A MICROLOCAL APPROACH TO EIGENFUNCTION CONCENTRATION

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ABSTRACT. We describe a new approach to understanding averages of high energy Laplace eigenfunctions, u_h, over submanifolds,

$$\left| \int_H u_h d\sigma_H \right|$$

where $H \subset M$ is a submanifold and σ_H the induced by the Riemannian metric on M. This approach can be applied uniformly to submanifolds of codimension $1 \leq k \leq n$ and in particular, gives a new approach to understanding $\|u_h\|_{L^\infty(M)}$. The method, developed in [GT17, Gal17, CGT18, CG17, CG18], relies on estimating averages by the behavior of u_h microlocally near the conormal bundle to H. By doing this, we are able to obtain quantitative improvements on eigenfunction averages under certain uniform non-recurrent conditions on the conormal directions to H. In particular, we do not require any global assumptions on the manifold (M, g).

1. INTRODUCTION

In this note, we describe a new approach to understanding concentration properties of high energy eigenfunctions. Although the methods in [GT17, Gal17, CGT18, CG17, CG18] (on which this note is based) apply to the quasimodes of a wide variety of pseudodifferential operators, we focus on the case of the Laplacian on a Riemannian manifold (M, g) of dimension n and consider only eigenfunctions i.e. solutions to

$$(-h^2 \Delta_g - 1)u_h = 0 \tag{1.1}$$

for concreteness. Consider a submanifold $H \subset M$. We are interested in averages of the form

$$\int_H u_h d\sigma_H$$

where σ_H denotes the volume measure induced on H from M. We note that, when $H = \{x\}$ is a point in M, this average is given precisely by $u_h(x)$. Thus, using our methods we are able to obtain control on L^∞ norms. We do not give the details of many proofs in this note, instead referring to the relevant papers. We review some of the previously existing results, state the new theorems, and describe the ideas central to the proofs.

Since the middle of the twentieth century [Ava56, Lev52, Hör68] many authors have been interested in the growth of eigenfunctions for self-adjoint elliptic operators. In particular, they prove that a solution to (1.1) satisfies,

$$\|u_h\|_{L^\infty(M)} \leq C h^{\frac{1-n}{2}} \|u_h\|_{L^2(M)}. \tag{1.2}$$

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If one considers the case of $(M, g) = (S^2, g_{\text{round}})$, the sphere with the round metric, it is not hard to construct the family of zonal harmonics, Z_h, with the property that

$$ch^{-\frac{1}{2}}\|Z_h\|_{L^2(S^2)} \leq \|Z_h\|_{L^\infty(S^2)} \leq Ch^{-\frac{1}{2}}\|Z_h\|_{L^2(S^2)}, \qquad (-h^2\Delta_{S^2} - 1)Z_h = 0,$$

and hence that the estimate (1.2) cannot be improved on a general manifold. Because of this, it is natural to try to understand situations in which (1.1) is sharp. It is also interesting to think of the question of L^∞ norms as averages over points and to generalize that question to averages over submanifolds $H \subset M$. While it is a more recent line of inquiry than that of L^∞ bounds, it dates at least to the early 1980's [Hej82, Goo83]. The analog of (1.1) was proved in [Zel92], where the author shows that if H has codimension k, then

$$\left| \int_H u_h d\sigma_H \right| \leq Ch^{\frac{1-k}{2}}\|u_h\|_{L^2(M)}. \tag{1.3}$$

Again, for every $1 \leq k \leq n$, there are examples on the sphere of dimension n which saturate the estimate (1.3) and it is natural to characterize situations in which (1.3) can be saturated.

1.1. **A review of previous L^∞ results.** Before we can state the results on a general manifold M, we need some concepts from geometry. Let T^*M denote the cotangent bundle to M, $H \subset M$ a submanifold with conormal bundle N^*H, and SN^*H, the unit conormal bundle to H,

$$SN^*H := \big\{ (x, \xi) \in N^*H \mid |\xi|_{g(x)} = 1 \big\},$$

where $|\cdot|_g$ denotes the metric induced on T^*M by g. Note that $SN^*\{x\} = S_x^*M$ where $S_x^*M := T_x^*M \cap S^*M$. Next, let $G^t : S^*M \to S^*M$ denote the geodesic flow.

We define the *first return time* $T_H : SN^*H \to [0, \infty]$ by

$$T_H(x, \xi) := \inf\{t > 0 \mid G^t(x, \xi) \in SN^*H\}.$$

We then define the *loop set of H*, $\mathcal{L}_H := T_H^{-1}([0, \infty))$. Finally, we let σ_{SN^*H} be the volume induced on SN^*H by the Liouville measure on T^*M.

Theorem 1 ([SZ02]). *Let (M, g) be a Riemannian manifold of dimension n. Suppose that $x \in M$ and $\sigma_{S_x^*M}(\mathcal{L}_x) = 0$. Then for u_h solving (1.1)*

$$|u_h(x)| = o\big(h^{\frac{1-n}{2}}\|u_h\|_{L^2(M)}\big).$$

We define the first return map $\eta_H : \mathcal{L}_H \to SN^*H$ by

$$\eta_H(x, \xi) := G^{T_H(x,\xi)}(x, \xi).$$

and let

$$\mathcal{L}_H^{\pm n} := \bigcap_{k=0}^{n} \eta_H^{\pm k}(SN^*H), \qquad \mathcal{L}_H^\infty := \bigcap_{n} \mathcal{L}_H^n.$$

Next, define the *recurrent set of H*

$$\mathcal{R}_H := \Big\{ (x, \xi) \in \mathcal{L}_H^\infty \mid (x, \xi) \in \Big[\bigcap_{n>0} \overline{\bigcup_{k \geq n} \eta_H^k(x, \xi)} \Big] \bigcap \Big[\bigcap_{n>0} \overline{\bigcup_{k \geq n} \eta_H^{-k}(x, \xi)} \Big] \Big\}.$$

In [STZ11], the authors show that $\mathcal{L}_x$ can be replaced by $\mathcal{R}_x$ in Theorem 1. Finally, in [SZ16a, SZ16b], the authors obtain still more restrictive assumptions in the case that (M, g) is real analytic. In fact, in the case of a real analytic surface, they are able to verify the conjecture that one can replace $\mathcal{R}_x$ by the set of directions ξ so that the geodesic through (x, ξ) is a smoothly closed loop.

If one wants to go beyond $o(1)$ improvements of (1.2), very few results are available. In [Bér77] (combined with [Bon16]), the author shows using the Hadamard parametrix that if (M, g) is a manifold without conjugate points, then

$$\|u_h\|_{L^\infty(M)} \leq C \frac{h^{\frac{1-n}{2}}}{\sqrt{\log h^{-1}}} \|u_h\|_{L^2(M)}. \tag{1.4}$$

The only polynomial improvements that the author is aware of appear in [IS95] where the authors study Hecke–Maas forms on certain arithmetic surfaces.

1.2. **A review of previous results on averages.** The study of when (1.3) is saturated is much more recent and, until the methods of this note were introduced, the only improvements on (1.3) available under no additional assumptions on (M, g) are:

Theorem 2 ([Wym17b])**.** *Suppose that* $\sigma_{SN^*H}(\mathcal{L}_H) = 0$. *Then*

$$\left| \int_H u_h d\sigma_H \right| = o\big(h^{\frac{1-k}{2}} \|u_h\|_{L^2(M)}\big)$$

The article [CS15] provides $o(1)$ improvements on (1.3) on surfaces of negative curvature when H is a geodesic.

On the other hand, on manifolds with non-positive curvature the Hadamard parametrix is available and as a result logarithmic improvements of the form

$$\left| \int_H u_h d\sigma_H \right| \leq C \frac{h^{\frac{1-k}{2}}}{\sqrt{\log h^{-1}}} \|u_h\|_{L^2(M)} \tag{1.5}$$

hold under a variety of assumptions on the pair (M, H) [SXZ16, Wym17a, Wym18]. However, none of these results give general dynamical conditions guaranteeing such improvements.

1.3. **Results of the microlocal techniques.** The question raised in all previous attempts to understand when (1.2) and (1.3) can be saturated can be thought of as 'In which geometries can saturation occur?'. The question raised in [GT17, CGT18, Gal17, CG17, CG18] is instead 'How does an eigenfunction that saturates (1.2) or (1.3) behave?' It then turns out that a sufficiently good understanding of the answer to the latter question yields answers to the former. In fact, by describing the behavior of eigenfunctions saturating (1.2), we will be able to extend *all* existing results. Moreover, our analysis of the eigenfunctions saturating these bounds demonstrates that the phenomena governing averages is identical to that governing L^∞ bounds.

1.3.1. *Defect measures.* We begin by describing some of the results of [CG17] where we rely on defect measures to describe the behavior of u_h. Recall that a defect measure is a positive Radon measure, μ, on T^*M associated to a sequence of functions $\{u_h\}_{0<h<h_0}$ so that for any $a \in C_c^\infty(T^*M)$,

$$\langle Op_h(a)u_h, u_h \rangle_{L^2(M)} \to \int a(x,\xi)d\mu$$

where $Op_h(a)$ denotes the quantization of the symbol a (see e.g. [DZ16, Appendix E] for a description of quantization procedures). See, for example, [Zwo12, Chapter 5] for a treatment of these measures. We recall that every L^2 bounded sequence of functions $\{u_h\}$ has a subsequence with a defect measure, μ and, moreover if u_h solves (1.1), then $\operatorname{supp}\mu \subset S^*M$ and μ is G^t invariant.

Suppose that μ is a finite radon measure invariant under the geodesic flow. Then we define for any Borel $A \subset SN^*H$,

$$\mu_H(A) := \lim_{\delta \to 0} \frac{1}{2\delta}\mu\Big(\bigcup_{|t|\leq\delta} G^t(A)\Big).$$

We write $\omega \perp \nu$ when the measures ω and ν are mutually singular.

We then have the following consequence of [CG17, Theorem 6] (see also [Gal17, Theorem 2] for the case $k = n$).

Theorem 3. *Let (M,g) be a smooth compact Riemannian manifold and $H \subset M$ a closed embedded submanifold of codimension k. Suppose that u_h solves (1.1) and has defect measure μ. Let $f \in L^1(SN^*H; \sigma_{SN^*H})$ so that*

$$\mu_H = f d\sigma_{SN^*H} + \omega, \qquad \omega \perp \sigma_{SN^*H}.$$

Then there is $C_{n,k} > 0$ depending only on (n,k) so that for $A \subset H$ with smooth boundary,

$$\Big| \int_A u_h d\sigma_H \Big| \leq C_{n,k} h^{\frac{1-k}{2}} \int_{\pi_H^{-1}(A)} \sqrt{f}d\sigma_{SN^*H} + o(h^{\frac{1-k}{2}})$$

where $\pi_H : SN^*H \to H$ is the natural projection.

Note that Theorem 3 can be interpreted as saying that *every* eigenfunction which maximizes either (1.2) or (1.3) must have a component which behaves $o(1)$ microlocally the same as the canonical example on S^n. In particular, in order that u_h maximize the L^∞ bounds, there must be a point where u_h behaves like the zonal harmonic, Z_h (See e.g. [GT17, Section 4] for a description of the defect measures of zonal harmonics.)

As an easy consequence of Theorem 3 together with the Poincaré recurrence theorem we are able to replace $\mathcal{L}_H$ in Theorem 2 by $\mathcal{R}_H$.

Corollary 4 ([CG17, Theorem 2])**.** *Suppose that $A \subset H$ has smooth boundary, $\sigma_{SN^*H}(\pi_H^{-1}(A) \cap \mathcal{R}_H) = 0$, and u_h solves (1.1) then*

$$\Big| \int_A u_h d\sigma_H \Big| = o\big(h^{\frac{1-k}{2}}\|u_h\|_{L^2(M)}\big).$$

Using geometric arguments to show that $\sigma_{SN^*H}(\mathcal{R}_H) = 0$ in a variety of settings, we are then able to recover all existing $o(1)$ improvements over (1.3) in [CG17, Theorem 4].

Theorem 5 ([CG17, Theorem 4],[CG18, Theorem 3]). *In all of the following situations, $\sigma_{SN^*H}(\mathcal{R}_H) = 0$.*

    A. *(M, g) has no conjugate points and H has codimension $k > \frac{n+1}{2}$.*

    B. *(M, g) has no conjugate points and H is a geodesic sphere.*

    C. *(M, g) is a surface with Anosov geodesic flow and H is any submanifold.*

    D. *(M, g) has constant negative curvature and H is any submanifold.*

    E. *(M, g) has Anosov flow, non-positive curvature, and $k > 1$.*

    F. *(M, g) has Anosov geodesic flow and non-positive curvature, and H is totally geodesic.*

    G. *(M, g) has Anosov geodesic flow and H is a subset M that lifts to a horosphere.*

1.3.2. *Towards quantitative estimates.* In order to pass to the quantitative estimates from [CG18], we will first describe some easy consequences of Theorem 3. We say that $A \subset T^*M$ is $[t, T]$ *non-self looping* if either

$$G^s(A) \cap A = \emptyset, \qquad s \in [t, T], \qquad \text{or} \qquad G^{-s}(A) \cap A = \emptyset, \qquad s \in [t, T] \qquad (1.6)$$

We have the following Corollary of Theorem 3.

Corollary 6. *Suppose that there is a an h-independent covering $\{B, \{G_\ell\}_\ell\}$ of SN^*H and $\{t_\ell\}_\ell$, $\{T_\ell\}_\ell \subset (1, \infty)$ with $t_\ell < T_\ell$ independent of h so that*

$$SN^*H = B \cup \bigcup_\ell G_\ell.$$

and G_ℓ is $[t_\ell, T_\ell]$ non-self looping an that u_h solves (1.1). Then, there is $C > 0$ so that for all u_h solving (1.1)

$$\left| \int_H u_h d\sigma_H \right| \leq C h^{\frac{1-k}{2}} \left(\sigma_{SN^*H}(B)^{\frac{1}{2}} + \sum_\ell \frac{\sigma_{SN^*H}(G_\ell)^{\frac{1}{2}} t_\ell^{\frac{1}{2}}}{T_\ell^{\frac{1}{2}}} + o(1) \right) \|u_h\|_{L^2(M)} \qquad (1.7)$$

In fact, Corollary 4 can be deduced from Corollary 6. To see this, let $\{U_i\}_{i=1}^\infty$ be a basis for the topology of SN^*H. Then let $T > 0$ and set

$$E_i^{\pm, T} := \{x \in U_i \mid G^t(x) \notin U_i, \pm t > T\}, \qquad E_i^T = \bigcup_\pm E_i^{\pm, T}, \qquad E_i^\infty = \bigcup_{T > 0} E_i^T.$$

Let

$$B_N = \left[\bigcap_i \left(SN^*H \setminus E_i^T \right) \right] \cup \left(\bigcup_{j=N}^\infty E_j^T \setminus \bigcup_{k=1}^{N-1} E_k^T \right), \qquad G_i = E_i^T.$$

Then, since G_i is $[T, S]$ non-self looping for any $S > T$, we apply (1.7) to obtain

$$\limsup_{h \to 0} \frac{h^{\frac{k-1}{2}}}{\|u_h\|_{L^2(M)}} \left| \int_H u_h d\sigma_H \right| \leq C \left(\sigma_{SN^*H}(B_N)^{\frac{1}{2}} + \sum_{i=1}^{N-1} \frac{\sigma_{SN^*H}(E_i^T)^{1/2} T^{\frac{1}{2}}}{S^{\frac{1}{2}}} \right)$$

Sending $S \to \infty$ gives

$$\limsup_{h \to 0} \frac{h^{\frac{k-1}{2}}}{\|u_h\|_{L^2(M)}} \left| \int_H u_h d\sigma_H \right| \leq C \sigma_{SN^*H}(B_N)^{\frac{1}{2}}$$

Sending $N \to \infty$ then gives

$$\limsup_{h \to 0} \frac{h^{\frac{k-1}{2}}}{\|u_h\|_{L^2(M)}} \Big| \int_H u_h d\sigma_H \Big| \leq C \sigma_{SN^*H} \Big(\bigcap_i \big(SN^*H \setminus E_i^T \big) \Big)^{\frac{1}{2}}$$

Finally, sending $T \to \infty$ gives

$$\limsup_{h \to 0} \frac{h^{\frac{k-1}{2}}}{\|u_h\|_{L^2(M)}} \Big| \int_H u_h d\sigma_H \Big| \leq C \sigma_{SN^*H} \Big(\bigcap_i \big(SN^*H \setminus E_i^\infty \big) \Big)^{\frac{1}{2}}$$

Now, suppose x is not recurrent. Then, there exists i, T so that $x \in U_i$, $G^t(x) \notin U_i$ either for $t > T$ or $-t > T$. In particular, $x \in E_i^T \subset E_i^\infty$. Therefore, if x is not recurrent, then $x \in \cup_i E_i^\infty$. In particular,

$$SN^*H \setminus \mathcal{R}_H \subset \bigcup_i E_i^\infty, \qquad \text{so} \qquad \bigcap_i \big(SN^*H \setminus E_i^\infty \big) \subset \mathcal{R}_H.$$

Therefore, if $\mathcal{R}_H$ has measure 0, then $\bigcap_i \big(SN^*H \setminus E_i^\infty \big)$ has measure 0 and we have obtained Corollary 4.

The fact that $\rho \in \mathcal{R}_H$ does not contain *any* quantitative information about how long it takes for the geodesic through ρ to return to a given neighborhood of ρ. Because of this, one should not expect to have a quantitative version of Corollary 4. However, Corollary 6 *is* quantitative and one might hope that it holds even with B, G_ℓ, and $[t_\ell, T_\ell]$ h-dependent. This is almost true, although we will require some additional structure of the sets B and G_ℓ (see Theorem 8).

1.3.3. *Quantitative Estimates.* In order to state our quantitative estimates, we will need to define a few additional objects. We will use the metric induced by the Sasaki metric on T^*M (see e.g. [Ebe73] for a description of the Sasaki metric) for convenience, but our results do not depend on the choice of metric on T^*M. First, fix $\mathcal{H} \subset T^*M$ a smooth hypersurface transverse to the geodesic flow so that $SN^*H \subset \mathcal{H}$. Define $\psi : \mathbb{R} \times \mathcal{H}_\Sigma \to T^*M$ by $\psi(t, q) = \varphi_t(q)$. Next, let

$$\tau_{\mathrm{inj}H} := \sup\{\tau \leq 1 : \psi|_{(-\tau, \tau) \times \mathcal{H}_\Sigma} \text{ is injective}\}.$$

Given $A \subset T^*M$, define

$$\Lambda_A^\tau := \bigcup_{|t| \leq \tau} G^t(A).$$

Then, for $r > 0$ and $A \subset \mathcal{H}$, define

$$\Lambda_A^\tau(r) := \Lambda_{A_r}^{\tau+r}, \qquad A_r := \{\rho \in \mathcal{H} \mid d(\rho, A) < r\}.$$

Finally, let $K_H > 0$ be a bound for the the sectional curvatures of H and for the second fundamental form of H.

Theorem 7. *Let $H \subset M$ be a closed embedded submanifold of codimension k. There exist $C_{n,k} > 0$ depending only on n, k, $\tau_0 > 0$ depending on $(M, g, \tau_{\mathrm{inj}H})$, and $R_0 = R_0(n, k, K_H)$ so that the following holds.*

Let $0 < \tau < \tau_0$, $0 \leq \delta < \frac{1}{2}$, $N > 0$, and $R_0 > R(h) \geq 5h^\delta$. Then, there exists a family $\{\gamma_j\}_{j=1}^{N_h}$ of geodesics through SN^*H, and a partition of unity $\{\chi_j\}_{j=1}^{N_h}$ for $\Lambda_{\Sigma_{H,p}}^{\tau}(h^\delta)$ with $\chi_j \in S_\delta \cap C_c^\infty(T^*M; [0,1])$,

$$\operatorname{supp}\chi_j \subset \Lambda_{\rho_j}^{\tau}(R(h)), \qquad \rho_j := \gamma_j \cap SN^*H,$$

so that for all $w \in C_c^\infty(H)$, $N > 0$ there is $C_N > 0$ and $h_0 > 0$ with the property that for any $0 < h < h_0$ and all u_h solving (1.1)

$$h^{\frac{k-1}{2}}\Big| \int_H w u_h d\sigma_H \Big| \leq C_{n,k} R(h)^{\frac{n-1}{2}} \sum_j \frac{\|Op_h(\chi_j)u_h\|_{L^2(M)}}{\tau^{\frac{1}{2}}} + C_N h^N \|u_h\|_{L^2(M)}.$$

Theorem 7 is a much finer analog of Theorem 3 and in particular can be interpreted as saying that *every* eigenfunction which maximizes either (1.2) or (1.3) must have a component which behaves the same as the canonical example on S^n microlocally on h^δ scales. In particular, in order that u_h maximize the L^∞ bounds, there must be a point where u_h behaves like a zonal harmonic at scale h^δ.

While at first it may seem difficult to use Theorem 7 in concrete situations, combining Theorem 7 with Egorov's theorem up to the Ehrenfest time (see e.g. [DG14]) we obtain a purely dynamical estimate which is readily applicable.

We define the *maximal expansion rate*

$$\Lambda_{\max} := \limsup_{|t| \to \infty} \frac{1}{|t|} \log \sup_{S^*M} \|dG^t(x,\xi)\|.$$

Then the Ehrenfest time at frequency h is

$$T_e(h) := \frac{\log h^{-1}}{2\Lambda_{\max}}.$$

Note that $\Lambda_{\max} \in [0,\infty)$ and if $\Lambda_{\max} = 0$, we may replace it by an arbitrarily small positive constant. We have the following quantitative version of Corollary 6.

Theorem 8 ([CG18, Theorem 5]). *Suppose that $H \subset M$ is a closed embedded submanifold of codimension k. Let $0 < \delta < \frac{1}{2}$, $N > 0$. There exist positive constants $h_0 = h_0(M,g,K_H)$, $\tau_0 = \tau_0(M,g,\tau_{injH})$, $R_0 = R_0(n,k,K_H)$ and $C_{n,k}$ depending only on n and k, and for each $0 < \tau < \tau_0$ there exists and $C_N = C_N(\tau,\delta,M,g) > 0$, so that the following holds.*

Let $R_0 > R(h) \geq 5h^\delta$, $\alpha < 1 - 2\limsup_{h\to 0} \frac{\log R(h)}{\log h}$, and suppose $\{\Lambda_{\rho_j}^{\tau}(R(h))\}_{j=1}^{N_h}$ is a cover of $\Lambda_{\Sigma_{H,p}}^{\tau}(h^\delta)$ that is the union of $C_{n,k}$ subsets of disjoint tubes (the existence is guaranteed by [CG18, Lemma 2.2]). In addition, suppose there exist $\mathcal{B} \subset \{1, \ldots N_h\}$ and a finite collection $\{\mathcal{G}_\ell\}_{\ell \in L} \subset \{1, \ldots N_h\}$ with

$$\{1, \ldots N(R(h))\} \subset \mathcal{B} \cup \bigcup_{\ell \in L} \mathcal{G}_\ell,$$

and so that for every $\ell \in L$ there exist $t_\ell(h) > 0$ and $T_\ell(h) \leq 2\alpha T_e(h)$ so that

$$\bigcup_{j \in \mathcal{G}_\ell} \Lambda_{\rho_j}^{\tau}(R(h)) \quad \text{is } [t_\ell(h), T_\ell(h)] \text{ non-self looping.}$$

Then, for all $w \in C_c^\infty(H)$, $N > 0$ there exists $C_N > 0$, $h_0 > 0$ so that for all for u_h solving (1.1) *and* $0 < h < h_0$,

$$h^{\frac{k-1}{2}}\Big|\int_H wu_h d\sigma_H\Big| \leq \frac{C_{n,k}\|w\|_{L^\infty}R(h)^{\frac{n-1}{2}}}{\tau^{\frac{1}{2}}}\left[|\mathcal{B}|^{\frac{1}{2}} + \sum_{\ell \in L}\frac{|\mathcal{G}_\ell|^{\frac{1}{2}}t_\ell^{\frac{1}{2}}(h)}{T_\ell^{\frac{1}{2}}(h)} + C_N h^N\right]\|u_h\|_{L^2(M)}.$$

$$(1.8)$$

Note that the term

$$R(h)^{n-1}|\mathcal{G}_\ell| \propto \sigma_{SN^*H}\Big(\bigcup_{j \in \mathcal{G}_\ell}\Lambda^\tau_{\rho_j}(R(h)) \cap SN^*H\Big).$$

and in particular when T_ℓ, $R(h)$ are h independent (1.8) implies (1.7). Since (1.7) implies Corollary 4, Theorem 8 should be thought of as a quantitative version of the non-recurrent condition. With this intuition in mind, we are able to construct effective covers by tubes in many geometric situations.

Theorem 9 ([CG18, Theorem 3]). *Let (M, g) be a smooth, compact Riemannian manifold of dimension n. Let $H \subset M$ be a closed embedded submanifold of codimension k. Suppose one of the following assumptions holds:*

  A. *(M, g) has no conjugate points and H has codimension $k > \frac{n+1}{2}$.*

  B. *(M, g) has no conjugate points and H is a geodesic sphere.*

  C. *(M, g) is a surface with Anosov geodesic flow.*

  D. *(M, g) has Anosov geodesic flow, non-positive curvature and $k > 1$.*

  E. *(M, g) has Anosov geodesic flow and non-positive curvature, and H is totally geodesic.*

  F. *(M, g) has Anosov geodesic flow and H is a subset of M that lifts to a horosphere in the universal cover.*

Then there exists $C > 0$ so that for all $w \in C_c^\infty(H)$ there is $h_0 > 0$ so that for $0 < h < h_0$ and u_h solving (1.1)

$$\Big|\int_H wu_h d\sigma_H\Big| \leq Ch^{\frac{1-k}{2}}\frac{\|u_h\|_{L^2(M)}}{\sqrt{\log h^{-1}}}.$$

$$(1.9)$$

Finally, there is some uniformity in the estimates from Theorem 7 and we can obtain L^∞ estimates. To state these estimates we need to recall a few notions from Riemannian geometry. A Jacobi field along a geodesic $\gamma(t)$ is a vector field along $\gamma(t)$ satisfying

$$D_t^2 J + R(J, \dot\gamma)\dot\gamma = 0$$

where D_t denotes the covariant derivative along γ and $R(\cdot, \cdot)(\cdot)$ denotes the Riemann curvature tensor (see e.g. [Lee06, Chapter 10]). We say that J is perpendicular of $\langle J, \dot\gamma\rangle_g = 0$ and $\langle D_t J, \dot\gamma\rangle_g = 0$.

For a geodesic γ, we say that γ has a conjugate point of multiplicity m at t_0 if there are perpendicular Jacobi fields $\{J_i\}_{i=1}^m$ so that $J_i(0) = 0$, $\{D_t J_i(0)\}_{i=1}^m$ are linearly independent, and $J_i(t_0) = 0$. Note that the maximum multiplicity of of a conjugate point is $n - 1$ where n is the dimension of the manifold M. Moreover, it is not hard

to see that there exists $\delta > 0$ so that for any geodesic γ and any $t_0 \in \mathbb{R}$, there are at most $n - 1$ conjugate points counted with multiplicity in $[t_0 - \delta, t_0 + \delta]$.

Define

$$\Lambda_x^{m,r,t} := \big\{ \gamma \in \Lambda : \gamma(0) = x, \exists \text{ at least } m \text{ conjugate points to } x \text{ in } \gamma([t - r, t + r]) \big\},$$

where we count conjugate points with multiplicity. Next, for a set $V \subset M$ write

$$\mathcal{C}_V^{m,r,t} := \bigcup_{x \in V} \{ \gamma(t) : \gamma \in \Lambda_x^{m,r,t} \}.$$

Note that the set $\mathcal{C}_x^{n-1,0,t}$ is the set of points that are *maximally conjugate to* x at time t. In particular, for $y \in \mathcal{C}_x^{n-1,0,t}$ there is a geodesic γ with $\gamma(0) = x$, $\gamma(t) = y$ and so that *all* of the perpendicular Jacobi fields vanish at t. One case where this happens is on the sphere where x and y are antipodal points. While the condition $x \notin \mathcal{C}_x^{n-1,0,t}$ for $t \geq t_0$ is enough to guarantee $o(1)$ improvements in L^∞ bounds, a notion of *uniform maximal self conjugacy* is necessary to have quantitative improvements.

Theorem 10 ([CG18, Theorem 1]). *Let $U \subset M$ and suppose that there is $T > 0, a > 0$ so that for all $x \in U$,*

$$d\Big(x, \mathcal{C}_x^{n-1,r_a(t),t}\Big) \geq r_a(t), \qquad t \geq T$$

where $r_a(t) = a^{-1} e^{-at}$. Then

$$\|u_h\|_{L^\infty(U)} \leq C \frac{h^{\frac{1-n}{2}}}{\sqrt{\log h^{-1}}} \|u_h\|_{L^2(M)}.$$

It is not hard to see that Theorem 8 implies even stronger estimates where we only assume certain volume control on the directions along which x is maximally self-conjugate.

We note at this point that Theorems 9 and 10 subsume *all* previous conditions known to give logarithmic improvements and 5 subsumes *all* previous conditions known to give $o(1)$ improvements.

2. The overall ideas of the proofs

2.1. **The microlocal estimate.** The first important observation in the proof of Theorem 7 is that the most localized that an eigenfunction can be is to an $h^{\frac{1}{2}}$ tube around a single length ~ 1 piece of geodesic. This is the case, for example, for the highest weight spherical harmonics on S^2 given by the restriction of $j^{\frac{n-1}{4}} (x_1 + ix_2)^j$ to the sphere, $x_1^2 + x_2^2 + x_3^2 = 1$. It is then natural to think of building an eigenfunction out of pieces localized to such tubes. Locally, these pieces are of the form

$$u_h(x) = h^{-\frac{1-n}{4}} e^{\frac{i}{h} x_1} e^{-\frac{|x|^2}{2h}} a(x)$$
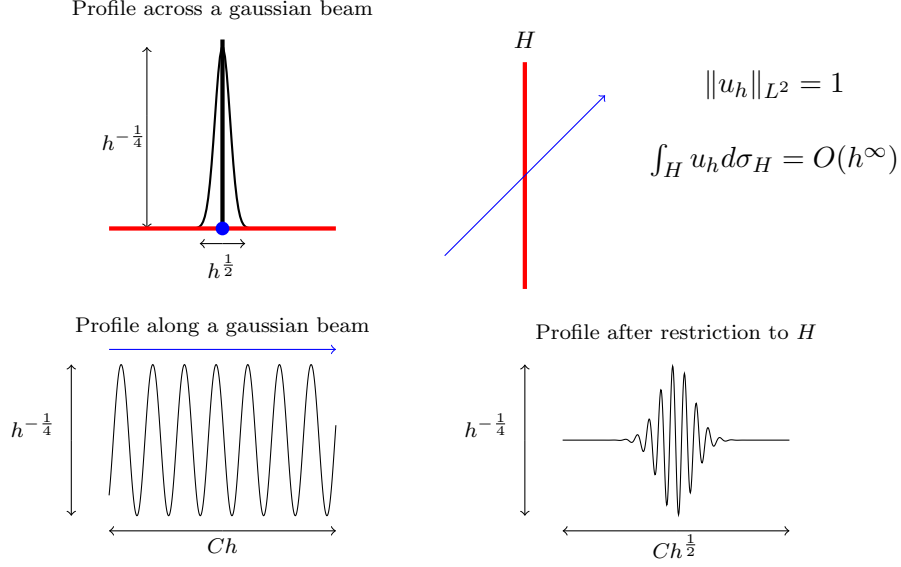
Profile across a gaussian beam

FIGURE 1.   Diagram when a gaussian beam passes over H non-normally

with the geodesic given by $\gamma = \{(x_1, 0, 1, 0) \mid |x_1| < 1\}$. Here, we have taken (x_1, x', ξ_1, ξ') as coordinates on T^*M. We will refer to this type of object as a *gaussian beam*.

The first step is then to understand how the average over H of an eigenfunction localized to such a tube behaves. This is a two step process. First, if the tube is passing over the hypersurface in a direction which is not normal to the hypersurface, then the contribution is $O(h^\infty)$. Such a restriction is shown in Figure 1. Since oscillation remains after restriction, the contribution from such a tube is $O(h^\infty)$. Once we have this in place, we need to study tubes passing normally over H as in Figure 2.

When we decompose eigenfunctions using tubes, we will use tubes of size $R(h) \geq h^\delta$ for some $0 \leq \delta < \frac{1}{2}$ so that the symbolic calculus works well. It remains to understand how eigenfunctions localized to such tubes behave when restricted to submanifolds. The key observation is that localization to a small tube implies better control on oscillation and that this control gives improved L^∞ estimates. In particular, imagine that we are working on $\mathbb{R}^n$ with coordinates (x_1, x', ξ_1, ξ') on $T^*\mathbb{R}^n$ and

$$\gamma = \{(x_1, 0, 1, 0) \mid |x_1| < 1\} \subset T^*\mathbb{R}^n.$$

Then, assume that a function u_h has frequencies only in $|\xi'| \leq R(h)$, i.e. with

$$\mathcal{F}_h(u_h)(\xi) := \int e^{-\frac{i}{h}\langle x, \xi \rangle} u_h(x) dx,$$

satisfying $\operatorname{supp} \mathcal{F}_h(u) \subset \{|\xi'| \leq R(h)\}$ modulo $O(h^\infty)$. Then

$$\|(hD_{x'})^m u_h\|_{L^2} \leq CR(h)^m \|u_h\|_{L^2}.$$

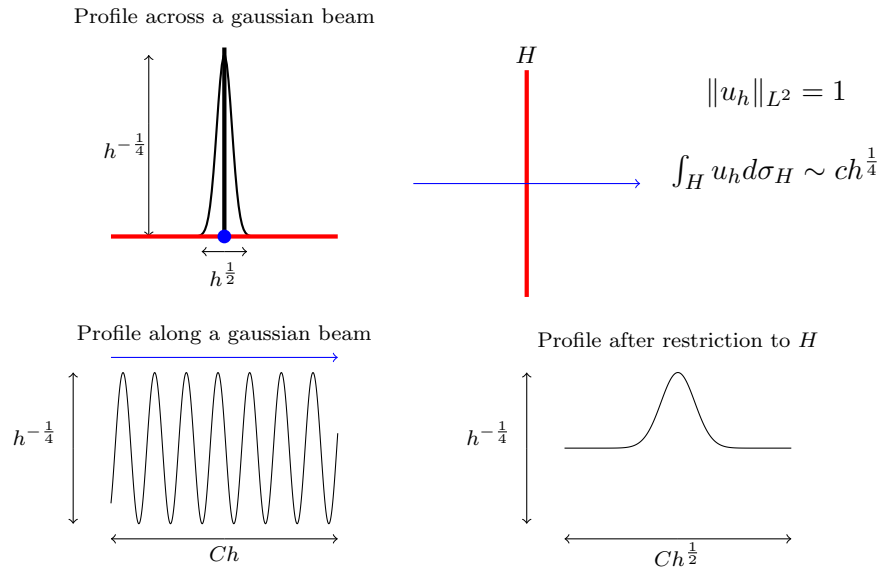In particular, u is oscillating at frequency $R(h)h^{-1}$ in the x' variables.

FIGURE 2. Diagram when a gaussian beam passes over H normally

Now, suppose that H is given by $\{x_1 = \bar{x} = 0\}$ where $\bar{x} \in \mathbb{R}^{k-1}$ and $x' = (\bar{x}, x'')$. By the standard Sobolev embedding $H^s(\mathbb{R}^m) \to L^\infty(\mathbb{R}^{k-1})$ for $s > \frac{k-1}{2}$, such oscillation then implies that

$$\|u_h(x_1, \cdot)\|_{L^\infty_{\bar{x}} L^2_{x''}} \leq CR(h)^{\frac{k-1}{2}} h^{\frac{1-k}{2}} \|u_h(x_1, \cdot)\|_{L^2_{x'}}.$$

Then, if we assume in addition $(-h^2\Delta - 1)u = 0$, a standard energy estimate (see e.g. [Zwo12, Chapter 7]) implies that for u localized close enough (h independently) to γ,

$$\|u_h(x_1, \cdot)\|_{L^2_{x'}} \leq C\|u_h\|_{L^2}.$$

In particular,

$$\|u_h\|_{L^\infty_{\bar{x}} L^2_{x''}} \leq CR(h)^{\frac{k-1}{2}} h^{\frac{1-k}{2}} \|u_h\|_{L^2}.$$

Finally, if u is also supported on $|x'| \leq R(h)$ modulo $O(h^\infty)$, then

$$\left| \int u_h(0, x'')dx'' \right| \leq CR(h)^{\frac{k}{2}} \|u_h(0, \cdot)\|_{L^2_{x''}} \leq CR(h)^{\frac{n-1}{2}} h^{\frac{1-k}{2}} \|u_h\|_{L^2}. \qquad (2.1)$$

In order to make this argument on a general manifold, we construct microlocal cut-offs, χ, to $R(h)$ sized tubes around geodesics (see Figure 3) which essentially commute with the Laplacian near H. We are then able to use the calculus of pseudodifferential operators to obtain the estimate (2.1).

In order to complete the proof of Theorem 7, we then cover SN^*H by tubes as in Figure 4. In the case of $k = n$, combining the estimates is just a matter of applying the triangle inequality. However, when $k < n$, we must once again use that, modulo $O(h^\infty)$, the cutoffs are supported *in physical space* at a distance $R(h)$ from a geodesic. Covering H by balls of radius $R(h)$ and applying the triangle inequality in each ball then gives the required estimate.
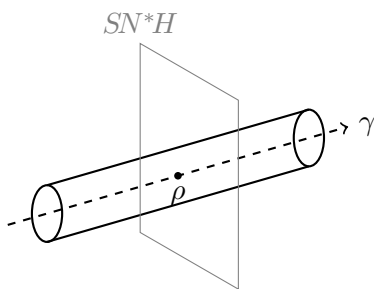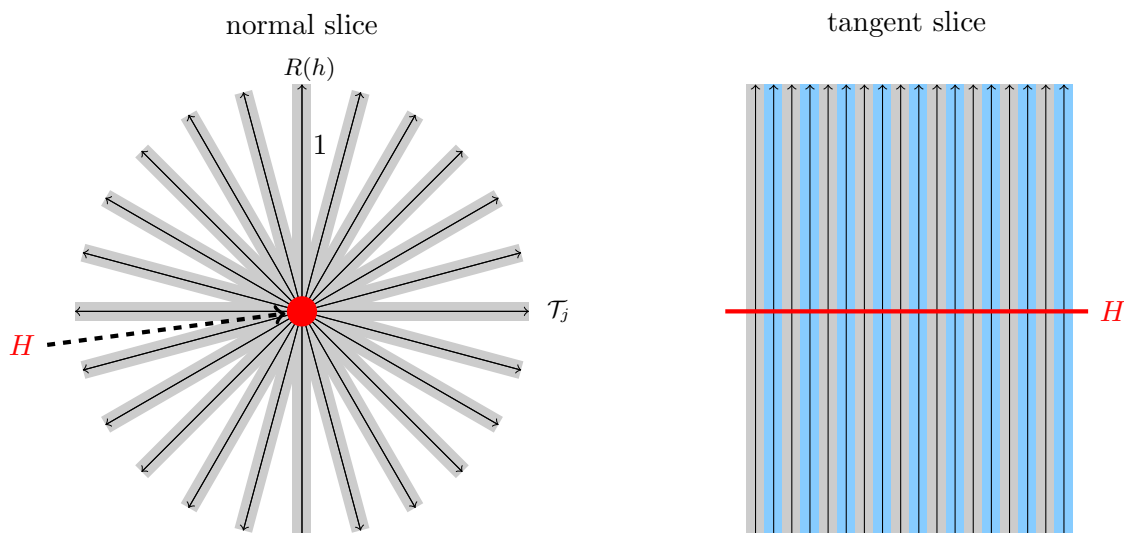
FIGURE 3. A single tube

FIGURE 4. The cover of SN^*H by tubes. (left) The projection onto a plane normal to H. (right) The projection of the tubes onto a plane tangent to H. Note that each pair of tubes (pointing up and down) on the right corresponds to a whole sphere of tubes in SN^*H as pictured on the left. The tubes alternate in color only to make it easier to distinguish adjacent tubes.

2.2. **From the microlocal estimate to a Theorem 8.** Passing from Theorem 7 to Theorem 8 is an application of Egorov's theorem to long times. In particular, observe that if $\chi \in C_c^\infty(T^*M; [0,1]) \cap S_\delta$ is $[t_0, T_0]$ non-self looping, then

$$\left| \frac{1}{T} \int_0^T \chi^2 \circ \varphi_t dt \right| \leq \frac{t_0}{T}. \tag{2.2}$$

Here, we say $\chi \in S_\delta$ if

$$\left| \partial_x^\alpha \partial_\xi^\beta \chi(x, \xi) \right| \leq C_{\alpha\beta} h^{-\delta(|\alpha|+|\beta|)}.$$

Together with Egorov's theorem to the Ehrenfest time (see e.g. [DG14, Proposition 3.8]), (2.2) implies that

$$\|Op_h(\chi)u_h\|_{L^2}^2 = \left\langle \frac{1}{T}\int_0^T e^{ith\Delta_g}Op_h(\chi)^*Op_h(\chi)e^{-ith\Delta_g}dt u_h, u_h \right\rangle_{L^2(M)}$$
$$\leq \frac{t_0}{T}\big(1+O(h^\varepsilon)\big)\|u_h\|_{L^2(M)}^2.$$

In particular, if $\bigcup_{j\in\mathcal{G}_\ell}\Lambda_{\rho_j}^\tau(R(h))$ is $[t_\ell, T_\ell]$ non-self looping, then since there are at most C χ_j with overlapping supports,

$$\sum_{j\in\mathcal{G}_\ell}\|Op_h(\chi_j)u_h\|_{L^2}^2 \leq C\Big\|\sum_{j\in\mathcal{G}_\ell}Op_h(\chi_j)u_h\Big\|_{L^2}^2 \leq \frac{t_\ell}{T_\ell}\big(1+O(h^\varepsilon)\big)\|u_h\|_{L^2(M)}^2. \qquad (2.3)$$

An application of Cauchy–Schwarz together with Theorem 7 and (2.3) then gives Theorem 8.

2.3. **Construction of effective covers.** There are two mechanisms used to construct the effective covers for Theorem 9; contraction and rotation.
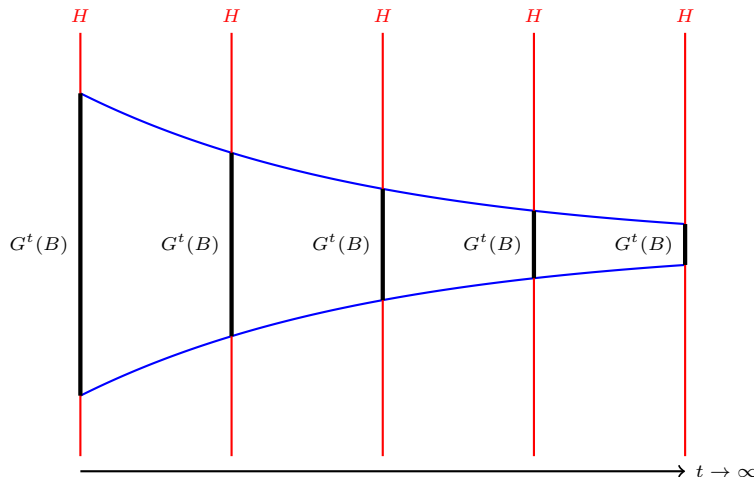

FIGURE 5. Contraction mechanism for constructing effective covers.

2.3.1. *Contraction.* In the contraction mechanism, pictured in Figure 5, we use the fact that a subset of SN^*H contracts under the flow either forward or backward in time. This is the case, for instance, when H is contained in a stable or unstable horosphere. Under this condition, we start with a macroscopic set A_0 and flow it forward in time. We remove all of the pieces of

$$B_0 := \bigcup_{t_0}^T G^t(A_0)$$

intersecting A_0. Since A_0 is contracting, we may choose t_0 large enough so that $\sigma_{SN^*H}(B_0) \leq \varepsilon\sigma_{SN^*H}(A_0)$. We then let $G_0 = A_0 \setminus B_0$ and $A_1 = B_0$. By construction G_0 is $[t_0, T]$ non-self looping. We can then repeat the process replacing T by $\frac{1}{2}T$

to obtain G_1 that is $[t_0, \frac{1}{2}T]$ non-self looping. Inductively repeating this process, we construct an effective non-self looping cover of SN^*H.
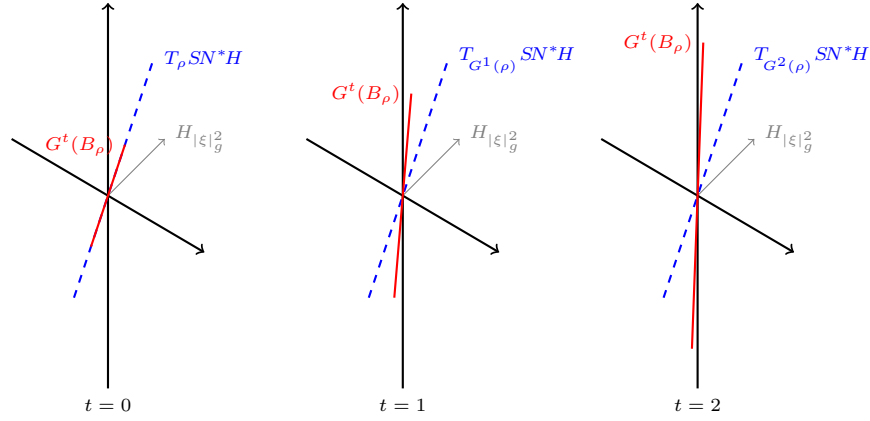

FIGURE 6.   Rotation mechanism for constructing effective covers.

2.3.2. *Rotation.* In the rotation mechanism, pictured in Figure 6, a ball of small radius $B_\rho \subset SN^*H$ rotates to become transverse to the plane of $T_{G^t(\rho)}SN^*H$ (when $G^t(\rho) \in SN^*H$) as $t \to \pm\infty$. In this situations, we can use the implicit function theorem to show that the intersection of $\bigcup_{t_0}^T \varphi_t(B_\rho)$ with SN^*H is a finite union of lower dimensional subsets. Covering these lower dimensional subsets by tubes with small volume, we are able to construct an effective cover.
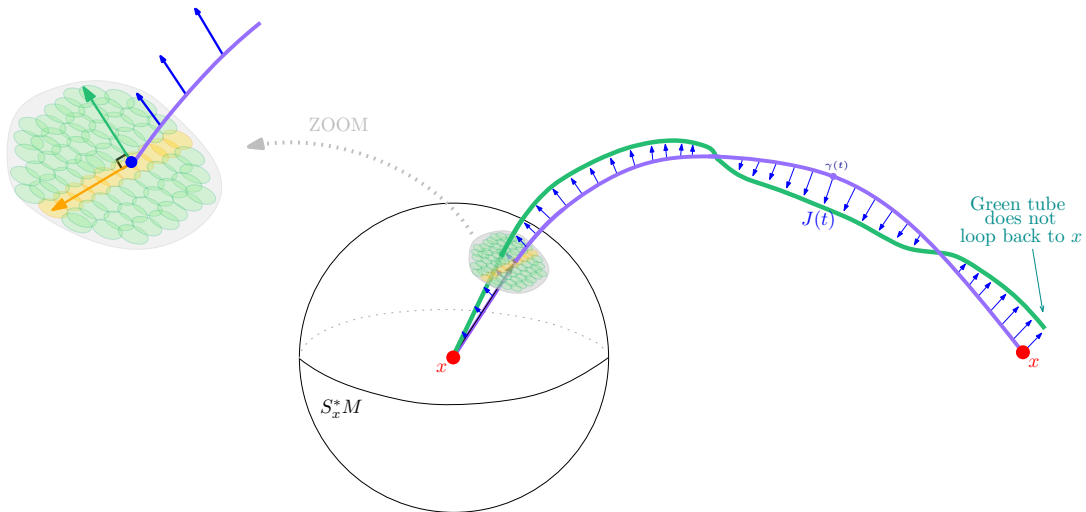

FIGURE 7.   The construction of a non-self looping collection of tubes under a non-uniformly maximal self conjugacy assumption.

2.3.3. *Effective covers with no uniformly maximal self-conjugate points.* We now sketch the construction of the cover by tubes that is used to prove Theorem 10. The crucial fact is that if J is a Jacobi field along γ with $J(0) = 0$ and $\Gamma : (-\varepsilon, \varepsilon) \times \mathbb{R} \to M$ is a map so that

$$\Gamma(s, 0) = \gamma(0), \qquad \gamma_s : t \mapsto \Gamma(s, t) \text{ is a geodesic}, \qquad \partial_s D_t \Gamma(0, 0) = D_t J(0),$$

then $\partial_s \gamma(0, t) = J(t)$. Said another way, if $J(t_0)$ is non-zero, then for $s \neq 0$ small $\gamma_s(t_0) \neq \gamma(0)$.

Translating this from the $SM \subset TM$ to $S^*M \subset T^*M$, this implies that there is a vector $V = (D_t J(0))^\sharp \in T_{\dot\gamma^\sharp} S_x^* M$ so that

$$d\pi dG^{t_0} V \neq 0$$

where $\pi : T^*M \to M$ denotes the projection. Using this together with the implicit function theorem, we find a submanifold $B \subset S_x^* M$ of dimension$< n - 1$ and a neighborhood W of $\dot\gamma^\sharp$ so that for t near t_0 and $\rho \in W \setminus B$, $G^t(\rho) \notin S_x^* M$. We can then cover B by $\sim R(h)^{2-n}$ tubes.

Since x is not maximally self-conjugate for $t > s_0$, we can repeat this argument near each point $\rho \in S_x^* M$ and then for approximately T values of t_0, we produce a large collection of tubes, $\mathcal{G}$ whose union is $[s_0, T]$ non-self looping and $\sim T R(h)^{2-n}$ possibly looping tubes $\mathcal{B}$.

In order to make this construction work, we must control the size of the neighborhood W near each ρ. It is precisely in this quantification where the uniformity in the non-maximally self conjugacy is used.

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