CONVERGENCE THEORY FOR TWO-LEVEL HYBRID SCHWARZ PRECONDITIONERS FOR HIGH-FREQUENCY HELMHOLTZ PROBLEMS

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Abstract. We give a novel convergence theory for two-level hybrid Schwarz domain-decomposition (DD) methods for finite-element discretisations of the high-frequency Helmholtz equation. This theory gives sufficient conditions for the preconditioned matrix to be close to the identity, and covers DD subdomains of arbitrary size, and arbitrary absorbing layers/boundary conditions on both the global and local Helmholtz problems. The assumptions on the coarse space are satisfied by the approximation spaces using problem-adapted basis functions that have been recently analysed as coarse spaces for the Helmholtz equation, as well as all spaces that are known to be quasi-optimal via a Schatz-type argument.

As an example, we apply this theory when the coarse space consists of piecewise polynomials; these are then the first rigorous convergence results about a two-level Schwarz preconditioner applied to the high-frequency Helmholtz equation with a coarse space that does not consist of problem-adapted basis functions.

1. Introduction.

1.1. Context and motivation.

Coarse grids are the key to parallel scalability of domain-decomposition (DD) methods for self-adjoint positive-definite problems (such as Laplace's equation); see, e.g., [45, 47], [12, Chapter 4]. However, the design of practical coarse spaces (with associated theory) for high-frequency wave problems, such as the high-frequency Helmholtz equation, is a longstanding open problem (see, e.g., the recent computational study [4] and the references therein).

For the Helmholtz equation with small wavenumber k, [6] analysed two-level additive Schwarz methods by treating Helmholtz as perturbation of Laplace, and [25] performed the corresponding analysis when k is complex valued with large $|k|$ and sufficiently large imaginary part.

For the Helmholtz equation with k real and large, the four recent papers [27, 29, 31, 15] all analyse hybrid Schwarz methods (where the coarse and subdomain solves are combined in a multiplicative way) with coarse spaces consisting of problem-adapted basis functions (coming from solving Helmholtz problems on subsets of the domain).

The present paper gives a novel convergence theory for two-level hybrid Schwarz methods under very general assumptions (e.g., the subdomains can have arbitrary size and boundary conditions). This convergence theory has three immediate applications.

1. The coarse spaces in [27, 29, 31, 15] all satisfy the assumptions in the theory of the present paper; therefore our theory gives results about hybrid Schwarz preconditioners using these coarse spaces, complementary to those in [27, 29, 31, 15]. Indeed, although the method in the present paper and the methods in [27, 29, 31, 15] are all hybrid, the particular hybrid method considered here is slightly different. However, the theory here covers arbitrary absorbing layers/boundary conditions on both the global and local Helmholtz problems, while the analyses in [27, 29, 31, 15] are all for global impedance boundary conditions and local Dirichlet (with no absorbing layers) or local impedance boundary conditions. Furthermore, while [27, 29] prove results about the field

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of values of the preconditioned matrix, the theory here proves the stronger result that the preconditioned matrix is close to the identity.

2. The theory in the present paper applies to problem-adapted approximation spaces used to solve the Helmholtz equation that have not yet been used as coarse spaces, e.g., the multiscale space of [8].
3. We use the theory in the present paper to prove the first convergence results about piecewise-polynomial coarse spaces for the high-frequency Helmholtz equation. We highlight that the recent computational study [4] comparing coarse spaces found piecewise-polynomial coarse spaces to be competitive – from the point of view of number of GMRES iterations – with coarse spaces involving problem-adapted basis functions. Piecewise-polynomial coarse spaces also have the advantage that they are much cheaper to use than those involving problem-adapted basis functions.

1.2. Informal statement of the main result.

1.2.1. The global and local Helmholtz problems. Let $\Omega \subset \mathbb{R}^d$ be bounded Lipschitz domain, and let $\mathcal{V}$ equal $H^1(\Omega)$, possibly with zero Dirichlet boundary conditions on part of $\partial\Omega$. Let

$$a(u,v) := \int_\Omega k^{-2}(A\nabla u)\cdot\overline{\nabla v} + k^{-1}(B\cdot\nabla u)\overline{v} - c^{-2}u\overline{v} - \mathrm{i}k^{-1}\int_{\partial\Omega}\theta u\overline{v} \tag{1.1}$$

be a sesquilinear form on $\mathcal{V}$, with $A \in L^\infty(\Omega;\mathbb{C}^{d\times d})$, $B \in L^\infty(\Omega;\mathbb{C}^d)$, and $c \in L^\infty(\Omega;\mathbb{C})$. We are primarily interested in solving Helmholtz problems that approximate scattering by either a penetrable obstacle (modelled by variable coefficients A, B, and c in the interior of Ω) and/or an impenetrable obstacle (via $\partial\Omega$ not being connected). Truncation of the unbounded domain exterior to the scatterer by a perfectly-matched layer (PML) [1, 11], other absorbing layers (such as a complex-absorbing potential [39]), or an impedance boundary condition can all be written in the form of (1.1) via appropriate choices of A, B, c, and θ.

Let $\{\Omega_\ell\}_{\ell=1}^N$ be an overlapping cover of Ω, and let $\{\chi_\ell\}_{\ell=1}^N$ be a partition of unity subordinate to $\{\Omega_\ell\}_{\ell=1}^N$. Let $\{\chi_\ell^>\}_{\ell=1}^N$ be a second set of cut-off functions with $\chi_\ell^>$ supported in Ω_ℓ and $\chi_\ell^> \equiv 1$ on $\mathrm{supp}\chi_\ell$ (i.e., $\chi_\ell^>$ is "bigger than" χ_ℓ; hence the notation). Let $\mathcal{V}_\ell$ equal $H^1(\Omega_\ell)\cap\mathcal{V}$, possibly with zero Dirichlet boundary conditions on part of $\partial\Omega_\ell$ (in addition to any imposed via $\mathcal{V}$). Let

$$a_\ell(u,v) := \int_{\Omega_\ell} k^{-2}(A_\ell\nabla u)\cdot\overline{\nabla v} + k^{-1}(B_\ell\cdot\nabla u)\overline{v} - c_\ell^{-2}u\overline{v} - \mathrm{i}k^{-1}\int_{\partial\Omega}\theta_\ell u\overline{v}. \tag{1.2}$$

We assume that

$$A_\ell \equiv A, \quad B_\ell \equiv B, \quad c_\ell \equiv c, \quad \text{and } \theta \equiv \theta_\ell \quad \text{on } \mathrm{supp}\chi_\ell^>,$$

so that $a(\cdot,\cdot) \equiv a_\ell(\cdot,\cdot)$ on $\mathrm{supp}\chi_\ell^>$, and, in particular, near $\partial\Omega\cap\partial\Omega_\ell$. This means that the local problems have the same boundary conditions on $\partial\Omega\cap\partial\Omega_\ell$ as the global problem, but away from $\mathrm{supp}\chi_\ell^>$ (i.e., near the boundaries of the subdomains that are not boundaries of Ω) the local problems can have different boundary conditions and/or absorbing layers to the global problem.

We allow the coefficients A, A_ℓ, B, B_ℓ, c, and c_ℓ to depend on k (e.g., for a Cartesian PML near $\partial\Omega$, $B \sim k^{-1}$), but we assume that A, A_ℓ, B, B_ℓ are all bounded above independent of k, and A, A_ℓ, c and c_ℓ are all bounded below independent of k.

Let

$$\|u\|^2_{H^1_k(\Omega)} := k^{-2}\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\,. \tag{1.3}$$

We note that many papers on the numerical analysis of the Helmholtz equation use the weighted H^1 norm $|||v|||^2 := \|\nabla v\|^2_{L^2(\Omega)} + k^2\,\|v\|^2_{L^2(\Omega)}$; we work with (1.3) instead, because weighting the jth derivative with k^{-j} makes the norm dimensionless and is easier to keep track of than weighting the jth derivative with k^{-j+1}.

The assumptions above imply that $a(\cdot,\cdot)$ is continuous and satisfies a Gårding inequality on $\mathcal{V}$ (equipped with the norm (1.3)), with the constants in these inequalities independent of k; similarly, $a_\ell(\cdot,\cdot)$ is continuous and satisfies a Gårding inequality on $\mathcal{V}_\ell$, again with constants independent of k.

1.2.2. The discretised problem and preconditioners. Let $\mathcal{V}_h \subset H^1_0(\Omega)$ (the *fine space*) be piecewise-polynomial Lagrange finite elements on a shape-regular mesh of size h, $\mathcal{V}_{\ell,h} := \mathcal{V}_h \cap \mathcal{V}_\ell$, and let $\mathcal{V}_0$ (the *coarse space*) be a subspace of $\mathcal{V}_h$.

Let A be the Galerkin matrix of $a(\cdot,\cdot)$ discretised in $\mathcal{V}_h$; we assume that h depends on the polynomial degree p and k in such a way that A is invertible (we discuss these conditions on h below Corollary 1.2).

Let R_0 be the restriction matrix from the degrees of freedom on $\mathcal{V}_h$ to the degrees of freedoms on $\mathcal{V}_0$, and let $\mathsf{A}_0 := \mathsf{R}_0\mathsf{A}\mathsf{R}_0^T$; i.e., A_0 is the Galerkin matrix of $a(\cdot,\cdot)$ discretised in $\mathcal{V}_0$. Let A_ℓ be the Galerkin matrix of $a_\ell(\cdot,\cdot)$ discretised in $\mathcal{V}_{\ell,h}$.

We define two restriction matrices from the degrees of freedom on $\mathcal{V}_h$ to the degrees of freedoms on $\mathcal{V}_{\ell,h}$, $\mathsf{R}_\ell^{\chi_\ell}$ and $\mathsf{R}_\ell^{\chi_\ell^>}$, where $\mathsf{R}_\ell^{\chi_\ell}$ is weighted by χ_ℓ, and $\mathsf{R}_\ell^{\chi_\ell^>}$ is weighted by $\chi_\ell^>$ (see (A.2) below); such restriction operators weighted by partition-of-unity functions appear widely (see, e.g., [12, §5.3], [21, §2], [26]). Let

$$\mathsf{B}_L^{-1} = \mathsf{B}_L^{-1}(\mathsf{A}) := \mathsf{R}_0^T\mathsf{A}_0^{-1}\mathsf{R}_0 + \left(\sum_{\ell=1}^N \big(\mathsf{R}_\ell^{\chi_\ell}\big)^T\mathsf{A}_\ell^{-1}\mathsf{R}_\ell^{\chi_\ell^>}\right)\big(\mathsf{I} - \mathsf{A}\mathsf{R}_0^T\mathsf{A}_0^{-1}\mathsf{R}_0\big). \tag{1.4}$$

We call B_L^{-1} a *hybrid* Schwarz preconditioner because the coarse and subdomain solves are combined in a multiplicative way; this idea was first introduced in [32, 33].

Let the real symmetric positive-definite matrix D_k be such that, for all $v_h \in \mathcal{V}_h$ with coefficient vectors $\mathbf{V}$,

$$\|v_h\|^2_{H^1_k(\Omega)} = \langle \mathsf{D}_k\mathbf{V},\mathbf{V}\rangle_2, \tag{1.5}$$

where $\langle\cdot,\cdot\rangle_2$ denotes the Euclidean inner product. Let † denote the adjoint with respect to the Euclidean inner product $\langle\cdot,\cdot\rangle$; i.e., $\mathsf{A}^\dagger = \overline{\mathsf{A}}^T$. A few lines of calculation then show that

$$\text{if}\quad \mathbf{W}_j = \mathsf{D}_k\mathbf{V}_j \quad\text{then}\quad \big\langle\mathbf{V}_1,\mathbf{V}_2\big\rangle_{\mathsf{D}_k} = \big\langle\mathbf{W}_1,\mathbf{W}_2\big\rangle_{\mathsf{D}_k^{-1}}\quad\text{and} \tag{1.6}$$

$$\big\langle\mathbf{V}_1,\mathsf{B}_L^{-1}(\mathsf{A})\mathsf{A}\mathbf{V}_2\big\rangle_{\mathsf{D}_k} = \big\langle\mathsf{A}^\dagger(\mathsf{B}_L^{-1}(\mathsf{A}))^\dagger\mathbf{W}_1,\mathbf{W}_2\big\rangle_{\mathsf{D}_k^{-1}}. \tag{1.7}$$

We therefore set $\mathsf{B}_R(\mathsf{A}) := (\mathsf{B}_L^{-1}(\mathsf{A}^\dagger))^\dagger$, i.e.,

$$\mathsf{B}_R^{-1}(\mathsf{A}) = \mathsf{R}_0^T\mathsf{A}_0^{-1}\mathsf{R}_0 + \big(\mathsf{I} - \mathsf{R}_0^T\mathsf{A}_0^{-1}\mathsf{R}_0\mathsf{A}\big)\left(\sum_{\ell=1}^N \big(\mathsf{R}_\ell^{\chi_\ell^>}\big)^T\mathsf{A}_\ell^{-1}\mathsf{R}_\ell^{\chi_\ell}\right),$$

so that, by (1.6) and (1.7) with A replaced by $\mathsf{A}^\dagger$,

$$\big\langle\mathbf{V}_1,\big(\mathsf{I} - \mathsf{B}_L^{-1}(\mathsf{A}^\dagger)\mathsf{A}^\dagger\big)\mathbf{V}_2\big\rangle_{\mathsf{D}_k} = \big\langle\big(\mathsf{I} - \mathsf{A}(\mathsf{B}_R^{-1}(\mathsf{A}))\big)\mathbf{W}_1,\mathbf{W}_2\big\rangle_{\mathsf{D}_k^{-1}}. \tag{1.8}$$

1.2.3. Informal statement of the main result.

THEOREM 1.1. *With the set-up above, suppose additionally that*

(a) The subdomain diameters are all proportional to k^{-1} and the subdomains all have generous overlap (i.e., the overlaps are also proportional to k^{-1}).

(b) The coarse space is such that the following holds: if the Helmholtz problem is solved using the Galerkin method in the coarse space, then

- *the H^1_k Galerkin error is bounded (independent of k) by the H^1_k norm of the solution, and*
- *the L^2 Galerkin error is bounded by a sufficiently-small (independent of k) multiple of the H^1_k norm of the solution.*

(We recall below that both these bounds hold if the Galerkin solution in the coarse space is proved to be quasi-optimal via the Schatz argument.)

Then

$$\left\| \mathsf{I} - \mathsf{B}_L^{-1}\mathsf{A} \right\|_{\mathsf{D}_k} \qquad \text{is small (independent of } k)$$

and thus there exists $0 < c < 1$ (independent of k) such that

- *the preconditioned fixed point iteration $\mathbf{x}^{n+1} = \mathbf{x}^n + \mathsf{B}_L^{-1}(\mathbf{b} - \mathsf{A}\mathbf{x}^n)$ for solving $\mathsf{A}\mathbf{x} = \mathbf{b}$ satisfies*

$$\|\mathbf{x} - \mathbf{x}^n\|_{\mathsf{D}_k} \leq c^n \left\|\mathsf{B}_L^{-1}\mathbf{b}\right\|_{\mathsf{D}_k}, \tag{1.9}$$

- *when GMRES is applied in the D_k inner product, the residual $\mathbf{r}^n := \mathbf{b} - \mathsf{A}\mathbf{x}^n$ satisfies*

$$\|\mathbf{r}^n\|_{\mathsf{D}_k} / \left\|\mathbf{r}^0\right\|_{\mathsf{D}_k} \leq c^n, \qquad \text{and} \tag{1.10}$$

- *when GMRES is applied in the Euclidean inner product and the fine mesh is quasi-uniform, the residual $\mathbf{r}^n$ satisfies*

$$\|\mathbf{r}^n\|_2 / \left\|\mathbf{r}^0\right\|_2 \leq c^n (hk)^{-1}. \tag{1.11}$$

Furthermore, if Assumption (b) also holds for the adjoint sesquilinear form, then, by (1.8), $\left\|\mathsf{I} - \mathsf{A}\mathsf{B}_R^{-1}\right\|_{\mathsf{D}_k^{-1}}$ is small, and so an analogous result holds for right preconditioning (in the D_k^{-1} inner product).

We make the following remarks.

(i) Since the Helmholtz solution operator at high frequency involves propagation at length scales independent of k, and subdomains of size k^{-1} cannot see this, the coarse space must resolve this propagation. Theorem 1.1 encodes this requirement in Assumption (b). We discuss below – in the context of piecewise polynomials – how one might seek to weaken Assumption (b), but highlight that the coarse spaces of [27, 29, 31, 15] all satisfy Assumption (b) (see §5.2 below).

(ii) Following on from (i), once the wave nature of the solution is resolved by the coarse space, one obtains the Galerkin solution at the accuracy of the fine space. One can therefore view the role of the subdomains as inexpensive high-accuracy interpolators, tailored to the specific wave problem.

(iii) The main ways in which Theorem 1.1 is a simplification of the main result (Theorem 3.1 below) are that

- Theorem 3.1 allows arbitrary-sized subdomains. We have considered subdomains of size k^{-1} in Theorem 1.1 since, in practice, one wants to take the subdomains small for parallel scalability. Indeed, with h chosen as a function of k and p to maintain accuracy as $k \to \infty$, the number of degrees of freedom in each k^{-1}-sized subdomain grows mildly with k (like $k^{d/p}$ or $k^{d/(2p)}$ for nontrapping problems, depending on the measure of accuracy used).

- Theorem 1.1 assumes that the fine space consists of piecewise polynomials; the only assumption on the fine space in Theorem 3.1 is a super-approximation result (Assumption 2.1 below) which is satisfied by piecewise polynomials, but also, in principle, other spaces.

(iv) The result (1.11) in Theorem 1.1 (about standard GMRES) follows from the result (1.10) about weighted GMRES via an inverse estimate, as recognised in [22, Corollary 5.8]. The numerical experiments in [25, Experiment 1], [3, §6] showed little difference in the number of weighted/unweighted GMRES iterations.

(v) The set up of Theorem 1.1 allows a wide variety of boundary conditions/absorbing layers on the global and subdomain problems (including, e.g., a PML). The existing two-level hybrid Schwarz analyses in [27, 29, 31, 15] all consider impedance boundary conditions on $\partial\Omega$ (which give k-independent errors when approximating the Sommerfeld radiation condition [18]) and either impedance or Dirichlet boundary conditions on $\partial\Omega_\ell$.

(vi) By (1.4),

$$\begin{aligned}\mathsf{I} - \mathsf{B}_L^{-1}\mathsf{A} &= (\mathsf{I} - \mathsf{R}_0^T\mathsf{A}_0^{-1}\mathsf{R}_0\mathsf{A}) - \left(\sum_{\ell=1}^N \left(\mathsf{R}_\ell^{\chi_\ell}\right)^T \mathsf{A}_\ell^{-1}\mathsf{R}_\ell^{\chi_\ell^>}\,\mathsf{A}\right)(\mathsf{I} - \mathsf{R}_0^T\mathsf{A}_0^{-1}\mathsf{R}_0\mathsf{A}) \\ &= \left(\mathsf{I} - \sum_{\ell=1}^N \left(\mathsf{R}_\ell^{\chi_\ell}\right)^T \mathsf{A}_\ell^{-1}\mathsf{R}_\ell^{\chi_\ell^>}\,\mathsf{A}\right)(\mathsf{I} - \mathsf{R}_0^T\mathsf{A}_0^{-1}\mathsf{R}_0\mathsf{A}).\end{aligned}$$

One way to understand heuristically why $\mathsf{I} - \mathsf{B}_L^{-1}\mathsf{A}$ is small is the following

- $\mathsf{I} - \mathsf{R}_0^T\mathsf{A}_0^{-1}\mathsf{R}_0\mathsf{A}$ is effective at removing frequencies $\lesssim k$, i.e., it acts like a high-pass filter, and
- $\mathsf{I} - \sum_{\ell=1}^N \left(\mathsf{R}_\ell^{\chi_\ell}\right)^T \mathsf{A}_\ell^{-1}\mathsf{R}_\ell^{\chi_\ell^>}\,\mathsf{A}$ is effective at removing frequencies $\gtrsim k$, i.e., it acts like a low-pass filter;

thus the product of these two operators is small.

Regarding the first point: this is because the Galerkin solution in the coarse space is quasi-optimal and the best approximation of a frequency $\lesssim k$ function is small [17].

Regarding the second point: at the continuous level, frequencies $\gg k$ do not propagate under the action of the Helmholtz solution operator (as a consequence of semiclassical ellipticity [13, Theorem E.33]). Thus, modulo frequencies $\lesssim k$, the solution can be well-approximated by a sum of local solves. Provided the local spaces $\mathcal{V}_{\ell,h}$ accurately approximate $\mathcal{V}_h$ restricted to Ω_ℓ, this property is inherited by the discrete solution operator. (We encode this accurate-approximation assumption in the super-approximation-type estimate, Assumption 2.1 below.)

1.2.4. The main result applied to piecewise-polynomial coarse spaces. Let $C_{\rm sol}$ denote the $L^2 \to H^1_k$ norm of the Helmholtz solution operator (recall that this $\sim kL$ when the problem is nontrapping [13, §4.6], where L is the characteristic length scale of Ω).

COROLLARY 1.2 (Piecewise-polynomial coarse space of fixed degree). *Suppose that $\mathcal{V}_h$ consists of degree-p piecewise polynomials on a mesh of size h and $\mathcal{V}_0 \subset \mathcal{V}_h$ with the coarse mesh size H and each element of the coarse mesh a union of elements of the fine mesh.*

Suppose that the subdomain diameters are all proportional to k^{-1} *and the subdomains all have generous overlap (i.e., the overlaps are also proportional to* k^{-1}*).*

If the domain and coefficients have suitable regularity,

$$(kH)^p C_{\rm sol} \text{ is sufficiently small,} \tag{1.12}$$

and, given a coarsening factor $C_{\rm coarse} > 1$*,*

$$h = H/C_{\rm coarse},$$

then the properties (1.9)*,* (1.10)*, and* (1.11) *hold.*

The precise statement of Corollary 1.2 is Corollary 6.1 below. We make the following remarks about Corollary 1.2/Corollary 6.1.

(i) The condition (1.12) on H means that the sequence of Galerkin solutions in the sequence of coarse spaces is quasi-optimal. One could hope to prove a result with the condition (1.12) replaced by "$(kH)^{2p}C_{\rm sol}$ sufficiently small", with this threshold ensuring that, for k-oscillatory data, the Galerkin solution in the coarse space has bounded relative error [20]. However, as discussed after Theorem 1.1, the coarse problem needs to sufficiently resolve the wave-nature of the solution. We note that the experiments in [4, Table 9] show that, when a two-level additive Schwarz preconditioner with a fixed number of points per wavelength in both fine and coarse space is used, the number of GMRES iterations is large if the coarse mesh does not contain a sufficient number of points per wavelength.

(ii) In the set up of Corollary 1.2,

$$\frac{\text{coarse-space dimension}}{\text{fine-space dimension}} \sim \left(\frac{h}{H}\right)^d \sim \left(\frac{1}{C_{\rm coarse}}\right)^d \text{ as } k \to \infty.$$

The coarse problem is still large – as expected from the requirement discussed in (i) that the coarse problem resolve the oscillations – and, in practice, is often solved using a one-level DD method (see, e.g., [46], [3, §6]). We do not attempt to analyse this set up in the present paper (in common with the two-level analyses in [27, 29, 31, 15]).

(iii) The follow-up paper [23] proves results about piecewise-polynomial coarse spaces with polynomial degree increasing like $\log k$. This analysis is restricted to a specific Helmholtz problem (where the radiation condition is approximated by a complex absorbing potential [39]), no partition of unity functions appear in the preconditioner, and the subdomains have zero Dirichlet boundary conditions and width a sufficiently small multiple of k^{-1} (so that the subdomain problems are coercive). The difficulty in applying the theory in the present paper with increasing polynomial degree is the super-approximation result in Assumption 2.1/Lemma 4.6; the current proof of this result gives a constant that blows up as the polynomial degree increases.

2. Statement of the abstract assumptions.

2.1. The space $\mathcal{V}$ and sesquilinear form $a(\cdot,\cdot)$. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and let $\mathcal{V}$ be a closed subspace of $H^1(\Omega)$.

Let $a : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ be a continuous sesquilinear form; i.e., there exists $C_{\rm cont} > 0$ such that

$$\big|a(u,v)\big| \le C_{\rm cont} \left\|u\right\|_{H^1_k(\Omega)} \left\|v\right\|_{H^1_k(\Omega)} \quad \text{for all } u, v \in \mathcal{V}.$$

Assume that, given $F \in \mathcal{V}^*$, there exists a unique solution to the variational problem

$$\text{find } u \in \mathcal{V} \text{ such that } a(u,v) = F(v) \text{ for all } v \in \mathcal{V}. \tag{2.1}$$

2.2. The space $\mathcal{V}_h$, subdomains Ω_ℓ, and spaces $\mathcal{V}_\ell$ and $\mathcal{V}_{\ell,h}$. Let $\{\mathcal{V}_h\}_{h>0}$ be a family of finite-dimensional subspaces of $\mathcal{V}$, indexed by $h>0$. Assume that there exists a unique solution to the variational problem

$$\text{find } u_h \in \mathcal{V}_h \text{ such that } a(u_h, v_h) = F(v_h) \quad \text{for all } v_h \in \mathcal{V}_h. \tag{2.2}$$

Let $\{\Omega_\ell\}_{\ell=1}^N$ (the *subdomains*) form an overlapping cover of Ω, with each Ω_ℓ a non-empty open Lipschitz domain with characteristic length scale H_ℓ (i.e., its diameter $\sim H_\ell$, its surface area $\sim H_\ell^{d-1}$, and its volume $\sim H_\ell^d$). Let

$$\Lambda := \max\big\{\#\Lambda(\ell) : \ell = 1, ..., N\big\}, \quad \text{where} \quad \Lambda(\ell) := \big\{\ell' : \Omega_\ell \cap \Omega_{\ell'} \neq \emptyset\big\}; \tag{2.3}$$

i.e., Λ is the maximum number of subdomains that can overlap any given subdomain.

Let $\mathcal{V}_\ell$ be a closed subspace of $\mathcal{V} \cap L^2(\Omega_\ell)$, and $\mathcal{V}_{\ell,h} \subset \mathcal{V}_\ell$ a finite dimensional subspace. Define the norm $\|\cdot\|_{H^1_k(\Omega_\ell)}$ in an analogous way to (1.3) with Ω replaced by Ω_ℓ. Let $\{\chi_\ell\}_{\ell=1}^N$ be a partition of unity subordinate to $\{\Omega_\ell\}_{\ell=1}^N$. Let $\{\chi_\ell^>\}_{\ell=1}^N$ be such that, for $\ell = 1, \ldots, N$, $\chi_\ell^> \in L^\infty(\Omega_\ell; [0,1])$ and $\chi_\ell^> \equiv 1$ on $\mathrm{supp}\chi_\ell$.

Assumption 2.1 (Super-approximation-type assumption). *There exist linear operators $\mathcal{I}_h^\ell : H^1(\Omega) \to \mathcal{V}_{\ell,h}$ and $\{\mu_\ell\}_{\ell=1}^N$ such that, for all $v_h \in \mathcal{V}_h$,*

$$\max\left\{\big\|(I-\mathcal{I}_h^\ell)(\chi_\ell v_h)\big\|_{H^1_k(\Omega_\ell)}, \big\|(I-\mathcal{I}_h^\ell)(\chi_\ell^> v_h)\big\|_{H^1_k(\Omega_\ell)}\right\} \le \mu_\ell \|v_h\|_{H^1_k(\Omega_\ell)}. \tag{2.4}$$

Lemma 4.6 below shows that (2.4) holds when $\mathcal{V}_h$ consists of piecewise polynomials on a shape-regular mesh; in this case $\mu_\ell := h_\ell/\delta_\ell$, where $h_\ell := \max_{K \subset \overline{\Omega_\ell}} h_K$ and δ_ℓ is such that $\|\partial^\alpha \chi_\ell\|_{L^\infty}, \|\partial^\alpha \chi_\ell^>\|_{L^\infty} \le C\delta_\ell^{-|\alpha|}$ (and thus δ_ℓ is related to the overlap of the subdomains).

Remark 2.2. *In the conventional two-level DD theory, the fine space is given by $\mathcal{V}_h = \sum_\ell \mathcal{V}_{\ell,h}$; thus $\mathcal{V}_0 \subset \sum_\ell \mathcal{V}_{\ell,h}$, and one can take a single interpolation operator $\mathcal{I}_h$ in Assumption 2.1, rather than one for each subdomain.*

However, the assumptions here also allow, e.g., $\mathcal{V}_h = (\cup_\ell \mathcal{V}_{\ell,h}) \cup \mathcal{V}_0$. That is, the local spaces need not contain *the restrictions to Ω_ℓ of the coarse space; instead the local spaces must* accurately approximate *the restrictions of the coarse space. For instance, if $\mathcal{V}_0$ consists only of functions oscillating at frequency $\lesssim k$ (i.e., such that $\|u\|_{H^\ell_k(\Omega)} \le C_\ell \|u\|_{H^1_k(\Omega)}$, $\ell = 2, \ldots, p+1$) then one can take as the local space piecewise polynomials on a sufficiently small mesh. The two-level method will then obtain the solution in the union of the local spaces and the coarse space.*

Let

$$\mu := \max_\ell \mu_\ell. \tag{2.5}$$

2.3. The local sesquilinear forms $a_\ell(\cdot,\cdot)$. Let $a_\ell : \mathcal{V}_\ell \times \mathcal{V}_\ell \to \mathbb{C}$ be a continuous sesquilinear form. Without loss of generality we assume that the continuity constant is the same as for $a(\cdot,\cdot)$; i.e.,

$$\big|a_\ell(u,v)\big| \le C_{\mathrm{cont}} \|u\|_{H^1_k(\Omega_\ell)} \|v\|_{H^1_k(\Omega_\ell)} \quad \text{for all } u, v \in \mathcal{V}.$$

Assumption 2.3. *$a_\ell(\cdot,\cdot)$ satisfies the discrete inf-sup condition*

$$\inf_{u_{\ell,h} \in \mathcal{V}_{\ell,h}} \sup_{v_{\ell,h} \in \mathcal{V}_{\ell,h}} \frac{\big|a(u_{\ell,h}, v_{\ell,h})\big|}{\|u_{\ell,h}\|_{H^1_k(\Omega_\ell)} \|v_{\ell,h}\|_{H^1_k(\Omega_\ell)}} \ge \gamma_\ell^{-1}. \tag{2.6}$$

Let

$$\gamma := \max_{\ell} \gamma_\ell.$$

ASSUMPTION 2.4 (a and a_ℓ agree "in the interior of Ω_ℓ").

$$a_\ell\big(u_h, \mathcal{I}_h^\ell(\chi_\ell^> v_{\ell,h})\big) = a\big(u_h, \mathcal{I}_h^\ell(\chi_\ell^> v_{\ell,h})\big) \quad \textit{for all } u_h \in \mathcal{V}_h, v_{\ell,h} \in \mathcal{V}_{\ell,h}. \tag{2.7}$$

The u_h in the first argument of a_ℓ on the left-hand side of (2.7) is understood as $u_h|_{\Omega_\ell}$, and thus the left-hand side of (2.7) is well-defined.

ASSUMPTION 2.5 (Commutator bound). *Given $k_0 > 0$ there exists $C_{\rm com} > 0$ and positive numbers $\{\delta_\ell\}_{\ell=1}^N$ such that for all $k \geq k_0$, all $u, v \in H^1(\Omega_\ell)$, and $\ell = 1, \ldots, N$,*

$$\big|a_\ell(\chi_\ell^> u, v) - a_\ell(u, \chi_\ell^> v)\big| \leq C_{\rm com}(k\delta_\ell)^{-1}\big(1 + (k\delta_\ell)^{-1}\big) \|u\|_{L^2(\Omega_\ell)} \|v\|_{H^1_k(\Omega_\ell)}. \tag{2.8}$$

Let

$$\delta := \min_{\ell} \delta_\ell. \tag{2.9}$$

2.4. The coarse space $\mathcal{V}_0$.

ASSUMPTION 2.6. *$\mathcal{V}_0$ is a finite-dimensional subspace of $\mathcal{V}_h$ and there exists an operator $Q_0 : \mathcal{V} \to \mathcal{V}_0$ satisfying*

$$\|(I - Q_0)v_h)\|_{L^2(\Omega)} \leq \sigma_{L^2} \|v_h\|_{H^1_k(\Omega)}, \quad \|(I - Q_0)v_h\|_{H^1_k(\Omega)} \leq \sigma_{H^1} \|v_h\|_{H^1_k(\Omega)}. \tag{2.10}$$

Note that the standard way to define Q_0 is as the Galerkin solution in $\mathcal{V}_0$:

$$a\big(Q_0 u, v_0\big) = a\big(u, v_0\big) \quad \text{for all } v_0 \in \mathcal{V}_0, \tag{2.11}$$

but the main abstract result (Theorem 3.1) only requires Assumption 2.6 (and thus covers, e.g., the LOD coarse space of [29], where the projection Q_0 is defined via a Petrov–Galerkin method).

3. The main abstract result and its proof.

3.1. Statement of the main abstract result. For $\ell = 1, \ldots, N$, let $Q_\ell : \mathcal{V} \to \mathcal{V}_{\ell,h}$ be defined by

$$a_\ell\big(Q_\ell u, v_{\ell,h}\big) = a\big(u, \mathcal{I}_h^\ell(\chi_\ell^> v_{\ell,h})\big) \quad \text{for all } v_{\ell,h} \in \mathcal{V}_{\ell,h}. \tag{3.1}$$

Let

$$Q = Q_0 + \sum_{\ell=1}^N \mathcal{I}_h^\ell\big(\chi_\ell Q_\ell (I - Q_0)\big), \tag{3.2}$$

where Q_0 is as in Assumption 2.6.

§A below shows that when Q_0 is defined by (2.11) (i.e., the Galerkin solution in the coarse space) then the matrix form of Q is $\mathsf{B}_L^{-1}\mathsf{A}$ with B_L^{-1} defined by (1.4).

THEOREM 3.1 (Bound on $I - Q$). *Under the assumptions in §2 and with Q defined by (3.2), for all $v_h \in \mathcal{V}_h$,*

$$\|(I - Q)v_h\|_{H^1_k(\Omega)} \leq 2\Lambda\Big((1 + \mu)\gamma C_{\rm com}(k\delta)^{-1}\big(1 + (k\delta)^{-1}\big)\sigma_{L^2} + \big(1 + (1 + \mu)(1+\gamma C_{\rm cont})\big)\mu\sigma_{H^1}\Big) \|v_h\|_{H^1_k(\Omega)}. \tag{3.3}$$

The key point is that the right-hand side of (3.3) can be made small by making both σ_{L^2} and μ small.

3.2. Proof of Theorem 3.1.

LEMMA 3.2 (Consequences of the definition of Λ). *For all $v \in L^2(\Omega)$ and $w \in H^1(\Omega)$*

$$\sum_{\ell=1}^{N} \|v\|^2_{L^2(\Omega_\ell)} \leq \Lambda \|v\|^2_{L^2(\Omega)} \quad \text{and} \quad \sum_{\ell=1}^{N} \|w\|^2_{H^1_k(\Omega_\ell)} \leq \Lambda \|w\|^2_{H^1_k(\Omega)}. \tag{3.4}$$

Furthermore, given $v_\ell \in \mathcal{V}_{\ell,h}$,

$$\left\| \sum_{\ell=1}^{N} v_\ell \right\|^2_{H^1_k(\Omega)} \leq 2\Lambda \sum_{\ell=1}^{N} \|v_\ell\|^2_{H^1_k(\Omega_\ell)}. \tag{3.5}$$

Proof. The bounds in (3.4) and follow immediately from the definition (2.3) of Λ. The bound (3.5) without an explicit expression for the constant is proved in [25, Lemma 4.2]. The definition of Λ implies that [25, Equation 4.8] holds with $\lesssim \sum_{\ell=1}^{N} \|v_\ell\|^2_{H^1_k(\Omega)}$ at the end replaced by $\leq \Lambda \sum_{\ell=1}^{N} \|v_\ell\|^2_{H^1_k(\Omega)}$. The result then follows, with the factor of 2 arising from use of the inequality

$$(a+b)^2 \leq 2(a^2+b^2) \quad \text{for all } a, b > 0, \tag{3.6}$$

at the end of [25, Proof of Lemma 4.2] (with this constant hidden in the notation $\lesssim$ in [25, Proof of Lemma 4.2]). ∎

LEMMA 3.3. *With Q_ℓ defined by (3.1), for all $u_h \in \mathcal{V}_h$,*

$$\begin{aligned} &\left\| Q_\ell u_h - \chi_\ell^{>} u_h \right\|_{H^1_k(\Omega_\ell)} \\ &\leq \Big(\gamma_\ell C_{\rm com} (k\delta_\ell)^{-1} \big(1 + (k\delta_\ell)^{-1}\big) \|u_h\|_{L^2(\Omega_\ell)} + \big(1 + \gamma_\ell C_{\rm cont}\big) \mu_\ell \|u_h\|_{H^1_k(\Omega_\ell)} \Big). \end{aligned} \tag{3.7}$$

REMARK 3.4. *The bound (3.7) is similar to that in [26, Lemma 3.8] except the latter contains only $\|u_h\|_{H^1_k(\Omega)}$ on the right-hand side.*

Proof of Lemma 3.3. First observe that it is sufficient to prove that

$$\begin{aligned} &\left\| Q_\ell u_h - \mathcal{I}_h^\ell(\chi_\ell^{>} u_h) \right\|_{H^1_k(\Omega_\ell)} \\ &\qquad \leq \gamma_\ell \Big(C_{\rm com} (k\delta_\ell)^{-1} \big(1 + (k\delta_\ell)^{-1}\big) \|u_h\|_{L^2(\Omega_\ell)} + C_{\rm cont} \mu_\ell \|u_h\|_{H^1_k(\Omega_\ell)} \Big), \end{aligned} \tag{3.8}$$

since then (3.7) follows from (2.4) and the triangle inequality.

By the definition of Q_ℓ (3.1) and the property (2.7),

$$\begin{aligned} &a_\ell\big(Q_\ell u_h - \mathcal{I}_h^\ell(\chi_\ell^{>} u_h), v_{\ell,h}\big) \\ &= a\big(u_h, \mathcal{I}_h^\ell(\chi_\ell^{>} v_{\ell,h})\big) - a_\ell\big(\mathcal{I}_h^\ell(\chi_\ell^{>} u_h), v_{\ell,h}\big) \\ &= a_\ell\big(u_h, \mathcal{I}_h^\ell(\chi_\ell^{>} v_{\ell,h})\big) - a_\ell\big(\mathcal{I}_h^\ell(\chi_\ell^{>} u_h), v_{\ell,h}\big) \\ &= a_\ell\big(u_h, \chi_\ell^{>} v_{\ell,h}\big) - a_\ell\big(\chi_\ell^{>} u_h, v_{\ell,h}\big) - a_\ell\big(u_h, (I - \mathcal{I}_h^\ell)(\chi_\ell^{>} v_{\ell,h})\big) \\ &\qquad\qquad + a_\ell\big((I - \mathcal{I}_h^\ell)(\chi_\ell^{>} u_h), v_{\ell,h}\big). \end{aligned}$$

Then, by (2.8) and (2.4),

$$\begin{aligned} \left| a_\ell\big(Q_\ell u_h - \mathcal{I}_h^\ell(\chi_\ell^{>} u_h), v_{\ell,h}\big) \right| &\leq C_{\rm com} (k\delta_\ell)^{-1} \big(1 + (k\delta_\ell)^{-1}\big) \|u_h\|_{L^2(\Omega_\ell)} \|v_{\ell,h}\|_{H^1_k(\Omega_\ell)} \\ &\qquad + C_{\rm cont} \mu_\ell \|u_h\|_{H^1_k(\Omega_\ell)} \|v_{\ell,h}\|_{H^1_k(\Omega_\ell)}; \end{aligned}$$

the result then follows from (2.6). ∎

LEMMA 3.5.

$$\big\|\mathcal{I}_h^\ell(\chi_\ell v_h)\big\|_{H^1_k(\Omega_\ell)} \leq (1+\mu_\ell)\,\|v_h\|_{H^1_k(\Omega_\ell)} \qquad \textit{for all } v_h \in \mathcal{V}_h. \tag{3.9}$$

Proof. The result follows immediately from (2.4) and the triangle inequality. ◻

Proof of Theorem 3.1. By the fact that $\{\chi_\ell\}_{\ell=1}^N$ is a partition of unity, and the fact that $\chi_\ell^> \equiv 1$ on $\mathrm{supp}\chi_\ell$, for all $u_h \in \mathcal{V}_h$,

$$u_h = \sum_{\ell=1}^N \Big(\mathcal{I}_h^\ell(\chi_\ell u_h) + (I-\mathcal{I}_h^\ell)\chi_\ell u_h\Big) = \sum_{\ell=1}^N \Big(\mathcal{I}_h^\ell(\chi_\ell \chi_\ell^> u_h) + (I-\mathcal{I}_h^\ell)\chi_\ell u_h\Big).$$

By this last equation (with $u_h = (I-Q_0)v_h$) and the definition of Q (3.2), for $v_h \in \mathcal{V}_h$,

$$\begin{aligned}
(Q-I)v_h &= -(I-Q_0)v_h + \sum_{\ell=1}^N \mathcal{I}_h^\ell\big(\chi_\ell Q_\ell (I-Q_0)v_h\big)\\
&= \sum_{\ell=1}^N \Big(-\mathcal{I}_h^\ell(\chi_\ell\chi_\ell^>(I-Q_0)v_h) + \mathcal{I}_h^\ell\big(\chi_\ell Q_\ell(I-Q_0)v_h\big) - (I-\mathcal{I}_h^\ell)\chi_\ell(I-Q_0)v_h\Big)\\
&= \sum_{\ell=1}^N \Big(\mathcal{I}_h^\ell\big(\chi_\ell(Q_\ell - \chi_\ell^>)(I-Q_0)v_h\big) - (I-\mathcal{I}_h^\ell)\chi_\ell(I-Q_0)v_h\Big).
\end{aligned}$$

Therefore, by (in this order) (3.5), (3.9), (2.4), (3.7), (3.6), the definitions of μ (2.5) and δ (2.9), and finally the two bounds in (3.4),

$$\begin{aligned}
&\big\|(Q-I)v_h\big\|^2_{H^1_k(\Omega)}\\
&\leq 2\Lambda \sum_{\ell=1}^N \Big(\big\|\mathcal{I}_h^\ell\big(\chi_\ell(Q_\ell-\chi_\ell^>)(I-Q_0)v_h\big)\big\|_{H^1_k(\Omega_\ell)} + \big\|(I-\mathcal{I}_h^\ell)\chi_\ell(I-Q_0)v_h\big\|_{H^1_k(\Omega_\ell)}\Big)^2\\
&\leq 2\Lambda \sum_{\ell=1}^N \Big((1+\mu_\ell)\big\|(Q_\ell-\chi_\ell^>)(I-Q_0)v_h\big\|_{H^1_k(\Omega_\ell)} + \mu_\ell\big\|(I-Q_0)v_h\big\|_{H^1_k(\Omega_\ell)}\Big)^2\\
&\leq 2\Lambda \sum_{\ell=1}^N \Big((1+\mu_\ell)\gamma_\ell C_{\mathrm{com}}(k\delta_\ell)^{-1}\big(1+(k\delta_\ell)^{-1}\big)\,\|(I-Q_0)v_h\|_{L^2(\Omega_\ell)}\\
&\qquad\qquad + \big(1+(1+\mu_\ell)(1+\gamma_\ell C_{\mathrm{cont}})\big)\mu_\ell\,\|(I-Q_0)v_h\|_{H^1_k(\Omega_\ell)}\Big)^2\\
&\leq 4\Lambda \sum_{\ell=1}^N \Big((1+\mu_\ell)^2\gamma_\ell^2 C^2_{\mathrm{com}}(k\delta_\ell)^{-2}\big(1+(k\delta_\ell)^{-1}\big)^2\,\|(I-Q_0)v_h\|^2_{L^2(\Omega_\ell)}\\
&\qquad\qquad + \big(1+(1+\mu_\ell)(1+\gamma_\ell C_{\mathrm{cont}})\big)^2\mu_\ell^2\,\|(I-Q_0)v_h\|^2_{H^1_k(\Omega_\ell)}\Big)\\
&\leq 4\Lambda(1+\mu)^2\gamma^2 C^2_{\mathrm{com}}(k\delta)^{-2}\big(1+(k\delta)^{-1}\big)^2 \sum_{\ell=1}^N \|(I-Q_0)v_h\|^2_{L^2(\Omega_\ell)}\\
&\qquad\qquad + 4\Lambda\big(1+(1+\mu)(1+\gamma C_{\mathrm{cont}})\big)^2\mu^2 \sum_{\ell=1}^N \|(I-Q_0)v_h\|^2_{H^1_k(\Omega_\ell)}\\
&\leq 4\Lambda^2(1+\mu)^2\gamma^2 C^2_{\mathrm{com}}(k\delta)^{-2}\big(1+(k\delta)^{-1}\big)^2\,\|(I-Q_0)v_h\|^2_{L^2(\Omega)}
\end{aligned}$$

$$+ 4\Lambda^2\big(1+(1+\mu)(1+\gamma C_{\rm cont})\big)^2\mu^2 \left\|(I-Q_0)v_h\right\|^2_{H^1_k(\Omega)}.$$

Then, by the two inequalities in (2.10),

$$\big\|(Q-I)v_h\big\|^2_{H^1_k(\Omega)} \leq 4\Lambda^2\Big((1+\mu)^2\gamma^2 C^2_{\rm com}(k\delta)^{-2}\big(1+(k\delta)^{-1}\big)^2\sigma^2_{L^2} + \big(1+(1+\mu)(1+\gamma C_{\rm cont})\big)^2\mu^2\sigma^2_{H^1}\Big)\left\|v_h\right\|^2_{H^1_k(\Omega)}.$$

The result (3.3) then follows by using that $a^2+b^2 \leq (a+b)^2$ for $a,b\geq 0$. ∎

4. Assumptions 2.1, 2.3, 2.4, and 2.5 are valid for Helmholtz problems with piecewise-polynomial fine spaces. §2 contains five main assumptions: Assumptions 2.1, 2.3, 2.4, 2.5, and 2.6. Here we show that the first four of these – i.e., those not involving the coarse space – are satisfied for Helmholtz problems with piecewise-polynomial fine spaces.

4.1. Definition of the finite-element space. We consider a partition of Ω into a family of conforming meshes $\mathcal{T}_h$ of (potentially curved) elements K. For $K\in\mathcal{T}_h$, let h_K be the diameter of K, and let $\mathcal{F}_K:\widehat{K}\to K$ be the mapping between the reference simplex $\widehat{K}$ and the element K. Fixing a polynomial degree $p\in\mathbb{Z}^+$, we associate with $\mathcal{T}_h$ the finite element space

$$\mathcal{V}_h := \Big\{v_h\in\mathcal{V} \;:\; v_h\circ\mathcal{F}_K\in\mathcal{P}_p(\widehat{K})\Big\}.$$

We consider *both* the case when the element maps are affine, and thus the mesh $\mathcal{T}^h$ is simplicial *and* the case when the elements are curved.

Assumption 4.1. *$\mathcal{T}^h$ is a family of conforming simplicial meshes of Ω that are shape-regular (in the sense of [9, Chapter 3, Page 111])*

Lemma 4.2. *Given $p\in\mathbb{Z}^+$, if Assumption 4.1 holds then there exists $C>0$ and a nodal interpolation operator $\mathcal{I}_h: H^2(\Omega)\cap\mathcal{V}\to\mathcal{V}_h$ such that, for all $K\in\mathcal{T}^h$ and for all $v\in H^{p+1}(K)\cap\mathcal{V}$,*

$$h_K^{-1}\big\|(I-\mathcal{I}_h)v\big\|_{L^2(K)} + \big|(I-\mathcal{I}_h)v\big|_{H^1(K)} \leq Ch_K^p|v|_{H^{p+1}(K)}. \tag{4.1}$$

References for the proof. See, e.g., [9, §3.1], [5, Theorem 4.4.4]. ∎

Assumption 4.3 (Curved elements). *Let L be the characteristic length scale of Ω. There exists $C>0$ such that, for all $h>0$, all $K\in\mathcal{T}^h$, and $1\leq|\alpha|\leq p+1$,*

$$\|\partial^\alpha\mathcal{F}_K\|_{L^\infty(\widehat{K})}\leq CL\left(\frac{h_K}{L}\right)^{|\alpha|} \qquad \textit{and} \qquad \|\partial^\alpha(\mathcal{F}_K^{-1})\|_{L^\infty(K)}\leq Ch_K^{-|\alpha|}. \tag{4.2}$$

Note that the bound (4.2) with $|\alpha|=1$ implies that the mesh is shape-regular (in the sense of [9, Chapter 3, Page 111]).

Remark 4.4 (Assumption 4.3 is satisfied for a piecewise C^{p+1} domain). *Given a piecewise C^{p+1} domain, element maps satisfying Assumption 4.3 are constructed in [2, §6], building on the 2-d results of [44, 48] and the isoparametric elements in general dimension of [28].*

Lemma 4.5 (Interpolation on curved meshes [10, 28, 2]). *Given $p\in\mathbb{Z}^+$, if Assumption 4.3 holds then there exists $C>0$ and a nodal interpolation operator $\mathcal{I}_h: H^2(\Omega)\cap\mathcal{V}\to\mathcal{V}_h$ such that for all $K\in\mathcal{T}^h$ and for all $v\in H^{p+1}(K)\cap\mathcal{V}$,*

$$h_K^{-1}\big\|(I-\mathcal{I}_h)v\big\|_{L^2(K)} + \big|(I-\mathcal{I}_h)v\big|_{H^1(K)} \leq Ch_K^p\left\|v\right\|_{H^{p+1}(K)}. \tag{4.3}$$

References for the proof. The result under Assumption 4.3 is proved in [10, Theorem 2]; see also [28, Theorem 1], [2, Theorem 5.1]. ∎

4.2. Satisfying Assumption 2.1 (super-approximation).

LEMMA 4.6. *Let $\mathcal{V}_h$ be as in §4.1 and $\{\Omega_\ell\}_{\ell=1}^N$ be as in §2.2. Suppose that* either *Assumption 4.1 or Assumption 4.3 holds. Suppose, additionally, that there exists $C_{\rm PoU}>0$ such that, for $\ell=1,\ldots,N$, $\chi_\ell,\chi_\ell^> \in C^{p,1}(K)$ with*

$$\max\left\{ \|\partial^\alpha \chi_\ell\|_{L^\infty(K)}, \left\|\partial^\alpha \chi_\ell^>\right\|_{L^\infty(K)}\right\} \leq C_{\rm PoU}\delta_\ell^{-|\alpha|} \tag{4.4}$$

for all $K\in\mathcal{T}^h$ all $0\leq|\alpha|\leq p+1$, and some $\{\delta_\ell\}_{\ell=1}^N$ with $\delta_\ell\geq h_\ell:=\max_{K\subset\overline{\Omega_\ell}} h_K$.

Then (2.4) holds with $\mathcal{I}_h^\ell=\mathcal{I}_h$, the interpolant from Lemma 4.2/Lemma 4.5 and there exists $C>0$ such that

$$\mu_\ell := C\left(\frac{h_\ell}{\delta_\ell}+\left(\frac{h_\ell}{\delta_\ell}\right)^p\frac{1}{k\delta_\ell}\right). \tag{4.5}$$

Proof. We give the proof under Assumption 4.3; the proof under Assumption 4.1 is almost identical (there is no L^2 norm on the right-hand side of (4.6) below). We emphasise that all the constants in this proof depend on p. By (4.3),

$$\left|(I-\mathcal{I}_h)(\chi_\ell v_h)\right|_{H^1(K)} \leq C h_K^p \|\chi_\ell v_h\|_{H^{p+1}(K)} \leq C' h_K^p\Big(|\chi_\ell v_h|_{H^{p+1}(K)}+\|\chi_\ell v_h\|_{L^2(K)}\Big). \tag{4.6}$$

Now,

$$|\chi_\ell v_h|_{H^{p+1}(K)} \leq C\sum_{m=0}^{p+1}|\chi_\ell|_{W^{m,\infty}(K)}|v_h|_{H^{p+1-m}(K)}. \tag{4.7}$$

Recall that $\mathcal{F}_K:\widehat{K}\to K$. Let $\widetilde{\mathcal{F}}_K=\mathcal{F}_k\circ T_{h_K^{-1}}$, where $T_{h_K^{-1}}(x)=h_K^{-1}x$; i.e., $T_{h_K^{-1}}: h_K\widehat{K}\to\widehat{K}$ via scaling by h_K^{-1}. Therefore $\widetilde{\mathcal{F}}_K: h_K\widehat{K}\to K$ and (4.2) implies that

$$\|\partial^\alpha\widetilde{\mathcal{F}}_K\|_{L^\infty(\widehat{K})}\leq C \quad\text{ and }\quad \|\partial^\alpha(\widetilde{\mathcal{F}}_K^{-1})\|_{L^\infty(h_K\widehat{K})}\leq C. \tag{4.8}$$

Since $v_h\circ\mathcal{F}_K$ is a polynomial of degree p, so is $v_h\circ\widetilde{\mathcal{F}}_K$ and thus $|v_h\circ\widetilde{\mathcal{F}}_K|_{H^{p+1}(h_K\widehat{K})}=0$. Therefore, by the chain rule and a standard inverse estimate on shape-regular meshes (see, e.g., [43, Theorem 4.76, Page 208]),

$$\begin{aligned}
\int_K|\partial^\alpha v_h(x)|^2\,{\rm d}x &= \int_K\left|\partial^\alpha\big(v_h\circ\widetilde{\mathcal{F}}_K\circ\widetilde{\mathcal{F}}_K^{-1}\big)(x)\right|^2{\rm d}x\\
&\leq C\int_K\sum_{|\beta|\leq|\alpha|}\left|\partial^\beta\big(v_h\circ\widetilde{\mathcal{F}}_K\big)(\widetilde{\mathcal{F}}_K^{-1}(x))\right|^2{\rm d}x\\
&\leq C'\sum_{0\leq j\leq\min\{p,|\alpha|\}}\left|v\circ\widetilde{\mathcal{F}}_K\right|^2_{H^j(h_K\widehat{K})}\\
&\leq C''\bigg(\|v\circ\widetilde{\mathcal{F}}_K\|^2_{L^2(h_K\widehat{K})}+\sum_{1\leq j\leq\min\{p,|\alpha|\}}h_K^{-2(j-1)}\left|v\circ\widetilde{\mathcal{F}}_K\right|^2_{H^1(h_K\widehat{K})}\bigg),
\end{aligned}$$

so that

$$|v_h|_{H^{|\alpha|}(K)}\leq C'''\Big(\|v_h\|_{L^2(K)}+h_K^{1-\min\{p,|\alpha|\}}|v_h|_{H^1(K)}\Big).$$

Combining this with (4.7) and (4.4), we obtain that

$$|\chi_\ell v_h|_{H^{p+1}(K)} \le C\bigg(\sum_{m=1}^{p} \delta_\ell^{-m} h_K^{m-p}\bigg)|v_h|_{H^1(K)} + C'\bigg(\sum_{m=0}^{p+1}\delta_\ell^{-m}\bigg)\|v_h\|_{L^2(K)}. \tag{4.9}$$

The combination of (4.6), (4.9), and the fact that $h_K \le h_\ell \le \delta_\ell$ then implies that

$$\begin{aligned} k^{-1}\big|(I-\mathcal{I}_h)(\chi_\ell v_h)\big|_{H^1(K)} &\le C\bigg(\frac{h_K}{\delta_\ell}k^{-1}|v_h|_{H^1(K)} + \bigg(\frac{h_K}{\delta_\ell}\bigg)^p \frac{1}{\delta_\ell k}\|v_h\|_{L^2(K)}\bigg) \\ &\le C\bigg(\frac{h_K}{\delta_\ell} + \bigg(\frac{h_K}{\delta_\ell}\bigg)^p\frac{1}{\delta_\ell k}\bigg)\|v_h\|_{H^1_k(K)}. \end{aligned} \tag{4.10}$$

Arguing in a very similar way, we obtain the inequality

$$\|(I-\mathcal{I}_h)(\chi_\ell v_h)\|_{L^2(K)} \le C\bigg(\frac{h_K}{\delta_\ell}\bigg)\|v_h\|_{L^2(K)}, \tag{4.11}$$

Adding (4.10) and (4.11), summing over K intersecting the support of χ_ℓ, and using that $h_K \le h_\ell$ for such elements, we obtain the bound involving χ_ℓ in (2.4) with μ_ℓ given by (4.5). The proof of the bound involving $\chi_\ell^>$ in (2.4) is identical. ∎

4.3. Satisfying Assumption 2.4 (a and a_ℓ agree "in the interior of Ω_ℓ").

LEMMA 4.7. *Suppose that $a(\cdot,\cdot)$ is defined by (1.1), $a_\ell(\cdot,\cdot)$ is defined by (1.2), and*

$$A_\ell \equiv A, \quad B_\ell \equiv B, \quad c_\ell \equiv c, \quad \text{and} \quad \theta \equiv \theta_\ell \quad \text{on } \operatorname{supp}\mathcal{I}_h(\chi_\ell^>).$$

Then Assumption 2.4 holds.

Proof. This follows immediately from the definitions of $a(\cdot,\cdot)$ and $a_\ell(\cdot,\cdot)$. ∎

4.4. Satisfying Assumption 2.5 (the commutator property (2.8).

LEMMA 4.8. *Suppose that, for $\ell = 1,\ldots,N$, $a_\ell(\cdot,\cdot)$ is defined by (1.2) with $A_\ell \in C^{0,1}(\Omega_\ell;\mathbb{C}^{d\times d})$, $B_\ell \in L^\infty(\Omega_\ell;\mathbb{C}^d)$, $c_\ell^{-2} \in L^\infty(\Omega_\ell,\mathbb{C})$ and with their norms bounded independently of k. Suppose that, for all $\ell = 1,\ldots,N$, $\chi_\ell^> \in C^{p,1}(K)\cap C^{1,1}(\Omega_\ell)$ satisfying (4.4). Then Assumption 2.5 holds.*

Proof. By the definition of $a_\ell(\cdot,\cdot)$ (1.2) and integration by parts (using that $\chi_\ell^> \in C^{1,1}(\Omega_\ell)$),

$$\begin{aligned} \big|a_\ell(\chi_\ell^> u, v) - a_\ell(u,\chi_\ell^> v)\big| &= \int_{\Omega_\ell} k^{-2}(A_\ell\nabla\chi_\ell^>)\cdot\big(u\overline{\nabla v} - \nabla u\overline{v}\big) + k^{-1}(B_\ell\cdot\nabla\chi_\ell^>)u\overline{v} \\ &= \int_{\Omega_\ell} k^{-2}(A_\ell\nabla\chi_\ell^>)\cdot u\overline{\nabla v} + k^{-2}u\nabla\big((A_\ell\nabla\chi_\ell^>)\overline{v}\big) \\ &\qquad + k^{-1}(B_\ell\cdot\nabla\chi_\ell^>)u\overline{v}. \end{aligned}$$

The bound (2.8) then follows by the bound (4.4), the definition of $\|\cdot\|_{H^1_k(\Omega_\ell)}$ (1.3), the assumption that the norms of A_ℓ and B_ℓ are independent of k, the fact that $C^{0,1}(\overline{D})$ can be identified with $W^{1,\infty}(D)$ for D Lipschitz [14, §4.2.3, Theorem 5], and the fact that we can choose the bound on these norms to be independent of ℓ (just by taking the maximum over ℓ). ∎

REMARK 4.9 (Constructing χ_ℓ and $\chi_\ell^>$ as in Lemma 4.8). *A partition of unity (PoU) $\{\chi_\ell\}_{\ell=1}^N$ as in Lemma 4.8 – with, in particular, each $\chi_\ell \in C^{p,1}(K) \cap C^{1,1}(\Omega_\ell)$ – can be constructed by modifying the standard construction appearing in, e.g., [12, Lemma 5.7], [45, §3.2]. The standard construction defines the PoU functions in terms of the distance function; now one convolves the distance function p times with a piecewise-linear hat function at scale δ_ℓ – since both the distance function and the hat function are $C^{0,1}$, the resulting PoU function is $C^{p,1}(\Omega_\ell)$. Analogous ideas can be used to construct $\{\chi_\ell^>\}_{\ell=1}^N$.*

4.5. Satisfying Assumption 2.3 (the discrete inf-sup condition). As described in the discussion after Theorem 1.1, we focus on the situation when $H_\ell \lesssim k^{-1}$ (both for brevity, and because this is the most interesting case).

When verifying Assumption 2.3, care is needed if $\partial\Omega$ is disconnected – i.e., there is an impenetrable obstacle – and the boundary conditions on the part of $\partial\Omega$ corresponding to the obstacle do not match the boundary conditions on the DD subdomains (e.g., if the obstacle has Neumann boundary conditions, but one wants to use a PML with Dirichlet boundary condition). For brevity, we therefore make the following simplifying assumption (but emphasise that more general situations can be considered).

ASSUMPTION 4.10. *One of the following holds.*
(i) $\mathcal{V} = H_0^1(\Omega)$ and $\mathcal{V}_\ell = H_0^1(\Omega_\ell)$.
(ii) $\partial\Omega$ is connected, $\mathcal{V} = H^1(\Omega)$, and $\mathcal{V}_\ell = H^1(\Omega_\ell)$.

LEMMA 4.11 (Discrete inf-sup condition for subdomain widths $\lesssim k^{-1}$). *Let $a_\ell(\cdot,\cdot)$ be given by (1.2). Suppose that there exists $c, s > 0$ such that $\Re A_\ell \geq c > 0$ (in the sense of quadratic forms) and $A_\ell \in C^{1+s}(\Omega_\ell)$. Suppose that $\mathcal{V}$ and $\mathcal{V}_\ell$ satisfy Assumption 4.10. Suppose that, given $F \in (\mathcal{V}_\ell)^*$, the solution to the variational problem*

$$\text{find } u_\ell \in \mathcal{V}_\ell \quad \text{such that} \quad a_\ell(u_\ell, v_\ell) = F(v_\ell) \quad \text{for all } v_\ell \in \mathcal{V}_\ell \tag{4.12}$$

is unique. Suppose that $\mathcal{V}_h$ satisfies either *Assumption 4.1* or *Assumption 4.3, and each subdomain Ω_ℓ is the union of elements of $\mathcal{T}^h$. Given $C_1, k_0 > 0$ there exists $C_2, C_3 > 0$ such that the following is true. If $k \geq k_0$ and, for $\ell = 1, \ldots, N$, $kH_\ell \leq C_1$ and $kh_\ell \leq C_2$, then $\gamma_\ell \geq C_3$ for $\ell = 1, \ldots, N$.*

Proof. The bound $\Re A_\ell \geq c > 0$ implies that $a_\ell(\cdot,\cdot)$ satisfies a Gårding inequality. The fact that the solution to (4.12) is unique, combined with Fredholm theory (see, e.g., [34, Theorem 2.33]), then implies that the solution to (4.12) exists.

The change of variables $x = H_\ell \widetilde{x}$ transforms (4.12) to a Helmholtz problem on an order-one domain with wavenumber kH_ℓ. Since $kH_\ell \leq C_1$, the wavenumber is then bounded independently of k. The $L^2 \to H_k^1$ norm of the solution operator for this problem is therefore bounded independently of k. If the solution u_ℓ of (4.12) is in $H^{1+\epsilon}(\Omega_\ell)$, for some $\epsilon > 0$, when $F(v_\ell) := \int_{\Omega_\ell} f v_\ell$ for $f \in L^2(\Omega_\ell)$, then the Schatz argument (Appendix B) combined with the interpolation result (4.1)/(4.3) and the fact that the subdomains are resolved by the mesh $\mathcal{T}^h$ imply that the sequence of Galerkin solutions is quasi-optimal, with quasi-optimality constant independent of k. The result [35, Theorem 4.2] then implies that $\gamma_\ell \geq C_3$ (using that the solution operator is bounded independently of k, and noting that [35] work with the weighted norm $|||\cdot|||_{H_k^1}$ defined in the paragraph after (1.3)).

It is therefore sufficient to prove that the solution of (4.12) is in $H^{1+\epsilon}(\Omega_\ell)$, for some $\epsilon > 0$, for $L^2(\Omega_\ell)$ data. If the Dirichlet trace of u is in $H^1(\partial\Omega_\ell)$, then this result with $\epsilon = 1/2$ follows from [36, Theorem 3.1, Equation 3.8], using that $A_\ell \in C^{1+s}(\Omega_\ell)$. If $\mathcal{V}$ and $\mathcal{V}_\ell$ satisfy the first condition in Assumption 4.10, then the Dirichlet trace of u on $\partial\Omega_\ell$ is zero, and thus in $H^1(\partial\Omega_\ell)$. If $\mathcal{V}$ and $\mathcal{V}_\ell$ satisfy the second condition in Assumption 4.10, then $\partial_n u_\ell = \mathrm{i}k\theta u_\ell$ on $\partial\Omega_\ell$. If $\theta = 0$, $\partial_n u_\ell = 0$, and if $\theta \neq 0$, $\partial_n u_\ell \in H^{1/2}(\partial\Omega_\ell)$. In both cases, therefore, $\partial_n u_\ell \in L^2(\partial\Omega_\ell)$. The result [37, §5.1.2.] (see also [34, Theorem 4.24]) then implies that the trace of u is in $H^1(\partial\Omega_\ell)$, and the proof is complete. ∎

5. Conditions under which Assumption 2.6 (i.e., the assumption on the coarse space) holds.

5.1. Sufficient conditions for Assumption 2.6 to hold via the Schatz argument. Given $f \in L^2(\Omega)$, let $\mathcal{S}^* f \in \mathcal{V}$ be the solution of the variational problem

$$a(v, \mathcal{S}^* f) = \int_\Omega v\overline{f} \quad \text{for all } v \in \mathcal{V}. \tag{5.1}$$

Lemma 5.1 (Sufficient conditions for Assumption 2.6 to hold via the Schatz argument). *Let*

$$\eta(\mathcal{V}_0) := \left\|(I - \Pi_0)\mathcal{S}^*\right\|_{L^2(\Omega)\to H^1_k(\Omega)}, \tag{5.2}$$

where $\Pi_0 : H^1_0(\Omega) \to \mathcal{V}_0$ is the orthogonal projection in the $H^1_k(\Omega)$ norm (1.3). Suppose that Q_0 is defined by (2.11). Then there exists $c > 0$ such that if

$$\eta \le c, \tag{5.3}$$

then $Q_0 : H^1_0(\Omega) \to \mathcal{V}_0$ is well-defined and such that

$$\|(I - Q_0)v\|_{L^2(\Omega)} \le \eta C_{\rm cont} \|(I - Q_0)v\|_{H^1_k(\Omega)} \tag{5.4}$$

and

$$\|(I - Q_0)v\|_{H^1_k(\Omega)} \le 2C_{\rm cont}\left\|(I - \Pi_0)v\right\|_{H^1_k(\Omega)}. \tag{5.5}$$

Therefore (since $\|I - \Pi_0\|_{H^1_k(\Omega)\to H^1_k(\Omega)} \le 1$) the first inequality in (2.10) holds with $\sigma_{L^2} = 2\eta(C_{\rm cont})^2$ and the second inequality in (2.10) holds with $\sigma_{H^1} = 2C_{\rm cont}$.

Proof. This follows from the Schatz argument (recapped as Theorem B.1 below). Recall that the notation $\eta(\mathcal{V}_0)$ was introduced in [40, Equation 7]. ∎

5.2. Approximation spaces in the literature satisfying Assumption 2.6.

Approximation space of [30], used as a coarse space in [27, 31]. The "discrete MS-GFEM" of [30, §4] (which creates a multiscale coarse space inside $\mathcal{V}_h$) satisfies Assumption 2.6. Indeed, [30, Lemma 3.13/Equation 3.62] gives conditions under which η defined by (5.2) is small, with these conditions then carried into the two-level hybrid Schwarz theory in [31] (see [31, Theorem 2.17]).

The coarse spaces of [27] and [31] are closely related – see discussion in [31, §1]. We note that [27, Equation 6.16] is precisely (5.4) (noting that [27] use the H^1 norm $\|\!|\cdot|\!\|_{H^1_k(\Omega)}$ defined in the paragraph after (1.3)) and the displayed equation immediately after [27, Equation 6.16] is the second inequality in (2.10).

Approximation space of [16], used as a coarse space in [15]. The wavelet-based multiscale method of [16] satisfies Assumption 2.6 by the bound on η (5.2) in [16, Theorem 4.4], with conditions for η being small then given in [16, Equation 4.11]. This approximation space is then used as a coarse space in [15], with the proof of [15, Proposition 5.1] proving that the two inequalities in (2.10) hold.

Approximation space of [8]. The multiscale method of [8] satisfies Assumption 2.6 by bound on η in [8, Equation 4.4].

The localised orthogonal decomposition (LOD) method [38], used as a coarse space in [29]. The multiscale method of [38] is used as a coarse space in [29], with [29, Equation 3.16] and [29, Equation 3.12] corresponding, respectively, to the first and second bounds in (2.10).

5.3. Piecewise polynomials. Let

$$C_{\rm sol} := \sup_{f\in L^2(\Omega)} \frac{\|\mathcal{S}^* f\|_{H^1_k(\Omega)}}{\|f\|_{L^2(\Omega)}}.$$

LEMMA 5.2 (Bound on $\eta(\mathcal{V}_0)$ for piecewise polynomials). *Suppose that, for $m \in \mathbb{Z}^+$, $\mathcal{V}_0$ consists of piecewise polynomials of degree $p \leq m$ on a shape-regular mesh and one of the following holds.*

(i) Ω is $C^{m,1}$, A, B, and c are all $C^{m-1,1}$, and Assumption 4.3 holds,

(ii) Ω is $C^{1,1}$, A, B, and c are all C^∞ and correspond to a radial PML, and Assumption 4.3 holds, or

(iii) Ω is a convex polygon/polyhedron, A, B, and c are all C^∞ and correspond to a radial complex absorbing potential with, in particular, $A \equiv I$ and $B \equiv 0$ near $\partial\Omega$, and Assumption 4.1 holds.

Given $k_0 > 0$ and $M > 0$ there exists $C > 0$ such that for all $k \geq k_0$,

$$\eta(\mathcal{V}_0) \leq C\big(kH + (kH)^p C_{\rm sol} + (kL)^{-M}\big). \tag{5.6}$$

Proof. Part (i) (without the term k^{-M} in (5.6)) is proved in [7]; see also [20, Theorem 1.7]. Part (ii) is proved in [19, Lemma 2.5], and Part (iii) is proved in [23] by adapting the results of [19, Theorem 1.5]. ∎

6. Rigorous statement of Corollary 1.2. The combination of Lemmas 4.6, 4.7, 4.8, and 5.2 give bounds on $\|\mathsf{I} - \mathsf{B}_L^{-1}\mathsf{A}\|_{\mathsf{D}_k}$ and $\|\mathsf{I} - \mathsf{A}\mathsf{B}_R^{-1}\|_{\mathsf{D}_k^{-1}}$ when the coarse space consists of piecewise polynomials – we now summarise these results here. For brevity we only consider

- the first situation in Assumption 4.10 ($\mathcal{V} = H^1_0(\Omega)$ and $\mathcal{V}_\ell = H^1_0(\Omega_\ell)$),
- $\theta \equiv \theta_\ell \equiv 0$ (i.e., no impedance boundary conditions), and
- Case (i) in Lemma 5.2 (covering the most general class of Helmholtz problems),

but we emphasise that it is straightforward to write down results about the omitted cases.

COROLLARY 6.1 (Theorem 3.1 applied with piecewise-polynomial coarse spaces).

(The domain and coefficients.) *For some $m \geq 2$, suppose that Ω is $C^{m,1}$, and A, B, and c are all $C^{m-1,1}$. Let $\mathcal{V} = H^1_0(\Omega)$ and let $a(\cdot,\cdot)$ be defined by (1.1) with $\theta \equiv 0$. Assume that the solution of the variational problem (2.1) exists and is unique. Let $C_{\rm sol}$ be defined by (5.1).*

(The fine space.) *Let $\mathcal{V}_h$ consist of piecewise-polynomials on a shape regular mesh $\mathcal{T}^h$ with polynomial degree $p \leq m$ with the element maps satisfying Assumption 4.3.*

(The subdomains.) *Let $\{\Omega_\ell\}_{\ell=1}^N$ be Lipschitz domains such that*

- *each subdomain Ω_ℓ is the union of elements of $\mathcal{T}^h$,*
- *$C_1 \leq kH_\ell \leq C_2$ for some $C_1, C_2 > 0$,*

- *the partition of unity* $\{\chi_\ell\}_{\ell=1}^N$ *and functions* $\{\chi_\ell^>\}_{\ell=1}^N$ *such that* $\chi_\ell^> \equiv 1$ *on* $\mathrm{supp}\chi_\ell$ *satisfy* $\chi_\ell, \chi_\ell^> \in C^{p+1}(K) \cap C^{1,1}(\Omega_\ell)$ *and*

$$\max\left\{ \|\partial^\alpha \chi_\ell\|_{L^\infty(K)}, \left\|\partial^\alpha \chi_\ell^>\right\|_{L^\infty(K)}\right\} \leq C_{\mathrm{PoU}} k^{|\alpha|} \tag{6.1}$$

for all $K \in \mathcal{T}^h$ *and all* $0 \leq |\alpha| \leq p+1$.

(The local sesquilinear forms.) *Let* $\mathcal{V}_\ell = H_0^1(\Omega_\ell)$. *Suppose that, for* $\ell = 1, \ldots, N$, $a_\ell(\cdot,\cdot)$ *is defined by* (1.2) *with*

- $A_\ell \in C^{1+s}(\Omega_\ell; \mathbb{C}^{d\times d})$ *for some* $s > 0$, $B_\ell \in L^\infty(\Omega_\ell; \mathbb{C}^d)$, $c_\ell^{-2} \in L^\infty(\Omega_\ell, \mathbb{C})$ *and with their norms bounded independently of* k,
- $\Re A_\ell \geq c > 0$ *(in the sense of quadratic forms), and*

$$A_\ell \equiv A, \quad B_\ell \equiv B, \quad c_\ell \equiv c, \quad \text{and} \quad \theta \equiv \theta_\ell \quad \text{on } \mathrm{supp}\,\mathcal{I}_h(\chi_\ell^>).$$

(The coarse space.) $\mathcal{V}_0 \subset \mathcal{V}_h$ *consists of piecewise polynomials of degree* $p \leq m$ *on a shape-regular mesh, with each coarse-grid element a union of fine-grid elements, and the coarse-space element maps satisfying Assumption 4.3.*

(The result.) *Given* $m, C_1, C_2, C_{\mathrm{PoU}}, s > 0$ *such that the above hold, and* $\epsilon > 0$ *and* $C_{\mathrm{coarse}} > 1$, *there exists* $k_0, c > 0$ *such that if* $k \geq k_0$,

$$(kH)^p C_{\mathrm{sol}} \leq c, \quad \text{and} \quad h \leq H/C_{\mathrm{coarse}}, \tag{6.2}$$

then the Galerkin solution (2.2) *exists and is quasi-optimal (with quasi-optimality constant independent of* k*) and*

$$\max\left\{ \left\|\mathsf{I} - \mathsf{B}_L^{-1}\mathsf{A}\right\|_{\mathsf{D}_k}, \left\|\mathsf{I} - \mathsf{A}\mathsf{B}_R^{-1}\right\|_{\mathsf{D}_k^{-1}} \right\} \leq \epsilon. \tag{6.3}$$

Proof. The combination of Lemmas 4.6, 4.7, 4.8, 4.11, and 5.1/5.2 (verifying Assumptions 2.1, 2.4, 2.5, 2.3, and 2.6 respectively) and the fact that $h \leq H/C_{\mathrm{coarse}}$ imply that (3.3) holds with γ independent of k, δ proportional to k^{-1},

$$\mu \leq \frac{h}{\delta} + C\left(\frac{h}{\delta}\right)^p \leq C'\big(kh + (kh)^p\big) \leq C''\big(C_{\mathrm{sol}}^{-1/p} + C_{\mathrm{sol}}^{-1}\big),$$

and $\sigma_{L^2} \leq C(kH + (kH)^p C_{\mathrm{sol}})$. Recall that $C_{\mathrm{sol}} \geq CkL$. The bound

$$\|(I-Q)v_h\|_{H_k^1(\Omega)} \leq \epsilon \|v_h\|_{H_k^1(\Omega)} \text{ for all } v_h \in \mathcal{V}_h \tag{6.4}$$

follows from (3.3) since σ_{L^2} is made sufficiently small by (6.2) and $\mu \to 0$ as $k \to \infty$. The bound (6.3) then follows from (6.4) by Appendix A and (1.5).

The result about quasi-optimality of the Galerkin solution follows from the Schatz argument (Appendix B) and the analogue of Lemma 5.2 with the coarse space replaced by the fine space.

We obtain the result about right preconditioning, via (1.8), by showing that the assumptions of Theorem 3.1 are satisfied for the adjoint sesquilinear form. Lemmas 4.6, 4.7, 4.8, 4.11 (verifying Assumptions 2.1, 2.4, 2.5, and 2.3, respectively) hold immediately for the adjoint sesquilinear form. To apply Lemma 5.2 (for Assumption 2.6) we observe that $\|\mathcal{S}^*\|_{L^2\to L^2} = \|\mathcal{S}\|_{L^2\to L^2}$, and thus the $L^2 \to H_k^1$ norms of these operators have the same dependence on kL (since an $L^2 \to L^2$ bound implies an $L^2 \to H_k^1$ bound by Green's identity; see, e.g., [24, Lemmas 3.10 and A.10]). The combination of Lemmas 5.2 and 5.1 therefore implies that the condition (6.2) ensures that Assumption 2.6 holds for the adjoint problem, and the proof is complete. ∎

Appendix A. The matrix form of the operator Q (3.2).

Additional notation for the fine space $\mathcal{V}_h$. Denote the nodes of $\mathcal{T}^h$ by $\mathcal{N}^h = \{x_j : j \in \mathcal{J}_h\}$, where $\mathcal{J}_h$ is a suitable index set. Let $\{\phi_j : j \in \mathcal{J}_h\}$ be the standard nodal basis for $\mathcal{V}^h$. Let

$$(\mathsf{A})_{ij} := a(\phi_j, \phi_i) \quad \text{for } i, j \in \mathcal{J}_h, \tag{A.1}$$

so that the Galerkin equations (2.2) are equivalent to the linear system $\mathsf{A}\mathbf{u} = \mathbf{f}$.

Restriction matrices on the fine grid. Denote the freedoms for $\mathcal{V}_{\ell,h}$ by $\mathcal{N}^h(\Omega_\ell) = \{x_j : j \in \mathcal{J}_h(\Omega_\ell)\}$, where $\mathcal{J}_h(\Omega_\ell)$ is a suitable index set. The basis for $\mathcal{V}_{\ell,h}$ can then be written as $\{\phi_j : j \in \mathcal{J}_h(\Omega_\ell)\}$. Let

$$(\mathsf{A}_\ell)_{ij} := a_\ell(\phi_j, \phi_i) \quad \text{for } i, j \in \mathcal{J}_h(\Omega_\ell);$$

i.e., A_ℓ is the Galerkin matrix of $a_\ell(\cdot,\cdot)$. Let

$$(\mathsf{R}_\ell^{\chi_\ell})_{jj'} := \delta_{jj'}\chi_\ell(x_j) \quad \text{and} \quad (\mathsf{R}_\ell^{\chi_\ell^>})_{jj'} := \delta_{jj'}\chi_\ell^>(x_j). \tag{A.2}$$

The coarse grid and associated restriction matrices. Let $\{\mathcal{T}^H\}$ be a sequence of shape-regular, simplicial meshes on $\overline{\Omega}$, with mesh diameter H. We assume that each element of $\mathcal{T}^H$ consists of the union of a set of fine-grid elements. Let $\mathcal{J}_H$ be the set of coarse mesh nodes, so that $\{\Phi_p : p \in \mathcal{J}_H\}$ is the nodal basis for $\mathcal{V}_0$. Since $\mathcal{V}_0 \subset \mathcal{V}_h$, there exists a matrix R_0 such that

$$\Phi_p = \sum_j (\mathsf{R}_0)_{pj}\phi_j. \tag{A.3}$$

Let

$$\mathsf{A}_0 := \mathsf{R}_0 \mathsf{A} \mathsf{R}_0^T. \tag{A.4}$$

In fact, since each element of $\mathcal{T}^H$ consists of the union of a set of fine-grid elements, if $\mathcal{I}_h$ is the nodal interpolation operator then

$$\Phi_p = \mathcal{I}_h\Phi_p = \sum_j \Phi_p(x_j^h)\phi_j, \tag{A.5}$$

and thus

$$(\mathsf{R}_0)_{pj} := \Phi_p(x_j^h), \quad j \in \mathcal{J}_h, \quad p \in \mathcal{J}_H. \tag{A.6}$$

LEMMA A.1. *$(\mathsf{A}_0)_{pq} = a(\Phi_q, \Phi_p)$; i.e., A_0 is the Galerkin matrix for the variational problem (2.2) discretised in $\mathcal{V}_0$ using the basis $\{\Phi_p : p \in \mathcal{J}_H\}$.*

Proof. By the definition (A.3) of R_0 and the definition (A.1) of A,

$$a(\Phi_q, \Phi_p) = \sum_j \sum_i (\mathsf{R}_0)_{qj} a(\phi_j, \phi_i)(\mathsf{R}_0)_{pi} = \sum_j \sum_i (\mathsf{R}_0)_{pi}(\mathsf{A})_{ij}(\mathsf{R}_0^T)_{jq} = (\mathsf{R}_0\mathsf{A}\mathsf{R}_0^T)_{pq},$$

and the result follows from (A.4). ∎

The matrix form of the $H_k^1(\Omega)$ inner product. Let

$$(\mathsf{S})_{\ell,m} = \int_\Omega \nabla\phi_\ell \cdot \nabla\phi_m, \qquad (\mathsf{M})_{\ell,m} = \int_\Omega \phi_\ell\phi_m, \quad \text{and} \quad \mathsf{D}_k := k^{-2}\mathsf{S} + \mathsf{M}. \tag{A.7}$$

It then follows that if $v_h, w_h \in \mathcal{V}_h$ with coefficient vectors $\mathbf{V}, \mathbf{W}$ then

$$(v_h, w_h)_{H_k^1(\Omega)} \;=\; \langle \mathbf{V}, \mathbf{W}\rangle_{\mathsf{D}_k}. \tag{A.8}$$

The matrix form of the operators Q_ℓ. The fact that the matrix form of Q is $\mathsf{B}_L^{-1}\mathsf{A}$, with B_L^{-1} defined by (1.4), is an immediate consequence of the following result combined with the definition of Q (3.2).

THEOREM A.2. *Let $v_h = \sum_{j\in\mathcal{J}_h} V_j\phi_j \in \mathcal{V}_h$. Then, for $\ell = 1,\ldots,N$,*

$$\mathcal{I}_h(\chi_\ell Q_\ell v_h) = \sum_{j\in\mathcal{J}_h(\Omega_\ell)} \left(\left(\mathsf{R}_\ell^{\chi_\ell}\right)^T \mathsf{A}_\ell^{-1}\mathsf{R}_\ell^{\chi_\ell^>}\mathsf{A}\mathbf{V}\right)_j \phi_j, \quad Q_0 v_h = \sum_{j\in\mathcal{J}_h}\left(\mathsf{R}_0^T\mathsf{A}_0^{-1}\mathsf{R}_0\mathsf{A}\mathbf{V}\right)_j \phi_j.$$

Proof. The proof of the first expression is very similar to the proof of [26, Theorem 2.10] (where only one type of weighted restriction matrix is used, in contrast to the two used here). The second expression is proved in [25, Theorem 5.4]. ∎

Appendix B. Recap of the Schatz argument and Aubin–Nitsche lemma.

THEOREM B.1 (The Schatz argument [41, 42]). *Suppose that the sequilinear form $b:\mathcal{H}\times\mathcal{H}\to\mathbb{C}$ is continuous, i.e.,*

$$\text{(B.1)}\qquad \left|b(u,v)\right| \le C_{\rm cont}\left\|u\right\|_{\mathcal{H}}\left\|v\right\|_{\mathcal{H}} \quad \text{for all } u,v\in\mathcal{H}$$

and satisfies the Gårding equality

$$\text{(B.2)}\qquad \Re b(v,v) \ge C_{G1}\left\|v\right\|^2_{\mathcal{H}} - C_{G2}\left\|v\right\|^2_{\mathcal{H}_0} \quad \text{for all } v\in\mathcal{H}$$

where $\mathcal{H}_0\subset\mathcal{H}$. Suppose that $\mathcal{H}_h$ is a finite-dimensional subspace of $\mathcal{H}$. Given $u\in\mathcal{H}$ and $u_h\in\mathcal{H}_h$ such that

$$\text{(B.3)}\qquad b(u-u_h,v_h) = 0 \quad \text{for all } v_h\in\mathcal{H}_h,$$

(B.4)

$$\text{if}\quad \left\|u-u_h\right\|_{\mathcal{H}_0} \le \frac{C_{G1}}{\sqrt{2C_{G2}}}\left\|u-u_h\right\|_{\mathcal{H}} \quad \text{then}\quad \left\|u-u_h\right\|_{\mathcal{H}} \le 2C_{\rm cont}\min_{v_h\in\mathcal{H}_h}\left\|u-v_h\right\|_{\mathcal{H}}.$$

Proof. By (B.2), (B.3), and (B.1), for all $v_h\in\mathcal{H}_h$,

$$\begin{aligned} C_{G1}\left\|u-u_h\right\|^2_{\mathcal{H}} &\le \Re b\big(u-u_h,u-u_h\big) + C_{G2}\left\|u-u_h\right\|^2_{\mathcal{H}_0}\\ &\le \Re b\big(u-u_h,u-v_h\big) + C_{G2}\left\|u-u_h\right\|^2_{\mathcal{H}_0}\\ &\le C_{\rm cont}\left\|u-u_h\right\|_{\mathcal{H}}\left\|u-v_h\right\|_{\mathcal{H}} + C_{G2}\left\|u-u_h\right\|^2_{\mathcal{H}_0}, \end{aligned}$$

and the result (B.4) follows. ∎

LEMMA B.2. (The Aubin–Nitsche lemma [9, Theorem 3.2.4]) *Under the assumptions of Theorem B.1, given $f\in\mathcal{H}_0$, let $\mathcal{S}^*f$ be the solution of the variational problem*

$$\text{(B.5)}\qquad b(w,\mathcal{S}^*f) = (w,f)_{\mathcal{H}_0} \quad \text{for all } w\in\mathcal{H}.$$

Let

$$\text{(B.6)}\qquad \eta(\mathcal{H}_h) := \sup_{f\in L^2(\Omega)}\min_{v_h\in\mathcal{H}_h}\frac{\left\|\mathcal{S}^*f - v_h\right\|_{H^1_k(\Omega)}}{\left\|f\right\|_{L^2(\Omega)}}.$$

Then

$$\text{(B.7)}\qquad \left\|u-u_h\right\|_{\mathcal{H}_0} \le C_{\rm cont}\eta(\mathcal{H}_h)\left\|u-u_h\right\|_{\mathcal{H}}.$$

Proof. By (B.5), (B.3), (B.1), and (B.6), for all $v_h\in\mathcal{H}_h$,

$$\begin{aligned} \left\|u-u_h\right\|^2_{\mathcal{H}_0} = b\big(u-u_h,\mathcal{S}^*(u-u_h)\big) &= b\big(u-u_h,\mathcal{S}^*(u-u_h)-v_h\big)\\ &\le C_{\rm cont}\left\|u-u_h\right\|_{\mathcal{H}}\left\|\mathcal{S}^*(u-u_h)-v_h\right\|_{\mathcal{H}}, \end{aligned}$$

and the result (B.7) follows. ∎

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