SEMICLASSICAL RESOLVENT BOUNDS FOR LONG RANGE LIPSCHITZ POTENTIALS

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Abstract. We give an elementary proof of weighted resolvent estimates for the semiclassical Schrödinger operator $-h^2\Delta + V(x) - E$ in dimension $n \neq 2$, where h, $E > 0$. The potential is real-valued, V and $\partial_r V$ exhibit long range decay at infinity, and may grow like a sufficiently small negative power of r as $r \to 0$. The resolvent norm grows exponentially in h^{-1}, but near infinity it grows linearly. When V is compactly supported, we obtain linear growth if the resolvent is multiplied by weights supported outside a ball of radius $CE^{-1/2}$ for some $C > 0$. This E-dependence is sharp and answers a question of Datchev and Jin.

1. Introduction and statement of results

Let $\Delta \coloneqq \sum_{j=1}^n \partial_j^2 \leq 0$ be the Laplacian on $\mathbb{R}^n$, $n \neq 2$. Let P denote the semiclassical Schrödinger operator

$$P = P(h) \coloneqq -h^2\Delta + V(x) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \qquad x \in \mathbb{R}^n,\ h > 0. \tag{1.1}$$

We use $(r, \theta) = (|x|, x/|x|) \in (0, \infty) \times \mathbb{S}^{n-1}$ for polar coordinates on $\mathbb{R}^n \setminus \{0\}$. Let c_0, $c_1 > 0$ and $0 \leq \delta < \sqrt{8} - 2$. Furthermore, let $p(r) > 0$ be bounded and decreasing to zero as $r \to \infty$, and suppose $0 < m(r) \leq 1$ with

$$\lim_{r\to\infty} m(r) = 0, \qquad (r+1)^{-1}m(r) \in L^1(0,\infty). \tag{1.2}$$

We assume the potential $V : \mathbb{R}^n \to \mathbb{R}$, satisfies

$$V \in L^n(\mathbb{R}^n;\mathbb{R}) + L^\infty(\mathbb{R}^n;\mathbb{R}), \tag{1.3}$$

$$V\mathbf{1}_{|x|<1} \leq c_1 r^{-\delta}, \tag{1.4}$$

$$V\mathbf{1}_{|x|\geq 1} \leq p(r). \tag{1.5}$$

In addition, we suppose there is function $\partial_r V \in L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ such that, for each $\theta \in \mathbb{S}^{n-1}$, the function $(0,\infty) \ni r \mapsto V(r,\theta) \coloneqq V(r\theta)$ has distributional derivative equal to $r \mapsto \partial_r V(r,\theta)$, and

$$\partial_r V(r,\theta)\mathbf{1}_{0<r<1} \leq c_1 r^{-1-\delta}, \tag{1.6}$$

$$\partial_r V(r,\theta)\mathbf{1}_{r\geq 1} \leq c_0 r^{-1}m(r). \tag{1.7}$$

The prototypes we have in mind for (1.2) are the long range cases

$$m = \log^{-1-\rho}(r+e), \qquad m = (r+1)^{-\rho}, \qquad\qquad \rho > 0. \tag{1.8}$$

When $n \geq 3$, (1.3) implies (1.1) is self-adjoint with respect to the domain $\mathcal{D}(P) = H^2(\mathbb{R}^n)$ [Ne64, Theorem 8] . If $n = 1$, then (1.1) is self-adjoint with respect to

$$\mathcal{D}(P) = \{u \in H^1(\mathbb{R}) \mid u' \in L^\infty(\mathbb{R}),\ Pu \in L^2(\mathbb{R})\},$$

see the proof in the Appendix. By [Ne64, Theorem 8], (1.1) is self adjoint with respect to $H^2(\mathbb{R}^n)$, provided $V \in L^p + L^\infty$ and $p \geq 2$, $p > n/2$, but for simplicity we work with (1.3).

For $E > 0$ and $s > 1/2$ fixed, and $h, \varepsilon > 0$, our goal is to establish h-dependent upper bounds on the weighted resolvent norms

$$g_s^{\pm}(h,\varepsilon) := \|\langle x\rangle^{-s}(P(h) - E \pm i\varepsilon)^{-1}\langle x\rangle^{-s}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}, \tag{1.9}$$

$$g_s^{\pm}(h,M,\varepsilon) := \|\langle x\rangle^{-s}\mathbf{1}_{|x|\geq M}(P(h) - E \pm i\varepsilon)^{-1}\mathbf{1}_{|x|\geq M}\langle x\rangle^{-s}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}. \tag{1.10}$$

Here, $\langle x\rangle = \langle r\rangle := (1 + r^2)^{1/2}$.

In our Theorem, we bound (1.9) and (1.10) and show that, if V is compactly supported, there are constants C_1, $h_0 > 0$, such that (1.10) grows linearly in h^{-1}, provided $M \geq C_1E^{-1/2}$, $\varepsilon > 0$ and $h \in (0, h_0]$.

Theorem 1. *Fix $E > 0$ and $s > 1/2$. Suppose $V : \mathbb{R}^n \to \mathbb{R}$ satisfies (1.3) through (1.7). There exist $M = M(E, p, c_0, c_1, \delta, m)$, $C_2 = C_2(E, s, p, c_0, c_1, \delta, m)$, $C_3 = C_3(E, s, p, c_0, c_1, \delta, m) > 0$ and $h_0 \in (0, 1]$ so that, for all $\varepsilon > 0$ and $h \in (0, h_0]$,*

$$\left\|\langle x\rangle^{-s}(P(h) - E \pm i\varepsilon)^{-1}\langle x\rangle^{-s}\right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq e^{C_3/h}, \tag{1.11}$$

and

$$\left\|\langle x\rangle^{-s}\mathbf{1}_{|x|\geq M}(P(h) - E \pm i\varepsilon)^{-1}\mathbf{1}_{|x|\geq M}\langle x\rangle^{-s}\right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq C_2/h. \tag{1.12}$$

Moreover, if $\operatorname{supp} V \subseteq B(0, R_0)$, then one can take $M = C_1(p, c_0, c_1, \delta, R_0)E^{-1/2}$.

The main novelty of the Theorem is in the compactly supported case, where M need not be larger than a constant times $E^{-1/2}$. This seems to be the first general bound of the form $g_s^{\pm}(h, M, \varepsilon) \leq Ch^{-1}$ for which M depends explicitly on E. Moreover, owing to a construction of Datchev and Jin [DaJi20, Theorem 1], this E-dependence of M is optimal. In particular, if $V \in C_0^{\infty}(\mathbb{R}^n; \mathbb{R})$, $n \geq 2$ is radial and $\min(V) < 0$, then there is $M \leq cE^{-1/2}$ with $g_s^{\pm}(h, M, \varepsilon) \geq e^{C/h}$. In addition, to the author's knowledge, this article is the first in this line of work to allow V to be unbounded. We have included this to illustrate the flexibility of our methods, but do not expect the growth of $r^{-\sqrt{8}+2}$ near $r = 0$ is optimal.

Burq [Bu98] was the first to show $g_s^{\pm} \leq e^{Ch^{-1}}$ for compactly supported perturbations of the Laplacian on $\mathbb{R}^n$. This bound was refined and extended many times [Vo00, Bu02, Sj02, CaVo02, Da14, Sh19, Vo20d] and is sharp in general, see [DDZ15]. Cardoso and Vodev [CaVo02], refining Burq's earlier work [Bu02], were the first to prove an exterior estimate of the form (1.12). They did so for smooth V on a large class of infinite volume Riemannian manifolds. Exterior estimates were subsequently established under a wide range of regularity and geometric conditions [Da14, Vo14, RoTa15, DadeH16, Sh19].

Stronger bounds on $g_s^{\pm}$ are known when V is smooth and conditions are imposed on the classical flow $\Phi(t) = \exp t(2\xi\partial_x - \partial_x V(x)\partial_\xi)$ (note that $\Phi(t)$ may be undefined in our case). The key dynamical object is the *trapped set* $\mathcal{K}(E)$ at energy $E > 0$, defined as the set of $(x, \xi) \in T^*\mathbb{R}^n$ such that $|\xi|^2 + V(x) = E$ and $|\Phi(t)(x, \xi)|$ is bounded as $|t| \to \infty$. If $\mathcal{K}(E) = \emptyset$, that is, if E is *nontrapping*, Robert and Tamura [RoTa87] showed $g_s^{\pm} \leq Ch^{-1}$. We may think of (1.12) as a low regularity analog; it says that applying cutoffs supported far away from zero removes the losses from (1.11) due to trapping.

Resolvent estimates such as (1.11) imply logarithmic local energy decay for the wave equation

$$\begin{cases} (\partial_t^2 - c^2(x)\Delta)u(x,t) = 0, & (x,t) \in (\mathbb{R}^n \setminus \Omega) \times (0,\infty),\ n \geq 2, \\ u(x,0) = u_0(x), & \\ \partial_t u(x,0) = u_1(x), & \\ u(t,x) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \end{cases} \tag{1.13}$$

where Ω is a compact (possibly empty) obstacle with smooth boundary, and the initial data are compactly supported. Such a decay rate was first proved by Burq [Bu98, Bu02] for c smooth.

Logarithmic decay was subsequently established (when $\Omega = \emptyset$) for Lipschitz c bounded from above and below [Sh18, Theorem 1]. See also [Be03, CaVo04, Bo11, Mo16, Ga19]. By leveraging (1.11), we expect [Sh18, Theorem 1] extends to certain c which tend to 0 at a point.

As shown in section XIII.7 of [ReSi78], the exterior bound (1.12) is related to exterior smoothing and Strichartz estimates for Schrödinger propagators, see also [BoTz07, MMT08] and Section 7.1 of [DyZw19]. Furthermore, Christiansen [Ch17] used an estimate like (1.12) to find a lower bound on the resonance counting function for compactly supported perturbations on the Laplacian on even-dimensional Riemannian manifolds.

To prove the Theorem, we adapt the Carleman estimate from [GaSh20], which was used to prove a resolvent estimate for L^∞ potentials. The key ingredients remain a weight $w(r)$ and phase $\varphi(r)$ that obey a crucial lower bound, see (3.9) below. The main technical innovation is that, by leveraging the additional regularity of V, we can decrease φ' to zero (outside of a compact set) in an explicit, E-dependent fashion. We then obtain (1.12) for any M such that $\mathbf{1}_{|x|\geq M}$ is supported in the set where φ is constant.

If we do not assume anything about the derivatives of V, for instance, if $V \in L^\infty_{\text{comp}}(\mathbb{R}^n;\mathbb{R})$, then the best known bound in general is $g_s^\pm \leq \exp(Ch^{-4/3}\log(h^{-1}))$ [KlVo19, Sh20], although Vodev [Vo20c] showed this can be improved to $g_s^\pm \leq \exp(Ch^{-4/3})$ if V is short range and radial. See also [Vo19a, Vo20a, Vo20d, GaSh20]. On the other hand, it is not known whether an exterior estimate like (1.12) holds for L^∞ potentials, except in dimension one [DaSh20], and there $\mathbf{1}_{|x|\geq M}$ and V need only have disjoint supports.

We remark that the Theorem should hold in dimension two also, provided $V \in L^\infty$ and $|\nabla V|$ is locally bounded near the origin. The extra difficulty in dimension two comes from the effective potential term, see (2.1) below, having a negative singularity at $r = 0$. This necessitates a stronger assumption on the derivatives of V, see [Vo20d, Theorem 4.2] for more details.

For more background on semiclassical resolvent estimates, we refer the reader to the introductions of [DaJi20, GaSh20].

ACKNOWLEDGEMENTS. The authors thank Kiril Datchev for helpful comments and for reading an early version of this article, and the two anonymous referees for their careful reading and insightful remarks. J. Shapiro was supported in part by the Australian Research Council through grant DP180100589.

2. NOTATION AND PRELIMINARY CALCULATIONS

Notation: Throughout, "prime" notation indicates differentiation with respect to the radial variable $r = |x|$, e.g., $u' = \partial_r u$.

As in most previous proofs of resolvent estimates for low regularity potentials, the backbone of the proof is a Carleman estimate. Our Carleman estimate is stated as Lemma 3.2.

We start from the identities

$$\begin{gathered} r^{\frac{n-1}{2}}(-\Delta)r^{-\frac{n-1}{2}} = -\partial_r^2 + \Lambda, \\ \Lambda := \frac{1}{r^2}\left(-\Delta_{\mathbb{S}^{n-1}} + \frac{(n-1)(n-3)}{4}\right) \geq 0, \end{gathered} \tag{2.1}$$

where $\Delta_{\mathbb{S}^{n-1}}$ denotes the negative Laplace-Beltrami operator on $\mathbb{S}^{n-1}$. Below, we construct an absolutely continuous phase function φ on $[0,\infty)$ which obeys $\varphi \geq 0$, $\varphi(0) = 0$, and $\varphi' \geq 0$. Using φ, we form the conjugated operator

$$\begin{aligned} P_\varphi^\pm(h) &:= e^{\varphi/h} r^{\frac{n-1}{2}}\left(P(h) - E \pm i\varepsilon\right) r^{-\frac{n-1}{2}} e^{-\varphi/h} \\ &= -h^2\partial_r^2 + 2h\varphi'\partial_r + h^2\Lambda + V - (\varphi')^2 + h\varphi'' - E \pm i\varepsilon. \end{aligned} \tag{2.2}$$

Let $u \in e^{\varphi/h} r^{(n-1)/2} C_0^\infty(\mathbb{R}^n)$ when $n \geq 3$, $u \in e^{\varphi/h} r^{(n-1)/2}\mathcal{D}(P)$ when $n = 1$. Define a spherical energy functional $F[u](r)$,

$$F(r) = F[u](r) := \|hu'(r,\cdot)\|^2 - \langle (h^2\Lambda + V - (\varphi')^2 - E)u(r,\cdot), u(r,\cdot)\rangle, \tag{2.3}$$

where $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the norm and inner product on $L^2(\mathbb{S}^{n-1}_\theta)$, respectively (when $n = 1$, $\|u(r,\theta)\|_{\mathbb{S}^0} := |u(r)| + |u(-r)|$). It is easy to compute (see e.g. [Da14, Sh19, Sh20, GaSh20]) that for $w \in C^0[0,\infty)$ and piecewise C^1, $(wF)'$, as a distribution on $(0,\infty)$, is given by

$$\begin{aligned}(wF)' = &-2w\operatorname{Re}\langle P^{\pm}_\varphi(h)u, u'\rangle \mp 2\varepsilon w \operatorname{Im}\langle u, u'\rangle + (2wr^{-1} - w')\langle h^2\Lambda u, u\rangle \\ &+ (4h^{-1}w\varphi' + w')\|hu'\|^2 + (w(E + (\varphi')^2 - V))'\|u\|^2 + 2w\operatorname{Re}\langle h\varphi'' u, u'\rangle.\end{aligned} \tag{2.4}$$

We will construct $w \geq 0$, $w(0) = 0$ such that

$$2wr^{-1} - w' \geq 0, \tag{2.5}$$

and use (2.1) to control the term involving Λ. Using (2.5) together with $2ab \geq -(\gamma a^2 + \gamma^{-1}b^2)$ for $\gamma > 0$, we find

$$\begin{aligned}(wF)' = &-2w\operatorname{Re}\langle P^{\pm}_\varphi(h)u, u'\rangle \mp 2\varepsilon w \operatorname{Im}\langle u, u'\rangle + (2wr^{-1} - w')\langle h^2\Lambda u, u\rangle \\ &+ (4h^{-1}w\varphi' + w')\|hu'\|^2 + (w(E + (\varphi')^2 - V))'\|u\|^2 + 2w\operatorname{Re}\langle h\varphi'' u, u'\rangle \\ \geq &-\frac{\gamma_1 w^2}{h^2 w'}\|P^{\pm}_\varphi(h)u\|^2 \mp 2\varepsilon w\operatorname{Im}\langle u, u'\rangle \\ &+ (4(1-\gamma_2^{-1})h^{-1}w\varphi' + (1 - \gamma_1^{-1} - \gamma_2^{-1})w')\|hu'\|^2 \\ &+ (w(E + (\varphi')^2 - V))'\|u\|^2 - \frac{\gamma_2(w\varphi'')^2}{w' + 4h^{-1}\varphi' w}\|u\|^2, \qquad \gamma_1, \gamma_2 > 0.\end{aligned}$$

Fix $\eta > 0$ and put $\gamma_1 = (1+\eta)/\eta$, $\gamma_2 = 1 + \eta$, yielding

$$\begin{aligned}(wF)' \geq &-\frac{(1+\eta)w^2}{\eta h^2 w'}\|P^{\pm}_\varphi(h)u\|^2 \mp 2\varepsilon w\operatorname{Im}\langle u, u'\rangle \\ &+ (w(E + (\varphi')^2 - V))'\|u\|^2 - \frac{(1+\eta)(w\varphi'')^2}{w' + 4h^{-1}\varphi' w}\|u\|^2.\end{aligned} \tag{2.6}$$

To complete the proof of the Carleman estimate, we seek to build w and φ so that the second line of (2.6) has a good lower bound. Indeed, putting

$$A(r) := (w(E + (\varphi')^2 - V))', \qquad B(r) := \frac{(w\varphi'')^2}{w' + 4h^{-1}\varphi' w}, \tag{2.7}$$

it suffices for w and φ to satisfy, for $\eta > 0$ fixed,

$$A(r) - (1+\eta)B(r) \geq \frac{E}{2}w', \qquad 0 < h \ll 1, \tag{2.8}$$

along with a few other properties (see (3.5) through (3.8)).

In order to construct the weight and phase functions for our Carleman estimates, we adapt the method in [GaSh20]. Whenever $w', \varphi' \neq 0$, put

$$\Phi := \frac{\varphi''}{\varphi'} = (\log|\varphi'|)', \qquad \mathcal{W} := \frac{w}{w'} = \frac{1}{(\log|w|)'}, \tag{2.9}$$

Then, as in [GaSh20, (2.10)],

$$A(r) - (1+\eta)B(r) \geq w'\Big[E + (\varphi')^2(1 + 2\mathcal{W}\Phi - (1+\eta)\mathcal{W}\Phi^2\min(\mathcal{W}, \tfrac{h}{4\varphi'})) - V - \mathcal{W}V'\Big]. \tag{2.10}$$

So when $|w'|$, $|\varphi'| > 0$, to show (2.8), it is enough to bound the bracketed expression in (2.10) from below by $E/2$.

3. Construction of the phase and weight functions

Throughout this section, we assume $E > 0$, $s > 1/2$ are fixed, and suppose V satisfies (1.4) through (1.7). Using (1.5) and (1.7), let

$$b := \sup\{|x| \geq 1 : V(x) + |x|V'(x) \geq \tfrac{E}{4}\} < \infty, \tag{3.1}$$

so that b is independent of h and

$$(V + |x|V')\mathbf{1}_{|x|\geq 1} \leq (V + |x|V')\mathbf{1}_{|x|\leq b} + \frac{E}{4}\mathbf{1}_{|x|>b}. \tag{3.2}$$

(Note that b can be chosen to depend only on p, m, c_0, and E, and that $b \leq R_0$ provided $\operatorname{supp} V \subseteq B(0, R_0)$.) Additionally, let

$$M > a \geq b, \qquad \tau_0 \geq 1, \tag{3.3}$$

be parameters, independent of h, to be specified in the proof of Lemma 3.1 below.

Let $\omega \in C_0^\infty((-3/4, 3/4); [0, 1])$ with $\omega = 1$ near $[-1/2, 1/2]$. The weight w and phase φ, which will be shown to satisfy (2.8), are functions of the radial variable $r = |x|$ only, and are defined by

$$\tilde{m}(r) := \min\left[\frac{E}{2c_0}m^{-1}(r), (r+1)^{2s-1}\right], \tag{3.4}$$

$$w(0) = 0,\ w'(0) = 1, \qquad \frac{w}{w'} = \mathcal{W} := \begin{cases} \frac{r(1+\omega(r))}{2} & 0 < r < M \\ \frac{r}{2}\tilde{m}(r) & r > M \end{cases}, \tag{3.5}$$

$$\varphi(0) = 0,\ \varphi'(\tfrac{1}{2}) = \tfrac{2}{3}\tau_0, \qquad \frac{\varphi''}{\varphi'} = \Phi := \begin{cases} -\frac{\delta}{2r} & 0 < r < \frac{1}{2} \\ -\frac{1}{r+1} & \frac{1}{2} < r < a \\ -\frac{2}{M-r} & a < r < M \end{cases}, \qquad \varphi' = 0, \qquad r \geq M. \tag{3.6}$$

Short computations yield,

$$w = \begin{cases} r & 0 < r < 1/2 \\ \frac{1}{2}e^{\int_{1/2}^r \frac{2}{s(1+\omega(s))}ds} & 1/2 \leq r < M \\ w(M)e^{\int_M^r \frac{2}{s\tilde{m}(s)}ds} & r \geq M \end{cases}, \qquad w' = \begin{cases} 1 & 0 < r < 1/2 \\ \frac{1}{r(1+\omega(r))}w & 1/2 < r < M \\ \frac{2w(M)e^{\int_M^r \frac{2}{s\tilde{m}(s)}ds}}{r\tilde{m}(r)} \geq (r+1)^{-2s} & r > M \end{cases}, \tag{3.7}$$

$$\varphi' = \begin{cases} 3^{-1}\cdot 2^{-\frac{\delta}{2}+1}\tau_0 r^{-\frac{\delta}{2}} & 0 < r < \frac{1}{2} \\ \frac{\tau_0}{r+1} & \frac{1}{2} \leq r < a \\ \varphi'(a)\left(\frac{M-r}{M-a}\right)^2 & a \leq r < M \\ 0 & r \geq M \end{cases}. \tag{3.8}$$

We now prove the crucial lower bound involving E, w and φ that is needed to prove the Carleman estimate.

Lemma 3.1. *Fix $0 < \eta < \frac{4-4\delta-\delta^2}{\delta^2}$ and let V satisfy (1.4), (1.5), (1.6) and (1.7). Then, using the notation of (2.7) and (3.4) through (3.8), there exist suitable M, a and τ_0 so that*

$$A - (1+\eta)B \geq \frac{E}{2}w', \qquad h \in (0, h_0],\ r > 0,\ r \neq \frac{1}{2},\ a,\ M. \tag{3.9}$$

Once Lemma 3.1 is proved, we can use a standard argument similar to that found, e.g., in [GaSh20, Sections 5,6] to prove the following Carleman estimate. This argument is contained in Section 4.

Lemma 3.2. *There are C, $h_0 > 0$ independent of h and ε so that*

$$\|\langle x\rangle^{-s}e^{\varphi/h}v\|_{L^2}^2 \leq \frac{C}{h^2}\|\langle x\rangle^{s}e^{\varphi/h}(P(h) - E \pm i\varepsilon)v\|_{L^2}^2 + \frac{C\varepsilon}{h}\|e^{\varphi/h}v\|_{L^2}^2, \tag{3.10}$$

for all $\varepsilon > 0$ and $h \in (0, h_0]$, and for all $v \in C_0^\infty(\mathbb{R}^n)$ ($n \geq 3$) or all $v \in \mathcal{D}(P)$ with $\langle x \rangle^s P v \in L^2(\mathbb{R})$ ($n = 1$).

From here, Theorem 1 follows from the argument in Section 5.

Proof of Lemma 3.1. **Case $0 < r < \frac{1}{2}$:**
Using (2.10), together with $\mathcal{W} = r$, $\Phi = -\frac{\delta}{2r}$, and $\varphi' = 3^{-1} \cdot 2^{-\frac{\delta}{2}+1} \tau_0 r^{-\frac{\delta}{2}}$, we have

$$\begin{aligned} A(r) - (1+\eta)B(r) &\geq w' \Big[E + (\varphi')^2 (1 + 2\mathcal{W}\Phi - (1+\eta)\mathcal{W}\Phi^2 \min(\mathcal{W}, \tfrac{h}{4\varphi'})) - V - \mathcal{W}V' \Big] \\ &\geq w' \left[E + 9^{-1} \cdot 2^{-\delta+2} \tau_0^2 r^{-\delta} \left(1 - \delta - \tfrac{1}{4}(1+\eta)\delta^2\right) - 2c_1 r^{-\delta} \right]. \end{aligned}$$

Since $\eta > 0$ is such that $1 - \delta - 4^{-1}(1+\eta)\delta^2 > 0$, choosing

$$\tau_0^2 \geq \frac{9 \cdot 2^{\delta-1} c_1}{(1 - \delta - 4^{-1}(1+\eta)\delta^2)}, \tag{3.11}$$

yields

$$A - (1+\eta)B \geq \frac{E}{2} w', \qquad h \in (0, 1],\ 0 < r < \frac{1}{2}.$$

The precise value, $(1+\eta)$ will not play a crucial role below, therefore, to ease notation, we put $K := 1 + \eta$ below.

Case $\frac{1}{2} < r < a$:
First, recall (2.10):

$$A(r) - KB(r) \geq w' \Big[E + (\varphi')^2 (1 + 2\mathcal{W}\Phi - K\mathcal{W}\Phi^2 \min(\mathcal{W}, \tfrac{h}{4\varphi'})) - V - \mathcal{W}V' \Big].$$

By (3.5) and (3.6),

$$1 + 2\mathcal{W}\Phi \geq \frac{1}{4(r+1)}, \qquad \tfrac{1}{2} < r < a.$$

Also by (3.5), $|\mathcal{W}| \leq r$ when $0 < r < a$, hence appealing to (3.2),

$$V + \mathcal{W}V' \leq (V + rV')\mathbf{1}_{|x| \leq b} + \frac{E}{4} \mathbf{1}_{|x| > b}.$$

Furthermore, using $|\mathcal{W}| \leq r$ again, by (3.6) $\Phi^2 = (r+1)^{-2}$, and by (3.8) $\varphi' = \tau_0 (r+1)^{-1}$,

$$(\varphi')^2 \mathcal{W}\Phi^2 \min(\mathcal{W}, \tfrac{h}{4\varphi'}) \leq \frac{hr\tau_0}{4(r+1)^3}, \qquad 0 < r < a.$$

From these estimates, and using once more that $\varphi' = \tau_0 (r+1)^{-1}$, we find,

$$\begin{aligned} &A(r) - KB(r) \\ &\quad \geq w' \Big[E + (\varphi')^2 \big(1 + 2\mathcal{W}\Phi\big) - \frac{Khr\tau_0}{4(r+1)^3} - (V + rV')\mathbf{1}_{r \leq b} - \frac{E}{4}\mathbf{1}_{r > b} \Big] \\ &\quad \geq w' \Big[\frac{3E}{4} + \Big(\frac{\tau_0^2}{4(r+1)^3} - 2c_1 r^{-\delta} \mathbf{1}_{0 < r < 1} - (V + rV')\mathbf{1}_{1 \leq r \leq b} \Big) - \frac{Khr\tau_0}{4(r+1)^3} \Big], \qquad \tfrac{1}{2} < r < a. \end{aligned} \tag{3.12}$$

We now further increase τ_0, if necessary, so that

$$\tau_0 \geq 2 \sup_{\frac{1}{2} \leq r \leq b} (r+1)^{3/2} \sqrt{2c_1 r^{-\delta} \mathbf{1}_{0 < r < 1} + (p(r) + c_0 m(r))\mathbf{1}_{1 \leq r \leq b}},$$

which makes the term in parenthesis in the third line of (3.12) is nonnegative. We then take $h_0 = h_0(K, \tau_0, E) \in (0, 1]$ sufficiently small to achieve

$$A - KB \geq \frac{E}{2} w', \qquad h \in (0, h_0],\ \frac{1}{2} < r < a. \tag{3.13}$$

Case $a < r < M$:
As in the previous case, we begin from (2.10). We use (3.5), (3.6) and (3.8) to see

$$(\varphi')^2(1+2\mathcal{W}\Phi) = (\varphi'(a))^2\Big(\frac{M-r}{M-a}\Big)^4\Big(1-\frac{2r}{M-r}\Big).$$

Next, we use $a \geq b$, $|\mathcal{W}| \leq r/2$ and (3.2) to obtain

$$V + \mathcal{W}V' \leq \frac{E}{4}\mathbf{1}_{>b}.$$

Then, again by (3.5), (3.6), and (3.8),

$$(\varphi')^2\mathcal{W}\Phi^2\min(\mathcal{W}, \tfrac{h}{4\varphi'}) \leq \frac{hr\varphi'(a)}{2(M-a)^2}.$$

Combining these bounds with (2.10) and the formula (3.8) for $\varphi'(a)$, we have

$$\begin{aligned} A(r) - KB(r) &\geq w'\Big[\frac{3E}{4} + (\varphi'(a))^2\Big(\frac{M-r}{M-a}\Big)^4\Big(1-\frac{2r}{M-r}\Big) - 2^{-1}Kh\varphi'(a)r\frac{1}{(M-a)^2}\Big] \\ &\geq w'\Big[\frac{3E}{4} - \frac{2\tau_0^2}{(a+1)^2}r\frac{(M-r)^3}{(M-a)^4} - 2^{-1}K\frac{\tau_0}{a+1}hr\frac{1}{(M-a)^2}\Big]. \end{aligned} \tag{3.14}$$

Now, choose $M = 2a$ and estimate, for $a < r < M$,

$$2r\frac{(M-r)^3}{(M-a)^4} \leq 4, \qquad r\frac{1}{(M-a)^2} \leq \frac{2}{a}.$$

Therefore, we choose

$$a = \max(\sqrt{20}\tau_0 E^{-1/2}, b)$$

and $h_0 = K^{-1}$, ensuring that the bracketed terms in the second line of (3.14) are bounded from below by $E/2$. This yields

$$A - KB \geq \frac{E}{2}w', \qquad h \in (0, h_0],\ a < r < M. \tag{3.15}$$

Case $r > M$:
In this final case we have $\varphi' = 0$, so appealing to (2.7), we have

$$A - KB = w'[E - V - \mathcal{W}V'].$$

By (3.2), $V \leq \frac{E}{4}$. By (3.4) and (3.5), $\mathcal{W}V' \leq c_0 m\tilde{m}/2 \leq E/4$. Hence,

$$A - KB = w'[E - V - \mathcal{W}V'] \geq w'\Big[\frac{3E}{4} - \frac{c_0 m\tilde{m}}{2}\Big] \geq \frac{E}{2}w', \qquad h \in (0,1],\ r > M.$$

This completes the proof of the Lemma.

□

4. Carleman estimate

Our goal in this section is to prove Lemma 3.2. This argument is standard; versions of it appear, for instance, in [GaSh20, Section 5] in [Da14, proof of Lemma 2.2], but we include it here for the reader's convenience.

Remark: In the proof of Lemma 3.2, we abuse notation slightly. In dimension $n \geq 3$, we put $\|u\| = \|u(r,\cdot)\|_{L^2(\mathbb{S}^{n-1}_\theta)}$, while we put $\|u\| = |u(r)| + |u(-r)|$ when $n = 1$. If $n \geq 3$, $\int_{r,\theta}$ denotes the integral over $(0,\infty) \times \mathbb{S}^{n-1}$ with respect to the measure $drd\theta$, while if $n = 1$, $\int_{r,\theta} u(x)$ denotes $\int_0^\infty u(r)dr + \int_0^\infty u(-r)dr = \int_{\mathbb{R}} u(x)dx$.

Proof of Lemma 3.2. Since $\langle x\rangle^{-2s} \le 1$, without loss of generality, we may assume $0 < \varepsilon \le h$.

The proof begins from (2.6). Then, applying (3.9), it follows that, for $h \in (0, h_0]$,

$$w'F + wF' \ge -\frac{3w^2}{h^2 w'}\|P^{\pm}_{\varphi}(h)u\|^2 \mp 2\varepsilon w \operatorname{Im}\langle u, u'\rangle + \frac{1}{3}w'\|hu'\|^2 + \frac{E}{2}w'\|u\|^2. \tag{4.1}$$

Now we integrate both sides of (4.1). We integrate $\int_0^\infty dr$ and use $wF, (wF)' \in L^1((0,\infty); dr)$ (when $n = 1$, this follows from the facts that w is bounded and $w/w' \le (r+1)^{2s}$). Since, $wF(0) = 0$, $\int_0^\infty (wF)' dr = 0$. Therefore,

$$\int_{r,\theta} w'\left(|u|^2 + |hu'|^2\right) \lesssim \frac{1}{h^2}\int_{r,\theta}(r+1)^{2s}|P^{\pm}_{\varphi}(h)u|^2 + 2\varepsilon\int_{r,\theta} w|uu'|, \qquad h \in (0, h_0]. \tag{4.2}$$

In addition to w bounded, $w/w' \le (r+1)^{2s}$, (3.7), also gives $w' \ge w$ on $r < \frac{1}{2}$ and $w' \ge (r+1)^{-2s}$. Therefore,

$$\int_{r,\theta}(r+1)^{-2s}\left(|u|^2 + |hu'|^2\right) \lesssim \frac{1}{h^2}\int_{r,\theta}(r+1)^{2s}|P^{\pm}_{\varphi}(h)u|^2 + \frac{\varepsilon}{h}\int_{r,\theta}|u|^2 + \frac{\varepsilon}{h}\int_{\substack{r,\theta \\ r\ge\frac{1}{2}}}|hu'|^2, \qquad h \in (0, h_0], \tag{4.3}$$

where we have used $0 < \varepsilon \le h$.

Moreover, letting $\chi \in C_0^\infty[0, \frac{1}{2})$ with $\chi \equiv 1$ on $[0, \frac{1}{4}]$ and $\psi(r) = 1 - \chi(r)$,

$$\begin{aligned}\operatorname{Re}\int_{r,\theta}(P^{\pm}_{\varphi}(h)\psi u)\overline{\psi u} &= \int_{r,\theta}|h(\psi u)'|^2 + \operatorname{Re}\int_{r,\theta}2h\varphi'(\psi u)'\overline{\psi u} + \int_{r,\theta}(h^2\Lambda\psi u)\overline{\psi u} \\ &\quad + \int_{r,\theta}h\varphi''|\psi u|^2 + \int_{r,\theta}\left(V - E - (\varphi')^2\right)|\psi u|^2,\end{aligned} \tag{4.4}$$

and

$$\int_{r,\theta}h\varphi''|\psi u|^2 = -\operatorname{Re}\int_{r,\theta}2\varphi' h(\psi u)'\overline{\psi u}. \tag{4.5}$$

These two identities, together with the facts that $\Lambda \ge 0$ and $|V - E - (\varphi')^2|$ is bounded on $\operatorname{supp}\psi$ (independent of h and ε) imply, that for all $h \in (0, 1]$, $\gamma > 0$,

$$\begin{aligned}\int_{\substack{r,\theta \\ r\ge\frac{1}{2}}}|hu'|^2 &\lesssim \int_{r,\theta}|u|^2 + \frac{\gamma}{2}\int_{r,\theta}(r+1)^{-2s}|u|^2 + \frac{1}{2\gamma}\int_{r,\theta}(r+1)^{2s}|P^{\pm}_{\varphi}(h)(\psi u)|^2 \\ &\le \int_{r,\theta}|u|^2 + \frac{\gamma}{2}\int_{r,\theta}(r+1)^{-2s}|u|^2 + \frac{1}{\gamma}\int_{r,\theta}(r+1)^{2s}|\psi P^{\pm}_{\varphi}(h)(u)|^2 + \frac{Ch^2}{\gamma}\int_{\substack{r,\theta \\ \frac{1}{4}\le r\le\frac{1}{2}}}|hu'|^2.\end{aligned} \tag{4.6}$$

To finish, we substitute (4.6) into the right side of (4.3). Recalling $0 < \varepsilon \le h$, choose $\gamma > 0$ small enough (independent of h and ε), and then h small enough, to absorb the second and fourth terms in the last line of (4.6) into the right side of (4.3). We obtain

$$\begin{aligned}&\int_{r,\theta}(r+1)^{-2s}(|u|^2 + |hu'|^2) \lesssim \\ &\qquad \frac{1}{h^2}\int_{r,\theta}(r+1)^{2s}|P^{\pm}_{\varphi}(h)u|^2 + \frac{\varepsilon}{h}\int_{r,\theta}|u|^2, \qquad h \in (0, h_0].\end{aligned} \tag{4.7}$$

The estimate (3.10) is now an easy consequence of (4.7).

$\square$

5. Resolvent estimates

In this section, we deduce Theorem 1 from Lemma 3.2. The same argument appears, e.g., in [GaSh20, Section 6] and [Da14, last proof of Section 2], but we include it here for completeness.

Proof of the Theorem. Since increasing s in only decreases the weighted resolvent norms (1.9) and (1.10), without loss of generality we may take $1/2 < s < 1$. Let $C, h_0 > 0$ be as in the statement of Lemma 3.2. Put $C_\varphi = C_\varphi(h) \coloneqq 2\max\varphi$. By (3.10), and since $2\varphi(r) = C_\varphi$ for $r \geq M$,

$$e^{-C_\varphi/h}\|\langle x\rangle^{-s}\mathbf{1}_{\leq M}v\|^2_{L^2} + \|\langle x\rangle^{-s}\mathbf{1}_{\geq M}v\|^2_{L^2} \leq \frac{C}{h^2}\|\langle x\rangle^{s}(P(h) - E \pm i\varepsilon)v\|^2_{L^2} + \frac{C\varepsilon}{h}\|v\|^2_{L^2}, \tag{5.1}$$

for all $\varepsilon \geq 0$ and $h \in (0, h_0]$, and all $v \in C_0^\infty(\mathbb{R}^n)$ $(n \geq 3)$ or all $v \in \mathcal{D}(P)$ with $\langle x\rangle^s Pv \in L^2(\mathbb{R})$ $(n = 1)$. Moreover, for any γ, $\gamma_0 > 0$,

$$\begin{aligned}
2\varepsilon\|v\|^2_{L^2} &= -2\operatorname{Im}\langle(P(h) - E \pm i\varepsilon)v, v\rangle_{L^2} \\
&\leq \gamma^{-1}\|\langle x\rangle^{s}\mathbf{1}_{\leq M}(P(h) - E \pm i\varepsilon)v\|^2_{L^2} + \gamma\|\langle x\rangle^{-s}\mathbf{1}_{\leq M}v\|^2_{L^2} \\
&\quad + \gamma_0^{-1}\|\langle x\rangle^{s}\mathbf{1}_{\geq M}(P(h) - E \pm i\varepsilon)v\|^2_{L^2} + \gamma_0\|\langle x\rangle^{-s}\mathbf{1}_{\geq M}v\|^2_{L^2}
\end{aligned} \tag{5.2}$$

Setting $\gamma = he^{-C_\varphi/h}/C$ and $\gamma_0 = h/C$, (5.1) and (5.2) imply, for some $\tilde{C} > 0$, all $\varepsilon \geq 0$, $h \in (0, h_0]$, and all $v \in C_0^\infty(\mathbb{R}^n)$ $(n \geq 3)$ or all $v \in \mathcal{D}(P)$ with $\langle x\rangle^s Pv \in L^2(\mathbb{R})$ $(n = 1)$,

$$\begin{aligned}
e^{-\tilde{C}/h}\|\langle x\rangle^{-s}\mathbf{1}_{\leq M}v\|^2_{L^2} + \|\langle x\rangle^{-s}\mathbf{1}_{\geq M}v\|^2_{L^2} &\leq e^{\tilde{C}/h}\|\langle x\rangle^{s}\mathbf{1}_{\leq M}(P(h) - E \pm i\varepsilon)v\|^2_{L^2} \\
&\quad + \frac{\tilde{C}}{h^2}\|\langle x\rangle^{s}\mathbf{1}_{\geq M}(P(h) - E \pm i\varepsilon)v\|^2_{L^2}.
\end{aligned} \tag{5.3}$$

The final task is to use (5.3) to deduce

$$\begin{aligned}
&e^{-\tilde{C}/h}\|\langle x\rangle^{-s}\mathbf{1}_{\leq M}(P(h) - E \pm i\varepsilon)^{-1}\langle x\rangle^{-s}f\|^2_{L^2} + \|\langle x\rangle^{-s}\mathbf{1}_{\geq M}(P(h) - E \pm i\varepsilon)^{-1}\langle x\rangle^{-s}f\|^2_{L^2} \\
&\qquad \leq e^{\tilde{C}/h}\|\mathbf{1}_{\leq M}f\|^2_{L^2} + \frac{\tilde{C}}{h^2}\|\mathbf{1}_{\geq M}f\|^2_{L^2}, \qquad \varepsilon > 0,\ h \in (0, h_0],\ f \in L^2,
\end{aligned} \tag{5.4}$$

from which Theorem 1 follows. If $n = 1$, (5.3) immediately implies (5.4) by setting $v = (P(h) - E \pm i\varepsilon)^{-1}\langle x\rangle^{-s}f$. To establish (5.4) when $n \geq 3$, we prove a simple Sobolev space estimate and then apply a density argument that relies on (5.3).

The operator

$$[P(h), \langle x\rangle^s]\langle x\rangle^{-s} = \left(-h^2\Delta\langle x\rangle^s - 2h^2(\nabla\langle x\rangle^s)\cdot\nabla\right)\langle x\rangle^{-s}$$

is bounded $H^2 \to L^2$. So, for $v \in H^2$ such that $\langle x\rangle^s v \in H^2$,

$$\begin{aligned}
\|\langle x\rangle^{s}(P(h) - E \pm i\varepsilon)v\|_{L^2} &\leq \|(P(h) - E \pm i\varepsilon)\langle x\rangle^s v\|_{L^2} + \|[P(h), \langle x\rangle^s]\langle x\rangle^{-s}\langle x\rangle^s v\|_{L^2} \\
&\leq C_{\varepsilon,h}\|\langle x\rangle^s v\|_{H^2},
\end{aligned} \tag{5.5}$$

for some constant $C_{\varepsilon,h} > 0$ depending on ε and h.

Given $f \in L^2$, the function $u = \langle x\rangle^s(P(h) - E \pm i\varepsilon)^{-1}\langle x\rangle^{-s}f \in H^2$ because

$$u = (P(h) - E \pm i\varepsilon)^{-1}(f - w), \qquad w = \langle x\rangle^s[P(h), \langle x\rangle^{-s}]\langle x\rangle^s\langle x\rangle^{-s}u,$$

with $\langle x\rangle^s[P(h), \langle x\rangle^{-s}]\langle x\rangle^s$ being bounded $L^2 \to L^2$ since $s < 1$.

Now, choose a sequence $v_k \in C_0^\infty$ such that $v_k \to \langle x\rangle^s(P(h) - E \pm i\varepsilon)^{-1}\langle x\rangle^{-s}f$ in H^2. Define $\tilde{v}_k \coloneqq \langle x\rangle^{-s}v_k$. Then, as $k \to \infty$,

$$\begin{aligned}
&\|\langle x\rangle^{-s}\tilde{v}_k - \langle x\rangle^{-s}(P(h) - E \pm i\varepsilon)^{-1}\langle x\rangle^{-s}f\|_{L^2} \\
&\qquad \leq \|v_k - \langle x\rangle^s(P(h) - E \pm i\varepsilon)^{-1}\langle x\rangle^{-s}f\|_{H^2} \to 0.
\end{aligned}$$

Also, applying (5.5),

$$\|\langle x\rangle^s(P(h) - E \pm i\varepsilon)\tilde{v}_k - f\|_{L^2} \leq C_{\varepsilon,h}\|v_k - \langle x\rangle^s(P(h) - E \pm i\varepsilon)^{-1}\langle x\rangle^{-s}f\|_{H^2} \to 0.$$

We then achieve (5.4) by replacing v by $\tilde{v}_k$ in (5.3) and sending $k \to \infty$.

□

Appendix A. Self-adjointness of P in one dimension

Lemma A.1 (generalization of the Lemma in [DaSh20])**.** *Suppose $V : \mathbb{R} \to \mathbb{R}$ may be written as $V = V_1 + V_\infty$ for $V_1 \in L^1(\mathbb{R})$ and $V_\infty \in L^\infty(\mathbb{R})$. Let $\mathcal{D}$ be the set of all $u \in H^1(\mathbb{R})$ such that $u' \in L^\infty(\mathbb{R})$ and $Pu \in L^2(\mathbb{R})$. Then P, with domain $\mathcal{D}$, is densely defined and self-adjoint on $L^2(\mathbb{R})$.*

Proof of Lemma. Let $\mathcal{D}_{\max}$ be the set of all $u \in L^2$ such that $u' \in L^1_{\text{loc}}$ and $Pu \in L^2$. By [Ze05, Lemma 10.3.1], $\mathcal{D}_{\max}$ is dense in $L^2(\mathbb{R})$. We begin by proving that $\mathcal{D}_{\max} = \mathcal{D}$. Indeed, for any $a > 0$ and $u \in \mathcal{D}_{\max}$, by integration by parts and Cauchy–Schwarz, we have

$$\int_{-a}^{a} |u'|^2 = u'\bar{u}|_{-a}^{a} - \int_{-a}^{a} u''\bar{u} \leq$$

$$2 \sup_{[-a,a]} |u'| \sup_{[-a,a]} |u| + h^{-2}\|V_1\|_{L^1} \sup_{[-a,a]} |u|^2 + h^{-2}(\|V_\infty\|_{L^\infty} + \|Pu\|_{L^2})\|u\|_{L^2},$$

$$\sup_{[-a,a]} |u|^2 = \sup_{x\in[-a,a]} \left(|u(0)|^2 + 2\operatorname{Re}\int_0^x u'\bar{u}\right) \leq |u(0)|^2 + 2\left(\int_{-a}^{a} |u'|^2\right)^{1/2} \|u\|_{L^2},$$

$$\sup_{[-a,a]} |u'|^2 \leq |u'(0)|^2 + 2h^{-2}\|V_1\|_{L^1} \sup_{[-a,a]} |u| \sup_{[-a,a]} |u'| + 2h^{-2}(\|V_\infty\|_{L^\infty} + \|Pu\|_{L^2})\left(\int_{-a}^{a} |u'|^2\right)^{1/2}.$$

This is a system of inequalities of the form $x^2 \leq 2yz + Ay^2 + B$, $y^2 \leq C + Dx$, $z^2 \leq E + Fyz + Gx$. After using the second to eliminate y, we obtain a system in x and z with quadratic left hand sides and subquadratic right hand sides. Hence x, y, and z are each bounded in terms of $A, B, \ldots, G$. Letting $a \to \infty$, we conclude that $u' \in L^2$, $u \in L^\infty$, and $u' \in L^\infty$. Hence $\mathcal{D}_{\max} = \mathcal{D}$.

Equip P with the domain $\mathcal{D}_{\max} = \mathcal{D} \subset L^2$. By integration by parts, $P \subset P^*$. But, by Sturm–Liouville theory, $P^* \subset P$: see [Ze05, Lemma 10.3.1]. Hence $P = P^*$.

□

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