

THE QUANTUM SABINE LAW FOR RESONANCES IN TRANSMISSION PROBLEMS

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ABSTRACT. We prove a quantum version of the Sabine law from acoustics describing the location of resonances in transmission problems. This work extends the work of the author to a broader class of systems. Our main applications are to scattering by transparent obstacles, scattering by highly frequency dependent delta potentials, and boundary stabilized wave equations. We give a sharp characterization of the resonance free regions in terms of dynamical quantities. In particular, we relate the imaginary part of resonances or generalized eigenvalues to the chord lengths and reflectivity coefficients for the ray dynamics, thus proving a quantum version of the Sabine law.

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1. INTRODUCTION

In this paper we study scattering in systems where the metric or potential has a singularity along an interface. Metric examples include scattering in media having sharp changes of index of refraction [CPV99, CPV01, PV99a], in dielectric microcavities [CW15] and in fiber optic cables [EG02]. Schrödinger operators with a distributional potential along a hypersurface can be used to model quantum corrals, concert halls, and other thin barriers [BZH10, CLEH95]. Such potentials are also used to understand leaky quantum graphs [Exn08].

Mathematically, an abrupt change in the index of refraction corresponds to a discontinuity in the metric along a hypersurface. Scattering in such situations has been studied in [Bel03, CPV99, CPV01, PV99a, PV99b] while scattering by certain distributional potentials has been studied in [Gal14, Gal16, GS14]. These types of problems have also been studied from the point of view of propagation of singularities [MT, Mil00, WH85] and quantum chaos [JSS15].

For a Schrödinger operator, P , on $L^2(\mathbb{R}^d)$ (d odd) it is often possible to prove that solutions, u , to

$$(\partial_t^2 + P)u = 0$$

have expansions roughly of the form

$$(1) \quad u \sim \sum_{\lambda \in \text{Res}} e^{-it\lambda} u_\lambda$$

where Res is the set of *scattering resonances* of P . Thus, the real and (negative) imaginary part of a scattering resonance correspond respectively to the frequency and decay rate of the associated resonance state, $e^{-it\lambda} u_\lambda$. This expression is similar to the expansion in terms of eigenvalues that one obtains when solving the wave equation on a compact manifold. Hence, for leaky systems, scattering resonances play the role of eigenvalues in the closed setting.

To get a quantitative heuristic for the decay of waves (the imaginary part of resonances), we imagine that the interface for our problem occurs at $\partial\Omega$ for some $\Omega \Subset \mathbb{R}^d$. We then think of solving the wave equation

$$(\partial_t^2 + P)u = 0, \quad u|_{t=0} = u_0, \quad u_t|_{t=0} = 0$$

with initial data u_0 a wave packet (that is a function localized in frequency and space up to the scale allowed by the uncertainty principle) localized at position $x_0 \in \Omega$ and frequency $\xi_0 \in S^{d-1}$. We also assume that P creates waves with speed c . The solution, u , then propagates along the billiard flow starting from (x_0, ξ_0) . At each intersection of the billiard flow with the boundary, the amplitude inside of Ω will decay by a factor, R , depending on the point and direction of intersection. Suppose that the billiard flow from (x_0, ξ_0) intersects the boundary at $(x_n, \xi_n) \in \partial\Omega \times S^{d-1}$, $n > 0$. Let $l_n = |x_{n+1} - x_n|$ be the distance between two consecutive intersections with the boundary (see Figure 1.1). Then the amplitude of the wave decays by a factor $\prod_{i=1}^n R_i$ in time $\sum_{i=1}^n c^{-1} l_i$ where $R_i = R(x_i, \xi_i)$. The energy scales as amplitude squared and since the imaginary part of a resonance gives the exponential decay rate of L^2 norm, this leads us to the heuristic that resonances should occur at

$$(2) \quad \text{Im } \lambda = \overline{\log |R|^2} / (2c^{-1} \bar{l})$$

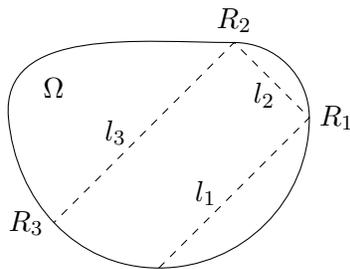


FIGURE 1.1. The figure shows the path of a wave packet along with the lengths between each intersection (l_i) and the reflection coefficient at each point of intersection with the boundary (R_i). After each reflection with the boundary, the amplitude of the wave packet inside Ω decays by a factor of R_i . If the speed of the wave is c , the time between reflections is given by $c^{-1}l_i$.

where the map $\bar{\cdot}$ is defined by $\bar{f} = \frac{1}{N} \sum_{i=1}^N f_i$. In the early 1900s, Sabine [Sab64] postulated that the decay rate of acoustic waves in a region with leaky walls is determined by the average decay over billiards trajectories. The expression (2) provides a precise statement of Sabine's idea and, because resonances are spectral quantity, we refer to such an expression as a quantum Sabine law. We will show in Theorem 4 that such a Sabine law holds for many different types of transmission problem.

Although the appearance of scattering resonances in (1) is intuitive, a more mathematically useful definition of a scattering resonance is as a pole of the meromorphic continuation of

$$(P - \lambda^2)^{-1}$$

from $\text{Im } \lambda \gg 1$. This description allows us to show that the existence of a scattering resonance at λ corresponds to the existence of a nonzero λ -outgoing solution to

$$(P - \lambda^2)u = 0.$$

By λ -outgoing we mean that there exists $g \in L^2_{\text{comp}}(\mathbb{R}^d)$ and $M \geq 0$ such that

$$u(x) = (R_0(\lambda)g)(x), \quad |x| \geq M.$$

Here, $R_0(\lambda)$ is the meromorphic continuation of $(-\Delta - \lambda^2)^{-1}$ from $\text{Im } \lambda \gg 1$ as an operator $R_0(\lambda) : L^2_{\text{comp}}(\mathbb{R}^d) \rightarrow L^2_{\text{loc}}(\mathbb{R}^d)$. (For a more complete description of mathematical scattering and further references, see [DZ])

We start by considering a few applications of our main theorem (see Theorem 4).

1.1. Transparent Obstacles. Our first application is to scattering by a transparent obstacle. That is, an obstacle with different refractive index than the ambient medium. In particular, let $\Omega \Subset \mathbb{R}^d$ be strictly convex with smooth boundary, $c \in \mathbb{R}_+ \setminus \{1\}$ be the speed of light in Ω , and $\aleph > 0$ be a coupling parameter. In [CPV99], Cardoso, Popov, and Vodev show that the set of

scattering resonances in this setting is given by λ such that there is a non-zero solution to

$$(3) \quad \begin{cases} (-c^2\Delta - \lambda^2)u_1 = 0 & \text{in } \Omega \\ (-\Delta - \lambda^2)u_2 = 0 & \text{in } \mathbb{R}^d \setminus \overline{\Omega} \\ u_1 = u_2 & \text{on } \partial\Omega \\ \partial_\nu u_1 - \aleph \partial_\nu u_2 = 0 & \text{on } \partial\Omega \\ u_2 \text{ is } \lambda\text{-outgoing.} \end{cases}$$

We denote the set of such λ by Λ . Here, ν denotes the outward unit normal to $\partial\Omega$.

Let $T^*\partial\Omega$ be the cotangent bundle to $\partial\Omega$ and $B^*\partial\Omega$ denote the coball bundle of $\partial\Omega$. Let $\pi_x : T^*\partial\Omega \rightarrow \partial\Omega$ be the projection to the base. Then define $r, l_N, r_N \in C^\infty(B^*\partial\Omega)$ and

$$l \in C^\infty(T^*\partial\Omega \times T^*\partial\Omega \setminus \{(x, \xi', x, \eta') \in T^*\partial\Omega \times T^*\partial\Omega\}) \cap C(T^*\partial\Omega \times T^*\partial\Omega)$$

by

$$(4) \quad \begin{aligned} r(x', \xi') &:= \frac{\sqrt{1 - |\xi'|_g^2} - \aleph \sqrt{c^2 - |\xi'|_g^2}}{\aleph \sqrt{c^2 - |\xi'|_g^2} + \sqrt{1 - |\xi'|_g^2}} & r_N(q) &:= \frac{\sum_{j=1}^N \log |r(\beta^j(q))|^2}{N} \\ l(q_1, q_2) &:= |\pi_x(q_1) - \pi_x(q_2)| & l_N(q) &:= \frac{\sum_{j=1}^N l(\beta^{j-1}(q), \beta^j(q))}{N} \end{aligned}$$

where $\beta : B^*\partial\Omega \rightarrow B^*\partial\Omega$ denotes the billiard ball map (see section 5) and $|\xi'|_g$ denotes the norm induced on the fibers of $T^*\partial\Omega$ by the metric on \mathbb{R}^d . Then r is the reflectivity for the transparent obstacle problem. Note that we take the branch of the square root so that $\sqrt{-1} = i$ and place the branch cut on the negative imaginary axis.

Remark 1.

- We will use ξ' to denote coordinates in the fiber of $T^*\partial\Omega$ and q to denote points in $T^*\partial\Omega$ throughout this paper.
- Note that the log in the definition of r_N appears because we measure exponential rates of decay and the reflection coefficient acts by multiplication.

Theorem 1. *Let $\Omega \Subset \mathbb{R}^d$ be strictly convex with smooth boundary and suppose that $0 < c \neq 1$, $\aleph > 0$. Then for all $M, \epsilon > 0$ there exists $\lambda_0 > 0$ such that for $\lambda \in \Lambda$ with $\text{Re } \lambda \geq \lambda_0$ and $\text{Im } \lambda \geq -M \log \text{Re } \lambda$,*

$$\sup_{N>0} \inf_{|\xi'|_g \leq 1} \frac{r_N}{2c^{-1}l_N} - \epsilon \leq \text{Im } \lambda \leq \inf_{N>0} \sup_{|\xi'|_g \leq 1} \frac{r_N}{2c^{-1}l_N} + \epsilon.$$

Moreover, for every \aleph, c as above, and $K > 0$, this bound is sharp in the region $\text{Im } \lambda \geq -K$ when $\Omega = B(0, 1) \subset \mathbb{R}^2$.

Remark 2.

- The lower bound in Theorem 1 is nontrivial, i.e. $|r(x', \xi')| > 0$, if either $c < 1$ and $\aleph < c^{-1}$, or $c > 1$ and $\aleph > c^{-1}$. This corresponds to *transverse electric waves* (TE). The opposite case, when there is no lower bound, corresponds to *transverse magnetic waves* (TM). In the TM case, the angle at which $r(x', \xi') = 0$ is called the *Brewster angle* ([Ida00, Chapter 13]). At this angle, there is complete transmission of the wave in the ray dynamics picture.

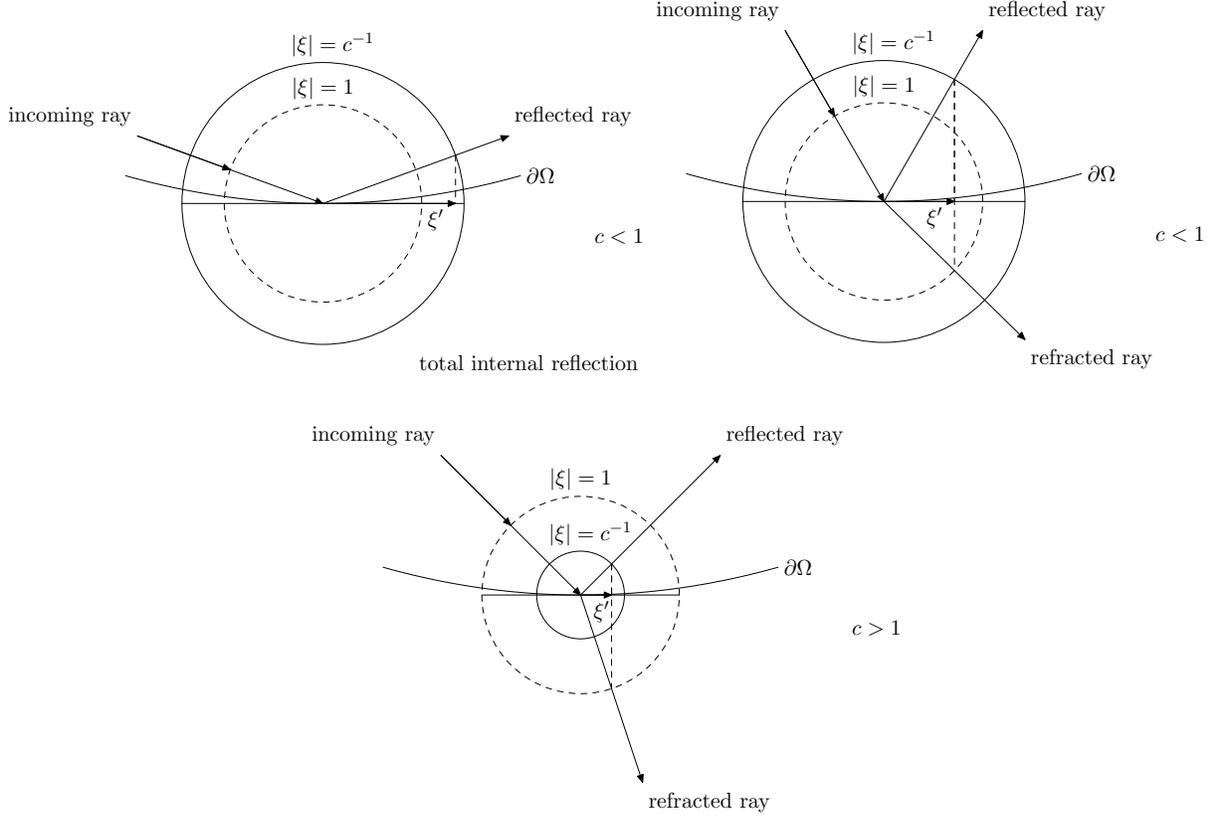


FIGURE 1.2. The figure shows the geometry of reflection and refraction at the boundary of an interface between a medium with speed of light c and one with speed of light 1. Total internal reflection occurs when the incoming ray does not project onto the ball of radius 1 in the ξ' variable.

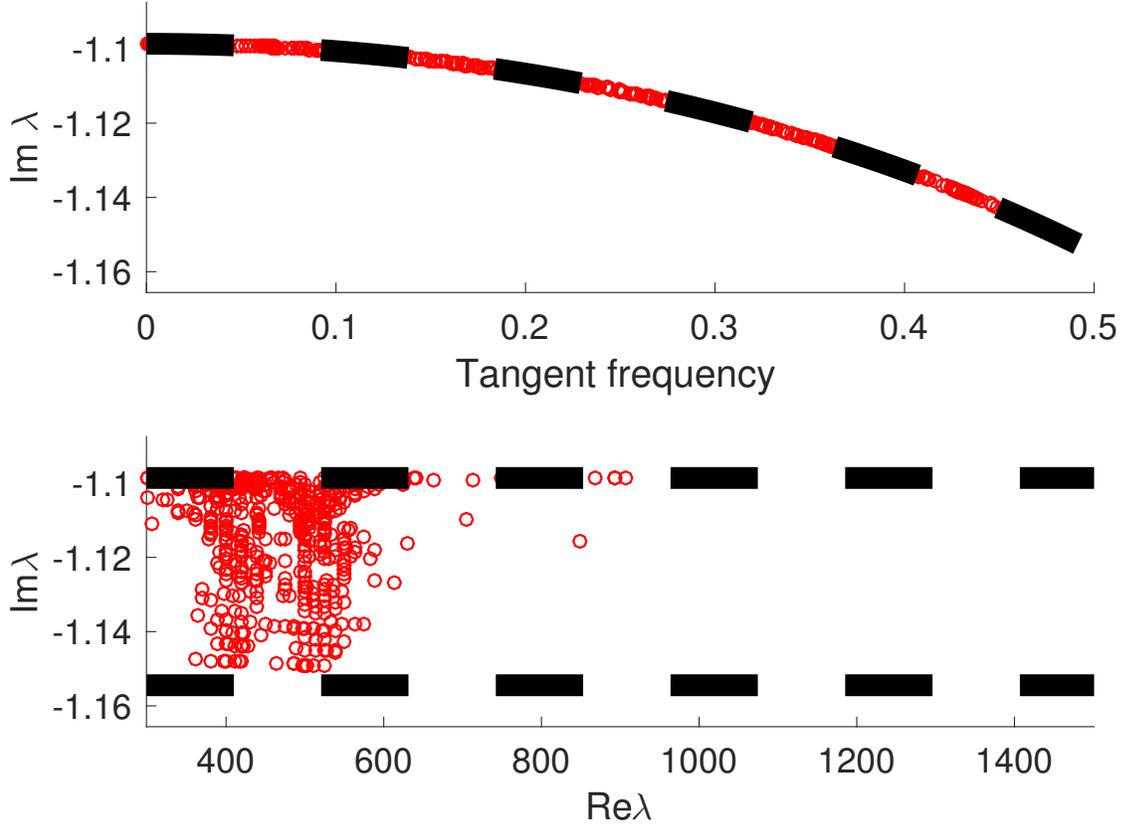
- The upper bound in Theorem 1 is nontrivial if $c > 1$. When $c < 1$, Popov and Vodev [PV99b] show that the presence of total internal reflection (see Figure 1.2) produces resonances $\{\lambda_n\}_{n=1}^\infty$ with $\operatorname{Re} \lambda_n \rightarrow \infty$ and $\operatorname{Im} \lambda_n = o((\operatorname{Re} \lambda_n)^{-\infty})$.
- The bounds for resonances given in Theorem 1 match our prediction (2).

Theorem 1 improves upon the results of Cardoso–Popov–Vodev [CPV99, CPV01] by giving sharp estimates on the sizes of the resonance free regions as well as expanding the range of parameters, \aleph , for which we have only a band of resonances.

1.2. Highly Frequency Dependent Delta Potentials. Let $\Psi^\infty(\partial\Omega)$ denote the set of semiclassical pseudodifferential operators of all orders whose seminorms are bounded by a constant independent of h so that $h^{-N}\Psi^\infty(\partial\Omega)$ denotes those whose seminorms are bounded by h^{-N} (see section 2 for more details).

We next consider operators of the form

$$(5) \quad -h^2\Delta + h(h\delta_{\partial\Omega} \otimes V) =: -h^2\Delta_{\partial\Omega,\delta}.$$



	Decay Rate	Tangential Frequency
Quantum Quantity	$\text{Im } \lambda$	$\frac{n}{\text{Re } \lambda}$
Classical Quantity	$\frac{cr_1}{2l_1}$	$c\xi'$

FIGURE 1.3. We show numerically computed resonances for the transparent obstacle problem with $c = 2$ and $\aleph = 1$ when $\Omega = B(0, 1) \subset \mathbb{R}^2$. (See Figure 13.1 for other values of c and \aleph .) In this case, we expand the solutions to (3) as $u_i(r, \theta) = \sum_n u_{i,n}(r)e^{in\theta}$ and solve for some of the resonances with $\text{Re } \lambda \sim 500$. In the lower graph, the red circles show $\text{Im } \lambda$ vs. $\text{Re } \lambda$. The dashed black lines show the upper and lower bounds for $\text{Im } \lambda$ (since \aleph is in the TE range with have both an upper and lower bound) from Theorem 1. Notice that by orthogonality of $e^{in\theta}$ and $e^{im\theta}$ for $m \neq n$, the pair $(u_{1,n}e^{in\theta}, u_{2,n}e^{in\theta})$ satisfies (3). In the top graph, the red circles show $\text{Im } \lambda$ vs. $n/\text{Re } \lambda$ for such pairs. The dashed curve shows a plot of $c\frac{r_1}{2l_1}(c\xi')$, the decay rate predicted for a billiards trajectory traveling with scaled tangent frequency $c\xi'$. See the table for the relationship between the points $(\text{Im } \lambda, n/\text{Re } \lambda)$ and $(cr_1/2l_1(c\xi'), c\xi')$ predicted by the quantum Sabine law.

where $h \in (0, 1]$, is a semiclassical parameter that should be thought of as the wavenumber (i.e. the inverse of the frequency), $V \in h^{-N}\Psi^\infty(\partial\Omega)$, and for $u, w \in C_c^\infty(\mathbb{R}^d)$

$$(6) \quad \langle (\delta_{\partial\Omega} \otimes V)u, w \rangle := \int_{\partial\Omega} (Vu)(x)w(x)d\sigma(x)$$

and σ is the surface measure of $\partial\Omega$. (See [GS14, Section 2.1] for the formal definition of this operator.) These operators are used as models for quantum corrals [BZH10, CLEH95] as well as concert halls, leaky quantum graphs [Exn08] and other thin barriers.

In a typical physical system, the interaction between a potential and wave depends on the frequency of the interacting wave. Therefore, we are motivated to consider h -dependent potentials V . Moreover, if one considers the delta interaction in 1 dimension

$$-\Delta + \delta(x_1) \otimes 1$$

and rescales to $y = hx$, we obtain

$$(7) \quad -h^2\Delta_y^2 + \delta(y_1/h) \otimes 1 = -h^2\partial_y^2 + h\delta(y_1) \otimes 1$$

which corresponds to $V = h^{-1}$ in (5). The operator (7) describes the *quantum point* interaction [Mil00].

Another motivation for highly frequency dependent delta potentials is the following wave equation

$$\begin{cases} (\partial_t^2 - \Delta + i(\delta_{\partial\Omega} \otimes (\langle a(x), \partial_x \rangle + a_0(x)\partial_t))u = F \text{ in } \mathbb{R}^d \\ F \in L_{\text{comp}}^2((0, \infty)_t \times \mathbb{R}^d), \quad u = 0 \text{ on } t < 0 \end{cases}$$

where $a, a_0 \in C^\infty(\partial\Omega; \mathbb{R})$, and the tensor product acts as in (6). Then, taking the time Fourier transform

$$\mathcal{F}_{t \rightarrow \lambda} u(x, \lambda) := \int_0^\infty e^{it\lambda} u(x, t) dt,$$

gives with $\lambda = z/h$,

$$(-h^2\Delta - z^2 + z(h\delta_{\partial\Omega} \otimes (\langle z^{-1}a, hD_x \rangle + a_0))\mathcal{F}_{t \rightarrow z/h} u = \mathcal{F}_{t \rightarrow z/h} F.$$

Remark 3. Note that we have switched the usual convention for the Fourier transform in our definition of $\mathcal{F}_{t \rightarrow \lambda}$ so that the integral converges absolutely for $\text{Im } \lambda > 0$.

In [GS14], Smith and the author show that the set of scattering resonances, $\Lambda(h)$, is equal to the set of z such that there is a non-zero solution to

$$\begin{cases} (-h^2\Delta - z^2)u_1 = 0 & \text{in } \Omega \\ (-h^2\Delta - z^2)u_2 = 0 & \text{in } \mathbb{R}^d \setminus \bar{\Omega} \\ u_1 = u_2 & \text{on } \partial\Omega \\ \partial_\nu u_1 - \partial_\nu u_2 + Vu_1 = 0 & \text{on } \partial\Omega \\ u_2 \text{ is } z/h \text{ outgoing.} \end{cases}$$

Denote by

$$(8) \quad \Lambda_{\log}(h) := \{z \in \Lambda(h) \mid z \in [1 - Ch, 1 + Ch] + i[-Mh \log h^{-1}, 0]\}.$$

For $V \in h^{-N}\Psi^\infty(\partial\Omega)$ with real valued symbol, $\sigma(V)$, the reflectivity, $r \in C^\infty(B^*\partial\Omega)$, is given by

$$r(x', \xi') := \frac{h\sigma(V)}{2i\sqrt{1 - |\xi'|_g^2 - h\sigma(V)}}$$

with $r_N(q)$ and $l_N(q)$ as in (4). For a more general definition of r see (18) and for r_N see (21).

Let $\Psi^m(\partial\Omega)$ denote the set of semiclassical pseudodifferential operators of order m (see Section 2) and $\Lambda_{\log}(h)$ be as in (8). Next, let

$$(9) \quad Ai(s) := \frac{1}{2\pi} \int e^{i(st+t^3/3)} dt, \quad A_-(s) := Ai(e^{2\pi i/3}s), \quad \Phi_-(s) := A'_-(s)/A_-(s),$$

$0 > \zeta_1 > \zeta_2 > \dots$ be the zeros of $Ai(s)$.

Finally, let $Q(x, \xi') \in C^\infty(T^*\partial\Omega)$ be the symbol of the second fundamental form to $\partial\Omega$. Then we have:

Theorem 2. *Let $\Omega \Subset \mathbb{R}^d$ be strictly convex with smooth boundary, $\alpha \geq -1$, and suppose that $V \in h^\alpha\Psi^1(\partial\Omega)$ is self adjoint with $\sigma(V) \geq 0$ and $\sigma(V) > c > 0$ in a neighborhood of $\{|\xi'|_g = 1\}$.*

(1) *Suppose that $\alpha > -5/6$. Then for all $\epsilon, N_1 > 0$ there exist $\epsilon_1 > 0, h_0 > 0$ such that for $0 < h < h_0$*

$$\Lambda_{\log}(h) \subset \left\{ \frac{\text{Im } z}{h} \leq \inf_{N \leq N_1} \sup_{|\xi'| < 1 - \epsilon_1} \frac{r_N}{2l_N} + \epsilon \right\}.$$

(2) *Suppose that $-5/6 \geq \alpha \geq -1$. Then for all $\epsilon > 0, M > 0$, there exists $h_0 > 0$ such that for $0 < h < h_0$*

$$\Lambda_{\log}(h) \subset \bigcup_{j=1}^M \left\{ B_{\min} - \epsilon \leq \frac{h^{2/3} \text{Im } z}{\text{Im } \Phi_-(\zeta_j)} \leq B_{\max} + \epsilon \right\} \cup \left\{ \frac{h^{2/3} \text{Im } z}{\text{Im } \Phi_-(\zeta_{M+1})} \geq B_{\min} - \epsilon \right\}$$

where

$$B_{\max} := \sup_{|\xi'|_g=1} \frac{2^{1/3}Q(x, \xi')^{4/3}}{|\sigma(V)(x, \xi')|^2}, \quad B_{\min} := \inf_{|\xi'|_g=1} \frac{2^{1/3}Q(x, \xi')^{4/3}}{|\sigma(V)(x, \xi')|^2}.$$

Moreover, these estimates are sharp in the case of $\Omega = B(0, 1) \subset \mathbb{R}^2$ with $V \equiv 1$.

Theorem 2 verifies several conjectures from [Gal14] and generalizes the results from [Gal16] to arbitrary convex domains. It also provides a second general class of examples that may have resonances with $-\text{Im } z/h \sim ch^\gamma$ for some $\gamma > 0$. That is, resonances converging to the real axis at a fixed polynomial rate, but no faster. Compared to the work in [Gal14, Theorem 5.4], Theorem 2 allows for potentials that depend more strongly on frequency. When the dependence is strong enough ($\alpha \leq -5/6$), the new phenomenon of a band structure appears.

Remark 4.

- Under the pinching condition,

$$\frac{B_{\min}}{B_{\max}} > \frac{\text{Im } \Phi_-(\zeta_j)}{\text{Im } \Phi_-(\zeta_{j+1})}$$

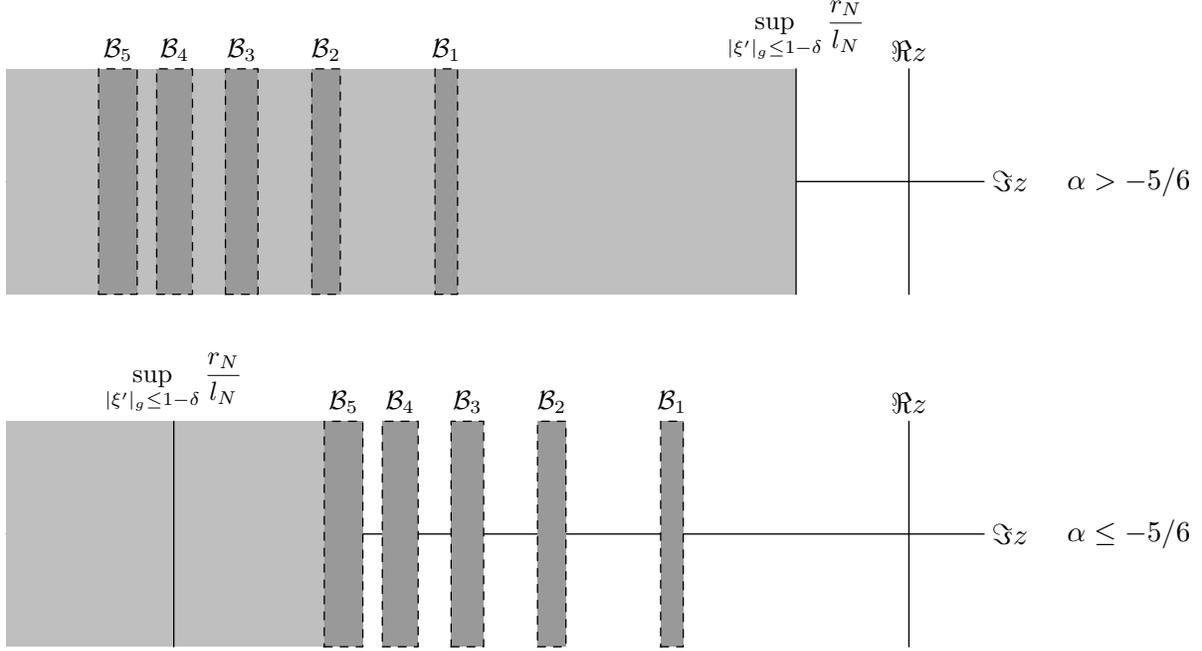


FIGURE 1.4. This figure shows a schematic representation of the resonance free regions from Theorem 2 for $\alpha > -5/6$ on the top and $\alpha \leq -5/6$ on the bottom. Resonances lie in the dark grey bands, $\mathcal{B}_j := \left\{ B_{\min} - \epsilon \leq \frac{h^{2/3} \operatorname{Im} z}{\operatorname{Im} \Phi_-(\zeta_j)} \leq B_{\max} + \epsilon \right\}$, or the light gray shaded region, but not in the white regions. Note that the bands start to group closer together as they go deeper into the complex plane. Thus, there will be only a finite number of bands if $\frac{B_{\max}}{B_{\min}} \neq 1$. See also Figures 1.6 and 1.7 for numerically computed resonances in the case of the disk where $\frac{B_{\max}}{B_{\min}} = 1$ when $V \equiv h^\alpha$.

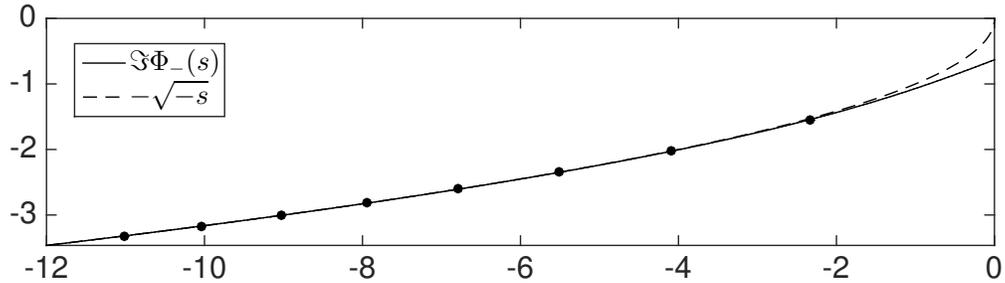


FIGURE 1.5. This figure shows $\operatorname{Im} \Phi_-$ in the solid line and $-\sqrt{-s}$ in the dashed line. The black dots are placed at $(\zeta_j, \operatorname{Im} \Phi_-(\zeta_j))$.

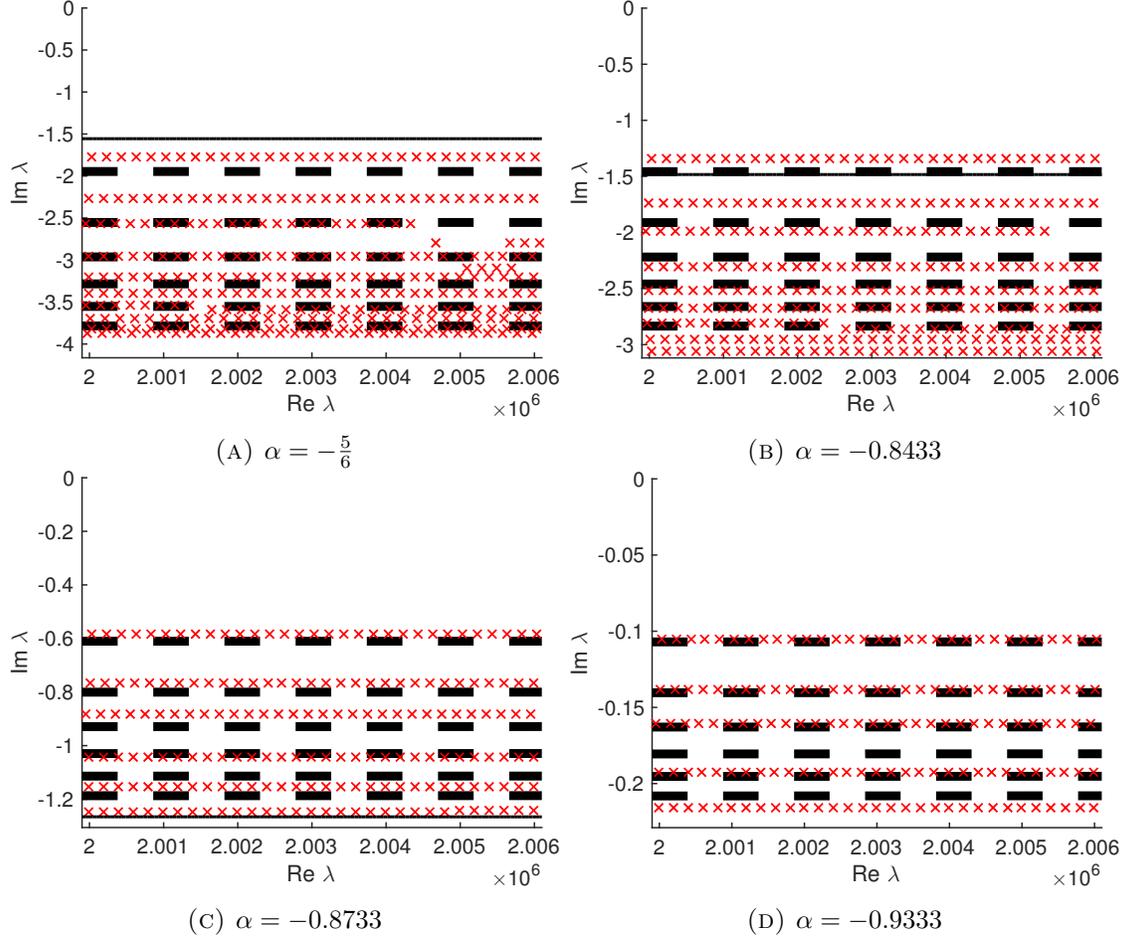


FIGURE 1.6. We show resonances for the delta potential on the circle with $\text{Re } \lambda \sim 10^6$, $V \equiv (\text{Re } \lambda)^{-\alpha}$ and several α . The plots show $\text{Im } \lambda$ vs. $\text{Re } \lambda$ in each case. The solid black line shows the (logarithmic) bound for resonances coming from non-glancing trajectories and the dashed black lines show the first few (polynomial) bands of resonances from near glancing trajectories. Since the solid black line is above the dashed black lines at $\alpha = -5/6$, it is necessary to go to still larger $\text{Re } \lambda$ to see the transition to resonances with fixed size imaginary parts. However, at $\alpha < -5/6$, we start to see better agreement with the bands of resonances predicted in Theorem 2.

there is a gap between the j^{th} and $(j + 1)^{\text{th}}$ band of resonances given by Theorem 2 for $\alpha \leq -5/6$. For a plot of $\text{Im } \Phi_-(s)$ see Figure 1.5.

- To see that the resonance bands in Theorem 2 for $\alpha \leq -5/6$ agree with those in [Gal16], observe that

$$\text{Im} \frac{A'_-(\zeta_j)}{A_-(\zeta_j)} = \text{Im} \frac{2\pi Ai'(\zeta_j)A'_-(\zeta_j)}{e^{5\pi i/6}} = -\frac{2\pi Ai'(\zeta_j)Ai'(\zeta_j)}{2|A_-(\zeta_j)Ai'(\zeta_j)|^3(2\pi)^3} = -\frac{1}{8\pi^2|A_-(\zeta_j)^3Ai'(\zeta_j)|}.$$

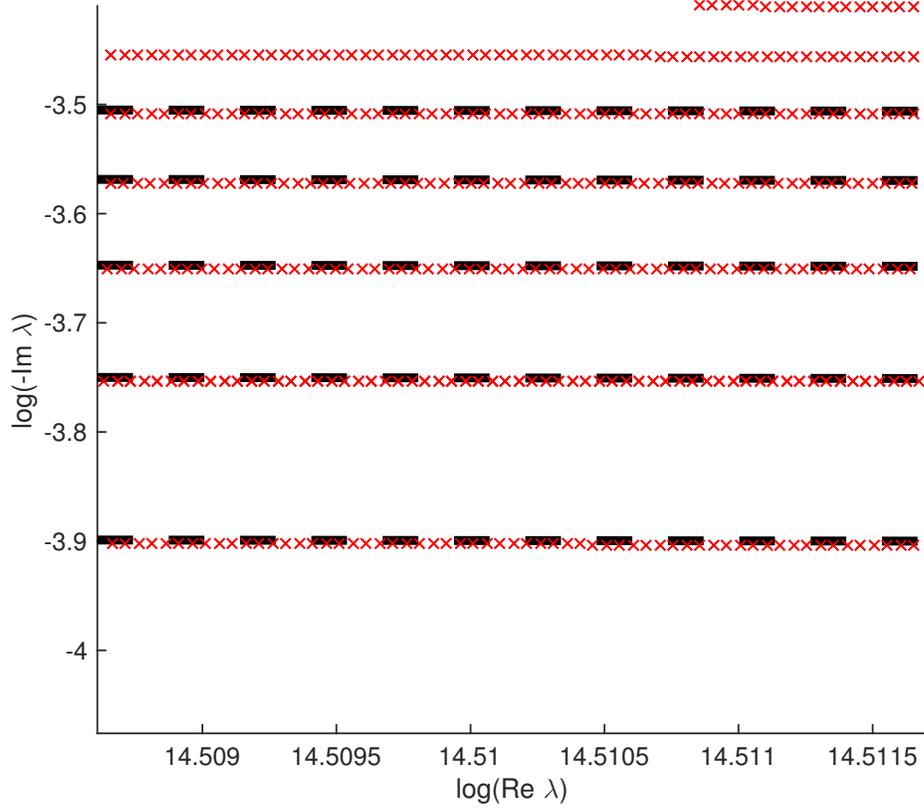


FIGURE 1.7. We show a plot of resonances for the delta potential on the disk with $V \equiv \text{Re } \lambda$. In particular, we show $\log(\text{Re } \lambda)$ vs. $\log(-\text{Im } \lambda)$ for $\text{Re } \lambda \sim 10^6$. The bands predicted by Theorem 2 are shown by the black dashed lines.

1.3. Boundary stabilization problem. Our final application of Theorem 4 is to a boundary stabilized wave equation

$$(10) \quad \begin{cases} (\partial_t^2 - \Delta)u = F \text{ in } \Omega \\ \partial_\nu u + a(x)\partial_t u = 0 \text{ on } \partial\Omega \\ F \in L^2_{\text{comp}}((0, \infty)_t \times \Omega) \\ u \equiv 0 \text{ on } t < -1 \end{cases}$$

with $0 \leq a(x) \in C^\infty(\partial\Omega; \mathbb{R})$. It is not hard to see that the energy

$$E(t) := \frac{1}{2} \left(\|\partial_t u\|^2 + \|\nabla u\|^2 \right)$$

for the corresponding initial value problem is nonincreasing. The study of (10) has a long history, see [BLR92] and the references therein. In [BLR92], Bardos, Lebeau, and Rauch give nearly sharp conditions on a to guarantee exponential decay of the energy.

Here, we impose the strongly dissipative condition $0 < a_0 \leq a$ and study the asymptotic ($|\operatorname{Re} \lambda| \gg 1$) spectral gap for the corresponding stationary problem. That is, taking the Fourier transform in time, we study

$$(11) \quad \begin{cases} (-\Delta - \lambda^2)\mathcal{F}_{t \rightarrow \lambda} u = \mathcal{F}_{t \rightarrow \lambda} F & \text{in } \Omega \\ (\partial_\nu - i\lambda a(x))\mathcal{F}_{t \rightarrow \lambda} u = 0 & \text{on } \partial\Omega. \end{cases}$$

In [CV10], the authors show the existence of a spectral gap in a much more general, but still strongly dissipative, situation. Here, we give estimates on the size of the gap. Let Λ denote the set of λ so that (11) has a nonzero solution. The reflectivity, $r \in C^\infty(B^*\partial\Omega)$, for this problem is given by

$$r(x', \xi') := \frac{\sqrt{1 - |\xi'|_g^2} - a(x')}{a(x') + \sqrt{1 - |\xi'|_g^2}}$$

and l_N, r_N as in (4).

Theorem 3. *Let $\Omega \Subset \mathbb{R}^d$ be strictly convex with smooth boundary and $a(x) \geq a_0 > 0$. Then for all $\epsilon, M > 0$ there exist $\lambda_0 > 0$ such that for $\lambda \in \Lambda$ with $|\operatorname{Re} \lambda| > \lambda_0$ and $\operatorname{Im} \lambda \geq -M \log |\operatorname{Re} \lambda|$,*

$$(12) \quad \sup_{N>0} \inf_{|\xi'|_g \leq 1} \frac{r_N}{2l_N} - \epsilon \leq \operatorname{Im} \lambda \leq \inf_{N>0} \sup_{|\xi'|_g \leq 1} \frac{r_N}{2l_N} + \epsilon.$$

Note that Theorem 3 can also be obtained from the results of Koch–Tataru [KT95]. Indeed, the result contained there actually implies a stronger estimate than (12) in the case of (11). We include this application to give a new proof of those results in this special case and to show that our analysis may be applied even to non-transmission problems. Moreover, note that the operator $a\partial_t$ can be replaced by a much more general pseudodifferential operator and our methods still apply.

1.4. The general setup - a generalized boundary damped wave equation. Theorems 1, 2, and 3 are a consequence of analysis of the boundary damped problem

$$(13) \quad \begin{cases} (-h^2\Delta - z^2)u = w & \text{in } \Omega \\ h\partial_\nu u + Bu = hv & \text{on } \partial\Omega \end{cases}$$

with $\operatorname{Re} z \sim 1$. Here, the operator B plays the role of damping waves upon interaction with the boundary and encodes the interaction with the exterior of Ω in the case of scattering problems.

Let $N_2(z/h)$ denote the outgoing Dirichlet to Neumann map for $\mathbb{R}^d \setminus \bar{\Omega}$. That is, the map given by $C^\infty(\Omega) \ni f \mapsto -\partial_\nu u$ where u solves

$$\begin{cases} (-h^2\Delta - z^2)u = 0 & \text{in } \mathbb{R}^d \setminus \bar{\Omega} \\ u|_{\partial\Omega} = f \\ u \text{ is } z/h \text{ outgoing.} \end{cases}$$

We assume that $B = hN_2(z/h) + hV(z)$ where V is in a certain second microlocal class of pseudodifferential operators which we specify later.

Remark 5. By replacing $\tilde{h} = hE$ and $B(h) = EB(\tilde{h}/E)$, $\tilde{z} = Ez$ we may work with $\operatorname{Re} z \sim E$. Notice that $z/h = \tilde{z}/\tilde{h}$ so operators that are functions of z/h do not change under this rescaling.

We first introduce some notation.. Let

$$D_M(h) := [1 - h, 1 + h] + i[-Mh \log h^{-1}, Mh \log h^{-1}].$$

Let $\gamma : H^s(\mathbb{R}^d) \rightarrow H^{s-1/2}(\partial\Omega)$, $s > 1/2$ be the restriction operator. Then the *single layer operator* is given by

$$G(z/h) := \gamma R_0(z/h) \gamma^*.$$

Recall that $R_0(\lambda)$ is the meromorphic continuation of $(-\Delta - \lambda^2)^{-1}$. From [Gal14, Lemma 4.25] [HZ04, Proposition 4.1] (see also Lemma 7.3), we have that

$$G(z/h) = G_\Delta(z/h) + G_B(z/h) + G_g(z/h) + \mathcal{O}_{\mathcal{D}'(\partial\Omega) \rightarrow C^\infty(\partial\Omega)}(h^\infty)$$

where G_Δ is pseudodifferential, G_B is a semiclassical Fourier integral operator associated to the billiard ball map (see section 2 for the definition of semiclassical Fourier integral operators), and G_g is microlocalized near $|\xi'|_g = 1$. Let $m \geq 0$ and $\Psi_{2/3}^{0,m}(|\xi'|_g = E')$ denote the set of pseudodifferential operators that are second microlocalized near $|\xi'|_g = E$ (see section 4).

We now introduce assumptions on V . For $a_1 \in \mathbb{R}$, $\alpha \geq -1$, $E' \in \mathbb{R} \setminus \{1\}$, $\delta > 0$, $M, M_1, M_2 > 0$, $0 < \epsilon < 1/2$. Let $\langle \cdot \rangle \in C^\infty(T^*\partial\Omega)$ be given by $\langle \xi' \rangle := (1 + |\xi'|_g^2)^{1/2}$. We assume that

$$(14) \quad V = a_1 N_2(z/h) + V_1, \quad V_1 \in h^\alpha \Psi_{2/3}^{0,m}(|\xi'|_g = E'), \quad V \text{ is elliptic on } ||\xi'|_g - 1| < \delta,$$

$$(15) \quad \left| 1 + \frac{h\sigma(V)}{2\sqrt{|\xi'|_g^2 - 1}} \right| \geq \delta \left(\left\langle \frac{h^{1+\alpha}}{\sqrt{|\xi'|_g^2 - 1}} \right\rangle + \langle \xi' \rangle^{m-1} \right) \quad |\xi'|_g > 1 + M_1 h^{2/3},$$

$$(15) \quad \left| 1 + \frac{hi\sigma(V)}{2\sqrt{1 - |\xi'|_g^2}} \right| \geq \delta \left\langle \frac{h^{1+\alpha}}{\sqrt{1 - |\xi'|_g^2}} \right\rangle \quad |\xi'|_g \leq 1 - h^\epsilon,$$

$$(16) \quad V(z) \text{ is an analytic family of operators for } z \in D_M(h),$$

$$(17) \quad \log \left(1 + \frac{h\sigma(V)}{\sqrt{|\xi'|_g^2 - 1}} \right) \text{ exists and is smooth on } T^*\partial\Omega \setminus \{|\xi'|_g \leq M_2\}.$$

We say that $\mathcal{AV}(a_1, \alpha, E', m, \delta, M, M_1, M_2, \epsilon)$ holds when (14)-(17) hold.

We now give a heuristic understanding of (14)-(17). The assumption (14) describes the structure of the operator V in particular, allowing us to include copies of $N_2(z/(hE'))$ which encodes the exterior behavior of waves at speed \sqrt{E}^{-1} . We assume that V is elliptic on $|\xi'|_g = 1$, the glancing set for the problem inside Ω , to simplify some of our analysis and guarantee that glancing effects play a nontrivial role in the analysis. Notice in particular that if $\text{WF}_h(V) \cap \{|\xi'|_g = 1\} = \emptyset$, then waves near glancing escape Ω essentially without reflection. This ellipticity assumption is not necessary for our analysis, but since the main advantage of the present paper over [Gal14] is the analysis near glancing, we include it to simplify our presentation.

Next, (15) guarantees that the problem is locally elliptic in the sense that if a singularity emerges from $(x', \xi') \in T^*\partial\Omega$, then there must be a singularity coming in to (x', ξ') . That is, the boundary cannot produce singularities spontaneously. Furthermore, this guarantees that there are no solutions microlocalized in the elliptic region $|\xi'|_g > 1$.

Finally, (16) and (17) are used to guarantee that the resolvent operator corresponding to (13) is meromorphic and hence that it makes sense to discuss its poles.

Remark 6.

- For the definition of ellipticity of V , see sections 2.3.2 and 4.1.1.
- These are not quite the most general assumptions we can make on V , but in practice all situations we have in mind fall into this category. For the most general assumptions, see (65) and for the statement of the Theorem in that case, see Theorem 5.
- We make the assumption that V is elliptic near glancing so there is no rapid loss of energy near glancing. We could remove this assumption, but there would be no new phenomena and the analysis near glancing would be more complicated.
- The final assumption (17) (used to prove that the underlying problem is Fredholm) is satisfied for example when $m < 1$, or when $m \geq 1$ and for some θ_0 fixed and $\sigma(\tilde{V})$ real valued

$$\sigma(V) = e^{i\theta_0} \sigma(\tilde{V}), \quad |\xi'|_g \geq M_2.$$

Let $\chi \in C_c^\infty(\mathbb{R})$ with $\chi \equiv 1$ near 0, and define

$$(18) \quad \chi_\epsilon \in C^\infty(T^*\partial\Omega), \quad \chi_\epsilon(x, \xi') := 1 - \chi\left(\frac{1 - |\xi'|_g}{h^\epsilon}\right),$$

$$(19) \quad R := -(I + G_\Delta^{1/2} V G_\Delta^{1/2})^{-1} G_\Delta^{1/2} V G_\Delta^{1/2} \text{Op}_h(\chi_\epsilon),$$

$$T(z) := G_\Delta^{-1/2}(z) G_B(z) G_\Delta^{-1/2}(z)$$

where G_B is the Fourier integral operator component of $G(z)$. (See section 2 for an explanation of the quantization procedure Op_h .) Note also that the inverse $(I + G_\Delta^{1/2} V G_\Delta^{1/2})^{-1}$ makes sense microlocally on $\text{supp } \chi_\epsilon$ by (15).

Let $\tilde{\sigma}$ denote the compressed shymbol (see [Gal14, Section 2.3] or Section 3). Then let $l_N, r_N(z) : B^*\partial\Omega \rightarrow \mathbb{R}$ (recall that $B^*\partial\Omega$ is the coball bundle of the boundary and β is the billiard ball map) be

$$(20) \quad l(q, q') := |\pi_x(q) - \pi_x(q')|, \quad l_N(q) := \frac{1}{N} \sum_{k=1}^N l(\beta^{k-1}(q), \beta^k(q))$$

$$(21) \quad r_N(q) := \frac{\text{Im } z}{h} l_N(q) + \frac{1}{2N} \log \tilde{\sigma}(((RT(z))^*)^N (RT(z))^N)(q).$$

The term $\frac{\text{Im } z}{h} l_N$ in (21) serves to cancel the growth of $T(z)$ in the definition of r_N .

Remark 7. Note that we use the notion of the compressed shymbol instead of a variable order symbol since we do not wish to make any apriori assumption on how the symbol of V varies from point to point. Moreover, the order of the symbol will vary also as a function of $\text{Im } z$.

In fact, for $0 < N$ independent of h we have

$$(22) \quad r_N(q) = \frac{1}{2N} \sum_{n=1}^N \log \left| \left(\tilde{\sigma}(R) \circ \beta^n(q) + \mathcal{O}(h^{I_R(q)+1-2\epsilon}) \right) \right|^2$$

where $I_R(q)$ is the local order of R at q (see [Gal14, Section 2.3] or Section 3). The expression (22) illustrates that r_N is the logarithmic average reflectivity over N iterations of the billiard ball map.

Let

$$(23) \quad \mathcal{P}(z) := \begin{pmatrix} -h^2 \Delta - z^2 \\ \partial_\nu + B \end{pmatrix} : H_h^{s+2}(\Omega) \rightarrow H_h^s(\Omega) \oplus H_h^{s+1/2-\max(m-1,0)}(\partial\Omega)$$

where H_h^m denotes the semiclassical Sobolev space with norm

$$(24) \quad \|u\|_{H_h^m} := \|\langle hD \rangle^m u\|_{L^2}.$$

(See [Zwo12, Section 7.1] for a more precise definition.) Let $\Phi_-(s)$ and ζ_j be as in (9) and $Q \in C^\infty(T^*\partial\Omega)$ be the symbol of the second fundamental form to $\partial\Omega$ (as in Theorem 2), and define $f_j(\cdot; h) \in C^\infty(T^*\partial\Omega)$ for $j = 1, 2, \dots$ by

$$f_j(q; h) := \frac{Q(q)((2hQ(q))^{1/3}(1+a_1)\operatorname{Im}\Phi_-(\zeta_j) + \sigma(h\operatorname{Im}V_1)(q))}{|\sigma(hV)(q)|^2}.$$

Let $S^*\partial\Omega$ denote the cosphere bundle of $\partial\Omega$ and

$$\mathcal{B}_{j,\pm}(\epsilon, C; h) := \left\{ z \in D_M(h) : \inf_{S^*\partial\Omega} (f_j(q; h) - Ch^{-\alpha})(1 \mp \epsilon) \leq \frac{\operatorname{Im} z}{h} \leq \sup_{S^*\partial\Omega} (f_j(q; h) + Ch^{-\alpha})(1 \pm \epsilon) \right\}.$$

Then Theorems 1, 2, and 3 are a consequence of the following:

Theorem 4. *Let $\Omega \Subset \mathbb{R}^d$ be strictly convex with smooth boundary. Fix $\epsilon > 0$, $M > 0$, $N_1, N_2 > 0$, $m \geq 0$ and suppose that $\mathcal{AV}(a_1, \alpha, E', m, \delta_0, M, M_1, M_2, \epsilon_0)$ holds. Then there exist $h_0 > 0$, $C, c, N > 0$, so that if $0 < h < h_0$, $z \in D_M(h)$,*

$$(25) \quad \frac{\operatorname{Im} z}{h} \leq \sup_{N \leq N_1} \inf_{|\xi'|_g \leq 1-h^{\epsilon_0}} \frac{r_N}{2l_N} (1 - \epsilon) \quad \text{or} \quad \frac{\operatorname{Im} z}{h} \geq \inf_{N \leq N_1} \sup_{|\xi'|_g \leq 1-h^{\epsilon_0}} \frac{r_N}{2l_N} (1 + \epsilon),$$

$\pm \operatorname{Im} z \geq 0$, $z \notin \cup_{j=1}^{N_2} \mathcal{B}_{j,\pm}(\epsilon, C; h)$, and

$$\frac{\operatorname{Im} z}{h} \geq \sup_{S^*\partial\Omega} \left(f_{N_2+1}(q; h) + Ch^{-\alpha} \right) (1 \pm \epsilon).$$

then $\mathcal{P}(z)$ is invertible and moreover if

$$\mathcal{P}(z)u = \begin{pmatrix} 0 \\ v \end{pmatrix},$$

then

$$(26) \quad \|u|_{\partial\Omega}\|_{H_h^m} \leq ch^{-N} \|v\|_{L^2}.$$

Observe that Theorem 4 (in particular, (25)) takes the same form as (2). Thus, the poles of $\mathcal{P}(z)^{-1}$ are controlled by the average reflectivity in the hyperbolic region. To see that this continues up to the glancing set and hence that Theorem 4 is a quantum version of the Sabine

law, observe that (2) matches (25). Moreover, using Lemma 5.1, that V is elliptic near $|\xi'|_g = 1$ and

$$\sigma(R) = \frac{\sigma(hV)}{2i\sqrt{1 - |\xi'|_g^2} - hV},$$

we have that for $q = (x, \xi') \in B^*\partial\Omega$ with $\sqrt{1 - |\xi'|_g^2} \ll h^{1+\alpha}$

$$(27) \quad \frac{\log |R(\beta(q))|^2}{2l(q, \beta(q))} = \frac{-Q(x, \xi')(\sqrt{1 - |\xi'|_g^2} - \text{Im } hV)}{|\sigma(hV)|^2} + O(h^{-\alpha-1}\sqrt{1 - |\xi'|_g^2}),$$

where, as above, $Q(x, \xi')$ is the symbol of the second fundamental form to $\partial\Omega$. Now,

$$\text{Im } \Phi_-(s) \sim -\sqrt{-s}, \quad s \rightarrow -\infty \quad (\text{see Figure 1.5}).$$

Therefore (27) matches the bounds in Theorem 4 modulo:

- (1) modes cannot concentrate closer than $h^{2/3}$ to $\{|\xi'|_g = 1\}$ (the glancing set)
- (2) a quantization involving the zeros of the Airy function happens at scale $h^{2/3}$ near glancing
- (3) replacing $-\sqrt{-s}$ by $\text{Im } \Phi_-(s)$.

1.5. Outline of the Proof. Proving Theorem 4 amounts to understanding the location of resonances, which correspond to z so that $\mathcal{P}(z)$ is not invertible. We proceed by proving the estimate (26) on solutions to (13) which implies an estimate on $\mathcal{P}(z)^{-1}$.

To avoid analyzing the microlocally complicated interior Dirichlet to Neumann map, we change the boundary condition. In particular, we have

$$(28) \quad (I + GV)\psi = Gv.$$

We then proceed similarly to [Gal14] and decompose the boundary microlocally into the hyperbolic, glancing, and elliptic regions given respectively by

$$\begin{aligned} \mathcal{H} &= \{(x, \xi') \in T^*\partial\Omega \mid |\xi'|_g \leq 1 - h^\epsilon\}, \\ \mathcal{G} &= \{(x, \xi') \in T^*\partial\Omega \mid ||\xi'|_g - 1| \leq h^\epsilon\}, \\ \mathcal{E} &= \{(x, \xi') \in T^*\partial\Omega \mid |\xi'|_g \geq 1 + h^\epsilon\}. \end{aligned}$$

Then, letting $\mathbb{1}_U$ be an operator microlocally equal to the identity on U and U' be a slight enlargement of U , we have

$$(I - \mathbb{1}_{U'})G\mathbb{1}_U = O_{\Psi^{-\infty}}(h^\infty)$$

where U is any of \mathcal{H} , \mathcal{G} , or \mathcal{E} . This allows us to work with each region separately.

For notational convenience, let $\psi = u|_{\partial\Omega}$ and recall that

$$(29) \quad \mathcal{P}(z)u = \begin{pmatrix} 0 \\ v \end{pmatrix}$$

where $\mathcal{P}(z)$ is as in (23). We first consider \mathcal{E} . Here, G is a pseudodifferential operator and our assumptions on V allow us to prove estimates on $\mathbb{1}_{\mathcal{E}}\psi$ in terms of v . We then consider the hyperbolic region, \mathcal{H} . Here the situation is more complicated because G consists of two pieces: G_B , a Fourier integral operator (FIO) associated to the billiard ball map, and G_Δ , a

pseudodifferential operator. Using the calculus of FIO's, we are able to reduce estimating solutions to (28) microlocally in \mathcal{H} to estimating solutions to

$$(I - (RT)^N)u = Av$$

for some A . Then, again using the calculus of FIOs, we see that $I - (RT)^N$ is microlocally invertible under the conditions given in (25).

Up to this point, the analysis in the present paper requires only minor changes from that in [Gal14]. However, the analysis near glancing is substantially different and heavily uses the microlocal model for G and $\mathcal{S}\ell := \mathbf{1}_\Omega R_0(z/h)\gamma^*$ near glancing given in [Gal14, Section 4.5]. The analysis in [Gal14, Chapter 5] uses only the microlocal model for G and does so simply to obtain a norm bound on G near glancing. Here we use the precise microlocal properties of G and $\mathcal{S}\ell$ near glancing.

We start by analyzing $I + GV$ as a second microlocal pseudodifferential operator on

$$\mathcal{G}_+ := \{(x, \xi') \in T^*\partial\Omega \mid 1 - Mh^{2/3} \leq |\xi'|_g \leq 1 + h^\epsilon\}$$

which is the microlocal region closest to glancing. When α is sufficiently small, ($\alpha < -2/3$) we see that $I + GV$ is elliptic on \mathcal{G}_+ outside of a union of $h^{2/3}$ thickened hypersurfaces given by

$$\mathcal{G}_N := \bigcup_{j=1}^N \mathcal{G}_j, \quad \mathcal{G}_j := \left\{ (x, \xi') \in T^*\partial\Omega \mid \left| \frac{|\xi'|_g^2 - 1}{(2Q(x, \xi'))^{2/3}} - h^{2/3}\zeta_j \right| \leq \delta h^{2/3} \right\}.$$

Since we have microlocal invertibility on \mathcal{G}_+ off of \mathcal{G}_N , resonance states must concentrate on \mathcal{G}_N . This is the quantization condition which occurs at scale $h^{2/3}$.

To get this quantization property, we have used the microlocal structure of G . To obtain estimates the remaining part of ψ , i.e, on $\psi_g := (\mathbf{1}_{\mathcal{G}_N} + \mathbf{1}_{\mathcal{G}_-})\psi$ where

$$\mathcal{G}_- := \{(x, \xi') \in T^*\partial\Omega \mid |\xi'|_g \leq 1 - Mh^{2/3}\}$$

we will use the microlocal structure of $\mathcal{S}\ell$.

We have that u solves (29). Integrating by parts in Ω , we have

$$(30) \quad \left(\frac{2 \operatorname{Re} z \operatorname{Im} z}{h} \|u\|_{L^2}^2 - \operatorname{Im} \langle B\psi, \psi \rangle \right) = -\operatorname{Im} \langle hv, \psi \rangle.$$

Then, letting $\mathcal{D}\ell$ denote the double layer potential and using a classical boundary layer formula together with the boundary condition from (13), we have

$$u = h^{-1} \mathcal{S}\ell h \partial_\nu u - \mathcal{D}\ell u = -(h^{-1} \mathcal{S}\ell B + \mathcal{D}\ell)\psi + \mathcal{S}\ell v = -\mathcal{S}\ell V\psi + \mathcal{S}\ell v.$$

So, we can write u in terms of ψ via the boundary layer potential, $\mathcal{S}\ell$. Another technical innovation in our proof is to use the model for $\mathcal{S}\ell$ near glancing to identify $\mathcal{S}\ell^* \mathcal{S}\ell$ as a second microlocal pseudodifferential operator on \mathcal{G} . We are then able to apply the sharp Gårding inequality to obtain upper and lower bounds on

$$\left(\frac{2 \operatorname{Re} z \operatorname{Im} z}{h} \|u_g\|_{L^2}^2 - \operatorname{Im} \langle B\psi_g, \psi_g \rangle \right)$$

where $u_g = -h^{-1} \mathcal{S}\ell V\psi_g$. Together with (30), this allows us to estimate ψ_g in terms of v .

Combining the estimates on \mathcal{E} , \mathcal{G} , and \mathcal{H} , we are able to estimate ψ in terms of v . In order to prove that condition (3) of Theorem 4 together with (25) implies (26), we refine our estimates on \mathcal{G} when $|\operatorname{Im} z| \geq ch^N$ for some $N > 0$.

Because we have polynomial bounds on the interior Dirichlet to Neumann map, $N_1(z/h)$, in this region and

$$(N_1 + N_2)G = I = G(N_1 + N_2),$$

we are able to show that if

$$(I + GV)\tilde{\psi} = w,$$

then there exists $v = (N_1 + N_2)w$ such that $(I + GV)\tilde{\psi} = Gv$ and hence there exists \tilde{u} solving (13) with v replaced by $(N_1 + N_2)w = \mathcal{O}_{L^2 \rightarrow L^2}(h^{-N})w$.

Returning to the original problem, $(I + GV)\psi = Gv$, we see that for δ small enough, \mathcal{G}_j are separated by $\delta h^{2/3}$. Hence, we can find ψ_j microlocalized $\delta h^{2/3}$ close to \mathcal{G}_j so that

$$(I + GV)\psi_j = w_j, \quad \|w_j\| \leq Ch^{-M}\|v\|.$$

Therefore, we can find u_j solving (13) with $u_j|_{\partial\Omega} = \psi_j$ and $v = v_j = h(N_1 + N_2)w_j$ and, repeating the analysis above using boundary layer operators, we can obtain estimates on ψ_j . Together with knowledge of the symbol of N_2 and that of $\mathcal{S}\ell^*\mathcal{S}\ell$, this finishes the proof of Theorem 4.

1.6. Organization of the paper. The paper is organized as follows. We start by introducing the necessary standard semiclassical tools as well as the shymbol from [Gal14] in Sections 2 and 3. Then in Section 4, we introduce the second microlocal calculus from [SZ99]. We conclude the preliminary material with Section 5 where we introduce the billiard ball flow and map.

As a guide for the general case, Section 6 analyzes the single and double layer potentials, respectively

$$(31) \quad \begin{aligned} \mathcal{S}\ell(\lambda)f(x) &:= \int_{\partial\Omega} R_0(\lambda)(x, y)f(y)dS(y) & x \in \Omega \\ \mathcal{D}\ell(\lambda)f(x) &:= \int_{\partial\Omega} \partial_{\nu_y} R_0(\lambda)(x, y)f(y)dS(y) & x \in \Omega \end{aligned}$$

and operators, respectively

$$\begin{aligned} G(\lambda)f(x) &:= \int_{\partial\Omega} R_0(\lambda)(x, y)f(y)dS(y) & x \in \partial\Omega \\ N(\lambda)f(x) &:= \int_{\partial\Omega} \partial_{\nu_y} R_0(\lambda)(x, y)f(y)dS(y) & x \in \partial\Omega \end{aligned}$$

in the special case of the Friedlander model. Section 7 contains the analysis of the boundary layer potentials and operators in the general strictly convex case. Next, Section 8 gives the proof of Theorem 4 including the Fredholm property and meromorphy of the resolvent for \mathcal{P} . Sections 10, 11, and 12 respectively contain the necessary material to deduce Theorems 1, 2, and 3 from Theorem 4. Finally, Section 13 gives the proof that Theorem 1 is sharp in the case of the unit disk.

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2. SEMICLASSICAL PRELIMINARIES

In this section, we review the methods of semiclassical analysis which are needed throughout the rest of our work. The theories of pseudodifferential operators, wavefront sets, and the local theory of Fourier integral operators are standard and our treatment follows that in [DG14] and [Zwo12]. We introduce the notion *shymbol* from [Gal14] which is a notion of sheaf-valued symbol that is sensitive to local changes in the semiclassical order of a symbol.

2.1. Notation. We review the relevant notation from semiclassical analysis in this section. For more details, see [DS99] or [Zwo12].

2.1.1. Big O notation. The $O(\cdot)$ and $o(\cdot)$ notations are used in this paper in the following ways: we write $u = O_{\mathcal{X}}(F)$ if the norm of u in the functional space \mathcal{X} is bounded by the expression F times a constant. We write $u = o_{\mathcal{X}}(F)$ if the norm of u has

$$\lim_{s \rightarrow s_0} \frac{\|u(s)\|_{\mathcal{X}}}{F(s)} = 0$$

where s is the relevant parameter. If no space \mathcal{X} is specified, then $u = O(F)$ and $u = o(F)$ mean

$$(32) \quad |u(s)| \leq C|F(s)| \quad \text{and} \quad \lim_{s \rightarrow s_0} \frac{|u(s)|}{F(s)} = 0$$

respectively.

2.1.2. Phase space. Let M be a d -dimensional manifold without boundary. Then we denote an element of the cotangent bundle to M , (x, ξ) where $\xi \in T_x^*M$.

2.2. Symbols and Quantization. We start by defining the exotic symbol class $f(h)S_{\delta}^m(M)$.

Definition 2.1. Let $a(x, \xi; h) \in C^{\infty}(T^*M \times [0, h_0))$, $f \in C^{\infty}((0, h_0))$, $m \in \mathbb{R}$, and $\delta \in [0, 1/2)$. Then, we say that $a \in f(h)S_{\delta}^m(T^*M)$ if for every $K \Subset M$ and ς, ϖ multiindices, there exists $C_{\varsigma\varpi K}$ such that

$$(33) \quad |\partial_x^{\varsigma} \partial_{\xi}^{\varpi} a(x, \xi; h)| \leq C_{\varsigma\varpi K} f(h) h^{-\delta(|\varsigma|+|\varpi|)} \langle \xi \rangle^{m-|\varpi|}$$

We denote $S_{\delta}^{\infty} := \cup_m S_{\delta}^m$, $S_{\delta}^{-\infty} := \cap_m S_{\delta}^m$ and when one of the parameters δ or m is 0, we suppress it in the notation.

We say that $a(x, \xi; h) \in S_{\delta}^{\text{comp}}(M)$ if $a \in S_{\delta}(M)$ and a is supported in some h -independent compact set.

This definition of a symbol is invariant under changes of variables (see for example [Zwo12, Theorem 9.4] or more precisely, the arguments therein).

2.3. Pseudodifferential operators. We follow [Zwo12, Section 14.2] to define the algebra $\Psi_\delta^m(M)$ of pseudodifferential operators with symbols in $S_\delta^m(M)$. (For the details of the construction of these operators, see for example [Zwo12, Sections 4.4, 14.12]. See also [Hör07, Chapter 18] or [GS94, Chapter 3].) Since we have made no assumption on the behavior of our symbols as $x \rightarrow \infty$, we do not have control over the behavior of $\Psi_\delta^k(M)$ near infinity in M . However, we do require that all operators $A \in \Psi_\delta^m(M)$ are *properly supported*. That is, the restriction of each projection map $\pi_x, \pi_{x'} : M \times M \rightarrow M$ to the support of $K_A(x, x'; h)$, the Schwartz kernel of A , is a proper map. For the construction of such a quantization procedure, see for example [Hör07, Proposition 18.1.22]. An element in $A \in \Psi_\delta^m(M)$ acts $H_{h,\text{loc}}^s(M) \rightarrow H_{h,\text{loc}}^{s-m}(M)$ where $H_{h,\text{loc}}^s(M)$ denotes the space of distributions locally in the semiclassical Sobolev space $H_h^s(M)$. The definition of these spaces can be found for example in [Zwo12, Section 7.1]. Finally, we say that a properly supported operator, A , with

$$A : \mathcal{D}'(M) \rightarrow C^\infty(M)$$

and each seminorm $O(h^\infty)$ is $O_{\Psi^{-\infty}}(h^\infty)$. We include operators that are $O_{\Psi^{-\infty}}(h^\infty)$ in all pseudodifferential classes.

With this definition, we have the semiclassical principal symbol map

$$(34) \quad \sigma : \Psi_\delta^m(M) \rightarrow S_\delta^m(M) / h^{1-2\delta} S_\delta^{m-1}(M)$$

and a non-canonical quantization map

$$\text{Op}_h : S_\delta^m(M) \rightarrow \Psi_\delta^m(M)$$

with the property that $\sigma \circ \text{Op}_h$ is the natural projection map onto

$$S_\delta^m(M) / h^{1-2\delta} S_\delta^{m-1}(M).$$

Henceforward, we will take $\sigma(A)$ to be any representative of the corresponding equivalence class in the right-hand side of (34). We do not include the sub-principal symbol because then the calculus of pseudodifferential operators would be more complicated. With this in mind, the standard calculus of pseudodifferential operators with symbols in S_δ^m gives for $A \in \Psi_\delta^{m_1}(M)$ and $B \in \Psi_\delta^{m_2}(M)$,

$$\begin{aligned} \sigma(A^*) &= \overline{\sigma(A)} + O_{S_\delta^{m_1-1}(M)}(h^{1-2\delta}) \\ \sigma(AB) &= \sigma(A)\sigma(B) + O_{S_\delta^{m_1+m_2-1}(M)}(h^{1-2\delta}) \\ \sigma([A, B]) &= -ih\{\sigma(A), \sigma(B)\} + O_{S_\delta^{m_1+m_2-2}(M)}(h^{2(1-2\delta)}). \end{aligned}$$

Here $\{\cdot, \cdot\}$ denotes the Poisson bracket and we take adjoints with respect to $L^2(M)$.

2.3.1. Wavefront sets and microsupport of pseudodifferential operators. In order to define a notion of wavefront set that captures both h -microlocal and C^∞ behavior, we define the *fiber radially compactified cotangent bundle*, \bar{T}^*M , by $\bar{T}^*M = T^*M \sqcup S^*M$ where

$$S^*M := (T^*M \setminus \{M \times 0\}) / \mathbb{R}_+$$

and the \mathbb{R}_+ action is given by $(t, (x, \xi)) \mapsto (x, t\xi)$. Let $|\cdot|_g$ denote the norm induced on T^*M by the Riemannian metric g . Then a neighborhood of a point $(x_0, \xi_0) \in S^*M$ is given by $V \cap \{|\xi|_g \geq K\}$ where V is an open conic neighborhood of $(x_0, \xi_0) \in T^*M$.

For each $A \in \Psi_\delta^m(M)$ there exists $a \in S_\delta^m(M)$ with $A = \text{Op}_h(a) + \mathcal{O}_{\Psi^{-\infty}}(h^\infty)$. Then the *semiclassical wavefront set* of A , $\text{WF}_{h,\Psi}(A) \subset \overline{T^*M}$, is defined as follows. A point $(x, \xi) \in \overline{T^*M}$ does not lie in $\text{WF}_{h,\Psi}(A)$ if there exists a neighborhood U of (x, ξ) such that each (x, ξ) derivative of a is $\mathcal{O}(h^\infty \langle \xi \rangle^{-\infty})$ in U . As in [Ale08], we write

$$\text{WF}_{h,\Psi}(A) =: \text{WF}_{h,\Psi}^f(A) \sqcup \text{WF}_{h,\Psi}^i(A)$$

where $\text{WF}_{h,\Psi}^f(A) = \text{WF}_h(A) \cap T^*M$ and $\text{WF}_{h,\Psi}^i(A) = \text{WF}_h(A) \cap S^*M$.

Operators with compact wavefront sets in T^*M are called *compactly microlocalized*. These are operators of the form

$$\text{Op}_h(a) + \mathcal{O}_{\Psi^{-\infty}}(h^\infty)$$

for some $a \in S_\delta^{\text{comp}}(M)$. The class of all compactly microlocalized operators in $\Psi_\delta^m(M)$ are denoted by $\Psi_\delta^{\text{comp}}(M)$.

We will also need a finer notion of microsupport on h -dependent sets.

Definition 2.2. An operator $A \in \Psi_\delta^{\text{comp}}(M)$ is *microsupported* on an h -dependent family of sets $V(h) \subset T^*M$ if we can write $A = \text{Op}_h(a) + \mathcal{O}_{\Psi^{-\infty}}(h^\infty)$, where for each compact set $K \subset T^*M$, each differential operator ∂^ζ on T^*M , and each N , there exists a constant $C_{\zeta NK}$ such that for h small enough,

$$\sup_{(x,\xi) \in K \setminus V(h)} |\partial^\zeta a(x, \xi; h)| \leq C_{\zeta NK} h^N.$$

We then write

$$\text{MS}_{h,\Psi}(A) \subset V(h).$$

The change of variables formula for the full symbol of a pseudodifferential operator [Zwo12, Theorem 9.10] contains an asymptotic expansion in powers of h consisting of derivatives of the original symbol. Thus definition 2.2 does not depend on the choice of the quantization procedure Op_h . Moreover, since we take $\delta < 1/2$, if $A \in \Psi_\delta^{\text{comp}}$ is microsupported inside some $V(h)$ and $B \in \Psi_\delta^m$, then AB , BA , and A^* are also microsupported inside $V(h)$. This implies the following.

Lemma 2.1. *Suppose that $A, B \in \Psi_\delta^{\text{comp}}$ and $\text{MS}_{h,\Psi}(A) \cap \text{MS}_{h,\Psi}(B) = \emptyset$. Then*

$$\text{WF}_{h,\Psi}(AB) = \emptyset.$$

For $A \in \Psi_\delta^{\text{comp}}(M)$, $(x, \xi) \notin \text{WF}_h(A)$ if and only if there exists an h -independent neighborhood u of (x, ξ) such that A is microsupported on the complement of u . However, A need only be microsupported on any h -independent neighborhood of $\text{WF}_{h,\Psi}(A)$, not on $\text{WF}_{h,\Psi}(A)$ itself. Also, notice that by Taylor's formula if $A \in \Psi_\delta^{\text{comp}}(M)$ is microsupported in $V(h)$ and $\delta' > \delta$, then A is also microsupported on the set of all points in $V(h)$ which are at least $h^{\delta'}$ away from the complement of $V(h)$.

Remark 8. Notice that since we are working with $A \in \Psi_\delta^{\text{comp}}(M)$ for $0 \leq \delta < 1/2$ we have $a \in S_\delta^{\text{comp}}(T^*M)$ and a can only vary on a scale $\sim h^{-\delta}$. This implies that the set $\text{MS}_{h,\Psi}(A)$ will respect the uncertainty principle.

2.3.2. Ellipticity and L^2 operator norm. For $A \in \Psi_\delta^m(M)$, define its *elliptic set* $\text{ell}(A) \subset T^*M$ as follows: $(x, \xi) \in \text{ell}(A)$ if and only if there exists a neighborhood U of (x, ξ) in $\overline{T^*}M$ and a constant C such that $|\sigma(A)| \geq C^{-1}\langle \xi \rangle^m$ in $U \cap T^*M$. The following statement is the standard semiclassical elliptic estimate; see [Hör07, Theorem 18.1.24] for the closely related microlocal case and for example [Dya12, Section 2.2] for the semiclassical case.

Lemma 2.2. *Suppose that $P \in \Psi_\delta^m(M)$ and $A \in \Psi_\delta^{m'}(M)$ with $\text{WF}_{h,\Psi}(A) \subset \text{ell}(P)$. Then for each $\chi \in C_c^\infty(M)$, there exist $Q_i \in \Psi_\delta^{m'-m}(M)$ such that*

$$\chi A = \chi Q_1 P + O_{\Psi_\delta^{-\infty}}(h^\infty) = \chi P Q_2 + O_{\Psi^{-\infty}}(h^\infty).$$

In particular, for each $s \in \mathbb{R}$ and $u \in H_h^{s+m'}$ there exists $C > 0$ such that for all $N > 0$, and $\chi_1 \in C^\infty(M)$ with $\chi_1 \equiv 1$ on $\text{supp } \chi$,

$$\|\chi A u\|_{H_h^s} \leq C \|\chi P u\|_{H_h^{s+m'-m}} + O(h^\infty) \|\chi_1 u\|_{H_h^{-N}}.$$

We also recall the estimate for the $L^2 \rightarrow L^2$ norm of a pseudodifferential operator (see for example [Zwo12, Chapter 13]).

Lemma 2.3. *Suppose that $A \in \Psi_\delta(M)$. Then there exists $C > 0$ such that*

$$\|A\|_{L^2 \rightarrow L^2} \leq \sup_{T^*M} |\sigma(A)| + Ch^{1-2\delta}.$$

2.4. Semiclassical microlocalization of distributions and operators.

2.4.1. Semiclassical wavefront sets and microsupport for distributions. An h -dependent family $u(h) : (0, h_0) \rightarrow \mathcal{D}'(M)$ is called *h -tempered* if for each open $U \Subset M$, there exist constants C and N such that

$$(35) \quad \|u(h)\|_{H_h^{-N}(U)} \leq Ch^{-N}.$$

For a tempered distribution u , we say that $(x_0, \xi_0) \in \overline{T^*}M$ does not lie in the wavefront set $\text{WF}_h(u)$, if there exists a neighborhood V of (x_0, ξ_0) such that for each $A \in \Psi(M)$ with $\text{WF}_{h,\Psi}(A) \subset V$, we have $Au = O_{C^\infty}(h^\infty)$. As above, we write

$$\text{WF}_h(u) = \text{WF}_h^f(u) \sqcup \text{WF}_h^i(u)$$

where $\text{WF}_h^i(u) = \text{WF}_h(u) \cap S^*M$. By Lemma 2.2, $(x_0, \xi_0) \notin \text{WF}_h(u)$ if and only if there exists compactly supported $A \in \Psi(M)$ elliptic at (x_0, ξ_0) such that $Au = O_{C^\infty}(h^\infty)$. The wavefront set of u is a closed subset of $\overline{T^*}M$. It is empty if and only if $u = O_{C^\infty(M)}(h^\infty)$. We can also verify that for u tempered and $A \in \Psi_\delta^m(M)$, $\text{WF}_h(Au) \subset \text{WF}_{h,\Psi}(A) \cap \text{WF}_h(u)$.

Definition 2.3. A tempered distribution u is said to be *microsupported* on an h -dependent family of sets $V(h) \subset T^*M$ if for $\delta \in [0, 1/2)$, $A \in \Psi_\delta(M)$, and $\text{MS}_{h,\Psi}(A) \cap V = \emptyset$, $\text{WF}_h(Au) = \emptyset$.

2.4.2. *Semiclassical wavefront sets of tempered operators.* An h -dependent family of operators $A(h) : \mathcal{S}(M) \rightarrow \mathcal{S}'(M')$ is called *h -tempered* if for each $U \Subset M$, there exists $N \geq 0$ and $k \in \mathbb{Z}^+$, such that

$$(36) \quad \|A(h)\|_{H_h^k(U) \rightarrow H_{h,\text{loc}}^{-k}(M')} \leq Ch^{-N}$$

For an h -tempered family of operators, we write that the wavefront set of A is given by

$$\text{WF}_h'(A) := \{(x, \xi, y, \eta) \mid (x, \xi, y, -\eta) \in \text{WF}_h(K_A)\}$$

where K_A is the Schwartz kernel of A .

Definition 2.4. A tempered operator A is said to be *microsupported* on an h -dependent family of sets $V(h) \subset T^*M \times T^*M'$, if for all $\delta \in [0, 1/2)$ and each $B_1 \in \Psi_\delta(M')$ and $B_2 \in \Psi_\delta(M)$ with $(\text{MS}_{h,\Psi}(B_1) \times \text{MS}_{h,\Psi}(B_2)) \cap V = \emptyset$, we have $\text{WF}_h(B_1AB_2) = \emptyset$. We then write

$$\text{MS}_h'(A) \subset V(h).$$

Remark 9. With the definitions above, we have for $A \in \Psi_\delta^m(M)$,

$$\text{WF}_h'(A) = \{(x, \xi, x, \xi) : (x, \xi) \in \text{WF}_{h,\Psi}(A)\}.$$

In addition, we have that if $A \in \Psi_\delta^{\text{comp}}$, then $\text{MS}_{h,\Psi}(A) \subset V(h)$ if and only if

$$\text{MS}_h'(A) \subset \{(x, \xi, x, \xi) : (x, \xi) \in V(h)\}.$$

Since there is a simple relationship between $\text{WF}_{h,\Psi}$ and WF_h , as well as $\text{MS}_{h,\Psi}$ and MS_h , we will only use the notation without Ψ from this point forward and the correct object will be understood from context.

2.5. Semiclassical Lagrangian distributions. In this subsection, we review some facts from the theory of semiclassical Lagrangian distributions. See [GS77, Chapter 6] or [VN06, Section 2.3] for a detailed account, and [Hör09, Section 25.1] or [GS94, Chapter 11] for the microlocal case. We do not attempt to define the principal symbol as a globally invariant object. Indeed, it is not always possible to do so in the semiclassical setting. When it is possible to do so, i.e. when the Lagrangian is exact, we define the symbol modulo the Maslov bundle. Taking symbols modulo the Maslov bundle makes the theory considerably simpler. We can make this simplification since for all of our symbolic computations, we work only in a single coordinate chart and, moreover, we always work with exact Lagrangians.

2.5.1. *Phase functions.* Let M be a manifold without boundary. We denote its dimension by d . Let $\varphi(x, \theta)$ be a smooth real-valued function on some open subset U_φ of $M \times \mathbb{R}^L$, for some L ; we call x the *base variable* and θ the *oscillatory variable*. As in [Hör07, Section 21.2], we say that φ is a *phase function* if the differentials $(\partial_{\theta_1}\varphi), \dots, d(\partial_{\theta_L}\varphi)$ on the *critical set*

$$(37) \quad C_\varphi := \{(x, \theta) \mid \partial_\theta\varphi = 0\} \subset U_\varphi$$

are independent. Note that

$$\Lambda_\varphi := \{(x, \partial_x\varphi(x, \theta)) \mid (x, \theta) \in C_\varphi\} \subset T^*M$$

is an immersed Lagrangian submanifold (we will shrink the domain of φ to make it embedded).

2.5.2. *Symbols.* Let $\delta \in [0, 1/2)$. A smooth function $a(x, \theta; h)$ is called a compactly supported symbol of type δ on U_φ , if it is supported in some compact h -independent subset of U_φ , and for each differential operator ∂^ς on $M \times \mathbb{R}^L$, there exists a constant C_ς such that

$$\sup_{U_\varphi} |\partial^\varsigma a| \leq C_\varsigma h^{-\delta|\varsigma|}.$$

As above, we write $a \in S_\delta^{\text{comp}}(U_\varphi)$ and denote $S^{\text{comp}} := S_0^{\text{comp}}$.

2.5.3. *Lagrangian distributions.* Given a phase function φ and a symbol $a \in S_\delta^{\text{comp}}(U_\varphi)$, consider the h -dependent family of functions

$$(38) \quad u(x; h) = (2\pi h)^{-(d+2L)/4} \int_{\mathbb{R}^L} e^{i\varphi(x, \theta)/h} a(x, \theta; h) d\theta.$$

We call u a *Lagrangian distribution* of type δ generated by φ and denote this by $u \in I_\delta^{\text{comp}}(\Lambda_\varphi)$.

By the method of non-stationary phase, if $\text{supp } a$ is contained in some h -dependent compact set $K(h) \subset U_\varphi$, then

$$(39) \quad \text{MS}_h(u) \subset \{(x, \partial_x \varphi(x, \theta)) \mid (x, \theta) \in C_\varphi \cap K(h)\} \subset \Lambda_\varphi.$$

Remark 10. We are using the fact that $a \in S_\delta(U_\varphi)$ for some $\delta < 1/2$ here.

2.5.4. *Principal Symbols.* We define the principal symbol of a Lagrangian distribution independently of the choice of φ . To do this, we will need to use half-densities on Λ_φ (see, for example [Zwo12, Chapter 9] for a definition).

Following [Hör09, Section 25.1], letting

$$\Phi = \begin{pmatrix} \varphi''_{xx} & \varphi''_{x\theta} \\ \varphi''_{\theta x} & \varphi''_{\theta\theta} \end{pmatrix},$$

Lemma 2.4. *Modulo Maslov factors, and a factor $e^{iA/h}$ for some constant $A \in \mathbb{R}$ depending on φ , the principal symbol*

$$\sigma(u) \in S_\delta^{\text{comp}}(\Lambda_\varphi; \Omega^{1/2}) / h^{1-2\delta} S_\delta^{\text{comp}}(\Lambda_\varphi; \Omega^{1/2})$$

is a half density given by

$$\sigma(u)(x, \xi) = |d\xi|^{1/2} a(x, \theta) e^{i\pi/4 \text{sgn } \Phi} |\det \Phi|^{-1/2}.$$

Remark 11. In the case that Λ_φ is exact the factor $e^{iA/h}$ can be removed.

Definition 2.5. Let $\Lambda \subset T^*M$ be an embedded Lagrangian submanifold. We say that an h -dependent family of functions $u(x; h) \in C_c^\infty(M)$ is a (compactly supported and compactly microlocalized) Lagrangian distribution of type δ associated to Λ , if it can be written as a sum of finitely many functions of the form (38), for different phase functions φ parametrizing open subsets of Λ , plus an $O_{C_c^\infty}(h^\infty)$ remainder. Denote by $I_\delta^{\text{comp}}(\Lambda)$ the space of all such distributions, and put $I^{\text{comp}}(\Lambda) := I_0^{\text{comp}}(\Lambda)$.

The action of a pseudodifferential operator on a Lagrangian distribution is given by the following Lemma, following from the method of stationary phase:

Lemma 2.5. *Let $u \in I_\delta^{\text{comp}}(\Lambda)$ and $P \in \Psi_\delta^m(M)$. Then $Pu \in I_\delta^{\text{comp}}(\Lambda)$ and*

$$\sigma(Pu) = \sigma(P)|_\Lambda \cdot \sigma(u) + O(h^{1-2\delta})_{S_\delta^{\text{comp}}(\Lambda)}.$$

2.6. Fourier integral operators. A special case of Lagrangian distributions are Fourier integral operators associated to canonical graphs. Let M be a manifold of dimension d . Consider a Lagrangian submanifold $\Lambda \subset T^*M \times T^*M$ given by

$$\Lambda = \{(\kappa(y, \eta), y, -\eta)\}$$

where κ is a symplectomorphism.

A compactly supported operator $U : \mathcal{D}'(M') \rightarrow C_c^\infty(M)$ is called a (semiclassical) *Fourier integral operator* of type δ associated to κ if its Schwartz kernel $K_U(x, x')$ lies in $I_\delta^{\text{comp}}(\Lambda)$. We write $U \in I_\delta^{\text{comp}}(C)$ where

$$C = \{(x, \xi, y, \eta) \mid (x, \xi, y, -\eta) \in \Lambda\}.$$

The numerology $h^{-(d+2L)/4}$ in (38) is explained by the fact that the normalization for Fourier integral operators is chosen so that

$$\|U\|_{L^2(M) \rightarrow L^2(M)} \sim 1$$

when C is the generated by a symplectomorphism.

We will need the following lemma from the calculus of Fourier integral operators

Lemma 2.6. *Let $A \in I_\delta^{\text{comp}}(M \times M, C)$ and $P \in \Psi_\delta^{\text{comp}}(M)$. Then, $A^*PA \in \Psi_\delta^{\text{comp}}(M)$ and*

$$\sigma(A^*PA)(q) = |\sigma(A)(q, \kappa(q))|^2 \sigma(P)(\kappa(q)).$$

3. THE SHYMBOL

It will be useful to calculate symbols of operators whose semiclassical order may vary from point to point in T^*M . One can often handle this type of behavior by using weights to compensate for the growth. However, this requires some a priori knowledge of how the order changes and limits the allowable size in the change of order. In this section, we will develop a notion of a sheaf valued symbol, the *shymbol*, that can be used to work in this setting without such a priori knowledge.

Let M be a compact manifold. Let $\mathcal{T}(T^*M)$ be the topology on T^*M . For $s \in \mathbb{R}$, denote the symbol map

$$\sigma_s : h^s \Psi_\delta^{\text{comp}} \rightarrow h^s S_\delta^{\text{comp}} / h^{s+1-2\delta} S_\delta^{\text{comp}}.$$

Suppose that for some $N > 0$ and $\delta \in [0, 1/2)$, $A \in h^{-N} \Psi_\delta^{\text{comp}}(M)$. We define a finer notion of symbol for such a pseudodifferential operator. Fix $0 < \epsilon \ll 1 - 2\delta$. For each open set $U \in \mathcal{T}(T^*M)$, define the ϵ -order of A on U

$$I_A^\epsilon(U) := \sup_{s \in \mathcal{S}_\epsilon} s + 1 - 2\delta$$

where

$$\mathcal{S}_\epsilon := \left\{ s \in \epsilon\mathbb{Z} \mid \begin{array}{l} \text{there exists } \chi \in C_c^\infty(T^*M), \chi|_U = 1, \\ \sigma_s(\text{Op}_h(\chi)A \text{Op}_h(\chi))|_U \equiv 0 \end{array} \right\}.$$

Then it is clear that for any $V \Subset U$ there exists $\chi \in C_c^\infty(U)$ with $\chi = 1$ on V such that $\text{Op}_h(\chi)A \text{Op}_h(\chi) \in h^{I_A^\epsilon(U)} \Psi_\delta^{\text{comp}}(M)$.

Give $\mathcal{T}(T^*M)$ the ordering that $U \leq V$ if $V \subset U$ with morphisms $U \rightarrow V$ if $U \leq V$. Notice that $U \leq V$ implies $I_A^\epsilon(U) \leq I_A^\epsilon(V)$. Then define the functor $F_A^\epsilon : \mathcal{T}(T^*M) \rightarrow \mathbf{Comm}$ (the category of commutative rings) by

$$F_A^\epsilon(U) = \begin{cases} h^{I_A^\epsilon(U)} S_\delta^{\text{comp}}(M)|_U / h^{I_A^\epsilon(U)+1-2\delta} S_\delta^{\text{comp}}(M)|_U & I_A^\epsilon(U) \neq \infty \\ \{0\} & I_A^\epsilon(U) = \infty \end{cases},$$

$$F_A^\epsilon(U \rightarrow V) = \begin{cases} h^{I_A^\epsilon(V)-I_A^\epsilon(U)}|_V & I_A^\epsilon(V) \neq \infty \\ 0 & I_A^\epsilon(V) = \infty \end{cases}.$$

Then F_A^ϵ is a presheaf on T^*M . We sheafify F_A^ϵ , still denoting the resulting sheaf by F_A^ϵ , and say that A is of ϵ -class F_A^ϵ . We define the *stalk* of the sheaf at q by $F_A^\epsilon(q) := \varinjlim_{q \in U} F_A^\epsilon(U)$.

Now, for every $U \subset \mathcal{T}(T^*M)$, $I_A^\epsilon(U) \neq \infty$, there exists $\chi_U \in C_c^\infty(T^*M)$ with $\chi_U \equiv 1$ on U such that $\sigma_{I_A^\epsilon(U)}(\text{Op}_h(\chi_U)A \text{Op}_h(\chi_U))|_U \neq 0$. Then we define the ϵ -*shymbol* of A to be the section of F_A^ϵ , $\tilde{\sigma}_{(\cdot)}^\epsilon(A) : \mathcal{T}(T^*M) \rightarrow F_A^\epsilon(\cdot)$, given by

$$\tilde{\sigma}_U^\epsilon(A) := \begin{cases} \sigma_{I_A^\epsilon(U)}(\text{Op}_h(\chi_U)A \text{Op}_h(\chi_U))|_U & I_A^\epsilon(U) \neq \infty \\ 0 & I_A^\epsilon(U) = \infty \end{cases}.$$

Define also the ϵ -*stalk shymbol*, $\tilde{\sigma}^\epsilon(A)_q$ to be the germ of $\tilde{\sigma}^\epsilon(A)$ at q as a section of F_A^ϵ .

Now, define

$$I_A^\epsilon(q) := \sup\{I_A^\epsilon(U) \mid q \in U\}.$$

We then define the simpler *compressed shymbol*. Let $U_n \downarrow \{q\}$ be a sequence of open sets.

$$(40) \quad \tilde{\sigma}^\epsilon(A) : T^*M \rightarrow \bigsqcup_q h^{I_A^\epsilon(q)} \mathbb{C} / h^{I_A^\epsilon(q)+1-2\delta} \mathbb{C} \quad \text{by}$$

$$\tilde{\sigma}^\epsilon(A)(q) := \begin{cases} 0 & I_A^\epsilon(q) = \infty \\ \lim_n \tilde{\sigma}_{U_n}^\epsilon(A)(q) & I_A^\epsilon(q) < \infty \end{cases}$$

The limit in (40) exists since if $I_A^\epsilon(q) < \infty$, then there exists $U \ni q$ such that for all $V \subset U$, $I_A^\epsilon(V) = I_A^\epsilon(U)$. This also shows that the limit is independent of the choice of sequence of $U_n \downarrow q$. It is easy to see from standard composition formulae that the compressed shymbol has

$$\tilde{\sigma}^\epsilon(AB)(q) = \tilde{\sigma}^\epsilon(A)(q) \tilde{\sigma}^\epsilon(B)(q), \quad A \in h^{-N} \Psi_\delta^{\text{comp}} \quad \text{and} \quad B \in h^{-M} \Psi_\delta^{\text{comp}}.$$

Moreover,

$$\tilde{\sigma}^\epsilon([A, B])(q) = -ih \{ \tilde{\sigma}^\epsilon(A)(q), \tilde{\sigma}^\epsilon(B)(q) \}.$$

The following lemma follows from standard formulas for the composition of FIOs combined with the definitions above:

Lemma 3.1. *Suppose that $A \in \Psi_\delta^{\text{comp}}$ and let T be a semiclassical FIO associated to the symplectomorphism κ with elliptic symbol $t \in S_\delta$. Then for $N > 0$ independent of h $(AT)_N := (T^*A^*)^N (AT)^N$ has*

$$\tilde{\sigma}^\epsilon((AT)_N)(q) = \prod_{i=1}^N \left(|\tilde{\sigma}^\epsilon(A)t|^2 \circ \kappa^i(q) + o \left(h^{I_{A_i}^\epsilon(\beta^k(q))+1-2\delta} \right) \right).$$

Proof. Fix $q \in T^*M$. Let $\chi_k \in C_c^\infty(T^*M)$ have $\chi_k = 1$ on $B\left(q, \frac{1}{k}\right)$, the open ball of radius k^{-1} around q , and $\text{supp } \chi_k \subset B\left(q, \frac{2}{k}\right)$. Then let $D := \text{Op}_h(\chi_k)(AT)_N \text{Op}_h(\chi_k)$. We have that

$$D = \text{Op}_h(\chi_k)(A_N T A_{N-1} T \dots A_1 T)^* (A_N T A_{N-1} T \dots A_1 T) \text{Op}_h(\chi_k) + \mathcal{O}_{\Psi_\delta^{\text{comp}}}(h^\infty)$$

where $A_i = \text{Op}_h(\psi_{k,i}) A \text{Op}_h(\psi_{k,i})$ with $C_c^\infty(T^*M) \ni \psi_{k,i} = 1$ in some neighborhood of $\beta^i(q)$ and is supported inside a neighborhood $U_{k,i}$ of $\beta^i(q)$ such that $U_{k,i} \downarrow q$. Then the result follows from standard composition formulae in Lemma 2.6. \square

Now, since $\epsilon > 0$ is arbitrary, we define the *semiclassical order of A at q* by $I_A(q) := \sup_{\epsilon > 0} I_A^\epsilon(q)$ with the understanding that $f = \mathcal{O}(h^{I_A(q)})$ means that for any $\epsilon > 0$,

$$|f(q)| \leq C_\epsilon h^{I_A(q) - \epsilon}.$$

Furthermore, we suppress the ϵ in the notation $\tilde{\sigma}^\epsilon(A)(q)$ and denote the *compressed shymbol*, $\tilde{\sigma}(A)(q)$, again with the understanding that for any $\epsilon > 0$,

$$\tilde{\sigma}(A)(q) \in h^{I_A(q) - \epsilon} \mathbb{C} / h^{I_A(q) + 1 - 2\delta - \epsilon} \mathbb{C}.$$

4. A SECOND MICROLOCAL CALCULUS

In the present work, it will be necessary to localize $h^{2/3}$ near the glancing submanifold in $T^*\partial\Omega$. In order to do this, we present the second microlocal calculus from [SZ99].

4.1. The local model. We start by considering the model case of $\Sigma_0 = \{\xi_1 = 0\} \subset T^*\mathbb{R}^d$. Suppose that U is a neighborhood of $(0, 0)$ and $a \in C_c^\infty(U)$. In that case, we write $a = a(x, \xi, \lambda; h)$ with $\lambda = h^{-\delta}\xi_1$. Suppose that $\epsilon < \min(1/2, \delta)$, and $\epsilon + \delta \leq 1$. We say that $a \in S_{\delta, \epsilon}^{k_1}(\Sigma_0)$ if and only if

$$(41) \quad \partial_x^s \partial_\xi^\varpi \partial_\lambda^k a(x, \xi, \lambda; h) = \mathcal{O}(h^{-\epsilon(|s| + |\varpi|)}) \langle h^\epsilon \lambda \rangle^{k_1 - k}.$$

We will write

$$a = \widetilde{\mathcal{O}}_\epsilon(\langle h^\epsilon \lambda \rangle^{k_1}) \quad \text{if and only if} \quad (41) \text{ holds.}$$

For such a , we define the exact quantization

$$\widetilde{\text{Op}}_h(a)u = \frac{1}{(2\pi h)^d} \int a\left(\frac{x+y}{2}, \xi, h^{-\delta}\xi_1; h\right) e^{\frac{i}{h}\langle x-y, \xi \rangle} u(y) dy d\xi.$$

Then,

Lemma 4.1. *Suppose that $a = \widetilde{\mathcal{O}}_\epsilon(\langle h^\epsilon \lambda \rangle^{k_1})$ and $b = \widetilde{\mathcal{O}}_\epsilon(\langle h^\epsilon \lambda \rangle^{k_2})$. Then,*

$$\widetilde{\text{Op}}_h(a) \circ \widetilde{\text{Op}}_h(b) = \widetilde{\text{Op}}_h(a \sharp b).$$

where

$$a \sharp b = e^{ihA(D)} (a|_{\lambda=h^{-\delta}\xi_1} b|_{\mu=h^{-\delta}\eta_1}) \Big|_{\substack{y=x \\ \xi=\eta}} = \widetilde{\mathcal{O}}_\epsilon(\langle h^\epsilon \lambda \rangle^{k_1+k_2})$$

where

$$A(D) = \frac{1}{2} \sigma((D_x, D_\xi), (D_y, D_\eta)) =: \frac{1}{2} \langle QD, D \rangle.$$

Moreover if $\epsilon + \delta < 1$,

$$a \# b = \sum_{k=0}^{\infty} \frac{i^k h^k}{k!} A(D)^k (a|_{\lambda=h^{-\delta}\xi_1} b|_{\mu=h^{-\delta}\eta_1}) \Big|_{\substack{y=x \\ \eta=\xi}} \pmod{h^\infty \Psi^{-\infty}}.$$

We say that $a(x, \xi, y, \eta, \lambda, \mu) = \widetilde{O}_\epsilon(\langle h^\epsilon \lambda \rangle^{k_1} \langle h^\epsilon \mu \rangle^{k_2})$ if

$$|\partial_x^{\varsigma_1} \partial_y^{\varsigma_2} \partial_\xi^{\varpi_1} \partial_\eta^{\varpi_2} \partial_\lambda^{m_1} \partial_\mu^{m_2} a| \leq C_{\varsigma\varpi m} h^{-\epsilon(|\varsigma|+|\varpi|)} \langle h^\epsilon \lambda \rangle^{k_1-m_1} \langle h^\epsilon \mu \rangle^{k_2-m_2}.$$

The only part of this lemma that is non-standard is the following. The rest follows from applying stationary phase.

Lemma 4.2.

$$e^{ihA(D)} : \widetilde{O}_\epsilon(\langle h^\epsilon \lambda \rangle^{k_1} \langle h^\epsilon \mu \rangle^{k_2}) \rightarrow \widetilde{O}_\epsilon(\langle h^\epsilon \lambda \rangle^{k_1} \langle h^\epsilon \mu \rangle^{k_2}).$$

Proof. We start by considering the case of one dimension. Let $w_1 = (x'_1, \xi'_1, y'_1, \eta'_1)$ and

$$\varphi_1(w_1) = \frac{1}{2}(\langle \xi'_1, y'_1 \rangle - \langle \eta'_1, x'_1 \rangle).$$

Then, with $z = (x_1, \xi_1, y_1, \eta_1)$,

$$c := (e^{ihA(D)} a)(z, \mu) = Ch^{-2} \int e^{-\frac{i}{h}\varphi_1(w)} a(w - z, \lambda - h^{-\delta}\xi_1, \mu - h^{-\delta}\eta_1) dw$$

Then, rescale $(x'_1, y'_1) = (\tilde{x}_1, \tilde{y}_1)h^{-(1-\delta)}$, and $(\xi'_1, \eta'_1) = (\tilde{\xi}_1, \tilde{\eta}_1)h^{-\delta}$. We have that with $\tilde{w} = (\tilde{x}_1, \tilde{\xi}_1, \tilde{y}_1, \tilde{\eta}_1)$,

$$\begin{aligned} c &= C \int e^{-i\varphi_1(\tilde{w})} \left(\chi(\tilde{w}) a(x_1 - h^{1-\delta}\tilde{x}_1, \xi_1 - h^\delta\tilde{\xi}_1, y_1 - h^{1-\delta}\tilde{y}_1, \eta_1 - h^\delta\tilde{\eta}_1, \lambda - \tilde{\xi}_1, \mu - \tilde{\eta}_1) \right. \\ &\quad \left. + (1 - \chi(\tilde{w})) a(x_1 - h^{1-\delta}\tilde{x}_1, \xi_1 - h^\delta\tilde{\xi}_1, y_1 - h^{1-\delta}\tilde{y}_1, \eta_1 - h^\delta\tilde{\eta}_1, \lambda - \tilde{\xi}_1, \mu - \tilde{\eta}_1) \right) d\tilde{w} \\ &=: A + B \end{aligned}$$

where $\chi \in C_c^\infty(\mathbb{R}^4)$ has $\chi \equiv 1$ on $B(0, 1)$ and $\text{supp } \chi \subset B(0, 2)$.

$$|\partial^\varsigma A(z, \lambda, \mu)| \leq C \sup_{|\tilde{w}| \leq 2} |\partial^\varsigma a(x_1 - h^{1-\delta}\tilde{x}_1, \xi_1 - h^\delta\tilde{\xi}_1, y_1 - h^{1-\delta}\tilde{y}_1, \eta_1 - h^\delta\tilde{\eta}_1, \lambda - \tilde{\xi}_1, \mu - \tilde{\eta}_1)|$$

and hence $A = \widetilde{O}_\epsilon(\langle h^\epsilon \lambda \rangle^{k_1} \langle h^\epsilon \mu \rangle^{k_2})$. Letting

$$L := \frac{-\langle \partial\varphi(\tilde{w}), D_{\tilde{w}} \rangle}{|\partial\varphi(\tilde{w})|^2}$$

and integrating by parts sufficiently many times shows also that $B = \widetilde{O}_\epsilon(\langle h^\epsilon \lambda \rangle^{k_1} \langle h^\epsilon \mu \rangle^{k_2})$.

To obtain the general case, we simply observe that

$$e^{ihA(D_x, D_\xi, D_y, D_\eta)} = e^{ihA(D_{x'}, D_{\xi'}, D_{y'}, D_{\eta'})} e^{ihA(D_{x_1}, D_{\xi_1}, D_{y_1}, D_{\eta_1})}$$

and use that

$$e^{ihA(D_{x'}, D_{\xi'}, D_{y'}, D_{\eta'})} : S_\epsilon \rightarrow S_\epsilon.$$

□

Now, rewriting the asymptotic expansion, and assuming that $|\xi_1| \leq C$ so that

$$h^{1-2\epsilon} \leq Ch^{1-\delta-\epsilon} \langle h^\epsilon \lambda \rangle^{-1}$$

we have if $\epsilon + \delta < 1$, taking $p_1 > \frac{1-2\epsilon}{1-\delta-\epsilon}$

$$\begin{aligned} a \sharp b(x, \xi, \lambda; h) &= \sum_{k=0}^{\infty} \frac{i^k h^k}{2^k k!} (\sigma(D_x, D_{\xi_1} + h^{-\delta} D_\lambda, D_{\xi'}, D_y, D_{\eta_1} + h^{-\delta} D_\mu, D_{\eta'}))^{k, ab} \Big|_{\substack{y=x, \eta=\xi \\ \lambda=\mu}} \\ &= ab + \frac{1}{2i} h^{1-\delta} (\partial_\lambda b \partial_{x_1} a - \partial_\lambda a \partial_{x_1} b) + \frac{h}{2i} \{a, b\} \\ &\quad + \sum_{k=2}^{p_1} \frac{i^k h^{k(1-\delta)}}{2^k k!} (\sigma(D_{x_1}, D_\lambda, D_{y_1}, D_\mu))^{k, ab} \Big|_{y=x, \eta=\xi, \lambda=\mu} + \widetilde{O}_\epsilon \left(h^{2-3\epsilon-\delta} \langle h^\epsilon \lambda \rangle^{k_1+k_2-1} \right) \end{aligned}$$

4.1.1. *Ellipticity and Boundedness in the local model.* We now present the analogs of microlocal elliptic estimates and the sharp Gårding inequalities in the second microlocal setting. Suppose that $\epsilon + \delta < 1$ and $a = \widetilde{O}_\epsilon(\langle h^\epsilon \lambda \rangle^{k_1})$. We define the *elliptic set* of a , $\text{ell}(a)$ by $(x, \xi, \lambda) \in \text{ell}(a)$ if there exists a neighborhood, U of (x, ξ, λ) and $c > 0$ so that $|a| > c \langle h^\epsilon \lambda \rangle^{k_1}$ on U .

Lemma 4.3. *Suppose that $p = \widetilde{O}_\epsilon(\langle h^\epsilon \lambda \rangle^{k_1})$, $b = \widetilde{O}_\epsilon(\langle h^\epsilon \lambda \rangle^{k_2})$ and that $\text{supp } b \subset \text{ell}(p)$. Then there exists $a_i = O_\epsilon(\langle h^\epsilon \lambda \rangle^{k_2-k_1})$, $i = 1, 2$ so that*

$$\widetilde{O}_{\text{Ph}}(a_1) \widetilde{O}_{\text{Ph}}(p) = \widetilde{O}_{\text{Ph}}(p) \widetilde{O}_{\text{Ph}}(a_2) + \mathcal{O}_{\Psi^{-\infty}}(h^\infty) = \widetilde{O}_{\text{Ph}}(b) + \mathcal{O}_{\Psi^{-\infty}}(h^\infty).$$

Proof. By elementary analysis, one sees that

$$\partial^s p^{-1} = p^{-1} \sum_{k=1}^{|s|} \sum_{\substack{\varsigma=\varpi^1+\dots+\varpi^k \\ |\varpi^j| \geq 1}} C_{\varpi^1, \dots, \varpi^k} \prod_{j=1}^k (p^{-1} \partial^{\varpi^j} p)$$

(see for example the proof of [Zwo12, Theorem 4.32]). Thus, since $|p| \geq c \langle h^\epsilon \lambda \rangle^{k_1}$ on $\text{supp } b$,

$$q_0 : bp^{-1} = \widetilde{O}_\epsilon(\langle h^\epsilon \lambda \rangle^{k_2-k_1}).$$

So,

$$\widetilde{O}_{\text{Ph}}(q_0) \widetilde{O}_{\text{Ph}}(p) = \widetilde{O}_{\text{Ph}}(b) + h^{1-\delta-\epsilon} \widetilde{O}_{\text{Ph}}(e_1) + \mathcal{O}_{\Psi^{-\infty}}(h^\infty)$$

where $e_1 = \widetilde{O}_\epsilon(\langle h^\epsilon \lambda \rangle^{k_2-1})$ with $\text{supp } e_1 \subset \text{ell}(p)$. Thus, setting $r_1 = h^{1-\delta-\epsilon} e_1$ and letting $q_1 = -r_1 p^{-1} = h^{1-\delta-\epsilon} \widetilde{O}_\epsilon(\langle h^\epsilon \lambda \rangle^{k_2-k_1-1})$. Continuing in this way, we obtain

$$q_n = h^{n(1-\delta-\epsilon)} \widetilde{O}_\epsilon(\langle h^\epsilon \lambda \rangle^{k_2-k_1-n})$$

so that with $a_1 \sim \sum q_i$,

$$\widetilde{O}_{\text{Ph}}(a_1) \widetilde{O}_{\text{Ph}}(p) = \widetilde{O}_{\text{Ph}}(b) + \mathcal{O}_{\Psi^{-\infty}}(h^\infty).$$

A similar argument, yields a_2 . □

Lemma 4.4. *Suppose that $a = \widetilde{O}_\epsilon(\langle h^\epsilon \lambda \rangle^{k_1})$. Then*

$$\|\widetilde{O}_{\text{Ph}}(a)\|_{L^2 \rightarrow L^2} \leq C \langle h^{(\epsilon-\delta)k_1} \rangle.$$

Proof. The proof for $h = 1$ follows from for example [Zwo12, Theorem 4.23]. Suppose that $u(x) \in \mathcal{S}$. The proof follows that in [Zwo12, Theorem 5.1]. We have that

$$\|\widetilde{\text{Op}}(a)\|_{L^2 \rightarrow L^2} \leq C \sup_{|\varsigma| \leq Md} |\partial^\varsigma a|.$$

So, we rescale $\tilde{\xi} = h^{-\frac{(1+\delta)}{2}} \xi$, $\tilde{x} = h^{-\frac{1-\delta}{2}} x$ and $\tilde{u}(\tilde{x}) = h^{\frac{(1-\delta)d}{4}} u(h^{\frac{1-\delta}{2}} \tilde{x})$. Then,

$$\widetilde{\text{Op}}_h(a)u(x) = h^{-\frac{d(1-\delta)}{4}} \widetilde{\text{Op}}(a_h)\tilde{u}(\tilde{x}).$$

where

$$a_h(\tilde{x}, \tilde{\xi}) := a(h^{\frac{1-\delta}{2}} \tilde{x}, h^{\frac{1+\delta}{2}} \tilde{\xi}, h^{\frac{1-\delta}{2}} \tilde{\xi}_1).$$

Therefore,

$$\begin{aligned} \|\widetilde{\text{Op}}_h(a)u\|_{L_x^2} &= \|\widetilde{\text{Op}}(a_h)\tilde{u}(\tilde{x})\|_{L_{\tilde{x}}^2} \leq \|\widetilde{\text{Op}}(a_h)\|_{L^2 \rightarrow L^2} \|\tilde{u}\|_{L_{\tilde{x}}^2} \\ &\leq C \sup_{|\varsigma| \leq Md} |\partial^\varsigma a_h| \|u\|_{L_x^2} \\ &\leq C \sup_{|(\varsigma, \varpi, k)| \leq Md} h^{(|\varsigma| + |\varpi| + k)\frac{1-\delta}{2}} |\partial_x^\varsigma \partial_\xi^\varpi \partial_\lambda^k a| \\ &\leq C \sup_{|(\varsigma, \varpi, k)| \leq Md} h^{(|\varsigma| + |\varpi| + k)\frac{1-\delta}{2}} (1 + \langle h^{\epsilon-\delta} \rangle^{k_1-k}) \end{aligned}$$

□

We now prove an analog of the Sharp Gårding inequality for the second microlocal operators.

Lemma 4.5. *Suppose that $a = \widetilde{\text{O}}_0(\langle \lambda \rangle^0)$ and $a \geq 0$. Then*

$$\langle \widetilde{\text{Op}}_h(a)u, u \rangle \geq -Ch^{1-\delta} \|u\|_{L^2}^2.$$

Proof. We again follow the proof in the classical case. (See for example [Zwo12, Theorem 4.32]).

Fix \tilde{h} sufficiently small and let $\gamma = h^\epsilon/\tilde{h}$. We will show that $q = (a + \gamma)^{-1}$ satisfies

$$(42) \quad \partial_x^\varsigma \partial_\xi^\varpi \partial_\lambda^k q = O(h^{-\epsilon} \tilde{h} (\tilde{h} h)^{-\frac{\epsilon}{2} (|\varsigma| + |\varpi| + k)} \langle \lambda \rangle^{-k}).$$

That is $q \in h^{-\epsilon} \tilde{h} S_{\delta+\epsilon/2, \epsilon/2}^0(\Sigma_0)$. We will then be able to invert $a + \gamma$ when $\epsilon \leq 1 - \delta$.

First, since $a \geq 0$ and $a = \widetilde{\text{O}}_0(\langle \lambda \rangle^0)$, $|\partial_\lambda a| \leq C \langle \lambda \rangle^{-1} a^{1/2}$. (see for example [Zwo12, Lemma 4.31]) Moreover, $|\partial_x a| + |\partial_\xi a| \leq Ca^{1/2}$. Then recall that

$$(43) \quad \partial^\varsigma (a + \gamma)^{-1} = (a + \gamma)^{-1} \sum_{k=1}^{|\varsigma|} \sum_{\substack{\varsigma = \varpi^1 + \dots + \varpi^k \\ |\varpi^j| \geq 1}} C_{\varpi^1, \dots, \varpi^k} \prod_{j=1}^k ((a + \gamma)^{-1} \partial_x^{\varpi^j, 1} \partial_\xi^{\varpi^j, 2} \partial_\lambda^{\varpi^j, 3} a).$$

Now,

$$|\partial_\lambda a| (a + \gamma)^{-1} \leq C \gamma^{-1/2} \langle \lambda \rangle^{-1}$$

and for $|\varpi| = 1$

$$(|\partial_x^\varpi a| + |\partial_\xi^\varpi a|) (a + \gamma)^{-1} \leq C \gamma^{-1/2}.$$

Moreover, for $|(\varsigma, \varpi, k)| \geq 2$,

$$|\partial_x^\varsigma \partial_\xi^\varpi \partial_\lambda^k a| (a + \gamma)^{-1} \leq C \gamma^{-1} \langle \lambda \rangle^{-k}.$$

So,

$$\left| \prod_{j=1}^k (a + \gamma)^{-1} \partial_x^{\varpi_{j,1}} \partial_\xi^{\varpi_{j,2}} \partial_\lambda^{\varpi_{j,3}} a \right| \leq C \prod_{|\varpi| \geq 2} \gamma^{-1} \langle \lambda \rangle^{-\varpi_{j,3}} \prod_{|\varpi|=1} \gamma^{-1/2} \langle \lambda \rangle^{-\varpi_{j,3}} \leq C \langle \lambda \rangle^{-s_3} \gamma^{-|\varsigma|/2}$$

Plugging this into (43) gives (42).

We now choose $\epsilon = 1 - \delta$. So, $a + \gamma \in S_{\delta,0}^0(\Sigma_0) \subset S_{\delta+\epsilon/2,\epsilon/2}^0(\Sigma_0)$. Then, write $a_1(x, \xi, \lambda_1)$ for the function so that

$$\widetilde{\text{Op}}_\delta(a) = \widetilde{\text{Op}}_{\delta+\epsilon/2}(a_1).$$

Write also $q_1 = (a_1 + \gamma)^{-1}$. So we can define $(a_1 + \gamma)\#q_1$. Then, using Taylor's formula and letting $w = (x, \xi)$, $z = (y, \eta)$,

$$\begin{aligned} (a_1 + \gamma)\#q_1 &= e^{ihA(D)}(a_1 + \gamma)|_{\lambda=h^{-\delta-\epsilon/2}\xi_1} q_1|_{\mu=h^{-\delta-\epsilon/2}\xi_1} \Big|_{w=z} \\ &= 1 + \int_0^1 (1-t) e^{ithA(D)} (ihA(D))^2 (a_1(w, h^{-\delta-\epsilon/2}\xi_1) q_1(z, h^{-\delta-\epsilon/2}\eta_1)) \Big|_{w=z} dt \\ &=: 1 + r(z). \end{aligned}$$

Note that we have used that $\{a_1 + \gamma, (a_1 + \gamma)^{-1}\} = 0$. Now, $(ihA(D))^2 (a_1 + \gamma)\#q_1 \in \tilde{h}S_{\delta+\epsilon/2,\epsilon/2}^0(\Sigma_0)$. So,

$$\|\widetilde{\text{Op}}_{\delta+\epsilon/2}(r)\|_{L^2 \rightarrow L^2} \leq C\tilde{h} \leq \frac{1}{2}$$

for \tilde{h} small enough. Thus, $\widetilde{\text{Op}}_{\delta+\epsilon/2}(q)$ is an approximate right (and similarly left) inverse for $\widetilde{\text{Op}}_\delta(a) + \gamma$. This implies that $(\widetilde{\text{Op}}_\delta(a) + \gamma + \gamma_1)^{-1}$ exists for any $\gamma_1 \geq 0$. Therefore,

$$\text{Spec}(\widetilde{\text{Op}}_\delta(a)) \subset [-\gamma, \infty).$$

Thus, by [Zwo12, Theorem C.8]

$$\langle \widetilde{\text{Op}}_\delta(a)u, u \rangle \geq -\gamma \|u\|_{L^2}^2.$$

□

Using the Sharp Gårding inequality, it is not hard to prove that

Lemma 4.6. *Suppose $a = \tilde{O}(\langle \lambda \rangle^0)$. Then,*

$$\langle \widetilde{\text{Op}}_\delta(a)^* \widetilde{\text{Op}}_\delta(a)u, u \rangle \leq (\sup |a| + Ch^{1-\delta}) \|u\|_{L^2}^2.$$

4.2. The global second microlocal calculus. Let $\Sigma \subset T^*M$ be a smooth compact hypersurface. Let V_i denote vector fields tangent to Σ and W_i denote any vector fields. Let $0 \leq \delta < 1$. We define the symbol class $S_\delta^{k_1, k_2}(M; \Sigma)$ by $a \in S_\delta^{k_1, k_2}(M; \Sigma)$ if and only if

$$(44) \quad \begin{cases} \text{near } \Sigma : V_1 \dots V_{l_1} W_1 \dots W_{l_2} a = o(h^{-\delta l_2} \langle h^{-\delta} d(\Sigma, \cdot) \rangle^{k_1}), \\ \text{away from } \Sigma : \partial_x^\varsigma \partial_\xi^\varpi a(x, \xi; h) = o(h^{-\delta k_1} \langle \xi \rangle^{k_2 - |\varpi|}). \end{cases}$$

where $d(\Sigma, \cdot)$ denotes the absolute value of any defining function of Σ that behaves like $\langle \xi \rangle$ near fiber infinity. Then we have the following

Lemma 4.7. *For $0 \leq \delta < 1$, there exists a class of operators, $\Psi_\delta^{k_1, k_2}(M; \Sigma)$, acting on $C^\infty(M)$ and maps*

$$\begin{aligned} \text{Op}_{h, \Sigma} : S_\delta^{k_1, k_2}(T^*M; \Sigma) &\rightarrow \Psi_\delta^{k_1, k_2}(M; \Sigma) \\ \sigma_\Sigma : \Psi_\delta^{k_1, k_2}(M; \Sigma) &\rightarrow S_\delta^{k_1, k_2}(T^*M; \Sigma) \Big/ h^{1-\delta} S_\delta^{k_1-1, k_2-1}(T^*M; \Sigma) \end{aligned}$$

such that

$$\sigma_\Sigma(A \circ B) = \sigma_\Sigma(A) \sigma_\Sigma(B),$$

$$0 \rightarrow h^{1-\delta} \Psi_\delta^{k_1-1, k_2-1}(M; \Sigma) \rightarrow \Psi_\delta^{k_1, k_2}(M; \Sigma) \xrightarrow{\sigma_\Sigma} S_\delta^{k_1, k_2}(T^*M; \Sigma) \Big/ h^{1-\delta} S_\delta^{k_1-1, k_2-1}(T^*M; \Sigma) \rightarrow 0$$

is a short exact sequence, and

$$\sigma_\Sigma \circ \text{Op}_{h, \Sigma} : S_\delta^{k_1, k_2}(T^*M; \Sigma) \rightarrow S_\delta^{k_1, k_2}(T^*X; \Sigma) \Big/ h^{1-\delta} S_\delta^{k_1-1, k_2-1}(T^*M; \Sigma)$$

is the natural projection map.

As in, [SZ99] near Σ it is possible to reduce all computations to the case where $\Sigma = \Sigma_0 := \{\xi_1 = 0\}$. We then have analogs of all the properties from the model case for the global calculus. We sometimes suppress M and T^*M in our notation, writing only $S_\delta^{k_1, k_2}(\Sigma)$ and $\Psi_\delta^{k_1, k_2}(\Sigma)$. We also sometimes suppress the Σ in $\text{Op}_{h, \Sigma}$ to simplify notation.

5. THE BILLIARD BALL FLOW AND MAP

Recall that $\Omega \Subset \mathbb{R}^d$ is an open set with smooth boundary $\partial\Omega$. We need notation for the billiard ball flow and billiard ball map. Write ν for the outward pointing unit normal to $\partial\Omega$. Then

$$S^*\mathbb{R}^d|_{\partial\Omega} = \partial\Omega_+ \sqcup \partial\Omega_- \sqcup \partial\Omega_0$$

where $(x, \xi) \in \partial\Omega_+$ if ξ is pointing out of Ω (i.e. $\nu(\xi) > 0$), $(x, \xi) \in \partial\Omega_-$ if it points inward (i.e. $\nu(\xi) < 0$), and $(x, \xi) \in \partial\Omega_0$ if $(x, \xi) \in S^*\partial\Omega$. The points $(x, \xi) \in \partial\Omega_0$ are called *glancing* points. Let $B^*\partial\Omega$ be the unit coball bundle of $\partial\Omega$ and denote by $\pi_\pm : \partial\Omega_\pm \rightarrow B^*\partial\Omega$ and $\pi : S^*\mathbb{R}^d|_{\partial\Omega} \rightarrow \overline{B^*\partial\Omega}$ the canonical projections onto $\overline{B^*\partial\Omega}$. Then the maps π_\pm are invertible. Finally, write

$$t_0(x, \xi) = \inf\{t > 0 : \exp_t(x, \xi) \in T^*\mathbb{R}^d|_{\partial\Omega}\}$$

where $\exp_t(x, \xi)$ denotes the lift of the geodesic flow to the cotangent bundle. That is, t_0 is the first positive time at which the geodesic starting at (x, ξ) intersects $\partial\Omega$.

We define the broken geodesic flow as in [DZ13, Appendix A]. Without loss of generality, we assume $t_0 > 0$. Fix $(x, \xi) \in S^*\mathbb{R}^d$ and denote $t_0 = t_0(x, \xi)$. If $\exp_{t_0}(x, \xi) \in \partial\Omega_0$, then the billiard flow cannot be continued past t_0 . Otherwise there are two cases: $\exp_{t_0}(x, \xi) \in \partial\Omega_+$ or $\exp_{t_0}(x, \xi) \in \partial\Omega_-$. We let

$$(x_0, \xi_0) = \begin{cases} \pi_-^{-1}(\pi_+(\exp_{t_0}(x, \xi))) \in \partial\Omega_-, & \text{if } \exp_{t_0}(x, \xi) \in \partial\Omega_+ \\ \pi_+^{-1}(\pi_-(\exp_{t_0}(x, \xi))) \in \partial\Omega_+, & \text{if } \exp_{t_0}(x, \xi) \in \partial\Omega_- \end{cases}.$$

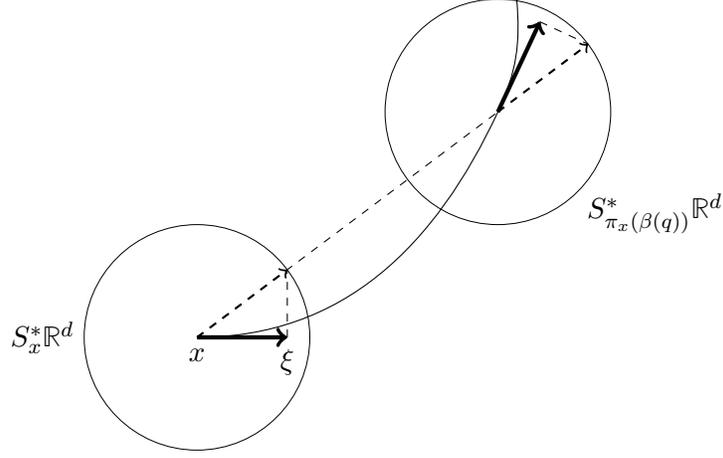


FIGURE 5.1. The figure shows how the billiard ball map is constructed. Let $q = (x, \xi) \in B^* \partial \Omega$. The solid black arrow on the left denotes the covector $\xi \in B_x^* \partial \Omega$ and that on the right $\xi(\beta(q)) \in B_{\pi_x(\beta(q))}^* \partial \Omega$. The center of the left circle is x and that of the right is $\pi_x(\beta(q))$.

We then define $\varphi_t(x, \xi)$, the *broken geodesic flow*, inductively by putting

$$\varphi_t(x, \xi) = \begin{cases} \exp_t(x, \xi) & 0 \leq t < t_0 \\ \varphi_{t-t_0}(x_0, \xi_0) & t \geq t_0 \end{cases}.$$

We introduce notation from [Saf87] for the billiard flow. Let K be the set of ternary fractions of the form $0.k_1 k_2, \dots$, where $k_j = 0$ or 1 and S denote the left shift operator

$$S(0.k_1 k_2 \dots) = 0.k_2 k_3 \dots$$

For $k \in K$, we define the billiard flow of type k , $G_k^t : S^* \mathbb{R}^d \rightarrow S^* \mathbb{R}^d$ as follows. For $0 \leq t \leq t_0$,

$$(45) \quad G_k^t(x, \xi) = \begin{cases} \varphi_t(x, \xi) & \text{if } k_1 = 0 \\ \exp_t(x, \xi) & \text{if } k_1 = 1 \end{cases}$$

Then, we define G_k^t inductively for $t > t_0$ by

$$(46) \quad G_k^t(x, \xi) = G_{S_k}^{t-t_0}(G_k^{t_0}(x, \xi)).$$

We call G_k^t the billiard flow of type k . By [Saf87, Proposition 2.1], G_k^t is measure preserving.

Remark 12.

- In [Saf87], geodesics could be of multiple types when total internal reflection occurred. However, in our situation, the metrics on either side of the boundary match, so there is no total internal reflection and geodesics are uniquely identified by their starting points and $k \in K$.
- In general, there exist situations where G_k^t intersects the boundary infinitely many times in finite time. However, since we work in convex domains, we need not consider this

situation. For a proof of this fact see the proof of Lemma 5.1. Note of course that the number of possible reflection in a given time T grows as one approaches glancing points.

Now, for $k \in K$ and $T > 0$, we define the set $\mathcal{O}_{T,k} \subset S^*\mathbb{R}^d$ to be the complement of the set of (x, ξ) such that one can define the flow G_k^t for $t \in [0, T]$. That is, $\mathcal{O}_{T,k}$ is the set for which the billiard flow of type k is glancing in time $0 \leq t \leq T$. Last, define the set

$$(47) \quad \mathcal{O}_T = \bigcup_{k \in K} \mathcal{O}_{k,T}.$$

The billiard ball map reduces the dynamics of G_0^k to the boundary. We define the billiard ball map as in [GU81]. Let $(x, \xi') \in B^*\partial\Omega$ and $(x, \xi) = \pi_-^{-1}(x, \xi') \in \partial\Omega_-$ be the unique inward pointing covector with $\pi(x, \xi) = (x, \xi')$. Then, the billiard ball map $\beta : B^*\partial\Omega \rightarrow \overline{B^*\partial\Omega}$ maps (x, ξ') to the projection onto $T^*\partial\Omega$ of the first intersection of the billiard flow with the boundary. That is,

$$(48) \quad \beta : (x, \xi') \mapsto \pi(\exp_{t_0(x, \xi)}(x, \xi)).$$

Remark 13.

- Just like the billiard flow, the billiard ball map is not defined for $(x, \xi') \in \pi(\partial\Omega_0) = S^*\partial\Omega$. However, since we consider convex domains, $\beta : B^*\Omega \rightarrow B^*\Omega$ and β^n is well defined on $B^*\partial\Omega$.
- Figure 5.1 shows the process by which the billiard ball map is defined.

The billiard ball map is symplectic. This follows from the fact that the Euclidean distance function $|x - x'|$ is locally a generating function for β ; that is, the graph of β in a neighborhood of $(x_0, \xi_0, y_0, \eta_0)$ is given by

$$(49) \quad \{(x, -d_x|x - y|, y, d_y|x - y|) : (x, y) \in \partial\Omega \times \partial\Omega\}.$$

We denote the graph of β by C_b . For strictly convex Ω , C_b is given globally by (49).

We also write

$$\beta_E := (x(\beta(x, \xi/\sqrt{E})), \sqrt{E}\xi(\beta(x, \xi/\sqrt{E}))) : B_E^*\partial\Omega \rightarrow B_E^*\partial\Omega$$

where $B_E^*\partial\Omega$ is the coball bundle of radius \sqrt{E} .

5.1. Dynamics in Strictly Convex Domains. We are interested in the behavior of the billiard ball map, $\beta(q)$ when $|\xi'(q)|_g$ is close to 1. Our interest in this region comes from a desire to understand how the reflection coefficients R from (18) behaves when a wave travels nearly tangent to a strictly convex boundary.

Fix $q = (x_0, \xi_0) \in B^*\partial\Omega$ so that $\partial\Omega$ is strictly convex near x_0 and $|\xi_0|_g^2$ is sufficiently close to 1. Let $\gamma : [0, \delta) \rightarrow \partial\Omega$ be the unique length minimizing geodesic connecting x_0 and $\pi_x(\beta(q))$. The existence and uniqueness of such a geodesic is guaranteed for $|\xi_0|_g^2$ close enough to 1 by the strict convexity of $\partial\Omega$. Indeed, this follows from the fact that $l(q, \beta(q)) \rightarrow 0$ as $|\xi_0|_g^2 \rightarrow 1$ and the fact that the exponential map is a diffeomorphism for small times.

Let $s \in [0, \delta)$ have $\gamma(s) = \pi_x(\beta(q))$. We first examine how the normal component to $\partial\Omega$ changes under the billiard ball map. Let Δ_{ξ_d} denote the change in the normal component under β . Then

$$\begin{aligned}\Delta_{\xi_d} &= \frac{((\gamma(s) - \gamma(0)) \cdot \nu(0) - (\gamma(0) - \gamma(s)) \cdot \nu(s))}{|\gamma(s) - \gamma(0)|} \\ &= \frac{(\gamma(s) - \gamma(0)) \cdot (\nu(0) + \nu(s))}{|\gamma(s) - \gamma(0)|}.\end{aligned}$$

Here $|\cdot|$ is the euclidean norm in \mathbb{R}^d and ν is the inward pointing unit normal.

First, note that

$$\begin{aligned}\gamma''(s) &= \kappa(s)\nu(s), & \nu'(s) \cdot \gamma'(s) &= -\kappa(s), \\ \gamma'(s) \cdot \nu(s) &= 0, & \|\gamma'(s)\| &= \|\nu(s)\| = 1\end{aligned}$$

where $\kappa(s)$ is the curvature of the geodesic γ as a curve in \mathbb{R}^d . Then, expanding in Taylor series gives

$$\begin{aligned}\Delta_{\xi_d} [s + O(s^2)] &= [\gamma'(0)s + \gamma''(0)\frac{s^2}{2} + \gamma^{(3)}(0)\frac{s^3}{6} + O(s^4)] \cdot [2\nu(0) + \nu'(0)s + \nu''(0)\frac{s^2}{2} + O(s^3)] \\ \Delta_{\xi_d} [1 + O(s)] &= 2\gamma'(0) \cdot \nu(0) + (\gamma' \cdot \nu)'(0)s + \left(2\gamma^{(3)}(0) \cdot \nu(0) + 3(\gamma' \cdot \nu)''(0)\right)\frac{s^2}{6} + O(s^3) \\ \Delta_{\xi_d} &= [2(\kappa'(0)\nu(0) - \kappa(0)\nu'(0)) \cdot \nu(0) - 3\kappa'(0)]\frac{s^2}{6} + O(s^3) \\ (50) \quad \Delta_{\xi_d} &= (2\kappa'(0) - 3\kappa(0))\frac{s^2}{6} + O(s^3) = -\kappa'(0)\frac{s^2}{6} + O(s^3).\end{aligned}$$

Next observe that

$$\sqrt{1 - |\xi'(q)|_g^2} = \frac{\gamma(s) - \gamma(0)}{|\gamma(s) - \gamma(0)|} \cdot \nu(0) = \frac{\kappa(0)}{2}s + O(s^2)$$

Now, using $\kappa(0) > c > 0$ for Ω strictly convex this implies

$$s = \frac{2\sqrt{1 - |\xi'(q)|_g^2}}{\kappa(0)} + O((1 - |\xi'|_g^2))$$

and therefore,

$$l(q, \beta(q)) = |\gamma(s) - \gamma(0)| = s + O(s^2) = \frac{2}{\kappa(0)}\sqrt{1 - |\xi'|_g^2} + O(1 - |\xi'|_g^2).$$

Summarizing, we have

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^d$ be strictly convex. Then, for $q \in B^*\partial\Omega$ sufficiently close to $S^*\partial\Omega$*

$$\begin{aligned}\sqrt{1 - |\xi'(\beta(q))|_g^2} &= \sqrt{1 - |\xi'(q)|_g^2} + O(1 - |\xi'(q)|_g^2) \\ l(q, \beta(q)) &= \frac{2}{\kappa(0)}\sqrt{1 - |\xi'|_g^2} + O(1 - |\xi'|_g^2).\end{aligned}$$

This implies that set of $O(h^\epsilon)$ near glancing points is stable under the billiard ball map. This also follows from the equivalence of glancing hypersurfaces [Mel76].

6. BOUNDARY LAYER OPERATORS AND POTENTIALS IN THE NON-NOMOGENEOUS
FRIEDLANDER MODEL

Our goal is to give microlocal descriptions of the boundary layer operators and potentials near a glancing point. We start by considering the non-homogeneous Friedlander model problem

$$(51) \quad \begin{aligned} ((hD_{x_d})^2 - \mu x_d + hD_{y_1})u &= 0, & u(0, y) &= f(y) \\ u|_{x_d > 0} &\text{ outgoing,} & \|u\|_{L^2((-\infty, 0] \times \mathbb{R}^{d-1})} &< \infty. \end{aligned}$$

Then, let $\mathcal{F}_h(u)$ denote the semiclassical Fourier transform in y ,

$$\mathcal{F}_h u(x_d, \eta) := \frac{1}{(2\pi h)^{d-1}} \int u(x_d, y) e^{-\frac{i}{h}\langle y, \eta \rangle} dy.$$

Rescaling $w = h^{-2/3}\mu^{1/3}x_d$ gives that

$$(h^{2/3}\mu^{-1/3}(D_w^2 - w + h^{-2/3}\mu^{-2/3}\eta_1)\mathcal{F}_h(u)(w, \eta) = 0, \quad \mathcal{F}_h(u)(0, \eta) = \mathcal{F}_h(f)(\eta).$$

Hence, using (51)

$$\mathcal{F}_h(u)(x_d, \eta) = \begin{cases} \frac{Ai(-h^{-2/3}\mu^{1/3}x_d + h^{-2/3}\mu^{-2/3}\eta_1)}{Ai(h^{-2/3}\mu^{-2/3}\eta_1)} \mathcal{F}_h(f)(\eta) & x_d < 0 \\ \frac{A_-(-h^{-2/3}\mu^{1/3}x_d + h^{-2/3}\mu^{-2/3}\eta_1)}{A_-(h^{-2/3}\mu^{-2/3}\eta_1)} \mathcal{F}_h(f)(\eta) & x_d > 0 \end{cases}.$$

So, the Dirichlet to Neumann map for the interior problem ($x_d < 0$) is given by

$$\mathcal{F}_h(N_1 f)(\eta) = -h^{-2/3}\mu^{1/3} \frac{Ai'(h^{-2/3}\mu^{-2/3}\eta_1)}{Ai(h^{-2/3}\mu^{-2/3}\eta_1)} \mathcal{F}_h(f)(\eta)$$

and that for the exterior problem ($x_d > 0$) by

$$\mathcal{F}_h(N_2 f)(\eta) = h^{-2/3}\mu^{1/3} \frac{A'_-(h^{-2/3}\mu^{-2/3}\eta_1)}{A_-(h^{-2/3}\mu^{-2/3}\eta_1)} \mathcal{F}_h(f)(\eta).$$

Remark 14. Since the goal of this section is only to present a simple model where the calculations are exact, we ignore the poles in N_1 . It is possible to find the single and double layer operators and potentials without using the Dirichlet to Neumann map N_1 (see [Gal14, Section 4.5] see also [Tay11, Section 7.11] for a general introduction to layer potential methods), but it simplifies the presentation to do so here.

So, letting $\Theta_h(\eta) = h^{-2/3}\mu^{-2/3}\eta_1$, the single layer operator is given by

$$\begin{aligned} \mathcal{F}_h(Gf)(\eta) &= \mathcal{F}_h((N_1 + N_2)^{-1}f)(\eta) = h^{2/3}\mu^{-1/3} \frac{Ai(\Theta_h)A_-(\Theta_h)}{A'_-(\Theta_h)Ai(\Theta_h) - Ai'(\Theta_h)A_-(\Theta_h)} \mathcal{F}_h(f)(\eta) \\ &= h^{2/3}\mu^{-1/3} 2\pi e^{\pi i/6} Ai(\Theta_h)A_-(\Theta_h) \mathcal{F}_h(f)(\eta) \end{aligned}$$

and the double layer operator is given by

$$\begin{aligned} \mathcal{F}_h(Nf)(\eta) &= \frac{1}{2} \mathcal{F}_h(f)(\eta) - \mathcal{F}_h(GN_2 f)(\eta) \\ &= \left(\frac{1}{2} - 2\pi e^{\pi i/6} Ai(\Theta_h)A'_-(\Theta_h) \right) \mathcal{F}_h(f)(\eta) \end{aligned}$$

Therefore, since

$$\gamma^+ \mathcal{S} \ell = G \quad \gamma^+ \mathcal{D} \ell = -\frac{1}{2} I + N$$

and both solve the Friedlander model equation away from $x_d = 0$,

$$\begin{aligned} \mathcal{F}_h(\mathcal{S} \ell f) &= h^{2/3} \mu^{-1/3} 2\pi e^{\pi i/6} \text{Ai}(-h^{-2/3} \mu^{1/3} x_d + \Theta_h) A_-(\Theta_h) \mathcal{F}_h(f)(\eta) \\ \mathcal{F}_h(\mathcal{D} \ell f) &= -2\pi e^{\pi i/6} \text{Ai}(-h^{-2/3} \mu^{1/3} x_d + \Theta_h) A'_-(\Theta_h) \mathcal{F}_h(f)(\eta) \end{aligned}$$

Now, consider the kernel of $\mathcal{S} \ell^* \mathcal{S} \ell$,

$$\begin{aligned} \mathcal{S} \ell^* \mathcal{S} \ell(x', y') &= \frac{4\pi^2 \mu^{-2/3} h^{4/3}}{(2\pi h)^{2d-2}} \iint_{-\infty}^0 \overline{\text{Ai}(-h^{-2/3} \mu^{1/3} w_1 + \Theta_h(\eta)) A_-(\Theta_h(\eta))} \\ &\quad \text{Ai}(-h^{-2/3} \mu^{1/3} w_1 + \Theta_h(\xi)) A_-(\Theta_h(\xi)) e^{\frac{i}{h}(\langle x' - w', \eta \rangle + \langle w' - y', \xi \rangle)} dw_1 d\xi dw' d\eta \\ &= \frac{4\pi^2 \mu^{-1} h^2}{(2\pi h)^{d-1}} \iint_{\Theta_h(\xi)}^{\infty} | \text{Ai}(s) |^2 | A_-(\Theta_h(\xi)) |^2 e^{\frac{i}{h} \langle x' - y', \xi \rangle} d\xi \\ &= \frac{h^2}{\mu} \frac{1}{(2\pi h)^{-d+1}} \int \Psi_{\mathcal{S} \ell}(\Theta_h(\xi)) e^{\frac{i}{h} \langle x' - y', \xi \rangle} ds d\xi \end{aligned}$$

Similarly,

$$\begin{aligned} \mathcal{S} \ell^* \mathcal{D} \ell(x', y') &= -\frac{h^{4/3}}{\mu^{2/3}} \frac{1}{(2\pi h)^{d-1}} \int \overline{\Psi_{\mathcal{D} \ell \mathcal{S} \ell}(\Theta_h(\xi))} e^{\frac{i}{h} \langle x' - y', \xi \rangle} d\xi \\ \mathcal{D} \ell^* \mathcal{S} \ell(x', y') &= -\frac{h^{4/3}}{\mu^{2/3}} \frac{1}{(2\pi h)^{d-1}} \int \Psi_{\mathcal{D} \ell \mathcal{S} \ell}(\Theta_h(\xi)) e^{\frac{i}{h} \langle x' - y', \xi \rangle} d\xi \\ \mathcal{D} \ell^* \mathcal{D} \ell(x', y') &= \frac{h^{2/3}}{\mu^{1/3}} \frac{1}{(2\pi h)^{d-1}} \int \Psi_{\mathcal{D} \ell}(\Theta_h(\xi)) e^{\frac{i}{h} \langle x' - y', \xi \rangle} d\xi \end{aligned}$$

where

$$(52) \quad \begin{aligned} \Psi_{\mathcal{S} \ell}(x) &:= 4\pi^2 \int_x^{\infty} | \text{Ai}(s) |^2 | A_-(x) |^2 ds &= 4\pi^2 | A_-(x) |^2 [(\text{Ai}'(x))^2 - x (\text{Ai}(x))^2] \\ \Psi_{\mathcal{D} \ell \mathcal{S} \ell}(x) &:= 4\pi^2 \int_x^{\infty} | \text{Ai}(s) |^2 A_-(x) \overline{A'_-(x)} ds &= 4\pi^2 A_-(x) \overline{A'_-(x)} [(\text{Ai}'(x))^2 - x (\text{Ai}(x))^2] \\ \Psi_{\mathcal{D} \ell}(x) &:= 4\pi^2 \int_x^{\infty} | \text{Ai}(s) |^2 | A'_-(x) |^2 ds &= 4\pi^2 | A'_-(x) |^2 [(\text{Ai}'(x))^2 - x (\text{Ai}(x))^2] \end{aligned}$$

since

$$\int_x^{\infty} (\text{Ai}(s))^2 ds = (\text{Ai}'(x))^2 - x (\text{Ai}(x))^2.$$

Using the Wronskian we have that $\Psi_{\mathcal{S} \ell}(\zeta_j) = 1$ where ζ_j is a zero of the Airy function, i.e. $\text{Ai}(\zeta_j) = 0$. Moreover, using asymptotics for the Airy function, as $x \rightarrow -\infty$,

$$\Psi_{\mathcal{S} \ell}(x) \sim 1, \quad \Psi_{\mathcal{D} \ell}(x) \sim -x, \quad \Psi_{\mathcal{D} \ell \mathcal{S} \ell} \sim i(-x)^{1/2}.$$

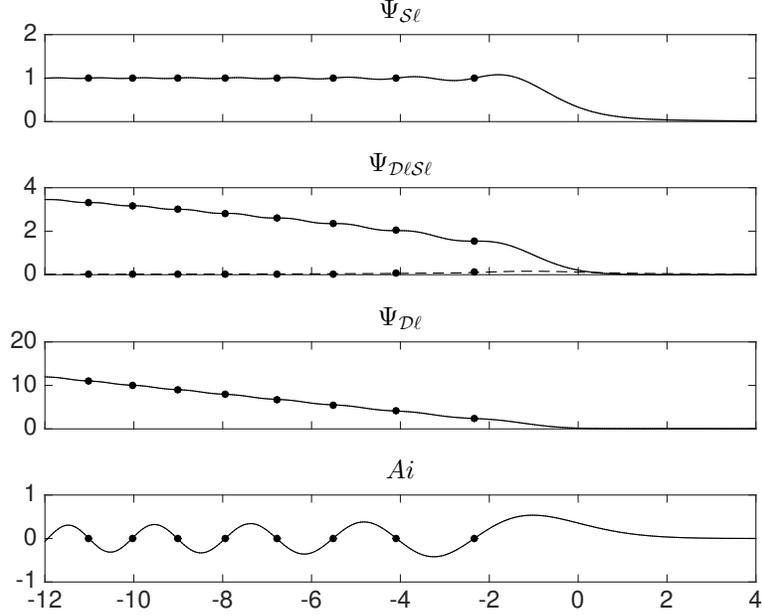


FIGURE 6.1. We plot the symbols of $\mathcal{S}l^*\mathcal{S}l$, $\mathcal{D}l^*\mathcal{S}l$ and $\mathcal{D}l^*\mathcal{D}l$. From top to bottom, the graphs show $\Psi_{\mathcal{S}l}$, $\Psi_{\mathcal{D}l\mathcal{S}l}$, $\Psi_{\mathcal{D}l}$. The bottom graph shows Ai for reference. In the graph of $\Psi_{\mathcal{D}l\mathcal{S}l}$, the imaginary part is shown in the solid line, and the real part in the dashed line. The black dots in each graph show $(\zeta_j, f(\zeta_j))$ where ζ_j are the zeros of $Ai(s)$ and f is one of $\Psi_{\mathcal{S}l}$, $\Psi_{\mathcal{D}l\mathcal{S}l}$, $\Psi_{\mathcal{D}l}$ or Ai as described at the top of each graph.

7. ANALYSIS OF THE BOUNDARY LAYER OPERATORS AND POTENTIALS NEAR GLANCING

Our next task is to show that analogs of all of the formulas for the boundary layer operators and potentials from Section 6 hold in the general case.

7.1. Preliminaries for the General Case. In order to make an analysis similar to that for the model case, we use the microlocal models for G , N , $\mathcal{S}l$, and $\mathcal{D}l$ developed in [Gal14, Section 4.5] We recall the results here. The idea is to write a parametrix for the solution to the problem

$$(-h^2\Delta - z^2)u = L^*\delta_{\partial\Omega} \otimes g_1 + \delta_{\partial\Omega} \otimes g_2$$

where f_i are microlocalized near glancing and $\delta_{\partial\Omega}$ denotes the surface measure on $\partial\Omega$. The parametrix for the problem will be a sum of oscillatory integrals of the form

$$(53) \quad \begin{aligned} H_1 F &= (2\pi h)^{-d+1} \int (f_0 Ai(h^{-2/3}\rho) + ih^{1/3} f_1 Ai'(h^{-2/3}\rho)) A_-(h^{-2/3}\Theta) e^{\frac{i}{h}\theta} \mathcal{F}_h(F)(\xi') d\xi' \\ H_2 F &= (2\pi h)^{-d+1} \int (f_0 Ai(h^{-2/3}\rho) + ih^{1/3} f_1 Ai'(h^{-2/3}\rho)) A'_-(h^{-2/3}\Theta) e^{\frac{i}{h}\theta} \mathcal{F}_h(F)(\xi') d\xi'. \end{aligned}$$

where f_i solve certain transport equations and ρ, θ certain eikonal equations. The boundary values of f_0 and f_1 are determined by the limiting behavior of $\mathcal{D}lg_1$ and $\mathcal{S}lg_0$ at $\partial\Omega$.

Let $z = 1 + i\mu$ with $|\mu| \leq Mh \log h^{-1}$. Then let $\epsilon(h) := \max(h, |\mu|)$. Let $(x_0, \xi_0) \in S^*\partial\Omega$ and suppose that in coordinates (x', x_d) near x_0 , with $\partial\Omega = \{x_d = 0\}$ and $x_d > 0$ in Ω ,

$$-h^2\Delta = \sum_{ij} a^{ij} h D_{x_i} h D_{x_j} + h \left(\sum_i b^i h D_{x_i} + c \right).$$

Then there exist

$$\rho(x, \xi'; h) = \rho_0 + \sum_j \rho_j \epsilon(h)^j, \quad \theta(x, \xi'; h) = \theta_0 + \sum_j \theta_j \epsilon(h)^j$$

solving the eikonal equations

$$\begin{cases} z^2 + O(h^\infty) = \langle ad\theta, d\theta \rangle - \rho \langle ad\rho, d\rho \rangle \\ O(h^\infty) = 2 \langle d\theta, d\rho \rangle \end{cases}$$

on $\rho_0 \leq 0$ and in Taylor series at $\rho_0 = 0, x_d = 0$. Here, ρ_0, θ_0 are real valued solving

$$\begin{cases} 1 = \langle ad\theta_0, d\theta_0 \rangle - \rho_0 \langle ad\rho_0, d\rho_0 \rangle \\ 0 = 2 \langle d\theta_0, d\rho_0 \rangle \end{cases}$$

on $\rho_0 \leq 0$ and in Taylor series at $\rho_0 = 0, x_d = 0$. We need a few additional properties of ρ and θ . In particular,

$$(54) \quad \rho_0|_{\partial\Omega} = \xi_1, \quad \partial_{x_d}\rho_0|_{\partial\Omega} > 0, \quad \partial_{x'\xi'}^2\theta_0|_{\partial\Omega} \neq 0$$

and $\theta_{0b} := \theta_0|_{\partial\Omega}$ has that

$$(55) \quad \kappa : (\partial_{\xi'}\theta_{0b}(x', \xi'), \xi') \mapsto (x', \partial_{x'}\theta_{0b}(x', \xi'))$$

is a symplectomorphism reducing the billiard ball map for the Friedlander model case to that for Ω . We also write $\theta_b = \theta|_{\partial\Omega}$. Next, let

$$\Theta := \rho|_{\partial\Omega} = \xi_1 + i\epsilon(h), \quad \Theta_0 := \rho_0|_{\partial\Omega} = \xi_1.$$

Finally, there exist

$$f_i \sim \sum_{j=0}^{\infty} f_{i,j} h^j, \quad i = 0, 1$$

with $f_{0b} := f_0|_{\partial\Omega}$ having $|f_{0b}| > c > 0$ and $g_1|_{\partial\Omega} = 0$ solving

$$(56) \quad \begin{cases} 2 \langle ad\theta_0, df_{0,n} \rangle + 2\rho_0 \langle ad\rho_0, df_{1,n} \rangle + \langle b, df_{0,n} \rangle \\ \quad + \langle ad\rho_0, d\rho_0 \rangle f_{1,n} - P_2\theta_0 f_{1,n} - \rho_0(P_2\rho_0) f_{1,n} = F_{1,n}(\theta, \rho, f_{i,m < n}, \mu) \\ \\ 2 \langle ad\rho_0, df_{0,n} \rangle - 2 \langle ad\theta_0, df_{1,n} - \langle b, df_{1,n} \rangle \rangle \\ \quad - (P_2\rho_0) f_{0,n} + (P_2\theta_0) f_{1,n} = F_{2,n}(\theta, \rho, f_{i,m < n}, \mu). \end{cases}$$

on $\rho_0 \leq 0$ and in Taylor series at $\rho_0 = 0, x_d = 0$ so that for H_i as in (53) $(-h^2\Delta - z^2)H_i F = \mathcal{O}_{\Psi^{-\infty}}(h^\infty)F$ whenever F is supported h^ϵ close to $\xi_1 = 0$. If $|\mu| \leq Ch$, then this also holds when F is supported δ close to $\xi_1 = 0$ for δ small enough.

7.1.1. *Identification of $\partial_{x_d}\rho_0|_{x_d=0}$ and $|\partial_{y'}\theta_{0b}|_g^2$.* It will be useful to have the value of $\partial_{x_d}\rho_0|_{x_d=0}$ and $|\partial_{y'}\theta_{0b}|_g^2$. To obtain these, we simply write the eikonal equations in normal geodesic coordinates. Recall that in normal geodesic coordinates (y', x_d) with $x_d > 0$ in Ω ,

$$-h^2\Delta = (hD_{x_d})^2 + R(y', hD_{y'}) + 2x_dQ(x_d, y', hD_{y'}) + hF(x_d, y')hD_{x_d}$$

where

$$R(y', D_{y'}) = -\Delta_{\partial\Omega} = \bar{g}^{-1/2} \sum_{ij} D_{y_i} \bar{g}^{1/2} g^{ij} D_{y_j}, \quad \bar{g} = (\det(g^{ij}))^{-1/2}$$

$$Q(0, y', D_{y'}) = \sum_{ij} D_{y_j} \bar{g}^{1/2} a_{ij} D_{y_i}$$

where $Q(y', \xi') = \sum_{ij} a_{ij}(y') \xi_i \xi_j$ is the second fundamental form of $\partial\Omega$ lifted to $T^*\partial\Omega$, $g^{ij} = g^{ij}(y')$ is the metric on $T^*\partial\Omega$, and $R(y', \xi') = \sum_{ij} g^{ij} \xi_i \xi_j$ is the symbol of $-h^2\Delta_{\partial\Omega}$.

Using the eikonal equations for ρ_0 and θ_0 in these coordinates,

$$\begin{cases} 1 = (\partial_{x_d}\theta_0)^2 + R(y', \partial_{y'}\theta_0) + 2x_d(Q(y', \partial_{y'}\theta_0) + O(x_d)) \\ \quad - \rho_0 \left[(\partial_{x_d}\rho_0)^2 + R(y', \partial_{y'}\rho_0) + 2x_d(Q(y', \partial_{y'}\rho_0) + O(x_d)) \right] \\ 0 = 2(\partial_{x_d}\theta_0 \partial_{x_d}\rho_0 + g^{ij} \partial_{y_i}\theta_0 \partial_{y_j}\rho_0 + 2x_d(a_{ij} \partial_{y_i}\theta_0 \partial_{y_j}\rho_0 + O(x_d))). \end{cases}$$

Now, we know that $\rho_0|_{x_d=0} = \xi_1$ and $\partial_{x_d}\rho_0|_{x_d=0} > 0$. So, evaluation at $x_d = 0$ shows

$$\begin{aligned} 1 &= (\partial_{x_d}\theta_0)^2 + R(y', \partial_{y'}\theta_0) - \xi_1(\partial_{x_d}\rho_0)^2 = R(y', \partial_{y'}\theta_0) - \xi_1(\partial_{x_d}\rho_0)^2 \\ 0 &= \partial_{x_d}\theta_0 \end{aligned}$$

Moreover, differentiating the first equation in x_d and the second in y' and evaluating at $x_d = 0$ shows

$$\begin{aligned} 0 &= 2g^{ij} \partial_{x_d y_j}^2 \theta_0 \partial_{y_i} \theta_0 + 2Q(y', \partial_{y'}\theta_0) - (\partial_{x_d}\rho_0)^3 - 2\xi_1 \partial_{x_d}^2 \rho_0 \partial_{x_d}\rho_0 \\ 0 &= 2(\partial_{y' x_d}^2 \theta_0 \partial_{x_d}\rho_0) \end{aligned}$$

Hence,

$$\begin{aligned} (\partial_{x_d}\rho_0)^3|_{x_d=0} &= 2Q(y', \partial_{y'}\theta_0) - 2(\xi_1 \partial_{x_d}^2 \rho_0 \partial_{x_d}\rho_0)|_{x_d=0} = 2Q(y; \partial_{y'}\theta_{0b}) + O(\xi_1) \\ R(y', \partial_{y'}\theta_{0b}) &= |\partial_{y'}\theta_{0b}|_g^2 = 1 + \xi_1(\partial_{x_d}\rho_0)^2|_{x_d=0} \end{aligned}$$

The implicit function theorem then implies that with $\xi' = \partial_{y'}\theta_{0b}$,

$$(57) \quad \xi_1 = \frac{|\xi'|_g^2 - 1}{(2Q(y, \xi'))^{2/3}} + O((|\xi'|_g^2 - 1)^2), \quad \partial_{x_d}\rho = 2Q(y, \xi') + O(|\xi'|_g^2 - 1).$$

Now, in coordinates $(x, \xi) = \kappa^{-1}(y, \eta)$ where κ is as in (55), we have

$$\beta(x, \xi) = (x_1 - 2\sqrt{-\xi_1}, x', \xi)$$

since κ reduces the Friedlander model to the billiard ball map for Ω . Let φ_i be a partition of unity on $1 - \epsilon \leq |\xi'|_g \leq 1 + \epsilon$ for some $\epsilon > 0$ small enough so that on $\text{supp } \varphi_i$ κ_i^{-1} , with κ_i given by (55), is well defined. Let

$$(58) \quad \Xi := \sum_i \varphi_i \xi_1(\kappa_i^{-1}(x, \xi)).$$

Then we have the following lemma given the existence of an *approximate interpolating Hamiltonian* for the billiard ball map. In particular, the lemma follows from the equivalence of glancing hypersurfaces [Mel76] (see [KP90, Proposition 3.1] for a proof, see also [MM82])

Lemma 7.1. *Let Ξ be as in (58). Then at $S^*\partial\Omega$, $\Xi = 0$, $|\mathrm{d}\Xi| > 0$ and $\Xi < 0$ in $B^*\partial\Omega$. Moreover,*

$$\begin{aligned} \Xi \circ \beta(q) - \Xi(q) &= O((|\xi'_g|^2 - 1)^\infty), \\ \beta(q) - \exp(-2\sqrt{-\Xi}H_\Xi)(q) &= O((|\xi'_g|^2 - 1)^\infty), \\ \Xi(x', \xi') &= \frac{|\xi'_g|^2 - 1}{(2Q(x', \xi'))^{2/3}} + O((|\xi'_g|^2 - 1)^2). \end{aligned}$$

7.1.2. Microlocal description of the boundary layer potentials and operators. We now recall the microlocal descriptions of the boundary layer potentials and operators near glancing from [Gal14, Section 4.5]. Let \mathcal{A}_i , \mathcal{A}'_i , \mathcal{A}_- , and \mathcal{A}'_- denote the Fourier multiplier with multiplier $A_i(\Theta_h)$, $A'_i(\Theta_h)$, $A_-(\Theta_h)$, and $A'_-(\Theta_h)$ where for convenience, we define

$$\Theta_h := h^{-2/3}\Theta, \quad \Theta_{0h} := h^{-2/3}\Theta_0, \quad \rho_h := h^{-2/3}\rho, \quad \rho_{0h} := h^{-2/3}\rho.$$

Next, let

$$\begin{aligned} Jf &:= (2\pi h)^{-d+1} \int f_{0b} e^{\frac{i}{h}(\theta_0 + \langle x' - y', \xi' \rangle)} f(y') dy' d\xi', \\ JCF &:= (2\pi h)^{-d+1} \int f_{0b} (\partial_{x_d} \rho + ih \partial_{x_d} g_1)|_{x_d=0} e^{\frac{i}{h}(\theta_0 + \langle x' - y', \xi' \rangle)} f(y') dy' d\xi', \\ JBF &:= (2\pi h)^{-d+1} \int f_{0b} \partial_{x_d} g_0|_{x_d=0} e^{\frac{i}{h}(\theta_0 + \langle x' - y', \xi' \rangle)} f(y') dy' d\xi'. \end{aligned}$$

Then J is an elliptic semiclassical Fourier integral operator quantizing the reduction of the Friedlander glancing pair to the glancing pair $\partial\Omega$, $S^*\mathbb{R}^d$ and it is not hard to check that $B, C \in \Psi(\partial\Omega)$ so that for any $\delta > 0$,

$$\sigma(JCJ^{-1}) = (2Q(x, \xi'))^{1/3} + O_{S_\delta}(h^{1-2\delta})$$

where Q is the second fundamental form lifted to the cotangent bundle, $T^*\partial\Omega$. Thus C is elliptic.

Lemma 7.2. *Suppose that $(x_0, \xi_0) \in S^*\partial\Omega$ and $\zeta \in C_c^\infty(\mathbb{R}^d)$ have $\zeta \equiv 1$ on $[-1, 1]$ with $\mathrm{supp} \zeta \subset [-2, 2]$. Then there exists $\delta > 0$ such that for any $M, \epsilon > 0$ if $|\mathrm{Im} z| \leq Mh \log h^{-1}$,*

$$\begin{aligned} GXg &= h^{2/3} \omega^{-1} \chi J \mathcal{A}_i \mathcal{A}_- C^{-1} J^{-1} Xg + O_{\Psi^{-\infty}}(h^\infty)g \\ NXg &= \left(\frac{1}{2} \mathrm{Id} - \omega^{-1} \chi J (\mathcal{A}_i \mathcal{A}'_- + h^{2/3} \mathcal{A}_- \mathcal{A}_i C^{-1} B) J^{-1} \right) Xg + O_{\Psi^{-\infty}}(h^\infty)g \\ (\mathcal{S}lXg)|_\Omega &= \omega^{-1} h^{2/3} A_{1,g} J C^{-1} J^{-1} Xg + O_{\mathcal{D}' \rightarrow C^\infty}(h^\infty)g \\ (\mathcal{D}lXg)|_\Omega &= -\omega^{-1} A_{2,g} Xg - h^{2/3} \omega^{-1} A_{1,g} J C^{-1} B J^{-1} Xg + O_{\mathcal{D}' \rightarrow C^\infty}(h^\infty)g \end{aligned}$$

where $\omega = \frac{e^{-\pi i/6}}{2\pi}$,

$$\begin{aligned} \chi &:= \zeta((3\delta)^{-1}|x - x_0|), \quad A_{1,g} := \chi H_1 J^{-1}, \quad A_{2,g} := \chi H_2 J^{-1} \\ X &:= \mathrm{Op}_h \left[\zeta \left(\delta^{-1}(|x - x_0| + |\xi' - \xi_0|_g) \right) \zeta \left(h^{-\epsilon} \delta^{-1} ||\xi'_g - 1| \right) \right]. \end{aligned}$$

If we only allow $|\mathrm{Im} z| \leq Mh$, then we can set $\epsilon = 0$.

A simple calculation shows that on $-Mh^{2/3} \leq \xi_1$, $AiA_-(\Theta_h) \in \Psi_{2/3}^{-1/2, -1/2}(\xi_1 = 0)$ and on $-Ch^\epsilon \leq \xi_1 \leq -Mh^{2/3}$, $AiA_-(\Theta_h) \in h^{-1/4+\epsilon/4}\Psi_{1-\epsilon/2}^{-1/2, -1/2}(\xi_1 = 0)$. Moreover, for $\xi_1 \geq Mh^{2/3}$,

$$2\pi e^{\pi i/6} AiA_-(\Theta_h) = \frac{h^{1/3}}{2\sqrt{\xi_1}}(1 + o(h(\xi_1)^{-3/2})).$$

So, using (57)

$$\begin{aligned} \sigma(Jh^{2/3}\omega^{-1}\chi AiA_-C^{-1}J^{-1}X) &= \frac{h}{2\sqrt{\xi_1(\kappa^{-1}(q))}}(1 + o(h(\xi_1)^{-3/2}))\frac{1}{\partial_{x_d}\rho(\kappa^{-1}(q))} \\ &\quad \zeta\left(\delta^{-1}(|x-x_0|+|\xi'-\xi_0|_g)\right)\zeta\left(h^{-\epsilon}\delta^{-1}\|\xi'\|_g-1\right) \\ (59) \quad &= \frac{h}{2\sqrt{|\xi'_g|^2-1}}(1 + o(h(|\xi'_g|^2-1)^{-3/2})) \\ &\quad \zeta\left(\delta^{-1}(|x-x_0|+|\xi'-\xi_0|_g)\right)\zeta\left(h^{-\epsilon}\delta^{-1}\|\xi'\|_g-1\right) \end{aligned}$$

Finally, we recall the decomposition of the boundary layer operators away from glancing from [Gal14, Lemma 4.27]. For a similar decomposition when $\text{Im } z = 0$ see [HZ04, Proposition 4.1].

Lemma 7.3. *Let $\Omega \subset \mathbb{R}^d$ be strictly convex with $\partial\Omega \in C^\infty$. Then for all $1/2 > \epsilon, \gamma > 0$, and $z = E + o(h^{1-\gamma})$ with $\text{Im } z \geq -Ch \log h^{-1}$. Then*

$$\begin{aligned} G(z/h) &:= G_\Delta(z) + G_B(z) + G_g(z) + \mathcal{O}_{\mathcal{D}' \rightarrow C^\infty}(h^\infty) \\ N(z/h) &:= N_\Delta(z) + N_B(z) + N_g(z) + \mathcal{O}_{\mathcal{D}' \rightarrow C^\infty}(h^\infty) \\ \partial_\nu \mathcal{D}l(z/h) &:= \partial_\nu \mathcal{D}l_\Delta(z) + \partial_\nu \mathcal{D}l_B(z) + \partial_\nu \mathcal{D}l_g(z) + \mathcal{O}_{\mathcal{D}' \rightarrow C^\infty}(h^\infty) \end{aligned}$$

where $G_\Delta \in h^{1-\frac{\epsilon}{2}}\Psi_\epsilon^{-1}$, $N_\Delta \in h^{1-2\epsilon}\Psi_\epsilon^{-1}$, $\partial_\nu \mathcal{D}l_\Delta \in h^{-1}\Psi_\epsilon^1$, and $G_B \in h^{1-\frac{\epsilon}{2}}e^{(\text{Im } z)-d_\Omega/h}I_\delta^{\text{comp}}(C_b)$, $N_B \in e^{(\text{Im } z)-d_\Omega/h}I_\delta^{\text{comp}}(C_b)$, and $\partial_\nu \mathcal{D}l_B \in h^{-1}e^{(\text{Im } z)-d_\Omega/h}I_\delta^{\text{comp}}(C_b)$ are FIOs associated to β_E where $\delta = \max(\epsilon, \gamma)$. Moreover,

$$\begin{aligned} \text{MS}_h'((\cdot)_B) &\subset \left\{ \begin{array}{l} (q, p) \in B_E^* \partial\Omega \times B_E^* \partial\Omega : \\ \min(E - |\xi'(q)|_g, E - |\xi'(p)|_g, l(q, p)) > ch^\epsilon \end{array} \right\} \\ \text{MS}_h'((\cdot)_g) &\subset \left\{ \begin{array}{l} (q, p) \in T^* \partial\Omega \times T^* \partial\Omega : \\ \max(|E - |\xi'(q)|_g|, |E - |\xi'(p)|_g|, l(q, p)) < ch^\epsilon \end{array} \right\} \end{aligned}$$

$$\sigma(G_\Delta) = \frac{ih}{2\sqrt{E^2 - |\xi'_g|^2}}, \quad \sigma(\partial_\nu \mathcal{D}l_\Delta) = \frac{ih^{-1}\sqrt{E^2 - |\xi'_g|^2}}{2},$$

$$\sigma(G_B e^{\frac{\text{Im } z}{h}} \text{Oph}(l(q, \beta_E(q)))) = \frac{he^{\frac{i}{h} \text{Re } \omega_0 l(q, \beta_E(q))}}{2(E^2 - |\xi'(\beta_E(q))|_g^2)^{1/4}(E^2 - |\xi'(q)|_g^2)^{1/4}} dq^{1/2},$$

$$\sigma(N_B e^{\frac{\text{Im } z}{h}} \text{Oph}(l(q, \beta_E(q)))) = \frac{-ie^{\frac{i}{h} \text{Re } \omega_0 l(q, \beta_E(q))}(E^2 - |\xi'(q)|_g^2)^{1/4}}{2(E^2 - |\xi'(\beta_E(q))|_g^2)^{1/4}} dq^{1/2},$$

$$\begin{aligned} \sigma(\partial_\nu \mathcal{D} \ell_{BE} \frac{\text{Im } z}{h} \text{Op}_h(l(q, \beta_E(q)))) = \\ \frac{h^{-1} e^{\frac{i}{h} \text{Re } \omega_0 l(q, \beta_E(q))} (E^2 - |\xi'(\beta_E(q))|_g^2)^{1/4} (E^2 - |\xi'(q)|_g^2)^{1/4}}{2} dq^{1/2}. \end{aligned}$$

where we take $\sqrt{z} = \sqrt{|z|} e^{\frac{1}{2} \text{Arg}(z)}$ for $-\pi/2 < \text{Arg}(z) < 3\pi/2$.

Remark 15. The decomposition in [HZ04] is slightly less precise than that in [Gal14] because the glancing pieces are microlocalized to a neighborhood of $S^* \partial \Omega \times S^* \partial \Omega$ rather than to a neighborhood of $S^* \partial \Omega \times S^* \partial \Omega \cap \Delta(T^* \partial \Omega)$ where $\Delta(T^* \partial \Omega)$ denotes the diagonal.

In particular, Lemma 7.3 together with (59) imply that there exists $M > 0$ so that for $\chi = \chi(|\xi'|_g) \in \Psi_{2/3}^{0,0}(|\xi'|_g = 1)$ with $\text{supp } \chi \subset \{|\xi'|_g \geq 1 + Mh^{2/3}\}$, $G \text{Op}_h(\chi) \in h^{2/3} \Psi_{2/3}^{-1/2, -1/2}(|\xi'|_g = 1)$ with

$$(60) \quad \sigma(G \text{Op}_h(\chi)) = \frac{h\chi(|\xi'|_g)}{2\sqrt{|\xi'|_g^2 - 1}} \left(1 + o(h(|\xi'|_g^2 - 1)^{-3/2})\right).$$

7.2. Analysis of $S\ell^* S\ell$, $\mathcal{D}\ell^* \mathcal{D}\ell$, and $\mathcal{D}\ell^* S\ell$ near glancing. Our next goal is to understand $S\ell^* S\ell$, $\mathcal{D}\ell^* \mathcal{D}\ell$, and $\mathcal{D}\ell^* S\ell$ microlocally near glancing points. To do this, we will use the microlocal description of $S\ell$ and $\mathcal{D}\ell$ from Lemma 7.2. In particular, let J_1 be a microlocally unitary FIO quantizing κ where κ is as in (55). Then we prove

Lemma 7.4. *Fix $z = 1 + i\mu$ with $|\mu| \leq Mh \log h^{-1}$. Then for any $\epsilon > 0$ and $\delta \leq 2/3$, for $\chi \in \Psi_{2/3}^{0,0}(|\xi'|_g = 1)$ self adjoint with $\text{WF}_h(\chi) \subset \{||\xi'|_g - 1| \leq h^\delta\}$,*

$$\begin{aligned} \chi S\ell^* S\ell \chi \in h^{2-\epsilon} \Psi_{1-\delta/2}^{0,0}(\{|\xi'|_g = 1\}), \quad \chi \mathcal{D}\ell^* S\ell \chi, \chi S\ell^* \mathcal{D}\ell \chi \in h^{\frac{3}{2}-\frac{\delta}{4}-\epsilon} \Psi_{1-\delta/2}^{0,1/2}(\{|\xi'|_g = 1\}), \\ \chi \mathcal{D}\ell^* \mathcal{D}\ell \chi \in h^{1-\frac{\delta}{2}-\epsilon} \Psi_{1-\delta/2}^{0,1}(\{|\xi'|_g = 1\}) \end{aligned}$$

Moreover,

$$\begin{aligned} \sigma(J_1^* \chi S\ell^* S\ell \chi J_1) &= \frac{h^2 \Psi_{S\ell}(h^{-2/3} \Theta_0(\xi')) \chi^2(\kappa(x', \xi'))}{2Q(\kappa(x', \xi'))}, \\ \sigma(J_1^* \chi \mathcal{D}\ell^* S\ell \chi J_1) &= \frac{h^{4/3} \Psi_{\mathcal{D}\ell S\ell}(h^{-2/3} \Theta_0(\xi')) \chi^2(\kappa(x', \xi'))}{(2Q(\kappa(x', \xi')))^{2/3}}, \\ \sigma(J_1^* \chi S\ell^* \mathcal{D}\ell \chi J_1) &= \frac{h^{4/3} \overline{\Psi_{\mathcal{D}\ell S\ell}(h^{-2/3} \Theta_0(\xi'))} \chi^2(\kappa(x', \xi'))}{(2Q(\kappa(x', \xi')))^{2/3}}, \\ \sigma(J_1^* \chi \mathcal{D}\ell^* \mathcal{D}\ell \chi J_1) &= \frac{h^{2/3} \Psi_{\mathcal{D}\ell}(h^{-2/3} \Theta_0(\xi')) \chi^2(\kappa(x', \xi'))}{(2Q(\kappa(x', \xi')))^{1/3}}. \end{aligned}$$

We prove this lemma using Lemma 7.2 to write a parametrix for $S\ell^* S\ell$. We then Taylor expand the Airy functions around their values at the boundary of Ω and estimate each of the terms. The higher order terms in the expansion will turn out to be lower order in h and the symbols will be found by computing the first term. The operators $\mathcal{D}\ell^* S\ell$, $S\ell^* \mathcal{D}\ell$ and $\mathcal{D}\ell^* \mathcal{D}\ell$ are handled similarly.

7.2.1. *Estimates on the Remainder Terms.* We first give estimates on the size of terms that will be lower order. These terms arise from a Taylor expansion of the integrand when computing $\mathcal{S} \ell^* \mathcal{S} \ell$ using the microlocal model from Lemma 7.2. In particular, consider an operator with kernel given by

$$R_{ijklmno} = (2\pi h)^{-2d+2} \iint_0^\infty b(w, x', y', \eta, \xi') h^{-2/3(j+k)} (\rho(w, \eta') - \rho(w, \xi'))^k (\Theta(\eta') - \Theta(\xi'))^j w_d^n \\ Ai^{(l)}(\rho_h(w, \xi')) A_-^{(m)}(\Theta_h(\xi')) \overline{Ai^{(o)}(\rho_h(w, \xi')) A_-^{(i)}(\Theta_h(\xi'))} \\ e^{\frac{i}{h}(\theta(w, \xi') - \theta(w, \eta') - \theta_b(y', \xi') + \theta_b(x', \eta'))} dw_d d\xi' d\eta' dw'$$

where $b \in S_\delta(\xi_1 = 0)$ is supported in $|\Theta(\xi')|, |\Theta(\eta')| \leq Ch^\delta$. First, observe that since $\partial_{x_d} \rho_0 > 0$ and for $t \gg 1$,

$$Ai(t) \leq C e^{-t^{3/2}},$$

we may assume that b is supported on $w_d < \epsilon$ for any $\epsilon > 0$ by introducing an $O(e^{-C/h})$ error. Next, notice that

$$\theta(w, \xi') - \theta(w, \eta') = \theta_b(w', \xi') - \theta_b(w', \eta') + w_d^2 \langle \xi' - \eta', r(w, \xi', \eta') \rangle.$$

So,

$$\partial_{w'} \theta(w, \xi') - \partial_{w'} \theta(w, \eta') = (\partial_{x' \xi'}^2 \theta_b(w', \eta') + w_d^2 \partial_{w'} r)(\xi' - \eta')$$

and, using that $\partial_{x' \xi'}^2 \theta_b \neq 0$, for w_d small enough, the phase is stationary precisely at $\xi' = \eta'$.

We first change variables so that $W_d = h^{-2/3} \rho_0(w, \xi')$. Then, $w_d = h^{2/3} e(W_d, w', \xi') (W_d - h^{-2/3} \Theta_0(\xi'))$ where e is elliptic. So, the kernel takes the form

$$R_{ijklmno} \\ = (2\pi h)^{-2d+2} h^{2/3} \iint_{h^{-2/3} \Theta_0(\xi')}^\infty b_1(h^{2/3} W_d, w', x', y', \eta, \xi') h^{-2/3(j+k-n)} \\ (\rho(w_d(W_d, w', \xi'), \eta') - h^{2/3} W_d - \epsilon(h) \rho_1)^k (\Theta(\eta') - \Theta(\xi'))^j (W_d - h^{-2/3} \Theta_0(\xi'))^n \\ \overline{Ai^{(l)}(W_d + h^{-2/3} \epsilon(h) \rho_1(w, \xi')) A_-^{(m)}(\Theta_h(\xi')) Ai^{(o)}(W_d + h^{-2/3} \epsilon(h) \rho_1(w, \xi')) A_-^{(i)}(\Theta_h(\xi'))} \\ e^{\frac{i}{h}(\theta_b(w', \xi') - \theta_b(w', \eta') - \theta_b(y', \xi') + \theta_b(x', \eta') + h^{4/3}(W_d - \Theta_0(\xi'))^2 \langle \xi' - \eta', r(W_d, w', x', \xi', \eta') \rangle)} dW_d d\xi' d\eta' dw'$$

Now, the integrand vanishes to order $|\xi' - \eta'|^{j+k}$, and the phase is stationary in w' precisely at $\xi' = \eta'$. Hence, integrating by parts $j+k$ times in w' and then applying stationary phase in the w', η' variables gives a finite sum of terms (possibly with additional positive powers of h) of the form

$$\frac{h^{2/3}}{(2\pi h)^{d-1}} \iint_{h^{-2/3} \Theta_0(\xi')}^\infty b_2(h^{2/3} W_d, x' + o((W_d - h^{-2/3} \Theta_0(\xi'))^2 h^{4/3}), x', y', \xi', \xi') \\ h^{1/3(j+k+2(n-p-q))} \epsilon(h)^{p+q} (W_d - h^{-2/3} \Theta_0(\xi'))^n Ai^{(l+p)}(W_d + h^{-2/3} \epsilon(h) \rho_1) A_-^{(m)}(\Theta_h(\xi')) \\ \overline{Ai^{(o+q)}(W_d + h^{-2/3} \epsilon(h) \rho_1) A_-^{(i)}(\Theta_h(\xi'))} e^{\frac{i}{h}(\theta_b(x', \xi') - \theta_b(y', \xi'))} dW_d d\xi'.$$

Note that we can apply stationary phase in the w', η' variables since $\partial_{x', \xi'}^2 \theta_{0b} \neq 0$. Next, change variables $\xi' \mapsto \Xi'(x', y', \xi')$ so that

$$\theta_{0b}(x', \xi') - \theta_{0b}(y', \eta') = \langle x' - y', \Xi'(x', y', \xi') \rangle.$$

To find such a change of variables, observe that $\Xi(x', x', \xi') = \partial_{x'} \theta_{0b}$ and hence $\partial_{\xi'} \Xi = \partial_{\xi', x'}^2 \theta_{0b} \neq 0$ so we can apply the implicit function theorem. Then, integrating in W_d and using the fact that on $\text{supp } b_2$, $|\Theta(\xi')| \leq Ch^\delta$, we obtain

$$R_{ijklmno} = (2\pi h)^{-d+1} \int b_3(x', y', \xi'; h) e^{\frac{i}{h} \langle x' - y', \xi' \rangle} d\xi'$$

where, letting

$$r = \frac{2n + l + p + o + q + m + i - 4}{2} + \frac{1}{4}(\delta_{o+q}^0 + \delta_{l+p}^0 + \delta_i^0 + \delta_m^0),$$

$$b_3 \in h^{2/3 + \frac{1}{3}(j+k+p+q+2n)} (\log h^{-1})^{p+q} h^{\max(r, 0)(\frac{1}{3} - \frac{\delta}{2})} S_{1-\delta/2}^{0,r}(\mathbb{R}^{d-1}; \{\xi_1 = 0\})$$

Hence, the operator $R^{ijklmno}$ with kernel $R_{ijklmno}$ has for any $\epsilon > 0$,

$$R^{ijklmno} \in h^{1/3(j+k-l-m-i-o+2) + \delta/2(l+m+i+2n+o) - \epsilon} \Psi_{1-\delta/2}^{0,0}(\{\xi_1 = 0\})$$

$$R^{ijklmno} \in h^{2/3 + \frac{1}{3}(j+k+2n)} h^{\max(r, 0)(\frac{1}{3} - \frac{\delta}{2})} \Psi_{1-\delta/2}^{0,r}(\mathbb{R}^{d-1}; \{\xi_1 = 0\}).$$

7.2.2. The Principal Part. By the analysis above, we see that when microlocalized near glancing points $\mathcal{S}\ell^* \mathcal{S}\ell$, $\mathcal{D}\ell^* \mathcal{D}\ell$, and $\mathcal{D}\ell^* \mathcal{S}\ell$ are pseudodifferential in a second microlocal class. We just need to compute the principal symbol of these operators. The symbols will turn out to be $\Psi_{\mathcal{S}\ell}$, $\Psi_{\mathcal{D}\ell}$, and $\Psi_{\mathcal{D}\ell \mathcal{S}\ell}$, respectively.

First, using the principle of stationary phase, we compute that

$$J^{-1} f = (2\pi h)^{-d+1} \int b_0(y', \xi') e^{\frac{i}{h} (\langle x', \xi' \rangle - \theta_b(y', \xi'))} f(y') dy' d\xi',$$

$$C^{-1} J^{-1} f = (2\pi h)^{-d+1} \int b_1(y', \xi') e^{\frac{i}{h} (\langle x', \xi' \rangle - \theta_b(y', \xi'))} f(y') dy' d\xi'.$$

where

$$b_0 = \frac{|\det \partial_{x', \xi'}^2 \theta_b(y', \xi')|}{g_{0b}(y', \xi')} + O_S(h) \quad \text{and} \quad b_1 = \frac{b_0(y', \xi')}{\partial_{x_d} \rho(y', \xi')} + O_S(h).$$

Denote the kernels of $\mathcal{S}\ell^* \mathcal{S}\ell$, $\mathcal{D}\ell^* \mathcal{D}\ell$, and $\mathcal{D}\ell^* \mathcal{S}\ell$, respectively by $K_{\mathcal{S}\ell}$, $K_{\mathcal{D}\ell}$, and $K_{\mathcal{D}\ell \mathcal{S}\ell}$ respectively. We explicitly consider $\mathcal{S}\ell^* \mathcal{S}\ell$ and we record the end result for the others. The kernel of $\mathcal{S}\ell$ is given by

$$\begin{aligned} \mathcal{S}\ell(x, y) &= \frac{2\pi e^{\pi i/6} h^{2/3}}{(2\pi h)^{d-1}} \int \left(g_0(x, \xi') Ai(\rho_h(x, \xi')) + ih^{1/3} g_1(x, \xi') Ai'(\rho_h(x, \xi')) \right) \\ &\quad A_-(\Theta_h(\xi')) b_1(y', \xi') e^{\frac{i}{h} (\theta(x, \xi') - \theta_b(y', \xi'))} d\xi' \end{aligned}$$

The kernel of $\mathcal{S}\ell^*\mathcal{S}\ell$ is given by

$$K_{\mathcal{S}\ell} = \frac{4\pi^2 h^{4/3}}{(2\pi h)^{2d-2}} \iint_0^\infty \left(g_0(w, \xi') Ai(\rho_h(w, \xi')) + i h^{1/3} g_1(w, \xi') Ai'(\rho_h(w, \xi')) \right) \\ \left(\overline{g_0(w, \eta') Ai(\rho_h(w, \eta'))} - i h^{1/3} \overline{g_1(w, \eta') Ai'(\rho_h(w, \eta'))} \right) \\ A_-(\Theta_h(\xi')) \overline{A_-(\Theta_h(\eta'))} b_1(y', \xi') \overline{b_1(x', \eta')} e^{\frac{i}{h}(\theta_b(x', \eta') - \theta(w, \eta') + \theta(w, \xi') - \theta_b(y', \xi'))} dw_d dw' d\xi' d\eta'$$

Taylor expanding the Airy functions around $\rho_h(w, \xi')$ and $\Theta_h(\xi')$ produces lower order terms of the form $h^{4/3} R_{0jkjk0}$, $(j, k) \neq (0, 0)$, $h^{5/3} R_{0jk(j+1)k10}$, $h^{5/3} R_{0jkjk11}$ and $h^2 R_{0jk(j+1)k21}$. In particular, $\mathcal{S}\ell^*\mathcal{S}\ell = A + O_{\Psi_{1-\delta/2}(\{\xi_1=0\})}(h^{2+\delta/2-\epsilon})$ where A has kernel

$$A(x, y) = \frac{4\pi^2 h^{4/3}}{(2\pi h)^{2d-2}} \iint_0^\infty g_0(w, \xi') Ai(\rho_h(w, \xi')) \overline{g_0(w, \eta') Ai(\rho_h(w, \eta'))} A_-(\Theta_h(\xi')) \overline{A_-(\Theta_h(\eta'))} \\ b_1(y', \xi') \overline{b_1(x', \eta')} e^{\frac{i}{h}(\theta_b(x', \eta') - \theta(w, \eta') + \theta(w, \xi') - \theta_b(y', \xi'))} dw_d dw' d\xi' d\eta'$$

Then, changing variables $W_d \mapsto h^{-2/3} \rho_0(w, \xi')$ and performing stationary phase as in the analysis of R_{jklmno} gives

$$A(x, y) = \frac{4\pi^2 h^2}{(2\pi h)^{d-1}} \iint_{h^{-2/3}\Theta_0(\xi')}^\infty \frac{a_0(x', \xi') \overline{a_0(x', \xi')} + O_{S_{1-\delta/2}}(h^{1/3+\delta/2-\epsilon})}{|\det \partial_{x', \xi'}^2 \theta(x', \xi')| |\partial_{x_d} \rho(x', \xi')|} |Ai(W_d + h^{-2/3} \epsilon(h) \rho_1)|^2 \\ |A_-(\Theta_h(\xi'))|^2 b_1(y', \xi') \overline{b_1(x', \xi')} e^{\frac{i}{h}(\theta_b(x', \xi') - \theta_b(y', \xi'))} dW_d dw' d\xi' d\eta'$$

Using that the phase is stationary at $x' = y'$ to integrate by parts in ξ' when terms of size $|x' - y'|$ appear, that for any $\epsilon > 0$,

$$Ai(W_d + h^{-2/3} \epsilon(h) \rho_1) = \begin{cases} Ai(W_d) + O_S(h^{-2/3} \epsilon(h) \langle W_d \rangle^{1/4}) & W_d \leq C \\ Ai(W_d) + O_S(h^{-2/3} \epsilon(h) \langle W_d \rangle^{1/4} e^{-2/3 W_d^{3/2}}) & W_d \geq C, \end{cases}$$

and using the definition of $\Psi_{\mathcal{S}\ell}$ gives for any $\epsilon > 0$,

$$A(x, y) = \frac{h^2}{(2\pi h)^{d-1}} \int \frac{|a_0(x', \xi')|^2 |b_1(x', \xi')|^2 \Psi_{\mathcal{S}\ell}(\Theta_{0h}(\xi')) + O_{S_{1-\delta/2}}(h^{\delta/2-\epsilon})}{|\det \partial_{x', \xi'}^2 \theta_{0b}(x', \xi')| |\partial_{x_d} \rho(x', \xi')|} e^{\frac{i}{h}(\theta_b(x', \xi') - \theta_b(y', \xi'))} d\xi'$$

Now, let J_1 be a microlocally unitary semiclassical FIO quantizing κ i.e.

$$J_1 f = (2\pi h)^{-d+1} \int c(x', \xi') e^{\frac{i}{h}(\theta_{0b}(x', \xi') - \langle y', \xi' \rangle)} d\xi'$$

where

$$c = |\det \partial_{x', \xi'}^2 \theta_{0b}(x', \xi')|^{1/2} + O(h).$$

Applying stationary phase gives

$$\begin{aligned}
 & J_1^* A J_1(x, y) \\
 &= \frac{h^2}{(2\pi h)^{d-1}} \int \frac{\bar{c}(w', \xi') c(z', \xi') |a_0(w', \xi')|^2 |b_1(w', \xi')|^2 \Psi_{S\ell}(\Theta_{0h}(\xi')) + O_{S_{1-\delta/2}}(h^{\delta/2-\epsilon})}{|\det \partial_{x', \xi'}^2 \theta_{0b}(w', \xi')|^2 |\partial_{x_d} \rho(w', \xi')| |\det \partial_{x', \xi'}^2 \theta_{0b}(z', \xi')|} \Bigg|_{\substack{y' = \partial_{\xi'} \theta_{0b}(z', \xi') \\ x' = \partial_{\xi'} \theta_{0b}(w', \xi')}} \\
 & \quad e^{\frac{i}{h} \langle (x' - y', \xi') + \theta_{1b}(w', \xi') - \theta_{1b}(z', \xi') \rangle} d\xi'
 \end{aligned}$$

Again, using integration by parts on terms that are $O(|x' - y'|)$, we can assume that $x' = y'$ in the amplitude and hence have

$$\begin{aligned}
 & J_1^* A J_1(x, y) \\
 &= \frac{h^2}{(2\pi h)^{d-1}} \int \frac{\bar{c}(w', \xi') c(w', \xi') |a_0(w', \xi')|^2 |b_1(w', \xi')|^2 \Psi_{S\ell}(\Theta_{0h}(\xi')) + O_{S_{1-\delta/2}}(h^{\delta/2-\epsilon})}{|\det \partial_{x', \xi'}^2 \theta_{0b}(w', \xi')|^2 |\partial_{x_d} \rho(w', \xi')| |\det \partial_{x', \xi'}^2 \theta_{0b}(w', \xi')|} \Bigg|_{x' = \partial_{\xi'} \theta_{0b}(w', \xi')} \\
 & \quad e^{\frac{i}{h} \langle x' - y', \xi' \rangle} d\xi'
 \end{aligned}$$

So, plugging in the definition of c and b_1 , we have

$$(61) \quad J_1^* S\ell^* S\ell J_1 = \frac{h^2}{(2\pi h)^{d-1}} \int \frac{\Psi_{S\ell}(\Theta_{0h}(\xi')) + O_{S_{1-\delta/2}}(h^{\delta/2-\epsilon})}{|\partial_{x_d} \rho(w', \xi')|^2 |\partial_{x_d} \rho(w', \xi')|} \Bigg|_{x' = \partial_{\xi'} \theta_{0b}(w', \xi')} e^{\frac{i}{h} \langle x' - y', \xi' \rangle} d\xi'$$

Similar computations give

$$(62) \quad \begin{aligned}
 J_1^* \mathcal{D}\ell^* S\ell J_1 &= \frac{h^{4/3}}{(2\pi h)^{d-1}} \int \frac{\Psi_{\mathcal{D}\ell S\ell}(\Theta_{0h}(\xi')) + O_{S_{1-\delta/2}^{0,1/2}}(h^{\frac{1}{6} + \delta/4 - \epsilon})}{(\partial_{x_d} \rho(w', \xi'))^2} \Bigg|_{x' = \partial_{\xi'} \theta_{0b}(w', \xi')} e^{\frac{i}{h} \langle x' - y', \xi' \rangle} d\xi' \\
 J_1^* S\ell^* \mathcal{D}\ell J_1 &= \frac{h^{4/3}}{(2\pi h)^{d-1}} \int \frac{\overline{\Psi_{\mathcal{D}\ell S\ell}(\Theta_{0h}(\xi'))} + O_{S_{1-\delta/2}^{0,1/2}}(h^{\frac{1}{6} + \delta/4 - \epsilon})}{(\partial_{x_d} \rho(w', \xi'))^2} \Bigg|_{x' = \partial_{\xi'} \theta_{0b}(w', \xi')} e^{\frac{i}{h} \langle x' - y', \xi' \rangle} d\xi' \\
 J_1^* \mathcal{D}\ell^* \mathcal{D}\ell J_1 &= \frac{h^{2/3}}{(2\pi h)^{d-1}} \int \frac{\Psi_{\mathcal{D}\ell}(\Theta_{0h}(\xi')) + O_{S_{1-\delta/2}^{0,1}}(h^{\frac{1}{3} - \epsilon})}{\partial_{x_d} \rho(w', \xi')} \Bigg|_{x' = \partial_{\xi'} \theta_{0b}(w', \xi')} e^{\frac{i}{h} \langle x' - y', \xi' \rangle} d\xi'.
 \end{aligned}$$

Hence, all of the above operators are second microlocal pseudodifferential operators with respect to the glancing surface $\{|\xi'_g| = 1\}$.

Plugging (57) into (61) and (62) gives that

$$\begin{aligned}
 \sigma(J_1^* S\ell^* S\ell J_1) &= \frac{h^2 \Psi_{S\ell}(h^{-2/3} \Theta_0(\xi'))}{2Q(\kappa(x', \xi'))} & \sigma(J_1^* \mathcal{D}\ell^* S\ell J_1) &= \frac{h^{4/3} \Psi_{\mathcal{D}\ell S\ell}(h^{-2/3} \Theta_0(\xi'))}{(2Q(\kappa(x', \xi')))^{2/3}} \\
 \sigma(J_1^* S\ell^* \mathcal{D}\ell J_1) &= \frac{h^{4/3} \overline{\Psi_{\mathcal{D}\ell S\ell}(h^{-2/3} \Theta_0(\xi'))}}{(2Q(\kappa(x', \xi')))^{2/3}} & \sigma(J_1^* \mathcal{D}\ell^* \mathcal{D}\ell J_1) &= \frac{h^{2/3} \Psi_{\mathcal{D}\ell}(h^{-2/3} \Theta_0(\xi'))}{(2Q(\kappa(x', \xi')))^{1/3}}
 \end{aligned}$$

where κ is as in (55).

8. PRELIMINARY ANALYSIS OF THE GENERALIZED BOUNDARY DAMPED EQUATION

We examine problems of the form

$$(63) \quad \begin{cases} (-h^2\Delta - z^2)u = w & \text{in } \Omega \\ h\partial_\nu u + Bu = hv & \text{on } \partial\Omega \\ u|_{\partial\Omega} = \psi \end{cases}$$

$$(64) \quad z \in [1 - ch, 1 + ch] + i[-Mh \log h^{-1}, Mh \log h^{-1}].$$

We then assume that $B = hN_2(z/h) + hV(z)$, with V analytic for z as in (64), $V \in h^\alpha(\Psi_{2/3}^{0,m}\{|\xi'|_g = E'\} \cup \Psi_{2/3}^{0,m}\{|\xi'|_g = 1\})$ for some $\alpha \geq -1$ and $m \in \mathbb{R}$.

Furthermore, suppose that for some $\delta > 0$, $M, M_1 > 0$, and $0 < \epsilon < \frac{1}{2}$

$$(65) \quad \begin{aligned} & V \text{ is elliptic, on } ||\xi'|_g - 1| < \delta, \\ & \left| 1 + \frac{h\sigma(V)}{2\sqrt{|\xi'|_g^2 - 1}} \right| \geq \delta \left(\left\langle \frac{h^{1+\alpha}}{\sqrt{|\xi'|_g^2 - 1}} \right\rangle + \langle \xi' \rangle^{m-1} \right) \quad |\xi'|_g > 1 + Mh^{2/3} \\ & \left| 1 + \frac{ih\sigma(V)}{2\sqrt{1 - |\xi'|_g^2}} \right| \geq \delta \left\langle \frac{h^{1+\alpha}}{\sqrt{1 - |\xi'|_g^2}} \right\rangle \quad |\xi'|_g \leq 1 - h^\epsilon \\ & \log \left(1 + \frac{h\sigma(V)}{\sqrt{|\xi'|_g^2 - 1}} \right) \text{ exists and is smooth on } T^*\partial\Omega \setminus \{|\xi'|_g \leq M_1\} \end{aligned}$$

The problem (63) is a highly generalized version of a standard boundary damped equation which was studied in the seminal work of Bardos–Lebeau–Rauch [BLR92] see also [KT95]. In order to study this problem from the spectral point of view, we must see that the inverse operator is meromorphic with finite rank poles. This is similar to the analysis in the case of the standard damped wave equation (see for example [Zwo12, Chapter 5] and references therein).

8.1. Meromorphy of the Resolvent. For $s > -1/2$, let

$$\mathcal{P}(z) := \begin{pmatrix} -h^2\Delta - z^2 \\ \gamma\partial_\nu + h^{-1}B(z)\gamma \end{pmatrix} : H^{s+2}(\Omega) \rightarrow H^s(\Omega) \oplus H^{s+1/2-\max(m-1,0)}(\partial\Omega).$$

We will show that $\mathcal{P}(z)^{-1}$ is a meromorphic family of operators with finite rank poles. Our analysis is similar in spirit to that for potential and black box scattering see for example [DZ, Chapters 2,3,4].

Then, when $(I + VG)^{-1} : H^s(\partial\Omega) \rightarrow H^{s+\max(m-1,0)}(\partial\Omega)$ exists,

$$\begin{aligned} (\mathcal{P}^{-1})^t &= \begin{pmatrix} [I - \mathcal{S}\ell(I + VG)^{-1}(\gamma\partial_\nu + h^{-1}B\gamma)]h^{-2}1_\Omega R_0(z/h)1_\Omega \\ \mathcal{S}\ell(I + VG)^{-1} \end{pmatrix} \\ &: H^s(\Omega) \oplus H^{s+1/2-\max(m-1,0)}(\partial\Omega) \rightarrow H^{s+2}(\Omega). \end{aligned}$$

To check that this is the inverse, we simply apply the jumps formulas from for example [Gal14, Lemma 4.1 and Proposition 4.1.1]. For the Sobolev mapping properties of $1_\Omega R_0 1_\Omega$, $\mathcal{S}l$, $\mathcal{D}l$, see for example [Eps07, Theorems 9, 10]. Now,

$$(I + VG)^{-1} = I - V(I + GV)^{-1}G, \quad (I + GV)^{-1} = I - G(I + VG)^{-1}V$$

therefore, $I + GV$ is invertible if and only if $I + VG$ is invertible. Thus, to check that \mathcal{P}^{-1} has a meromorphic continuation from $\text{Im } z > 0$, it is enough to check that for $(I + GV)^{-1}$. To see this, we first show that $I + GV$ is a holomorphic family of Fredholm operators with index 0 on the domain of R_0 . The condition (65) and Lemma 7.3 imply that for M sufficiently large and $0 \leq \chi_0 \in C_c^\infty(\mathbb{R})$ with $\chi_0 \equiv 1$ on $|x| \leq M$ and $\text{supp } \chi_0 \subset \{|x| \leq M+1\}$, $(I + GV)(1 - \chi(|hD'|_g)) \in \Psi^{\max(m-1, 0)}(\partial\Omega)$ is elliptic on $|\xi'|_g \geq M+1$ with symbol

$$f := \sigma((I + GV)(1 - \chi_0(|hD'|_g))) = \left(1 + \frac{h\sigma(V)}{2\sqrt{|\xi'|_g^2 - 1}}\right) (1 - \chi_0(|\xi'|_g)).$$

Then, for $k = 1, 2$, let $0 \leq \chi_k \in C_c^\infty(\mathbb{R})$ with $\chi_k \equiv 1$ on $|x| \leq M+1$ and $\text{supp } \chi_k \subset \text{supp } \chi_{k+1}$ with $\text{supp } \chi_2 \subset \{|x| \leq M+2\}$. Then, by assumption, $\log \frac{f}{|f|}$ is well defined on $\text{supp } \chi_2(|\xi'|_g)$ and hence for $K > 0$ large enough

$$q = f + K\chi_2(|\xi'|_g) \left(\frac{f}{|f|}\right)^{1 - \chi_1(|\xi'|_g)} \in S^{m-1} \quad \text{has } |q| \geq c\langle \xi' \rangle^{m-1}.$$

Now, $\text{Op}_h(q) : H_h^{s+\max(m-1, 0)}(\partial\Omega) \rightarrow H_h^s(\partial\Omega)$ is invertible for h small enough and

$$\begin{aligned} \text{Op}_h(q)(I + GV) &= I + A_1 & (I + GV)\text{Op}_h(q) &= I + A_2 \\ \text{Op}_h(q)(I + VG) &= I + A_3 & (I + VG)\text{Op}_h(q) &= I + A_4 \end{aligned}$$

with $A_i : H_h^s(\partial\Omega) \rightarrow H_h^{s-1}(\partial\Omega)$. Therefore, both $I + GV$ and $I + VG$ are Fredholm with index 0. The analysis below will show that there exists z_0 with $\text{Im } z > 0$ so that $I + GV$ is injective. Therefore, $(I + GV)^{-1}$ exists at z_0 and by the analytic Fredholm Theorem has a meromorphic continuation to \mathbb{C} when d is odd and to the logarithmic cover of $\mathbb{C} \setminus \{0\}$ when d is even.

Write

$$(66) \quad (I + VG)\varphi = v.$$

Note that if φ has (66), then $u = \mathcal{S}l\varphi$ solves (67) with $w = 0$ and $\psi = G\varphi$. That is,

$$(67) \quad \begin{cases} (-h^2\Delta - z^2)u = 0 & \text{in } \Omega \\ h\partial_\nu u + Bu = hv & \text{on } \partial\Omega \\ u|_{\partial\Omega} = \psi \end{cases}$$

Similarly, if

$$(68) \quad (I + GV)\psi = Gv,$$

then

$$u = -\mathcal{S}lV\psi + \mathcal{S}lv$$

solves (67). Now, suppose that u solves (67). Then

$$u = h^{-1}\mathcal{S}lh\partial_\nu u - \mathcal{D}lu|_{\partial\Omega} = -h^{-1}\mathcal{S}lBu|_{\partial\Omega} - \mathcal{D}lu|_{\partial\Omega} + \mathcal{S}lv = -\mathcal{S}lVu|_{\partial\Omega} + \mathcal{S}lv$$

where we have used that in Ω ,

$$\mathcal{S}lN_2 + \mathcal{D}l = 0 \quad \text{and hence} \quad (h^{-1}\mathcal{S}lB + \mathcal{D}l) = \mathcal{S}lV.$$

Therefore, taking $x \rightarrow \partial\Omega$ gives

$$u|_{\partial\Omega} = -GVu|_{\partial\Omega} + Gv \quad \Rightarrow \quad (I + GV)u|_{\partial\Omega} = Gv.$$

That is, $\psi := u|_{\partial\Omega}$ solves (68). Finally, if ψ solves (68), then $\varphi := v - V\psi$ solves (66).

Lemma 8.1. *The following are equivalent*

- (1) u solves (67)
- (2) $u = \mathcal{S}l(v - Vu|_{\partial\Omega})$
- (3) $u|_{\partial\Omega} = \psi$ solves (68)
- (4) $v - Vu|_{\partial\Omega} = \varphi$ solves (66).

Note also that since $I + VG$ is Fredholm with index 0, it is not invertible if and only if there exists a nonzero solution ψ to $(I + VG)\psi = 0$. Hence, together with Lemma 8.1 we have proved the following

Lemma 8.2. *The operator \mathcal{P}^{-1} is meromorphic on the domain of $R_0(\lambda)$ and the following are equivalent*

- (1) $\mathcal{P}^{-1}(z)$ has a pole at z_0 .
- (2) There exists a nonzero solution ψ to $(I + G(z_0)V(z_0))\psi = 0$.
- (3) There exists a nonzero solution φ to $(I + V(z_0)G(z_0))\varphi = 0$.
- (4) There exists a nonzero solution u to (67) with $v = 0$.

9. MICROLOCAL ANALYSIS OF THE GENERALIZED BOUNDARY DAMPED WAVE EQUATION

We now proceed to study the poles of $\mathcal{P}(z)^{-1}$. It is convenient to study (68) because then the solution to (67) has $u|_{\partial\Omega} = \psi$. From now on, we do so without comment.

9.1. Brief outline of the computations. The analysis in the next few sections proceeds as follows. We first study the elliptic region where there is no propagation and hence the analysis is relatively simple. Then, we study the hyperbolic region where standard propagation occurs. In this case, we use the decomposition of G (Lemma 7.3) to rewrite (68) in terms of the reflectivity operator, R from (18) and transition operator T from (19). We use the symbolic calculus of FIO's to show that this new operator has a microlocal inverse on the hyperbolic set. However, we must show that this inverse preserves the hyperbolic set up to a small remainder. This is done in Lemma 9.2.

Putting these two regions together leaves the glancing region to be analyzed. Here, we apply the microlocal models of G and $\mathcal{S}l$ near glancing from Lemmas 7.2 and 7.4. We start by using (68) together with the model for G near glancing to further localize ψ near certain 'almost glancing hypersurfaces'. Using that $\mathcal{S}lV\psi$ solves (67) with $v = 0$, we obtain estimates on $\text{Im } z$ from the description of $\mathcal{S}l^*\mathcal{S}l$ near glancing.

9.2. Elliptic Region. Fix $0 < \epsilon < \frac{1}{2}$ and $0 < c_1 < c_2 < c$. We first estimate solutions to (68) in the elliptic region $\mathcal{E} := \{|\xi'|_g \geq 1 + ch^\epsilon\}$.

Let $\chi_1 \in S_\epsilon(|\xi'|_g = 1)$ have $\chi_1 \equiv 1$ on $|\xi'|_g \geq \{1 + c_2h^\epsilon\}$ and $\text{supp } \chi_1 \subset \{|\xi'|_g \geq 1 + c_1h^\epsilon\}$. Also, let $\chi_2 \in S_\epsilon(|\xi'|_g = 1)$ have $\text{supp } \chi_2 \subset \{|\xi'|_g \geq 1 + c_2h^\epsilon\}$ and $\chi_2 \equiv 1$ on $\{|\xi'|_g \geq 1 + ch^\epsilon\}$. Let $X_1 = \text{Op}_h(\chi_1)$ and $X_2 = \text{Op}_h(\chi_2)$.

Let ψ solve (68). Then, we have

$$(I + GV)X_1\psi = [GV, X_1]\psi + X_1Gv.$$

Now, by Lemma 7.3, $GVX_1 = G_\Delta V X_1 + \mathcal{O}_{\Psi^{-\infty}}(h^\infty)$ where $G_\Delta \in h^{2/3}\Psi_{2/3}^{-1/2, -1}(|\xi'|_g = 1)$. By our assumptions on V and Lemma 4.3, there exists

$$A \in h^{\max(-2/3-\alpha, 0)}\Psi_{2/3}^{1/2, \min(0, 1-m)}(|\xi'|_g = 1) \cup \Psi_{2/3}^{0, \min(0, 1-m)}(|\xi'|_g = E')$$

so that $A(I + G_\Delta V) = X_2$ and $\text{MS}_h(A) \subset \{\chi_1 \equiv 1\}$. So,

$$X_2\psi = A[G_\Delta V, X_1]\psi + AX_1Gv + \mathcal{O}_{\Psi^{-\infty}}(h^\infty)(\psi + v)$$

and hence,

$$\begin{aligned} \|X_2\psi\|_{H_h^m} &\leq C(\|A[G_\Delta V, X_1]\psi\|_{L^2} + \|AX_1G_\Delta v\|_{H_h^m} + \mathcal{O}(h^\infty)(\|\psi\|_{H_h^{-N}} + \|v\|_{H_h^{-N}})) \\ &\leq C(h^{1-\epsilon/2}\|v\|_{L^2} + \mathcal{O}(h^\infty)\|\psi\|). \end{aligned}$$

Summarizing,

Lemma 9.1. *For all $0 < \epsilon < 1/2$, $c > 0$, and $N > 0$, there exists $h_0 = h_0(\epsilon, c) > 0$ such that for $0 < h < h_0$, $\chi \in S_\epsilon^{0,0}(|\xi'|_g = 1)$ with $\text{supp } \chi \subset \{|\xi'|_g \geq 1 + ch^\epsilon\}$, and ψ solving (68)*

$$\|\text{Op}_h(\chi)\psi\|_{H_h^m} \leq C(h^{1-\epsilon/2}\|v\|_{L^2} + \mathcal{O}(h^\infty)\|\psi\|_{H_h^{-N}}).$$

9.3. Hyperbolic Region. Recall from Lemma 7.3 that

$$G = G_\Delta + G_B + G_g + \mathcal{O}_{L^2 \rightarrow C^\infty}(h^\infty).$$

First suppose that $\text{MS}_h(X) \subset \{|\xi'|_g \leq 1 - ch^\epsilon\}$ for some $0 < \epsilon < 1/2$. Then, suppose that

$$(I + GV)X\psi = f$$

and let $G_\Delta^{-1/2}$ be a microlocal inverse for $G_\Delta^{1/2}$ on

$$\mathcal{H} := \{|\xi'|_g \leq 1 - r_{\mathcal{H}}h^\epsilon\}$$

where $r_{\mathcal{H}} \ll c$. Then

$$\begin{aligned} (I + GV)X_1\psi &= (I + (G_\Delta + G_B)V)X_1\psi + \mathcal{O}(h^\infty)\psi \\ &= (I + G_\Delta^{1/2}(I + G_\Delta^{-1/2}G_B G_\Delta^{-1/2})G_\Delta^{1/2}V)X_1\psi + \mathcal{O}(h^\infty)\psi = f. \end{aligned}$$

Thus, f is microlocalized on \mathcal{H} and, following the formal algebra in [Zal10, Section 2] multiplying by $G_\Delta^{1/2}V$, we have

$$G_\Delta^{1/2}V X_1\psi = -G_\Delta^{1/2}V G_\Delta^{1/2}(I + G_\Delta^{-1/2}G_B G_\Delta^{-1/2})G_\Delta^{1/2}V X_1\psi + \mathcal{O}(h^\infty)\psi + G_\Delta^{1/2}V f.$$

Remark 16. By Lemma 5.1, a microlocal inverse on \mathcal{H} will be a microlocal inverse on $\text{MS}_h(G_B X_1)$.

Writing $\varphi = G_\Delta^{1/2} V X_1 \psi$ and $T = G_\Delta^{-1/2} G_B G_\Delta^{-1/2}$, we have

$$(I + G_\Delta^{1/2} V G_\Delta^{1/2})\varphi = -G_\Delta^{1/2} V G_\Delta^{1/2} T\varphi + O(h^\infty)\psi + G_\Delta^{1/2} V f.$$

Hence, letting

$$R := -(I + G_\Delta^{1/2} V G_\Delta^{1/2})^{-1} G_\Delta^{1/2} V G_\Delta^{1/2},$$

we have

$$\varphi = RT\varphi + O(h^\infty)\psi - R G_\Delta^{-1/2} f.$$

Here, T is an FIO associated to the billiard map such that

$$\sigma\left(\exp\left(\frac{\text{Im } z}{h} \text{Op}_h(l(q, \beta(q)))\right) T\right)(\beta(q), q) = \exp\left(\frac{i \text{Re } \omega_0 l(\beta(q), q)}{h}\right) e^{-i\pi/4} dq^{1/2} \in S_\epsilon$$

and $R \in \Psi_\epsilon \cup \Psi_{2/3}^{0,0}(|\xi'|_g = E')$ is as in (18).

Thus by the wavefront set calculus we have for $N > 0$ independent of h ,

$$(69) \quad (I - (RT)^N)\varphi = O(h^\infty)\psi - \sum_{m=0}^{N-1} (RT)^m R G_\Delta^{-1/2} f$$

and by Egorov's theorem (Lemma 2.6), we have

$$(70) \quad (RT)_N := ((RT)^*)^N (RT)^N = \text{Op}_h(a_N) + O_{\Psi-\infty}(h^\infty)$$

where $a_N \in S_\epsilon \cup S_{2/3}^{0,0}(|\xi'|_g = E')$. Moreover, with $\delta = \max(2\epsilon, 2/3)$ for u with $\text{MS}_h(u) \subset \mathcal{H}$, by the Sharp Gårding inequality, Lemma 4.5, and Lemma 4.6,

$$\begin{aligned} \inf_{\mathcal{H}} \left(|\tilde{\sigma}((RT)_N)(q)| + O(h^{I_{(RT)_N}(q)+1-\delta}) \right) \|u\|_{L^2} &\leq \| (RT)^N u \|_{L^2}^2 \\ \| (RT)^N u \|_{L^2}^2 &\leq \sup_{\mathcal{H}} \left(|\tilde{\sigma}((RT)_N)(q)| + O(h^{I_{(RT)_N}(q)+1-\delta}) \right) \|u\|_{L^2}. \end{aligned}$$

Let

$$\varkappa_1 := 1 - \sqrt{\sup_{\mathcal{H}} \tilde{\sigma}((RT)_N)}}, \quad \varkappa_2 := \sqrt{\inf_{\mathcal{H}} \tilde{\sigma}((RT)_N)} - 1.$$

Finally, let $\varkappa = \max(\varkappa_1, \varkappa_2)$. Then, we have

Lemma 9.2. *Suppose that $\varkappa > h^{\gamma_1}$ where $\gamma_1 < \min(1/2 - \epsilon, 1/6)$. Let $c > r_{\mathcal{H}}$ and $g \in L^2$ have $\text{MS}_h(g) \subset \{1 - Ch^\epsilon \leq |\xi'|_g \leq 1 - ch^\epsilon\}$. If*

$$(I - (RT)^N)u = g,$$

then for any $\delta > 0$,

$$\text{MS}_h(u) \subset \{1 - (C + \delta)h^\epsilon \leq |\xi'|_g \leq 1 - (c - \delta)h^\epsilon\}.$$

In particular, there exists an operator A with $\|A\|_{L^2 \rightarrow L^2} \leq 2\varkappa^{-1}$,

$$A(I - (RT)^N) = I \text{ microlocally on } \mathcal{H}$$

and if $\text{MS}_h(g) \subset \{1 - Ch^\epsilon \leq |\xi'|_g \leq 1 - ch^\epsilon\}$, then

$$\text{MS}_h(Ag) \subset \{1 - (C + \delta)h^\epsilon \leq |\xi'|_g \leq 1 - (c - \delta)h^\epsilon\}.$$

Proof. In the case that $\varkappa_2 > h^{\gamma_1}$, we write

$$(I - (RT)^N) = -(RT)^N(I - (RT)^{-N})$$

microlocally on \mathcal{H} and invert by Neumann series to see that for any g , $(I - (RT)^N)u = g$ has a unique solution modulo h^∞ with $\|u\| \leq \varkappa^{-1}\|g\|$. On the other hand, if $\varkappa_1 > h^{\gamma_1}$, $\|(RT)^N\| \leq 1 - \varkappa_1$, and we have that for any g , $(I - (RT)^N)u = g$ has a unique solution with $\|u\| \leq \varkappa_1^{-1}\|g\|$.

We will consider the case of $\varkappa_1 > h^{\gamma_1}$, the case of $\varkappa_2 < h^{\gamma_1}$ being similar with $(RT)^N$ replace by $(RT)^{-N}$. Inversion by Neumann series already shows that we can solve $(I - (RT)^N)u_1 = g$ with $\|u_1\| \leq \varkappa^{-1}\|g\|$. To complete the proof of the lemma, we need to show that this inverse has the required microsupport property. For this, we need a fine almost invariance result near the glancing set. In particular, by Lemma 7.1, that there exists an approximate first integral $\Xi(x, \xi) \in C^\infty(\overline{B^*\partial\Omega})$ so that $\Xi = 0$, $|d\Xi| > 0$ on $S^*\partial\Omega$, $\Xi < 0$ in $B^*\partial\Omega$ and

$$(71) \quad \Xi(\beta(q)) - \Xi(q) = r(q)$$

with $r(q) \in C^\infty(B^*\partial\Omega)$ vanishing to infinite order at $S^*\partial\Omega$. (See also [KP90, MM82, PV99b]) In particular, we have that in neighborhood of $S^*\partial\Omega$,

$$\Xi(x', \xi') = e(x', \xi)(|\xi'|_g^2 - 1)$$

with $e > c > 0$.

For $k \geq 1$, let $\chi_k = \chi_k(\zeta)$ with $\chi_{k+1} \equiv 1$ on $\text{supp } \chi_k$ and $\chi_1 \equiv 1$ on $\text{MS}_h(g)$ so that

$$\text{supp } \chi_k \subset \{1 - (C + \delta)h^\epsilon \leq |\xi'|_g \leq 1 - (c - \delta)h^\epsilon\}.$$

Let $X_k = \text{Op}_h(\chi_k)$. Finally, let $\chi_\infty \in S_\epsilon$ with $\chi_\infty \equiv 1$ on $\bigcup_k \text{supp } \chi_k$ and

$$\text{supp } \chi_\infty \subset \{1 - (C + 2\delta)h^\epsilon \leq |\xi'|_g \leq 1 - (c - 2\delta)h^\epsilon\}.$$

Then (71) implies that

$$|\chi_k(\beta(q)) - \chi_k(q)| = o(h^\infty).$$

Suppose that u is the unique solution of

$$(I - (RT)^N)u = g.$$

We will show that u is microlocalized as described in the lemma. Letting $u_1 = u$, we have

$$(I - (RT)^N)X_1u_1 = g + o(h^\infty)g + [X_1, (RT)^N]X_\infty u_1 =: g + g_1.$$

Let $\delta = \max(2\epsilon, 2/3)$. Then

$$[X_1, T] = T(T^{-1}X_1T - X_1) = Th^{1-\delta}B$$

with $B \in \Psi_\epsilon$. In fact,

$$(72) \quad T^{-1}X_1T = \text{Op}_h(\chi_1(\beta(q)) + \mathcal{O}_{\Psi_\epsilon}(h^{1-2\epsilon})).$$

Hence, since $X_\infty u$ is microlocalized h^ϵ close to glancing,

$$\text{MS}_h([X_1, (RT)^N]X_\infty u_1) \subset \{\chi_2 \equiv 1\}$$

and $g_1 := [X_1, (RT)^N]X_\infty u_1$ has

$$\|g_1\| \leq Ch^{1-\delta}\varkappa^{-1}\|g\|_{L^2}.$$

Now, let u_2 have

$$(I - (RT)^N)u_2 = -g_1, \quad \|u_2\| \leq \varkappa^{-1}\|g_1\| \leq Ch^{1-\delta}\varkappa^{-2}\|g\|$$

So,

$$(I - (RT)^N)(X_1u + u_2) = g + o(h^\infty)g.$$

Continuing in this way, let

$$(I - (RT)^N)u_k = -g_{k-1}, \quad g_{k-1} = [X_{k-1}, (RT)^N]X_\infty u_{k-1}.$$

Then,

$$\|u_k\| \leq \varkappa^{-2k}(h^{k(1-\delta)})\|g\|_{L^2}.$$

Moreover, letting $\tilde{u} \sim \sum_k X_k u_k$, we have $X_\infty \tilde{u} = \tilde{u} + o(h^\infty)\tilde{u}$ and

$$(I - (RT)^N)\tilde{u} = g + o(h^\infty)g$$

which implies $\tilde{u} - u = o(h^\infty)$ and hence that $(I - (RT)^N)$ has a microlocal inverse, A , with the properties claimed in the lemma. \square

We now suppose that ψ solves (68) and use (69) to obtain estimates on ψ . Let $\chi_k \in S_\epsilon$ with $\chi_k \equiv 1$ on $\{|\xi'|_g \leq 1 - 2kch^\epsilon\}$ and $\text{supp } \chi_k \subset \{|\xi'|_g \leq 1 - (2k-1)ch^\epsilon\}$. Then

$$(I + GV)X_1\psi = -[X_1, GV]\psi + X_1Gv =: \psi_1 + \tilde{v}$$

where $\text{MS}_h(\psi_1) \subset \mathcal{H} \cap \{|\xi'|_g \geq 1 - 3c/2h^\epsilon\}$. Then with $\varphi = G_\Delta^{1/2}VX_1\psi$,

$$(I - (RT)^N)\varphi = o(h^\infty)\psi - \sum_{m=0}^{N-1} (RT)^m RG_\Delta^{-1/2}(\psi_1 + \tilde{v})$$

and hence by Lemma 9.2, when $\varkappa \geq h^{\gamma_1}$ for $\gamma_1 < \min(1/2 - \epsilon, 1/6)$,

$$\varphi = o(h^\infty)\psi - \sum_{m=0}^{N_1} A(RT)^m RG_\Delta^{-1/2}(\psi_1 + \tilde{v})$$

and, using the microsupport statement from Lemma 9.2,

$$X_2\varphi = - \sum_{m=0}^{N-1} A(RT)^m RG_\Delta^{-1/2}\tilde{v} + o_{\Psi^{-\infty}}(h^\infty)(\psi + v).$$

Hence,

$$\begin{aligned} \|X_2\varphi\|_{L^2} &\leq \varkappa^{-1} \left\| \sum_{m=0}^{N-1} (RT)^m RG_\Delta^{-1/2}X_1Gv \right\| + o(h^\infty)(\|\psi\| + \|v\|) \\ &\leq C\varkappa^{-1}e^{ND_\Omega(\text{Im } z) - /h} h^{1/2 - \epsilon/2} \|v\| + o(h^\infty)\|\psi\|. \end{aligned}$$

Then, since $\varphi = G_\Delta^{1/2}VX_1\psi$, $VX_1\psi = G_\Delta^{-1/2}\varphi + o(h^\infty)\psi$ and

$$\begin{aligned} X_3\psi &= -X_3GV\psi + X_3Gv = -X_3GVX_1\psi + X_3Gv + o(h^\infty)\psi \\ &= -X_3GG_\Delta^{-1/2}\varphi + X_3Gv + o(h^\infty)\psi = -X_3GG_\Delta^{-1/2}X_2\varphi + X_3Gv + o(h^\infty)\psi. \end{aligned}$$

Hence,

$$\begin{aligned} \|X_3\psi\| &\leq \|X_3GG_{\Delta}^{-1/2}X_2\varphi\| + \|X_3Gv\| + O(h^\infty)\|\psi\| \\ &\leq C(\varkappa^{-1}h^{1-\epsilon}e^{(N+1)D_{\Omega}(\text{Im } z)-/h}\|v\| + O(h^\infty)\|\psi\|) \end{aligned}$$

Next, we examine when $\varkappa \geq ch^{\gamma_1}$. If this is not the case, then

$$\liminf_{h \rightarrow 0} \frac{\inf |\tilde{\sigma}((RT)_N)(q)| - 1|}{h^{\gamma_1}} = 0.$$

So, let

$$|\tilde{\sigma}(RT)_N(q)| = e^{e(q)}.$$

Taking logs and renormalizing we have

$$\frac{2\text{Im } z}{h}Nl_N(q) - \frac{2\text{Im } z}{h}Nl_N(q) + \log |\tilde{\sigma}((RT)_N)(q)| = e(q).$$

This implies

$$\begin{aligned} -\frac{\text{Im } z}{h} &= -l_N^{-1}(q) \left[\frac{\text{Im } z}{h}l_N(q) + \frac{1}{2N} \log |\tilde{\sigma}((RT)_N)(q)| + e(q) \right] \\ &= -l_N^{-1}(q)(r_N(q) + e(q)). \end{aligned}$$

where r_N as in (21). Thus, if $\varkappa \leq ch^{\gamma_1}$, for any $c > 0$,

$$\inf_{\mathcal{H}} -l_N^{-1}(r_N + ch^{\gamma_1}) \leq -\frac{\text{Im } z}{h} \leq \sup_{\mathcal{H}} -l_N^{-1}(r_N - ch^{\gamma_1}).$$

Now, writing

$$RT = \left[R \exp \left(-\frac{\text{Im } z}{h} \text{Op}_h(l(q), \beta(q)) \right) \right] \left[\exp \left(\frac{\text{Im } z}{h} \text{Op}_h(l(q), \beta(q)) \right) T \right]$$

and applying Lemma 3.1 shows that

$$\begin{aligned} r_N(q) &:= \tilde{\sigma}((RT)_N)(q) \\ &= \exp \left(-\frac{2\text{Im } z}{h} \sum_{n=0}^{N-1} l(\beta^n(q), \beta^{n+1}(q)) \right) \prod_{i=1}^N \left(|\tilde{\sigma}(R)(\beta^i(q))|^2 + O(h^{I_R(\beta^i(q))+1-2\epsilon}) \right). \end{aligned}$$

Summarizing the discussion, we have

Lemma 9.3. *Let $0 < \epsilon < 1/2$, $\gamma_1 < \min(1/2 - \epsilon, 1/6)$, $c > 0$, $M > 0$ and suppose that $\chi \equiv 1$ on $\{|\xi'|_g \leq 1 - Ch^\epsilon\}$ and $\text{supp } \chi \subset \{|\xi'| \leq 1 - ch^\epsilon\}$. Suppose further that ψ solves (68). Then there exists $h_0 > 0$ small enough so that if $0 < h < h_0$ and*

$$(73) \quad -\frac{\text{Im } z}{h} < \inf_{\mathcal{H}} -l_N^{-1}(r_N + ch^{\gamma_1}) \text{ or } -\frac{\text{Im } z}{h} > \sup_{\mathcal{H}} -l_N^{-1}(r_N - ch^{\gamma_1}),$$

where l_N and r_N are as in (20) and (21) respectively, then

$$(74) \quad \|\text{Op}_h(\chi)\psi\|_{L^2} \leq C(h^{1-\epsilon-\gamma_1}e^{(N+1)D_{\Omega}(\text{Im } z)-/h}\|v\|_{L^2} + O(h^\infty)\|\psi\|_{H_h^{-M}})$$

9.4. Glancing Region. Let $\chi \in S_\epsilon(|\xi'|_g = 1)$ with $\chi \equiv 1$ on $\{||\xi'|_g - 1| \leq ch^\epsilon\}$ and $\text{supp } \chi \subset \{||\xi'|_g - 1| \leq Ch^\epsilon\}$. Then

$$(I + GV) \text{Op}_h(\chi)\psi = [GV, \text{Op}_h(\chi)]\psi + \text{Op}_h(\chi)Gv.$$

Let φ_i be a partition of unity on $S^*\partial\Omega$. We then use the microlocal model for G near glancing.

$$\sum_i (I + h^{2/3}J_i\omega^{-1}\mathcal{A}_-\mathcal{A}_iC^{-1}J_i^{-1}V)\varphi_i \text{Op}_h(\chi)\psi = \text{Op}_h(\chi)Gv + [GV, \text{Op}_h(\chi)]\psi + o(h^\infty)(\psi).$$

First, observe that if $\alpha > -2/3$, then our model shows that $(I + GV)$ is an elliptic pseudodifferential operator on $\text{supp } \chi$ and hence

Lemma 9.4. *Suppose that $\alpha > -2/3$. Then under the assumptions of Lemma 9.3, there exists $N > 0$ so that*

$$\|\psi\|_{L^2} \leq Ch^{-N}\|v\|_{L^2}.$$

Throughout the rest of our analysis near glancing, it will be convenient to use Ξ from Lemma 7.1. Then

$$\Xi(x', \xi') := (|\xi'|_g^2 - 1)(2Q(x', \xi'))^{-2/3} + o((|\xi'|_g^2 - 1)^2).$$

Moreover, $\xi_1(\kappa^{-1}(x', \xi')) = \Xi(x', \xi') + o((|\xi'|_g^2 - 1)^\infty)$ where κ is the symplectomorphism (55) reducing the billiard ball map for the Friedlander model to that for Ω near $(x', \xi') \in S^*\partial\Omega$. In particular, notice that if $\chi \in S_\epsilon^{0,0}(\xi_1 = 1)$ with $\text{supp } \chi \subset \{ah_1^\epsilon \leq 1 - |\xi'|_g^2 \leq bh^\epsilon\}$, then

$$\begin{aligned} \sigma(J_i \text{Op}_h(\chi(\Xi))J_i^{-1}) &= \chi(\xi_1) \\ \text{MS}_h(i\text{Op}_h(\chi(\Xi))J_i^{-1}) &\subset \{ah_1^\epsilon \leq \xi_1 \leq bh^{\epsilon_2}\}. \end{aligned}$$

Now, the assumption that on $|\xi'|_g - 1 > Mh^{2/3}$,

$$\left| 1 + \frac{h\sigma(V)}{2\sqrt{|\xi'|_g^2 - 1}} \right| \geq \delta \left\langle \frac{h^{1+\alpha}}{\sqrt{|\xi'|_g^2 - 1}} \right\rangle$$

(see (65)) together with Lemma 4.3 and (60) imply that $I + GV$ is microlocally invertible on $|\xi'|_g \geq 1 + Mh^{2/3}$.

When $\alpha < -2/3$, we can localize further. In particular, fix $M_1 > 0$. Then since V is elliptic and $\alpha < -2/3$, $I + GV$ is an elliptic pseudodifferential operator when for some $\delta > 0$ and all $1 \leq j \leq M_1$,

$$|h^{-2/3}\Xi(x', \xi') + \zeta_j| \geq \delta, \quad h^{-2/3}\Xi(x', \xi') + \zeta_{M_1+1} \geq \delta$$

So, there exists $C > 0$ such that, letting $\chi_2 \in S_{2/3}(|\xi'|_g = 0)$ have $\text{supp } \chi_2 \subset |\xi_1| \leq Ch^\epsilon$ and

$$(75) \quad \chi_2 \equiv 1 \text{ on } \begin{cases} |\xi_1| \leq CMh^{2/3} & \alpha = -2/3 \\ |\xi_1 h^{-2/3} + \zeta_j| \leq \delta, Ch^\epsilon \leq \xi_1 \leq h^{2/3}\zeta_{M_1} + \delta h^{2/3} & \alpha < -2/3, \end{cases}$$

we have

Lemma 9.5. *Let χ_2 be as in (75). Then*

$$\|(1 - \text{Op}_h(\chi_2(\Xi))) \text{Op}_h(\chi_1)\psi\| \leq Ch^{-1/3+\epsilon/2-\alpha}(\|\text{Op}_h(\chi)Gv\| + \|[GV, \text{Op}_h(\chi)]\psi\| + o(h^\infty)\|\psi\|$$

and hence, under the assumptions of Lemma 9.3,

$$\|(1 - \text{Op}_h(\chi_2(\Xi))) \text{Op}_h(\chi_1)\psi\| \leq Ch^{-1/3+\epsilon/2}(h^{2/3} + h^{1-\epsilon-\gamma_1}e^{(N+1)D_\Omega(\text{Im } z)-/h})\|v\| + o(h^\infty)\|\psi\|$$

9.4.1. *Flux formula.* With χ_2 as in (75), define

$$\psi_{ng} := (1 - \text{Op}_h(\chi_2(\Xi))) \text{Op}_h(\chi_1)\psi.$$

and $\psi_g := \psi - \psi_{ng}$.

By an integration by parts, we have for a solution u to (67),

$$(76) \quad \left(\frac{2 \text{Re } z \text{Im } z}{h} \|u\|_{L^2}^2 - \text{Im} \langle B\psi, \psi \rangle \right) = -\text{Im} \langle hv, \psi \rangle.$$

On the other hand,

$$(77) \quad u = h^{-1} \mathcal{S} \ell h \partial_\nu u - \mathcal{D} \ell u = -(h^{-1} \mathcal{S} \ell B + \mathcal{D} \ell) \psi + \mathcal{S} \ell v = -\mathcal{S} \ell V \psi + \mathcal{S} \ell v$$

Since we already have estimates for ψ_{ng} , we write

$$u = (-\mathcal{S} \ell V \psi_g) + (\mathcal{S} \ell(v - V \psi_{ng})) =: u_g + u_{ng}.$$

Now, [HT15, Theorem 1.1] together with an application of the Phragmén Lindelöf principle implies

$$\begin{aligned} \|\mathcal{S} \ell(v - V \psi_{ng})\| &= \|u_{ng}\| \leq h^{5/6} e^{D_\Omega(\text{Im } z)-/h} (\|v\| + h^\alpha \|\psi_{ng}\|_{H_h^m}) \\ \|\mathcal{S} \ell V \psi_g\| &= \|u_g\| \leq Ch^{5/6+\alpha} e^{D_\Omega(\text{Im } z)-/h} \|\psi_g\| \end{aligned}$$

Then,

$$\begin{aligned} \|u\|^2 - \|u_g\|^2 &= 2 \text{Re} \langle u_g, u_{ng} \rangle + \|u_{ng}\|^2 \\ &\leq \delta \|u_g\|^2 + (1 + 2\delta^{-1}) \|u_{ng}\|^2 \\ &\leq C\delta h^{5/3+2\alpha} e^{2D_\Omega(\text{Im } z)-/h} \|\psi_g\|^2 + (1 + 2\delta^{-1}) \|u_{ng}\|^2 \end{aligned}$$

$$\begin{aligned} |\langle B\psi, \psi \rangle - \langle B\psi_g, \psi_g \rangle| &= |\langle B\psi_g, \psi_{ng} \rangle + \langle B\psi_{ng}, \psi_g \rangle + \langle B\psi_{ng}, \psi_{ng} \rangle| \\ &\leq C(\delta \|\psi_g\|^2 + C(1 + \delta^{-1}) \|\psi_{ng}\|_{H_h^m}^2). \end{aligned}$$

Now, rewrite (76) as

$$\frac{2 \text{Re } z \text{Im } z}{h} \|u_g\|^2 - \text{Im} \langle B\psi_g, \psi_g \rangle = \text{Im} \langle hv, \psi \rangle + \frac{2 \text{Re } z \text{Im } z}{h} (\|u_g\|^2 - \|u\|^2) + \text{Im}(\langle B\psi_g, \psi_g \rangle - \langle B\psi, \psi \rangle).$$

Plugging our estimates in together gives

$$\begin{aligned}
(78) \quad & \left| \frac{2 \operatorname{Re} z \operatorname{Im} z}{h} \|u_g\|^2 + \operatorname{Im} \langle -hN_2 \psi_g, \psi_g \rangle + \langle -h \operatorname{Im} V \psi_g, \psi_g \rangle \right| \\
& \leq Ch(\delta_1^{-1} \|v\|^2 + \delta_1 \|\psi\|^2) + C |\operatorname{Im} z| h^{-1} (\delta_2 h^{5/3+2\alpha} e^{2D_\Omega(\operatorname{Im} z)_- / h} \|\psi_g\|^2 + (1 + \delta_2^{-1}) \|u_{ng}\|^2) \\
& \quad + C(\delta_3 \|\psi_g\|^2 + (1 + \delta_3^{-1}) \|\psi_{ng}\|_{H_h^m}^2) \\
& \leq C(\delta_1 h + |\operatorname{Im} z| h^{2/3+2\alpha} e^{2D_\Omega(\operatorname{Im} z)_- / h} \delta_2 + \delta_3) \|\psi_g\|_{L^2}^2 \\
& \quad + C(h\delta_1 + \delta_3^{-1} + (1 + \delta_2^{-1}) |\operatorname{Im} z| h^{2/3+2\alpha} e^{2D_\Omega(\operatorname{Im} z)_- / h}) \|\psi_{ng}\|_{H_h^m}^2 \\
& \quad + C(h\delta_1^{-1} + \delta_2^{-1} |\operatorname{Im} z| h^{2/3+2\alpha} e^{2D_\Omega(\operatorname{Im} z)_- / h}) \|v\|_{L^2}^2
\end{aligned}$$

In particular, we have

Lemma 9.6. *For all $\gamma_1 \in \mathbb{R}$, $c > 0$, there exists $C > 0$ so that if*

$$(79) \quad \left| \frac{2 \operatorname{Re} z \operatorname{Im} z}{h} \|u_g\|^2 + \operatorname{Im} \langle -hN_2 \psi_g, \psi_g \rangle + \langle -h \operatorname{Im} V \psi_g, \psi_g \rangle \right| \geq ch^{\gamma_1} \|\psi_g\|^2.$$

then

$$\begin{aligned}
\|\psi_g\|^2 & \leq C(h^{\gamma_1} + h^{-\gamma_1} + (1 + |\operatorname{Im} z| h^{2/3+2\alpha-\gamma_1} e^{2D_\Omega(\operatorname{Im} z)_- / h}) |\operatorname{Im} z| h^{2/3+2\alpha} e^{2D_\Omega(\operatorname{Im} z)_- / h}) \|\psi_{ng}\|_{H_h^m}^2 \\
& \quad + C(h^{2-\gamma_1} + |\operatorname{Im} z|^2 h^{4/3+2\alpha-\gamma_1} e^{4D_\Omega(\operatorname{Im} z)_- / h}) \|v\|_{L^2}^2
\end{aligned}$$

9.4.2. *Estimates on the glancing set.* We now obtain estimates of the form (79) using the description of the single and double layer potentials from section 7. First, observe that

$$\|u_g\|_{L^2(\Omega)}^2 = \langle \mathcal{B} \psi_g, \psi_g \rangle_{L^2(\partial\Omega)}$$

where by Lemma 7.4

$$\mathcal{B} := V^* \mathcal{S} \ell^* \mathcal{S} \ell V \in h^{2+2\alpha} \Psi_{1-\epsilon/2}(|\xi'|_g = 1)$$

is elliptic and has symbol given by

$$\sigma(\mathcal{B}) = \frac{|\sigma(hV)|^2}{2Q} \left(\Psi_{\mathcal{S}\ell}(\alpha_0 h) \circ \kappa^{-1} \right).$$

Take $\epsilon, \epsilon_1 > 0$ small enough and let

$$(80) \quad \mathcal{L}_\alpha := \begin{cases} \{ \left| |\xi'|_g - 1 \right| \leq h^\epsilon, |\Xi + h^{2/3} \zeta_j| < \epsilon_1 h^{2/3} \text{ or } \Xi \leq -M_1 h^{2/3} \} & \alpha < -2/3 \\ \{ \left| |\xi'|_g - 1 \right| \leq CM h^{2/3} \} & \alpha \geq -2/3 \end{cases}$$

where C and M are as in (75).

Now, define

$$\frac{2 \operatorname{Re} z \operatorname{Im} z}{h} \|u_g\|^2 + \operatorname{Im} \langle -hN_2 \psi_g, \psi_g \rangle + \langle -h \operatorname{Im} V \psi_g, \psi_g \rangle = \langle A \psi_g, \psi_g \rangle$$

where

$$A := \frac{2 \operatorname{Re} z \operatorname{Im} z}{h} \mathcal{B} - \operatorname{Im}(hN_2 + hV_2).$$

Then, applying the Sharp Gårding inequality (see Lemma 4.5) along with bounds on the norm of pseudodifferential operators (see Lemma 4.6), we obtain

$$(81) \quad \inf_{\mathcal{L}_\alpha} \left(\frac{2 \operatorname{Re} z \operatorname{Im} z}{h} \frac{|\sigma(hV)|^2}{2Q} \Psi_{S\ell}(h^{-2/3}\Xi)(1 + O(h^{\epsilon/2})) \right. \\ \left. - h(\operatorname{Im} \sigma(N_2) + \sigma(\operatorname{Im} V)) - ch^{1/3+\epsilon/2} - ch^{4/3+\alpha} \right) \|\psi_g\|^2 \leq \langle A\psi_g, \psi_g \rangle$$

$$(82) \quad \langle A\psi_g, \psi_g \rangle \leq \sup_{\mathcal{L}_\alpha} \left(\frac{2 \operatorname{Re} z \operatorname{Im} z}{h} \frac{|\sigma(hV)|^2}{2Q} \Psi_{S\ell}(h^{-2/3}\Xi)(1 + O(h^{\epsilon/2})) \right. \\ \left. - h(\operatorname{Im} \sigma(N_2) + \sigma(\operatorname{Im} V)) + ch^{1/3+\epsilon/2} + ch^{4/3+\alpha} \right) \|\psi_g\|^2$$

Notice that for all $\delta > 0$, there exists M_1 large enough and ϵ_1 small enough so that

$$1 - \delta \leq \Psi_{S\ell}(h^{-2/3}\Xi) \leq 1 + \delta, \quad (x, \xi) \in \mathcal{L}_\alpha \quad (\alpha < -2/3).$$

So, we have

Lemma 9.7. *For all $\delta > 0$ there exists $h_0 > 0$, $N, M > 0$, $C, c > 0$ such that for $0 < h < h_0$ if $\pm \operatorname{Im} z \geq 0$ and one of the following holds*

$$(83) \quad \frac{-\operatorname{Im} z}{h} \leq \inf_{\mathcal{L}_\alpha} - \frac{h(\operatorname{Im} \sigma(N_2) + \sigma(\operatorname{Im} V) + c(h^{1/3+\alpha} + h^{-1/3+\epsilon/2}))Q}{|\sigma(hV)|^2 \Psi_{S\ell}(h^{-2/3}\Xi)} (1 \pm \delta) \\ \frac{-\operatorname{Im} z}{h} \geq \sup_{\mathcal{L}_\alpha} - \frac{h(\operatorname{Im} \sigma(N_2) + \sigma(\operatorname{Im} V) - c(h^{1/3+\alpha} + h^{-1/3+\epsilon/2}))Q}{|\sigma(hV)|^2 \Psi_{S\ell}(h^{-2/3}\Xi)} (1 \mp \delta)$$

then

$$(84) \quad \|\psi_g\|_{L^2} \leq Ch^{-N} (\|v\|_{L^2} + \|\psi_{ng}\|_{H_h^m}) + O(h^\infty) \|\psi\|_{H_h^{-M}}.$$

If $\alpha < -2/3$, we can replace the conditions (83) with

$$\frac{-\operatorname{Im} z}{h} \leq \inf_{\mathcal{L}_\alpha} - \frac{h(\operatorname{Im} \sigma(N_2) + \sigma(\operatorname{Im} V) + c(h^{1/3+\alpha} + h^{-1/3+\epsilon/2}))Q}{|\sigma(hV)|^2} (1 \pm \delta) \\ \frac{-\operatorname{Im} z}{h} \geq \sup_{\mathcal{L}_\alpha} - \frac{h(\operatorname{Im} \sigma(N_2) + \sigma(\operatorname{Im} V) - c(h^{1/3+\alpha} + h^{-1/3+\epsilon/2}))Q}{|\sigma(hV)|^2} (1 \mp \delta).$$

9.5. Further localization away from the real axis when $\alpha < -2/3$. We now focus our attention on the region $|\operatorname{Im} z| \geq ch^N$ for some $N > 0$ and $\alpha < -2/3$. In this region, we are able to decompose $\psi = u|_{\partial\Omega}$ into pieces, ψ_j , concentrating at $\Xi = \zeta_j h^{2/3}$, that still have

$$(I + GV)\psi_j = Gv_j$$

with the norm of v_j controlled by the norm of v .

We again use the representation of G near glancing. With χ and φ_i as above

$$(I + \sum_i h^{2/3} J_i \omega^{-1} \mathcal{A}_i \mathcal{A}_i C^{-1} J_i^{-1} V \varphi_i) \operatorname{Op}_h(\chi) \psi = \operatorname{Op}_h(\chi) Gv + [GV, \operatorname{Op}_h(\chi)] \psi + O(h^\infty) \psi.$$

Fix $\epsilon_1 > 0$ small enough and let $\chi_j \equiv 1$ on $|\xi_1 + \zeta_j| \leq \epsilon_1 h^{2/3}$ with $\text{supp } \chi_j \subset |\xi_1 + \zeta_j| \leq 2\epsilon_1 h^{2/3}$ and let $L_j = \text{Op}_h(\chi_j(\Xi))$. Then

$$\begin{aligned} \sum_i (I + h^{2/3} J_i \omega^{-1} \mathcal{A}_i \mathcal{A}_i C^{-1} J_i^{-1} V) \varphi_i L_j \text{Op}_h(\chi) \psi = \\ L_j \text{Op}_h(\chi) G v + L_j [GV, \text{Op}_h(\chi)] \psi + [GV, L_j] \text{Op}_h(\chi) \psi + o(h^\infty) \psi \end{aligned}$$

Now, $[GV, L_j]$ is a pseudodifferential operator with support on the complement of \mathcal{L}_α . Therefore by Lemma 9.5 there exists $M > 0$ so that

$$\|[GV, L_j] \text{Op}_h(\chi) \psi\| \leq h^{-M} \|v\| + o(h^\infty) \|\psi\|.$$

So,

$$(I + GV) L_j \text{Op}_h(\chi) \psi = w$$

with

$$\|w\| \leq h^{-M} \|v\| + o(h^\infty) \|\psi\|.$$

Now, $G^{-1} = N_1 + N_2$ and since $|\text{Im } z| \leq M h \log h^{-1}$,

$$\|h(N_1 + N_2)\|_{H_h^1 \rightarrow L^2} \leq \frac{C}{|\text{Im } z|}.$$

Hence, using that $|\text{Im } z| \geq ch^N$, we have

$$(I + GV) L_j \text{Op}_h(\chi) \psi = G G^{-1} w = G(N_1 + N_2) w =: G v_j$$

so that for some $M > 0$,

$$\|v_j\| \leq h^{-M} \|v\| + o(h^\infty) \|\psi\|.$$

So, formulas (76) and (77) hold with ψ replaced by $L_j \text{Op}_h(\chi) \psi$ and v replaced by v_j . Let $\psi_j = L_j \text{Op}_h(\chi) \psi$,

$$\mathcal{L}_j := \{|\Xi(x', \xi) + h^{2/3} \zeta_j| < 2\epsilon_1 h^{2/3}\},$$

and u_j be the solution to

$$\begin{cases} (-h^2 \Delta - z^2) u_j = 0 & \text{in } \Omega \\ (h \partial_\nu + B) u_j = v_j & \text{on } \partial \Omega \\ u_j|_{\partial \Omega} = \psi_j \end{cases}$$

Next, fix $\delta > 0$ and take ϵ_1 small enough, $\pm \text{Im } z \geq 0$. Then following the arguments above,

$$(85) \quad \inf_{\mathcal{L}_j} \left(\frac{2 \text{Im } z}{h(1 \pm \delta)} \frac{|\sigma(hV)|^2}{2Q} - h(\text{Im } \sigma(N_2) + \sigma(\text{Im } V)) - ch^{1/3+\epsilon/2} - ch^{4/3+\alpha} \right) \|\psi_j\|^2 \leq \langle A \psi_j, \psi_j \rangle$$

$$(86) \quad \langle A \psi_j, \psi_j \rangle \leq$$

$$\sup_{\mathcal{L}_j} \left(\frac{2 \text{Im } z}{h(1 \mp \delta)} \frac{|\sigma(hV)|^2}{2Q} - h(\text{Im } \sigma(N_2) + \sigma(\text{Im } V)) + ch^{1/3+\epsilon/2} + ch^{4/3+\alpha} \right) \|\psi_j\|^2$$

and

$$(87) \quad |\langle A \psi_j, \psi_j \rangle| \leq C(\delta^{-1} \|v_j\| + \delta \|\psi_j\|).$$

In particular, using that

$$\sigma(hN_2) = (2hQ)^{1/3} \frac{A'_-(h^{-2/3}\Xi)}{A_-(h^{-2/3}\Xi)},$$

we have

Lemma 9.8. *Suppose that $\pm \operatorname{Im} z \geq ch^M$, $\alpha < -2/3$. Fix $j > 0$. Then there exist $h_0 > 0$, $N, C > 0$ such that if one of the following holds*

$$\begin{aligned} \frac{-\operatorname{Im} z}{h} &\leq \inf_{\mathcal{L}_j} - \frac{h(h^{-2/3}(2Q)^{1/3} \operatorname{Im} \frac{A'_-(-\zeta_j)}{A_-(-\zeta_j)} + \sigma(\operatorname{Im} V) + ch^{1/3+\alpha})Q}{|\sigma(hV)|^2} (1 \pm \delta) \\ \frac{-\operatorname{Im} z}{h} &\geq \sup_{\mathcal{L}_j} - \frac{h(h^{-2/3}(2Q)^{1/3} \operatorname{Im} \frac{A'_-(-\zeta_j)}{A_-(-\zeta_j)} + \sigma(\operatorname{Im} V) - ch^{1/3+\alpha})Q}{|\sigma(hV)|^2} (1 \mp \delta) \end{aligned}$$

then

$$\|\psi_j\| \leq Ch^{-N} \|v\| + O(h^\infty) \|\psi\|.$$

With these estimates in hand, for any $M > 0$, let

$$(88) \quad \mathcal{L}'_M := \{-2h^\epsilon \leq \Xi \leq (-\zeta_{M+1} + 2\epsilon)h^{2/3}\}.$$

and let $\chi'_2 = \chi'_2(\xi_1) \in S_{2/3}$ have $\chi_2 \equiv 1$ on

$$\{-h^\epsilon \leq \xi_1 \leq (-\zeta_{M+1} + \epsilon)h^{2/3}\}$$

and $\operatorname{supp} \chi_2 \subset \mathcal{L}'_M$. Then define

$$\psi'_g = \operatorname{Op}_h(\chi_2(\Xi)) \operatorname{Op}_h(\chi_1) \psi$$

and $\psi'_{ng} = \psi - \psi'_g$. Then (81) and (82) still hold with \mathcal{L} replaced by \mathcal{L}'_M and we have

Lemma 9.9. *For all $\delta > 0$ there exists $h_0 > 0$, $N, M > 0$, $C > 0$ such that for $0 < h < h_0$ if $\pm \operatorname{Im} z \geq 0$ and one of the following holds*

$$\begin{aligned} \frac{-\operatorname{Im} z}{h} &\leq \inf_{\mathcal{L}'_M} - \frac{h(\operatorname{Im} \sigma(N_2) + \sigma(\operatorname{Im} V) + ch^{1/3+\alpha})Q}{|\sigma(hV)|^2} (1 \pm \delta) \\ \frac{-\operatorname{Im} z}{h} &\geq \sup_{\mathcal{L}'_M} - \frac{h(\operatorname{Im} \sigma(N_2) + \sigma(\operatorname{Im} V) - ch^{1/3+\alpha})Q}{|\sigma(hV)|^2} (1 \mp \delta) \end{aligned}$$

then

$$\|\psi'_g\|_{L^2} \leq Ch^{-N} (\|v\|_{L^2} + \|\psi'_{ng}\|_{H_h^m}) + O(h^\infty) \|\psi\|_{H_h^{-N}}.$$

So, combining Lemmas 9.1, 9.3, 9.4, 9.5, 9.7, 9.8, and 9.9 gives

Theorem 5. *Let ψ be a solution to (68). Fix $\delta > 0$, $0 < \epsilon < 1/2$, $\gamma_1 < \min(\frac{1}{2} - \epsilon, \frac{1}{6})$, $M_1, M_2 > 0$. Then there exists $h_0 > 0$ and $N > 0$ such that for $0 < h < h_0$ if*

$$-\frac{\operatorname{Im} z}{h} < \inf_{\mathcal{H}} -l_N^{-1}(r_N + ch^{\gamma_1}) \quad \text{or} \quad -\frac{\operatorname{Im} z}{h} > \sup_{\mathcal{H}} -l_N^{-1}(r_N - ch^{\gamma_1}),$$

$\pm \operatorname{Im} z \geq 0$ and one of the following holds

$$(89) \quad \begin{aligned} \frac{-\operatorname{Im} z}{h} &\leq \inf_{\mathcal{L}_\alpha} - \frac{h(\operatorname{Im} \sigma(N_2) + \sigma(\operatorname{Im} V) + c(h^{1/3+\alpha} + h^{-1/3+\epsilon/2}))Q}{|\sigma(hV)|^2 \Psi_{S\ell}(h^{-2/3}\Xi)} (1 \pm \delta) \\ \frac{-\operatorname{Im} z}{h} &\geq \sup_{\mathcal{L}_\alpha} - \frac{h(\operatorname{Im} \sigma(N_2) + \sigma(\operatorname{Im} V) - c(h^{1/3+\alpha} + h^{-1/3+\epsilon/2}))Q}{|\sigma(hV)|^2 \Psi_{S\ell}(h^{-2/3}\Xi)} (1 \mp \delta) \end{aligned}$$

then

$$(90) \quad \|\psi\|_{L^2} \leq Ch^{-N} \|v\|_{L^2}.$$

and $\mathcal{P}(z)$ is invertible. Moreover, if $\alpha < -2/3$ then (89) can be replaced by

$$(91) \quad \begin{aligned} \frac{-\operatorname{Im} z}{h} &\leq \inf_{\mathcal{L}_\alpha} - \frac{h(\operatorname{Im} \sigma(N_2) + \sigma(\operatorname{Im} V) + c(h^{1/3+\alpha} + h^{-1/3+\epsilon/2}))Q}{|\sigma(hV)|^2} (1 \pm \delta) \\ \frac{-\operatorname{Im} z}{h} &\geq \sup_{\mathcal{L}_\alpha} - \frac{h(\operatorname{Im} \sigma(N_2) + \sigma(\operatorname{Im} V) - c(h^{1/3+\alpha} + h^{-1/3+\epsilon/2}))Q}{|\sigma(hV)|^2} (1 \mp \delta). \end{aligned}$$

Finally, if $\pm \operatorname{Im} z \geq ch^{M_1}$ and $\alpha < -2/3$, then (90) holds and $\mathcal{P}(z)$ is invertible if

$$\begin{aligned} \frac{-\operatorname{Im} z}{h} &\leq \inf_{\mathcal{L}'_{M_2}} - \frac{h(\operatorname{Im} \sigma(N_2) + \sigma(\operatorname{Im} V) + ch^{1/3+\alpha})Q}{|\sigma(hV)|^2} (1 \pm \delta) \\ \frac{-\operatorname{Im} z}{h} &\geq \sup_{\mathcal{L}'_{M_2}} - \frac{h(\operatorname{Im} \sigma(N_2) + \sigma(\operatorname{Im} V) - ch^{1/3+\alpha})Q}{|\sigma(hV)|^2} (1 \mp \delta) \end{aligned}$$

and one of the following holds for $1 \leq j \leq M_2$

$$\begin{aligned} \frac{-\operatorname{Im} z}{h} &\leq \inf_{\mathcal{L}_j} - \frac{h(h^{-2/3}(2Q)^{1/3} \operatorname{Im} \frac{A'_-(-\zeta_j)}{A_-(-\zeta_j)} + \sigma(\operatorname{Im} V) + ch^{1/3+\alpha})Q}{|\sigma(hV)|^2} (1 \pm \delta) \\ \frac{-\operatorname{Im} z}{h} &\geq \sup_{\mathcal{L}_j} - \frac{h(h^{-2/3}(2Q)^{1/3} \operatorname{Im} \frac{A'_-(-\zeta_j)}{A_-(-\zeta_j)} + \sigma(\operatorname{Im} V) - ch^{1/3+\alpha})Q}{|\sigma(hV)|^2} (1 \mp \delta) \end{aligned}$$

In particular, this implies Theorem 4.

10. APPLICATION TO TRANSPARENT OBSTACLES

In the case of transparent obstacles, we want to consider (3), repeated here for the reader's convenience,

$$\begin{cases} (-c^2\Delta - \lambda^2)u_1 = 0 & \text{in } \Omega \\ (-\Delta - \lambda^2)u_2 = 0 & \text{in } \mathbb{R}^d \setminus \bar{\Omega} \\ u_1 = u_2 & \text{on } \partial\Omega \\ \partial_\nu u_1 - \varkappa \partial_\nu u_2 = 0 & \text{on } \partial\Omega \\ u_2 \text{ is } \lambda\text{-outgoing} \end{cases}$$

Thus, writing $\lambda = cz/h$, in the language of (67),

$$B = hN_2(z/h) + \varkappa hN_2(cz/h) - hN_2(z/h)$$

where N_2 is the outgoing Dirichlet to Neumann map for the exterior problem (see section 1.4)

Thus, $V = \Re N_2(cz/h) - N_2(z/h)$ has

$$V \in h^{-2/3}(\Psi_{2/3}^{1/2,1}(|\xi'|_g = c) \cup \Psi_{2/3}^{1/2,1}(|\xi'|_g = 1)) \subset h^{-1}(\Psi_{2/3}^{0,1}(|\xi'|_g = c) \cup \Psi_{2/3}^{0,1}(|\xi'|_g = 1)).$$

In order to fit the transparent obstacle problem into the framework of Theorem 5 with $\alpha = -1$, we only need to check that V is elliptic near $|\xi'|_g = 1$ and that $1 + \frac{h\sigma(V)}{2\sqrt{|\xi'|_g^2 - 1}}$ has the required properties. We start by calculating the symbols of \mathcal{B} , B , and V . Let Ξ^E be the function given by Lemma 7.1 when we replace 1 by E in the eikonal equation for ρ_0 and θ_0 .

$$g_E(x, \xi') := (2Q(x, \xi'))^{1/3} \frac{A'_-(h^{-2/3}\Xi^E)}{A_-(h^{-2/3}\Xi^E)}$$

Then,

$$\sigma(B) = \sigma(h\Re N_2(cz/h)) = \begin{cases} -i\Re\sqrt{c^2 - |\xi'|_g^2} & |\xi'|_g \leq c - h^\epsilon \\ \Re h^{1/3}g_c(x, \xi') & ||\xi'|_g - c| \leq h^\epsilon \\ \Re\sqrt{|\xi'|_g^2 - c^2} & |\xi'|_g \geq c + h^\epsilon \end{cases}$$

$$\sigma(hV) = \begin{cases} i(\sqrt{1 - |\xi'|_g^2} - \Re\sqrt{c^2 - |\xi'|_g^2}) & |\xi'|_g \leq \min(1, c) - h^\epsilon \\ i\sqrt{1 - |\xi'|_g^2} + \Re\sqrt{|\xi'|_g^2 - c^2} & c + h^\epsilon \leq |\xi'|_g \leq 1 - h^\epsilon \\ -i\Re\sqrt{c^2 - |\xi'|_g^2} - \sqrt{|\xi'|_g^2 - 1} & 1 + h^\epsilon \leq |\xi'|_g \leq c - h^\epsilon \\ \Re\sqrt{|\xi'|_g^2 - c^2} - \sqrt{|\xi'|_g^2 - 1} & |\xi'|_g \geq \max(1, c) + h^\epsilon \\ h^{1/3}\Re g_c + i\sqrt{1 - |\xi'|_g^2} & ||\xi'|_g - c| \leq h^\epsilon, |\xi'|_g \leq 1 - h^\epsilon \\ h^{1/3}\Re g_c - \sqrt{|\xi'|_g^2 - 1} & ||\xi'|_g - c| \leq h^\epsilon, |\xi'|_g \geq 1 + h^\epsilon \\ -i\Re\sqrt{c^2 - |\xi'|_g^2} - h^{1/3}g_1 & ||\xi'|_g - 1| \leq h^\epsilon, |\xi'|_g \leq c - h^\epsilon \\ \Re\sqrt{|\xi'|_g^2 - c^2} - h^{1/3}g_1 & ||\xi'|_g - 1| \leq h^\epsilon, |\xi'|_g \geq c + h^\epsilon \end{cases}$$

$$\sigma(\mathcal{B}) = \frac{\Re^2|c^2 - |\xi'|_g^2|}{2Q} \Psi_{S\ell}(h^{-2/3}\Xi)(1 + o(1)), \quad ||\xi'|_g - 1| \leq h^\epsilon$$

Now, we compute

$$1 + \frac{h\sigma(V)}{2\sqrt{|\xi'|_g^2 - 1}} = \begin{cases} \frac{1}{2} + \frac{\aleph}{2} \frac{\sqrt{c^2 - |\xi'|_g^2}}{\sqrt{1 - |\xi'|_g^2}} & |\xi'|_g \leq \min(1, c) - h^\epsilon \\ \frac{1}{2} + i\frac{\aleph}{2} \frac{\sqrt{|\xi'|_g^2 - c^2}}{\sqrt{1 - |\xi'|_g^2}} & c + h^\epsilon \leq |\xi'|_g \leq 1 - h^\epsilon \\ \frac{1}{2} - i\frac{\aleph}{2} \frac{\sqrt{c^2 - |\xi'|_g^2}}{\sqrt{|\xi'|_g^2 - 1}} & 1 + Mh^{2/3} \leq |\xi'|_g \leq c - h^\epsilon \\ \frac{1}{2} + \frac{\aleph}{2} \frac{\sqrt{|\xi'|_g^2 - c^2}}{\sqrt{|\xi'|_g^2 - 1}} & \max(c + h^\epsilon, 1 + Mh^{2/3}) \leq |\xi'|_g \\ \frac{1}{2} + i\frac{\aleph}{2} \frac{h^{1/3}g_c}{\sqrt{1 - |\xi'|_g^2}} & |c - |\xi'|_g| \leq h^\epsilon, |\xi'|_g \leq 1 - h^\epsilon \\ \frac{1}{2} + \frac{\aleph}{2} \frac{h^{1/3}g_c}{\sqrt{|\xi'|_g^2 - 1}} & |c - |\xi'|_g| \leq h^\epsilon, |\xi'|_g \geq 1 + Mh^{2/3} \end{cases}$$

Thus, we can see that V is elliptic near $|\xi'|_g = 1$ and the transparent obstacle problem fits into the framework of Theorem 5.

In order to finish the proof of Theorem 1, we just need to check a few symbolic properties. First, notice $V = \aleph N_2(cz/h) - N_2(z/h)$. Thus,

$$\sigma(N_2(z/h) + V) = \aleph \sigma(N_2(cz/h)) = -ih^{-1} \aleph \sqrt{c^2 - |\xi'|_g^2}$$

where we take $\sqrt{-1} = i$. Putting this in (83) gives that (84) holds when $c > 1$ and

$$\frac{-\operatorname{Im} z}{h} \leq \inf_{|\xi'(q)|_g=1} -\frac{Q}{\aleph \sqrt{c^2 - 1}} (1 \pm \delta) \quad \text{or} \quad \frac{-\operatorname{Im} z}{h} \geq \sup_{|\xi'(q)|_g=1} -\frac{Q}{\aleph \sqrt{c^2 - 1}} (1 \mp \delta).$$

or when $c < 1$ and

$$\frac{-\operatorname{Im} z}{h} \geq \delta.$$

Next, observe that

$$\sigma(R) = \begin{cases} \frac{-\sqrt{1 - |\xi'|_g^2} + \aleph \sqrt{c^2 - |\xi'|_g^2}}{\sqrt{1 - |\xi'|_g^2} + \aleph \sqrt{c^2 - |\xi'|_g^2}} & |\xi'|_g \leq \min(1, c) - h^\epsilon \\ \frac{i\aleph h^{1/3}g_c - \sqrt{1 - |\xi'|_g^2}}{\sqrt{1 - |\xi'|_g^2} + i\aleph h^{1/3}g_c} & |c - |\xi'|_g| \leq h^\epsilon, |\xi'|_g \leq 1 - h^\epsilon \\ \frac{-\sqrt{1 - |\xi'|_g^2} + i\aleph \sqrt{|\xi'|_g^2 - c^2}}{\sqrt{1 - |\xi'|_g^2} + i\aleph \sqrt{|\xi'|_g^2 - c^2}} & c + h^\epsilon \leq |\xi'|_g \leq 1 - h^\epsilon \end{cases}$$

The following geometric lemma completes the proof of Theorem 1.

Lemma 10.1. *Fix $N > 0$ and let $(x_0, \xi_0) \in S^* \partial \Omega$ and suppose that $\{(x_n, \xi_n)\} \subset B^* \partial \Omega$ has $(x_n, \xi_n) \rightarrow (x_0, \xi_0)$. Then*

$$l_N^{-1} r_N \rightarrow \begin{cases} \frac{Q(x_0, \xi_0)}{\aleph \sqrt{c^2 - 1}} & c > 1 \\ 0 & c < 1 \end{cases}.$$

Proof. The conclusion for $c < 1$ is clear since for $|\xi'|_g > c$, $\log |\sigma(R)|^2 = 0$. So, we need only consider the case $c > 1$. First, write

$$|\sigma(R)|^2(x, \xi') = 1 - \frac{4\sqrt{1 - |\xi'|_g^2}}{\sqrt{1 - |\xi'|_g^2} + \aleph \sqrt{c^2 - |\xi'|_g^2}} + o(1 - |\xi'|_g^2).$$

So,

$$(92) \quad \log |\sigma(R)|^2(x, \xi') = -\frac{4\sqrt{1 - |\xi'|_g^2}}{\sqrt{1 - |\xi'|_g^2} + \aleph\sqrt{c^2 - |\xi'|_g^2}} + O(1 - |\xi'|_g^2).$$

Now, by Lemma 5.1

$$\sqrt{1 - |\xi'(\beta(q))|_g^2} = \sqrt{1 - |\xi'(q)|_g^2} + O(1 - |\xi'|_g^2), \quad l(q, \beta(q)) = \frac{2}{\kappa(0)}\sqrt{1 - |\xi'|_g^2} + O(1 - |\xi'|_g^2)$$

where $\kappa(s)$ is the curvature of the unique length minimizing geodesic, γ in $\partial\Omega$ connecting $\pi_x(q)$ and $\pi_x(\beta(q))$ at the point $\gamma(s)$. Thus, we have that for q sufficiently close to glancing,

$$\frac{\log |\sigma(R)(\beta(q))|^2}{2l(q, \beta(q))} = -\frac{\kappa(0)}{\aleph\sqrt{c^2 - |\xi'|_g^2}} + O(\sqrt{1 - |\xi'|_g^2}).$$

Moreover, since $\sqrt{1 - |\xi'(\beta(q))|_g^2} = \sqrt{1 - |\xi'|_g^2} + O(1 - |\xi'|_g^2)$ and $\kappa(s) = \kappa(0) + O(s) = \kappa(0) + O(\sqrt{1 - |\xi'|_g^2})$, we have

$$\frac{r_N}{l_N} = -\frac{\kappa(0)}{\aleph\sqrt{c^2 - |\xi'|_g^2}} + O(1 - |\xi'|_g^2).$$

All that remains to prove is that $\kappa(0) = Q(x, \xi') + o(1)$ as $|\xi'|_g \rightarrow 1$. This follows from the fact that the curvature of the geodesic on $\partial\Omega$ passing through x in the direction ξ' is $Q(x, \xi')$ together with the fact that

$$\gamma'(0) - \frac{\xi'}{|\xi'|_g} = o(1).$$

To see this we simply use the fact that a billiards trajectory approaches a geodesic as $|\xi'|_g \rightarrow 1$ (see for example [PS92]). \square

Together, this discussion proves Theorem 1.

11. APPLICATION TO δ POTENTIALS

For the application to δ potentials, we consider

$$(-h^2\Delta + h^2V \otimes \delta_{\partial\Omega} - z^2)u = 0, \quad u \text{ is } z/h \text{ outgoing.}$$

It is shown in [GS14] that this is equivalent to $u = u_1 \oplus u_2$ where $u_1 = u1_\Omega$ and $u_2 = u1_{\mathbb{R}^d \setminus \bar{\Omega}}$ solving

$$(93) \quad \begin{cases} (-h^2\Delta - z^2)u = 0 & \text{in } \mathbb{R}^d \setminus \partial\Omega \\ u_1 - u_2 = 0 & \text{on } \partial\Omega \\ h\partial_\nu u_1 - h\partial_\nu u_2 + hVu_1 = 0 & \text{on } \partial\Omega \\ u_2 \text{ is } z/h\text{-outgoing} \end{cases}$$

In this case, $V = V$ (indeed this is the motivation for our notation). For our purposes, we will assume that $V \in h^\alpha\Psi^1$ is self adjoint and hence, $\text{Im } V = 0$. Moreover, we assume that $\alpha \geq -1$

and $\sigma(V) \geq ch^\alpha$ on $|\xi'|_g = 1$ and for any $\delta > 0$, there exists $c > 0$ so that $\frac{h\sigma(V)}{2\sqrt{|\xi'|_g^2 - 1}} > -1 + c$ on $|\xi'|_g \geq 1 + \delta$. This clearly implies all of the assumptions (65). Theorem 5 then yields Theorem 2 as a Corollary.

12. APPLICATION TO BOUNDARY STABILIZATION

The application to the boundary stabilization problem (11) is similar to that for the transmission problem. In particular, note that

$$1 + \frac{i\sigma(hV)}{2\sqrt{1 - |\xi'|_g^2}} = -\frac{1}{2} + \frac{a}{2\sqrt{1 - |\xi'|_g^2}}$$

and the fact that $a \geq a_0 > 0$ implies the ellipticity of V . Finally, an argument identical to that in Lemma 10.1 together with Theorem 5 gives Theorem 3.

13. OPTIMALITY FOR THE TRANSPARENT OBSTACLE PROBLEM ON THE CIRCLE

For the optimality of Theorem 2, see [Gal16]. We now show that Theorem 1 is optimal in the case of the unit disk in \mathbb{R}^2 . In this case, (3) reads

$$\begin{cases} (-c^2\Delta - \lambda^2)u_1 = 0 & \text{in } B(0, 1) \\ (-\Delta - \lambda^2)u_2 = 0 & \text{in } \mathbb{R}^d \setminus \overline{B(0, 1)} \\ u_1 = u_2 & \text{on } |x| = 1 \\ \partial_r u_1 - \aleph \partial_r u_2 = 0 & \text{on } |x| = 1 \\ u_2 \text{ is } \lambda\text{-outgoing} \end{cases}.$$

We now expand u_i in Fourier series, writing

$$u_i(r, \theta) = \sum_n u_{i,n}(r) e^{in\theta}.$$

Then,

$$\left(-c^2\partial_r^2 - \frac{c^2}{r}\partial_r + \frac{c^2n^2}{r} - \lambda^2\right)u_{1,n}(r) = 0 \quad \left(-\partial_r^2 - \frac{1}{r}\partial_r + \frac{n^2}{r} - \lambda^2\right)u_{2,n}(r) = 0$$

Multiplying by r^2 and rescaling by $x_1 = \lambda c^{-1}r$ for $u_{1,n}$ and $x_2 = \lambda r$ for u_2 , we see that $u_{i,n}(x_i)$ solves Bessel's equation. Together with the outgoing condition for u_2 and the fact that u_1 is in L^2 , this implies that

$$u_{1,n} = K_n J_n(\lambda c^{-1}r), \quad u_{2,n} = C_n H_n^{(1)}(\lambda r).$$

Then, the boundary conditions imply that either $K_n = C_n = 0$ or $C_n \neq 0$ and

$$\frac{K_n}{C_n} = \frac{H_n^{(1)}(\lambda)}{J_n(c^{-1}\lambda)}, \quad K_n c^{-1} \lambda J_n'(\lambda c^{-1}) - C_n \aleph \lambda H_n^{(1)'}(\lambda) = 0.$$

Rewriting this (and assuming $\lambda \neq 0$) we have

$$(94) \quad f(\lambda) := c^{-1} J_n'(c^{-1}\lambda) H_n^{(1)}(\lambda) - \aleph H_n^{(1)'}(\lambda) J_n(c^{-1}\lambda) = 0$$

Throughout this section we will refer to microlocalization of the Fourier modes $e^{in\theta}$. Notice that for a Fourier mode $u_n = (u_{1,n}(r) \oplus u_{2,n})e^{in\theta}$, the component of the frequency tangent to $\partial B(0, 1)$ is given by n and the rest of the oscillations are normal to the boundary. Naively taking the Fourier transform, we see that if $(-\Delta - \lambda^2)u = 0$, then the Fourier support of u is contained in $|\xi|^2 = \lambda^2$. Therefore, since $|\operatorname{Im} \lambda| \ll |\operatorname{Re} \lambda|$ the total frequency of the mode is given by $|\operatorname{Re} \lambda|$ and the fraction of frequency tangent to the boundary is given by $n/\operatorname{Re} \lambda$. This can be reinterpreted in terms of the semiclassical wavefront set (with $\operatorname{Re} \lambda = h^{-1}$) of the mode as saying that

$$\operatorname{WF}_h(u_n|_{\partial\Omega}) \subset \{|\xi'_g| = hn\}.$$

For this reason, we refer to modes with $n \ll |\operatorname{Re} \lambda|$ as normal to the boundary, those with $\epsilon|\operatorname{Re} \lambda| < n < (c^{-1} - \epsilon)|\operatorname{Re} \lambda|$ transverse, and $(c^{-1} - \epsilon)|\operatorname{Re} \lambda| < n$ glancing.

13.1. Asymptotics of Bessel and Hankel functions. We collect here some properties of the Airy and Bessel functions that are used in the analysis for the unit disk. These formulae can be found in, for example [OLBC10, Chapter 9,10].

Recall that the Bessel of order n functions are solutions to

$$z^2 y'' + zy' + (z^2 - n^2)y = 0.$$

We consider the two independent solutions $H_n^{(1)}(z)$ and $J_n(z)$.

We now record some asymptotic properties of Bessel functions. Consider n fixed and $z \rightarrow \infty$

$$J_n(z) = \left(\frac{1}{2\pi z}\right)^{1/2} \left(e^{i(z - \frac{n}{2}\pi - \frac{1}{4}\pi)} + e^{-i(z - \frac{n}{2}\pi - \frac{1}{4}\pi)} + O(|z|^{-1}e^{|\operatorname{Im} z|})\right)$$

$$H_n^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \left(e^{i(z - \frac{n}{2}\pi - \frac{1}{4}\pi)} + O(|z|^{-1}e^{|\operatorname{Im} z|})\right)$$

$$J'_n(z) = i \left(\frac{1}{2\pi z}\right)^{1/2} \left(e^{i(z - \frac{n}{2}\pi - \frac{1}{4}\pi)} - e^{-i(z - \frac{n}{2}\pi - \frac{1}{4}\pi)} + O(|z|^{-1}e^{|\operatorname{Im} z|})\right)$$

$$H_n^{(1)'}(z) = i \left(\frac{2}{\pi z}\right)^{1/2} \left(e^{i(z - \frac{n}{2}\pi - \frac{1}{4}\pi)} + O(|z|^{-1}e^{|\operatorname{Im} z|})\right)$$

$$(95) \quad J'_n(c^{-1}z)H_n^{(1)}(z) = \frac{i\sqrt{c}}{\pi z} \left(e^{i((c^{-1}+1)z - n\pi - \frac{1}{2}\pi)} - e^{-i(c^{-1}-1)z} + O(|z|^{-1}e^{(c^{-1}+1)|\operatorname{Im} z|})\right)$$

$$(96) \quad J_n(c^{-1}z)H_n^{(1)'}(z) = \frac{i\sqrt{c}}{\pi z} \left(e^{i((c^{-1}+1)z - n\pi - \frac{1}{2}\pi)} + e^{-i(c^{-1}-1)z} + O(|z|^{-1}e^{(c^{-1}+1)|\operatorname{Im} z|})\right)$$

Next, we record asymptotics that are uniform in n and z as $n \rightarrow \infty$. Let $\zeta = \zeta(z)$ be the unique smooth solution on $0 < z < \infty$ to

$$(97) \quad \left(\frac{d\zeta}{dz}\right)^2 = \frac{1 - z^2}{\zeta z^2}$$

with

$$\lim_{z \rightarrow 0} \zeta = \infty, \quad \lim_{z \rightarrow 1} \zeta = 0, \quad \lim_{z \rightarrow \infty} \zeta = -\infty.$$

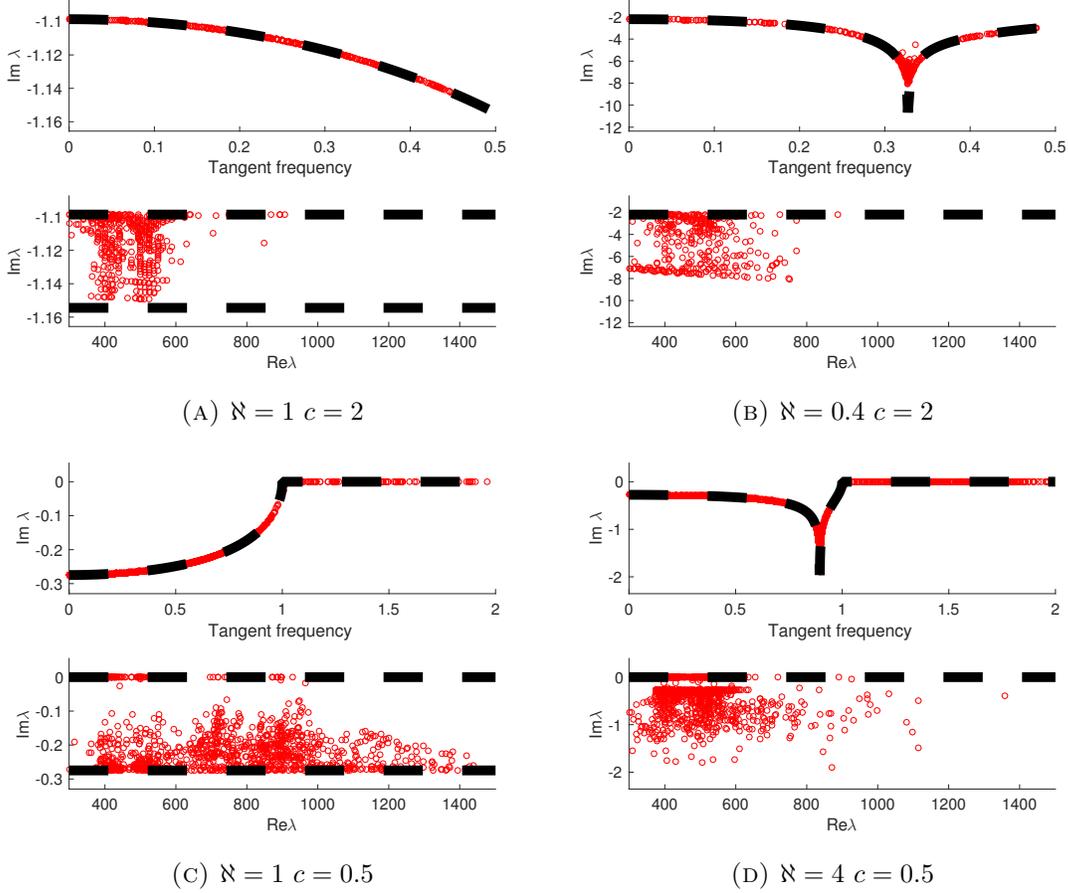


FIGURE 13.1. We show numerically computed resonances for the transparent obstacle problem with various c and \aleph when $\Omega = B(0, 1) \subset \mathbb{R}^2$. In this case, we expand the solutions to (3) as $u_i(r, \theta) = \sum_n u_{i,n}(r)e^{in\theta}$ and solve for some of the resonances with $\text{Re } \lambda \sim 800$. In the lower graphs of each of the four subfigures, the blue circles show $\text{Im } \lambda$ vs. $\text{Re } \lambda$. The red lines show the upper and lower bounds for $\text{Im } \lambda$ when \aleph corresponds to TE waves and the upper bounds on $\text{Im } \lambda$ when \aleph corresponds to TM waves from Theorem 1. Notice that by orthogonality of $e^{in\theta}$ and $e^{im\theta}$ for $m \neq n$, the pair $(u_{1,n}e^{in\theta}, u_{2,n}e^{in\theta})$ satisfies (3). In the top graph of each subfigure, the blue circles show $\text{Im } \lambda$ vs. $n/\text{Re } \lambda$ for such pairs. That is, we plot $\text{Im } \lambda$ vs. the scaled tangent frequency of the resonance state. The red curve shows a plot of $c \frac{r_1}{2l_1}(c\xi)$, the decay rate predicted for a billiards trajectory traveling with scaled tangent frequency $c\xi$. The large spikes in the top graphs occur at the Brewster angle when \aleph corresponds to TM waves.

Then

$$(98) \quad \begin{aligned} \frac{2}{3}(-\zeta)^{3/2} &= \sqrt{z^2 - 1} - \operatorname{arcsec}(z) & 1 < z < \infty \\ \frac{2}{3}(\zeta)^{3/2} &= \log\left(\frac{1 + \sqrt{1 - z^2}}{z}\right) - \sqrt{1 - z^2} & 0 < z < 1 \end{aligned}$$

$$(99) \quad \frac{1 - z^2}{\zeta z^2} \rightarrow \sqrt[3]{2} \quad z \rightarrow 0$$

Let

$$Ai(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\frac{1}{3}t^3 + st)} dt$$

for $s \in \mathbb{R}$ be the Airy function solving $Ai''(z) - zAi(z) = 0$. Then, $A_-(z) = Ai(e^{2\pi i/3}z)$ is another solution of the Airy equation.

For z fixed as $n \rightarrow \infty$

$$(100) \quad \begin{aligned} J_n(nz) &= \left(\frac{4\zeta}{1 - z^2}\right)^{1/4} \left(\frac{Ai(n^{2/3}\zeta)}{n^{1/3}} + o(Ei(5/3, 7/3))\right) \\ H_n^{(1)}(nz) &= 2e^{-\pi i/3} \left(\frac{4\zeta}{1 - z^2}\right)^{1/4} \left(\frac{A_-(n^{2/3}\zeta)}{n^{1/3}} + o(E_-(5/3, 7/3))\right) \\ J_n'(nz) &= -\frac{2}{z} \left(\frac{1 - z^2}{4\zeta}\right)^{1/4} \left(\frac{Ai'(n^{2/3}\zeta)}{n^{2/3}} + o(Ei(8/3, 4/3))\right) \\ H_n^{(1)'}(nz) &= \frac{4e^{2\pi i/3}}{z} \left(\frac{1 - z^2}{4\zeta}\right)^{1/4} \left(\frac{A'_-(n^{2/3}\zeta)}{n^{2/3}} + o(E_-(8/3, 4/3))\right) \end{aligned}$$

$$J_n'(c^{-1}nz)H_n^{(1)}(nz) = \frac{4e^{2\pi i/3}c}{z} \left(\frac{(1 - c^{-2}z^2)\zeta(z)}{\zeta(c^{-1}z)(1 - z^2)}\right)^{1/4} \left(\frac{Ai'(n^{2/3}\zeta(c^{-1}z))}{n^{2/3}} + o(Ei(8/3, 4/3)(c^{-1}z))\right) \left(\frac{A_-(n^{2/3}\zeta)}{n^{1/3}} + o(E_-(5/3, 7/3)(z))\right)$$

$$J_n(c^{-1}nz)H_n^{(1)'}(nz) = \frac{4e^{2\pi i/3}}{z} \left(\frac{(1 - z^2)\zeta(c^{-1}z)}{\zeta(z)(1 - c^{-2}z^2)}\right)^{1/4} \left(\frac{Ai(n^{2/3}\zeta(c^{-1}z))}{n^{1/3}} + o(Ei(5/3, 7/3)(c^{-1}z))\right) \left(\frac{A'_-(n^{2/3}\zeta(z))}{n^{2/3}} + o(E_-(8/3, 4/3)(z))\right)$$

where

$$\begin{aligned} E_-(s, t) &= |A'_-(n^{2/3}\zeta)|n^{-s} + |A_-(n^{2/3}\zeta)|n^{-t} \\ Ei(s, t) &= |Ai'(n^{2/3}\zeta)|n^{-s} + |Ai(n^{-2/3}\zeta)|n^{-t} \end{aligned}$$

We now record some facts about the Airy functions Ai and A_- . For $s \in \mathbb{R}$,

$$Ai(s) = e^{-\pi i/3} A_-(s) + e^{\pi i/3} \overline{A_-(s)}$$

and hence

$$(101) \quad \operatorname{Im}(e^{-5\pi i/6} A_-(s)) = -\frac{Ai(s)}{2}$$

Next, we record asymptotics for Airy functions as $z \rightarrow \infty$ in the sector $|\operatorname{Arg} z| < \pi/3 - \delta$. Many of these asymptotic formulae hold in larger regions, but we restrict our attention to this sector. Let $\eta = 2/3 z^{3/2}$ where we take principal branch of the square root. Then

$$(102) \quad \begin{aligned} A_-(z) &= \frac{e^{-\pi i/6} e^\eta}{2\sqrt{\pi} z^{1/4}} (1 + O(|z|^{-3/2})) & A_-(-z) &= \frac{e^{\pi i/12} e^{i\eta}}{2\sqrt{\pi} z^{1/4}} \\ A'_-(z) &= \frac{e^{-\pi i/6} z^{1/4} e^\eta}{2\sqrt{\pi}} (1 + O(|z|^{-3/2})) & A'_-(-z) &= \frac{e^{-5\pi i/12} z^{1/4} e^{i\eta}}{2\sqrt{\pi}} \\ Ai(z) &= \frac{z^{-1/4} e^{-\eta}}{2\sqrt{\pi}} (1 + O(|z|^{-3/2})) & Ai(-z) &= \frac{z^{-1/4}}{2\sqrt{\pi}} \left(e^{i\eta - i\pi/4} + e^{-i\eta + i\pi/4} + O(|z|^{-3/2} e^{|\operatorname{Im} \eta|}) \right) \\ Ai'(z) &= -\frac{z^{1/4} e^{-\eta}}{2\sqrt{\pi}} (1 + O(|z|^{-3/2})) & Ai'(-z) &= \frac{z^{1/4}}{2i\sqrt{\pi}} \left(e^{i\eta - i\pi/4} - e^{-i\eta + i\pi/4} + O(|z|^{-3/2} e^{|\operatorname{Im} \eta|}) \right) \end{aligned}$$

13.2. Resonances normal to the boundary (fixed n). First, we fix $n \geq 0$ and examine solutions with $\operatorname{Re} \lambda \rightarrow \infty$. We assume that $\aleph \neq c^{-1}$. Consider (94) and apply the asymptotics (95) and (96) with $\operatorname{Im} \lambda \leq 0$

$$(c^{-1} - \aleph) e^{i((c^{-1}+1)\lambda - n\pi - \frac{1}{2}\pi)} - (c^{-1} + \aleph) e^{-i(c^{-1}-1)\lambda} + O(|z|^{-1} e^{(c^{-1}+1)|\operatorname{Im} z|}) = 0.$$

So, ignoring the error term for now, we have

$$\frac{1 - \aleph c}{1 + \aleph c} e^{i(2c^{-1}\lambda_0 - n\pi - \frac{1}{2}\pi)} = 1.$$

So,

$$c^{-1} \operatorname{Im} \lambda_0 = \frac{1}{2} \log \left| \frac{1 - \aleph c}{1 + \aleph c} \right|, \quad c^{-1} \operatorname{Re} \lambda_0 = \frac{2 - \operatorname{sgn}(1 - \aleph c) + 2n + 4k}{4} \pi$$

Taking λ_0 as above, we have $f(\lambda_0) = O(|\operatorname{Re} \lambda_0|^{-1})$, $|f'(\lambda_0)| \geq c$, and $|f''(\lambda)| \leq C$ for $|\lambda - \lambda_0| < \delta$ for some $\delta > 0$. We now recall Newton's method (see for example [Gal16, Lemma 4.1])

Lemma 13.1. *Suppose that $z_0 \in \mathbb{C}$. Let $\Omega := \{z \in \mathbb{C} : |z - z_0| \leq \epsilon\}$ and suppose $f : \Omega \rightarrow \mathbb{C}$ is analytic. Suppose that*

$$|f(z_0)| \leq a, \quad |\partial_z f(z_0)| \geq b, \quad \sup_{z \in \Omega} |\partial_z^2 f(z)| \leq d.$$

Then if

$$(103) \quad a + d\epsilon^2 < eb < c < 1$$

there is a unique solution z to $f(z) = 0$ in Ω .

Using this, we have that there exists a unique solution λ_1 to $f(\lambda_1) = 0$ with $|\lambda_1 - \lambda_0| = \mathcal{O}(|\operatorname{Re} \lambda_0|^{-1})$.

13.3. Resonances with non-zero tangent frequency ($\epsilon \operatorname{Re} \lambda \leq n \leq (\min(1, c) - \epsilon) \operatorname{Re} \lambda$). In this case, we write

$$f(\lambda) := c^{-1} J'_n \left(n \frac{c^{-1} \lambda}{n} \right) H_n^{(1)}(\lambda) - \aleph H_n^{(1)'}(\lambda) J_n \left(n \frac{c^{-1} \lambda}{n} \right) = 0.$$

Write $z = \frac{\lambda}{n}$. Then taking $n \leq (c - \epsilon) \operatorname{Re} \lambda$ and ignoring error terms, $f(\lambda_0) = 0$ implies

$$(104) \quad \left[\left(\frac{1 - c^{-2} z_0^2}{1 - z_0^2} \right)^{1/2} - \aleph \right] e^{\frac{4in}{3} ((-\zeta(c^{-1} z_0))^{3/2} - i\pi/2)} = \left[\left(\frac{1 - c^{-2} z_0^2}{1 - z_0^2} \right)^{1/2} + \aleph \right] \\ -i \frac{\sqrt{c^{-2} z_0^2 - 1} - \aleph \sqrt{z_0^2 - 1}}{\sqrt{c^{-2} z_0^2 - 1} + \aleph \sqrt{z_0^2 - 1}} = e^{-\frac{4in}{3} ((-\zeta(c^{-1} z_0))^{3/2})}$$

Fix $\max(c, 1) + \delta < r < \infty$ with $\delta < c^2$ so that

$$\sqrt{c^{-2} r^2 - 1} - \aleph \sqrt{r^2 - 1} \neq 0.$$

Let

$$g(s, n, k) := \sqrt{c^{-2} s^2 - 1} - \operatorname{arcsec}(c^{-1} s) + \frac{4k - \operatorname{sgn}(\sqrt{c^{-2} s^2 - 1} - \aleph \sqrt{s^2 - 1})}{4n} \pi.$$

Then, fix $q \in \mathbb{Z}_+$ $p \in \mathbb{Z}$ and let $n = qm$ and $k = pm$ so that

$$g(s, qm, pm) = \sqrt{c^{-2} s^2 - 1} - \operatorname{arcsec}(c^{-1} s) + \frac{p}{q} \pi - \frac{\operatorname{sgn}(\sqrt{c^{-2} s^2 - 1} - \aleph \sqrt{s^2 - 1})}{4mq} \pi.$$

Then, for any $\epsilon > 0$ small enough, there exists p_ϵ, q_ϵ so that

$$|g(r, qm, pm)| < \epsilon + \mathcal{O}(m^{-1}) \\ \partial_s g(r, q_\epsilon m, p_\epsilon m) = \frac{\sqrt{c^{-2} r^2 - 1}}{r} \geq C\sqrt{\delta}, \quad \partial_s^2 g(r, q_\epsilon m, p_\epsilon m) = -\frac{r^{-3}}{\sqrt{c^{-2} - r^{-2}}} \leq \frac{C}{\sqrt{\delta}}$$

Therefore, taking ϵ small enough and m large enough (depending on $r - c$), there is a solution r_m to $g(r_m, q_\epsilon m, p_\epsilon m) = 0$ with $|r - r_m| < C\epsilon$.

With this r_m , let

$$\lambda_0 = mqr_m + i \frac{r_m}{2\sqrt{c^{-2} r_m^2 - 1}} \log \left| \frac{\sqrt{c^{-2} r_m^2 - 1} - \aleph \sqrt{r_m^2 - 1}}{\sqrt{c^{-2} r_m^2 - 1} + \aleph \sqrt{r_m^2 - 1}} \right|$$

and $z_0 = \lambda_0/mq$. Let

$$H(z, n) = \exp \left(-\frac{4in}{3} ((-\zeta(c^{-1} z))^{3/2}) \right) + i \frac{\sqrt{c^{-2} z^2 - 1} - \aleph \sqrt{z^2 - 1}}{\sqrt{c^{-2} z^2 - 1} + \aleph \sqrt{z^2 - 1}}.$$

Then, accounting for the errors omitted to obtain (104) there is a function $a(z, n) = O(n^{-2/3})$, analytic in z so that nz is a resonance if and only if

$$H(z, n) = a(z, n) \left[1 + \exp \left(-\frac{4in}{3} (-\zeta(c^{-1}z))^{3/2} \right) \right]$$

Now, using (98)

$$\begin{aligned} -\frac{4imq}{3} (-\zeta(c^{-1}z_0))^{3/2} &= -2imq(\sqrt{c^{-2}z_0^2 - 1} - \operatorname{arcsec}(c^{-1}z_0)) \\ &= -2imq(\sqrt{c^{-2}r_m^2 - 1} - \operatorname{arcsec}(c^{-1}r_m)) + i\frac{\sqrt{c^{-2}r_m^2 - 1}}{r_m} \operatorname{Im} z_0 + O((\operatorname{Im} z_0)^2) \\ &= i(2mpi - \frac{\operatorname{sgn}(\sqrt{c^{-2}r_m^2 - 1} - \Re\sqrt{r_m^2 - 1})}{2})\pi \\ &\quad + \log \left| \frac{\sqrt{c^{-2}r_m^2 - 1} - \Re\sqrt{r_m^2 - 1}}{\sqrt{c^{-2}r_m^2 - 1} + \Re\sqrt{r_m^2 - 1}} \right| + O((mq)^{-1}) \end{aligned}$$

So,

$$\begin{aligned} \exp \left(-\frac{4imq}{3} (-\zeta(c^{-1}z_0))^{3/2} \right) &= \\ &= -\operatorname{sgn}(\sqrt{c^{-2}r_m^2 - 1} - \Re\sqrt{c^{-2}r_m^2 - 1})i \left| \frac{\sqrt{c^{-2}r_m^2 - 1} - \Re\sqrt{r_m^2 - 1}}{\sqrt{c^{-2}r_m^2 - 1} + \Re\sqrt{r_m^2 - 1}} \right| (1 + O((mq)^{-1})). \end{aligned}$$

So, $H(z_0, mq) = O((mq)^{-1})$. Moreover, $|z_0 - z| \leq 1$,

$$|\partial_z H(z, mq)| \geq cmq.$$

Hence, by the implicit function theorem, there exists a resonance z_1 with

$$\begin{aligned} z_1 &= z_0 + O \left(\frac{\sup_{|z-z_0| \leq 1} |a(z, mq) [1 + \exp(-\frac{4imq}{3} (-\zeta(c^{-1}z))^{3/2})]|}{\inf_{|z-z_0| \leq 1} |\partial_z H(z, mq)|} \right) \\ &= z_0 + O((mq)^{-5/3}) \end{aligned}$$

Thus, there is a resonance, λ_1 with

$$\lambda_1 = mqr_m + i\frac{r_m}{2\sqrt{c^{-2}r_m^2 - 1}} \log \left| \frac{\sqrt{c^{-2}r_m^2 - 1} - \Re\sqrt{r_m^2 - 1}}{\sqrt{c^{-2}r_m^2 - 1} + \Re\sqrt{r_m^2 - 1}} \right| + O((mq)^{-2/3}).$$

Now, notice that if $|\xi'|_g^{-1}c = r$, then on $B(0, 1)$, $l((x, \xi'), \beta(x, \xi')) = 2\sqrt{1 - r^{-2}c^2}$. So,

$$\begin{aligned} l_N^{-1}r_N(x, \xi') &= \frac{1}{4\sqrt{1 - r^{-2}c^2}} \log \left| \frac{\sqrt{1 - r^{-2}c^2} - \Re\sqrt{c^2 - r^{-2}c^2}}{\sqrt{1 - r^{-2}c^2} + \Re\sqrt{c^2 - r^{-2}c^2}} \right|^2 \\ &= \frac{c^{-1}r}{2\sqrt{c^{-2}r^2 - 1}} \log \left| \frac{\sqrt{c^{-2}r^2 - 1} - \Re\sqrt{r^2 - 1}}{\sqrt{c^{-2}r^2 - 1} + \Re\sqrt{r^2 - 1}} \right|. \end{aligned}$$

Now, by construction for any r with $\max(1, c) < r < \infty$ so that $\sqrt{c^{-2}r^2 - 1} - \aleph\sqrt{r^2 - 1} \neq 0$ and δ small enough, we have $|r - r_m| < \delta$ so, taking m large enough,

$$|c^{-1} \operatorname{Im} \lambda - l_N^{-1} r_N(x, \xi')| \leq C\delta.$$

This shows that Theorem 1 is sharp. Moreover, when $c < 1$, [PV99b] shows that there are sequences of resonances converging to the real axis that have $n \approx c^{-1} \operatorname{Re} \lambda$.

Remark 17. Notice also that

$$\frac{mq}{c^{-1} \operatorname{Re} \lambda} = cr_m^{-1} = |\xi'|_g.$$

Thus, since (94) with parameter n corresponds to a resonant state with $u|_{\partial\Omega} = Ae^{in\theta}$, the semiclassical tangent frequency of the resonance state is $cn/\operatorname{Re} \lambda$ when we take $\operatorname{Re} z \sim c$. Plugging this into $cl_N^{-1} r_N(x, \xi')$ gives the decay rate of the resonance state. See also Figures 1.3 and 13.1 for numerically computed resonances in this case.

APPENDIX A. LIST OF NOTATION

For the convenience of the reader, we include a list of some of the notation used in this paper.

- Ω : strictly convex domain with smooth boundary – Section 1.1
- $l(q_1, q_2)$: chord length – (20)
- $l_N(q)$: average chord length – (20)
- $|\xi'|_g$ metric induced on $T^*\partial\Omega$ – Section 1.1
- $\beta : B^*\partial\Omega \rightarrow B^*\partial\Omega$: the billiard ball map – Section 5
- $\Psi_\delta^m(M)$: semiclassical pseudodifferential operator classes – Section 2
- $S_\delta^m(T^*M)$: symbol classes – (33)
- $\sigma : \Psi_\delta^m(M) \rightarrow S_\delta^m(T^*M)$: the symbol map – (34)
- Ai, A_i, Φ_-, ζ_i : Airy related functions – Section 1.2, (9)
- $Q(x', \xi') \in C^\infty(T^*\partial\Omega)$: the symbol of the second fundamental form – Section 1.2
- $N_2(z/h)$: the outgoing Dirichlet to Neumann Map – Section 1.4
- $G(z/h)$: the single layer operator – Section 1.4
- G_B, G_Δ : decomposition of G – Lemma 7.3
- $\Psi_\delta^{k_1, k_2}(M; \Sigma), S_\delta^{k_1, k_2}(M; \Sigma)$: second microlocal operators and symbols – Section 4
- R : the reflection operator – (18)
- T : the transition operator – (19)
- Op_h : quantization operator – Section 2
- r_N : the average reflectivity – (21)
- $\tilde{\sigma}$: the compressed shymbol – Section 3
- $I_A(q)$: the order of A at q – Section 3
- H_h^m : semiclassical Sobolev spaces – (24)
- $\mathcal{S}\ell, \mathcal{D}\ell$, respectively the single and double layer operators – (31)
- $\mathcal{O}(\cdot)$ and $\mathcal{o}(\cdot)$ – (32)
- WF_h the semiclassical wavefront set – Definition 2.3
- $\Psi_{\mathcal{S}\ell}, \Psi_{\mathcal{D}\ell\mathcal{S}\ell}, \Psi_{\mathcal{D}\ell}$ symbols of layer potentials – (52)

REFERENCES

- [Ale08] Ivana Alexandrova. Semi-classical wavefront set and Fourier integral operators. *Canad. J. Math.*, 60(2):241–263, 2008.
- [Bel03] Mourad Bellassoued. Carleman estimates and distribution of resonances for the transparent obstacle and application to the stabilization. *Asymptot. Anal.*, 35(3-4):257–279, 2003.
- [BLR92] Claude Bardos, Gilles Lebeau, and Jeffrey Rauch. Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J. Control Optim.*, 30(5):1024–1065, 1992.
- [BZH10] M. Barr, M. Zaletel, and E. Heller. Quantum corral resonance widths: Lossy scattering as acoustics. *Nano Letters*, (10):3253–3260, 2010.
- [CLEH95] M. Crommie, C. Lutz, D. Eigler, and E. Heller. Quantum corrals. *Physica D: Nonlinear Phenomena*, 83(1-3):98–108, 1995.
- [CPV99] Fernando Cardoso, Georgi Popov, and Georgi Vodev. Distribution of resonances and local energy decay in the transmission problem. II. *Math. Res. Lett.*, 6(3-4):377–396, 1999.
- [CPV01] Fernando Cardoso, Georgi Popov, and Georgi Vodev. Asymptotics of the number of resonances in the transmission problem. *Comm. Partial Differential Equations*, 26(9-10):1811–1859, 2001.
- [CV10] Fernando Cardoso and Georgi Vodev. Boundary stabilization of transmission problems. *J. Math. Phys.*, 51(2):023512, 15, 2010.
- [CW15] Hui Cao and Jan Wiersig. Dielectric microcavities: Model systems for wave chaos and non-hermitian physics. *Reviews of Modern Physics*, 87(1):61, 2015.
- [DG14] Semyon Dyatlov and Colin Guillarmou. Microlocal limits of plane waves and Eisenstein functions. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(2):371–448, 2014.
- [DS99] Mouez Dimassi and Johannes Sjöstrand. *Spectral asymptotics in the semi-classical limit*, volume 268 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1999.
- [Dya12] Semyon Dyatlov. Asymptotic distribution of quasi-normal modes for Kerr–de Sitter black holes. *Ann. Henri Poincaré*, 13(5):1101–1166, 2012.
- [DZ] S. Dyatlov and M. Zworski. *Mathematical theory of scattering resonances*.
- [DZ13] Semyon Dyatlov and Maciej Zworski. Quantum ergodicity for restrictions to hypersurfaces. *Nonlinearity*, 26(1):35–52, 2013.
- [EG02] Barry J Elliott and Mike Gilmore. *Fiber Optic Cabling*. Newnes, 2002.
- [Eps07] Charles L. Epstein. Pseudodifferential methods for boundary value problems. In *Pseudo-differential operators: partial differential equations and time-frequency analysis*, volume 52 of *Fields Inst. Commun.*, pages 171–200. Amer. Math. Soc., Providence, RI, 2007.
- [Exn08] Pavel Exner. Leaky quantum graphs: a review. In *Analysis on graphs and its applications*, volume 77 of *Proc. Sympos. Pure Math.*, pages 523–564. Amer. Math. Soc., Providence, RI, 2008.
- [Gal14] J. Galkowski. Distribution of resonances in scattering by thin barriers. *arXiv preprint, arxiv : 1404.3709, to appear in Mem. Amer. Math. Soc.*, 2014.
- [Gal16] Jeffrey Galkowski. Resonances for thin barriers on the circle. *Journal of Physics A: Mathematical and Theoretical*, 49(12):125205, 2016.
- [GS77] Victor Guillemin and Shlomo Sternberg. *Geometric asymptotics*. American Mathematical Society, Providence, R.I., 1977. Mathematical Surveys, No. 14.
- [GS94] Alain Grigis and Johannes Sjöstrand. *Microlocal analysis for differential operators*, volume 196 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1994. An introduction.
- [GS14] J. Galkowski and H. Smith. Restriction bounds for the free resolvent and resonances in lossy scattering. *Int. Math. Res. Not*, 2014.
- [GU81] V. Guillemin and G. Uhlmann. Oscillatory integrals with singular symbols. *Duke Math. J.*, 48(1):251–267, 1981.
- [Hör07] Lars Hörmander. *The analysis of linear partial differential operators. III*. Classics in Mathematics. Springer, Berlin, 2007. Pseudo-differential operators, Reprint of the 1994 edition.
- [Hör09] Lars Hörmander. *The analysis of linear partial differential operators. IV*. Classics in Mathematics. Springer-Verlag, Berlin, 2009. Fourier integral operators, Reprint of the 1994 edition.

- [HT15] Xiaolong Han and Melissa Tacy. Sharp norm estimates of layer potentials and operators at high frequency. *J. Funct. Anal.*, 269(9):2890–2926, 2015. With an appendix by Jeffrey Galkowski.
- [HZ04] Andrew Hassell and Steve Zelditch. Quantum ergodicity of boundary values of eigenfunctions. *Comm. Math. Phys.*, 248(1):119–168, 2004.
- [Ida00] Nathan Ida. *Engineering electromagnetics*, volume 2. Springer, 2000.
- [JSS15] Dmitry Jakobson, Yuri Safarov, and Alexander Strohmaier. The semiclassical theory of discontinuous systems and ray-splitting billiards. *Amer. J. Math.*, 137(4):859–906, 2015. With an appendix by Yves Colin de Verdière.
- [KP90] Valery Kovachev and Georgi Popov. Invariant tori for the billiard ball map. *Trans. Amer. Math. Soc.*, 317(1):45–81, 1990.
- [KT95] Herbert Koch and Daniel Tataru. On the spectrum of hyperbolic semigroups. *Comm. Partial Differential Equations*, 20(5-6):901–937, 1995.
- [Mel76] R. B. Melrose. Equivalence of glancing hypersurfaces. *Invent. Math.*, 37(3):165–191, 1976.
- [Mil00] Luc Miller. Refraction of high-frequency waves density by sharp interfaces and semiclassical measures at the boundary. *Journal de mathématiques pures et appliquées*, 79(3):227–269, 2000.
- [MM82] Shahla Marvizi and Richard Melrose. Spectral invariants of convex planar regions. *J. Differential Geom.*, 17(3):475–502, 1982.
- [MT] R. Melrose and M. Taylor. *Boundary problems for wave equations with grazing and gliding rays*.
- [OLBC10] Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert, and Charles W. Clark, editors. *NIST handbook of mathematical functions*. U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010. With 1 CD-ROM (Windows, Macintosh and UNIX).
- [PS92] Vesselin M. Petkov and Luchezar N. Stoyanov. *Geometry of reflecting rays and inverse spectral problems*. Pure and Applied Mathematics (New York). John Wiley & Sons, Ltd., Chichester, 1992.
- [PV99a] Georgi Popov and Georgi Vodev. Distribution of the resonances and local energy decay in the transmission problem. *Asymptot. Anal.*, 19(3-4):253–265, 1999.
- [PV99b] Georgi Popov and Georgi Vodev. Resonances near the real axis for transparent obstacles. *Comm. Math. Phys.*, 207(2):411–438, 1999.
- [Sab64] W. C. Sabine. *Collected papers on acoustics*. Dover Publications, New York, NY, 1964.
- [Saf87] Yu. G. Safarov. On the second term of the spectral asymptotics of the transmission problem. *Acta Appl. Math.*, 10(2):101–130, 1987.
- [SZ99] Johannes Sjöstrand and Maciej Zworski. Asymptotic distribution of resonances for convex obstacles. *Acta Math.*, 183(2):191–253, 1999.
- [Tay11] Michael E. Taylor. *Partial differential equations II. Qualitative studies of linear equations*, volume 116 of *Applied Mathematical Sciences*. Springer, New York, second edition, 2011.
- [VN06] San Vũ Ngọc. *Systèmes intégrables semi-classiques: du local au global*, volume 22 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 2006.
- [WH85] Winfried RE Weiss and George A Hagedorn. Reflection and transmission of high frequency pulses at an interface. *Transport Theory and Statistical Physics*, 14(5):539–565, 1985.
- [Zal10] M. Zaletel. The Sabine law and a trace formula for lossy billiards. *Unpublished note*, 2010.
- [Zwo12] Maciej Zworski. *Semiclassical analysis*, volume 138 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.

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