SEMICLASSICAL RESOLVENT BOUNDS FOR LONG RANGE LIPSCHITZ POTENTIALS

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Abstract. We give an elementary proof of weighted resolvent estimates for the semiclassical Schrödinger operator \(-\hbar^2\Delta + V(x) - E\) in dimension \(n \neq 2\), where \(\hbar, E > 0\). The potential is real-valued and \(V\) and \(\partial_r V\) exhibit long range decay. The resolvent norm grows exponentially in \(\hbar^{-1}\), but near infinity it grows linearly. When \(V\) is compactly supported, we obtain linear growth if the resolvent is multiplied by weights supported outside a ball of radius \(CE^{-1/2}\) for some \(C > 0\). This \(E\)-dependence is sharp and answers a question of Datchev and Jin.

1. Introduction and Statement of Result

Let \(\Delta := \sum_{i=1}^n \partial_i^2 \leq 0\) be the Laplacian on \(\mathbb{R}^n\), \(n \neq 2\). We consider the semiclassical Schrödinger operator of the form

\[
P = P(h) := -\hbar^2 \Delta + V : L^2_2(\mathbb{R}^n) \to L^2_2(\mathbb{R}^n), \quad h > 0.
\]

We use \((r, \theta) = (|x|, x/|x|) \in (0, \infty) \times \mathbb{S}^{n-1}\) to denote polar coordinates on \(\mathbb{R}^n \setminus \{0\}\). We suppose the potential satisfies \(V \in L^\infty(\mathbb{R}^n; \mathbb{R})\) with

\[
V(x) \leq p(|x|)
\]

for some function \(p(r) > 0\) decreasing to zero as \(r \to \infty\). We also suppose there exist \(c_0 > 0\) and a function \(0 < m(r) \leq 1\) so that

\[
\lim_{r \to \infty} m(r) = 0, \quad (r + 1)^{-1} m(r) \in L^1(0, \infty)
\]

and

\[
\partial_r V(x) \leq c_0 (r + 1)^{-1} m(r).
\]

The prototypes we have in mind for (1.2) are the long range cases

\[
m = \log^{-1-\rho}(r + e), \quad m = (r + 1)^{-\rho}, \quad \rho > 0.
\]

By the Kato-Rellich Theorem, the operator \(P\) is self-adjoint with respect to the domain \(H^2(\mathbb{R}^n)\). Therefore, the resolvent \((P - z)^{-1}\) is bounded \(L^2_2(\mathbb{R}^n) \to L^2_2(\mathbb{R}^n)\) for all \(z \in \mathbb{C} \setminus \mathbb{R}\). For \(E > 0\) and \(s > 1/2\) fixed, and \(h, \varepsilon > 0\), our goal is to establish \(h\)-dependent upper bounds on the weighted resolvent norms

\[
g^+_{\varepsilon}(h, \varepsilon) := \|\langle x \rangle^{-s} (P(h) - E \pm i\varepsilon)^{-1} \langle x \rangle^{-s} \|_{L^2_2(\mathbb{R}^n) \to L^2_2(\mathbb{R}^n)}, \quad \rho > 0.
\]

\[
g^\pm_{\varepsilon}(h, \rho, \varepsilon) := \|\langle x \rangle^{-s} 1_{|x| \geq M} (P(h) - E \pm i\varepsilon)^{-1} 1_{|x| \geq M} \langle x \rangle^{-s} \|_{L^2_2(\mathbb{R}^n) \to L^2_2(\mathbb{R}^n)},
\]

Here, \(\langle x \rangle = \langle r \rangle := (1 + r^2)^{1/2}\).

In our main theorem, we give estimates on both (1.4) and (1.5) and show that for \(V\) compactly supported there are constants \(C_1, h_0 > 0\), such that (1.5) grows linearly in \(h^{-1}\), provided \(M \geq C_1 E^{-1/2}, \varepsilon > 0\) and \(h \in (0, h_0]\).

Theorem. Fix \(E > 0\) and \(s > 1/2\). Suppose \(V \in L^\infty(\mathbb{R}^n; \mathbb{R})\) satisfies (1.1) and (1.2). There exist \(M = M(E, p, c_0, m)\), \(C_2 = C_2(E, s, p, c_0, m)\), \(C_3 = C_3(E, s, p, m) > 0\) and \(h_0 \in (0, 1]\) so that, for all \(\varepsilon > 0\) and \(h \in (0, h_0]\),

\[
\|\langle x \rangle^{-s} (P(h) - E \pm i\varepsilon)^{-1} \langle x \rangle^{-s} \|_{L^2_2(\mathbb{R}^n) \to L^2_2(\mathbb{R}^n)} \leq C_3 \frac{1}{h},
\]
and

$$\|\langle x \rangle^{-s}1_{|x| \geq M}(P(h) - E + i \varepsilon)^{-1}1_{|x| \geq M}(\langle x \rangle^{-s})\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \leq C_2/h. \quad (1.7)$$

Moreover, if supp $V \subseteq B(0, R_0)$, then $M \leq C_1(p, c_0, R_0)E^{-1/2}$.

The main novelty of the Theorem is in the compactly supported case, where we have that, for (1.7) to hold, $M$ need not be larger than a constant times $E^{-1/2}$. This seems to be the first bound of the form $g^\pm_s(h, M, \varepsilon) \leq Ch^{-1}$ for which $M$ depends explicitly on $E$. Moreover, owing to a construction of Datchev and Jin [DaJi20, Theorem 1], this $E$-dependence of $M$ is optimal. In particular, if $V \in C^\infty_0(\mathbb{R}^n; \mathbb{R})$, $n \geq 2$ is radial and $\min(V) < 0$, then there is $M \leq cE^{-1/2}$ with $g^\pm_s(h, M, \varepsilon) \geq e^{C/h}$.

Cardoso and Vodev [CaVo02], refining earlier work of Burq [Bu98], were the first to prove an exterior estimate of the form (1.7). They did so for smooth $V$ on a large class of infinite volume Riemannian manifolds. Exterior estimates were subsequently established under a wide range of regularity and geometric conditions [Da14, Vo14, RoTa15, DadeH16, Sh19].

For the semiclassical Schrödinger operator on $\mathbb{R}^n$, the conditions (1.1) and (1.2) on $V$ appear to be the most general yet under which it is known that (1.6) and (1.7) hold. Burq [Bu98] was the first to show $g^\pm_s \leq e^{Ch^{-1}}$ for compactly supported perturbations of the Laplacian on $\mathbb{R}^n$. This bound was refined and extended many times [Vo00, Bu02, Sj02, CaVo02, Da14, Sh19, Vo20b] and is sharp in general, see [DDZ15].

Stronger bounds on $g^\pm_s$ are known when $V$ is smooth and conditions are imposed on the classical flow $\Phi(t) = \exp t(2\xi_\partial_x - \partial_x V(x)\partial_x)$ (note that $\Phi(t)$ may be undefined in our case). The key dynamical object is the trapped set $\mathcal{K}(E)$ at energy $E > 0$, defined as the set of $(x, \xi) \in T^\ast \mathbb{R}^n$ such that $|\xi|^2 + V(x) = E$ and $|\Phi(t)(x, \xi)|$ is bounded as $|t| \rightarrow \infty$. If $\mathcal{K}(E) = \emptyset$, that is, if $E$ is nontrapping, Robert and Tamura [RoTa87] showed $g^\pm_s \leq Ch^{-1}$. We may think of (1.7) as a low regularity analog; it says that applying cutoffs supported far away from zero removes the losses from (1.6) due to trapping.

Resolvent estimates such as (1.6) and (1.7) are useful for several applications. Burq [Bu98, Bu02] used the exponential bound to show logarithmic local energy decay for solutions to the wave equation. This technique was subsequently used in many settings [Be03, CaVo04, Mo16, Sh18, Ga19]. As shown in section XIII.7 of [ReSi78], the exterior resolvent bound is related to exterior smoothing and Strichartz estimates for Schrödinger propagators, see also [BoTz07, MMT08] and Section 7.1 of [DyZw19]. Furthermore, Christiansen [Ch17] used an estimate like (1.7) to find a lower bound on the resonance counting function for compactly supported perturbations on the Laplacian on even-dimensional Riemannian manifolds.

To prove the Theorem, we adapt the Carleman estimate from [GaSh20], which was used to prove a resolvent estimate for $L^\infty$ potentials. The key ingredients remain a weight $w(r)$ and phase $\varphi(r)$ that obey a crucial lower bound, see (3.7) below. The main technical innovation is that, by leveraging the additional regularity of $V$, we can decrease $\varphi^\prime$ to zero (outside of a compact set) in an explicit, $E$-dependent fashion. We then obtain (1.7) for any $M$ such that $1_{|x| \geq M}$ is supported in the set where $\varphi$ is constant.

If we do not assume anything about the derivatives of $V$, for instance, if $V \in L^\infty_{\text{comp}}(\mathbb{R}_p; \mathbb{R})$, then the best known bound in general is $g^\pm_s \leq \exp(Ch^{-4/3}\log(h^{-1}))$ [KIVo19, Sh20], although Vodev [Vo20c] showed this can be improved to $g^\pm_s \leq \exp(Ch^{-4/3})$ if $V$ is short range and radial. See also [Vo19a, Vo20b, Vo20a, GaSh20]. On the other hand, it is not known whether an exterior estimate like (1.7) holds for $L^\infty$ potentials, except in dimension one [DaSh20], and there $1_{|x| \geq M}$ and $V$ need only have disjoint supports.

We remark that the Theorem should hold in dimension two, too, provided the left side of (1.2) is replaced by $|\nabla V|$. The extra difficulty in dimension two comes from the effective potential term,
see (2.1) below, having a negative singularity at \( r = 0 \). This necessitates a stronger assumption on the derivatives of \( V \), see [Sh19] for more details.

For more background on semiclassical resolvent estimates, we refer the reader to the introductions of [DaJi20, GaSh20].

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2. Notation and Preliminary Calculations

Throughout the paper, we use “prime” notation to indicate differentiation with respect to the radial variable \( r = |x| \), e.g., \( u' = \partial_r u \). As in most previous proofs of resolvent estimates for low regularity potentials, the backbone of the proof is a Carleman estimate. We start from the identities

\[
\begin{align*}
  r^{\frac{n-1}{2}} (-\Delta)^{\frac{n-1}{4}} &= -\partial_r^2 + \Lambda, \\
  \Lambda &:= \frac{1}{r^2} \left( -\Delta_{S^{n-1}} + \frac{(n-1)(n-3)}{4} \right) \geq 0, \\
\end{align*}
\]

where \( \Delta_{S^{n-1}} \) denotes the negative Laplace-Beltrami operator on \( S^{n-1} \). Then, for a phase \( \varphi \) that we construct below, we form the conjugated operator

\[
P_{\varphi}^\pm (h) := e^{\varphi/h} r^{\frac{n-1}{2}} (P(h) - E \pm i\varepsilon) r^{-\frac{n-1}{2}} e^{-\varphi/h} \\
= -h^2 \partial_r^2 + 2h \varphi' \partial_r + h^2 \Lambda + V - (\varphi')^2 + h \varphi'' - E \pm i\varepsilon.
\]

Let \( u \in e^{\varphi/h} r^{(n-1)/2} C^\infty_0 (\mathbb{R}^n) \), define a spherical energy functional \( F[u](r) \),

\[
F(r) = F[u](r) := \| hu'(r, \cdot) \|^2 - \langle (h^2 \Lambda + V - (\varphi')^2 - E) u(r, \cdot), u(r, \cdot) \rangle,
\]

where \( \| \cdot \| \) and \( \langle \cdot, \cdot \rangle \) denote the norm and inner product on \( L^2(S^{n-1}_0) \), respectively (or in the \( n = 1 \) case, we take (2.3) to simply be a pointwise energy). It is easy to compute (see e.g. [Da14, Sh19, Sh20, GaSh20]) that for \( w \in C^0 \) and piecewise \( C^1 \), \((wF)'\), as a distribution on \((0, \infty)\), is given by

\[
(wF)' = -2w \Re \langle P_{\varphi}^\pm (h) u, u' \rangle + 2\varepsilon w \Im \langle u, u' \rangle + (2wr^{-1} - w')(h^2 \Lambda u, u) \\
+ (4h^{-1}(w' + w')\|hu'\|^2 + (w(E + (\varphi')^2 - V))'\|u\|^2 + 2w \Re \langle h\varphi'' u, u' \rangle.
\]

We will construct \( w \) such that

\[
2wr^{-1} - w' \geq 0,
\]

and use (2.1) to control the term involving \( \Lambda \). Using (2.5) together with \( 2ab \geq -(\gamma a^2 + \gamma^{-1}b^2) \) for \( \gamma > 0 \), we find

\[
wF + wF' \geq -\frac{3w^2}{h^2 w'} \|P_{\varphi}^\pm (h) u\|^2 + 2\varepsilon w \Im \langle u, u' \rangle + \frac{1}{3} (w' + 4h^{-1} \varphi') \|hu'\|^2 \\
+ (w(E + (\varphi')^2 - V))'\|u\|^2 - \frac{3(w\varphi'')^2}{w' + 4h^{-1} \varphi' w} \|u\|^2.
\]

To complete the proof of the Carleman estimate, we seek to build \( w \) and \( \varphi \) so that the second line of (2.6) has a good lower bound. Indeed, putting

\[
A(r) := (w(E + (\varphi')^2 - V))', \quad B(r) := \frac{(w\varphi'')^2}{w' + 4h^{-1} \varphi' w},
\]

it suffices for \( w \) and \( \varphi \) to satisfy, for \( K > 0 \) fixed,

\[
A(r) - KB(r) \geq \frac{E}{2} w', \quad 0 < h \ll 1,
\]

along with a few other properties (see (3.3) through (3.6)).
In order to construct the weight and phase functions for our Carleman estimates, we adapt the method in [GaSh20]. Whenever $|w'|, |\varphi'| > 0$, put
\[
\Phi := \frac{\varphi''}{\varphi} = (\log \varphi)' , \quad W := \frac{w}{w'} = \frac{1}{(\log w)'},
\] (2.9)

Then, as in [GaSh20, (2.10)],
\[
A(r) - KB(r) \geq w \left[ E + (\varphi')^2 (1 + 2W \Phi - KW \Phi^2 \min(W, \frac{h}{1+2s})) - V - WV' \right].
\] (2.10)

So when $|w'|, |\varphi'| > 0$, to show (2.8), it is enough to bound the bracketed expression in (2.10) from below by $E/2$.

3. Construction of the phase and weight functions

Throughout this section, we assume $E > 0$, $s > 1/2$ are fixed and suppose $V$ satisfies (1.1) and (1.2). Using (1.1) and (1.2), let
\[
b := \max \left( \sup \{ r \mid V(r) + (r + 1)\partial_r V(r) \geq \frac{E}{4} \}, 1 \right)
\]
so that $b$ is independent of $h$ and
\[
V + (r + 1)\partial_r V \leq (|V| + (r + 1)\partial_r V)1_{0 < r < M} + E/4 1_{b > r}.
\] (3.1)

(Noted that $b$ can be chosen to depend only on $p$, $m$, $c_0$, and $E$, and that $b \leq R_0$ provided $\text{supp} V \subseteq B(0, R_0)$.) Additionally, let
\[
M > a \geq b, \quad \tau_0 \geq 1,
\]
be parameters, independent of $h$, to be specified in the proof of Lemma 3.1 below.

Let $\omega \in C_0^\infty((-3/4, 3/4); [0, 1])$ with $\omega = 1$ near $[-1/2, 1/2]$. The weight $w$ and phase $\varphi$, which will be shown to satisfy (2.8), are functions of the radial variable $r = |x|$ only, and are defined by
\[
\hat{m}(r) := \min \left( \frac{E}{2c_0} m^{-1}(r), (r + 1)^{2s-1} \right),
\] (3.2)
\[
w(0) = 0, \quad w'(0) = 1, \quad \frac{w}{w'} = W := \begin{cases} 
\frac{r(1+\omega(r))}{r+1} & 0 < r < M \\
\frac{r+1}{2} \hat{m}(r) & r \geq M
\end{cases},
\] (3.3)
\[
\varphi(0) = 0, \quad \varphi'(0) = \tau_0, \quad \frac{\varphi''}{\varphi} = \Phi := \begin{cases} 
\frac{-1}{r+1} & 0 < r < a \\
\frac{-1}{M-r} & a \leq r < M \\
0 & r \geq M
\end{cases}, \quad \varphi' = 0 \quad r \geq M.
\] (3.4)

Short computations yield,
\[
w = \begin{cases} 
\frac{r}{2} e^{\int_{r/2}^{r} 2 (1/(1+\omega(s)))^2 ds} & 0 < r < 1/2 \\
w(M) e^{\int_{r}^{r} 2 (1/(1+\omega(s)))^2 ds} & r \geq M
\end{cases}, \quad w' = \begin{cases} 
1 & 0 < r < 1/2 \\
\frac{1}{r(1+\omega(r))} w & 1/2 < r < M \\
\frac{w(M) e^{\int_{r}^{r} 2 (1/(1+\omega(s)))^2 ds}}{(r+1)/2} & r \geq M
\end{cases}
\] (3.5)
\[
\varphi' = \begin{cases} 
\varphi'(a) \left( \frac{M-a}{M-a} \right)^2 & 0 < r < a \\
\varphi'(a) \left( \frac{M-a}{M-a} \right)^2 & a \leq r < M \\
0 & r \geq M
\end{cases}.
\] (3.6)

We now prove the crucial lower bound involving $E$, $w$ and $\varphi$ that is needed to prove the Carleman estimate.
Lemma 3.1. Fix $K > 0$ and let $V$ satisfy (1.1) and (1.2). Then, using the notation of (2.7) and (3.2) through (3.6), there exist suitable $M$, $a$ and $\tau_0$ so that

$$A - KB \geq \frac{E}{2} w', \quad h \in (0, h_0], \ r \neq a, M. \quad (3.7)$$

Once Lemma 3.1 is proved, we can use the standard argument found, e.g., in [GaSh20, Sections 5.6] to prove the following Carleman estimate.

Lemma 3.2. There are $C, h_0 > 0$ independent of $h$ and $\varepsilon$ so that

$$\| (x)^{-s} e^{\varphi / h} v \|_{L^2}^2 \leq \frac{C}{h^2} \| (x)^{s} e^{\varphi / h} (P(h) - E \pm i\varepsilon) v \|_{L^2}^2 + \frac{C \varepsilon}{h} \| e^{\varphi / h} v \|_{L^2}^2, \quad (3.8)$$

for all $v \in C_0^\infty(\mathbb{R}^n)$, $\varepsilon \geq 0$ and $h \in (0, h_0]$.

From here, (1.6) and (1.7) follow from the last proof of [Da14, Section 2].

Proof of Lemma 3.1. Case $0 < r < a$:

First, recall (2.10):

$$A(r) - KB(r) \geq w' \left[ E + \frac{(\varphi')^2 (1 + 2 W \Phi - K W \Phi^2 \min(W, \frac{h}{4 \varphi'}))}{4} - V - W V' \right].$$

By (3.3) and (3.4),

$$4 + 2 W \Phi \geq \frac{1}{4(r + 1)}, \quad 0 < r < a.$$

Also by (3.3), $|W| \leq r$ when $0 < r < a$, hence appealing to (3.1),

$$V + W V' \leq (|V| + |(r + 1) \partial_r V|) 1_{\leq b} + E \frac{1}{4} 1_{> b}.$$

Furthermore, using $|W| \leq r$ again, by (3.4) $\Phi^2 = (r + 1)^{-2}$, and by (3.6) $\varphi' = \tau_0 (r + 1)^{-1}$,

$$(\varphi')^2 W \Phi^2 \min(W, \frac{h}{4 \varphi'}) \leq \frac{h \tau_0}{(r + 1)^3}, \quad 0 < r < a.$$

From these estimates, and using once more that $\varphi' = \tau_0 (r + 1)^{-1}$, we find,

$$A(r) - KB(r) \geq w' \left[ E + \frac{(\varphi')^2 (1 + 2 W \Phi - K h \tau_0 (r + 1)^3 - |V| + |(r + 1) \partial_r V|) 1_{\leq b} - E \frac{1}{4} 1_{> b}}{4} \right].$$

We now choose

$$\tau_0 := 2 \sup_{0 \leq r \leq b} \frac{r}{(r + 1)^{3/2} \sqrt{|V| + (r + 1)|\partial_r V|}},$$

so that the second term in line two of (3.9) is nonnegative. We then take $h_0 = h_0(K, \tau_0, E) \in (0, 1]$ sufficiently small to achieve

$$A - KB \geq \frac{E}{2} w', \quad h \in (0, h_0], \ 0 < r < a. \quad (3.10)$$

Case $a < r < M$:

As in the previous case, we begin from (2.10). We use (3.3), (3.4) and (3.6) to see

$$(\varphi')^2 (1 + 2 W \Phi) = (\varphi'(a))^2 \left( \frac{M - r}{M - a} \right)^4 \left( 1 - \frac{2r}{M - r} \right).$$

Next, we use $a \geq b$, $|W| \leq r/2$ and (3.1) to obtain

$$V + W V' \leq E \frac{1}{4} 1_{> b}.$$
Then, again by (3.3), (3.4), and (3.6),

$$(\varphi')^2 W \Phi^2 \min(W, \frac{\hbar}{4\varphi}) \leq \frac{hr\varphi'(a)}{2(M-a)^2}.$$  

Combining these bounds with (2.10) and the formula (3.6) for $\varphi'(a)$, we have

$$A(r) - KB(r) \geq w' \left[ \frac{3E}{4} + (\varphi'(a))^2 \frac{M-r}{M-a} \right] (1 - \frac{2r}{M-r}) - 2^{-1} K h \varphi'(a) \frac{1}{(M-a)^2}.$$  

(3.11)

Now, choose $M = 2a$ and estimate, for $a < r < M$,

$$2r (\frac{M-r}{M-a})^3 \leq 4, \quad \frac{r}{(M-a)^2} \leq \frac{2}{a}.$$  

Therefore, we choose

$$a = \max(\sqrt{20} \tau_0 E^{1/2}, b)$$  

and $h_0 = K^{-1}$, ensuring that the bracketed terms in the second line of (3.11) are bounded from below by $E/2$. This yields

$$A - KB \geq \frac{E}{2} w', \quad h \in (0, h_0), \ a < r < M. \quad (3.12)$$

**Case $r > M$:**

In this final case we have $\varphi' = 0$, so appealing to (2.7), we have

$$A - KB = w'[E - V - W V'].$$

By (3.1), $V \leq \frac{E}{4}$. By (3.2) and (3.3), $W V' \leq c_0 \tilde{m}/2 \leq E/4$. Hence,

$$A - KB = w'[E - V - W V'] \geq w' \left[ \frac{3E}{4} - \frac{c_0 \tilde{m}}{2} \right] \geq \frac{E}{2} w', \quad h \in (0, 1), \ r > M.$$  

This completes the proof of the Lemma.

\[\square\]

**References**


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