A Survey of Results in Poincaré Recurrence Estimation Jeffrey Galkowski

Abstract

The Poincaré Recurrence theorem states that any probability measure preserving map has almost everywhere recurrence. However, it gives no information on how quickly this recurrence occurs. In the last twenty years, significant advances have been made in tools for estimating the Poincaré recurrence times in measure preserving dynamical systems. These methods connect the long term behavior of recurrence times to the Hausdorff dimension of the ambient space. We will discuss several of these results and examine examples including billiards on some interesting domains.

1 Introduction

The notion of Poincaré recurrence dates back to the late 1800's, when Poincaré realized that, a map being measure preserving (i.e $\mu(T^{-1}B) = \mu(B)$) guarantees recurrence almost everywhere. More precisely,

Theorem 1 Let (X, \mathcal{N}, μ) be a probability space, $T : X \to X$ measure preserving, and $A \in \mathcal{N}$, then for μ almost every $x \in A$ we have

$$\{n|T^n x \in A\}$$
 is infinite.

This theorem has the following corollary,

Corollary 1 Let X be a metric probability space. For μ almost every $x \in X$

$$\liminf_{n \to \infty} d(T^n x, x) = 0$$

which can be seen by taking successively smaller balls around x and applying theorem 1. Although this corollary tells us that orbits return arbitrarily close to their origination point, it provides no information on the frequency with which recurrence occurs. It is only recently that advances have been made to estimate the recurrence times of measure preserving maps.

These advances have been accomplished by relating the notions of local and global Hausdorff dimension of the ambient space X to the recurrence time. The first occurence of these ideas is in [4], where Boshernitzan relates the Hausdorff dimension of X to globally determine the long term recurrence behavior. In [1],[2], Barreira strengthens these results by relating the lower and upper pointwise μ dimensions of $x \in X$ to the recurrence behavior near that point. He goes on to further improve the results by restricting the relationship between the map and the measure.

In the following sections, we will discuss the results of both Barreira and Boshernitzan, and attempt to elucidate the surprising nature of these results. Then, we will examine some examples to demonstrate that the systems to which the results apply are plentiful.

2 Notation

We will use a combination of the notations used by Barreira and Boshernitzan that are summarized below.

Definition 1 m_{α} is defined to be the Hausdorff- α measure on a metric space (X, d).

Definition 2 A measure preserving system (m.p.s.) is a probability space (X, \mathcal{N}, μ) together with a measure preserving map $T: X \to X$.

Definition 3 A metric measure preserving system (m.m.p.s.) is an m.p.s. with a metric d such tath the open sets relative to d are in \mathcal{N} .

Definition 4 The self return time of a point x to the ball B(x,r) is

$$\tau_r(x) = \inf\{n \in \mathbb{N} | d(T^n x, x) < r\}.$$

Definition 5 The return time of a point $y \in B(x,r)$ to the ball B(x,r) is

$$\tau_r(y, x) = \inf\{n \in \mathbb{N} | d(T^n y, x) < r\}$$

Definition 6 The lower and upper recurrence rates of x are respectively

$$R^{l}(x) = \liminf_{r \to 0} \frac{\log \tau_{r}}{-\log r} \quad R^{u}(x) = \limsup_{r \to 0} \frac{\log \tau_{r}}{-\log r}$$

Definition 7 The lower and upper pointwise dimensions of μ at a point $x \in X$ are respectively

$$d^l_\mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \quad d^u_\mu(x) = \limsup_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}$$

Definition 8 The Hausdorff dimension of a probability measure μ on X is

$$\dim_H \mu = \inf\{\dim_H Z | \mu(Z) = 1\},\$$

where $\dim_H Z$ is the Hausdorff dimension of $Z \subset X$.

Definition 9 We say that the measure μ has long return time with respect to T if, for μ -almost every $x \in X$ and for $\epsilon > 0$ small enough,

$$\liminf_{r \to 0} \frac{\log \mu(A_{\epsilon}(x,r))}{\log \mu(B(x,r))} > 1$$

where

$$A_{\epsilon} = \{ x \in B(x,r) | \tau_r(y,x) \le \mu(B(x,r))^{-1+\epsilon} \}.$$

Definition 10 We say that the measure μ is weakly diametrically regular (w.d.r) on a set $Z \subset X$ if $\exists \eta > 1$ such that for μ -almost every $x \in Z$ and every $\epsilon > 0$, $\exists \delta > 0$ such that if $r < \delta$ then

$$\mu(B(x,\eta r)) \le \mu(B(x,r))r^{-\epsilon}.$$
(1)

We will now prove the following small consequence of 10.

Lemma 1 If μ is w.d.r. on $Z \subset X$, then $\forall \eta > 1 \exists \delta > 0$ for μ almost every $x \in Z$ and every $\epsilon > 0$ such that $\forall r < \delta$ equation (1) holds.

Proof:

Since μ is w.d.r, $\exists \eta > 1$ such that for μ -almost everywhere $x \in Z$ and every $\epsilon > 0, \exists \delta > 0$ such that $r < \delta \Rightarrow (1)$.

Claim:

$$\mu(B(x,\eta^k r)) \leq \mu(B(x,r))r^{-k\epsilon}$$
 for $r < \frac{\delta}{n^{k-1}}$.

We proceed by induction. The base case k = 1 is clear. Assume that the inductive hypothesis is true for k - 1. Then, we have

$$\mu(B(x,\eta^{k-1}r)) \le \mu(B(x,r))r^{-(k-1)\epsilon}$$
 for $r < \frac{\delta}{\eta^{k-2}}$.

Therefore, if $\eta r < \frac{\delta}{\eta^{k-2}} \Rightarrow r < \frac{\delta}{\eta^{k-1}}$ then let $y = \eta^{k-1}r < \delta$

$$\begin{split} \mu(B(x,\eta y)) &\leq & \mu(B(x,y))y^{-\epsilon} \\ &= & \mu(B(x,\eta^{k-1}r))y^{-\epsilon} \\ &\leq & \mu(B(x,r))r^{-(k-1)\epsilon}(\eta^{k-1}r)^{-\epsilon} \\ &\leq & \mu(B(x,r))r^{-k\epsilon} \end{split}$$

where the last step follows from the fact that $\eta > 1$. Thus, we have proved the claim.

Now, let $\gamma > 1$ then $\eta^k > \gamma$ for some finite k > 0. Fix such a k. Then $B(x, \gamma r) \subset B(x, \eta^k r) \Rightarrow \mu(B(x, \gamma r)) \leq \mu(x, \eta^k r) \leq \mu(B(x, r))r^{-k\epsilon}$ for $r < \frac{\delta}{\eta^{k-1}}$. Thus, letting $\epsilon = \frac{\beta}{k}$ we have $\forall \beta > 0$ $r < \alpha = \frac{\delta}{\eta^{k-1}} \Rightarrow (1)$ as desired. \Box

The statement in lemma 1 strengthens the notion of w.r.d. and will help us obtain bounds on \mathbb{R}^u and \mathbb{R}^l

3 Global Recurrence Behavior

We will begin by examining the results for recurrence that apply globally to an m.p.s. X. In [4], Boshernitzan derives the following theorems that relates recurrence time to Hausdorff dimension.

Theorem 2 Let (X, \mathcal{N}, μ, T) be an m.p.s, (Y, d) a metric space and $f : X \to Y$ be measurable (i.e. U open $\Rightarrow f^{-1}(U)$ measurable). If, for some $\alpha > 0$, m_{α} is σ -finite on Y. Then for μ -almost every $x \in X$

$$\liminf_{n \to \infty} \{ n^{\frac{1}{\alpha}} d(f(x), f(T^n x)) \} < \infty$$
(2)

and, if $m_{\alpha}(Y) = 0$, then

$$\liminf_{n \to \infty} \{ n^{\frac{1}{\alpha}} d(f(x), f(T^n x)) \} = 0.$$
(3)

In other words, $\min_{1 \le n \le N} \{ d(f(x), f(T^n x)) \} \approx (\frac{c}{N})^{\frac{1}{\alpha}}$ for some constant $c \ge 0$ and N large enough. We can easily adjust this theorem to answer a question about how quickly the map T recurs by taking f to be the identity. By doing so, we arrive at the following theorem of Boshernitzan.

Theorem 3 Let $(X, \mathcal{N}, \mu, T, d)$ be an m.m.p.s, If, for some $\alpha > 0$, m_{α} is σ -finite on Y. Then for μ -almost every $x \in X$

$$\liminf_{n \to \infty} \{ n^{\frac{1}{\alpha}} d(x, T^n x) \} < \infty$$
(4)

and, if $m_{\alpha}(Y) = 0$, then

$$\liminf_{n \to \infty} \{ n^{\frac{1}{\alpha}} d(x, T^n x) \} = 0.$$
⁽⁵⁾

We can see from these theorems that, to within a constant multiple, we can obtain an upper bound on the global long term behavior of the recurrence of any any μ -invariant map T based only upon the Hausdorff measures on the ambient space. In fact, this statement can be strengthened.

Lemma 2 Let (Y, d) a metric space, then $\alpha < \dim_H Y \Rightarrow m_\alpha$ is not σ -finite.

Proof:

Suppose $\exists \alpha < \dim_H Y$ such that m_α is σ -finite on Y. Then $\exists A_i$ with $Y \subset \bigcup A_i$ and $m_\alpha(A_i) = c_i < \infty$. Then $\forall \alpha > \alpha \ m_{\alpha'}(A_i) = 0[8]$. Let $\alpha < \alpha' < \dim_H Y$. Then

$$m_{\alpha'}(Y) \leq \sum M_{\alpha'}(A_i) = 0$$

and since $\dim_H Y = \inf\{a|m(a)E = 0\}$ we have $\dim_H Y \leq \alpha' < \dim_H$, a contradiction. Therefore, \exists no such α .

Based on this we see that $\alpha = \dim_H X$ is the best one can hope to do with theorem 2 and, in fact, $\forall \alpha > \dim_H Y$

$$\liminf n^{\frac{1}{\alpha}} d(f(x), f(T^n x)) = 0.$$

So, for every $\epsilon > 0$, $c > 0 \min_{1 \le n \le N} \{ d(f(x), f(T^n x)) \} < (\frac{c}{N})^{\frac{1}{\dim_H Y} - \epsilon}$ for N large enough. Thus, we have obtained an upper bound, to within a constant, on the behavior of the recurrence time for any μ -preserving map T based solely upon Hausdorff dimension of the ambient space. In fact, we have obtained a much stronger result. We have that for any measurable function $f: X \to Y$ an upper bound for the recurrence of $f_n = f \circ T^n$ can be obtained by only looking

at the Hausdorff dimension of the space Y.

These statements are remarkable since, apriori, one would expect that the specific nature of a map would play a large role in its recurrence times and, even if the nature of the map did not affect this upper bound significantly, one would expect that, since the space of measurable functions has widely varying behavior, different measurable functions f would have widely varying recurrence behavior. However, we can now see that no matter how different their behavior, two μ -preserving maps T, T' with measurable functions f, f' respectively must have the same upper bound on their behavior. Because of this, we might suspect that these upper bounds are very far from optimal, however, [4] is able to find examples where these bounds are in fact optimal. We will see in the work of Barreira in [1] that in some more restricted domains and families of maps we are able to obtain both upper and lower bounds on the recurrence behavior of the map T given only information about the measure structure on X.

4 Local Recurrence

In this section, we restrict our attention to Borel measurable transformations T on separable metric spaces X and, in many cases, $X \subset \mathbb{R}^d$. This appears to be a large restriction, however, by the Whitney embedding theorem if $X \subset M$ with M a finite dimensional smooth manifold then X can be smoothly embedded into \mathbb{R}^d for some d > 0 and there we can apply the theorems in this section.

We will now discuss theorems that relate the upper and lower recurrence times to the pointwise dimension of the manifold X. Note that in this section, we will not discuss recurrence of measurable functions of an m.p.s., but will assume that X is an m.m.p.s. and discuss the recurrence of the map T.

In [1] Barreira is able to obtain local upper bounds on the lower and upper recurrence rates without additional assumptions on T or the space X. He does this in terms of the lower and upper pointwise dimensions of the space X.

Theorem 4 Let $(X, \mathcal{N}, \mu, T, d)$ be an m.m.p.s. with T Borel measurable and μ w.r.d., then for μ - almost every $x \in X$

$$R^l \le d^l_\mu \text{ and } R^u \le d^u_\mu. \tag{6}$$

The theorem can be better understood in the following form. For r small enough we obtain $\tau_r(x,x) \geq r^{-d_{\mu}^l}$. This is obtained by applying the bound on $R^l(x)$. Thus, we have obtained a lower bound on the recurrence time of the map T. Notice that this first theorem gives us a lower bound on the first recurrence to B(x,r), while the information from the global recurrence theorem 3 together with lemma 2 gives us an upper bound on this quantity when it is rewritten in the form $\tau_r(x,x) \leq C_x r^{-\dim_H X}$ for r small enough.

By examining $x \in X$ locally and looking only at the map T instead of measurable functions $f: X \to Y$, Barreira[1] improves the upper bound obtained from global considerations in the following theorem.

Theorem 5 If $(X, \mathcal{N}, \mu, T, d)$ is an m.m.p.s. with T a Borel measurable transformation and μ w.d.r. then (5) holds with f the identity $\forall \alpha > d^l_{\mu}(x)$.

This statement is in fact stronger than that in theorem 3 since, using Young's criteria from [9], one can show $\dim_H X \ge d^l_{\mu}(x)$ for μ almost every $x \in X$. Thus, if we have information about μ locally, we will in general gain a better upper bound on the recurrence time for T by examining the local dimension of μ .

The final result that we will discuss tightens both the upper and lower bound using local information about the T invariant measure μ given additional information about the relationship between the measure μ and the map T.

Theorem 6 [1] Let $(X, \mathcal{N}, \mu, T, d)$ be an m.m.p.s. with T Borel measurable and μ w.d.r. If μ has long return time with respect to T, and $d^l_{\mu}(x) > 0$ for μ almost every $x \in X$, then for μ almost every $x \in X$

$$R^{l}(x) = d^{l}_{\mu}(x) \text{ and } R^{u}(x) = d^{u}_{\mu}(x).$$
(7)

Thus, if μ has a long return time with respect to T, we can improve from the bounds

$$r^{-d_{\mu}^{l}} \leq \tau_{r}(x,x) \leq C_{x} r^{-\dim_{H} X}$$

to the bounds

$$r^{-d^l_{\mu}} \le \tau_r(x, x) \le r^{-d^u_{\mu}X}.$$

From a further theorem in [1] we see that the class of systems with long return time includes all those equilibrium measures supported on locally maximal hyperbolic sets.

Since these results seem to be very strong, one may be led to believe that measures that are w.d.r. are not very common, however, the following lemma shows that each one of these theorems applies to Borel measurable subsets of \mathbb{R}^d for any finite d.

Lemma 3 [1] Any Borel probability measure on \mathbb{R}^d is w.d.r.

Proof:

Let μ be a Borel probability measure on \mathbb{R}^d . Claim: It is sufficient to show that for μ almost every $x \in \mathbb{R}^d$

$$\mu(B(x, 2^{-n})) \le n^2 \mu(B(x, 2^{-n-1})) \tag{8}$$

for sufficiently large $n \in \mathbb{N}$.

Fix $\epsilon > 0$. Let $2^{-n-2}r \leq 2^{-n-1}$. Then we have

$$\mu(B(x,2r)) \le \mu(B(x,2^{-n})) \le n^2 \mu(b(x,2^{-n-1}))$$

for n > N large enough i.e. $r < 2^{-N-1}$. But $X \subset \mathbb{R}^d$, we have

$$\frac{\mu(B(x,2r))}{\mu(B(x,2^{-n-1}))} \ge \frac{\mu(x,2^{-n-2})}{\mu(B(x,2^{-n-1}))} \ge (n-1)^{-2} \ge n^{-2}$$

Thus, we obtain $\mu(B(x,2r)) \leq n^4 \mu(B(x,r))$. Thus, we need only find n > 0 such that $n^4 \leq r^{-\epsilon}$. But, we have $r^{-\epsilon} \geq 2^{(n+1)\epsilon}$. Thus, clearly $\exists N_2$ such that for $n \geq N_2 \Rightarrow 2^{(n+1)\epsilon} > n^4$. Therefore, $\forall n \geq N_2 \Rightarrow \forall r < 2^{-N_2-1}$ we have (1) and the claim is proven.

Now, we prove the lemma. For every $n > 0, \delta > 0$ let

$$K_n(\delta) = \{ x \in \text{supp}\mu | \mu(B(x, 2^{-n-1})) < \delta\mu(B(x, 2^{-n})) \}.$$

Now, take a maximal 2^{-n-2} separated set $E \subset K_n(\delta)$. Then, we have

$$\mu(K_n(\delta)) \le \sum_{x \in E} \mu(B(x, 2^{-n-1})) \le \sum_{x \in E} \delta\mu(B(x, 2^{-n}))$$

Now, since E is 2^{-n-2} -separated subset of \mathbb{R}^d , $\exists M > 0$ (depending only on d) such that $\exists E_i 1 \leq i \leq M$ with $E_i \ 2^{-n}$ separated and $E = \bigcup_{i=1}^M E_i$. Now, since $\bigcup_{x \in E_i} B(x, 2^{-n})$ is disjoint,

$$\mu(K_n(\delta)) \le \sum_{i=1}^M \sum_{x \in E_i} \delta \mu(B(x, 2^{-n})) \le M\delta.$$

Therefore, since

$$\sum_{n>0} \mu(K_n(n^{-2})) \le M \sum_{n>0} n^{-2} < \infty$$

we have by the Borel-Cantelli lemma that for n large enough (8) holds μ almost everywhere in X.

5 Proofs of the Theorems

5.1 Proof of global theorem 2

Proof:

Our plan for the proof, givein in [4] is to first prove the result for the case $m_{\alpha}(Y) < \infty$ by finding subsets of $F_n \subset X$ where $\mu(F_n) \to 1$ and the claim (2) holds in F_p then argue that we can manipulate the F_p in such a way to obtain F with the same property and $\mu(F) = 1$. We will then reduce the case $m_{\alpha}(Y) = \infty$ to that of $m_{\alpha}(Y) < \infty$. Case 1 $m_{\alpha}(Y) < \infty$

Claim 1: Let $V \subset X$ be measurable, $V \in \mathcal{N}$ For fixed $t \ge 1$, let

$$V(t) = \{ x \in V | T^i x \notin V, \forall i, 1 \le i \le t \}.$$

Then $\mu(V(t)) < \frac{1}{t}$.

We will show that $T^{-i}V(t) \cap T^{-j}V(t)$ for $i \neq j, 0 \leq i \leq t$ and therefore, since T is μ preserving, each has the same measure $\Rightarrow \mu(V(t)) < \frac{\mu(X)}{t} = \frac{1}{t}$. Let $0 \leq j < i \leq t$. Suppose $\exists x \in T^{-i}V(t) \cap T^{-j}V(t)$. Then, we have $T^i x \in V(t) \subset V$ and $T^j x \in V(t) \subset V$. Therefore, if $y = T^i x, y \in V$ but $T^{j-i}y \in V(t) \subset V$ and j - i < t therefore, $y = T^i x \notin V(t)$, a contradiction. Thus, we have proven the first claim.

Claim 2: Let $m_{\alpha}(Y) < c < \infty$. Then $\forall \epsilon > 0$ and $p \ge 1 \exists$ a measurable set $F = F(p, \epsilon) \subset X$, with $\mu(F) > 1 - \frac{1}{p}$, such that $\forall x \in F \exists$ an integer k such that

$$d(f(x), f(T^k x)) < \min\left(\left(\frac{4cp^2}{k}\right)^{\frac{1}{\alpha}}, \epsilon\right)$$

By the definition of the Hausdorff α measure, we can find a countable cover of $Y = \bigcup_{i \ge 1} U_i$, with U_i having diam $(U_i) = r_i < \min(1, \epsilon)$ and $\sum_{i\ge 1} r_i^{\alpha} < c$. Without loss of generality, we may assume U_i are Borel and disjoint up to sets of measure 0. We may assume the sets are Borel since for every set $U \subset Y$ with $m_{\alpha}(U) < \infty \exists U \subset W \subset Y$ such that Wis Borel and $m_{\alpha}(U) = m_{\alpha}(W)$. We then make the sets disjoint by taking $\tilde{U}_i = U_i \setminus \bigcup_{1 \le j \le i} U_i$, which will still be Borel.

Now, denote $V_i = f^{-1}(U_i), v_i = \mu(V_i)$. We examine the set

$$J = \{i \ge 1 | 2cpv_i \le r_i^{\alpha}\}.$$

Then, we have that

$$\sum_{i \in J} \le \frac{1}{2cp} \sum_{i \in J} r_i^{\alpha} < \frac{c}{2cp} = \frac{1}{2p} \text{ and } v_i > \frac{1}{2cp} r_i^{\alpha} \text{ for } i \notin J.$$

Let

$$t_i = \frac{4cp^2}{r_i^{\alpha}} = 4p^2 \frac{c}{r_i^{\alpha}} > 1.$$

Then, by the first claim, $\mu(V_i(t_i)) < \frac{1}{t_i}$ where

$$V_i(t_i) = \{ x \in V_i | T^k x \notin V_i, \forall k, 1 \le k \le t_i \}.$$

Thus, for $i \notin J$, we have

$$\mu(V_i(t_i)) < \frac{r_i^{\alpha}}{4cp^2} \le \frac{2cp\mu(V_i)}{4cp^2} = \frac{\mu(V_i)}{2p}.$$

Therefore, since U_i disjoint $\Rightarrow V_i$ disjoint and thus $\sum_i \mu(V_i) \leq 1$, we have

$$\sum_{i \notin J} \mu(V_i(t_i)) < \frac{1}{2p}.$$

Now, for the set $G = (\bigcup_{i \in J} V_i) \bigcup (\bigcup_{i \notin J} V_i(t_i))$, we obtain

$$\mu(G) \le \sum_{i \in J} \mu(V_i) + \sum_{i \notin J} \mu(V_i(t_i)) \le \sum_{i \in J} v_i + \sum_{i \notin J} \frac{v_i}{2p} < \frac{1}{2p} + \frac{1}{2p} = \frac{1}{p}$$

Now, let $F = X \setminus G$. Then $\mu(F) > 1 - \frac{1}{p}$ and $x \in F \Rightarrow x \in V_i \setminus V_i(t_i)$ for some *i*. Therefore, $\exists k, 1 \leq k \leq t_i$ such that $x, T^k x \in V_i$ and thus

$$d(f(x), f(T^k x)) < \operatorname{diam}(U_i) = r_i = \left(\frac{4cp^2}{t_i}\right)^{\frac{1}{\alpha}} \le \left(\frac{4cp^2}{k}\right)^{\frac{1}{\alpha}}$$
(9)

since $t_i = \frac{4cp^2}{r_i^{\alpha}}$. Thus, since $r_i < \epsilon$ we have proved claim 2.

Claim 3: Let $m_{\alpha}(Y) < c < \infty$. For every $p \ge 1$, let

$$F'(p) = \left\{ x \in X | \liminf_{n \ge 1} \{ n^{\frac{1}{\alpha}} d(f(x), f(T^n x)) \} \le (4cp^2)^{\frac{1}{\alpha}} \right\}.$$

Then $\mu(F'(p)) \ge 1 - \frac{2}{p}$. Let $(\epsilon_n)_{n\ge 1}$ be a decreasing sequence of positive numbers $\epsilon_n \downarrow 0$. Let $F_i = F(p, \epsilon_i)$ found in claim 2. Then, since $\mu(F_i) \ge 1 - \frac{1}{p}$ for all i, we have $\mu(F(p)) \ge 1 - \frac{1}{p}$, where

$$F(p) = \bigcap_{k \ge 1} \left(\bigcup_{i \ge k} F_i \right).$$

 $x \in F(p) \Rightarrow x \in F_i(p)$ for infinitely many i, and $x \in F_i(p) \Rightarrow \exists k$ such that (9) holds. Let

$$k_i = \inf\{k|(9) \text{ holds with } \epsilon = \epsilon_i\}.$$

Then, clearly $k_i \to \infty$ or $\exists k$ such that $f(T^k x) = f(x)$. **Case 1** If $\exists k_i \to \infty$ such that $d(f(x), f(T^{k_i} x)) \leq \left(\frac{4cp^2}{k_i}\right)^{\frac{1}{\alpha}}$, then $x \in F'(p)$ as desired.

Case 2 If $\exists k$ such that $f(T^k x) = f(x)$ then either $\exists N > 0$ such that for $k > N f(T^k x) \neq f(x)$ or $\exists k_i \to \infty$ such that $d(f(x), f(T^{k_i} x)) \leq \left(\frac{4cp^2}{k_i}\right)^{\frac{1}{\alpha}}$ as in the first case.

Now, suppose $\exists N > 0$ such that for k > N $f(T^k x) \neq f(x)$. Then, if $T^N x \in F(p)$ we obtain a sequence of $k_i \to \infty$ as above and $x \in F'(p)$. Therefore, we have reduced the set $x \in F(p)$ such that $x \notin F'(p)$ to the set

$$A \subset \{x \in X | \exists K_x > 0 \text{ such that } T^{K_x} x \notin F(p)\}.$$

Now, clearly if $T^{K}x \notin F(p)$ then, $\forall n > K \ T^{n}x \notin F(p)$. Therefore, $\forall n > \inf_{x \in F(p)} \{K_x\} > 0$, we have $A \subset T^{-n}(X \setminus F(p)) \Rightarrow \mu(A) \leq \frac{1}{p}$ since T is measure preserving. Therefore, since $\mu(F'(p)) = \mu(F(p) \setminus A) \geq (F(p) \setminus A) \geq 1$ $\mu(F(p)) - \mu(A) \ge 1 - \frac{2}{p}$ as desired.

Claim 4: Completion of the proof for $m_{\alpha}(Y) < \infty$. Let $F' = \bigcup_{p \geq \max(1, \frac{1}{4c})}$. Then by claim 4, $\mu(F') = 1$ and $\forall x \in F$, (4) holds.

Now, if $m_{\alpha}(Y) = 0$, then we show that 3 holds. Let $(c_p) = \frac{1}{4p^3}$. Then $c_p > 0 = m_{\alpha}(Y)$. Therefore, by claim 4,

$$\tilde{F}'(p) = \left\{ x \in X | \liminf_{n \ge 1} \{ n^{\frac{1}{\alpha}} d(f(x), f(T^n x)) \le (4c_p p^2)^{\frac{1}{\alpha}} = (\frac{1}{p})^{\frac{1}{\alpha}} \right\}$$

has $\mu(\tilde{F}'(p)) \ge 1 - \frac{2}{p}$. But, $\tilde{F}'(p) \supset \tilde{F}'(p+1) \Rightarrow \bigcap_{p=1}^{N} \tilde{F}'(p) = \tilde{F}(N)$ and then

$$\tilde{F}' = \bigcap p \ge 1\tilde{F}'(p)$$

has $\mu(\tilde{F}') \geq 1 - \frac{2}{p} \ \forall p \geq 1$ and for every $x \in \tilde{F}'$

$$\liminf_{n\geq 1} \{ n^{\frac{1}{\alpha}} d(f(x), f(T^n x)) \} < \frac{1}{p} \ \forall p \geq 1.$$

In other words, 3 holds for $x \in \tilde{F}'$ and the claim is proven.

Now, we show that the case $m_{\alpha}(Y) = \infty$ reduces to that of $m_{\alpha}(Y) < \infty$.

Since m_{α} is σ -finite on Y. Y can be covered by a countable family of sets U_i with $m_{\alpha}(U_i) < \infty$. As above, we may assume U_i are Borel. Now, since $f: X \to Y$ is measurable, the sets $K_i = f^{-1}(U_i)$ are measurable. Therefore, since $\mu(\bigcup K_i \setminus X) = 0$, we need only verify $\mu(E_i) = 0 \ \forall i$, where

$$E_{i} = \left\{ x \in K_{i} | \liminf_{n \ge 1} \{ n^{\frac{1}{\alpha}} d(f(x), f(T^{n}x)) \} = \infty \right\}.$$

Suppose that for some $E_i \subset K_i$, $\mu(E_i) > 0$. Then $\mu(K_i) \ge \mu(E_i) > 0$. For almost every $x \in K_i$, the set $N(x) = \{n \ge 1 | T^n(x) \in K_i\}$ of return times to K_i is infinite by theorem 1. Therefore

$$N(x) = \{n_1(x) < n_2(x) < \dots\}.$$

Let $S: K \to K$ be the induced transformation: $S^k(x) = T^{n(x)}(x)$ with $n(x) = n_1(x)$. Since $f(K_i) = U_i$ with $m_\alpha(U) < \infty$ we use the first part of this proof (with Y = U, T = S) to obtain $\mu(E') = 0$ where

$$E' = \left\{ x \in K_i | \liminf_{n \ge 1} \{ k^{\frac{1}{\alpha}} d(f(x), f(S^k(x))) \} = \infty \right\}.$$

Now,

$$E \setminus E' = \left\{ x \in K | \liminf_{k \ge 1} \{ k^{\frac{1}{\alpha}} d(f(x), f(S^k(x))) \} < \infty \text{ and } \liminf_{n \ge 1} \{ n^{\frac{1}{\alpha}} d(f(x), f(T^n(x))) \} = \infty \right\}$$

But $S^k = T^{nk}$ therefore, if

$$\liminf_{k\geq 1} \{k^{\frac{1}{\alpha}} d(f(x), f(S^k(x)))\} < \infty$$

then

$$\liminf_{k \ge 1} \{ n(x)^{\frac{1}{\alpha}} k^{\frac{1}{\alpha}} d(f(x), f(T^{nk}(x))) \} < \infty$$

and therefore,

$$\liminf_{n \to \infty} 1\{n^{\frac{1}{\alpha}}d(f(x), f(T^n(x)))\} < \infty.$$

Thus, $\mu(E \setminus E') = 0 \Rightarrow \mu(E) = 0$, a contradiction. Thus, the reduction is complete.

5.2 Local Theorems

We will follow [1] to prove these theorems. We will need the following lemmas to prove the theorems on local recurrence behavior.

Lemma 4 Let μ be a finite Borel measure on the separable metric space X, and $G \subset supp\mu$ a measurable set. Given r > 0, \exists a countable set $E \subset G$ such that

- 1. $B(x,r) \bigcap B(y,r) = \emptyset$ for any two distinct points $x, y \in E$
- 2. $\mu(G \setminus \bigcup_{x \in E} B(x, 2r)) = 0$

Proof:

Order the the collection of subsets of G satisfying the first property by inclusion. Clearly this collection is nonempty since any single point set in G is in it. Then by Zorn's lemma since G is an upper bound for the collection, \exists a maximal set $E \subset G$. Now, since $\mu(B(x, r)) > 0$ for each $x \in E \subset \text{supp}\mu$, the set E is at most countable.

Lemma 5 Let $(X, \mathcal{N}, \mu, T, d)$ be an m.m.p.s. with T Borel measurable. Then if μ is w.d.r. on a measurable set $Z \subset X$ with $\mu(Z) > 0$, (6) holds for μ -almost every $x \in Z$.

Proof:

Observe that the function $\delta(x, \cdot)$ in definition 10 can be made measurable for any fixed x. Fix $\epsilon > 0$ and choose $\delta > 0$ small enough so that

$$G = \{x \in Z | \delta(x, \epsilon) > \delta\}$$

has measure $\mu(G) > \mu(Z) - \epsilon$. Now, for any $r, \lambda > 0$ and $x \in X$ consider

$$A_{r,x} = \{ y \in B(x, 4r) | \tau_{4r}(y, x) \ge \lambda^{-1} \mu(B(x, 4r))^{-1} \}.$$

Then, Chebyshev's inequality gives us

$$\mu(A_{r,x}) \le \lambda \mu(B(x,4r)) \int_{B(x,4r)} \tau_{4r}(y,x) d\mu(y).$$

Now, since μ is invariant, it can be decomposed into a convex combination of ergodic measures[7] and we can apply Kac's lemma [5] tells us that

$$\int_{B(x,4r)} \tau_{4r}(y,x) d\mu(y) = 1.$$

Since $B(x, 2r) \subset B(x, 4r)$, we have

$$\mu(\{y \in B(x,2r) | \tau_{4r}(y,x)\mu(B(x,4r)) \ge \lambda^{-1}\}) \le \lambda \mu(B(x,4r)).$$

Moreover

$$\tau_{4r}(y,x)\mu(B(x,4r)) \ge \tau_{8r}(y,y)\mu(B(y,2r))$$

whenever d(x, y) < 2r and thus

$$\mu(\{y \in B(x,2r) | \tau_{8r}(y,y)\mu(B(y,2r)) \ge \lambda^{-1}\}) \ge \lambda\mu(B(x,4r)).$$
(10)

By lemma 4 \exists a countable maximal r separated set $E \subset G$. By (10) with $\lambda = r^{2\epsilon}$ and lemma 1 with $\eta = 4$ we obtain

$$\begin{split} \mu(D_{\epsilon}(r)) &:= & \mu(\{y \in G | \tau_{8r}(y, y) \mu(B(y, 2r)) \ge r^{-2\epsilon}\}) \\ &\leq & \sum_{x \in E} \mu(\{y \in B(x, 2r) | \tau_{8r}(y, y) \mu(B(y, 2r)) \ge r^{-2e}\}) \\ &\leq & r^{2\epsilon} \sum_{x \in E} \mu(B(x, 4r)) \\ &\leq & r^{\epsilon} \sum_{x \in E} \mu(B(x, r)) \le r^{\epsilon}. \end{split}$$

We then observer that

$$\sum_{n>-\log\delta} D_{\epsilon}(e^{-n}) \leq \sum_{n>-\log\delta} e^{-\epsilon n} < \infty$$

Then, by the Borel-Cantelli lemma, we have that for μ -almost every $x \in G$,

$$\frac{\log \tau_{8e^{-n}}(x,x)}{n} \le 2\epsilon + \frac{\log \mu(B(x,2e^{-n}))}{-n}$$

for n large enough. Thus, the identities in the lemma follow since $\epsilon>0$ was arbitrary and

$$\begin{array}{l} d_{\mu}^{l}(x) = \liminf \inf \frac{\log \mu(B(x,ae^{-n}))}{-n}, \quad d_{\mu}^{u}(x) = \limsup \frac{\log \mu(B(x,ae^{-n}))}{-n}\\ R^{l}(x) = \liminf \inf \frac{\log \tau_{ae^{-n}}(x)}{n} \qquad R^{u}(x) = \limsup \frac{\log \tau_{ae^{-n}}(x)}{n} \end{array} .$$

Lemma 6 Given $x \in X$, we have $R^{l}(x) \leq d \Leftrightarrow$ for every $\epsilon > 0$,

$$\liminf_{n \to \infty} n^{\frac{1}{d+\epsilon}} d(T^n x, x) = 0 \tag{11}$$

Proof:

Assume that $R^l(x) \leq d$. Then given $\epsilon > 0 \exists (r_n)$ such that $r_n \to 0$, and $\tau_{r_n}(x) < r_n^{-(d+\epsilon)}$ for all n. Let $m_n = \tau_{r_n}(x)$. If m_n is bounded, then x is periodic and clearly (11) holds. Now, if m_n is unbounded, we have $d(T^{m_n}x, x) < r_n$ and

$$m_n^{\frac{1}{d+2\epsilon}} d(T^{m_n} x, x) < \tau_{r_n}(x)^{\frac{1}{1+2\epsilon}} r_n < r_n^{-\frac{d+\epsilon}{d+2\epsilon}} r_n = r_n^{\frac{\epsilon}{d+\epsilon}}$$

Therefore

$$\liminf_{n \to \infty} n^{\frac{1}{d+2\epsilon}} d(T^n x, x) \le \liminf_{n \to \infty} m_n^{\frac{1}{d+2\epsilon}} d(T^{m_n} x, x) = 0.$$

Thus, (11) holds for every $\epsilon > 0$.

Now, assume (11) holds for all $\epsilon > 0$. Let $r_n = 2d(T^n x, x)$. We have that $\tau_{r_n}(X) \leq n$, and thus

$$\liminf_{n \to \infty} \tau_{r_n}(x)^{\frac{1}{d+\epsilon}} r_n = 0$$

Thus, \exists a divergings sequnce of positive integers k_n such that $\tau_{r_{k_n}}(x)^{\frac{1}{d+\epsilon}}r_{k_n} < 1$ for every n. Therefore,

$$R^{l}(x) \leq \liminf_{n \to \infty} \frac{\log \tau_{r_{n}}(x)}{-\log r_{n}} \leq \liminf_{n \to \infty} \frac{\log r_{k_{n}}^{d+\epsilon}}{-\log r_{k_{n}}} = d + \epsilon$$

Since ϵ was arbitrary, we have our result.

Now, to prove theorem 4 we simply apply lemma 5.

To prove theorem 5 we apply theorem 4 and lemma 6.

We will have to prove something more to prove the strongest result, theorem 6. **Proof:**

By theorem 4 we have $R^l(x) \leq d^l_{\mu}(x)$ and $R^u(x) \leq d^u_{\mu}(x)$ for μ -almost every $x \in X$. We need to obtain the reverse inequalities.

Since μ is w.r.d. and μ has long return time with respect to T and $d^l_{\mu}(x) > 0$ for μ -almost every $x \in X$, if $\epsilon > 0$ is small enough, we have that $\exists a, \gamma, \rho > 0$ and $G \subset X$ with $\mu(G) > 1 - \epsilon$ such that if $x \in G$ and $r \in (0, \rho)$

$$\mu(A_{\epsilon}(x,2r)) \le \mu(B(x,2r))^{1+\gamma},\tag{12}$$

$$\mu(B(x,2r)) \le \mu(B(x,\frac{r}{2}))r^{-a\frac{\gamma}{2}},\tag{13}$$

$$\mu(B(x,r)) \le r^a \tag{14}$$

where $A_{\epsilon}(x, 2r)$ is as in definiton 9. Now, consider

$$A_{\epsilon}(r) := \{ y \in G | \tau_r(y) \le \mu(B(y, 3r))^{-1_{\epsilon}} \}.$$

Then, if d(x, y) < r(a), we have $\tau_r(y, y) \ge \tau_{2r}(y, x)(b)$. Then, since $B(x, 2r) \subset B(y, 3r)$, if $x \in G$ then we obtain

$$\begin{array}{ll}
\mu(B(x,r) \bigcap A_{\epsilon}(r)) &\leq \mu(\{y \in B(x,r) | \tau_{2r}(y,x) \leq \mu(B(x,3r))^{-1+\epsilon}\}) & (a),(b) \\
&\leq \mu(A_{\epsilon}(x,2r)) & \mu(B(x,3r)) \geq \mu(B(x,2r)) \\
&\leq \mu(B(x,2r))^{1+\gamma} & (12) \\
&\leq \mu(B(x,\frac{r}{2}))r^{-a\frac{\gamma}{2}}(2r)^{a\gamma} & (13)(14)
\end{array}$$

Then, if $E \subset G$ is a maximal $\frac{r}{2}$ -separated set given by lemma (4), we have

$$\begin{array}{lll} \mu(A_{\epsilon}(r)) & \leq & \sum_{x \in E} \mu(B(x,r) \bigcap A_{\epsilon}(r)) \\ & \leq & \sum_{x \in E} \mu(B(x,\frac{r}{2})) r^{-a\frac{\gamma}{2}} (2r)^{a\gamma} \\ & \leq & 2^{a\gamma} r^{a\frac{\gamma}{2}} \end{array}.$$

Here, the last step follows from (14). Then, the Borel-Cantelli lemma gives us that for μ -almost every $x \in G$ we have

$$\tau_{e^{-n}}(x) > \mu(B(x, 3e^{-n}))^{-1+\epsilon}$$

for all n large enough. Then, (5.2) gives us that

$$R^{l}(x) \geq (1-\epsilon)d^{l}_{\mu}(x)$$
 and $R^{u}(x) \geq (1-\epsilon)d^{u}_{\mu}(x)$

for μ -almost every $x \in G$. Then, the desired result follows because ϵ was arbitrary.

6 Examples

6.1 Billiards

Let $\Omega \subset \mathbb{R}^d$ for some $d < \infty$ be convex and bounded. Let X be the pahse space $\Omega x S^{d-1}$. Then let $\phi_t : X \to X$ be the billiards flow on X which is well defined for all t since the domain is convex. Then, fix some time t_0 and let $T : \Omega \to \Omega$ $Tx = \phi_{t_0} x$. Then, by theorem 2, with $f : X \to Y$, the projection from X to the domain Y, we have

$$\liminf n^{\frac{1}{d}} d(PT^n x, Px) < \infty.$$

Now, viewing x as a point in the billiards domain Ω . For small $r \tau_r(x,x) \leq C_x r^{-d}$ for some constant C_x . This result applies in a remarkable amount of generality. We have shown that for any domain Ω , no matter how complicated the boudary, for Lebesgue almost every $x_0 \in \Omega$ the billiard will return to $B(x_0, r)$ in time $\approx Cr^{-d}t_0$.

6.2 Baker's Map

Let $T : [0,1]^2 \to [0,1]^2$ be the standard bakers map. We then have that T is Lebesgue measure preserving. And thus, by theorem 3 we have that for Lebesgue almost every $x \in [0,1]^2$, we have

$$\tau_r(x,x) \le C_x r^{-2}$$

for r small enough. Further, by theorem 4, since $d^u_{\mu}(x) = d^l_{\mu}(x) = 2$ for μ the Lebesgue measure, $\tau_r(x, x) \ge r^{-2}$. So, we have bounded the recurrence time for small r between

$$r^{-2} \le \tau_r(x, x) \le C_x r^{-2}$$

for some constant C_x depending on x.

6.3 Arnold's Cat Map

Let T be the standard cat map on $[0, 1]^2$. i.e.

$$Tx = \left(\begin{array}{cc} 1 & 1\\ 1 & 2 \end{array}\right) \left(\begin{array}{c} x_1\\ x_2 \end{array}\right) \mod 1$$

In [6], Dyson computes upper and lower bounds of recurrence times for a discrete cat map. In other words, he maps the space $[0,1]^2$ to a square lattice of points with x_1, x_2 in 0, 1, ..., N-1 for some N > 0 and computes bounds on recurrence in terms of N. Let $f : [0,1]^2 \to Y = \{0, 1, ..., N-1\}^2$ be the projection onto this discrete set given by

$$f(x_1, x_2) = (\sup_{0 \le n} \{ \frac{n}{N} \le x_1 \}, \sup_{0 \le i} \{ \frac{n}{i} \le x_2 \}).$$

Since T is Lebesgue measure preserving, we may apply theorem 3 to obtain for r small enough, $\tau_r(x) \leq C_x r^{-2}$ this is consistent with the results of Dyson since in one of his theorems he obtains an upper bound on recurrence time of $\frac{N^2}{2}$. Taking $r = \frac{1}{N}$ as an approximation to the region which is mapped by f onto the same point, this heuristically agrees with our result.

7 Final Comments

Based on the results from the papers of Barreira [1][3] and Bashernitzan[4], we are able to obtain both upper and lower bounds on the behavior of recurrence in spaces that, at least locally, have finite measure theoretic dimension. In fact, if a space Y has finite Hausdorff dimension, then we are able to place an upper bound on the recurrence time for measurable functions from an m.p.s. to that space Y conditional only on the dimension of Y. Finally, in some more restricted cases, we are able to place precise bounds on the behavior of recurrence in measure preserving systems. Since the conditions in many of these theorems are strictly measure theoretic, we have been able to relate the topological notion of recurrence to measure theoretic notions of pointwise and Hausdorff dimensions.

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