré lemma:

$$\Psi(x) = \int_0^1 \zeta(tx) \cdot x dt.$$

Then

$$\begin{aligned} \partial_{x_j} \Psi(x) &= \int_0^1 \left(\sum_{k=1}^n t z_k \partial_{x_j} \zeta_k(tx) + \zeta_j(tx) \right) dt \\ &= \int_0^1 \left(\sum_{k=1}^n t z_k \partial_{x_k} \zeta_j(tx) + \zeta_j(tx) \right) dt + \mathcal{O}(|x|^\infty) \\ &= \int_0^1 \partial_t (t \zeta_j(tx)) dt + \mathcal{O}(|x|^\infty) = \zeta_j(x) + \mathcal{O}(|\operatorname{Im} \zeta(x)|^\infty), \end{aligned} \tag{5.20}$$

in the last argument we used (5.19) again. We now define $\Psi(z)$ as an almost analytic extension of Ψ. From [MeSj74, Proposition 1.7(ii)] we obtain (5.18). □

The strict positivity of Λ implies that $\operatorname{Im} \Psi_{xx}(0)$ is positive definite:

$$\begin{gathered} T_{(0,0)}\{z, \Psi_z(z)\} = \{(Z, \Psi_{xx}(0)Z) : Z \in \mathbb{C}^n\}, \\ \operatorname{Im}\langle \Psi_{xx}(0)Z, \bar{Z}\rangle = \tfrac{1}{i}\sigma((Z, \Psi_{xx}(0)Z), (\bar{Z}, \overline{\Psi}_{xx}, Z)) \geq c|Z|^2. \end{gathered}$$

The eikonal equation is satisfied in the following sense: for $z \in \mathbb{C}^n$, $|z| < \epsilon$,

$$\widetilde{p}(z, \Psi_z) = \mathcal{O}(|\operatorname{Im} z|^\infty + |\operatorname{Im} \Psi_z|^\infty) = \mathcal{O}(|\operatorname{Im} z|^\infty + |\operatorname{Im} \Psi|^\infty), \tag{5.21}$$

(We can replace $\operatorname{Im} \Psi_z$ with $\operatorname{Im} \Psi$ as for $\operatorname{Im} z = 0$, $\operatorname{Im} \Psi \geq 0$ and hence $|\operatorname{Im} \Psi_x| \leq C|\operatorname{Im} \Psi|^{\frac{1}{2}}$.)

Proof of (5.21). We have for $s \in \mathbb{C}$,

$$\widehat{sH_{\widetilde{p}}}\widetilde{p} = \overline{s\partial_\zeta \widetilde{p}} \cdot \partial_{\bar{z}} \widetilde{p} - \overline{s\partial_z \widetilde{p}} \cdot \partial_{\bar{\zeta}} \widetilde{p} = \mathcal{O}(|\operatorname{Im} z|^\infty + |\operatorname{Im} \zeta|^\infty).$$

Since $\widetilde{p}|_{\Lambda_0} = 0$, we see that $\widetilde{p}(z, \Psi_z) \sim 0$ at $\Lambda_{\mathbb{R}}$. □

Hence to find u satisfying (5.14) we take

$$u(x) := e^{i\Psi(x)/h} a(x, h). \tag{5.22}$$

Almost analytic extension of a will make a natural appearance in the transport equation.

5.2.2. *Transport equations.* We write the amplitude as $a = a_0 + ha_1 + \cdots$, $a_j \in S$ and to find the transport equations we apply the method of com plex stationary phase [MeSj74, Theorem 2.3, p.148] to

$$
\begin{aligned}
Pu(x) &= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} P(x,\xi,h) e^{\frac{i}{h}(\langle x-y,\xi\rangle + \Psi(y))} a(y,h) dy d\xi \\
&= e^{\frac{i}{h}\Psi(x)} \Bigg[\widetilde{p}(x,\Psi_x) a(x,h) + \frac{h}{i} \left(\sum_{j=1}^{k} \partial_{\zeta_k} \widetilde{p}(x,\Psi_x) \partial_{x_k} + \tfrac{1}{2} \sum_{k=1}^{n} \partial^2_{x_k \xi_k} \widetilde{p}(x,\Psi_x) \right) a(x,h) \\
&\quad + h \left(\widetilde{p}_1(x,\Psi_x) - i \sum_{k,\ell=1}^{n} \Psi_{x_k x_\ell}(x) \partial^2_{\xi_k \xi_\ell} \widetilde{p}(x,\Psi_x) \right) a(x,h) + \mathcal{O}(h^2)_{\mathscr{S}} \Bigg].
\end{aligned}
$$

The first term is estimated using (5.21) and the transport equation become

$$
\begin{gathered}
V_p \widetilde{a}_k(z) + \tfrac{1}{2} \mathrm{div} V_p \, \widetilde{a}_k(z) + i c_\Psi \widetilde{a}_k(z) = F_{k-1}(\widetilde{a}_0, \cdots, \widetilde{a}_{k-1}), \quad F_{-1} \equiv 0, \\
V_p := (\pi_\Lambda)_* H_{\tilde{p}} = \partial_{\zeta_k} \widetilde{p}(x,\Psi_x) \partial_{z_k}, \quad \pi_\Lambda : \Lambda = \{(z, \Psi_z(z))\} \to \mathbb{C}^n, \\
c_\Psi(z) := \widetilde{p}_1(z,\Psi_z) - i \sum_{k,\ell=1}^{n} \Psi_{z_k z_\ell}(z) \partial^2_{\zeta_k \zeta_\ell} \widetilde{p}(z,\Psi_z).
\end{gathered}
\tag{5.23}
$$

We now solve these equations using the "almost analytic" flow of V_p:

$$
\begin{gathered}
z(t,w') := \exp(\widehat{tV_p})(0,w'), \quad |\operatorname{Im}[z(t,w') - w']| \sim |t|, \\
w' \in B_{\mathbb{C}^{n-1}}(0,\epsilon), \quad t \in \mathbb{C}, \quad |t| < \epsilon.
\end{gathered}
\tag{5.24}
$$

So, for instance,

$$
\widetilde{a}_0(z) := \exp g_0(z), \quad g_0(z(t,w')) := -\int_0^1 t b_0(z(ts,w')) ds, \quad b_0 := \tfrac{1}{2} \mathrm{div} V_p + i c_\Psi.
$$

We now calculate the action of V_p on the $g_0(z(t,w'))$ using almost analyticity of b_0 and the properties of $z(t,w')$ in (5.24):

$$
\begin{aligned}
V_p g_0(z(t,w')) &= -\int_0^1 \sum_{k=0}^{\infty} \frac{s^k}{k!} V_p \widehat{tV_p}^k t b_0(z(0,w')) + \mathcal{O}(|t|^\infty) ds \\
&= -\int_0^1 \sum_{k=0}^{\infty} \frac{s^k}{k!} \widehat{tV_p}^{k+1} b_0(z(0,w')) + \mathcal{O}(|t|^\infty) + \mathcal{O}(|\operatorname{Im} z(0,w')|^\infty) ds \\
&= -\sum_{k=0}^{\infty} \frac{1}{(k+1)!} \widehat{tV_p}^{k+1} b_0(z(0,w')) + \mathcal{O}(|t|^\infty + O(|\operatorname{Im} z(0,w')|^\infty) \\
&= -b_0(z(t,w')) + \mathcal{O}(|t|^\infty + |\operatorname{Im} z(0,w')|^\infty) \\
&= -b_0(z(t,w')) + \mathcal{O}(|\operatorname{Im} \Psi|^\infty + |\operatorname{Im} z|^\infty).
\end{aligned}
$$

This gives (5.23) with $k = 0$. Similarly we obtain solutions to the remaining transport equations. We obtain a by taking an asymptotic sum and multiplying it by $\chi(x)$ where

$\chi \in C_c^\infty(B_{\mathbb{R}^n}(0,\epsilon)$, $\chi \equiv 1$ near 0. Then returning to (5.22) we see that

$$P(x, hD_x, h)(e^{i\Psi(x)/h}a(x,h)) = \mathcal{O}(h^\infty + e^{-|\operatorname{Im}\Psi(x)|/Ch}|\operatorname{Im}\Psi(x)|^\infty)_{C_c^\infty} = \mathcal{O}(h^\infty)_{C_c^\infty},$$
$$\operatorname{WF}_h\left(e^{i\Psi(x)/h}a(x,h)\right) = \{(0,0)\}. \tag{5.25}$$

6. Projector in the case of deformations

We now present a version of [Sj96, §2] in the case of the usual FBI transform on $\mathbb{R}^n$. It is based on deformation of $T^*\mathbb{R}^n$ to a I-Lagrangian, $\mathbb{R}$-symplectic submanifold of $T^*\mathbb{C}^n$. In §8 we will show that this approach, described in §§3,4, is equivalent to the approach using weights.

The FBI transform and weights used in [Sj96, §2] are different from the ones used in [Ma02] and §3. The procedure of [Sj96], and earlier of [HeSj86], involves deformation of $Tu(x,\xi)$ to an I-Lagrangian, $\mathbb{R}$-symplectic submanifold of $\mathbb{C}^{2n}$:

$$\Lambda = \Lambda_G := \{(x + i\partial_\xi G(x,\xi), \xi - i\partial_x G(x,\xi) : (x,\xi) \in \mathbb{R}^{2n}\}, \tag{6.1}$$

where $G \in C_c^\infty(\mathbb{R}^{2n})$ is assumed to be small in C^2. This means that for the symplectic form (2.18) on $\mathbb{C}^{2n} = T^*\mathbb{C}^n$ we have

$$\operatorname{Im}\sigma|_\Lambda = 0, \quad \sigma_\Lambda := \operatorname{Re}\sigma|_\Lambda \text{ is non-degenerate.}$$

Smallness of G is needed for the second property. We also note that Λ is a (maximally) totally real submanifold of $\mathbb{C}^{2n} \simeq \mathbb{R}^{4n}$, $T_\rho\Lambda \cap iT_\rho\Lambda = \{0\}$.

We parametrize Λ by (x,ξ) using (6.1) and define

$$T_\Lambda u(x,\xi) = Tu(x + iG_\xi(x,\xi), \xi - iG_x(x,\xi)). \tag{6.2}$$

A natural weight associated to G is given by $H(x,\xi)$ satisfying

$$d_{x,\xi}H = -\operatorname{Im}\zeta \cdot dz|_\Lambda. \tag{6.3}$$

Since

$$-\operatorname{Im}\zeta \cdot dz|_\Lambda = -\operatorname{Im}(\xi - iG_x)d(x + iG_\xi) = (G_x - (\xi \cdot G_\xi)_x) \cdot dx - (G_{\xi\xi}\xi) \cdot d\xi,$$

H is given by (we choose $H = 0$ for $G = 0$)

$$H(x,\xi) = G(x,\xi) - \xi \cdot G_\xi(x,\xi). \tag{6.4}$$

Lemma 6. *For $u \in \mathscr{S}(\mathbb{R}^{2n})$ define*

$$S_\Lambda u(y) := \bar{c}h^{-\frac{3n}{4}} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(\langle y - x - iG_\xi, \xi - iG_x\rangle + \frac{i}{2}(x + iG_\xi - y)^2)} b(x,\xi)u(x,\xi)dxd\xi,$$
$$b(x,\xi)dx \wedge d\xi = d(\xi - iG_x) \wedge d(x + iG_\xi). \tag{6.5}$$

Then

$$S_\Lambda T_\Lambda v = v, \quad v \in L^2(\mathbb{R}^n), \tag{6.6}$$

and

$$T_\Lambda S_\Lambda = \mathcal{O}(1) : L^2_\Lambda \to L^2_\Lambda, \quad L^2_\Lambda := L^2(\mathbb{R}^{2n}, e^{-2H/h} dx d\xi). \tag{6.7}$$

Remark. The weight H defined by (6.3) is precisely the unique weight (up to an additive constant) for which (6.7) holds – see (6.9) in the proof below.

Proof. To prove (6.6) we write out the composition and deform the contour. The phase in the composition is given by

$$\langle \xi - iG_x, y - y' \rangle + \tfrac{i}{2} \left((x + iG_x - y)^2 + (x + iG_x - y')^2 \right).$$

If $z = x + iG_\xi(x,\xi)$, $\zeta = \xi - iG_x(x,\xi)$, then our choice of b shows that $b(x,\xi) dx d\xi = d\zeta \wedge dz$ and, by deforming the contour from Λ_G to $\Lambda_0 := \mathbb{R}^{2n}$ (note that Λ_G and Λ_0 coincide outside of a compact set),

$$\begin{aligned} S_\Lambda T_\Lambda u(y) &= c\bar{c} h^{-\frac{3n}{2}} \iint_{\Lambda_G} e^{\frac{i}{h}((y-y')\zeta + \frac{i}{2}((y-z)^2 + (y'-z)^2))} u(y') d\zeta dz dy' \\ &= c\bar{c} h^{-\frac{3n}{2}} \iint_{\Lambda_0} e^{\frac{i}{h}((y-y')\zeta + \frac{i}{2}((y-z)^2 + (y'-z)^2))} u(y') d\zeta dz dy \\ &= T^* T u(y) = u(y). \end{aligned}$$

To prove (6.7), we complete the squares in the phase arising in the composition of $T_\Lambda S_\Lambda$ to obtain the phase

$$\begin{aligned} \psi_\Lambda = \tfrac{1}{2}(\langle z, \zeta \rangle - \langle z', \zeta' \rangle) + \tfrac{1}{2}(\langle z, \zeta' \rangle - \langle z', \zeta \rangle) + \tfrac{i}{2}((\zeta - \zeta')^2 + (z - z')^2)), \\ z := x + iG_\xi, \ z' := x' + iG_{\xi'}, \\ \zeta := \xi - iG_x, \ \zeta := \xi' - iG_{x'}, \end{aligned} \tag{6.8}$$

and where $G_{\bullet'} := G_{\bullet'}(x', \xi')$.

We calculate (noting that as Λ is totally real we can use holomorphic differentials by taking almost analytic extensions), $d \operatorname{Im} \psi_\Lambda = (\partial + \bar{\partial}) \operatorname{Im} \psi_\Lambda = \operatorname{Im} \partial \psi_\Lambda$, where d denote the differential with respect to (x, ξ, x', ξ') and ∂ the holomorphic differential with respect to (z, ζ, z', ζ'). Using the expression above and restricting to $z = z'$ and $\zeta = \zeta'$ we see that

$$d \operatorname{Im} \psi_\Lambda |_{(x,\xi)=(x',\xi')} = \operatorname{Im}(\zeta dz - \zeta dz') = (-d_{x,\xi} H + d_{x',\xi'} H) |_{(x,\xi)=(x',\xi')}. \tag{6.9}$$

This means that

$$\operatorname{Im} \psi_\Lambda = -H(x,\xi) + \mathcal{O}((x - x')^2 + (\xi - \xi')^2) + H(x', \xi'), \tag{6.10}$$

and as G (and H) are small in C^2 the comparison with the case $G = H = 0$ gives

$$\operatorname{Im} \psi_\Lambda = -H(x,\xi) + (\tfrac{1}{2} - \mathcal{O}(\|G\|_{C^2}))((\xi - \xi')^2 + (x - x')^2) + H(x', \xi').$$

The Schur criterion now gives the boundedness in (6.7). $\square$

Remark. A more pedestrian way of seeing (6.10) follows from a direct calculation and from using the formula (6.4):

$$\begin{aligned}2\operatorname{Im}\psi_\Lambda &= \xi G_\xi - \xi' G_{\xi'} - xG_x + x'G_{x'} + \xi' G_\xi - xG_{x'} + x'G_x - \xi G_{\xi'}\\ &\quad + (\xi-\xi')^2 - (G_x - G'_x)^2 + (x-x')^2 - (G_\xi - G_{\xi'})^2\\ &= 2\xi G_\xi - 2\xi' G_\xi + 2(\xi'-\xi)G_\xi + 2(x'-x)G_x\\ &\quad + (\xi'-\xi)(G_{\xi'} - G_\xi) + (x'-x)(G_{x'} - G_x)\\ &\quad + (\xi-\xi')^2 - (G_x - G'_x)^2 + (x-x')^2 - (G_\xi - G_{\xi'})^2\\ &= -2G(x,\xi) + 2\xi G(x,\xi) + 2G(x',\xi') - 2\xi' G_{\xi'}(x',\xi')\\ &\quad + (1 - \mathcal{O}(\|G\|_{C^2})((\xi-\xi')^2 + (x-x')^2)\\ &= -2H(x,\xi) + 2H(x',\xi') + (1 - \mathcal{O}(\|G\|_{C^2})((\xi-\xi')^2 + (x-x')^2).\end{aligned}$$

We now move to construct the orthogonal projector

$$\Pi_\Lambda(L^2_\Lambda) = T_\Lambda(L^2(\mathbb{R}^n)), \quad \Pi_\Lambda^{*,H} = \Pi_\Lambda, \quad \Pi_\Lambda^2 = \Pi_\Lambda, \tag{6.11}$$

and describe its structure. That is done similarly to the construction of Π_φ in §3. The complication comes from a more involved form of the operators ζ_j which requires the use of the almost analytic methods reviewed in §5.

We start by defining operators which annihilate the deformed FBI transform. We first recall that the holomorphic extension of T satisfies

$$Z_j T \equiv 0, \quad Z_j = hD_{z_j} - \zeta_j - ihD_{\zeta_j}.$$

Hence,

$$Z_j^\Lambda(x,\xi,hD_x,hD_\xi)T_\Lambda \equiv 0, \quad Z_j^\Lambda(x,\xi,hD_x,hD_\xi) := \left(hD_{z_j} - \zeta_j - ihD_{\zeta_j}\right)|_\Lambda, \tag{6.12}$$

where

$$\begin{bmatrix} hD_z|_\Lambda \\ hD_\zeta|_\Lambda \end{bmatrix} = \begin{bmatrix} I + iG_{x\xi}(x,\xi) & iG_{\xi\xi}(x,\xi) \\ -iG_{xx}(x,\xi) & I - iG_{\xi,x}(x,\xi) \end{bmatrix}^{-1} \begin{bmatrix} hD_x \\ hD_\xi \end{bmatrix}.$$

Since Z_j's commute we have (with $\alpha = (x,\xi)$) $[Z_j^\Lambda(\alpha, hD_\alpha), Z_k^\Lambda(\alpha, hD_\alpha)] = 0$.

We now repeat the construction outlined in [Sj96] and presented in the slightly simpler setting in §3. Again, the argument proceeds in the following steps:

- construction of a uniformly bounded operator (as $h \to 0$) $B_\Lambda : L^2_\Lambda \to L^2_\Lambda$ such that $Z_j^\Lambda B_\Lambda = \mathcal{O}(h^\infty)_{L^2_\Lambda \to L^2_\Lambda}$, $B_\Lambda^{*,H} = B_\Lambda$ and $B_\Lambda^2 = B_\Lambda + \mathcal{O}(h^\infty)_{L^2_\Lambda \to L^2_\Lambda}$;
- characterization of the *unique* properties of the Schwartz kernel of B_Λ: uniqueness of the phase and the determination of the amplitude from its restriction to the diagonal;
- finding a projector $P_\lambda = \mathcal{O}(1)_{L^2_\Lambda \to L^2_\Lambda}$ onto the image of T_Λ.

- choosing $f \in S(1)$, $f \geq 1/C$ so that $A := P_\Lambda M_f P_\Lambda^{*,H}$ (in the notation of §2), satisfies $A = B_\Lambda + \mathcal{O}(h^\infty)_{L^2_\Lambda \to L^2_\Lambda}$; this relies on the uniqueness properties in the construction of B_Λ;
- expressing Π_Λ as a suitable contour integral of the resolvent of A and using it to show that $\Pi_\Lambda = B_\Lambda + \mathcal{O}(h^\infty)_{L^2_\Lambda \to L^2_\Lambda}$.

To construct B_Λ we postulate an ansatz

$$\begin{gathered} B_\Lambda u(\alpha) = h^{-n} \int_{T^*\mathbb{T}^n} e^{i\psi(\alpha,\beta)/h - 2H(\beta)/h} a(\alpha,\beta) u(\beta) dm_\Lambda(\beta), \\ dm_\Lambda(\beta) := (\sigma|_\Lambda)^n/n! = d\alpha, \ \beta = \operatorname{Re}\alpha, \quad \alpha \in \Lambda, \end{gathered} \tag{6.13}$$

and as in (2.9)

$$\begin{aligned} Z_j^\Lambda(\alpha, hD_\alpha)\left(e^{i\psi(\alpha,\beta)/h} a(\alpha,\beta)\right) &= \mathcal{O}(h^\infty + |\alpha-\beta|^\infty), \\ \widetilde{Z}_j^\Lambda(\beta, hD_\beta)\left(e^{i\psi(\alpha,\beta)/h} a(\alpha,\beta)\right) &= \mathcal{O}(h^\infty + |\alpha-\beta|^\infty), \end{aligned} \tag{6.14}$$

where $\widetilde{Z}_j^\Lambda$, $j = 1, \cdots, n$ are defined in (6.16) below. The equations (6.14) are consistent with $\psi(\alpha,\beta) = -\overline{\psi(\beta,\alpha)}$and $a(\alpha,\beta) = \overline{a(\beta,\alpha)}$ – see §6.1.

Notation. Suppose Q is a differential operator with holomorphic coefficients defined near Λ. We write

$$Q(\alpha, hD_\alpha) = Q(z, \zeta, hD_z, hD_\zeta),$$

and $\overline{Q(\alpha, hD_\alpha)}$ for the corresponding anti-holomorphic operator. The operator Q can be restricted to the totally real submanifold Λ and that restriction is denoted by Q^Λ. If we parametrize Λ by $\alpha \in T^*\mathbb{R}^n$ we write $Q^\Lambda = Q^\Lambda(\alpha, hD_\alpha)$. This operator then has an almost analytic extension to a neighbourhood of Λ and we denote it by the same letter. We also consider the anti-holomorphic operator $u \mapsto \overline{Q^t \bar{u}}$,

$$\int_\Lambda u(\alpha)[Qv](\alpha) d\alpha = \int_\Lambda [Q^t u](\alpha) v(\alpha) d\alpha,$$

and denote its restriction of Λ by $\bar{Q}^\Lambda$. The reason for this notation is the fact that, as function on $\Lambda \simeq T^*\mathbb{R}^n$,

$$\sigma(\overline{Q}^\Lambda) = \overline{\sigma(Q^\Lambda)}. \tag{6.15}$$

We use the same letter to denote its almost analytic extension to a neighbourhood of Λ. We also define $\widetilde{Q}^\Lambda$, $\sigma(\widetilde{Q}^\Lambda)(\alpha, \alpha^*) = \overline{\sigma(Q^\Lambda)}(\alpha, -\alpha^*)$. Here σ refers to the semiclassical principal symbol.

We illustrate this in a simple example: $\Lambda = \{(x, \xi - ig'(x)) : (x,\xi) \in \mathbb{R}^2\}$, $g \in C^\infty(\mathbb{R};\mathbb{R})$. If $Q = hD_z - \zeta - ihD_\zeta$ then

$$\begin{aligned} Q^\Lambda &= hD_x + ig''(x) hD_\xi - \xi + ig'(x) - ihD_\xi, \\ \bar{Q}^\Lambda &= hD_x - ig''(x) hD_\xi - \xi - ig'(x) + ihD_\xi, \\ \widetilde{Q}^\Lambda &= -hD_x + ig''(x) hD_\xi - \xi - ig'(x) - ihD_\xi, \end{aligned}$$

with the operators extended to a neighbourhood of Λ by taking holomorphic derivatives and an almost analytic extension of g. $\square$

We note again that the weight H does not appear in $\widetilde{Z}_j^\Lambda$. To see this we first compute $(Z_j^\Lambda)^{*,H}$:

$$\begin{aligned}\langle Z_j^\Lambda u, v\rangle_{L^2_\Lambda} &= \int_{T^*\mathbb{T}^n} Z_j^\Lambda(\alpha, hD_\alpha)u(\alpha)\overline{v(\alpha)}e^{-2H(\alpha)}dm_\Lambda(\alpha)\\ &= \int_\Lambda Z_j(\alpha, hD_\alpha)u(\alpha)\overline{v(\alpha)}e^{-2H(\alpha)/h}d\alpha\\ &= \int_\Lambda u(\alpha)((Z_j(\alpha, hD_\alpha))^t(\overline{v(\alpha)}e^{-2H(\alpha)})d\alpha\\ &= \int_\Lambda u(\alpha)\overline{\left(e^{2H(\alpha)/h}\overline{Z}_j(\alpha, hD_\alpha)e^{-2H(\alpha)}\right) v(\alpha)}e^{-2H(\alpha)})d\alpha\end{aligned}$$

where (see the remark about notation above) $\overline{Z}_j(\alpha, hD_\alpha)v(\alpha) := \overline{(Z_j(\alpha, hD_\alpha)^t\overline{v(\alpha)}}$. Hence

$$\left(Z_j^\Lambda(\alpha, hD_\alpha)\right)^{*,H} = e^{2H(\alpha)/h}\overline{Z}_j^\Lambda(\alpha, hD_\alpha)e^{-2H(\alpha)}, \quad \overline{Z}_j^\Lambda = \overline{Z}_j|_\Lambda.$$

We then have

$$\begin{aligned}0 \equiv (Z_j^\Lambda B_\Lambda)^*u(\alpha) &= B_\Lambda^*(Z_j^\Lambda)^*u(\alpha) = B_\Lambda(Z_j^\Lambda)^*u(\alpha)\\ &= \int_{T^*\mathbb{T}^n} K_\Lambda(\alpha,\beta)e^{-2H(\beta)/h}(Z_j^\Lambda)^*u(\beta)dm_\Lambda(\beta)\\ &= \int_\Lambda K_\Lambda(\alpha,\beta)\overline{Z}_j(\beta, hD_\beta)\left(e^{-2H(\beta)/h}u(\beta)\right)d\beta\\ &= \int \left(\widetilde{Z}_j(\beta, hD_\beta)K_\Lambda(\alpha,\beta)\right)u(\beta)e^{-2H(\beta)/h}d\beta\\ &= \int \widetilde{Z}_j^\Lambda(\beta, hD_\beta)K_\Lambda(\alpha,\beta)e^{-2H(\beta)/h}u(\beta)dm_\Lambda(\beta),\end{aligned}$$

where

$$\widetilde{Z}_j(\beta, hD_\beta)v(\beta) := \overline{Z}_j(\beta, hD_\beta)^t v(\beta) = \overline{Z_j(\beta, hD_\beta)\overline{v(\beta)}}, \quad \widetilde{Z}_j^\Lambda := \widetilde{Z}_j|_\Lambda. \tag{6.16}$$

Explicitly we have

$$\widetilde{Z}_j(z,\zeta, hD_x, hD_\zeta) = -h\bar{D}_{z_j} - \bar{\zeta}_j - ih\bar{D}_{\zeta_j}, \tag{6.17}$$

where

$$\begin{bmatrix} h\bar{D}_z|_\Lambda \\ h\bar{D}_\zeta|_\Lambda\end{bmatrix} = \begin{bmatrix} I - iG_{x\xi}(x,\xi) & -iG_{\xi\xi}(x,\xi)\\ iG_{xx}(x,\xi) & I + iG_{\xi,x}(x,\xi)\end{bmatrix}^{-1}\begin{bmatrix} hD_x \\ hD_\xi\end{bmatrix}.$$

Also, $[\widetilde{Z}_j^\Lambda, \widetilde{Z}_k^\Lambda] = 0$.

6.1. **Eikonal equations.** Let ζ_j^Λ and $\widetilde{\zeta}_j^\Lambda$ be the principal symbols of Z_j^Λ and $\widetilde{Z}_j^\Lambda$ respectively. The eikonal equations we want to solve are

$$\zeta_j^\Lambda(\alpha, d_\alpha\psi(\alpha,\beta)) = \mathcal{O}(|\alpha-\beta|^\infty), \quad \widetilde{\zeta}_j^\Lambda(\beta, d_\beta\psi(\alpha,\beta)) = \mathcal{O}(|\alpha-\beta|^\infty), \quad \alpha,\beta\in\Lambda. \tag{6.18}$$

We recall that ζ_j^Λ are restrictions to $T^*\Lambda$ of holomorphic functions on $T^*\mathbb{C}^{2n}$: $\zeta_j = x_j^* - \xi_j - i\xi_j^*$, $(x,\xi,x^*,\xi^*)\in\mathbb{C}^{2n}\times\mathbb{C}^{2n}$. We now put

$$\overline{\zeta}_j^\Lambda(\alpha,\alpha^*) := \widetilde{\zeta}_j^\Lambda(\alpha,-\alpha^*)$$

which is the principal symbol of $\overline{Z}_j^\Lambda$.

From the geometric point of view, so that we remain in the same framework as in §2.2, it is convenient to construct the phase function corresponding to $B_H := e^{-H/h}B_\Lambda e^{H/h}$. That means that properties of B_Λ on $L^2(\Lambda)$ are equivalent to the properties of B_H on L^2, that is we want

$$B_H = B_H^*, \quad B_H^2 = B_H. \tag{6.19}$$

We have

$$B_H u(\alpha) = h^{-n}\int e^{\frac{i}{h}\psi_H(\alpha,\beta)}a(\alpha,\beta)u(\beta)d\beta,$$
$$\psi_H(\alpha,\beta) := iH(\alpha) + \psi(\alpha,\beta) + iH(\beta).$$

To simplify the notation we first assume that Λ (and consequently H defined by (6.3) and ζ_j^H, $\bar{\zeta}_j^H$) are analytic. We will replace that by almost analyticity by proceeding as in §5.

To construct ψ_H we consider $\mathscr{C}_H$, the relation associated to it:

$$\mathscr{C}_H = \{(\alpha, d_\alpha\psi_H(\alpha,\beta), \beta, -d_\beta\psi_H(\alpha,\beta)) : (\alpha,\beta)\in \mathrm{nbhd}_{\mathbb{C}^{4n}}(\mathrm{Diag}(\Lambda\times\Lambda))\}. \tag{6.20}$$

In view of (6.19) we must have

$$\mathscr{C}_H\circ\mathscr{C}_H = \mathscr{C}_H, \quad \overline{\mathscr{C}}_H^t = \mathscr{C}_H. \tag{6.21}$$

(Here $\overline{\mathscr{C}}_H^t := \{(\bar\rho,\bar\rho') : (\rho',\rho)\in\mathscr{C}_H\}$, and $\rho\mapsto\bar\rho$ is defined after the almost analytic identification of Λ with $T^*\mathbb{R}^n$.)

We define

$$\begin{aligned}\zeta_j^H(\alpha,\alpha^*) &:= \zeta_j^\Lambda(\alpha,\alpha^* - idH(\alpha)),\\ \bar{\zeta}_j^H(\alpha,\alpha^*) &:= \bar{\zeta}_j^\Lambda(\alpha,\alpha^* + idH(\alpha)) = \overline{\zeta_j^H(\bar\alpha,\bar\alpha^*)},\end{aligned} \tag{6.22}$$

so that the formal analogue of (6.18) is given by

$$\zeta_j^H(\alpha, d_\alpha\psi_H(\alpha,\beta)) = 0, \quad \bar{\zeta}_j^H(\alpha, -d_\beta\psi_H(\alpha,\beta)) = 0. \tag{6.23}$$

(Here again the $\bar{\alpha}$ and $\bar{\alpha}^*$ are defined after an identification of Λ with $T^*\mathbb{R}^n$). We construct $\mathscr{C}_H$ geometrically – see §2.2 for the simpler linear algebraic treatment in the case of the FBI transform without weights. In view of (6.20) and (6.23) we must have

$$\begin{aligned}\mathscr{C}_H \subset S \times \overline{S}, \quad S &:= \{\rho : \zeta_j^H(\rho) = 0, \ \rho \in \mathrm{nbhd}_{\mathbb{C}^{4n}}(T^*\Lambda)\}, \\ \overline{S} := \{\bar{\rho} : \rho \in S\} &= \{\rho : \bar{\zeta}_j^H(\rho) = 0, \ \rho \in \mathrm{nbhd}_{\mathbb{C}^{4n}}(T^*\Lambda)\}.\end{aligned}$$

If follows that the complex vector fields $H_{\pi_L^*\zeta_j^H}$ and $H_{\pi_R\bar{\zeta}_j^H}$ ($\pi_L(\rho,\rho') := \rho$, $\pi_R(\rho,\rho') := \rho'$)) are tangent to $\mathscr{C}_H$. By checking the case of $T^*\Lambda = T^*\mathbb{R}^n$ (no deformation and hence $H \equiv 0$) we see, as in §2.2 that $S \cap \bar{S}$ is a symplectic submanifold (with respect to the complex symplectic form) of complex dimension $4n$. The independence of $H_{\zeta_k^H}$, $H_{\bar{\zeta}_j^H}$, $j,k = 1,\cdots n$ (again easily seen in the unperturbed case) shows that

$$B_{\mathbb{C}^n}(0,\epsilon) \times B_{\mathbb{C}^n}(0,\epsilon) \times (S \cap \bar{S}) \ni (t,s,\rho) \mapsto (\exp\langle t, H_{\zeta^H}\rangle(\rho), \exp\langle s, H_{\bar{\zeta}_H}\rangle(\rho)) \in \mathbb{C}^{8n},$$

is a bi-holomorphic map to an embedded (complex) $4n$ dimensional submanifold. This and idempotence (first condition in (6.21)) imply that

$$\mathscr{C}_H = \left\{ (\exp\langle t, H_{\zeta^H}\rangle(\rho), \exp\langle s, H_{\bar{\zeta}^H}\rangle(\rho)) : \rho \in S \cap \bar{S}, \quad t,s \in B_{\mathbb{C}^n}(0,\epsilon) \right\},$$

where $\langle t, H_{\bullet^H}\rangle := \sum_{k=1}^n t_k H_{\bullet_k^H}$, $\bullet = \zeta, \bar{\zeta}$.

The second condition in (6.21) is automatically satisfied (this makes sense since $\mathscr{C}_H \subset S \times \bar{S}$ came from demanding that $0 = (\zeta_j^H B)^* = B(\zeta_j^H)^*)$. Since $\pi : \mathscr{C}_H \to \mathrm{nbhd}_{\mathbb{C}^{4n}}(\Lambda \times \Lambda)$ is surjective we have have a parametrization given by (6.20) with ψ_H determined up to an additive constant. We claim that we can choose that constant so that

$$\psi_H(\alpha,\alpha) = 0. \tag{6.24}$$

To see this we note that (from $\mathscr{C}_H = \overline{\mathscr{C}}^t_H$)

$$d_\alpha \psi_H(\alpha,\beta)|_{\alpha=\beta} = -\overline{d_\beta \psi_H(\alpha,\beta)}|_{\alpha=\beta}, \ \alpha \in \Lambda,$$

and hence

$$\begin{aligned} d_\alpha(\psi_H(\alpha,\alpha)) &= d_\alpha \psi_H(\alpha,\beta)|_{\alpha=\beta} + d_\beta \psi_H(\alpha,\beta)|_{\alpha=\beta} \\ &= 2i \operatorname{Im} d_\alpha \psi_H(\alpha,\beta)|_{\alpha=\beta}, \quad \alpha \in \Lambda. \end{aligned} \tag{6.25}$$

To find $\operatorname{Im} d_\alpha \psi_H(\alpha,\beta)|_{\alpha=\beta}$ it is convenient to go to the origins of the symbols ζ_j^H (6.22) : Z_j^Λ's, with symbols ζ_j^Λ annihilate the phase in T_Λ and hence

$$\begin{gathered} S_\alpha := S \cap T_\alpha^*\Lambda^{\mathbb{C}} = \left\{ (\alpha, d_\alpha\varphi(\alpha,y) + idH(\alpha)) \ : \ y \in \mathbb{C}^n \right\}, \\ \varphi(\alpha,y) := \langle z-y, \zeta\rangle + i(z-y)^2/2, \\ z = \alpha_x + iG_\xi(\alpha_x,\alpha_\xi), \quad \zeta = \alpha_\xi - G_x(\alpha_x,\alpha_\xi). \end{gathered}$$

In the case $G = 0$ (and hence $H = 0$), S_α and $\bar{S}_\alpha := \bar{S} \cap T^*_\alpha \Lambda^{\mathbb{C}}$ intersect transversally in one point and that has to remain true under perturbations. Hence we are looking for a solution to

$$d_\alpha \varphi(\alpha, y) + i dH(\alpha) = \overline{d_\alpha \varphi(\alpha, y')} - i dH(\alpha). \tag{6.26}$$

Now, at $y = y' = \alpha_x$ we have $d_\alpha \varphi(\alpha, y) = \zeta dz|_\Lambda$ and in view of the definition of dH in (6.3), (6.26) holds. It follows that for $\alpha \in \Lambda$

$$S_\alpha \cap \bar{S}_\alpha = \{(\alpha, \operatorname{Re}(\zeta dz|_\Lambda)\} \in T^*\Lambda.$$

Next, by analytic continuation (replaced by almost analytic continuation below), it follows (since intersection of S and $\bar{S}$ is transversal and we have the right dimension) that

$$\mathscr{J} := S \cap \bar{S} = \{(\alpha, \omega(\alpha) + \bar{\omega}(\alpha)) : \alpha \in \operatorname{nbhd}_{\mathbb{C}^{2n}}(\Lambda)\}, \quad \omega(\alpha)|_\Lambda = \tfrac{1}{2} \operatorname{Re}(\zeta dz|_\Lambda), \tag{6.27}$$

where we recall that $\bar{\omega}(\alpha) = \overline{\omega(\bar{\alpha})}$. But this shows that $\pi^{-1}(\operatorname{diag}(\Lambda \times \Lambda)) \cap \mathscr{C}_H$ is real which means that $\operatorname{Im} d_\alpha \psi_H(\alpha, \beta)\big|_{\beta=\alpha} = 0$ for $\alpha \in \Lambda$ showing that $\psi_H(\alpha, \alpha)$ is a constant which can be chosen be 0. This gives (6.24).

Remark. Vanishing of $\operatorname{Im} d_\alpha \psi_H(\alpha, \beta)$ also shows that

$$-\operatorname{Im} \psi_H(\alpha, \beta) = \mathcal{O}(|\alpha - \beta|^2)$$

and since G is assumed to be small, the case of $G = 0$ shows that

$$-\operatorname{Im} \psi_H(\alpha, \beta) \le -|\alpha - \beta|^2/C, \quad C > 0. \tag{6.28}$$

This shows that B_H given by (6.13) is bounded on L^2. ☐

We now comment on the general case and explain how to use almost analytic extensions off Λ. We first identify Λ with $T^*\mathbb{R}^n$ using (6.1) and extending G almost analytically to $\mathbb{C}^{4n}$. We then define $\mathscr{J}$ by (6.27) using an almost analytic extension of $\omega(\alpha)|_\Lambda$ (where $\bar{\omega}(\alpha) = \overline{\omega(\bar{\alpha})}$). We are now basically in the same situation as in §5.2.1, except for a larger number of vector fields, with Λ replaced by $\mathscr{C}_H$ and Λ_0 by $\{(\rho, \rho) : \rho \in \mathscr{J}\}$. Hence we define

$$\mathscr{C}_H = \left\{ \left(\exp \langle \widehat{t, H_{\zeta^H}} \rangle(\rho), \exp \langle \widehat{s, H_{\bar{\zeta}^H}} \rangle(\rho) \right) : \rho \in \mathscr{J}, \quad t, s \in B_{\mathbb{C}^n}(0, \epsilon) \right\}.$$

Almost the same arguments as in §5.2.1 show that

$$|\operatorname{Im} \exp \langle \widehat{t, H_{\zeta^H}} \rangle(\rho)| \ge |t|/C, \quad |\operatorname{Im} \exp \langle \widehat{s, H_{\bar{\zeta}^H}} \rangle(\rho)| \ge |s|/C, \quad \rho \in \mathscr{J}. \tag{6.29}$$

In fact, for $\zeta_j := z_j^* - \zeta_j - i\zeta_j^*$ and $\bar{\zeta}_j := z_j^* - \zeta_j + i\zeta_j^*$ we have $\{\zeta_j, \bar{\zeta}_k\} = 2i\delta_{jk}$. On $T^*\Lambda$, $\bar{\zeta}_k^H = \overline{\zeta_k^H}$ and $\{\zeta_j^H, \bar{\zeta}_k^H\}/2i$ is positive definite. By taking a linear combination of ζ_j^H's we can then arrange that, at a given point, $\{\zeta_j^H, \bar{\zeta}_k^H\}/2i = \delta_{jk}$. We can then make a linear symplectic change of variables at any point of $T^*\Lambda$ giving new variables (x, y, ξ, η), $x, y, \xi, \eta \in \mathbb{R}^n$, centered at $0 \in \mathbb{R}^{4n}$, such that

$$\zeta_j^H = c(\eta_j + i y_j) + \mathcal{O}(|x|^2 + |y|^2 + |\xi|^2 + |\eta|^2), \quad c > 0,$$

and this holds also for almost continuations of ζ_j^H. That means that near 0,

$$\mathscr{J} = \{(z,0,\zeta,0) + F(z,\zeta) : (z,\zeta) \in \mathrm{nbhd}_{\mathbb{C}^{2n}}(0)\}, \quad F = \mathcal{O}(|z|^2 + |\zeta|^2), \tag{6.30}$$

We also note that for $(z,\zeta) \in \mathbb{R}^{2n}$ (which corresponds to the interection with $T^*\Lambda$), $\mathscr{J}$ is real. This means that in (6.30),

$$\operatorname{Im} F(z,\zeta) = \mathcal{O}((|\operatorname{Im} z| + |\operatorname{Im} \zeta|)(|z| + |\zeta|)).$$

Hence,

$$\begin{aligned} |\operatorname{Im} \exp \langle \widehat{t, H_{\zeta^H}} \rangle ((z,0,\zeta,0) + F(z,\zeta))| &= |(\operatorname{Im} z, c\operatorname{Im} t, \operatorname{Im} \zeta, c\operatorname{Re} t)| \\ &\quad + \mathcal{O}((|\operatorname{Im} z| + |\operatorname{Im} \zeta|)(|z| + |\zeta|) + |t|^2) \\ &\geq |t|/C, \quad \text{if } |z|, |\zeta| \ll 1, \end{aligned}$$

with the corresponding estimate for $\bar{\zeta}^H$. Lemma 4 and (6.29) now show the almost analyticity of $\mathscr{C}_H$. As in the proof of (5.18) we now obtain $\psi_H = \psi_H(\alpha,\beta)$ such that,

$$d_{\bar{\alpha},\bar{\beta}} \psi_H(\alpha,\beta) = \mathcal{O}\left(d((\alpha,\beta), \operatorname{diag}(\Lambda \times \Lambda))^\infty\right).$$

Restricting $\psi_H(\alpha,\beta)$ to $\Lambda \times \Lambda$ gives (6.18).

We now return to our original ψ in (6.13), $\psi(\alpha,\beta) = -iH(\alpha) + \psi_H(\alpha,\beta) - iH(\beta)$. Our construction shows that

$$\text{(6.18) holds}, \quad \psi(\alpha,\alpha) = -2iH(\alpha), \quad \psi(\alpha,\beta) = -\overline{\psi(\beta,\alpha)}, \quad \alpha,\beta \in \Lambda. \tag{6.31}$$

Remark. Although we motivated our construction using the self-adjointness and idempotence properties of the operator B_Λ (or equivalently B_H), the construction shows that ψ is uniquely determined, up to $\mathcal{O}(|\alpha-\beta|^\infty)$, by (6.31). $\square$

We also record that

$$\begin{gathered} \mathrm{c.v.}_\beta \left(\psi(\alpha,\beta) + 2iH(\beta) + \psi(\beta,\alpha)\right) = \psi(\alpha,\alpha), \\ -H(\alpha) - \operatorname{Im}\psi(\alpha,\beta) - H(\beta) \leq -|\alpha-\beta|^2/C, \quad C > 0. \end{gathered} \tag{6.32}$$

6.2. **Transport equations.** We now return to (6.14) and consider the transport equations satisfied by a. The analysis is similar to that in §5.2.2 and we start with a formal discussion (valid when all the objects are analytic). In view of the eikonal equations we have, as in (5.23),

$$a(\alpha,\beta) \sim \sum_{k=0}^\infty h^k a_k(\alpha,\beta),$$

where, with ζ_{j1}^{Λ} the second term in the expansion of the symbol of Z_j^{Λ}, we want to solve

$$\begin{aligned} V_j a_k(\alpha,\beta) + c_j(\alpha,\beta) a_k(\alpha,\beta) &= F_{k-1}^j(a_0,\cdots,a_{k-1})(\alpha,\beta), \quad F_{-1}^j \equiv 0, \\ V_j := \langle V_j(\alpha,\beta), \partial_\alpha\rangle, \quad & V_j(\alpha,\beta)_\ell := \partial_{\alpha_\ell^*}\zeta_j^{\Lambda}(\alpha, d_\alpha\psi(\alpha,\beta)), \\ c_j(\alpha,\beta) &:= \tfrac12 \sum_{\ell=1}^{2n} \partial_{\alpha_\ell} V_j(\alpha,\beta) + \zeta_{j1}(\alpha, d_\alpha\psi(\alpha,\beta)) \\ & \quad - i \sum_{k,\ell=1}^{2n} \partial^2_{\alpha_k\alpha_\ell}\psi(\alpha,\beta)\partial^2_{\alpha_k^*\alpha_\ell^*}\zeta_j^{\Lambda}(\alpha, d_\alpha\psi(\alpha,\beta)). \end{aligned} \tag{6.33}$$

We have similar expressions coming from the applications $\widetilde{Z}_j^{\Lambda}(\beta, hD_\beta)$ with V_j, c_j, F_k^j replaced by $\widetilde{V}_j$, $\widetilde{c}_j$, $\widetilde{F}_k^j$, and with the roles of α and β switched. A key observation here is that $H_{\zeta_j^{\Lambda}(\alpha)}$ and $H_{\bar\zeta_j^{\Lambda}(\beta)}$ are tangent to $\mathscr{C}$ and commute and that V_j and $-\widetilde{V}_j$ are these vector fields in the parametrization of $\mathscr{C}$ by (α,β). Hence,

$$[V_j, V_k] = 0, \quad [V_j, \widetilde{V}_k] = 0, \quad [\widetilde{V}_k, \widetilde{V}_k] = 0. \tag{6.34}$$

We note that, for any $b(\alpha,\beta) \in S^0$,

$$\begin{aligned} Z_j^{\Lambda}(\alpha, hD_\alpha)\left(e^{\frac{i}{h}\psi(\alpha,\beta)} b(\alpha,\beta)\right) &= h e^{\frac{i}{h}\psi(\alpha,\beta)}((V_j + c_j) b(\alpha,\beta) + \mathcal{O}(h)), \\ \widetilde{Z}_j^{\Lambda}(\beta, hD_\beta)\left(e^{\frac{i}{h}\psi(\alpha,\beta)} b(\alpha,\beta)\right) &= h e^{\frac{i}{h}\psi(\alpha,\beta)}((\widetilde{V}_j + \widetilde{c}_j) b(\alpha,\beta) + \mathcal{O}(h)). \end{aligned} \tag{6.35}$$

Moreover, solving (6.33) means that

$$\begin{aligned} Z_j^{\Lambda}(\alpha, hD_\alpha)\left(e^{\frac{i}{h}\psi(\alpha,\beta)} \sum_{k=0}^{K-1} h^k a_k(\alpha,\beta)\right) &= h^{K+1} e^{\frac{i}{h}\psi(\alpha,\beta)} F_{K-1}^j(\alpha,\beta), \\ \widetilde{Z}_j^{\Lambda}(\beta, hD_\beta)\left(e^{\frac{i}{h}\psi(\alpha,\beta)} \sum_{k=0}^{K-1} h^k a_k(\alpha,\beta)\right) &= h^{K+1} e^{\frac{i}{h}\psi(\alpha,\beta)} \widetilde{F}_{K-1}^j(\alpha,\beta). \end{aligned} \tag{6.36}$$

Since

$$\begin{gathered} [Z_j^{\Lambda}(\alpha, hD_\alpha), Z_k^{\Lambda}(\alpha, hD_\alpha)] = 0, \quad [\widetilde{Z}_j^{\Lambda}(\beta, hD_\beta), \widetilde{Z}_k^{\Lambda}(\beta, hD_\beta)] = 0, \\ [Z_j^{\Lambda}(\alpha, hD_\alpha), \widetilde{Z}_k^{\Lambda}(\beta, hD_\beta)] = 0, \end{gathered}$$

we have from (6.34) and (6.35),

$$V_j c_k = V_k c_j, \quad V_k \widetilde{c}_j = \widetilde{V}_j c_k, \quad \widetilde{V}_k \widetilde{c}_j = \widetilde{V}_j \widetilde{c}_k. \tag{6.37}$$

Similarly, (6.36) gives

$$\begin{gathered} (V_j + c_j) F_{K-1}^{\ell} = (V_k + c_k) F_{K-1}^j, \quad (\widetilde{V}_j + \widetilde{c}_j)\widetilde{F}_{K-1}^{\ell} = (\widetilde{V}_k + \widetilde{c}_k)\widetilde{F}_{K-1}^j, \\ (V_j + c_j)\widetilde{F}_{K-1}^{\ell} = (\widetilde{V}_k + \widetilde{c}_k) F_{K-1}^j. \end{gathered} \tag{6.38}$$

Equations (6.37) and (6.38) provide compatibility conditions for solving (6.33):

$$(V_j + c_j)a_k = F^j_{k-1}, \quad (\widetilde{V}_\ell + \widetilde{c}_\ell)a_k = F^\ell_{k-1}, \quad a_k(\alpha, \alpha) = b_k(\alpha),$$

where the b_k's are prescribed. In fact, since the V_ℓ's and $\widetilde{V}_j$'s are independent when $\alpha = \beta$ (as complex vectorfields),

$$\mathbb{C}^{2n} \times \mathbb{C}^n \times \mathbb{C}^n \ni (\rho, t, s) \mapsto (\alpha, \beta) = \Big(\exp\langle V, t\rangle(\rho), \exp\langle \widetilde{V}, s\rangle(\rho)\Big) \in \mathbb{C}^{2n} \times \mathbb{C}^{2n},$$

$$\langle V, t\rangle := \sum_{j=1}^n t_j V_j, \quad \langle \widetilde{V}, s\rangle := \sum_{\ell=1}^n s_j \widetilde{V}_\ell,$$

is a local bi-holomorphic map onto of $\mathrm{nbhd}_{\mathbb{C}^{4n}}(\mathrm{diag}(\Lambda \times \Lambda))$ (almost analytic in the general case). In view of this and of (6.34), (6.37), the following integrating factor, $g = g(\alpha, \beta)$, is well defined (in the analytic case) on $\mathrm{nbhd}_{\mathbb{C}^{4n}}(\mathrm{diag}(\Lambda \times \Lambda))$:

$$g(e^{\langle V,t\rangle}(\rho), e^{\langle \widetilde{V},s\rangle}(\rho)) := -\sum_{j=1}^n \int_0^1 (t_j c_j + s_j \widetilde{c}_j)|_{(\alpha,\beta)=(e^{\tau\langle V,t\rangle}(\rho), e^{\tau\langle \widetilde{V},s\rangle}(\rho))} d\tau,$$

and satisfies

$$V_j g(\alpha, \beta) = c_j(\alpha, \beta), \quad \widetilde{V}_j g(\alpha, \beta) = \widetilde{c}_j(\alpha, \beta), \quad j = 1, \cdots, n.$$

We then define $a_k(\alpha, \beta)$ inductively as follows: at $(\alpha, \beta) = (e^{\langle V,t\rangle}(\rho), e^{\langle \widetilde{V},s\rangle}(\rho))$,

$$\begin{aligned} a_k(\alpha, \beta) &= e^{g(\alpha,\beta)} b_k(\rho) \\ &\quad + e^{g(\alpha,\beta)} \int_0^1 e^{-g(\gamma,\gamma')}(t_j F^j_{k-1}(\gamma, \gamma') + s_j \widetilde{F}^j_{k-1}(\gamma, \gamma'))|_{(\gamma,\gamma')=(e^{\tau\langle V,t\rangle}(\rho), e^{\tau\langle \widetilde{V},s\rangle}(\rho))} d\tau. \end{aligned}$$

The compatibility relations (6.38) then show that (6.33) hold.

We now modify this discussion to the C^∞ case using almost analytic extensions as in §5.2.2 and that provides solutions of (6.33) for $(\alpha, \beta) \in \Lambda \times \Lambda$ valid to infinite order at $\mathrm{diag}(\Lambda \times \Lambda)$ with any initial data on the diagonal. In the time honoured tradition of [HeSj86] and [Sj96] we omit the tedious details.

Combination of §6.1 and 6.2 gives (2.9) with arbitrary $a(\alpha, \alpha) \sim \sum_k b_k(\alpha) h^k$.

6.3. **Construction of the projector.** We now proceed as in §3.3 and obtain the initial values, $b_k(\alpha)$ in the construction of the amplitude. Thus let B_Λ be given by (6.13) with phase and amplitude satisfying (6.14) and (6.28) with $a(\alpha, \alpha)$ to be chosen. We also note that (6.14) determine $a(\alpha, \beta)$ (up to $\mathcal{O}(|\alpha - \beta|^\infty + h^\infty)$) from $a(\alpha, \alpha) \sim \sum_k b_k(\alpha) h^k$.

To find b_k's we proceed by computing the expansion of B_Λ^2 using stationary phase. Since $Z_j^\Lambda B_\Lambda^2 = \mathcal{O}(h^\infty)_{L^2 \to L^2}$ and B_Λ^2 is self-adjoint, the integration kernel of B_Λ^2 is again determined by its values on the diagonal (for the phase see the remark after

(6.31)). The stationary phase argument (3.13)-(3.16) gives the desired $b_k's$ and we obtain $a(\alpha,\beta)$ with

$$a(\alpha,\beta)\sim\sum_{k=0}^{\infty}h^k a_k(\alpha,\beta),\quad |a_0(\alpha,\alpha)|>1/C,$$

such that B_Λ given by (2.9) satisfies

$$B_\Lambda^{*,H}=B_\Lambda,\quad B_\Lambda=\mathcal{O}(1):L^2_\Lambda\to L^2_\Lambda,\quad B_\Lambda^2=B_\Lambda. \tag{6.39}$$

We now proceed as in §3.4 and show that the exact orthogonal projector (6.11) satisfies

$$\Pi_\Lambda=B_\Lambda+\mathcal{O}(h^\infty)_{L^2_\Lambda\to L^2_\Lambda}. \tag{6.40}$$

The only difference in the argument is the construction of the the exact projector P_Λ:

$$P_\Lambda(L^2_\Lambda(\Lambda))=T_\Lambda(L^2(\mathbb{R}^n)),\quad P_\Lambda^2=P_\Lambda,\quad P_\Lambda=\mathcal{O}(1)_{L_H(\Lambda)\to L_H(\Lambda)}.$$

But a bounded projection was already provided by Lemma 6 and in its notation we can take

$$P_\Lambda=T_\Lambda S_\Lambda.$$

For $f\in S(\Lambda)$, $f(\alpha)\sim\sum_{k=0}^\infty f_k(\alpha)h^k$, $f_0(\alpha)>1/C$, we now define

$$A_f:=P_\Lambda f P_\Lambda^{*,H},\quad A_f u(\alpha)=:h^{-n}\int_\Lambda e^{\frac{i}{h}\psi_1(\alpha,\beta)}a_f(\alpha,\beta)u(\beta)e^{-2H(\beta)/h}dm_\Lambda(\beta).$$

As in §3.4 we claim that $\psi_1=\psi$ (modulo $\mathcal{O}(|\alpha-\beta|^\infty)$). Indeed, since $A_f^{*,H}=A_f$ and $P_\Lambda=T_\Lambda S_\Lambda$, the arguments leading to (6.18) apply and those eikonal equations hold for ψ_1 as well. Hence, ψ_1 is fully determined by its value on the diagonal and we find that using ψ_Λ in (6.8) and (6.9)

$$\psi_1(\alpha,\alpha)+2iH(\alpha)=\text{c.v.}_\beta\left(\psi_\Lambda(\alpha,\beta)-\overline{\psi_\Lambda(\alpha,\beta)}\right)=0.$$

But this means that (6.31) holds for ψ_1 and hence $\psi_1(\alpha,\beta)=\psi(\alpha,\beta)+\mathcal{O}(|\alpha-\beta|^\infty)$. Choosing f so that $A_f=B_\Lambda+\mathcal{O}(h^\infty)_{L^2_\Lambda\to L^2_\Lambda}$ as in §3.4 and arguing as in that section gives (6.40).

7. Pseudodifferential operators

We now discuss the action of pseudodifferential operators

$$Pu(x)=\frac{1}{(2\pi ih)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}p(x,\xi)e^{\frac{i}{h}\langle x-y,\xi\rangle}u(y)dyd\xi, \tag{7.1}$$

where p has a holomorphic extension satisfying

$$|p(z,\zeta)|\le M,\quad |\operatorname{Im} z|\le a,\quad |\operatorname{Im}\zeta|\le b, \tag{7.2}$$

for some $a,b,M>0$.

We start with the following

Lemma 7. *Suppose that P is given by (7.1). Then*

$$
\begin{gathered}
T_\Lambda P S_\Lambda = \mathcal{O}(1) : L^2_\Lambda \to L^2_\Lambda, \quad T_\Lambda P S_\Lambda = c_0 h^{-n} \int_\Lambda K_P(\alpha,\beta) u(\beta) d\beta, \\
K_P(\alpha,\beta) = e^{\frac{i}{h}\psi_\Lambda(\alpha,\beta)} a_P(\alpha,\beta) + r(\alpha,\beta), \\
a_P(\alpha,\beta) \sim \sum_{j=0}^\infty h^j a_P^j(\alpha,\beta), \quad a_P^0(\alpha,\alpha) = p|_\Lambda(\alpha),
\end{gathered}
\tag{7.3}
$$

where ψ_Λ is given by (6.8) and

$$
|r(\alpha,\beta)| \leq C e^{-\langle \alpha-\beta\rangle/Ch}.
$$

Proof. Formally,

$$
K_P(\alpha,\beta) = \frac{|c|^2}{h^{\frac{n}{2}}} \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^n} e^{\frac{i}{h}(\varphi(\alpha,y) + \langle y-y',\eta\rangle - \varphi^*(\beta,y'))} p(y,\eta) dy' d\eta dy,
$$

and the critical points of the phase

$$
\begin{gathered}
(y,y',\eta) \mapsto \varphi(\alpha,y) + \langle y-y',\eta\rangle - \varphi^*(\beta,y'), \\
\varphi(\alpha,y) = \langle \alpha_x - y, \alpha_\xi\rangle + \tfrac{i}{2}(\alpha_x-y)^2, \quad \varphi^*(\beta,y) = \overline{\varphi(\bar\beta,y)}, \quad \alpha,\beta\in\Lambda,
\end{gathered}
$$

are

$$
y = y' = y_c = \tfrac{1}{2}(\alpha_x+\beta_x) + \tfrac{i}{2}(\beta_\xi - \alpha_\xi), \quad \eta = \eta_c = \tfrac{1}{2}(\alpha_\xi+\beta_\xi) + \tfrac{i}{2}(\alpha_x - \beta_x).
$$

The critical value of the phase is given by ψ_Λ in (6.8). This gives a formal argument for (7.3).

To justify this, we first shift contours by

$$
\eta \mapsto \eta + i\epsilon \frac{y-y'}{\langle y-y'\rangle},
$$

which changes the phase to

$$
\langle \alpha_x - y, \alpha_\xi\rangle + \frac{i}{2}(\alpha_x-y)^2 + \langle y-y',\eta\rangle + \langle y'-\beta_x,\beta_\xi\rangle + \frac{i}{2}(\beta_x-y')^2 + i\epsilon\frac{(y-y')^2}{\langle y-y'\rangle}
$$

Next, we shift contours in y and y':

$$
y \mapsto y + i\epsilon\frac{\eta-\alpha_\xi}{\langle \eta-\alpha_\xi\rangle}, \qquad y' \mapsto y' + i\epsilon\frac{\beta_\xi-\eta}{\langle \beta_\xi-\eta\rangle}.
$$

This results in the phase

$$
\begin{gathered}
\langle \alpha_x - y, \alpha_\xi\rangle + \frac{i}{2}(\alpha_x-y)^2 + \langle y-y',\eta\rangle + \langle y'-\beta_x,\beta_\xi\rangle + \frac{i}{2}(\beta_x-y')^2 \\
+ i\epsilon\frac{(\eta-\alpha_\xi)^2}{\langle \eta-\alpha_\xi\rangle} + i\epsilon\frac{(\eta-\beta_\xi)^2}{\langle \eta-\beta_\xi\rangle} + i\epsilon\frac{(y-y')^2}{\langle y-y'\rangle} - \epsilon\Big[\Big\langle \alpha_x - y, \frac{\alpha_\xi-\eta}{\langle \alpha_\xi-\eta\rangle}\Big\rangle + \Big\langle \beta_x-y', \frac{\eta-\beta_\xi}{\langle \beta_\xi-\eta\rangle}\Big\rangle\Big] \\
+ O\Big(\epsilon^2\Big[\frac{|y-y|^2}{\langle y-y'\rangle} + \frac{|\eta-\alpha_\xi|^2}{\langle \eta-\alpha_\xi\rangle^2} + \frac{|\beta_\xi-\eta|^2}{\langle \beta_\xi-\eta\rangle^2}\Big]\Big).
\end{gathered}
$$

Therefore, choosing $\epsilon > 0$ small enough (not depending on G), we observe that the imaginary part of the phase satisfies

$$\begin{aligned}\operatorname{Im}\Phi \geq \operatorname{Im}\left[\langle \alpha_x - y, \alpha_\xi\rangle + \frac{i}{2}(\alpha_x - y)^2 + \langle y - y', \eta\rangle + \langle y' - \beta_x, \beta_\xi\rangle + \frac{i}{2}(\beta_x - y')^2\right] \\ + \operatorname{Im}\left[i\epsilon\frac{(\eta-\alpha_\xi)^2}{\langle \eta - \alpha_\xi\rangle} + i\epsilon\frac{(\eta-\beta_\xi)^2}{\langle \eta - \beta_\xi\rangle} + i\epsilon\frac{(y-y')^2}{\langle y - y'\rangle}\right] - \frac{M}{4}\epsilon\left[|\alpha_x - y|^2 + |\beta_x - y'|^2\right] \\ - \frac{\epsilon}{M}\frac{|\alpha_\xi - \eta|^2}{|\langle \alpha_\xi - \eta\rangle|^2} + \frac{|\eta - \beta_\xi|^2}{|\langle \beta_\xi - \eta\rangle|^2}] - C\epsilon^2\frac{|y-y|^2}{\langle y - y'\rangle} \\ \geq c|\alpha_x - y|^2 + c|\beta_x - y'|^2 + c\epsilon|\eta - \alpha_\xi| + c\epsilon|\eta - \beta_\xi| + c\epsilon|y - y'| - C\|G\|_{C^1}.\end{aligned}$$

In the last line we have used that G is compactly supported to see that

$$|\operatorname{Im}(\langle \alpha_x, \alpha_\xi\rangle - \langle \beta_x, \beta_\xi\rangle)| \leq C\|G\|_{C^1}.$$

Now, suppose that $|\alpha - \beta| > \delta$. Then,

$$|\alpha_x - y| + |\beta_x - y'| + |y - y'| + |\alpha_\xi - \eta| + |\beta_\xi - \eta| > \delta$$

and, choosing $\|G\|_{C^1}$ small enough depending on δ, the integral is controlled by $Ce^{-\langle \alpha-\beta\rangle/Ch}$.

In particular, we have, modulo an acceptable error,

$$K_P(\alpha,\beta) = \frac{|c|^2}{h^{\frac{n}{2}}}\frac{1}{(2\pi h)^n}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^n} e^{\frac{i}{h}(\varphi(\alpha,y)+\langle y-y',\eta\rangle-\varphi^*(\beta,y'))}p(y,\eta)\chi(\delta^{-1}|\alpha-\beta|)\,dy'd\eta dy$$

where $\chi \in C_c^\infty(-2,2)$, $\chi \equiv 1$ on $[-1,1]$.

Since we are now working in a small neighborhood of the diagonal, the contour shift,

$$y \mapsto y + y_c(\alpha,\beta) \qquad y' \mapsto y' + y_c(\alpha,\beta), \qquad \eta \mapsto \eta + \eta_c(\alpha,\beta)$$

is justified. The phase after this contour shift is given by

$$\frac{i}{4}[(\alpha_\xi - \beta_\xi)^2 + (\alpha_x - \beta_x)^2] + \frac{1}{2}(\alpha_x - \beta_x, \alpha_\xi + \beta_\xi) + \frac{i}{2}y^2 + \frac{i}{2}y'^2 + \langle y - y', \eta\rangle.$$

The stationary point of the phase is now at $y = y' = \eta = 0$ and the imaginary part of the phase is always larger than at the critical point. Therefore, we may apply the method of steepest decent to obtain the expansion in (7.3). □

We now proceed as in the proof of Theorem 2 to obtain

Theorem 3. *Suppose that P is given by (2.5) where the symbol p enjoys a holomorphic extension satisfying*

$$|p(z,\zeta)| \leq M, \quad |\operatorname{Im} z| \leq a, \quad |\operatorname{Im}\zeta| \leq b.$$

For $G \in C^\infty_{\mathrm{c}}(T^*\mathbb{R}^n)$ with $\|B\|_{C^2}$ sufficiently small we define

$$\Lambda = \Lambda_G := \{(x + iG_\xi(x,\xi), \xi - iG_x(x,\xi)) : (x,\xi) \in T^*\mathbb{R}^n), \quad L^2_\Lambda := L^2(\Lambda, e^{-2H(\alpha)/h}d\alpha),$$

where H is given by (6.4). Let $T_\Lambda u := Tu|_\Lambda$ (see (6.2)).

Then, for $u, v \in L^2(\mathbb{R}^n)$,

$$\langle TPu, Tv\rangle_{L^2_\Lambda} = \langle M_{P_\Lambda} Tu, Tv\rangle_{L^2_\Lambda} + \langle R_\Lambda Tu, Tv\rangle_{L^2_\Lambda}, \tag{7.4}$$

where $R_\Lambda = \mathcal{O}(h^\infty)_{L^2_\Lambda \to L^2_\Lambda}$ and

$$P_\Lambda(z,\zeta,h) = p|_\Lambda(z,\zeta) + hp^1_\Lambda(z,\zeta) + \cdots, \quad (z,\zeta)\in\Lambda.$$

8. Weights vs. deformations

To show that the approaches of §3 and §6 are the same, we want to find $\varphi = \varphi(x,\xi) \in C^\infty_{\rm c}(\mathbb{R}^{2n})$ such that

$$TS_\Lambda = \mathcal{O}(1) = L^2_\Lambda \to L^2_\varphi, \quad T_\Lambda S = \mathcal{O}(1) : L^2_\varphi \to L^2_\Lambda. \tag{8.1}$$

Let φ_G be the phase in T_Λ and $\widetilde{\varphi}_G$ be the phase in S_Λ. We need

$$\begin{aligned} \varphi(x,\xi) &= \varphi_{\max}(x,\xi) = \varphi_{\min}(x,\xi), \\ \varphi_{\max}(x,\xi) &:= \max_{(x',\xi')\in\mathbb{R}^{2n}} (-\operatorname{Im} \mathrm{c.v}_y(\varphi_0(x,\xi,y) + \widetilde{\varphi}_G(x',\xi',y)) + H(x',\xi')) \\ \varphi_{\min}(x,\xi) &:= \min_{(x',\xi')\in\mathbb{R}^{2n}} (-H(x',\xi') + \operatorname{Im} \mathrm{c.v}_y(\widetilde{\varphi}_0(x,\xi,y) + \varphi_G(x',\xi',y)))\,. \end{aligned} \tag{8.2}$$

We start by noting that

$$\varphi_G(x,\xi,y) = \Phi(z,\zeta,y)|_{(z,\zeta)\in\Lambda_G}, \quad \Phi(z,\zeta,y) = (z-y)\zeta + \tfrac{i}{2}(z-y)^2,$$

and that

$$\widetilde{\varphi}(x,\xi,y) = -\bar{\Phi}(z,\zeta,y)|_{(z,\zeta)\in\Lambda_G}, \quad \bar{\Phi}(z,\zeta) = \overline{(\bar{z},\bar{\zeta})}.$$

The critical value of $y \mapsto \Phi(z,\zeta,y) - \bar{\Phi}(x,\xi,y)$ is given by

$$y_{\rm c} = y_{\rm c}(x,\xi,z,\zeta) = \tfrac12(x + z + i(\xi - \zeta)),$$

while the critical value of $y \mapsto \Phi(x,\xi,y) - \bar{\Phi}(x,z,\zeta)$ is given by

$$\bar{y}_{\rm c} = \bar{y}_{\rm c}(x,\xi,z,\zeta) = \tfrac12(x + z + i(\zeta - \xi)).$$

To find the maximum in (8.2) we first note that with $z = x' + iG_{\xi'}$ and $\zeta = \xi' - iG_{x'}$,

$$\begin{aligned} \operatorname{Im} \bar{\Phi}(x',\xi',y_c) &= \tfrac12 \operatorname{Im}\left(i(\xi-\zeta)\xi' - i(x' - \tfrac12(x+z+i(\zeta-\xi)))^2\right) \\ &= -\tfrac14(3(\xi')^2 + (x')^2) + \mathcal{O}(\langle\xi'\rangle + \langle x'\rangle) \to -\infty, \quad (x',\xi') \to \infty. \end{aligned}$$

Hence,

$$-\operatorname{Im}[\Phi(x,\xi,y_c) - \bar{\Phi}(x',\xi',y_c)] + H(x',\xi') \to -\infty, \qquad (x',\xi') \to \infty$$

We then calculate (again with $z = x' + iG_{\xi'}$ and $\zeta = \xi' - iG_{x'}$)

$$d_{x',\xi'}(-\operatorname{Im} \mathrm{c.v.}_y(\Phi(x,\xi,y) - \bar{\Phi}(z,\zeta,y)) = \operatorname{Im} \partial_{z,\zeta}\bar{\Phi}(z,\zeta,y)|_{y=\bar{y}_{\rm c}}.$$

Since $d_{x',\xi'}H = -\operatorname{Im}\zeta dz|_\Lambda$ this means that the critical z, ζ are given by solving

$$\operatorname{Im}(\partial_{z,\zeta}\bar{\Phi}(z,\zeta,y)|_{y=\bar{y}_c} - \zeta dz)|_\Lambda = 0.$$

For the minimum we similarly obtain

$$\operatorname{Im}(\partial_{z,\zeta}\Phi(z,\zeta,y)|_{y=y_c} + \zeta dz)|_\Lambda = 0.$$

This shows that the critical points

$$(z_c, \zeta_c) = (z_c(x,\xi), \zeta_c(x,\xi)),$$

are the same for the maximum and minimum at (8.2). The maxima and minima are non-degenerate as that is the case when $G = 0$ and hence holds for $\|G\|_{C^2}$ sufficiently small.

We now need to show that the critical values $\varphi_{\max}$ and $\varphi_{\min}$ are also equal. For that we compute the differentials:

$$\begin{aligned} d_{x,\xi}\varphi_{\max}(x,\xi) &= -\operatorname{Im} d_{x,\xi}\Phi(x,\xi,y)|_{y=\bar{y}_c(x,\xi,z_c,\zeta_c)} = \operatorname{Im} d_{x,\xi}\bar{\Phi}(x,\xi,y)|_{y=y_c(x,\xi,z_c,\zeta_c)} \\ &= d_{x,\xi}\varphi_{\min}(x,\xi). \end{aligned}$$

Hence $\varphi_{\max}$ and $\varphi_{\min}$ differ by a constant. Since G and H vanish outside a compact set the critical values are both 0 when $H = G = 0$, we conclude the the constant is equal to 0. This gives us

Theorem 4. *There exist ϵ_0, C_0 such that if $G \in C_c^\infty(\mathbb{R}^{2n})$ and $\|G\|_{C^2} < \epsilon_0$ then*

$$\|T_\Lambda v\|_{L^2_\Lambda}/C_0 \le \|Tv\|_{L^2_\varphi} \le C_0\|T_\Lambda v\|_{L^2_\Lambda}, \quad v \in L^2(\mathbb{R}^n),$$
$$L^2_\Lambda := L^2(\Lambda, e^{-2H(x,\xi)/h}dxd\xi), \quad L^2_\varphi := L^2(T^*\mathbb{R}^n, e^{-\varphi(x,\xi)/h}dxd\xi),$$

where Λ, T_Λ are given in (6.1),(6.2), H is defined by (6.3), and φ is given (implicitely) by (8.2).

Proof. We have shown that for φ given by (8.2) we have (8.1). Hence,

$$\|T_\Lambda v\|_{L^2_\Lambda} = \|T_\Lambda STv\|_{L^2_\Lambda} \le \|T_\Lambda S\|_{L^2_\varphi \to L^2_\Lambda}\|Tv\|_{L^2_\varphi} \le C_0\|Tv\|_{L^2_\varphi},$$

with the other bound derived similarly. $\square$

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