Helmholtz boundary integral methods and the pollution effect

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August 15, 2025

Abstract

This paper is concerned with solving the Helmholtz exterior Dirichlet and Neumann problems with large wavenumber k and smooth obstacles using the standard second-kind boundary integral equations (BIEs) for these problems. We consider Galerkin and collocation methods – with subspaces consisting of *either* piecewise polynomials (in 2-d for collocation, in any dimension for Galerkin) *or* trigonometric polynomials (in 2-d) – as well as a fully discrete quadrature (a.k.a., Nyström) method based on trigonometric polynomials (in 2-d).

For each of these methods, we address the fundamental question: how quickly must N, the dimension of the approximation space, grow with k to maintain accuracy as $k \to \infty$?

For the methods involving piecewise-polynomials, we give sufficient conditions for k-uniform quasi-optimality. For the Galerkin method we show that these conditions are, in fact, necessary and sufficient. In particular we prove that the Galerkin method *suffers from the pollution effect*; i.e., N growing like k^{d-1} is often not sufficient for k-uniform quasi-optimality. For the Dirichlet BIEs, pollution occurs when the obstacle is trapping – and we also give numerical experiments illustrating this – but for the Neumann BIEs pollution occurs even when the obstacle is a ball.

For all the methods involving trigonometric polynomials, we show that, up to potential factors of k^ϵ for any $\epsilon > 0$, there is no pollution (even for trapping obstacles).

These are the first results about k-explicit convergence of collocation or Nyström methods applied to the Dirichlet BIEs, the first results about k-explicit convergence of any method used to solve the standard second-kind Neumann BIEs, and the first results proving that a boundary integral method applied to the Helmholtz equation suffers from the pollution effect.

Keywords: Helmholtz equation, boundary integral equation, Galerkin, collocation, Nyström, high-order, high frequency, pollution effect.

AMS: 65N35, 65N38, 65R20, 35J05

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1 Introduction

1.1 Motivation and context

1.1.1 Boundary integral equations for the Helmholtz equation.

Boundary integral equations (BIEs) are a popular way to solve acoustic, electromagnetic, and elastic scattering problems with constant wave speed and bounded scatterer. In this paper, we consider solving the Helmholtz equation $\Delta u + k^2 u = 0$ posed in the exterior of a smooth obstacle, with either Dirichlet or Neumann boundary conditions, using the standard second-kind BIEs.

Let $\Omega^- \subset \mathbb{R}^d$, $d \geq 2$ be a bounded open set such that its open complement $\Omega^+ := \mathbb{R}^d \setminus \overline{\Omega^-}$ is connected. Let $\Gamma := \partial\Omega^-$ and assume that Γ is C^∞. The second-kind BIE formulations reformulate solving the scattering problem as: given $f \in L^2(\Gamma)$, find $v \in L^2(\Gamma)$ such that

$$\mathcal{A}v = f, \qquad \mathcal{A} := c_0(I + L), \qquad c_0 \in \mathbb{C} \setminus \{0\}, \tag{1.1}$$

where the operator $L : L^2(\Gamma) \to L^2(\Gamma)$ is compact. Theorem A.2 below recaps the standard result that the Helmholtz exterior Dirichlet and Neumann problems can be reformulated as integral equations of the form (1.1) where f is given in terms of the known Dirichlet/Neumann boundary data and $\mathcal{A}$ is one of the boundary integral operators (BIOs)

$$A'_k := \frac{1}{2}I + K'_k - \mathrm{i}\eta_D S_k \quad \text{and} \quad A_k := \frac{1}{2}I + K_k - \mathrm{i}\eta_D S_k \tag{1.2}$$

for the Dirichlet problem and

$$B_{k,\mathrm{reg}} := \mathrm{i}\eta_N \left(\frac{1}{2}I - K_k\right) + S_{\mathrm{i}k} H_k \quad \text{and} \quad B'_{k,\mathrm{reg}} := \mathrm{i}\eta_N \left(\frac{1}{2}I - K'_k\right) + H_k S_{\mathrm{i}k} \tag{1.3}$$

for the Neumann problem. The operators S_k, K_k, K'_k, and H_k are the single-, double-, adjoint-double-layer and hypersingular operators defined by (A.4) and (A.5). Standard mapping properties of these operators (see, e.g., [26, Theorems 2.17 and 2.18]) imply that $A'_k, A_k, B_{k,\mathrm{reg}}, B'_{k,\mathrm{reg}}$ are bounded on $L^2(\Gamma)$. Furthermore, if $\eta_D, \eta_N \in \mathbb{R}\setminus\{0\}$, then each of $A'_k, A_k, B_{k,\mathrm{reg}}, B'_{k,\mathrm{reg}}$ is invertible on $L^2(\Gamma)$; moreover they are each equal to a multiple of the identity plus a compact operator on $L^2(\Gamma)$ (see, e.g., [26, §2.6] for A'_k, A_k and [45, Theorem 2.2] for $B_{k,\mathrm{reg}}, B'_{k,\mathrm{reg}}$). [1]

The BIEs involving the operators A'_k and A_k were introduced in [15, 78, 95]. The subscript "reg" on the Neumann boundary-integral operators indicates that these are not the "combined-field" (or "combined-potential") Neumann BIEs introduced by [24] (denoted by B'_k and B_k in, e.g., [26, §2.6]); the BIOs introduced in [24] are given by (1.3) with $S_{\mathrm{i}k}$ removed, and thus are not bounded on $L^2(\Gamma)$ because $H_k : L^2(\Gamma) \to H^{-1}(\Gamma)$. The idea of preconditioning H_k with an order -1 operator goes back to [23] (see, e.g., the discussion in [3]), with the use of $S_{\mathrm{i}k}$ proposed in [18], and then advocated for in [14, 109] (for more details, see the discussion in, e.g., [45, §2.1.1]). In fact, the results of the present paper hold for a wider class of regularising operators, of which $S_{\mathrm{i}k}$ is the prototypical example; see [45, Assumption 1.1]. For simplicity, however, here we only consider $B_{k,\mathrm{reg}}$ and $B'_{k,\mathrm{reg}}$ defined by (1.3) involving $S_{\mathrm{i}k}$.

Assumption 1.1 *There exists $C > 0$ such that $C^{-1} \leq |\eta_N| \leq C$, $C^{-1} \leq k^{-1}|\eta_D| \leq C$, where $\eta_N, \eta_D \in \mathbb{R}$ are the parameters in the definition of the operators, A_k, A'_k, $B_{k,\mathrm{reg}}$ and $B'_{k,\mathrm{reg}}$.*

Remark 1.2 *The question of how to choose the parameters η_D and η_N has been the subject of much research, starting with [74, 69, 2] and then continuing with [7, 27, 25, 10, 8, 45]. The choices in Assumption 1.1 are the most commonly-used and most rigorously-justified, with our results below contributing to this – Theorem 2.18 below shows that other choices of η_D are worse from the point of view of the pollution effect.*

Since the operator $\mathcal{A}$ is a compact perturbation of the identity, $\|\mathcal{A}^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ is bounded below by a k-independent constant (see Lemma A.3 below). We make the following assumption on the growth of $\|\mathcal{A}^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$.

Assumption 1.3 (Polynomial boundedness of $\mathcal{A}^{-1}$) *There exists $P_{\mathrm{inv}} \geq 0$ and $\mathcal{J} \subset [0,\infty)$ such that, given $k_0 > 0$, there exists $C > 0$ such that*

$$\rho(k) := \|\mathcal{A}^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq C k^{P_{\mathrm{inv}}} \quad \textit{for all } k \geq k_0 \textit{ with } k \in \mathbb{R}\setminus\mathcal{J}.$$

When the obstacle Ω^- is strongly trapping, Assumption 1.3 does not hold for $\mathcal{J} = \emptyset$. Indeed, the norm $\rho(k)$ grows exponentially through an increasing sequence of ks (see [10, Theorem 2.8] for the Dirichlet BIEs and [45, Theorem 2.6] for the Neumann BIEs). However, this assumption is satisfied for any Ω^- for "most" frequencies by the results of [77]. More precisely, for any smooth (or even Lipschitz) Ω^-, given $k_0, \delta > 0$, there exists $\mathcal{J} \subset [k_0,\infty)$ with $|\mathcal{J}| \leq \delta$ such that Assumption 1.3 holds for $k \in \mathbb{R}\setminus\mathcal{J}$ with P_{inv} independent of δ; see [77, Corollary 1.4] for this stated for the Dirichlet BIEs and [45, Theorem 2.3(iii)] for the Neumann BIEs. Moreover, the results of [77, Theorem 3.5] given $N > 0$ there exists P_{inv}, $\mathcal{J}$, and $C > 0$ such that Assumption 1.3 holds and $|\mathcal{J} \cap (k,\infty)| \leq Ck^{-N}$.

1.1.2 Approximation methods

We consider solving the second-kind BIEs (1.1) via projection methods; i.e., given $s \geq 0$, a finite dimensional space $V \subset L^2(\Gamma)$ with $\dim V =: N$, and projection $P_V : H^s(\Gamma) \to V$, we approximate the solution of the problem (1.1) by: given $f \in H^s(\Gamma)$, find $v_N \in V$ such that

$$(I + P_V L)v_N = c_0^{-1} P_V f. \tag{1.4}$$

Specifically we study

- Galerkin methods with subspaces consisting of either piecewise polynomials or, in 2-d, trigonometric polynomials,
- collocation methods in 2-d with subspaces consisting of either piecewise polynomials or trigonometric polynomials in 2-d, and

[1] In the real-valued $L^2(\Gamma)$ inner product, A'_k and A_k are each other's adjoints (hence the $'$ notation); similarly for $B_{k,\mathrm{reg}}, B'_{k,\mathrm{reg}}$.

- a fully-discrete quadrature (a.k.a., Nyström) method based on trigonometric polynomials in 2-d.

For the Nyström method, we further replace (1.4) by: given $f \in L^2(\Gamma)$, find $v_N \in V$ such that

$$(I + P_V L_V)v_N = c_0^{-1} P_V f, \tag{1.5}$$

where $L_V : V \to H^s(\Gamma)$ is a discrete approximation of L.

We study the question:

Question 1.4 *For the BIEs* (1.2) *and* (1.3)*, how quickly must N increase with k to maintain accuracy of the computed solutions as $k \to \infty$?*

The theory of projection methods for fixed (i.e. k-independent) compact perturbations of the identity as $N \to \infty$ is well established. see, e.g., the books [5, 72, 100]. However, to our knowledge, the only rigorous answers to Question 1.4 in the literature have been for the Galerkin method with piecewise-polynomials applied to the Dirichlet BIEs; see [21, 7, 81, 90, 53, 46, 48].

While the present paper focuses on Question 1.4, another important consideration in the implementation of (1.4) and (1.5) for the standard second-kind Helmholtz BIEs is that the kernels of the integral operators are singular and long-range – thus requiring high-order quadrature methods for their accurate discretization, and fast algorithms for applying or inverting the resulting dense linear systems. Over the last four decades, a variety of high order quadrature methods (see, e.g., [70, 37, 111, 16, 68, 19, 102, 110, 1, 104, 33]) fast algorithms for applying the discretized matrices (see, e.g., [97, 54, 20, 96, 79, 86, 67]) and fast direct methods for constructing compressed representations of the matrix and/or its inverse (see, e.g., [56, 57, 9, 87, 12, 51, 60, 31, 106, 66]) have been developed, resulting in accurate, robust, and highly performant linear complexity (up to log factors) computational tools for solving time harmonic wave scattering problems.

1.1.3 The pollution effect

For a scattering problem, the solution v to the BIE (1.1) oscillates at frequency $\lesssim k$ with few additional properties. The Weyl law (see e.g. [112, Chapter 14]) or the Nyquist–Shannon–Whittaker sampling theorem then implies that the dimension of this space is $\sim k^{d-1}$. Thus, it is certainly not possible to achieve accuracy for general scattering data with a space of dimension $\ll k^{d-1}$. Motivated by this, we say that a numerical BIE method *suffers from the pollution effect* if $N \gg k^{d-1}$ is required to maintain accuracy of the computed solutions as $k \to \infty$. We make this precise via the following definition (we work in $L^2(\Gamma)$ for concreteness, but note that the definition for higher-order spaces on Γ is analogous).

Definition 1.5 (The pollution effect in $L^2(\Gamma)$) *Let $\mathcal{V}$ be a collection of subspaces of $L^2(\Gamma)$. For $V \in \mathcal{V}$ and $k > 0$, let $\mathcal{A}_V^{-1}(k) : L^2(\Gamma) \to V$ an approximation of $\mathcal{A}^{-1}(k)$. The pair* $(\mathcal{V}, \{\mathcal{A}_V^{-1}\}_{V\in\mathcal{V}})$ suffers from the pollution effect in $L^2(\Gamma)$ *if*

$$\inf_{\Lambda>0} \limsup_{k\to\infty} \sup_{\substack{V\in\mathcal{V}\\ \dim V \geq \Lambda k^{d-1}}} \sup_{f\in V'} \inf \left\{ C_{\rm qo} : \left\|\mathcal{A}^{-1}f - \mathcal{A}_V^{-1}f\right\|_{L^2(\Gamma)} \leq C_{\rm qo} \min_{w\in V} \left\|\mathcal{A}^{-1}f - w\right\|_{L^2(\Gamma)} \right\} = \infty. \tag{1.6}$$

If the right-hand side of (1.6) is finite, then there exists k_0, Λ, and $C_{\rm qo}$ such that for all $k \geq k_0$, $V \in \mathcal{V}$ with $\dim V \geq \Lambda k^{d-1}$, and $f \in L^2(\Gamma)$,

$$\left\|\mathcal{A}^{-1}f - \mathcal{A}_V^{-1}f\right\|_{L^2(\Gamma)} \leq C_{\rm qo} \min_{w\in V} \left\|\mathcal{A}^{-1}f - w\right\|_{L^2(\Gamma)}; \tag{1.7}$$

i.e., k-uniform quasi-optimality is achieved (for all possible data) with a choice of subspace dimension proportional to k^{d-1}.

Finite element methods for the Helmholtz equation famously suffer from the pollution effect– i.e. $\gg k^d$ degrees of freedom are required to maintain accuracy [64, 65, 6]. However, there is a common belief in both engineering [82, 84, 83] and numerical analysis [7, 80, 53] that boundary integral methods do not suffer from the pollution effect. Indeed, it is standard in engineering to

Method	Scatterer	Approx. Space	d	$N \gtrsim$	$C_{\mathrm{qo}} \lesssim$
Galerkin	any	poly. degree p	≥ 2	$k^{(d-1)}\rho^{\frac{d-1}{2(p+1)}}$	$1+\rho(k/N^{\frac{1}{d-1}})^{p+1}$
Galerkin	any	trig. poly.	2	k	1
Collocation	any	poly. degree p	2	$k\rho^{\frac{1}{p+1}}$	$1+\rho(k/N)$
Collocation	any	trig. poly.	2	k	$1+\rho(k/N)^{\infty}$
Dirichlet Nyström	convex	trig. poly.	2	k	1
Dirichlet Nyström	nontrapping	trig. poly.	2	$k(\log k)^{+0}$	1
Dirichlet Nyström	any	trig. poly.	2	k^{1+0}	1
Neumann Nyström	any	trig. poly.	2	k^{1+0}	1

Table 2.1: In the table above we summarize the results estimating the quasioptimality constant C_{qo} in (1.7) in various settings (for Galerkin methods (1.7) holds in $L^2(\Gamma)$, but for collocation and Nyström methods our results establish the analogue of (1.7) in $H^s_k(\Gamma)$ for certain s). In the above expressions $+0$ represents the fact that for any $\epsilon > 0$ the statement holds with $+0$ replaced by $+\epsilon$. Similarly, $(\cdot)^{\infty}$ means that ∞ can be replaced by any power $M > 0$.

ask whether a specific number of points per wavelength (e.g., six [82]) suffices to obtain accurate solutions.

In earlier work [48], the first and third authors showed that the Galerkin method with piecewise polynomials applied to Dirichlet BIEs for a nontrapping obstacle does not suffer from the pollution effect. In this article, we both improve the analysis there – giving less stringent conditions for k-uniform quasioptimality – and show that i) for trapping obstacles the Galerkin method with piecewise polynomials applied to Dirichlet BIEs *suffers from the pollution effect* and, ii) even for nontrapping obstacles, the Galerkin method with piecewise polynomials applied to Neumann BIEs *suffers from the pollution effect.* We also present numerical experiments demonstrating the pollution effect for the Dirichlet BIEs for two trapping situations.

2 Statement of the main results

We study three methods for solving BIEs numerically: the Galerkin method, the collocation method, and the Nyström method. Our results typically depend on several things: the choice of the approximation space, the growth of the solution operator, and, occasionally, the geometry of the scatterer. A summary of our results is given in Table 2.1.

2.1 Choices of projections

The most natural choice of projection from the Hilbert space structure of $L^2(\Gamma) \to L^2(\Gamma)$ is the Galerkin projection.

Definition 2.1 (Galerkin projection and Galerkin solution) *Given a subspace $V \subset L^2(\Gamma)$, the* Galerkin projection onto V, $P_V^G : L^2(\Gamma) \to V$ *is the orthogonal projection onto V; in particular,*

$$\left\|(I - P_V^G)v\right\|_{L^2(\Gamma)\to L^2(\Gamma)} = \min_{w\in V} \|v - w\|_{L^2(\Gamma)}. \tag{2.1}$$

We call the solution to (1.4) with $P_V = P_V^G$ the Galerkin solution of *(1.1).*

Computing the Galerkin projection involves numerically approximating integrals. It is therefore usually less computationally expensive to use a projection based on point values.

To define this type of projection, we need to use points that determine the finite dimensional space V in an appropriate sense.

Definition 2.2 *Let $V \subset C^0(\Gamma)$ and $\{x_j\}_{j=1}^N \subset \Gamma$. We say that $\{x_j\}_{j=1}^N$ are* unisolvent for V *if for any $\{a_j\}_{j=1}^N \subset \mathbb{C}$, there is a unique $v_h \in V$ such that $v_h(x_j) = a_j$ for $j = 1, \dots, N$.*

We can now define the collocation projection.

Definition 2.3 *Given a subspace $V \subset C^0(\Gamma)$ and points $\{x_j\}_{j=1}^N \subset \Gamma$ that are unisolvent for V, we define the* collocation projection for $\{x_j\}_{j=1}^N$, $P_V^C : C^0(\Gamma) \to V$, *by*

$$(P_V^C v)(x_j) = v(x_j), \quad j = 1, \ldots, N.$$

Note that P_N^C is well defined since $\{x_j\}_{j=1}^N$ are unisolvent for V. We call the solution to (1.4) with $P_V = P_V^C$ the collocation solution of (1.1).

2.2 Sufficient conditions for quasi-optimality with piecewise-polynomial approximation spaces

Our results about piecewise polynomial spaces require some further technical assumptions that, roughly speaking, guarantee that the piecewise-polynomial spaces are maximally dense in Sobolev spaces.

Assumption 2.4 (Assumption on piecewise-polynomial V_h for the Galerkin method)
(i) $(\mathcal{T}_h)_{0<h\leq h_0}$ is a sequence of meshes of Γ (in the sense of, e.g., [101, Definition 4.1.2]), such that, for all $\tau \in \mathcal{T}_h$ there exists a reference map $\chi_\tau : \widehat{\tau} \to \tau$ where $\widehat{\tau}$ is a reference element.
(ii)

$$V_h := \Big\{ v : \Gamma \to \mathbb{C} : \text{ for all } \tau \in \mathcal{T}_h,\ v \circ \chi_\tau \text{ is a polynomial of degree } p \Big\}.$$

(iii) Given $p \geq 0$ there exists $C > 0$ such that for all $0 < h \leq h_0$ there exists a operator $\mathcal{I}_h$ such that, for $0 \leq t \leq p+1$, $\mathcal{I}_h : H^t(\Gamma) \to V_N$ with

$$\|v - \mathcal{I}_h v\|_{L^2(\Gamma)} \leq C h^t \|v\|_{H^t(\Gamma)}. \tag{2.2}$$

To state our results, we use the k-weighted Sobolev space norms; i.e., the standard Sobolev space $H_k^s(\Gamma)$ but with each derivative of weighted by k^{-1} (see §3.1 below for a precise definition) – these weighted Sobolev spaces are the natural spaces in which to study functions oscillating with frequency $\sim k^{-1}$ (such as Helmholtz solutions). Note that $H_k^0(\Gamma) = L^2(\Gamma)$.

Theorem 2.5 (Galerkin method with piecewise polynomials) *Suppose $\mathcal{A}$ is given by one of (1.2) or (1.3), Assumptions 1.1 and 1.3 hold. Let $p \geq 0$, $V_h \subset L^2(\Gamma)$ satisfy Assumption 2.4, and $k_0 > 0$. Then there are $c, C > 0$, such that if*

$$(hk)^{2p+2}\rho \leq c, \qquad hk \leq c, \tag{2.3}$$

and $k \geq k_0$, $k \notin \mathcal{J}$ the Galerkin solution, v_h, to (1.1) exists, is unique, and satisfies the quasi-optimal error bound

$$\|v - v_h\|_{L^2(\Gamma)} \leq \Big(1 + C(hk)^{p+1}\rho + Chk + Ck^{-1}\Big) \big\|(I - P_{V_h}^G)v\big\|_{L^2(\Gamma)}. \tag{2.4}$$

Moreover, if the right-hand side f corresponds to plane-wave scattering (i.e., is given as in Theorem A.2), then

$$\frac{\|v - v_h\|_{L^2(\Gamma)}}{\|v\|_{L^2(\Gamma)}} \leq \Big(1 + C(hk)^{p+1}\rho + Chk + Ck^{-1}\Big) C(hk)^{p+1} \tag{2.5}$$

(i.e., the relative error can be made arbitrarily-small by decreasing $(hk)^{2p+2}\rho$).

Remark 2.6 (Preasymptotic h-BEM and h-FEM error estimates) *Theorem 2.5, which is shown to be sharp by Theorem 2.10 below, is the h-BEM analogue of preasymptotic error estimates for the h-FEM. Indeed, for the h-FEM, if $(hk)^{2p}\mathscr{R}$ is sufficiently small, then the Galerkin solution exists, is unique, and satisfies a quasi-optimal error estimate with constant proportional to $1 + (hk)^p\mathscr{R}$, where $\mathscr{R}$ is the $L^2 \to L^2$ norm of the solution operator, and such that $\mathscr{R} \sim k$ for nontrapping problems. Thus, for k-oscillating data, the relative error of the h-FEM is controllably small when $(hk)^{2p}\mathscr{R}$ is small. This threshold for bounded relative error was famously* *identified for 1-d problems in [64, 65], with the associated bound first proved in [34] for constant-coefficient Helmholtz problems in 2-d and 3-d with an impedance boundary condition, and then proved for general Helmholtz problems in [49]. Theorem 2.5 is the* full *analogue of these results, with $(hk)^{2p}\mathscr{R}$ replaced by $(hk)^{2p+2}\rho$. Moreover, (2.4) shows that the in the limit $k \to \infty$ with h chosen so that $(hk)^{p+1}\rho \to 0$, the Galerkin solution is asymptotically* optimal*; i.e., $\|v - v_h\|_{L^2(\Gamma)} \to \|(I - P^G_{V_h})v\|_{L^2(\Gamma)}$.*

For the collocation solution of (1.1) with piecewise polynomial spaces, we specialize to $d = 2$ (this is for technical reasons related to the failure of the Sobolev embedding $H^1 \to L^\infty$ when $d \geq 3$). We also require an analogue of Assumption 2.7.

Assumption 2.7 (Assumption on piecewise-polynomial V_h for the collocation method) *$d = 2$, Parts (i) and (ii) of Assumption 2.4 hold and*

(iii) Given $p \geq 1$ and points $\{x_j\}_{j=1}^N \subset \Gamma$ that are unisolvent for V_h there exists $C > 0$ such that for all $0 < h \leq h_0$ the collocation projection P^C_V satisfies for $0 \leq q \leq 1$ and $1 \leq t \leq p+1$,

$$\left|(I - P^C_{V_h})v\right|_{H^q(\Gamma)} \leq Ch^{t-q}\|v\|_{H^t(\Gamma)}. \tag{2.6}$$

Theorem 2.8 (Collocation method with piecewise polynomials) *Suppose $\mathcal{A}$ is given by one of (1.2) or (1.3) and Assumptions 1.1 and 1.3 hold. Let $p \geq 1$, $V_h \subset L^2(\Gamma)$ satisfy Assumption 2.7, $P^C_{V_h}$ be the collocation projection for $\{x_j\}_{j=1}^{N_h}$, and $k_0 > 0$. There exist $c, C > 0$ such that if*

$$(hk)^{p+1}\rho \leq c, \qquad hk \leq c \tag{2.7}$$

and $k \geq k_0$, $k \notin \mathcal{J}$ then the collocation solution, v_h, to (1.1) exists, is unique, and satisfies the quasi-optimal error bound

$$\|v - v_h\|_{H^1_k(\Gamma)} \leq \Big(\|I - P^C_{V_h}\|^2_{H^1_k(\Gamma)\to H^1_k(\Gamma)} + Chk\rho + Ck^{-1}\Big)\left\|(I - P^C_{V_h})v\right\|_{H^1_k(\Gamma)}. \tag{2.8}$$

Moreover, if the right-hand side f corresponds to plane-wave scattering (i.e., is given as in Theorem A.2), then

$$\frac{\|v - v_h\|_{H^1_k(\Gamma)}}{\|v\|_{H^1_k(\Gamma)}} \leq C\Big(\|I - P^C_{V_h}\|^2_{H^1_k(\Gamma)\to H^1_k(\Gamma)} + Chk\rho + Ck^{-1}\Big)(hk)^p \tag{2.9}$$

(i.e., the relative error can be made arbitrarily-small by decreasing $(hk)^{p+1}\rho$).

Remark 2.9 (Implementing the operator product for the Neumann BIEs (1.3)) *Because of the operator product $S_{\mathrm{i}k}H_k$ in $B_{k,\mathrm{reg}}$, in practice, approximations to the solution of $B_{k,\mathrm{reg}}v = f$ are computing by applying the projection method, not to $B_{k,\mathrm{reg}}v = f$, but to the system*

$$\begin{pmatrix} \mathrm{i}\eta_N(\frac{1}{2}I - K) & S_{\mathrm{i}k} \\ H_k & -1 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}; \tag{2.10}$$

similarly for $B'_{k,\mathrm{reg}}$. For simplicity, when studying $B_{k,\mathrm{reg}}$ and $B'_{k,\mathrm{reg}}$ in this paper we consider the idealised situation of (1.4) with $P_V = P^G_V$ or $P_V = P^C_V$; i.e., we ignore the issue of discretising the operator product in Galerkin or collocation methods. We emphasise, however, that we do *analyse discretising the operator product in the Nyström method.*

Comparison with previous results for the Galerkin method. Of the investigations [7, 81, 53, 103, 46, 48] into proving quasioptimality of the Galerkin method with piecewise polynomials applied to A'_k and A_k, the best results are the following. The result of [48, Lemma 3.1] that if Γ is smooth and $\rho(hk)$ is sufficiently small, then the Galerkin method is quasioptimal with constant proportional to ρ. The result of [81, Theorem 3.17] that if Γ is analytic then the Galerkin method applied to A'_k is quasioptimal (with constant independent of k) if $(hk)^{p+1}k^5\rho$ is sufficiently small [81, Equation 3.22]; a similar result holds for A_k with k^5 replaced by k^6 [81, Equation 3.26]. These quasi-optimality constants are larger than that in (2.4) and the thresholds for existence are more-restrictive than that in (2.3).

When Γ is either a circle or sphere, k is sufficiently large, and $\eta_N \geq Ck^{1/3}$ with C sufficiently large, then $B_{k,\mathrm{reg}}$ and $B'_{k,\mathrm{reg}}$ are coercive on $L^2(\Gamma)$ with constant independent of k by [14, Theorems 3.6 and 3.9]. Céa's lemma then implies that the Galerkin method applied to these operators is quasioptimal, with quasioptimality constant $\gtrsim k^{1/3}$. This is in contrast to Theorem 2.5 where the quasioptimality constant ~ 1, albeit under the thresholds (2.3). Note that the choice $\eta_N \sim k^{1/3}$ is *not* used in practice, indeed, [14, §5] and [18, Equation 23] recommend using $\eta_N \sim 1$, stating that, out of all the possible choices of η_N, this gives a "nearly optimal number"/"'small number" of iterations when GMRES is used to solve the Galerkin linear systems.

Spaces satisfying Assumption 2.4 and 2.7. Boundary element spaces satisfying one of Assumptions 2.4 and 2.7 are constructed when $d = 3$ in [101, Chapter 4]. Indeed, discontinuous boundary element spaces satisfying Assumption 2.4 are constructed in [101, Theorem 4.3.19], and continuous boundary element spaces satisfying Assumption 2.7 are constructed in [101, Theorem 4.3.22(a)] (a subtlety is that [101] work in 3-d, but the constructions and arguments there establish (2.6) when $d = 2$ in fact for $0 \leq q < 3/2$, $t > 1/2$, and' $q \leq t \leq p + 1$).

2.3 Pollution in the Galerkin method with piecewise polynomials

2.3.1 Quasimodes imply pollution

Recall that *quasimodes* for the exterior Dirichlet/Neumann Helmholtz problem are functions u in Ω^+ such that, informally, $(-k^{-2}\Delta - 1)u$ is "small" and compactly supported. More precisely,

$$\|(-k^{-2}\Delta - 1)u\|_{L^2(\Omega^+)} = o(k^{-1})\|u\|_{L^2(B(0,R)\cap\Omega^+)}, \text{ either } \gamma^+ u = 0 \text{ or } \partial_\nu^+ u = 0 \text{ on } \partial\Omega^+,$$

and u satisfies the Sommerfeld radiation condition (A.1). The factor of k^{-1} in the bound is natural, since $(-k^{-2}\Delta - 1)(\chi e^{ikx\cdot\xi_0}) = O(k^{-1})$ for any k-independent $\chi \in C_c^\infty$ and ξ_0 with $|\xi_0| = 1$; thus functions u such that $(-k^{-2}\Delta - 1)u = O(k^{-1})$ can be constructed for any Ω^-. A quasimode (in the sense above) exists if and only if the obstacle is trapping [11], [35, Theorem 7.1].

In the BIE context, a quasimode is a function $v \in L^2(\Gamma)$ such that $\|\mathcal{A}v\|_{L^2(\Gamma)} = o(1)\|v\|_{L^2(\Gamma)}$. For the Dirichlet BIEs (with η_D satisfying Assumption 1.1), quasimodes exists if and only if Ω^- is trapping. However, for the Neumann BIEs, quasimodes exist even when Ω^- is the unit ball, since $\|\mathcal{A}^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \sim k^{1/3}$ in this case (see, e.g,. [45, Theorem 2.3]).

Our results showing the existence of pollution are based on the following theorem, which states, roughly, that the existence of a sufficiently good quasimode for the BIE that oscillates at frequency $\sim k$ guarantees pollution. To state this carefully, we let Δ_Γ denote the surface Laplacian on Γ.

Theorem 2.10 (Quasimodes imply pollution) *Suppose that Assumption 1.1 holds. Let $\Xi_0 > 1$, $0 < \epsilon < \frac{1}{10}$, $0 < \epsilon_0 < \Xi_0$, and $\mathcal{J} \subset (0, \infty)$ unbounded such that Assumption 1.3 holds and $V_h \subset L^2(\Gamma)$ satisfy Assumption 2.4, $p \geq 0$. Then there is $k_0 > 0$ such that for all $\chi \in C_c^\infty((-(1+\epsilon)^2, (1+\epsilon)^2))$ with $\chi \equiv 1$ on $[-1, 1]$ there is $c > 0$ such that if there are $k_n \to \infty$, $k_0 < k_n \notin \mathcal{J}$, $u_n, f_n \in L^2(\Gamma)$, $\beta_n, \alpha_n \in (0, 1]$ such that*

$$\beta_n \|\mathcal{A}^{-1}(k_n)\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq c,$$

and

$$\mathcal{A}(k_n)u_n = f_n, \qquad \|f_n\|_{L^2(\Gamma)} \leq \alpha_n, \qquad \|u_n\|_{L^2(\Gamma)} = 1, \tag{2.11}$$

$$\|(1 - \chi(\Xi_0^{-2}k_n^{-2}\Delta_\Gamma))f\|_{L^2(\Gamma)} \leq \beta_n\|f_n\|_{L^2(\Gamma)}, \qquad \|\chi(\epsilon_0^{-2}k_n^{-2}\Delta_\Gamma)f_n\|_{L^2(\Gamma)} \leq \beta_n\|f_n\|_{L^2(\Gamma)}, \tag{2.12}$$

$$\textit{either} \quad p = 0 \quad \textit{or} \quad k^{-p-1}\alpha_n\|1_{(0,k_0^2]}(-\Delta_\Gamma)(\mathcal{A}^*)^{-1}u_n\|_{L^2(\Gamma)} \leq c, \tag{2.13}$$

then, for any h and n such that and $(I + P_h^G L(k_n))$ has an inverse,

$$\|(I + P_{V_h}^G L(k_n))^{-1}(I - P_{V_h}^G)\|_{L^2(\Gamma)\to L^2(\Gamma)} \geq \frac{1}{2} + c\begin{cases}(hk_n)^{-p-1}, & 1 \leq (hk_n)^{2(p+1)}\alpha_n^{-1} \\ (hk_n)^{p+1}\alpha_n^{-1}, & \alpha_n^{-1}(hk_n)^{2(p+1)} \leq 1.\end{cases} \tag{2.14}$$

We make the following immediate remarks about Theorem 2.10:

- The assumptions (2.11) and (2.12) are the precise versions of, respectively, the statement that u_n is a sufficiently good quasimode for $\mathcal{A}$, and that f_n oscillates at frequency $\sim k_n$.
- When $p \geq 1$, the assumption (2.13) guarantees that u_n is not close to any solution of $\mathcal{A}v = \tilde{f}$ where $\tilde{f}$ is in the span of finitely many low frequency eigenfunctions of the surface Laplacian. This finite-dimensional assumption is required for technical reasons when $p \geq 1$ and we expect the theorem to hold without it even in that case. In the concrete examples below (see Theorems 2.13 and 2.15) the second part of (2.13) imposes an extra restriction on growth of the solution operator when $p \geq 1$. Since the space of obstructions is finite-dimensional, this extra assumption could also be avoided by constructing sufficiently many good quasimodes.
- If $\alpha_n \leq C\rho^{-1}$, i.e. f_n achieves the full growth of the solution operator, then Theorem 2.10 implies that

$$\|(I + P^G_{V_h}L)^{-1}(I - P^G_{V_h})\|_{L^2(\Gamma)\to L^2(\Gamma)} \geq \frac{1}{2} + c\begin{cases}(h\hbar)^{-(p+1)}, & 1 \leq (hk)^{2(p+1)}\rho \\ (hk)^{p+1}\rho, & (hk)^{2(p+1)}\rho \leq c.\end{cases} \tag{2.15}$$

  In particular, if (2.15) holds, then Theorem 2.5 is optimal since the Galerkin solution, v_h to $\mathcal{A}v = f$ satisfies

$$v - v_h = (I + P^G_{V_h}L)^{-1}(I - P^G_{V_h})(I - P^G_{V_h})v.$$

- By Definition 1.5, (2.14) with $\alpha = o(1)$ implies that the h-BEM suffers from the pollution effect. In particular, if $hk = \epsilon$, then (2.14) implies that the quasioptimality constant is *at least* $c\epsilon^{-p-1}$ and hence that decreasing ϵ actually makes the quasioptimality constant worse until the point that $\epsilon\alpha^{-1} \lesssim 1$.
- The proof of Theorem 2.10 is sketched in §2.6; the key technical ingredient is the lower bound from [40] on how well piecewise polynomials approximate a function with frequency $\sim k$.

While it is relatively straightforward to show that a strong quasimode for the Helmholtz equation implies the existence of a strong quasimode for the BIEs [10], [26, §5.6.2] (for the Dirichlet BIEs) and [45, Theorem 2.6] (for the Neumann BIEs), it is more challenging to determine the properties of the BIE quasimode from the properties of the quasimode of the Helmholtz equation; §13 below does this in the Dirichlet case, resulting in concrete examples of pollution for the Dirichlet BIEs (Theorems 2.12, 2.13, 2.15). We expect that the results of §13 can be extended to the Neumann case; we do not pursue this here, but show below that pollution occurs for the Neumann BIEs even for the unit disk.

2.3.2 Pollution for the Dirichlet BIEs

Throughout this section, $V_h \subset L^2(\Gamma)$ satisfies Assumption 2.4 for some $p \geq 0$. We start with an example that shows *quantitative* pollution for the Dirichlet BIEs.

Definition 2.11 (Four-diamond domain) *We say that Ω^- is a* four-diamond domain *if there exists $0 < \epsilon < \pi/2$ such that $\Omega^- = \cup_{i=1}^4 \Omega_i \Subset \mathbb{R}^2$, where Ω_i are open, convex, disjoint, and have smooth boundary so that*

$$\partial\big((-\pi,\pi)\times(-\pi,\pi)\big) \setminus \big(\cup_{a,b\in\pm 1} B((a\pi,b\pi),\epsilon)\big) \subset \Gamma \subset \mathbb{R}^2 \setminus \big(\cup_{a,b\in\pm 1} B((a\pi,b\pi),\epsilon/2)\big),$$
$$\Omega^- \cap (-\pi,\pi)\times(-\pi,\pi) = \emptyset.$$

Figure 2.1 shows an example of a four-diamond domain.

Theorem 2.12 (Pollution for the four-diamond domain when $p = 0$) *Let $p = 0$, Ω^- be a four-diamond domain, η_D satisfy Assumption 1.1, $\mathcal{J}$ satisfy Assumption 1.3 with $\mathcal{A} = A_k$ or A'_k, and $k_n := \sqrt{2}n$. Then there is $c > 0$, such that for all $k \in \mathbb{R}\setminus\mathcal{J}$ and n such that $k_n^{-1} \leq \delta \leq 1$, $|k - k_n| < \delta^{-1}k_n^{-1}$, and all $h > 0$ there is $f \in L^2(\Gamma)$ such that the Galerkin solution, v_h, to (1.1) with $\mathcal{A}$ given by either A_k or A'_k, if it exists, satisfies*

$$\frac{\|v_h - v\|_{L^2(\Gamma)}}{\|(I - P^G_{V_h})v\|_{L^2(\Gamma)}} \geq \frac{1}{2} + c\begin{cases}(hk_n)^{-1}, & 1 \leq (hk_n)^2\delta k_n, \\ (hk_n)\delta k_n, & \delta k_n(hk_n)^2 \leq 1.\end{cases}$$

Theorem 2.13 (Pollution for the four-diamond domain when $p \geq 1$) *Let Ω^- be a four-diamond domain, η_D satisfy Assumption 1.1, $\mathcal{A} = A_k$ or A'_k, and $k_n := \sqrt{2}n$.*

If there are $C_1 > 0, k_0 > 0$ such that

$$\left\|\mathcal{A}^{-1}\right\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq C_1 k^2 \quad \text{for all } k \geq k_0 \tag{2.16}$$

there are $c, C > 0$ such that for all $k \geq k_0$ and n such that $Ck_n^{-1} \leq \delta \leq 1$ and $|k - k_n| < \delta^{-1}k_n^{-1}$ and for all $h > 0$ there is $f \in L^2(\Gamma)$ such that the Galerkin solution, v_h, to (1.1) with $\mathcal{A}$ given by either A_k or A'_k, if it exists, satisfies

$$\frac{\|v_h - v\|_{L^2(\Gamma)}}{\|(I - P^G_{V_h})v\|_{L^2(\Gamma)}} \geq \frac{1}{2} + c \begin{cases} (hk_n)^{-p-1}, & 1 \leq (hk_n)^{2(p+1)}\delta k_n, \\ (hk_n)^{p+1}\delta k_n, & \delta k_n (hk_n)^{2(p+1)} \leq 1. \end{cases}$$

Remark 2.14 (Discussion of Theorems 2.12 and 2.13)

1. *When Ω^- consists of two aligned squares (or rounded squares) – where the trapping is the same nature (*parabolic*) as for a four-diamond domain (albeit on a smaller set in phase space) – the bound (2.16) is proved in [28, Corollary 1.14]. Furthermore, from both quasimode considerations and the bounds of [30, 29] in the setting of scattering by metrics of revolution, [28] conjecture that for Ω^- as in Theorem 2.12,*

$$\left\|A_k^{-1}\right\|_{L^2(\Gamma)\to L^2(\Gamma)} + \left\|(A'_k)^{-1}\right\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq Ck. \tag{2.17}$$

   If the bound (2.17) holds, then Theorems 2.12 and 2.13 show that Theorem 2.5 is optimal in this case and $\mathcal{J} = \emptyset$.

2. *We emphasise that the phenomenon behind Theorems 2.12 and 2.13 is* not *sparse in frequency. Indeed, both Theorems 2.12 and 2.13 hold with $\delta = 1$ for a set of k with infinite measure (since $\sum_{n=1}^{\infty} n^{-1} = \infty$). Furthermore, although we prove Theorem 2.12 with $k_n = \sqrt{2}n$, it is easy to generalize our construction in the proof of Theorem 2.12 to take $k_n = \sqrt{m_n^2 + \ell_n^2}$ for any $m_n, \ell_n \in \mathbb{Z}$ with $c < |\frac{m_n}{\ell_n}| < C$ and such that, if $m_n/\ell_n = p_n/q_n$ with p_n and q_n relatively prime, then $|q_n| \leq C$.*

3. *Theorems 2.12 and 2.13 are illustrated by numerical experiments in Figures 2.1 and 2.2, with Figures 2.3 and 2.4 illustrating pollution for a cavity domain. (The set up for all the numerical experiments in this section is described in §15.)*

Next, we show that in a wide variety of trapping problems, one has pollution *at all frequencies*. To avoid technicalities, we informally define the trapped set, K, as set of points x and directions ξ such that the billiard trajectory through (x, ξ) in Ω^+ remains in a compact set for all time. (We refer the reader to §13.1 for the careful definitions).

Theorem 2.15 (Qualitative pollution for a large class of trapping domains) *Suppose that Assumptions 1.1 and 1.3 hold, that $K \neq \emptyset$ and that for every point x and direction ξ such that $(x, \xi) \in K$, the line*

$$\mathscr{L}_{(x,\xi)} := x + \mathbb{R}\xi$$

is not tangent to Γ. Let $C_1, k_0 > 0$. Suppose that

$$\text{either} \quad p = 0 \quad \text{or} \quad p \geq 1 \;\text{ and }\; \left\|A_k^{-1}\right\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq C_1 k^{p+1} \text{ for all } k \geq k_0. \tag{2.18}$$

Then for all $k_n \to \infty$ $k_n \notin \mathcal{J}$, and $h > 0$ there are $f_n \in L^2(\Gamma)$ and $\alpha_n \to 0$ such that the Galerkin solution, v_h, to (1.1) with $\mathcal{A}$ given by A_{k_n}, if it exists, satisfies

$$\frac{\|v_h - v\|_{L^2(\Gamma)}}{\|(I - P^G_{V_h})v\|_{L^2(\Gamma)}} \geq \frac{1}{2} + c \begin{cases} (hk_n)^{-p-1}, & 1 \leq (hk_n)^{2(p+1)}\alpha_n^{-1}, \\ (hk_n)^{p+1}\alpha_n^{-1}, & \alpha_n^{-1}(hk_n)^{2(p+1)} \leq 1. \end{cases} \tag{2.19}$$

The assumptions in Theorem 2.15 are satisfied, for example, by two aligned squares with rounded corners (since (2.18) holds by the bound (2.16) proved in [28]).

There is an analogous result for A'_k, but it requires slightly stronger hypotheses that are harder state in a non-technical way so we postpone it to Theorem 13.34.

2.3.3 Pollution for the Dirichlet and Neumann BIEs on the unit disk

Theorem 2.16 (Pollution for the Neumann BIEs on the unit disk) *Let $\Omega_- = B(0,1) \subset \mathbb{R}^2$. For all $C > 0$, $k_n \to \infty$, and $C^{-1} \le \eta_N \le C$ there are $f_n \in L^2(\Gamma)$ such that the Galerkin solution, v_h. to (1.1) with $\mathcal{A}$ given by $B_{k_n,\mathrm{reg}}$ or $B'_{k_n,\mathrm{reg}}$ satisfies*

$$\frac{\|v_h - v\|_{L^2(\Gamma)}}{\|(I - P^G_{V_h})v\|_{L^2(\Gamma)}} \ge \frac{1}{2} + c\begin{cases}(hk_n)^{-p-1}, & 1 \le (hk_n)^{2(p+1)}k_n^{\frac{1}{3}},\\ (hk_n)^{p+1}k_n^{\frac{1}{3}}, & k_n^{\frac{1}{3}}(hk_n)^{2(p+1)} \le 1.\end{cases}$$

In particular, Theorem 2.5 is optimal in this case.

Remark 2.17 *We expect that the more sophisticated microlocal description of boundary layer operators in [39] can be used to extend Theorem 2.16 to any domain with smooth boundary in any dimension.*

Next, for the Dirichlet problem, we demonstrate pollution for any $k^{-N} \le \eta_D = o(k)$.

Theorem 2.18 (Pollution for the Dirichlet BIEs on the unit disk with non-standard coupling parameters) *Let $\Omega_- = B(0,1) \subset \mathbb{R}^2$. There is $k_n \to \infty$, such that for all $N > 0$, $C > 0$, $k^{-N} \le \eta_D \le Ck$ there are $f_n \in L^2(\Gamma)$ such that the Galerkin solution, v_h. to (1.1) with $\mathcal{A}$ given by A_{k_n} or A'_{k_n} satisfies*

$$\frac{\|v_h - v\|_{L^2(\Gamma)}}{\|(I - P^G_{V_h})v\|_{L^2(\Gamma)}} \ge \frac{1}{2} + c\begin{cases}(hk_n)^{-p-1}, & 1 \le (hk_n)^{2(p+1)}k_n\eta_n^{-1},\\ (hk_n)^{p+1}k_n\eta_n^{-1}, & k_n\eta_n^{-1}(hk_n)^{2(p+1)} \le 1.\end{cases}$$

In particular, Theorem 2.5 is optimal in this case.

Remark 2.19 *The specific examples in Theorems 2.12, 2.15, 2.16, and 2.18 show that Theorem 2.10 is effective in demonstrating pollution in concrete situations. Moreover, although it is difficult to prove a general statement for all domains simultaneously, given a trapping domain it is usually possible to construct quasimodes that, together with Theorem 2.10 demonstrate pollution.*

Remark 2.20 *When Γ is the sphere and $\eta = k^{-2/3}$ [7] conjecture that the Galerkin method applied to A_k and A'_k with piecewise polynomials does not suffer from the pollution effect. More precisely, [7, Proposition 3.14] show that if a specific bound on a combination of Bessel and Hankel functions holds ([7, Equation 3.20]) then the Galerkin solution is k-uniformly quasioptimal when hk is sufficiently small.*

In contrast, Theorem 2.18 proves that when $\eta = k^{-2/3}$, the Galerkin method applied to A_k and A'_k with piecewise polynomials does *suffer from the pollution effect. Thus the analogue of the bound [7, Equation 3.20] for the disk cannot hold.*

Remark 2.21 *The recent preprint [52] explores pollution numerically for a variety of Helmholtz integral operators on the disk.*

2.4 Sufficient conditions for quasi-optimality with trigonometric-polynomial approximation spaces

Let $d = 2$ and assume that Ω^- is connected (the results can be extended to multiply-connected domains in a straightforward way). Without loss of generality, assume that Γ is given by $x = \gamma(t)$ for $\gamma : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^2$ a smooth curve with $0 < |\dot{\gamma}(t)| \le c_{\max}$. The subspace $\mathscr{T}_N$ is defined in the t-variable by

$$\mathscr{T}_N := \operatorname{span}\big\{\exp(\mathrm{i}mt) : |m| \le N\big\}; \tag{2.20}$$

note that the dimension of $\mathscr{T}_N$ is then $2N + 1$. We consider the collocation projection $P^C_{\mathscr{T}_N}$ with evenly-spaced points in the t variable; i.e.,

$$t_j = 2\pi j/(2N + 1), \quad j = 0, 1, \ldots, 2N. \tag{2.21}$$

Piecewise constant elements

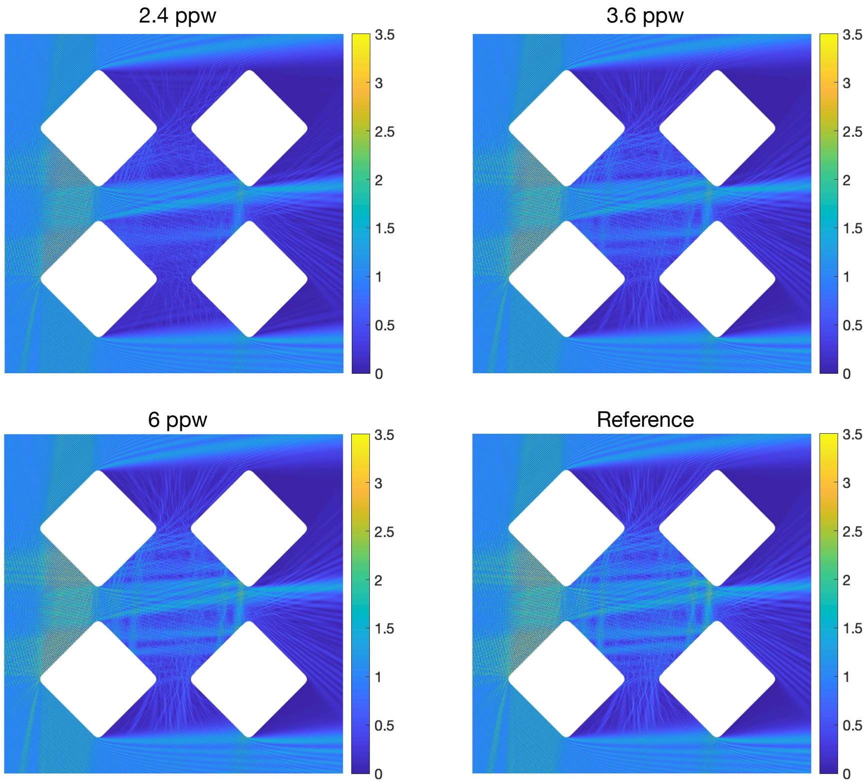

Figure 2.1: The plots depict the absolute value of the total field at $k = 40\sqrt{2}$ when the plane wave $e^{ik\langle\omega,x\rangle}$, with $\omega = (\cos(5\pi/180), \sin(5\pi/180))$, is incident on a sound-soft domain consisting of four nearly square obstacles. The plot shows the Galerkin solutions at 2.4 (top left), 3.6 (top right), and 6 (bottom left) points per wavelength and piecewise constant elements (i.e., $p = 0$), with the reference solution (bottom right) computed with $p = 11$. These numbers of points per wavelength correspond to points before, on, and after the peak of the quasioptimality constant in Figure 2.2. Many of the features of the Galerkin and reference plots are similar. However, in the Galerkin solutions with low numbers of points per wavelength the trapped rays are much less pronounced. The plots for piecewise linear elements (i.e., $p = 1$) look qualitatively similar.

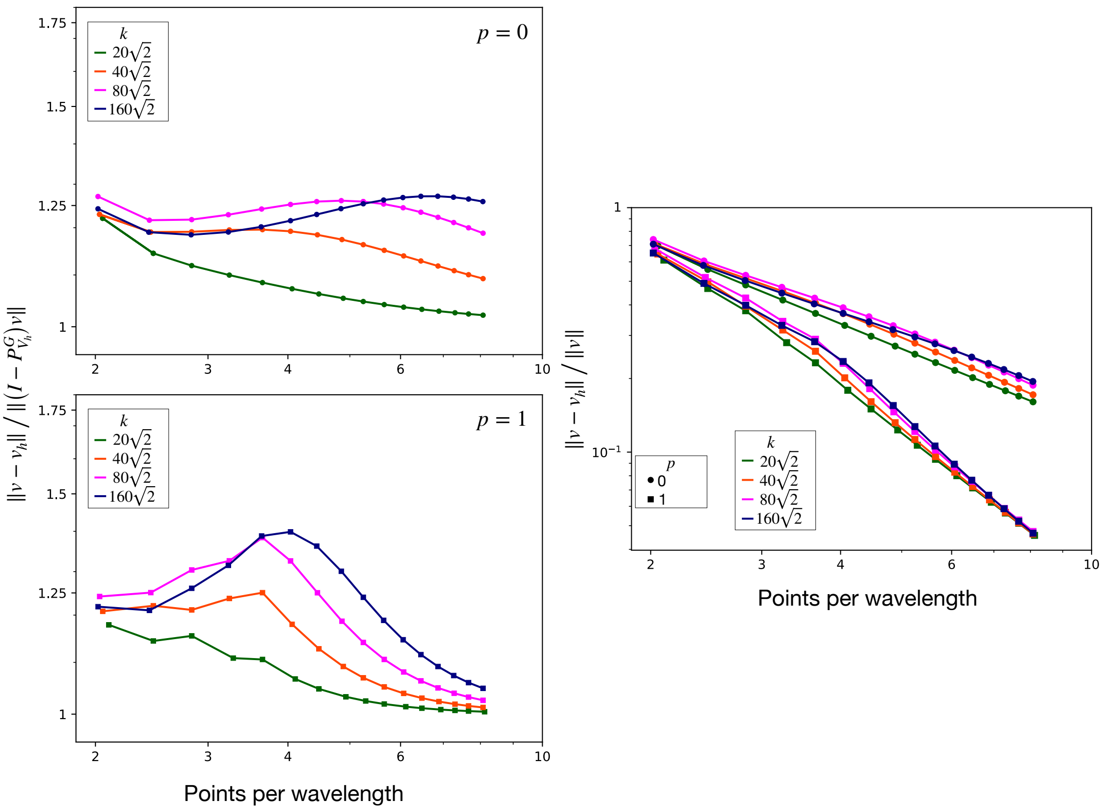

Figure 2.2: The left plots shows the implied quasioptimality constant for piecewise constant (top) and piecewise linear (bottom) elements when the plane wave $e^{ik\langle\omega,x\rangle}$ with $\omega = (\cos(5\pi/180), \sin(5\pi/180))$ is incident on a sound-soft domain consisting of four nearly square obstacles (see Figure 2.1). The right plot shows the corresponding L^2 relative error in the density. The reference and computed scattered solutions at $k = 40\sqrt{2}$ are shown in Figure 2.1. The presence of pollution in this example can be seen from both plots: in the left hand plot, one observes a peak in the implied quasioptimality constant that shifts up and to the right as k increases. In the right hand plot, the error increases for a fixed number of points per wavelength as k increases.

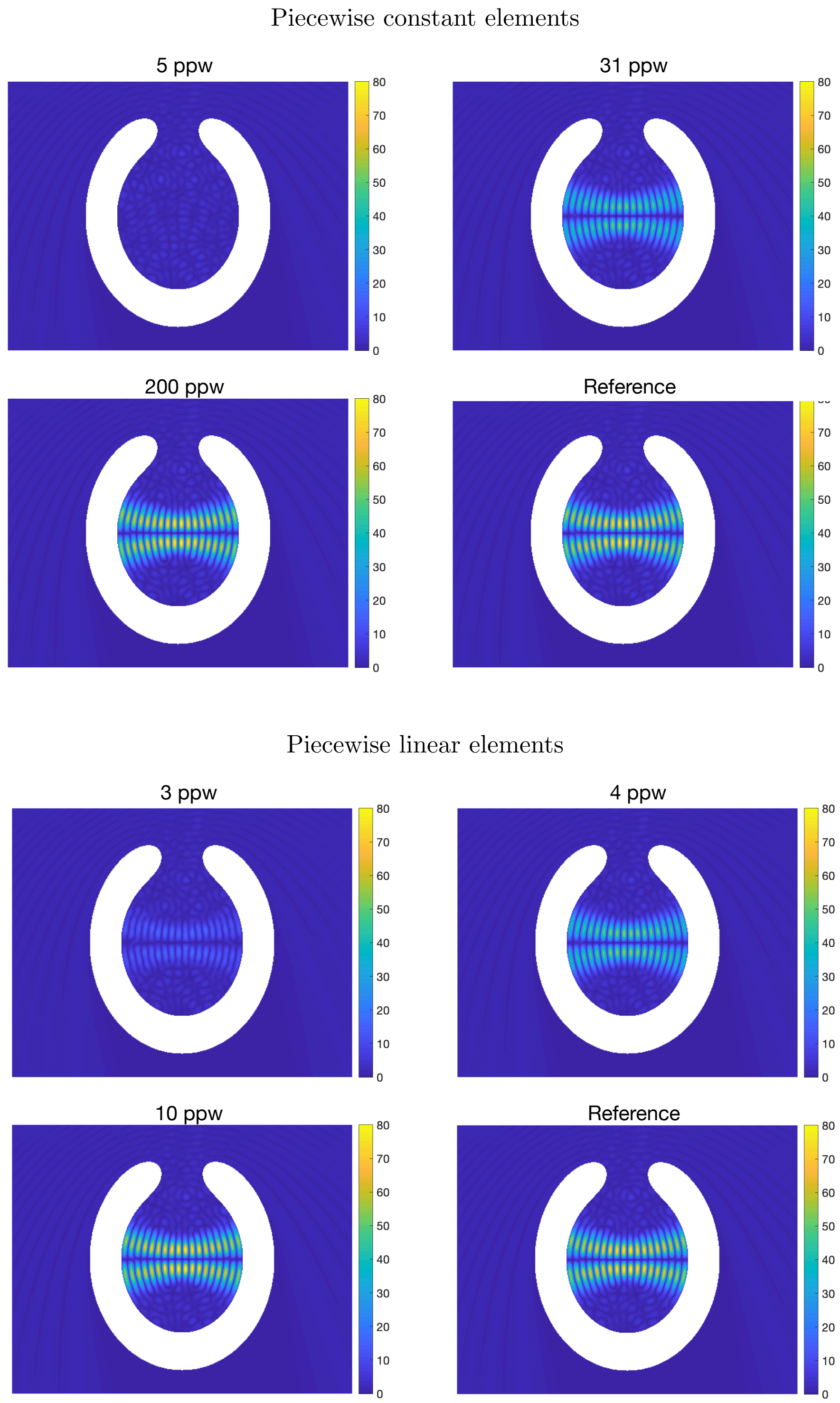

Figure 2.3: The plots show the absolute value of the total field for the reference solution (bottom right) at $k = 37.213$ when the plane wave $e^{ik\langle\omega,x\rangle}$ with $\omega = (\cos(-\pi/2 + 0.2), \sin(-\pi/2 + 0.2))$ is incident on a sound-soft cavity as well as the corresponding Galerkin solutions at various numbers of points per wavelength with both piecewise constant (top) and piecewise linear (bottom) elements. These numbers of points per wavelength correspond to points before, on, and after the peak of the quasioptimality constant in Figure 2.4. At low numbers of points per wavelength the Galerkin solutions fail almost entirely to resolve the behaviour of the total field.

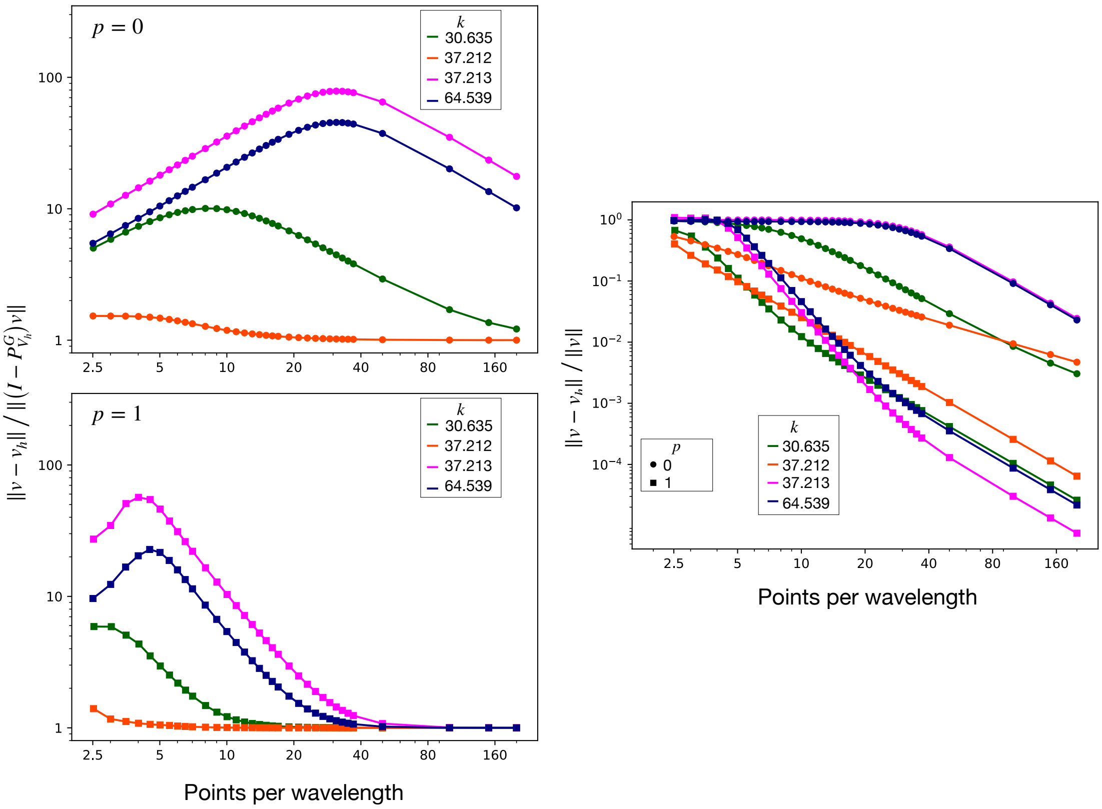

Figure 2.4: The left plots shows the implied quasioptimality constant for piecewise constant (top) and piecewise linear (bottom) elements when the plane wave $e^{ik\langle\omega,x\rangle}$ with $\omega = (\cos(-\pi/2 + 0.2), \sin(-\pi/2 + 0.2))$ is incident on a sound-soft cavity domain (see Figure 2.3). The right plot shows the corresponding L^2 relative error in the density. As in Figure 2.2, the pollution effect can be seen in both plots. While the magnitude of the effect can depend very strongly on the frequency (compare the curves at $k = 37.212$ and $k = 37.213$), the qualitative behavior is much less sensitive to the frequency chosen.

We assume that there are an odd number of points since the explicit expression for $P^C_{\mathscr{T}_N} v$ is simpler in this case (see (6.8)), but the results below also hold when there are an even number of points. In both cases the points are unisolvent by, e.g., [72, §11.3], [5, §3.2.2].

In the case of trigonometric polynomials, we change the L^2 norm used to define $P^G_{\mathscr{T}_N}$. In particular, rather than the measure induced from $\mathbb{R}^2$, we use in (2.1) the norm

$$\|u\|^2_{L^2(\Gamma)} := \int_0^{2\pi} |(u\circ\gamma)(t)|^2 dt. \tag{2.22}$$

Theorem 2.22 (Galerkin method with trigonometric polynomials) *Suppose $d = 2$, $\mathcal{A}$ is given by one of (1.2) or (1.3), and Assumptions 1.1 and 1.3 hold. Let V_N be given by (2.20), $k_0 > 0$, and $\epsilon > 0$. Then there exist $C_1, C_2 > 0$, such that if $\Xi \geq 1$,*

$$N \geq (\Xi c_{\max} + \epsilon)k \tag{2.23}$$

and $k \geq k_0$, $k \notin \mathcal{J}$, then the Galerkin solution, v_N to (1.1) exists, is unique, and satisfies the quasi-optimal error bound

$$\|v - v_N\|_{L^2(\Gamma)} \leq \left(1 + C_1\Xi^{-1} + C_2 k^{-1}\right) \left\|(I - P^G_{\mathscr{T}_N})v\right\|_{L^2(\Gamma)}. \tag{2.24}$$

Moreover, if the right-hand side f corresponds to plane-wave scattering (i.e., is given as in Theorem A.2), then

$$\frac{\|v - v_N\|_{L^2(\Gamma)}}{\|v\|_{L^2(\Gamma)}} = O(k^{-\infty}). \tag{2.25}$$

Remark 2.23 *We use the notation that $a = O(k^{-\infty})$ if a decays faster than any algebraic power of k; i.e., given $k_0, N > 0$, there exists $C(N, k_0)$ such that*

$$|a| \leq C(N, k_0)\, k^{-N} \quad \textit{for all } k \geq k_0.$$

As in the case of piecewise polynomials, our estimates are weaker for collocation than Galerkin.

Theorem 2.24 (Collocation method with trigonometric polynomials) *Suppose $d = 2$, $\mathcal{A}$ is given by one of (1.2) or (1.3), and Assumptions 1.1 and 1.3 hold. Let V_N be given by (2.20), $P^C_{\mathscr{T}_N}$ be the collocation projection (with $2N+1$ evenly-spaced points, as described above), $s > 1/2$, and $k_0 > 0$. Then there exists $C > 0$ such that if*

$$N \geq (c_{\max} + \epsilon)k \tag{2.26}$$

and $k \geq k_0$, $k \notin \mathcal{J}$, then the collocation solution, v_N, of (1.1) exists, is unique, and satisfies the quasi-optimal error bound

$$\|v - v_N\|_{H^s_k(\Gamma)} \leq \left(\|I - P^C_{\mathscr{T}_N}\|_{H^s_h \to H^s_h} + C(k/N) + Ck^{-1} + C\rho\Big(\frac{k}{N}\Big)^s\right) \left\|(I - P^C_{\mathscr{T}_N})v\right\|_{H^s_k(\Gamma)}. \tag{2.27}$$

Moreover, if the right-hand side f corresponds to plane-wave scattering (i.e., is given as in Theorem A.2), then

$$\frac{\|v - v_N\|_{H^s_k(\Gamma)}}{\|v\|_{H^s_k(\Gamma)}} = O(k^{-\infty}). \tag{2.28}$$

Remark 2.25 (The Nyquist sampling rate) *When Γ has length 2π, the number of degrees of freedom per wavelength for $\mathscr{T}_N$ (2.20) equals $(2N+1)/k$. The Nyquist–Shannon–Whittaker sampling theorem then indicates that one requires $N \geq k - 1/2$ to recover a function of frequency $\leq k$ using this many degrees of freedom. When $c_{\max} = 1$, (2.23) and (2.26) become $N \geq (1+\epsilon)k$; i.e., the Nyquist sampling rate is asymptotically sufficient to obtain existence of the Galerkin and collocation solutions.*

2.5 Results for the Nyström method with Kress quadrature

The quadrature we use for the Nyström method is based on trigonometric polynomials in $2d$ and is described in detail in § 9. This quadrature falls under the class of quadrature methods introduced by [85, 75] and often referred to as "Kress quadrature" due to its use by Kress in [70, 71].

We first consider the Dirichlet problem with the standard splitting of kS_k, K_k, K_k' described in Lemma 9.3 below. In particular, let $\mathcal{A} = \frac{1}{2}(I + L) = A_k$ or A_k' and for any $N \geq 0$, define L_N according to these splittings and Definition 9.2. The discrete approximation of L is then given by (1.5) with

$$L_V = L_{\mathscr{T}_N} = L_N \tag{2.29}$$

where $\mathscr{T}_N$ is the space of trigonometric polynomials (see (2.21)).

Theorem 2.26 (Dirichlet Nyström) *Suppose that $d = 2$ and Assumption 1.1 holds. Consider the Nyström method with Kress quadrature (defined in §9) for the Dirichlet BIE.*
(i) Suppose Γ is convex with non-vanishing curvature. Then there is $C_1 > 0$ such that for all $s > \frac{1}{2}$, there are $C > 0$, $k_0 > 0$ such that if $k \geq k_0$,

$$N \geq C_1 k$$

the solution, v_N, to (2.29) exists, is unique, and satisfies the quasi-optimal error bound

$$\|v - v_N\|_{H_\hbar^s(\Gamma)} \leq C\Big(\big\|(I - P_{\mathscr{T}_N}^C)v\big\|_{H_k^s(\Gamma)} + \big\|P_{\mathscr{T}_N}^C(L_N - L)P_{\mathscr{T}_N}^C v\big\|_{H_k^s(\Gamma)}\Big).$$

(ii) Suppose Assumption 1.3 holds with $P_{\text{inv}} = 0$. Then for all $\epsilon > 0$, there is $s_0 > 0$ and $C_1 > 0$ such that for all $s > s_0$, there are $C_2 > 0$, $k_0 > 0$ such that if $k \geq k_0$, $k \notin \mathcal{J}$,

$$N \geq C_1 k(\log k)^\epsilon$$

the solution, v_N, to (2.29) exists, is unique, and satisfies the quasi-optimal error bound

$$\|v - v_N\|_{H_\hbar^s(\Gamma)} \leq C_2\Big(\big\|(I - P_{\mathscr{T}_N}^C)v\big\|_{H_k^s(\Gamma)} + \big\|P_{\mathscr{T}_N}^C(L_N - L)P_{\mathscr{T}_N}^C v\big\|_{H_k^s(\Gamma)}\Big).$$

(iii) Suppose Assumption 1.3 holds. Then for all $\epsilon > 0$, there are $s_0 > 0$, $C_1 > 0$ such that for all $M > 0$, $s > s_0$, there are $C_2 > 0$, $k_0 > 0$ such that if $k \geq k_0$, $k \notin \mathcal{J}$,

$$N \geq C_1 k^{1+\epsilon}$$

the solution, v_N, to (2.29) exists, is unique, and satisfies the quasi-optimal error bound

$$\|v - v_N\|_{H_\hbar^s(\Gamma)} \leq C_2\Big(\|(I - P_{\mathscr{T}_N}^C)v\|_{H_k^s(\Gamma)} + \|P_{\mathscr{T}_N}^C(L_N - L)P_{\mathscr{T}_N}^C v\|_{H_k^s(\Gamma)} + k^{-M}\|v\|_{H_\hbar^s(\Gamma)}\Big).$$

Moreover, in all three cases, if the right-hand side f corresponds to plane-wave scattering (i.e., is given as in Theorem A.2), then

$$\frac{\|v - v_N\|_{H_k^s(\Gamma)}}{\|v\|_{H_k^s(\Gamma)}} = O(k^{-\infty}). \tag{2.30}$$

We next consider the Neumann problem with Kress quadrature of kS_k, K_k, K_k' described in Lemma 9.3, that for kS_{ik} K_{ik}, K_{ik}' described in Lemma 9.4, and that for $k^{-1}(H_k - H_{ik})$ described in Lemma 9.5. In particular, let $\mathcal{A} = (i\frac{\eta_N}{2} - \frac{1}{4})(I + L) = B_{k,\text{reg}}$ or $B_{k,\text{reg}}'$ and for any $N \geq 0$, define L_N according to the splittings above and Definition 9.2 with the composition $L_a L_b$ approximated by $\mathfrak{L}_a^N P_{\mathscr{T}_N}^C \mathfrak{L}_b^N$, where $P_{\mathscr{T}_N}^C$ denotes the collocation projection with trigonometric polynomials. The discrete approximation is then given by

$$L_{\mathscr{T}_N} = L_N. \tag{2.31}$$

Theorem 2.27 (Neumann Nyström) *Suppose that $d = 2$ and Assumptions 1.1 and 1.3 hold. Consider the Nyström method with Kress quadrature (defined in §9). For all $\epsilon > 0$, there are $s_0 > 0$, $C_1 > 0$ such that for all $M > 0$, $s > s_0$, there are $C_2 > 0$, $k_0 > 0$ such that if $k \geq k_0$, $k \notin \mathcal{J}$,*

$$N \geq C_1 k^{1+\epsilon}$$

the solution, v_N, to (2.31) exists, is unique, and satisfies the quasi-optimal error bound

$$\|v - v_N\|_{H^s_\hbar(\Gamma)} \leq C_2\Big(\big\|(I - P^C_{\mathscr{T}_N})v\big\|_{H^s_\hbar(\Gamma)} + \big\|P_{\mathscr{T}_N}(L_N - L)P_{\mathscr{T}_N} v\big\|_{H^s_k(\Gamma)} + k^{-M}\|v\|_{H^s_k(\Gamma)}\Big). \quad (2.32)$$

Moreover, if the right-hand side f corresponds to plane-wave scattering (i.e., is given as in Theorem A.2), then (2.30) holds.

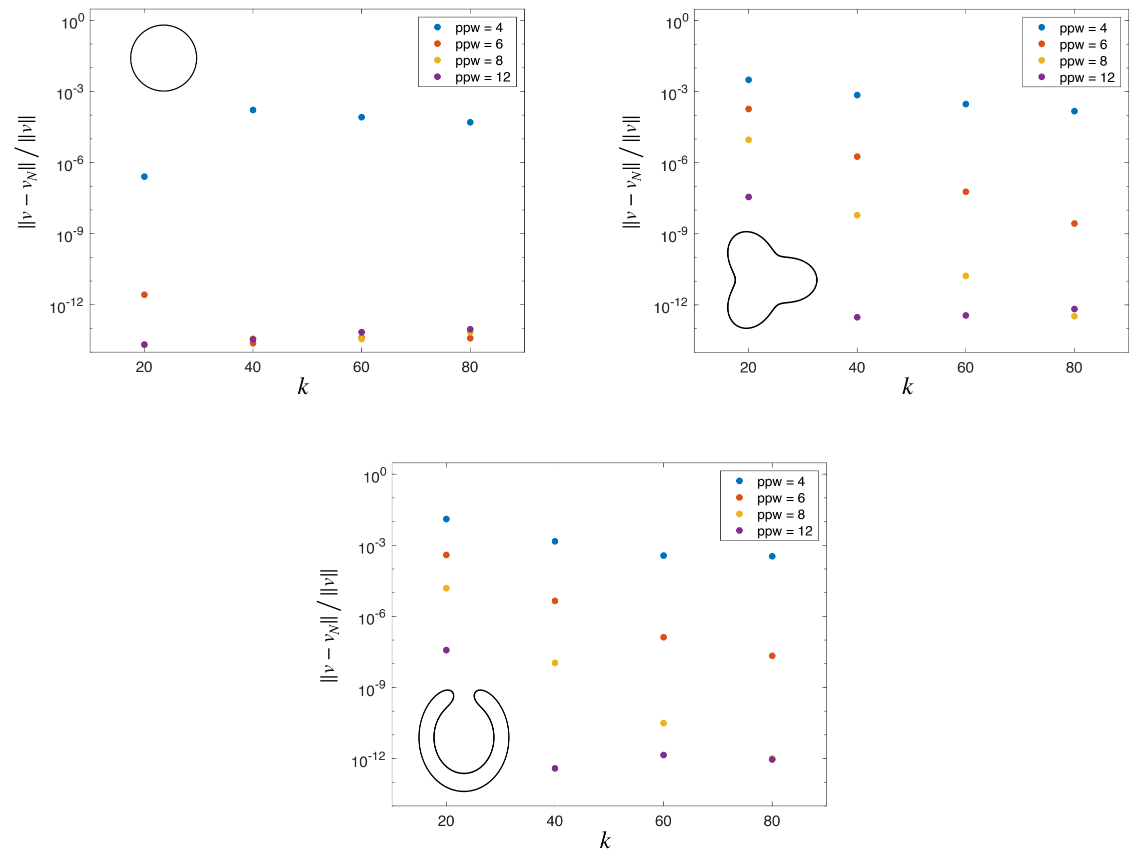

Figure 2.5: The plots show the relative L^2 error when the Nyström method is applied to the plane-wave sound-soft scattering problem in the exterior of a circle (top left), a star-shaped domain (top right), and cavity (bottom) at various frequencies and numbers of points per wavelength. As the number of points per wavelength increases or the frequency increases, the relative error decreases superalgebraically, which is consistent with Theorem 2.26.

Comparison with previous results involving trigonometric polynomials. The analysis of Galerkin, collocation, and Nystrom methods with trigonometric polynomials applied to Helmholtz BIEs on smooth boundaries in the limit $N \to \infty$ with k fixed has attracted a lot of attention in the literature; see, e.g., [73] [89] [98] [5, Chapter 7] [100, Chapters 9 and 10] and the references therein. These results prove superalgebraic convergence for smooth data, and Theorems 2.22, 2.24, 2.26, and 2.27 are their k-explicit analogues. Furthermore, e.g., [98] shows that the convergence is exponential if both the data and the boundary are analytic. For real analytic Γ, it should be possible to use methods from analytic microlocal analysis to improve Theorems 2.22, 2.24, 2.26, and 2.27 by replacing the superalgebraic decay by exponential decay, but we do not consider this here.

2.6 Discussion of the proof techniques

The k-explicit upper bounds on the projection-method errors. The current state-of-the-art analyses of k-explicit convergence of numerical methods for the Helmholtz rely on frequency splitting; specifically one splits into high ($\gg k$) and low ($\lesssim k$) frequencies and uses fact that Helmholtz solution operator is "well behaved" on high frequencies.

This idea was first used in the analysis of the hp-FEM for the constant-coefficient Helmholtz equation in [91, 92], with the frequency splitting carried out via the Fourier transform in $\mathbb{R}^d$. This philosophy was then applied to analyse the hp-BEM in [81, 90], with frequency splitting in $\mathbb{R}^d$ combined with trace operators to produce appropriate frequency splittings on Γ.

Semiclassical pseudodifferential operators give a natural and intrinsic way to achieve frequency splittings on general domains. Such operators were used to analyse the hp-FEM for the variable-coefficient Helmholtz equation in [76, 41, 42].

The first author's thesis [39] used the fact that the Helmholtz solution operator is "well behaved" (more precisely, semiclassically elliptic) on high frequencies to show that the high-frequency components of S_k, K_k, K_k', and H_k are semiclassical pseudodifferential operators. This fact was then used in [48] to show that the Galerkin method with piecewise polynomials applied to the Dirichlet BIEs does not suffer from the pollution effect when Ω^- is nontrapping. Conceptually, these results are linked to earlier convergence analyses of BIEs that used the integral operators' structure as homogeneous pseudodifferential operators to obtain results for fixed k as $N \to \infty$ (see, e.g., the books [100, 63, 55] and the references therein).

In the present paper, we use the results of [39] again to give sufficient conditions for the Galerkin solution to exist with an optimal bound on the quasioptimality constant – see Theorem 2.5. As discussed in Remark 2.6, Theorem 2.5 is analogous to the preasymptotic estimates for the h-FEM proved in [49], with these h-FEM results using frequency splittings defined by functional calculus and thus intrinsic to the operators considered, together with an elliptic-projection type argument. The results in the present paper are obtained by carefully analyzing the data to approximate solution maps from high to high, high to low, low to high, and low to low frequencies and using a norm on L^2 that is tailored to the meshwidth and the frequency splitting – see (4.34) and (4.35) below. Specifically, we weight the high frequency component of the function by $(hk)^{-t}\rho^{-1/2}$ for some $t \geq 0$ tailored to the particular projection method. This weighting allows us to take full advantage of the better behaviour of the BIEs at high frequency.

Our analyses of methods based on trigonometric polynomials use analogous frequency splittings together with the fact that trigonometric polynomials naturally respect frequency decomposition (see Lemma 7.6 below).

The proofs of pollution for the Galerkin method with piecewise polynomials. For the projection method

$$v - v_N = (I + P_N L)^{-1}(I - P_N)(I - P_N)v$$

(see (4.27) below). Therefore, to prove pollution, we need to show that $(I + P_N L)^{-1}(I - P_N)$ is large. A simple calculation shows that if

$$v := (I + L)^{-1}(I - P_N)\widetilde{f} \tag{2.33}$$

then

$$(I + P_N L)v = (I - P_N)v = (I - P_N)^2 v$$

i.e., we need to find v with $\|v\|_{L^2} = 1$ such that (2.33) holds and $(I - P_N)v$ is "small".

Under the assumption that there is a quasimode for the BIE, there is a "small" f and a u with $\|u\|_{L^2(\Gamma)} = 1$ such that $u = (I + L)^{-1}f$. Moreover, both f and u are k-oscillating and thus for the Galerkin projection with a piecewise polynomial space satisfying Assumption 2.4, $\|(I-P^G_{V_h})u\|_{L^2(\Gamma)} \le C(hk)^{p+1}\|u\|_{L^2(\Gamma)}$. If there is $\widetilde{f}$ such that $f=(I-P_N)\widetilde{f}$, then

$$\big\|(I+P^G_{V_h}L)^{-1}(I-P^G_{V_h})\big\|_{L^2(\Gamma)\to L^2(\Gamma)} \ge c(hk)^{-p-1}.$$

However, this cannot be true for all h since it contradicts the upper bound on the quasioptimality constant in Theorem 2.5; i.e., we cannot solve $f=(I-P_N)\widetilde{f}$.

Motivated by this discussion, our goal is to find $\widetilde{f}$ that is as close as possible to satisfying $(I-P_N)\widetilde{f}=f$. Notice that, since $v=(I+L)^{-1}(I-P_N)\widetilde{f}$ and $(I+L)^{-1}$ is $O(1)$ on high frequencies (see Lemma 4.9), low-frequency errors cost more than high-frequency errors. Therefore, we aim to solve $(I-P_N)\widetilde{f}=f+e$, where e has as few low-frequencies as possible.

The lower bounds on Galerkin piecewise-polynomial approximations of k-oscillating functions from [40] imply that if f has frequencies between $\epsilon_0 k$ and $\Xi_0 k$ then there exists $\widetilde{f}$ such that

$$\big\|(I-P^G_{V_h})\widetilde{f}\big\|_{L^2(\Gamma)} \le C(hk)^{-p-1}\big\|f\big\|_{L^2(\Gamma)} \tag{2.34}$$

and

$$(I-P^G_{V_h})\widetilde{f} = f + \chi_{\mathscr{L}}(I-P^G_{V_h})\widetilde{f} + \chi_{\mathscr{H}}(I-P^G_{V_h})\widetilde{f}$$

(see (12.10) and Corollary 12.8). Here $\chi_{\mathscr{L}}$ is a finite-rank operator (which can be taken to be zero when $p=0$) and

$$\big\|\chi_{\mathscr{L}}\big\|_{H^{-M}(\Gamma)\to H^M(\Gamma)} \le C_M \quad \text{and} \quad \big\|(I+L)^{-1}\chi_{\mathscr{H}}\big\|_{L^2(\Gamma)\to L^2(\Gamma)} \le C$$

(i.e., $\chi_{\mathscr{L}}$ and $\chi_{\mathscr{H}}$ are low- and high-frequency cutoffs, respectively). Thus

$$v=(I+L)^{-1}f+(I+L)^{-1}\chi_{\mathscr{L}}(I-P^G_{V_h})\widetilde{f}+(I+L)^{-1}\chi_{\mathscr{H}}(I-P^G_{V_h})\widetilde{f}. \tag{2.35}$$

In the rest of this sketch, we ignore the contribution from $\chi_{\mathscr{L}}$; including this contribution leads to the finite-dimensional condition (2.13). Using (2.35), (2.34), and the fact that $u=(I+L)^{-1}f$ (with $\|u\|_{L^2(\Gamma)}=1$), we see that $\|v\|_{L^2(\Gamma)} \ge 1/2$ if $(hk)^{-p-1}\|f\|_{L^2(\Gamma)} \ll 1$. Finally, by (2.33),

$$(I-P^G_{V_h})v=(I-P^G_{V_h})\chi_{\mathrm{H}}(I+L)^{-1}(I-P^G_{V_h})\widetilde{f}+(I-P^G_{V_h})\chi_{\mathrm{L}}v$$

where χ_{H} and χ_{L} are high- and low-frequency cutoffs, so that

$$\big\|(I-P^G_{V_h})v\big\|_{L^2(\Gamma)} \le C(hk)^{-p-1}\|f\|_{L^2(\Gamma)}+(hk)^{p+1}\|v\|_{L^2(\Gamma)}.$$

In summary,

$$\begin{aligned}\big\|(I+P^G_{V_h}L)^{-1}(I-P^G_{V_h})\big\|_{L^2(\Gamma)\to L^2(\Gamma)} &\ge \frac{\|v\|_{L^2(\Gamma)}}{\|(I-P^G_{V_h})v\|_{L^2(\Gamma)}}\\ &\ge c\begin{cases}(hk)^{-p-1}, & \|f\|_{L^2(\Gamma)} \le (hk)^{2(p+1)},\\ (hk)^{p+1}\|f\|^{-1}_{L^2(\Gamma)}, & (hk)^{2(p+1)} \le \|f\|_{L^2(\Gamma)} \ll (hk)^{p+1}.\end{cases}\end{aligned}$$

Since $\|v\|_{L^2(\Gamma)}/\|(I-P^G_{V_h})v\|_{L^2(\Gamma)} \ge 1$, by reducing c if necessary,

$$\big\|(I+P^G_{V_h}L)^{-1}(I-P^G_{V_h})\big\|_{L^2(\Gamma)\to L^2(\Gamma)} \ge \frac{1}{2}+c\begin{cases}(hk)^{-p-1}, & 1 \le (hk)^{2(p+1)}\|f\|^{-1}_{L^2(\Gamma)},\\ (hk)^{p+1}\|f\|^{-1}_{L^2(\Gamma)}, & \|f\|^{-1}_{L^2(\Gamma)}(hk)^{2(p+1)} \le 1,\end{cases}$$

which is (2.14).

2.7 Outline of the rest of the paper

§3 recaps standard results about semiclassical pseudodifferential operators.

§4 gives sufficient conditions for quasioptimality of projection method under fairly-general assumptions about $\mathcal{A}$.

§5 proves the Galerkin and collocation results for piecewise polynomials (Theorems 2.5 and 2.8).

§6 defines the projections for trigonometric polynomials in 2-d.

§7 bounds $\|(I - P^G_{\mathscr{T}_N})v\|_{H^s_k(\Gamma)}$ when v is the BIE solution corresponding to the plane-wave scattering problem.

§8 proves the Galerkin and collocation results for trigonometric polynomials (Theorems 2.22 and 2.24).

§9 defines the Nyström method analysed in Theorems 2.26 and 2.27.

§10 gives an abstract result about the convergence of the Nyström method and §11 uses this to prove Theorem 2.26 and 2.27.

§12 proves Theorem 2.10 and §13 proves Theorems 2.12 and 2.15 and §14 proves Theorems 2.16 and 2.18.

§15 describes the set up in the numerical experiments in §2.

Appendix A defines the scattering problems and BIEs in (1.2) and (1.3).

Appendix B proves the propagation-of-singularities result of Theorem 13.12.

3 Review of semiclassical pseudodifferential operators

In this section, we review standard results about semiclassical pseudodifferential operators, with our default references being [112] and [35, Appendix E]. Recall that semiclassical pseudodifferential operators are pseudodifferential operators with a large/small parameter, where behaviour with respect to this parameter is explicitly tracked in the associated calculus. In our case, the small parameter is k^{-1}, and we let $\hbar := k^{-1}$; normally the small parameter is denoted by h, but we use $\hbar$ to avoid a notational clash with the meshwidth of the h-BEM. The notation $\hbar$ is motivated by the fact that the semiclassical parameter is often related to Planck's constant, which is written as $2\pi\hbar$ see, e.g., [112, S1.2], [35, Page 82].

The counterpart of "semiclassical" involving differential/pseudodifferential operators without a small parameter is usually called "homogeneous", and the homogeneous analogues of these results can be found in, e.g., [108, Chapter 7], [100, Chapter 7], [63, Chapters 6].

3.1 Weighted Sobolev spaces

We first define weighted Sobolev spaces on $\mathbb{R}^d$, and then use these to define analogous weighted Sobolev spaces on Γ. The *semiclassical Fourier transform* is defined by

$$(\mathcal{F}_\hbar u)(\xi) := \int_{\mathbb{R}^d} \exp\big(-\mathrm{i}x\cdot\xi/\hbar\big)u(x)\,dx,$$

with inverse

$$(\mathcal{F}_\hbar^{-1}u)(x) := (2\pi\hbar)^{-d}\int_{\mathbb{R}^d} \exp\big(\mathrm{i}x\cdot\xi/\hbar\big)u(\xi)\,d\xi; \tag{3.1}$$

see [112, §3.3]; i.e., the semiclassical Fourier transform is just the usual Fourier transform with the transform variable scaled by $\hbar$. These definitions imply that, with $D := -\mathrm{i}\partial$,

$$\mathcal{F}_\hbar\big((\hbar D)^\alpha)u\big) = \xi^\alpha \mathcal{F}_\hbar u \quad \text{ and } \quad \|u\|_{L^2(\mathbb{R}^d)} = \frac{1}{(2\pi\hbar)^{d/2}}\|\mathcal{F}_\hbar u\|_{L^2(\mathbb{R}^d)};$$

see, e.g., [112, Theorem 3.8]. Let

$$H^s_\hbar(\mathbb{R}^d) := \Big\{u\in\mathcal{S}'(\mathbb{R}^d) \text{ such that } \langle\xi\rangle^s(\mathcal{F}_\hbar u)\in L^2(\mathbb{R}^d)\Big\}, \tag{3.2}$$

where $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$, $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz space (see, e.g., [88, Page 72]), and $\mathcal{S}'(\mathbb{R}^d)$ its dual. Define the norm

$$\|u\|^2_{H^m_\hbar(\mathbb{R}^d)} = \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\langle\xi\rangle^{2m}|\mathcal{F}_\hbar u(\xi)|^2\,d\xi; \tag{3.3}$$

for example, with $m=1$,

$$\|u\|^2_{H^1_\hbar(\mathbb{R}^d)} = \hbar^2\,\|\nabla u\|^2_{L^2(\mathbb{R}^d)} + \|u\|^2_{L^2(\mathbb{R}^d)} = k^{-2}\,\|\nabla u\|^2_{L^2(\mathbb{R}^d)} + \|u\|^2_{L^2(\mathbb{R}^d)}\,. \tag{3.4}$$

Working in a weighted H^1 norm with the derivative weighted by k^{-1} is ubiquitous in the literature on the numerical analysis of the Helmholtz equation, except that usually one works with the weighted H^1 norm squared being $\|\nabla u\|^2_{L^2}+k^2\|u\|_{L^2}$. Here we work with (3.4)/(3.3) since weighting the jth derivative by k^{-j} is easier to keep track of than weighting it by k^{-j+1}, especially when working with higher-order derivatives.

We define the norm for weighted Sobolev spaces on Γ as follows

$$\|u\|_{H^s_\hbar(\Gamma)} := \|(1-\hbar^2\Delta_\Gamma)^{s/2}u\|_{L^2(\Gamma)}, \tag{3.5}$$

where Δ_Γ denotes the Laplacian on Γ and we use the spectral theorem to define powers of $(1-\hbar^2\Delta_\Gamma)$. For example,

$$\|u\|^2_{H^1_\hbar(\Gamma)} = \hbar^2\,\|\nabla_\Gamma u\|^2_{L^2(\Gamma)} + \|u\|^2_{L^2(\Gamma)}\,,$$

where ∇_Γ is the surface gradient operator, defined in terms of a parametrisation of the boundary by, e.g., [26, Equation A.14]. Note that these same norms can be defined via interpolation from the integer powers of $(1-\hbar^2\Delta_\Gamma)$. The weighted spaces $H^s_\hbar(\Gamma)$ can also be defined by charts; see, e.g., [88, Pages 98 and 99] for the unweighted case and [93, §5.6.4] or [35, Definition E.20] for the weighted case (but note that [93, §5.6.4] uses a different weighting with k to us); these other definitions all lead to equivalen norms.

The unweighted $H^s(\Gamma)$ norms are defined as above with $\hbar=1$. We use below that, given $\hbar_0>0$ and $s>0$, there exists $C>0$ such that, for all $0<\hbar\le\hbar_0$,

$$\hbar^s\,\|w\|_{H^s(\Gamma)} \le C\,\|w\|_{H^s_\hbar(\Gamma)}\,. \tag{3.6}$$

3.2 Phase space, symbols, quantisation, and semiclassical pseudodifferential operators

For simplicity of exposition, we begin by discussing semiclassical pseudodifferential operators on $\mathbb{R}^d$, and then outline in §3.4 below how to extend the results from $\mathbb{R}^d$ to Γ.

The set of all possible positions x and momenta (i.e. Fourier variables) ξ is denoted by $T^*\mathbb{R}^d$; this is known informally as "phase space". Strictly, $T^*\mathbb{R}^d := \mathbb{R}^d\times(\mathbb{R}^d)^*$, i.e. the cotangent bundle to $\mathbb{R}^d$, but for our purposes, we can consider $T^*\mathbb{R}^d$ as $\{(x,\xi): x\in\mathbb{R}^d, \xi\in\mathbb{R}^d\}$.

A symbol is a function on $T^*\mathbb{R}^d$ that is also allowed to depend on $\hbar$, and can thus be considered as an $\hbar$-dependent family of functions. Such a family $a=(a_\hbar)_{0<\hbar\le\hbar_0}$, with $a_\hbar\in C^\infty(T^*\mathbb{R}^d)$, is a *symbol of order m*, written as $a\in S^m(T^*\mathbb{R}^d)$, if for any multiindices α,β

$$|\partial_x^\alpha\partial_\xi^\beta a_\hbar(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{m-|\beta|} \quad \text{for all } (x,\xi)\in T^*\mathbb{R}^d \text{ and for all } 0<\hbar\le\hbar_0, \tag{3.7}$$

(where recall that $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$) and $C_{\alpha,\beta}$ does not depend on $\hbar$; see [112, p. 207], [35, §E.1.2].

For $a\in S^m$, we define the *semiclassical quantisation* of a, denoted by $a(x,\hbar D):\mathcal{S}(\mathbb{R}^d)\to\mathcal{S}(\mathbb{R}^d)$, by

$$a(x,\hbar D)v(x) := (2\pi\hbar)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\exp\big(\mathrm{i}(x-y)\cdot\xi/\hbar\big)\,a(x,\xi)v(y)\,dyd\xi \tag{3.8}$$

where $D:=-\mathrm{i}\partial$; see, e.g., [112, §4.1] [35, Page 543]. We also write $a(x,\hbar D)=\mathrm{Op}_\hbar(a)$. The integral in (3.8) need not converge, and can be understood *either* as an oscillatory integral in the sense of [112, §3.6], [61, §7.8], *or* as an iterated integral, with the y integration performed first; see [35, Page 543].

Conversely, if A can be written in the form above, i.e. $A=a(x,\hbar D)$ with $a\in S^m(T^*\mathbb{R}^d)$, we say that A is a *semiclassical pseudo-differential operator of order m* and we write $A\in\Psi^m_\hbar(\mathbb{R}^d)$. We use the notation $a\in\hbar^l S^m$ if $\hbar^{-l}a\in S^m$; similarly $A\in\hbar^l\Psi^m_\hbar$ if $\hbar^{-l}A\in\Psi^m_\hbar$. We define $\Psi^{-\infty}_\hbar := \cap_m\Psi^{-m}_\hbar$.

Theorem 3.1 (Composition and mapping properties of semiclassical pseudo-differential operators [112, Theorem 8.10], [35, Propositions E.17, E.19, and E.24]) *If $A \in \Psi_\hbar^{m_1}$ and $B \in \Psi_\hbar^{m_2}$, then*

(i) $AB \in \Psi_\hbar^{m_1+m_2}$.

(ii) For any $s \in \mathbb{R}$, A is bounded uniformly in $\hbar$ as an operator from $H_\hbar^s$ to $H_\hbar^{s-m_1}$.

A key fact we use below is that if $\psi \in C_c^\infty(\mathbb{R})$ then, given $t \in \mathbb{R}$, $N > 0$ and $\hbar_0 > 0$ there exists $C > 0$ such that for all $\hbar \leq \hbar_0$,

$$\left\|\psi(|\hbar D|^2)\right\|_{H_\hbar^t(\mathbb{R}^d) \to H_\hbar^{t+N}(\mathbb{R}^d)} \leq C; \tag{3.9}$$

this can easily be proved using the semiclassical Fourier transform, since $\psi(|\hbar D|^2)$ is a Fourier multiplier (i.e., $\psi(|\hbar D|^2)$ is defined by (3.8) with $a(x,\xi) = \psi(|\xi|^2)$, which is independent of x).

3.3 The principal symbol map $\sigma_\hbar$

Let the quotient space $S^m/\hbar S^{m-1}$ be defined by identifying elements of S^m that differ only by an element of $\hbar S^{m-1}$. For any m, there is a linear, surjective map

$$\sigma_\hbar^m : \Psi_\hbar^m \to S^m/\hbar S^{m-1},$$

called the *principal symbol map*, such that, for $a \in S^m$,

$$\sigma_\hbar^m\big(\mathrm{Op}_\hbar(a)\big) = a \quad \mod \hbar S^{m-1}; \tag{3.10}$$

see [112, Page 213], [35, Proposition E.14] (observe that (3.10) implies that $\ker(\sigma_\hbar^m) = \hbar\Psi_\hbar^{m-1}$). When applying the map $\sigma_\hbar^m$ to elements of $\Psi_\hbar^m$, we denote it by $\sigma_\hbar$ (i.e. we omit the m dependence) and we use $\sigma_\hbar(A)$ to denote one of the representatives in S^m (with the results we use then independent of the choice of representative).

Key properties of the principal symbol that we use below are that

$$\sigma_\hbar(AB) = \sigma_\hbar(A)\sigma_\hbar(B) \quad \text{and} \quad \sigma_\hbar(A^*) = \overline{\sigma_\hbar(A)}, \tag{3.11}$$

see [35, Proposition E.17], and for $A \in \Psi_\hbar^0$,

$$\|A\|_{L^2 \to L^2} \leq \sup|\sigma_\hbar(A)| + C\hbar, \tag{3.12}$$

see [112, Theorem 13.13].

3.4 Extension of the above results from $\mathbb{R}^d$ to Γ

While the definitions above are written for operators on $\mathbb{R}^d$, semiclassical pseudodifferential operators and all of their properties above have analogues on compact manifolds (see e.g. [112, §14.2], [35, §E.1.7]). Roughly speaking, the class of semiclassical pseudodifferential operators of order m on a compact manifold Γ, $\Psi_\hbar^m(\Gamma)$, are operators that, in any local coordinate chart, have kernels of the form (3.8) where the function $a \in S^m$ modulo a remainder operator R that has the property that

$$\|R\|_{H_\hbar^{-N}(\Gamma) \to H_\hbar^N(\Gamma)} \leq C_N \hbar^N; \tag{3.13}$$

we say that an operator R satisfying (3.13) is $O(\hbar^\infty)_{\Psi_\hbar^{-\infty}(\Gamma)}$.

Semiclassical pseudodifferential operators on manifolds continue to have a natural principal symbol map

$$\sigma_\hbar : \Psi_\hbar^m \to S^m(T^*\Gamma)/\hbar S^{m-1}(T^*\Gamma) \tag{3.14}$$

where now $S^m(T^*\Gamma)$ is the class of functions on $T^*\Gamma$, the cotangent bundle of Γ, that satisfy the estimate (3.7) with x replaced by a local coordinate variable x' and ξ replaced by ξ' – the dual variables to x'. The property (3.11) holds as before. Furthermore, there is a noncanonical quantisation map $\mathrm{Op}_\hbar : S^m(T^*\Gamma) \to \Psi^m(\Gamma)$ (involving choices of cut-off functions and coordinate charts) that satisfies

$$\sigma_\hbar(\mathrm{Op}_\hbar(a)) = a.$$

and for all $A \in \Psi^m_\hbar(\Gamma)$, there is $a \in S^m(T^*\Gamma)$ such that

$$A = \mathrm{Op}_\hbar(a) + O(\hbar^\infty)_{\Psi_\hbar^{-\infty}}.$$

Let g be the metric induced on $T^*\Gamma$ from the standard metric on $\mathbb{R}^d$. Then, in exact analogy with (3.9), if $\psi \in C_c^\infty(\mathbb{R})$ then, given $t \in \mathbb{R}$, $N > 0$ and $\hbar_0 > 0$ there exists $C > 0$ such that for all $0 < \hbar \le \hbar_0$,

$$\left\|\psi(|\hbar D'|_g^2)\right\|_{H_\hbar^t(\Gamma)\to H_\hbar^{t+N}(\Gamma)} \le C, \tag{3.15}$$

where $D' := -\mathrm{i}\partial_{x'}$.

Finally, we record the following consequence of (3.12) for bounds on $H^s_\hbar(\Gamma)$ norms.

Lemma 3.2 *Suppose that $A \in \Psi^m_\hbar(\Gamma)$. Then, for all $s \in \mathbb{R}$ there is $C_s > 0$ such that*

$$\|A\|_{H_\hbar^s(\Gamma)\to H_\hbar^{s-m}(\Gamma)} \le \sup_{(x',\xi')\in T^*\Gamma} |\sigma(A)(x',\xi')(1+|\xi'|_g^2)^{-m/2}| + C_s\hbar.$$

Proof. By (3.5),

$$\begin{aligned}
\|Au\|_{H_\hbar^{s-m}(\Gamma)} &= \|(1-\hbar^2\Delta_\Gamma)^{(s-m)/2}Au\|_{L^2(\Gamma)} \\
&= \|(1-\hbar^2\Delta_\Gamma)^{(s-m)/2}A(1-\hbar^2\Delta_\Gamma)^{-s/2}(1-\hbar^2\Delta_\Gamma)^{s/2}u\|_{L^2(\Gamma)} \\
&\le \|(1-\hbar^2\Delta_\Gamma)^{(s-m)/2}A(1-\hbar^2\Delta_\Gamma)^{-s/2}\|_{L^2(\Gamma)\to L^2(\Gamma)}\|u\|_{H_\hbar^s(\Gamma)}.
\end{aligned}$$

Now, $(1-\hbar^2\Delta_\Gamma)^{s/2} \in \Psi^s_\hbar(\Gamma)$ with symbol $(1+|\xi|_g^2)^{s/2}$ and hence, by (3.12),

$$\|(1-\hbar^2\Delta_\Gamma)^{(s-m)/2}A(1-\hbar^2\Delta_\Gamma)^{-s/2}\|_{L^2(\Gamma)\to L^2(\Gamma)} \le \sup_{(x,\xi)\in T^*\Gamma}(1+|\xi'|_g^2)^{-m/2}|\sigma(A)|(x',\xi') + C_s\hbar,$$

as claimed. ∎

3.5 Wavefront set and ellipticity

Definition 3.3 (Ellipticity) *$B \in \Psi^\ell_\hbar(\Gamma)$ is* elliptic *on a set $U \subset T^*\Gamma$ if*

$$\liminf_{\hbar\to 0}\inf_{(x',\xi')\in U}\left|\sigma_\hbar(B)(x',\xi')\langle\xi'\rangle^{-\ell}\right| > 0, \tag{3.16}$$

where $\langle\xi'\rangle := (1+|\xi'|_g^2)^{\frac{1}{2}}$.

Definition 3.4 (Wavefront set of a pseudodifferential operator) *The wavefront set $\mathrm{WF}_\hbar(A)$ of $A \in \Psi^m_\hbar(\Gamma)$ is defined as follows: $(x_0,\xi_0) \in (\mathrm{WF}_\hbar(A))^c$ if there exists $B \in \Psi^{-m}_\hbar(\Gamma)$, elliptic in a neighbourhood of (x_0,ξ_0) such that*

$$BA = O(\hbar^\infty)_{\Psi_\hbar^{-\infty}(\Gamma)}.$$

We make three remarks.

(i) Definition 3.4 implies that

$$\mathrm{WF}_\hbar(A) = \emptyset \quad\Longleftrightarrow\quad A = O(\hbar^\infty)_{\Psi_\hbar^{-\infty}(\Gamma)}. \tag{3.17}$$

(ii) Strictly speaking, $\mathrm{WF}_\hbar(A)$ is a subset of the fiber-radially compactified cotangent bundle $\overline{T}^*\Gamma$ (see [35, §E.1.3]), but this notion is not important in what follows.

(iii) One can show that Definition 3.4 is equivalent to [35, Definition E.27] by using the composition formula [35, E.1.21]. This composition formula also implies that

$$\mathrm{WF}_\hbar(AB) \subset \mathrm{WF}_\hbar(A) \cap \mathrm{WF}_\hbar(B) \tag{3.18}$$

and if a is independent of $\hbar$ then

$$\mathrm{WF}_\hbar\big(\mathrm{Op}_\hbar(a)\big) \subset \operatorname{supp} a. \tag{3.19}$$

The related adjoint formula [35, E.1.22] implies that

$$\mathrm{WF}_\hbar(A^*) = \mathrm{WF}_\hbar(A) \tag{3.20}$$

(compare (3.18) and (3.20) to (3.11)).

We use repeatedly below the corollary of (3.17) and (3.18) that

$$\text{if} \quad \mathrm{WF}_\hbar(A) \cap \mathrm{WF}_\hbar(B) = \emptyset \quad \text{then} \quad AB = O(\hbar^\infty)_{\Psi_\hbar^{-\infty}(\Gamma)}; \tag{3.21}$$

i.e., pseudodifferential operators act microlocally (i.e., pseudo locally in phase space).

Theorem 3.5 (Elliptic parametrix) *Suppose that $A \in \Psi_\hbar^m(\Gamma)$ and that $B \in \Psi_\hbar^\ell(\Gamma)$ is elliptic on $\mathrm{WF}_\hbar(A)$. Then there exist $Q, Q' \in \Psi_\hbar^{m-\ell}(\Gamma)$ such that*

$$A = BQ + O(\hbar^\infty)_{\Psi_\hbar^{-\infty}(\Gamma)} = Q'B + O(\hbar^\infty)_{\Psi_\hbar^{-\infty}(\Gamma)}.$$

Reference for the proof. This is proved in [35, Proposition E.32]. ■

Corollary 3.6 (Elliptic estimates) *(i) If $B \in \Psi_\hbar^\ell(\Gamma)$ is* elliptic *on $T^*\Gamma$ then there exists $\hbar_0 > 0$ such that, for all $0 < \hbar \le \hbar_0$, $B^{-1} : H_\hbar^{s-\ell}(\Gamma) \to H_\hbar^s(\Gamma)$ exists and is bounded (with norm independent of $\hbar$) for all s. Furthermore, if $B \in \Psi_\hbar^0$, then $B^{-1} \in \Psi_\hbar^0$ and*

$$\sigma_\hbar^0(B^{-1}) = (\sigma_\hbar^0(B))^{-1}.$$

(ii) Suppose that $A \in \Psi_\hbar^m(\Gamma)$ and that $B \in \Psi_\hbar^\ell(\Gamma)$ is elliptic on $\mathrm{WF}_\hbar(A)$. Then, given $s, M, N \in \mathbb{R}$, there exists $C, \hbar_0 > 0$ such that, for all $0 < \hbar \le \hbar_0$,

$$\|Au\|_{H_\hbar^{s-m}(\Gamma)} \le C\Big(\|Bu\|_{H_\hbar^{s-m-\ell}(\Gamma)} + \hbar^M \|u\|_{H_\hbar^{s-m-N}(\Gamma)}. \Big) \tag{3.22}$$

Proof. The first statement in Part (i) follows from Part (ii) with $A = I$, $B = B$, and $A = I$, $B = B^*$. In both cases, the $O(\hbar^M)$ error term on the right-hand side is absorbed on the left-hand side, making $\hbar_0$ smaller if necessary, so that the bound (3.22) is just the statement of injectivity of B and its analogue with $B = B^*$ the statement of surjectivity of B. For the statement about $B \in \Psi_\hbar^0$, observe that by Theorem 3.5, there is $Q \in \Psi_\hbar^0(\Gamma)$ such that

$$I = QB + R, \qquad R = O(\hbar^\infty)_{\Psi_\hbar^{-\infty}(\Gamma)}.$$

In particular, since $(I-R)^{-1} = I + R(I-R)^{-1}$, $B^{-1} = Q + R(I-R)^{-1}Q = Q + O(\hbar^\infty)_{\Psi_\hbar^{-\infty}(\Gamma)} \in \Psi_\hbar^0(\Gamma)$. Furthermore, the symbol formula (3.11) then implies that $\sigma_\hbar^0(B^{-1}) = 1/\sigma_\hbar^0(B)$ as claimed.

Part (ii) follows from Theorem 3.5 and the definition (3.13) of $O(\hbar^\infty)_{\Psi_\hbar^{-\infty}(\Gamma)}$. ■

3.6 Recap of the results of [39, 45]

The key ingredient for the proof of the main results is the following result from [45, Theorem 4.3], adapted from the results in [39, Chapter 4].

Theorem 3.7 (The high-frequency components of S_k, K_k, K_k', and H_k are semiclassical pseudodifferential operators) *Let $\chi \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}(1-\chi) \cap [-1,1] = \emptyset$. Then*

$$\begin{gathered}
\big(I - \chi(|\hbar D'|_g^2)\big)S_k,\ \ S_k\big(I - \chi(|\hbar D'|_g^2)\big) \in \hbar\Psi_\hbar^{-1}(\Gamma),\\
\big(I - \chi(|\hbar D'|_g^2)\big)K_k',\ \ K_k'\big(I - \chi(|\hbar D'|_g^2)\big) \in \hbar\Psi_\hbar^{-1}(\Gamma),\\
\big(I - \chi(|\hbar D'|_g^2)\big)K_k,\ \ K_k\big(I - \chi(|\hbar D'|_g^2)\big) \in \hbar\Psi_\hbar^{-1}(\Gamma),\\
\big(I - \chi(|\hbar D'|_g^2)\big)H_k,\ \ H_k\big(I - \chi(|\hbar D'|_g^2)\big) \in \hbar^{-1}\Psi_\hbar^{1}(\Gamma).
\end{gathered}$$

Moreover,

$$\sigma_\hbar\big((I-\chi(|k^{-1}D'|^2_g))S_k\big)=\sigma_\hbar\big(S_k(1-\chi(|k^{-1}D'|^2_g)\big)=\frac{\big(1-\chi(|\xi'|^2_g)\big)}{2k\sqrt{|\xi'|^2_g-1}} \tag{3.23}$$

and

$$\sigma_\hbar\big((I-\chi(|k^{-1}D'|^2_g))H_k\big)=\sigma_\hbar\big(H_k(1-\chi(|k^{-1}D'|^2_g)\big)=-\big(1-\chi(|\xi'|^2_g)\big)\frac{k\sqrt{|\xi'|^2_g-1}}{2}. \tag{3.24}$$

Remark 3.8 *The statement of Theorem 3.7 in [45, Theorem 4.4] differs from Theorem 3.7 in the following two ways. First, [45, Theorem 4.4] has $\chi\in C^\infty_c(\mathbb{R})$ satisfying* $\operatorname{supp}(1-\chi)\cap[-2,2]=\emptyset$*; however, the result holds under the condition that* $\operatorname{supp}(1-\chi)\cap[-1,1]=\emptyset$ *(with this condition appearing in the key ingredient for the proof of [45, Theorem 4.4], namely [45, Lemma 4.1]). Second, [45, Theorem 4.4] is stated with high-frequency cut-offs of the form $1-\chi(|\hbar D'|^2_g)$, whereas above we have $1-\chi(|\hbar D'|^2_g)$ – either choice is possible, since $|\xi'|_g=1$ iff $|\xi'|^2_g=1$.*

3.7 Pseudodifferential Properties of functions of $-\hbar^2\Delta_\Gamma$

Theorem 3.7 is stated with the high-frequency cutoffs given by $(I-\chi(|\hbar D'|^2_g))$ where $\chi\in C^\infty_c(\mathbb{R})$ with $\operatorname{supp}(1-\chi)\cap[-1,1]=\emptyset$. The arguments in the rest of the paper also use high-frequency cutoffs defined in terms of functions of $-\hbar^2\Delta_\Gamma$; i.e., $(I-\chi(-\hbar^2\Delta_\Gamma))$ for χ as above. The following results show how one can replace instances of $(I-\chi(-\hbar^2\Delta_\Gamma))$ by $(I-\widetilde{\chi}(|\hbar D'|^2_g))$, where $\widetilde{\chi}$ is smaller than χ. Note that this replacement was used in [48] to present the results of Theorem 3.7 without explicitly using pseudodifferential operators; see [45, Remark 4.6].

Lemma 3.9 (Pseudodifferential properties of $(I-\chi(-\hbar^2\Delta_\Gamma))$) *If $\chi\in C^\infty_c(\mathbb{R})$ with* $\operatorname{supp}(1-\chi)\cap[-\Xi,\Xi]=\emptyset$*, then $(I-\chi(-\hbar^2\Delta_\Gamma))\in\Psi^0_\hbar(\Gamma)$ with*

$$\mathrm{WF}_\hbar\big(I-\chi(-\hbar^2\Delta_\Gamma)\big)\subset\overline{\big\{(x',\xi'):1-\chi(|\xi'|^2_g)>0\big\}}\subset\big\{(x',\xi'):|\xi'|^2_g>\Xi\big\}.$$

Proof. First recall from the Helffer-Sjöstrand functional calculus (see e.g. [112, Theorem 14.9]) that $\chi(-\hbar^2\Delta_\Gamma)\in\Psi^{-\infty}_\hbar(\Gamma)$ which implies $(I-\chi(-\hbar^2\Delta_\Gamma))\in\Psi^0_\hbar(\Gamma)$. By the definition of the wavefront set (Definition 3.4), to prove the lemma, we need to show that for any $(x'_0,\xi'_0)\in T^*\Gamma$ such that $\chi(|\xi'_0|^2_{g(x'_0)})=1$, there is $A\in\Psi^0_\hbar(T^*\Gamma)$ elliptic at (x_0,ξ_0) such that

$$\|A(I-\chi(-\hbar^2\Delta_\Gamma))v\|_{H^N_\hbar}\le C_N\hbar^N\|v\|_{H^{-N}_\hbar}. \tag{3.25}$$

To do this, fix such an (x'_0,ξ'_0) and let $\widetilde{\chi}\in C^\infty_c(\mathbb{R})$ with $\operatorname{supp}\widetilde{\chi}\cap\operatorname{supp}(1-\chi)=\emptyset$ and $\widetilde{\chi}(|\xi'_0|^2_{g(x'_0)})=1$. Then, by [112, Theorem 14.9] again, $\widetilde{\chi}(-\hbar^2\Delta_\Gamma)\in\Psi^{-\infty}_\hbar(T^*\Gamma)$ with principal symbol

$$\sigma_\hbar\big(\widetilde{\chi}(-\hbar^2\Delta_\Gamma)\big)=\widetilde{\chi}(|\xi'|^2_g).$$

We now show that (3.25) is satisfied with $A=\widetilde{\chi}(-\hbar^2\Delta_\Gamma)$. Indeed, By the definition of $\widetilde{\chi}$,

$$\widetilde{\chi}(-\hbar^2\Delta_\Gamma)\big(I-\chi(-\hbar^2\Delta_\Gamma)\big)=0,$$

so (3.25) certainly holds. Furthermore, since $\widetilde{\chi}(|\xi'_0|_{g(x'_0)})=1$, $A=\widetilde{\chi}(-\hbar^2\Delta_\Gamma)$ is elliptic at (x'_0,ξ'_0). Since (x'_0,ξ'_0) with $\chi(|\xi'_0|_{g(x'_0)})=1$ was arbitrary, the proof is complete. ∎

Corollary 3.10 (Replacing $I-\chi(-\hbar^2\Delta_\Gamma)$ by $(I-\widetilde{\chi}(|\hbar D'|^2_g))$) *Suppose $\chi,\widetilde{\chi}\in C^\infty_c(\mathbb{R})$ with both* $\operatorname{supp}(1-\chi)\cap[-\Xi,\Xi]=\emptyset$ *and* $\operatorname{supp}(1-\widetilde{\chi})\cap[-\Xi,\Xi]=\emptyset$ *and* $\operatorname{supp}\widetilde{\chi}\cap\operatorname{supp}(1-\chi)=\emptyset$*. Then*

$$\begin{aligned}I-\chi(-\hbar^2\Delta_\Gamma)&=\big(I-\widetilde{\chi}(|\hbar D'|^2_g)\big)\big(I-\chi(-\hbar^2\Delta_\Gamma)\big)+O(\hbar^\infty)_{\Psi^{-\infty}_\hbar(\Gamma)},\\ I-\chi(-\hbar^2\Delta_\Gamma)&=\big(I-\chi(-\hbar^2\Delta_\Gamma)\big)\big(I-\widetilde{\chi}(|\hbar D'|^2_g)\big)+O(\hbar^\infty)_{\Psi^{-\infty}_\hbar(\Gamma)}.\end{aligned} \tag{3.26}$$

Proof of Corollary 3.10. By Lemma 3.9 and the assumption that $\operatorname{supp}\widetilde{\chi}\cap\operatorname{supp}(1-\chi)=\emptyset$,

$$\mathrm{WF}_\hbar\big(I-\chi(-\hbar^2\Delta_\Gamma)\big)\cap\mathrm{WF}_\hbar(\widetilde{\chi}(|\hbar D'|_g^2))=\emptyset$$

Therefore,

$$\begin{aligned}\big(I-\widetilde{\chi}(|\hbar D'|_g^2)\big)\big(I-\chi(-\hbar^2\Delta_\Gamma)\big)&=\big(I-\chi(-\hbar^2\Delta_\Gamma)\big)+\widetilde{\chi}(|\hbar D'|_g^2)\big(I-\chi(-\hbar^2\Delta_\Gamma)\big)\\&=\big(I-\chi(-\hbar^2\Delta_\Gamma)\big)+O(\hbar^\infty)_{\Psi_\hbar^{-\infty}(\Gamma)},\end{aligned}$$

and

$$\begin{aligned}\big(I-\chi(-\hbar^2\Delta_\Gamma)\big)\big(I-\widetilde{\chi}(|\hbar D'|_g^2)\big)&=\big(I-\chi(-\hbar^2\Delta_\Gamma)\big)+\big(I-\chi(-\hbar^2\Delta_\Gamma)\big)\widetilde{\chi}(|\hbar D'|_g^2)\\&=\big(I-\chi(-\hbar^2\Delta_\Gamma)\big)+O(\hbar^\infty)_{\Psi_\hbar^{-\infty}(\Gamma)}.\end{aligned}$$

■

4 Convergence of projection methods in an abstract framework

4.1 New abstract result on convergence of the projection method

Assumption 4.1 (Assumptions on $\mathcal{A}$ and its high-frequency components)
(i) $\mathcal{A}:=I+L$ is bounded and invertible on $H^s(\Gamma)$ for all $s\in\mathbb{R}$, with $L:H^s(\Gamma)\to H^{s+1}(\Gamma)$.

(ii) For any $\chi\in C_c^\infty(\mathbb{R};[0,1])$ with $\operatorname{supp}(1-\chi)\cap[-1,1]=\emptyset$, $(I-\chi(|\hbar D'|_g^2))L$ and $L(I-\chi(|\hbar D'|_g^2))$ are both in $\Psi_\hbar^{-1}(\Gamma)$.

(iii) There is $L_{\max}>0$ such that for any $\chi\in C_c^\infty(\mathbb{R};[0,1])$ with $\operatorname{supp}(1-\chi)\cap[-1,1]=\emptyset$,

$$\sup_{T^*\Gamma}\big|1+\sigma_\hbar^0\big((I-\chi(|\hbar D'|_g^2))L\big)(x',\xi')\big|^{-1}\leq L_{\max}$$

Theorem 4.2 (New abstract result on convergence of the projection method) *Suppose that $\mathcal{A}$ satisfies Assumptions 4.1 and 1.3. Let $k_0>0$, $C_1>0$, let V_N be a finite-dimensional subspace of $H_\hbar^s(\Gamma)$ and let $P_N:H_\hbar^s(\Gamma)\to V_N$ be a projection satisfying*

$$\|(I-P_N)\|_{H_\hbar^t\to H_\hbar^q}\leq C_1\left(\frac{k}{N}\right)^{t-q}\tag{4.1}$$

for all $N\geq k$, and (t,q) equal to each of

$$(t_{\max},s),\quad(s+1,s),\quad(s,q_{\min}),\quad(s+1,q_{\min}),\quad(s,s),\tag{4.2}$$

with $q_{\min}\leq s\leq t_{\max}-1$.

Given $\epsilon>0$, there are $c,C>0$ such that the following holds. For all $k\geq k_0$, $k\notin\mathcal{J}$, $f\in H_k^s(\Gamma)$, and N satisfying

$$\left(\frac{k}{N}\right)^{t_{\max}-q_{\min}}\rho\leq c\tag{4.3}$$

the solution $v_N\in V_N$ to

$$(I+P_NL)v_N=P_Nf\tag{4.4}$$

exists, is unique, and satisfies the quasi-optimal error estimate

$$\begin{aligned}&\|v-v_N\|_{H_\hbar^s(\Gamma)}\\&\quad\leq\left(L_{\max}\|I-P_N\|^2_{H_\hbar^s\to H_\hbar^s}+C\left(\left(\frac{k}{N}\right)^{s-q_{\min}}\rho+k^{-1}+k/N\right)\right)\|(I-P_N)v\|_{H_\hbar^s(\Gamma)},\end{aligned}\tag{4.5}$$

where $v\in H_\hbar^s(\Gamma)$ is the solution of

$$\mathcal{A}v:=(I+L)v=f.\tag{4.6}$$

Moreover, for $\chi \in C_c^\infty(\mathbb{R};[0,1])$ *with* $\operatorname{supp}(1-\chi)\cap[-1,1]=\emptyset$,

$$\begin{aligned}&\big\|(I-1_{[-1-\epsilon,1+\epsilon]}(-\hbar^2\Delta_\Gamma))(v-v_N)(v-v_N)\big\|_{H^s_\hbar(\Gamma)}\\&\leq\left(L_{\max}\|I-P_N\|^2_{H^s_\hbar\to H^s_\hbar}+C\left(\left(\frac{k}{N}\right)^{t_{\max}-q_{\min}}\rho+k^{-1}+k/N\right)\right)\|(I-P_N)v\|_{H^s_\hbar(\Gamma)}. \quad (4.7)\end{aligned}$$

4.2 Results about microlocal properties of $\mathcal{A}$ under Assumption 4.1 needed for the proof of Theorem 4.2

Lemma 4.3 (Wavefront sets of $(I-\chi(|\hbar D'|_g^2))L$ and $L(I-\chi(|\hbar D'|_g^2))$) *If* $\mathcal{A}$ *satisfies Assumption 4.1, then for any* $\chi\in C_c^\infty(\mathbb{R};[0,1])$ *with* $\operatorname{supp}(1-\chi)\cap[-1,1]=\emptyset$

$$\mathrm{WF}_\hbar\Big(\big(I-\chi(|\hbar D'|_g^2)\big)L\Big)\cup\mathrm{WF}_\hbar\Big(L\big(I-\chi(|\hbar D'|_g^2)\big)\Big)\subset\overline{\big\{(x',\xi'):1-\chi(|\xi'|_g^2)>0\big\}}. \quad (4.8)$$

Proof. Let

$$(x_0,\xi_0)\in\Big(\overline{\big\{(x',\xi'):1-\chi(|\xi'|_g^2)>0\big\}}\Big)^c$$

Let B be elliptic in a neighbourhood of (x_0,ξ_0) and such that

$$\mathrm{WF}_\hbar(B)\Subset\Big(\overline{\big\{(x',\xi'):1-\chi(|\xi'|_g^2)>0\big\}}\Big)^c \quad (4.9)$$

(such a B exists since the set on the right is open). By the definition of $\mathrm{WF}_\hbar$ (Definition 3.4) it is sufficient to prove that

$$B(I-\chi(|\hbar D'|_g^2))L=O(\hbar^\infty)_{\Psi_\hbar^{-\infty}(\Gamma)}\quad\text{and}\quad BL(I-\chi(|\hbar D'|_g^2))=O(\hbar^\infty)_{\Psi_\hbar^{-\infty}(\Gamma)}. \quad (4.10)$$

By (3.19),

$$\mathrm{WF}_\hbar\big(I-\chi(|\hbar D'|_g^2)\big)\subset\overline{\big\{(x',\xi'):1-\chi(|\xi'|_g^2)>0\big\}}. \quad (4.11)$$

Therefore, by (4.9), $\mathrm{WF}_\hbar(B)\cap\mathrm{WF}_\hbar(I-\chi(|\hbar D'|_g^2))=\emptyset$, and thus the first equation in (4.10) holds by (3.21).

To prove the second equation in (4.10), choose $\widetilde{\chi}\in C_c^\infty(\mathbb{R}^d;[0,1])$ with $\widetilde{\chi}\equiv1$ on $[-1,1]$, $\operatorname{supp}\widetilde{\chi}\cap\operatorname{supp}(1-\chi)=\emptyset$, and

$$\mathrm{WF}_\hbar(B)\subset\Big(\overline{\big\{(x',\xi'):1-\widetilde{\chi}(|\xi'|_g^2)>0\big\}}\Big)^c;$$

such a choice is possible because of (4.9) (i.e., because there is space between $\mathrm{WF}_\hbar(B)$ and where $\chi\not\equiv1$). Observe that this choice of $\widetilde{\chi}$ and the property (4.11) (with χ replaced by $\widetilde{\chi}$) imply that

$$\mathrm{WF}_\hbar(B)\cap\mathrm{WF}_\hbar\big(I-\widetilde{\chi}(|\hbar D'|_g^2)\big)=\emptyset. \quad (4.12)$$

Then, using that $\operatorname{supp}\widetilde{\chi}\cap\operatorname{supp}(1-\chi)=\emptyset$, (3.19), (3.21), and Part (ii) of Theorem 3.1,

$$BL\big(I-\chi(|\hbar D'|_g^2)\big)=BL\big(I-\chi(|\hbar D'|_g^2)\big)\big(I-\widetilde{\chi}(|\hbar D'|_g^2)\big)+O(\hbar^\infty)_{\Psi_\hbar^{-\infty}(\Gamma)}. \quad (4.13)$$

By (3.18) and the assumption that $L\big(I-\chi(|\hbar D'|_g^2)\big)\in\Psi_\hbar^{-1}(\Gamma)$ (in Part (ii) of Assumption 4.1),

$$\mathrm{WF}_\hbar\Big(BL\big(I-\chi(|\hbar D'|_g^2)\big)\Big)\subset\mathrm{WF}_\hbar(B).$$

This inclusion combined with (4.12), (3.21), and (4.13), imply that $BL\big(I-\chi(|\hbar D'|_g^2)\big)=O(\hbar^\infty)_{\Psi_\hbar^{-\infty}(\Gamma)}$, which proves the second equation in (4.10) and thus completes the proof. ∎

Lemma 4.4 (Ellipticity of $(I-\chi(|\hbar D'|_g^2))\mathcal{A}$ on high frequencies) *If* $\mathcal{A}$ *satisfies Assumption 4.1, then for any* $\chi\in C_c^\infty(\mathbb{R};[0,1])$ *with* $\operatorname{supp}(1-\chi)\cap[-1,1]=\emptyset$

(a) $(I-\chi(|\hbar D'|_g^2))\mathcal{A}$ *and* $\mathcal{A}(I-\chi(|\hbar D'|_g^2))\in\Psi_\hbar^0(\Gamma)$ *are elliptic on* $\{(x',\xi'):1-\chi(|\xi'|_g^2)>0\}$, *and*

(b) $I+(I-\chi(|\hbar D'|_g^2))L$ *and* $I+L(I-\chi(|\hbar D'|_g^2))\in\Psi_\hbar^0(\Gamma)$ *are elliptic on* $T^*\Gamma$.

Proof. Part (b) follows immediately from Part (iii) of Assumption 4.1 and the definition of ellipticity (Definition 3.3). For Part (a), given χ as in the statement, let $\widetilde{\chi} \in C_c^\infty(\mathbb{R};[0,1])$ with $\operatorname{supp}(1-\widetilde{\chi}) \cap [-1,1] = \emptyset$ and $\operatorname{supp}(1-\chi)\cap \operatorname{supp}\widetilde{\chi} = \emptyset$. Then, by Part (a), $I + (1-\widetilde{\chi}(|\hbar D'|_g^2))L$ is elliptic on $T^*\Gamma$, and thus $(1-\chi(|\hbar D'|_g^2))(I+(1-\widetilde{\chi}(|\hbar D'|_g^2))L)$ is elliptic on $\{(x',\xi') : 1-\chi(|\xi'|_g^2) > 0\}$. Since $(1-\chi)(1-\widetilde{\chi}) = (1-\chi)$, Part (a) follows. ■

Corollary 4.5 (Ellipticity of $(I-\chi(|\hbar D'|_g^2))\mathcal{A}^*$ on high frequencies) *If $\mathcal{A}$ satisfies Assumption 4.1, then $(I-\chi(|\hbar D'|_g^2))\mathcal{A}^*$ and $\mathcal{A}^*(I-\chi(|\hbar D'|_g^2)) \in \Psi_\hbar^0(\Gamma)$ are elliptic on $\{(x,\xi') : 1-\chi(|\xi'|_g^2) > 0\}$.*

Proof. We prove the result for $(I-\chi(|\hbar D'|_g^2))\mathcal{A}^*$, the proof for $\mathcal{A}^*(I-\chi(|\hbar D'|_g^2))$ is analogous. Since $(I-\chi(|\hbar D'|_g^2))\mathcal{A}^* = (\mathcal{A}(I-\chi(|\hbar D'|_g^2))^*)^*$, by the second equation in (3.11) and the definition of ellipticity (Definition 3.3), it is sufficient to prove that

$$\mathcal{A}\big(I-\chi(|\hbar D'|_g^2)\big)^* \quad \text{is elliptic on} \quad \big\{(x',\xi') : 1-\chi(|\xi'|_g^2) > 0\big\}. \tag{4.14}$$

Choose $\widetilde{\chi} \in C_c^\infty(\mathbb{R};[0,1])$ with $\widetilde{\chi} \equiv 1$ on $[-1,1]$ and $\operatorname{supp}\widetilde{\chi} \cap \operatorname{supp}(1-\chi) = \emptyset$. Then, by (3.20) and (3.21),

$$\mathcal{A}\big(I-\chi(|\hbar D'|_g^2)\big)^* = \mathcal{A}\big(I-\widetilde{\chi}(|\hbar D'|_g^2)\big)\big(I-\chi(|\hbar D'|_g^2)\big)^*. \tag{4.15}$$

By Lemma 4.4, $\mathcal{A}(I-\widetilde{\chi}(|\hbar D'|_g^2))$ is elliptic on $\{(x,\xi') : 1-\widetilde{\chi}(|\xi'|_g^2) > 0\}$. Since $\{1-\chi > 0\} \subset \{1-\widetilde{\chi} > 0\}$, $\mathcal{A}(I-\widetilde{\chi}(|\hbar D'|_g^2))$ is elliptic on $\{(x,\xi') : 1-\chi(|\xi'|_g^2) > 0\}$. Then (4.14) follows by (4.15), both equations in (3.11), and the definition of ellipticity (Definition 3.3). ■

Recall that, since the operator $\mathcal{A}$ is a compact perturbation of the identity, ρ is bounded below by a k-independent constant (see Lemma A.3 below).

Lemma 4.6 (Norm of $\mathcal{A}^{-1}$ on $H_\hbar^s(\Gamma)$ bounded by norm on $L^2(\Gamma)$) *Suppose that $\mathcal{A}$ satisfies Assumption 4.1. Then, given $s \in \mathbb{R}$ and $k_0 > 0$, there exists $C > 0$ such that, for all $k \geq k_0$, $\mathcal{A}$ is invertible on $H_\hbar^s(\Gamma)$ with*

$$\big\|\mathcal{A}^{-1}\big\|_{H_\hbar^s(\Gamma)\to H_\hbar^s(\Gamma)} \leq C\rho. \tag{4.16}$$

Proof. Because $\mathcal{A}$ is invertible on $H^s(\Gamma)$ for all $k > 0$ by Part (i) of Assumption 4.1, it is sufficient to prove the result for $k \geq k_0$ with k_0 sufficiently large. We first prove this result for $s > 0$. Proving the bound (4.16) is equivalent to proving that if $\mathcal{A}\phi = g$ with $\phi, g \in H_\hbar^s(\Gamma)$, then

$$\|\phi\|_{H_\hbar^s(\Gamma)} \leq C\rho\, \|g\|_{H_\hbar^s(\Gamma)}\,. \tag{4.17}$$

Let $\chi, \widetilde{\chi} \in C_c^\infty(\mathbb{R};[0,1])$ with both $\operatorname{supp}(1-\chi)\cap[-1,1] = \emptyset$ and $\operatorname{supp}(1-\widetilde{\chi})\cap[-1,1] = \emptyset$, and $\operatorname{supp}\chi \cap \operatorname{supp}(1-\widetilde{\chi}) = \emptyset$. Thus, by (4.11) and Lemma 4.4,

$$(I-\chi(|\hbar D'|_g^2))\mathcal{A} \quad \text{is elliptic on} \quad \mathrm{WF}_\hbar\big(I-\widetilde{\chi}(|\hbar D'|_g^2)\big). \tag{4.18}$$

The idea of the proof is to write

$$\phi = \big(1-\widetilde{\chi}(|\hbar D'|_g^2)\big)\phi + \widetilde{\chi}(|\hbar D'|_g^2)\phi,$$

estimate the high-frequency components of ϕ (i.e., $(1-\widetilde{\chi}(|\hbar D'|_g^2))\phi$) using ellipticity of $(I-\chi(|\hbar D'|_g^2))\mathcal{A}$ on high frequencies and estimate the low-frequency components (i.e., $\widetilde{\chi}(|\hbar D'|_g^2)\phi$) by the property (3.15) of frequency cut-offs.

Indeed, by (4.18) and (3.22), given $s \in \mathbb{R}$ and $M > 0$, there exists $C, C', \hbar_0 > 0$ such that, for all $0 < \hbar \leq \hbar_0$,

$$\begin{aligned}\big\|(1-\widetilde{\chi}(|\hbar D'|_g^2))\phi\big\|_{H_\hbar^s(\Gamma)} &\leq C\Big(\big\|(I-\chi(|\hbar D'|_g^2))g\big\|_{H_\hbar^s(\Gamma)} + \hbar^M\|\phi\|_{H_\hbar^s(\Gamma)}\Big)\\ &\leq C'\Big(\|g\|_{H_\hbar^s(\Gamma)} + \hbar^M\|\phi\|_{H_\hbar^s(\Gamma)}\Big).\end{aligned} \tag{4.19}$$

For the bound on the low-frequencies, by (3.15), given $s \geq 0$ and $\hbar_0 > 0$, there exists $C'' > 0$ such that, for all $0 < \hbar \leq \hbar_0$,

$$\left\|\widetilde{\chi}(|\hbar D'|_g^2)\phi\right\|_{H_\hbar^s(\Gamma)} \leq C'' \left\|\phi\right\|_{L^2(\Gamma)} \leq C''\rho \left\|g\right\|_{L^2(\Gamma)}$$
$$\leq C''\rho \left\|g\right\|_{H_\hbar^s(\Gamma)}; \tag{4.20}$$

the result (4.17) then follows by combining (4.19) and (4.20) and reducing $\hbar_0$ if necessary to absorb the $\hbar^M \left\|\phi\right\|_{L^2(\Gamma)}$ term into the left-hand side.

Since $\|\mathcal{A}^{-1}\|_{H_\hbar^s(\Gamma)\to H_\hbar^s(\Gamma)} = \|(\mathcal{A}^*)^{-1}\|_{H_\hbar^{-s}(\Gamma)\to H_\hbar^{-s}(\Gamma)}$, the result for $s < 0$ follows by applying the above argument to $\mathcal{A}^*$ using Corollary 4.5. ■

We record the following simple corollary of Lemma 4.6.

Corollary 4.7 *Suppose that $\mathcal{A}$ satisfies Assumption 4.1. Then, given $s \in \mathbb{R}$ and $k_0 > 0$, there exists $C > 0$ such that, for all $k \geq k_0$,*

$$\|L(I+L)^{-1}\|_{H_\hbar^s \to H_\hbar^s} \leq C\rho, \tag{4.21}$$

Proof. Since

$$L(I+L)^{-1} = I - (I+L)^{-1}$$

the result follows from (4.16). ■

A result analogous to Lemma 4.6 about $\mathcal{A}$ (as opposed to $\mathcal{A}^{-1}$) also holds. Although we only use it in §7 below, we state and prove it here because of its similarity with Lemma 4.6.

Lemma 4.8 (Norm of $\mathcal{A}$ on $H_\hbar^s(\Gamma)$ bounded by norm on $L^2(\Gamma)$) *Suppose that $\mathcal{A}$ satisfies Assumption 4.1. Then, given $s \in \mathbb{R}, k_0 > 0$, there exists $C > 0$ such that*

$$\|\mathcal{A}\|_{H_\hbar^s(\Gamma)\to H_\hbar^s(\Gamma)} \leq C\Big(1 + \|\mathcal{A}\|_{L^2(\Gamma)\to L^2(\Gamma)}\Big) \quad \textit{for all } k \geq k_0.$$

Proof. Let $\chi \in C_c^\infty(\mathbb{R};[0,1])$ with $\chi \equiv 1$ on a neighbourhood of $[-1,1]$. If $v \in H^s(\Gamma)$ with $s > 0$, then by (3.15), the fact that $\big(I - \chi(|\hbar D'|_g^2)\big)\mathcal{A} \in \Psi_\hbar^0(\Gamma)$ (by, e.g., Part (i) of Lemma 4.4), and Part (ii) of Theorem 3.1, there exist $C_1, C_2 > 0$ such that, for all $k \geq k_0$,

$$\|\mathcal{A}v\|_{H_\hbar^s(\Gamma)} \leq \left\|\chi(|\hbar D'|_g^2)\mathcal{A}v\right\|_{H_\hbar^s(\Gamma)} + \left\|\big(I - \chi(|\hbar D'|_g^2)\big)\mathcal{A}v\right\|_{H_\hbar^s(\Gamma)},$$
$$\leq C_1 \left\|\mathcal{A}\right\|_{L^2(\Gamma)\to L^2(\Gamma)} \left\|v\right\|_{L^2(\Gamma)} + C_2 \left\|v\right\|_{H_\hbar^s(\Gamma)};$$

the result (with $s > 0$) then follows since $\|v\|_{L^2(\Gamma)} \leq \|v\|_{H_\hbar^s(\Gamma)}$.

Similarly, if $v \in H^s(\Gamma)$ with $s < 0$, then there exist $C_1, C_2 > 0$ such that, for all $k \geq k_0$,

$$\|\mathcal{A}v\|_{H_\hbar^s(\Gamma)} \leq \left\|\mathcal{A}\chi(|\hbar D'|_g^2)v\right\|_{H_\hbar^s(\Gamma)} + \left\|\mathcal{A}\big(I - \chi(|\hbar D'|_g^2)\big)v\right\|_{H_\hbar^s(\Gamma)},$$
$$\leq \left\|\mathcal{A}\chi(|\hbar D'|_g^2)v\right\|_{L^2(\Gamma)} + C_2 \left\|v\right\|_{H_\hbar^s(\Gamma)} \leq C_1 \left\|\mathcal{A}\right\|_{L^2(\Gamma)\to L^2(\Gamma)} \left\|v\right\|_{H_\hbar^s(\Gamma)} + C_2 \left\|v\right\|_{H_\hbar^s(\Gamma)},$$

which is the result with $s < 0$. ■

Finally, we use the following description of $\mathcal{A}^{-1}$ acting on high frequencies.

Lemma 4.9 (The high-frequency components of $\mathcal{A}^{-1}$) *Suppose that $\mathcal{A}$ satisfies Assumptions 4.1 and 1.3. Let $k_0 > 0$ and $\chi_1, \chi_2 \in C_c^\infty$ with* $\operatorname{supp}(1 - \chi_j) \cap [-1,1] = \emptyset$ *and* $\operatorname{supp}\chi_1 \cap \operatorname{supp}(1 - \chi_2) = \emptyset$.

Then for all $k > k_0$, $k \notin \mathcal{J}$, $I + L(I - \chi_1(|\hbar D'|_g^2))$ and $I + (I - \chi_1(|\hbar D'|_g^2))L$ are both in $\Psi_\hbar^0(\Gamma)$, are elliptic on $T^*\Gamma$,

$$(I+L)^{-1}\big(I - \chi_2(|\hbar D'|_g^2)\big) = \Big(I + L\big(I - \chi_1(|\hbar D'|_g^2)\big)\Big)^{-1}\big(I - \chi_2(|\hbar D'|_g^2)\big) + O(\hbar^\infty)_{\Psi_\hbar^{-\infty}} \tag{4.22}$$

and

$$\big(I-\chi_2(|\hbar D'|_g^2)\big)(I+L)^{-1}=\big(I-\chi_2(|\hbar D'|_g^2)\big)\Big(I+\big(I-\chi_1(|\hbar D'|_g^2)\big)L\Big)^{-1}+O(\hbar^\infty)_{\Psi_\hbar^{-\infty}}. \quad (4.23)$$

Furthermore, given $s\in\mathbb{R}$ and $k_0>0$, there is $C>0$ such that for all $k\geq k_0$

$$\begin{aligned}\Big\|\Big(I+L\big(I-\chi_1(|\hbar D'|_g^2)\big)\Big)^{-1}\Big\|_{H_\hbar^s(\Gamma)\to H_\hbar^s(\Gamma)}&\leq L_{\max}+Ck^{-1},\\ \Big\|\Big(I+\big(I-\chi_2(|\hbar D'|_g^2)\big)L\Big)^{-1}\Big\|_{H_\hbar^s(\Gamma)\to H_\hbar^s(\Gamma)}&\leq L_{\max}+Ck^{-1}.\end{aligned} \quad (4.24)$$

Proof. We prove the result (4.22) with the cutoffs on the right; the proof of (4.23) with the cutoffs on the left is analogous. By Lemma 4.4, $I+L(I-\chi_1(|\hbar D'|_g^2))\in\Psi_\hbar^0(\Gamma)$ is elliptic on $T^*\Gamma$ and hence, by Part (i) of Corollary 3.6, is invertible with inverse satisfying

$$|\sigma_\hbar^0\big((I+L(I-\chi_1(|\hbar D'|_g^2)))^{-1}\big)|=1/|\sigma_\hbar^0\big(I+L(I-\chi_1(|\hbar D'|_g^2))\big)|.$$

In particular, by Part (iii) of Assumption 4.1, and Lemma 3.2, the first estimate in (4.24) holds. Then

$$(I+L)^{-1}-\Big(I+L\big(I-\chi_1(|\hbar D'|_g^2)\big)\Big)^{-1}=-(I+L)^{-1}L\chi_1(|\hbar D'|_g^2)\Big(I+L\big(I-\chi_1(|\hbar D'|_g^2)\big)\Big)^{-1}.$$

Multiplying by $(1-\chi_2(|\hbar D'|_g^2))$, we have that

$$\begin{aligned}(I+L)^{-1}&\big(1-\chi_2(|\hbar D'|_g^2)\big)-\Big(I+L\big(I-\chi_1(|\hbar D'|_g^2)\big)\Big)^{-1}\big(1-\chi_2(|\hbar D'|_g^2)\big)\\ &=-(I+L)^{-1}L\chi_1(|\hbar D'|_g^2)\Big(I+L\big(I-\chi_1(|\hbar D'|_g^2)\big)\Big)^{-1}\big(1-\chi_2(|\hbar D'|_g^2)\big).\end{aligned}$$

By Lemma 4.6 and Assumption 1.3, $(I+L)^{-1}:H_\hbar^s(\Gamma)\to H_\hbar^s(\Gamma)$ is polynomially bounded for all $s\in\mathbb{R}$. Then, the fact that $\operatorname{supp}\chi_1\cap\operatorname{supp}(1-\chi_2)=\emptyset$ and the properties (3.18), (3.21), and (4.11) imply that the right-hand side of the last displayed equality is $O(\hbar^\infty)_{\Psi_\hbar^{-\infty}}$, and the result follows. ∎

Combining Lemma 4.9 with Corollary 3.10, we obtain the following result.

Corollary 4.10 *Suppose that $\mathcal{A}$ satisfies Assumptions 4.1 and 1.3 and $\chi\in C_c^\infty(\mathbb{R};[0,1])$ with $\operatorname{supp}(1-\chi)\cap[-1,1]=\emptyset$. Given $s\in\mathbb{R}$ and $k_0>0$ there exists $C>0$ such that for all $k\geq k_0$*

$$\big\|L(I+L)^{-1}\big(1-\chi(\hbar^2\Delta_\Gamma)\big)\big\|_{H_\hbar^s\to H_\hbar^{s+1}}\leq C.$$

Proof. Let $\widetilde{\chi}\in C_c^\infty(\mathbb{R};[0,1])$ be such that $\operatorname{supp}(1-\widetilde{\chi})\cap[-1,1]=\emptyset$ and $\operatorname{supp}(1-\chi)\cap\operatorname{supp}\widetilde{\chi}=\emptyset$. Then, by Corollary 3.10 and Assumption 1.3

$$L(I+L)^{-1}\big(1-\chi(\hbar^2\Delta_\Gamma)\big)=L(I+L)^{-1}\big(1-\widetilde{\chi}(|\hbar D'|_g^2)\big)\big(1-\chi(\hbar^2\Delta_\Gamma)\big)+O(\hbar^\infty)_{\Psi_\hbar^{-\infty}}.$$

On the one hand,

$$\begin{aligned}\widetilde{\chi}(|\hbar D'|_g^2)&L(I+L)^{-1}\big(1-\widetilde{\chi}(|\hbar D'|_g^2)\big)\big(1-\chi(\hbar^2\Delta_\Gamma)\big)\\ &=\widetilde{\chi}(|\hbar D'|_g^2)\big(I-(I+L)^{-1}\big)\big(1-\widetilde{\chi}(|\hbar D'|_g^2)\big)\big(1-\chi(\hbar^2\Delta_\Gamma)\big),\end{aligned}$$

and the bound

$$\big\|\widetilde{\chi}(|\hbar D'|_g^2)L(I+L)^{-1}\big(1-\chi(\hbar^2\Delta_\Gamma)\big)\big\|_{H_\hbar^s\to H_\hbar^{s+1}}\leq C \quad (4.25)$$

then follows from (4.22), (4.24), and (3.15).

On the other hand,

$$\big\|\big(I-\widetilde{\chi}(|\hbar D'|_g^2)\big)L(I+L)^{-1}\big(1-\widetilde{\chi}(|\hbar D'|_g^2)\big)\big(1-\chi(\hbar^2\Delta_\Gamma)\big)\big\|_{H_\hbar^s\to H_\hbar^{s+1}}\leq C \quad (4.26)$$

by Part (ii) of Assumption 4.1, (4.22), and (4.24). The result then follows by combining (4.25) and (4.26). ∎

4.3 The idea of the proof of Theorem 4.2

The proof of Theorem 4.2 starts by using the following lemma.

Lemma 4.11 (Quasi-optimality in terms of the norm of the discrete inverse) *If $P_N : H^s_\hbar(\Gamma) \to V_N$ is a projection and $I + P_N L : H^s_\hbar(\Gamma) \to H^s_\hbar(\Gamma)$ is invertible, then the solution $v_N \in V_N$ to (4.4) exists, is unique, and satisfies*

$$v - v_N = (I + P_N L)^{-1}(I - P_N)(I - P_N)v. \tag{4.27}$$

Proof. We first consider the equation (4.4) as an equation in $H^s_\hbar(\Gamma)$. Since $I + P_N L : H^s_\hbar(\Gamma) \to H^s_\hbar(\Gamma)$ is invertible, the solution v_N to (4.4) in $H^s_\hbar(\Gamma)$ exists and is unique as an element of $H^s_\hbar(\Gamma)$. Applying $(I - P_N)$ to (4.4), we see that $(I - P_N)v_N = 0$ and thus the solution $v_N \in V_N$; i.e., the equation (4.4) has a unique solution in V_N. Then, by (4.6) and (4.4),

$$(I + P_N L)(v - v_N) = (I + P_N L)v - P_N f = v + P_N L v - P_N\big((I + L)v\big) = (I - P_N)v.$$

Therefore, since $(I - P_N) = (I - P_N)^2$,

$$(v - v_N) = (I + P_N L)^{-1}(I - P_N)v = (I + P_N L)^{-1}(I - P_N)(I - P_N)v.$$

∎

To obtain conditions under which $I + P_N L : H^s_\hbar(\Gamma) \to H^s_\hbar(\Gamma)$ is invertible (and thus under which we can use Lemma 4.11), we write

$$(I + P_N L) = I + L + (P_N - I)L = \Big(I + (P_N - I)L(I + L)^{-1}\Big)(I + L), \tag{4.28}$$

and study when $I + (P_N - I)L(I + L)^{-1}$ is invertible. We then use that

$$\Big(I + (P_N - I)L(I + L)^{-1}\Big)^{-1}(I - P_N) = (I - P_N)\Big(I + (P_N - I)L(I + L)^{-1}\Big)^{-1}(I - P_N)$$

i.e., we can add a $(I - P_N)$ on the left. Indeed, if $(I + (P_N - I)L(I + L)^{-1})v = (I - P_N)f$, then $P_N v = 0$ so that $v = (I - P_N)v$. Therefore the operator on the right-hand side of (4.27) can be written as

$$(I + P_N L)^{-1}(I - P_N) = (I + L)^{-1}(I - P_N)\Big(I + (P_N - I)L(I + L)^{-1}\Big)^{-1}(I - P_N). \tag{4.29}$$

The first natural attempt is to show that $I + (P_N - I)L(I + L)^{-1}$ is invertible is to show that

$$\|(P_N - I)L(I + L)^{-1}\|_{H^s_\hbar \to H^s_\hbar} < \frac{1}{2}, \tag{4.30}$$

in which case $\|(I + (P_N - I)L(I + L)^{-1})^{-1}\|_{H^s_\hbar \to H^s_\hbar} \leq 2$.

To understand when the condition (4.30) holds, given $\epsilon > 0$, we define low- and high-frequency projectors,

$$\Pi_{\mathscr{L}} := 1_{[-1-\epsilon, 1+\epsilon]}(-\hbar^2 \Delta_\Gamma) \quad \text{and} \quad \Pi_{\mathscr{H}} := (I - \Pi_{\mathscr{L}}), \tag{4.31}$$

and decompose $H^s_\hbar$ into an orthogonal sum $\Pi_{\mathscr{L}} H^s_\hbar \oplus \Pi_{\mathscr{H}} H^s_\hbar$. That is, we write

$$(P_N - I)L(I + L)^{-1} = \begin{pmatrix} \Pi_{\mathscr{L}}(P_N - I)L(I + L)^{-1}\Pi_{\mathscr{L}} & \Pi_{\mathscr{L}}(P_N - I)L(I + L)^{-1}\Pi_{\mathscr{H}} \\ \Pi_{\mathscr{H}}(P_N - I)L(I + L)^{-1}\Pi_{\mathscr{L}} & \Pi_{\mathscr{H}}(P_N - I)L(I + L)^{-1}\Pi_{\mathscr{H}} \end{pmatrix}. \tag{4.32}$$

Using the results in §4.2, we show below (see Lemma 4.12) that

$$\big\|(P_N - I)L(I + L)^{-1}\big\|_{H^s_\hbar \to H^s_\hbar} \leq C \begin{pmatrix} (k/N)^{t_{\max} - q_{\min}}\rho & (k/N)^{s+1-q_{\min}} \\ (k/N)^{t_{\max} - s}\rho & (k/N) \end{pmatrix}, \tag{4.33}$$

where here and below we use the notation that

$$\|A\|_{H_\hbar^s\to H_\hbar^s}\leq C_1\begin{pmatrix} a_{\mathscr{L}\mathscr{L}} & a_{\mathscr{L}\mathscr{H}}\\ a_{\mathscr{H}\mathscr{L}} & a_{\mathscr{H}\mathscr{H}}\end{pmatrix},\quad \text{if}\quad A=\begin{pmatrix}\Pi_{\mathscr{L}}A\Pi_{\mathscr{L}} & \Pi_{\mathscr{L}}A\Pi_{\mathscr{H}}\\ \Pi_{\mathscr{H}}A\Pi_{\mathscr{L}} & \Pi_{\mathscr{H}}A\Pi_{\mathscr{H}}\end{pmatrix},$$

with

$$\|\Pi_{\mathscr{L}}A\Pi_{\mathscr{L}}\|_{H_\hbar^s\to H_\hbar^s}\leq C_1 a_{\mathscr{L}\mathscr{L}}\qquad \|\Pi_{\mathscr{H}}A\Pi_{\mathscr{L}}\|_{H_\hbar^s\to H_\hbar^s}\leq C_1 a_{\mathscr{H}\mathscr{L}}$$
$$\|\Pi_{\mathscr{L}}A\Pi_{\mathscr{H}}\|_{H_\hbar^s\to H_\hbar^s}\leq C_1 a_{\mathscr{L}\mathscr{H}}\qquad \|\Pi_{\mathscr{H}}A\Pi_{\mathscr{H}}\|_{H_\hbar^s\to H_\hbar^s}\leq C_1 a_{\mathscr{H}\mathscr{H}}.$$

(Once can then easily check that if $\|A\|\leq C_1\mathscr{A}$ and $\|B\|\leq C_2\mathscr{B}$, then $\|AB\|\leq C_1C_2\mathscr{A}\mathscr{B}$.)

The largest matrix entry on the right-hand side of (4.33) is $(k/N)^{t_{\max}-s}$, and demanding that this be sufficiently small is a more restrictive condition than (4.3).

We instead change the norm on $H_\hbar^s$ using an invertible operator $\mathcal{C}:H_\hbar^s\to H_\hbar^s$. More precisely, we observe that

$$I+(P_N-I)L(I+L)^{-1}=\mathcal{C}^{-1}\Big(I+\mathcal{C}(P_N-I)L(I+L)^{-1}\mathcal{C}^{-1}\Big)\mathcal{C}.$$

We now choose $\mathcal{C}$ to be a diagonal matrix with entries chosen so that the bottom-left and top-right entries of $\mathcal{C}(P_N-I)L(I+L)^{-1}\mathcal{C}^{-1}$ are equal; i.e., compared to $(P_N-I)L(I+L)^{-1}$ we make the bottom left entry smaller, at the cost of making the top right entry bigger. The choice of $\mathcal{C}$ that achieves this is

$$\mathcal{C}:=\begin{pmatrix} I & 0\\ 0 & (k/N)^{s-t_{\max}/2+1/2-q_{\min}/2}\rho^{-1/2}I\end{pmatrix},\tag{4.34}$$

with then

$$\begin{aligned}&\|\mathcal{C}(P_N-I)L(I+L)^{-1}\mathcal{C}^{-1}\|_{H_\hbar^s\to H_\hbar^s}\\ &\qquad\leq C\begin{pmatrix}(k/N)^{t_{\max}-q_{\min}}\rho & \Big((k/N)^{t_{\max}-q_{\min}}\rho\Big)^{1/2}(k/N)^{1/2}\\ \Big((k/N)^{t_{\max}-q_{\min}}\rho\Big)^{1/2}(k/N)^{1/2} & (k/N)\end{pmatrix};\end{aligned}\tag{4.35}$$

observe that the largest entry in the matrix on the right-hand side is the top left entry, which is $(k/N)^{t_{\max}-q_{\min}}\rho$ as desired. The condition (4.3) then ensures that

$$\big\|\mathcal{C}(P_N-I)L(I+L)^{-1}\mathcal{C}^{-1}\big\|_{H_\hbar^s\to H_\hbar^s}<\frac{1}{2},\tag{4.36}$$

and so $I+(P_N-I)L(I+L)^{-1}$ is invertible and

$$\Big(I+(P_N-I)L(I+L)^{-1}\Big)^{-1}=\mathcal{C}^{-1}\Big(I+\mathcal{C}(P_N-I)L(I+L)^{-1}\mathcal{C}^{-1}\Big)^{-1}\mathcal{C};\tag{4.37}$$

The result then follows by decomposing $(I+L)^{-1}(I-P_N)$ into high- and low-frequency components similar to in (4.32) (see Lemma 4.13 below).

4.4 Proof of Theorem 4.2

First observe that with $\Pi_{\mathscr{L}}$ and $\Pi_{\mathscr{H}}$ defined by (4.31), by the definition of $\|\cdot\|_{H_\hbar^s}$ (3.5),

$$\|\Pi_{\mathscr{L}}\|_{H_\hbar^{-N}\to H_\hbar^N}\leq C,\qquad\text{so that}\qquad \|\Pi_{\mathscr{H}}\|_{H_\hbar^t\to H_\hbar^t}\leq C.\tag{4.38}$$

Lemma 4.12 (Bounds on the decomposition of $(P_N-I)L(I+L)^{-1}$) *Let $k_0>0$, $q_{\min}\leq s\leq t_{\max}-1$, $\mathcal{A}$ satisfy Assumptions 4.1 and 1.3, $P_N:H_\hbar^s(\Gamma)\to V_N$ be a projection satisfying (4.1) with (t,q) the pairs*

$$(t_{\max},s),\quad (t_{\max},q_{\min}),\quad (s+1,q_{\min})\quad \textit{and}\quad (s+1,s),$$

Then there is $C > 0$ such that for all $k > k_0$, $k \notin \mathcal{J}$, and $N \geq k$.

$$
\begin{aligned}
\|\Pi_{\mathscr{L}}(I-P_N)L(I+L)^{-1}\Pi_{\mathscr{L}}\|_{H^s_\hbar\to H^s_\hbar} &\leq C(k/N)^{t_{\max}-q_{\min}}\rho,\\
\|\Pi_{\mathscr{H}}(I-P_N)L(I+L)^{-1}\Pi_{\mathscr{L}}\|_{H^s_\hbar\to H^s_\hbar} &\leq C(k/N)^{t_{\max}-s}\rho,\\
\|\Pi_{\mathscr{L}}(I-P_N)L(I+L)^{-1}\Pi_{\mathscr{H}}\|_{H^s_\hbar\to H^s_\hbar} &\leq C(k/N)^{s+1-q_{\min}},\\
\|\Pi_{\mathscr{H}}(I-P_N)L(I+L)^{-1}\Pi_{\mathscr{H}}\|_{H^s_\hbar\to H^s_\hbar} &\leq C(k/N).
\end{aligned}
\tag{4.39}
$$

Proof. For the term involving two low frequency projections, by (4.38), Corollary 4.7 and (4.1) with $(t,q) = (t_{\max}, q_{\min})$,

$$
\begin{aligned}
&\|\Pi_{\mathscr{L}}(I-P_N)L(I+L)^{-1}\Pi_{\mathscr{L}}\|_{H^s_\hbar\to H^s_\hbar}\\
&\leq \|\Pi_{\mathscr{L}}\|_{H^{q_{\min}}_\hbar\to H^s_\hbar}\|(I-P_N)\|_{H^{t_{\max}}_\hbar\to H^{q_{\min}}_\hbar}\|L(I+L)^{-1}\|_{H^{t_{\max}}_\hbar\to H^{t_{\max}}_\hbar}\|\Pi_{\mathscr{L}}\|_{H^s_\hbar\to H^{t_{\max}}_\hbar}\\
&\leq C(k/N)^{t_{\max}-q_{\min}}\rho.
\end{aligned}
$$

For the low-to-high frequency term, by (4.38), Corollary 4.7 and (4.1) with $(t,q) = (t_{\max}, s)$,

$$
\begin{aligned}
&\|\Pi_{\mathscr{H}}(I-P_N)L(I+L)^{-1}\Pi_{\mathscr{L}}\|_{H^s_\hbar\to H^s_\hbar}\\
&\leq \|\Pi_{\mathscr{H}}\|_{H^s_\hbar\to H^s_\hbar}\|(I-P_N)\|_{H^{t_{\max}}_\hbar\to H^s_\hbar}\|L(I+L)^{-1}\|_{H^{t_{\max}}_\hbar\to H^{t_{\max}}_\hbar}\|\Pi_{\mathscr{L}}\|_{H^s_\hbar\to H^{t_{\max}}_\hbar}\\
&\leq C(k/N)^{t_{\max}-s}\rho.
\end{aligned}
$$

For the high-to-low frequency term, let $\chi \in C^\infty_c((-2,2))$ with $\mathrm{supp}(1-\chi)\cap[-1,1] = \emptyset$ and observe that, by the definition of $\Pi_{\mathscr{H}}$ (4.31), $(1-\chi(-\hbar^2\Delta_\Gamma))\Pi_{\mathscr{H}} = \Pi_{\mathscr{H}}$. Using this, along with (4.38), Corollary 4.7 and (4.1) with $(t,q) = (s+1, q_{\min})$, we obtain that

$$
\begin{aligned}
&\|\Pi_{\mathscr{L}}(I-P_N)L(I+L)^{-1}\Pi_{\mathscr{H}}\|_{H^s_\hbar\to H^s_\hbar}\\
&\leq \|\Pi_{\mathscr{L}}\|_{H^{q_{\min}}_\hbar\to H^s_\hbar}\|(I-P_N)\|_{H^{s+1}_\hbar\to H^{q_{\min}}_\hbar}\|L(I+L)^{-1}(1-\chi(\hbar^2\Delta_\Gamma))\|_{H^s_\hbar\to H^{s+1}_\hbar}\|\Pi_{\mathscr{H}}\|_{H^s_\hbar\to H^s_\hbar}\\
&\leq C(k/N)^{s+1-q_{\min}}\|L(I+L)^{-1}(1-\chi(\hbar^2\Delta_\Gamma))\|_{H^s_\hbar\to H^{s+1}_\hbar}\\
&\leq C(k/N)^{s+1-q_{\min}}.
\end{aligned}
$$

Finally, for the high-to-high frequency term, by (4.38), Corollary 4.7 and (4.1) with $(t,q) = (s+1, s)$,

$$
\begin{aligned}
&\|\Pi_{\mathscr{H}}(I-P_N)L(I+L)^{-1}\Pi_{\mathscr{H}}\|_{H^s_\hbar\to H^s_\hbar}\\
&\leq \|\Pi_{\mathscr{H}}\|_{H^s_\hbar\to H^s_\hbar}\|(I-P_N)\|_{H^{s+1}_\hbar\to H^s_\hbar}\|L(I+L)^{-1}(1-\chi(\hbar^2\Delta_\Gamma))\|_{H^s_\hbar\to H^{s+1}_\hbar}\|\Pi_{\mathscr{H}}\|_{H^s_\hbar\to H^s_\hbar}\\
&\leq C(k/N).
\end{aligned}
$$

∎

Lemma 4.13 (Bounds on the decomposition of $(I+L)^{-1}(I-P_N)$) *Let $k_0 > 0$, $q_{\min} \leq s \leq t_{\max}$, $\mathcal{A}$ satisfy Assumptions 4.1 and 1.3, $P_N : H^s_\hbar(\Gamma) \to V_N$ be a projection satisfying (4.1) with (t,q) the pairs*

$$
(t_{\max}, s), \quad (t_{\max}, q_{\min}), \quad (s, q_{\min}) \quad \textit{and} \quad (s,s),
$$

Then there is $C > 0$ such that for all $k > k_0$, $k \notin \mathcal{J}$, and $N \geq k$

$$
\begin{aligned}
\|\Pi_{\mathscr{L}}(I+L)^{-1}(I-P_N)\Pi_{\mathscr{L}}\|_{H^s_\hbar\to H^s_\hbar} &\leq C(k/N)^{t_{\max}-q_{\min}}\rho,\\
\|\Pi_{\mathscr{H}}(I+L)^{-1}(I-P_N)\Pi_{\mathscr{L}}\|_{H^s_\hbar\to H^s_\hbar} &\leq C(k/N)^{t_{\max}-s},\\
\|\Pi_{\mathscr{L}}(I+L)^{-1}(I-P_N)\Pi_{\mathscr{H}}\|_{H^s_\hbar\to H^s_\hbar} &\leq C(k/N)^{s-q_{\min}}\rho,\\
\|\Pi_{\mathscr{H}}(I+L)^{-1}(I-P_N)\Pi_{\mathscr{H}}\|_{H^s_\hbar\to H^s_\hbar} &\leq \big(L_{\max} + Ck^{-1}\big)\|I-P_N\|_{H^s_\hbar\to H^s_\hbar}.
\end{aligned}
\tag{4.40}
$$

Proof. For the low-to-low frequency term, by (4.38), Corollary 4.7 and (4.1) with $(t,q) = (t_{\max}, q_{\min})$,

$$
\|\Pi_{\mathscr{L}}(I+L)^{-1}(I-P_N)\Pi_{\mathscr{L}}\|_{H^s_\hbar\to H^s_\hbar}
$$

$$\leq \|\Pi_{\mathscr{L}}\|_{H_\hbar^{q_{\min}}\to H_\hbar^s}\|(I+L)^{-1}\|_{H_\hbar^{q_{\min}}\to H_\hbar^{q_{\min}}}\|(I-P_N)\|_{H_\hbar^{t_{\max}}\to H_\hbar^{q_{\min}}}\|\Pi_{\mathscr{L}}\|_{H_\hbar^s\to H_\hbar^{t_{\max}}}$$
$$\leq C(k/N)^{t_{\max}-q_{\min}}\rho.$$

For the low-to-high frequency term, let $\chi \in C_c^\infty((-2,2);[0,1])$ with $\operatorname{supp}(1-\chi)\cap[-1,1]=\emptyset$ and observe that

$$\Pi_{\mathscr{H}} = \Pi_{\mathscr{H}}(1-\chi(-\hbar^2\Delta_\Gamma)). \tag{4.41}$$

By Corollary 3.10, (4.23), (4.24),

$$\|(1-\chi(-\hbar^2\Delta_\Gamma))(I+L)^{-1}\|_{H_\hbar^s\to H_\hbar^s} \leq L_{\max}+Ck^{-1}; \tag{4.42}$$

we record for later that analogous arguments also show that

$$\|(I+L)^{-1}(1-\chi(-\hbar^2\Delta_\Gamma))\|_{H_\hbar^s\to H_\hbar^s} \leq L_{\max}+Ck^{-1}. \tag{4.43}$$

Therefore, by (4.41), (4.42), (4.38), and (4.1) with $(t,q)=(t_{\max},s)$, we obtain that

$$\|\Pi_{\mathscr{H}}(I+L)^{-1}(I-P_N)\Pi_{\mathscr{L}}\|_{H_\hbar^s\to H_\hbar^s}$$
$$\leq \|\Pi_{\mathscr{H}}\|_{H_\hbar^s\to H_\hbar^s}\|(1-\chi(-\hbar^2\Delta_\Gamma))(I+L)^{-1}\|_{H_\hbar^s\to H_\hbar^s}\|(I-P_N)\|_{H_\hbar^{t_{\max}}\to H_\hbar^s}\|\Pi_{\mathscr{L}}\|_{H_\hbar^s\to H_\hbar^{t_{\max}}}$$
$$\leq C(k/N)^{t_{\max}-s}.$$

For the high-to-low frequency term, by (4.38), Corollary 4.7 and (4.1) with $(t,q)=(s,q_{\min})$,

$$\|\Pi_{\mathscr{L}}(I+L)^{-1}(I-P_N)\Pi_{\mathscr{H}}\|_{H_\hbar^s\to H_\hbar^s}$$
$$\leq \|\Pi_{\mathscr{L}}\|_{H_\hbar^{q_{\min}}\to H_\hbar^s}\|(I-P_N)\|_{H_\hbar^s\to H_\hbar^{q_{\min}}}\|(I+L)^{-1}\|_{H_\hbar^s\to H_\hbar^s}\|\Pi_{\mathscr{H}}\|_{H_\hbar^s\to H_\hbar^s}$$
$$\leq C(k/N)^{s-q_{\min}}\|(I+L)^{-1}\|_{H_\hbar^s\to H_\hbar^s}$$
$$\leq C(k/N)^{s-q_{\min}}\rho.$$

Finally, for the high-to-high frequency term, by (4.42), and the fact that $\|\Pi_{\mathscr{H}}\|_{H_\hbar^s\to H_\hbar^s}=1$,

$$\|\Pi_{\mathscr{H}}(I+L)^{-1}(I-P_N)\Pi_{\mathscr{H}}\|_{H_\hbar^s\to H_\hbar^s}$$
$$\leq \|\Pi_{\mathscr{H}}\|_{H_\hbar^s\to H_\hbar^s}\|(1-\chi(\hbar^2\Delta_\Gamma))(I+L)^{-1}\|_{H_\hbar^s\to H_\hbar^s}\|(I-P_N)\|_{H_\hbar^s\to H_\hbar^s}\|\Pi_{\mathscr{H}}\|_{H_\hbar^s\to H_\hbar^s}$$
$$\leq \big(L_{\max}+C\hbar\big)\|(I-P_N)\|_{H_\hbar^s\to H_\hbar^s}.$$

$\blacksquare$

Proof of Theorem 4.2. By Lemma 4.11, it is enough to study $(I+P_NL)^{-1}(I-P_N)$ and hence we use (4.29). Decomposing $H_\hbar^s=\Pi_{\mathscr{L}}H_\hbar^s\oplus\Pi_{\mathscr{H}}H_\hbar^s$, letting $\mathscr{C}$ be as in (4.34), and using Lemma 4.13, we obtain (4.35). Therefore, under the condition (4.3), (4.36) holds, and then $I+(P_N-I)L(I+L)^{-1}$ is invertible with (4.37).

We now claim that

$$\big\|\big(I+\mathcal{C}(P_N-I)L(I+L)^{-1}\mathcal{C}^{-1}\big)^{-1}\big\|_{H_\hbar^s\to H_\hbar^s}$$
$$\leq \begin{pmatrix} 1+C(k/N)^{t_{\max}-q_{\min}}\rho & C\Big((k/N)^{t_{\max}-q_{\min}}\rho\Big)^{1/2}(k/N)^{1/2} \\ C\Big((k/N)^{t_{\max}-q_{\min}}\rho\Big)^{1/2}(k/N)^{1/2} & 1+Ck/N \end{pmatrix}. \tag{4.44}$$

To see this, let

$$M := \begin{pmatrix} \delta & \delta^{1/2}(k/N)^{1/2} \\ \delta^{1/2}(k/N)^{1/2} & (k/N) \end{pmatrix}$$

and observe that $M^2=(\delta+k/N)M$. Thus

$$\sum_{n=0}^\infty M^n = I+\Big(\sum_{n=0}^\infty(\delta+k/N)^n\Big)M \leq I+CM$$

if both δ and k/N are sufficiently small. Applying this last inequality with $\delta = (k/N)^{t_{\max}-q_{\min}}\rho$, we obtain (4.44).

Then, by (4.37) and the definition of $\mathcal{C}$ (4.34),

$$\left\|\big(I + (P_N - I)L(I+L)^{-1}\big)^{-1}\right\|_{H^s_\hbar \to H^s_\hbar} \leq \begin{pmatrix} 1 + C(k/N)^{t_{\max}-q_{\min}}\rho & C(k/N)^{s+1-q_{\min}} \\ C(k/N)^{t_{\max}-s}\rho & 1 + Ck/N \end{pmatrix}. \tag{4.45}$$

By (4.1),

$$\|(I-P_N)\|_{H^{t_{\max}}_\hbar \to H^{q_{\min}}_\hbar} \leq (k/N)^{t_{\max}-q_{\min}}, \qquad \|(I-P_N)\|_{H^s_\hbar \to H^{q_{\min}}_\hbar} \leq (k/N)^{s-q_{\min}},$$
$$\|(I-P_N)\|_{H^{t_{\max}}_\hbar \to H^s_\hbar} \leq (k/N)^{t_{\max}-s},$$

so that, by (4.38),

$$\|(I-P_N)\|_{H^s_\hbar \to H^s_\hbar} \leq \begin{pmatrix} C(k/N)^{t_{\max}-q_{\min}} & C(k/N)^{s-q_{\min}} \\ C(k/N)^{t_{\max}-s} & \|I-P_N\|_{H^s_\hbar \to H^s_\hbar} \end{pmatrix}. \tag{4.46}$$

Thus, by (4.45), (4.46), and the fact that $(k/N)^{t_{\max}-q_{\min}}\rho \leq c$ by (4.3),

$$\begin{aligned} &\left\|\big(I + (P_N - I)L(I+L)^{-1}\big)^{-1}(I-P_N)\right\|_{H^s_\hbar \to H^s_\hbar} \\ &\quad\leq \begin{pmatrix} C(k/N)^{t_{\max}-q_{\min}} & C(k/N)^{s-q_{\min}} \\ C(k/N)^{t_{\max}-s} & \|I-P_N\|_{H^s_\hbar \to H^s_\hbar} + C(k/N) + C(k/N)^{t_{\max}-q_{\min}}\rho \end{pmatrix}, \end{aligned} \tag{4.47}$$

where we have bounded $\|I-P_N\|_{H^s_\hbar \to H^s_\hbar}$ by a constant (via (4.1)) when it is multiplied by a term that is small. By Lemma 4.13,

$$\|(I+L)^{-1}(I-P_N)\|_{H^s_\hbar \to H^s_\hbar} \leq \begin{pmatrix} C(k/N)^{t_{\max}-q_{\min}}\rho & C(k/N)^{s-q_{\min}}\rho \\ C(k/N)^{t_{\max}-s} & (L_{\max} + C\hbar)\|I-P_N\|_{H^s_\hbar \to H^s_\hbar}. \end{pmatrix}. \tag{4.48}$$

Combining (4.29), (4.47), and (4.48), we obtain that

$$\begin{aligned} &\|(I+P_N L)^{-1}(I-P_N)\|_{H^s_\hbar \to H^s_\hbar} \\ &\leq \|(I+L)^{-1}(I-P_N)\|_{H^s_\hbar \to H^s_\hbar}\|I + (P_N-I)L(I+L)^{-1}(I-P_N)\|_{H^s_\hbar \to H^s_\hbar} \\ &\leq C\begin{pmatrix} C(k/N)^{t_{\max}-q_{\min}}\rho & C(k/N)^{s-q_{\min}}\rho \\ C(k/N)^{t_{\max}-s} & (L_{\max}+C\hbar)\|I-P_N\|^2_{H^s_\hbar \to H^s_\hbar} + C\big((k/N) + (k/N)^{t_{\max}-q_{\min}}\rho\big). \end{pmatrix}, \end{aligned}$$

where we have bounded $(L_{\max} + C\hbar)\|I-P_N\|_{H^s_\hbar \to H^s_\hbar}$ by a constant when it is multiplied by a term that is small. The bounds (4.5) and (4.7) now follow by combining the last displayed bound with Lemma 4.11.

5 Proofs of the Galerkin and collocation results for piecewise polynomials

5.1 Checking that the Dirichlet and Neumann BIEs in §1.1.1 satisfy Assumption 4.1

5.1.1 The Dirichlet BIEs.

We let

$$L := 2\big(K_k - \mathrm{i}\eta_D S_k\big) \quad \text{or} \quad 2\big(K'_k - \mathrm{i}\eta_D S_k\big)$$

so that $\mathcal{A} = 2A_k$ or $2A'_k$, respectively. The properties of S_k, K_k, and K_k as standard (i.e., not semiclassical) pseudodifferential operators imply that $L : H^s(\Gamma) \to H^{s+1}(\Gamma)$ for all $s \in \mathbb{R}$ (see, e.g., [93, Theorem 4.4.1]); thus $\mathcal{A}$ is bounded on $H^s(\Gamma)$. Under Assumption 1.1, $\eta_D \in \mathbb{R}$ with $|\eta|$ proportional to k, and then $\mathcal{A}$ is injective on $H^s(\Gamma)$; see, e.g., [26, Proof of Theorem 2.27]. Fredholm theory then implies that $\mathcal{A}$ is invertible on $H^s(\Gamma)$, and Part (i) of Assumption 4.1 holds.

Part (ii) of Assumption 4.1 then follows from Theorem 3.7.

By Theorem 3.7 and the results about the principal symbol in §3.4,

$$\sigma^0_\hbar\Big((1-\chi(|\hbar D'|^2_g))K_k\Big) = \sigma^0_\hbar\Big((1-\chi(|\hbar D'|^2_g))K'_k\Big) = 0,$$

so that

$$\sigma^0_\hbar\Big((1-\chi(|\hbar D'|^2_g))L\Big) = -2\mathrm{i}\eta\,\sigma^0_\hbar\Big((1-\chi(|\hbar D'|^2_g))S_k\Big) = -\mathrm{i}\left(\frac{\eta_D}{k}\right)\frac{(1-\chi(|\xi'|^2_g))}{\sqrt{|\xi'|^2_g-1}}.$$

Since η_D is real with modulus proportional to k (by Assumption 1.1), Part (iii) of Assumption 4.1 follows with $L_{\max} = 1$.

5.1.2 The Neumann BIEs.

We give the proof for $B_{k,\mathrm{reg}}$; the proof for $B'_{k,\mathrm{reg}}$ is very similar. Let

$$\widetilde{L} := -\mathrm{i}\eta_N K_k + S_{\mathrm{i}k}H_k + \frac{1}{4}I \quad \text{so that} \quad B_{k,\mathrm{reg}} = \left(\frac{\mathrm{i}\eta_N}{2}-\frac{1}{4}\right)I + \widetilde{L}.$$

We then let

$$L := \left(\frac{\mathrm{i}\eta_N}{2}-\frac{1}{4}\right)^{-1}\widetilde{L} \quad \text{so that} \quad \left(\frac{\mathrm{i}\eta_N}{2}-\frac{1}{4}\right)^{-1}B_{k,\mathrm{reg}} = I + L;$$

i.e., $\mathcal{A} = (\mathrm{i}\eta_N/2 - 1/4)^{-1}B_{k,\mathrm{reg}}$. The first of the Calderón relations

$$S_kH_k = -\frac{1}{4}I + (K_k)^2 \quad \text{and} \quad H_kS_k = -\frac{1}{4}I + (K'_k)^2 \tag{5.1}$$

(see, e.g., [26, Equation 2.56]) implies that

$$\widetilde{L} = -\mathrm{i}\eta_N K_k + \big(S_{\mathrm{i}k} - S_k\big)H_k + (K_k)^2, \tag{5.2}$$

and then $\widetilde{L} : H^s(\Gamma) \to H^{s+1}(\Gamma)$ for all $s \in \mathbb{R}$ by the properties of K_k, H_k, and $S_{\mathrm{i}k} - S_k$ as standard pseudodifferential operators (see, e.g., [93, Theorem 4.4.1] and [13, Theorems 2.1 and 2.2]); thus $B_{k,\mathrm{reg}}$ is bounded on $H^s(\Gamma)$. Under Assumption 1.1, $\eta_N \in \mathbb{R}\setminus\{0\}$ is independent of k, and then $B_{k,\mathrm{reg}}$ is injective on $H^s(\Gamma)$; see [45, Theorem 2.2]. Invertibility of $B_{k,\mathrm{reg}}$ on $H^s(\Gamma)$ then follows from Fredholm theory, and Part (i) of Assumption 4.1 holds.

We now show that Parts (ii) and (iii) of Assumption 4.1 are satisfied. By Theorem 3.7,

$$\sigma_\hbar\big((1-\chi(|\hbar D'|^2_g))\widetilde{L}\big) = \sigma_\hbar\big((1-\chi(|\hbar D'|^2_g))S_{\mathrm{i}k}H_k\big) + \frac{(1-\chi(|\xi'|^2_g))}{4}.$$

Now, by, e.g., [45, Lemma 4.8], $S_{\mathrm{i}k} = \hbar\widetilde{S}$ where $\widetilde{S} \in \Psi^{-1}_\hbar(\Gamma)$ with

$$\sigma_\hbar(\widetilde{S}) = \frac{1}{2\sqrt{|\xi'|^2_g+1}}; \tag{5.3}$$

observe that $S_{\mathrm{i}k}$ is then elliptic on $T^*\Gamma$, and hence invertible by Part (i) of Corollary 3.6. Let $\widetilde{\chi} \in C^\infty_c(\mathbb{R}^d;[0,1])$ be such that $\widetilde{\chi} \equiv 1$ on $[-1,1]$ and $\operatorname{supp}\widetilde{\chi} \cap \operatorname{supp}(1-\chi) = \emptyset$. Then, by (5.3), (3.24), (3.11), and (3.21)

$$\sigma_\hbar\big((1-\chi(|\hbar D'|^2_g))S_{\mathrm{i}k}H_k\big) = \sigma_\hbar\big((1-\chi(|\hbar D'|^2_g))S_{\mathrm{i}k}\big(1-\widetilde{\chi}(|\hbar D'|^2_g)\big)H_k\big) = -\frac{1-\chi(|\xi'|^2_g)}{4}\sqrt{\frac{|\xi'|^2_g-1}{|\xi'|^2_g+1}},$$

so that

$$\sigma_\hbar\big((1-\chi(|\hbar D'|_g^2))\widetilde{L}\big) = \frac{1-\chi(|\xi'|_g^2)}{4}\left(1-\sqrt{\frac{|\xi'|_g^2-1}{|\xi'|_g^2+1}}\right) = \frac{1-\chi(|\xi'|_g^2)}{2(|\xi'|_g^2+1)}\left(1+\sqrt{1-\frac{2}{|\xi'|_g^2+1}}\right)^{-1}. \tag{5.4}$$

Since (5.4) is real on $\{(x',\xi') : 1-\chi(|\xi'|_g^2) > 0\}$ and $\eta_N \neq 0$, Part (iii) of Assumption 4.1 holds with $L_{\max} = 1$. The expression (5.4) implies that $(1-\chi(|\hbar D'|_g^2))\widetilde{L}$ is the sum of an operator in $\Psi_\hbar^{-2}(\Gamma)$ and an operator in $\hbar\Psi_\hbar^{-1}(\Gamma)$, and thus Part (ii) of Assumption 4.1 holds.

Remark 5.1 (More general regularising operators than $S_{\mathrm{i}k}$) *The properties of $S_{\mathrm{i}k}$ that are used in the above arguments to show that $B_{k,\mathrm{reg}}$ and $B'_{k,\mathrm{reg}}$ satisfy Assumption 4.1 are that $S_{\mathrm{i}k} \in \hbar\Psi_\hbar^{-1}(\Gamma)$, is elliptic, and its semiclassical principal symbol is real (with these assumptions equal to [45, Assumption 1.1]). The results in this paper therefore hold with $S_{\mathrm{i}k}$ replaced by any other operator satisfying these assumptions (such as the quantisation of the principal symbol of $S_{\mathrm{i}k}$, which is considered in [14]).*

Remark 5.2 (Regularisation understood via the Calderón relations) *The smoothing property in Part (ii) of Assumption 4.1 can also be checked by using the expression (5.2) and the fact that*

$$\big(1-\chi(|\hbar D'|_g^2)\big)(S_k - S_{\mathrm{i}k}) \quad \textit{and} \quad (S_k - S_{\mathrm{i}k})\big(1-\chi(|\hbar D'|_g^2)\big) \in \hbar\Psi_\hbar^{-3}(\Gamma).$$

with real principal symbols, which follows from the results in [45, §3,4].

5.2 Proof of Theorem 2.5 (Galerkin method with piecewise polynomials)

Lemma 5.3 *Let $p \geq 0$ and V_h be a space of piecewise polynomials on a mesh of width h satisfying Assumption 2.4. Then the approximation property (4.1) holds with $N = h$ and $P^G_{V_h}$ defined by (2.1), for all the pairs (t,q) in (4.2) with $s = 0$, $t_{\max} = p+1$, and $q_{\min} = -p-1$.*

Proof. Multiplying (2.2) by k^{-q}, then using that $k^{-t}\|v\|_{H^t(\Gamma)} \leq \|v\|_{H^t_k(\Gamma)}$ by (3.6), we find that

$$\|v - \mathcal{I}_h v\|_{L^2(\Gamma)} \leq C(hk)^t \|v\|_{H^t_k(\Gamma)} \qquad \text{for } 0 \leq t \leq p+1.$$

The minimisation property (2.1) (with $s = 0$) then implies that

$$\big\|I - P^G_{V_h}\big\|_{H^t_\hbar(\Gamma)\to L^2(\Gamma)} \leq C(hk)^t \qquad \text{for } 0 \leq t \leq p+1. \tag{5.5}$$

i.e., (4.1) holds for $(t,q) = (p+1,0)$ and $(t,q) = (1,0)$. Now, by (5.5),

$$\begin{aligned}\big\|(I-P^G_{V_h})u\big\|_{H_\hbar^{-1}(\Gamma)} &:= \sup_{v\in H^1_\hbar(\Gamma)} \frac{\big|\big((I-P^G_{V_h})u, v\big)_{L^2(\Gamma)}\big|}{\|v\|_{H^1_\hbar(\Gamma)}} \\ &= \sup_{v\in H^1_\hbar(\Gamma)} \frac{\big|\big((I-P^G_{V_h})u, (I-P^G_{V_h})v\big)_{L^2(\Gamma)}\big|}{\|v\|_{H^1_\hbar(\Gamma)}} \leq C\left(\frac{k}{N}\right)^{p+2}\|u\|_{H^{p+1}_\hbar(\Gamma)},\end{aligned} \tag{5.6}$$

so that (4.1) holds for $(t,q) = (p+1,-1)$. Since $I - P^G_{V_h}$ is the L^2-orthogonal projection, $(I-P^G_N)^* = I - P^G_N$; thus (4.1) then holds for $(t,q) = (0,-p-1)$ (from from (5.5)) and $(t,q) = (1,-p-1)$ (from (5.6)). ∎

With Lemma 5.3 in hand, we apply Theorem 4.2 with $s = 0$, $t_{\max} = p+1$, and $q_{\min} = -p-1$. We see that if c is sufficiently small, then the condition (2.3) ensures that (4.3) holds, and then the quasi-optimal error estimate (2.4) follows from (4.5), together with the facts that $\|I - P^G_{V_h}\|_{L^2(\Gamma)\to L^2(\Gamma)} = 1$, $L_{\max} = 1$.

To prove the bound (2.5) on the relative error, observe that the combination of (4.1) with $(t,q) = (p+1,0)$ and (7.2) imply that

$$\big\|(I-P^G_{V_h})v\big\|_{L^2(\Gamma)} \leq C(hk)^{p+1}\|v\|_{H^{p+1}_k(\Gamma)} \leq CC_{\mathrm{osc}}(hk)^{p+1}\|v\|_{L^2(\Gamma)}. \tag{5.7}$$

Inputting (5.7) into the quasi-optimal error estimate (2.4), we find (2.5).

5.3 Proof of Theorem 2.8 (collocation method with piecewise polynomials)

Lemma 5.4 *Let $d = 2$, $p \geq 1$ and V_h be a space of piecewise polynomials on a mesh of width h satisfying Assumption 2.7. Then the approximation property (4.1) holds with $P^C_{V_h}$ given by the interpolation operator, for the pairs (t, q) equal to (4.2) with $\frac{1}{2} < s \leq 1$, $t_{\max} = p + 1$, $q_{\min} = 0$.*

Proof. Applying (2.6) with first $q = 0$ and then $q = 1$, and using that $k^{-t}|v|_{H^t(\Gamma)} \leq \|v\|_{H^t_k(\Gamma)}$ by (3.6), we obtain that

$$\|v - \mathcal{I}_h v\|_{L^2(\Gamma)} \leq C(hk)^t \|v\|_{H^t_\hbar(\Gamma)} \tag{5.8}$$

and

$$k^{-1}|v - \mathcal{I}_h v|_{H^1(\Gamma)} \leq C(hk)^{t-1} \|v\|_{H^t_\hbar(\Gamma)} \tag{5.9}$$

for $\frac{1}{2} < t \leq p + 1$. Combining (5.9) and (5.8) to obtain a bound on $\|v - \mathcal{I}_h v\|_{H^1_k(\Gamma)}$, and then interpolating the result with (5.8), we obtain that

$$\|v - \mathcal{I}_h v\|_{H^q_k(\Gamma)} \leq C(hk)^{t-q} \|v\|_{H^t_\hbar(\Gamma)} \tag{5.10}$$

for $0 \leq q \leq 1$ and $\frac{1}{2} < t \leq p+1$. Setting $t_{\max} = p+1$ and $q_{\min} = 0$, we see that the approximation property (4.1) holds for the pairs (t, q) equal to (4.2) only when $\frac{1}{2} < s \leq 1$ (for the first pair in (4.2) we need $s \leq 1$, for the second pair we need $0 \leq s \leq 1$, and for the third pair we need $s > \frac{1}{2}$). ■

With Lemma 5.4 in hand, the rest of the proof is the essentially the same as the proof of Part (i) of Theorem 2.5 (i.e., using Theorem 4.2), except that now $s = 1$ instead of $s = 0$. ■

6 Definitions of the projections for trigonometric polynomials in 2-d

6.1 Recap of Fourier series results and the definition of $H^s_\hbar(0, 2\pi)$ in terms of Fourier series.

Given $f \in L^1(0, 2\pi)$, let

$$\widehat{f}(n) := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \exp(-int) f(t)\, dt.$$

Recall that if $f \in L^2(0, 2\pi)$ then

$$f(t) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \exp(int)\, \widehat{f}(n) \quad \text{and} \quad \|f\|^2_{L^2(0,2\pi)} = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2. \tag{6.1}$$

Let

$$\langle f, g \rangle_{H^s_\hbar(0,2\pi)} := \sum_{n=-\infty}^{\infty} \widehat{f}(n)\, \overline{\widehat{g}(n)} \langle n\hbar \rangle^{2s}$$

(where recall that $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$) and

$$|||f|||^2_{H^s_\hbar(0,2\pi)} := \langle f, f \rangle_{H^s_\hbar} := \sum_{n=-\infty}^{\infty} \big|\widehat{f}(n)\big|^2 \langle n\hbar \rangle^{2s}. \tag{6.2}$$

Given $s \in \mathbb{R}$ there exist $C_1, C_2 > 0$ such that, for all $v \in H^s_\hbar(\Gamma)$,

$$C_1 \|v\|_{H^s_\hbar(\Gamma)} \leq |||v \circ \gamma|||_{H^s_\hbar(0,2\pi)} \leq C_2 \|v\|_{H^s_\hbar(\Gamma)}, \tag{6.3}$$

where $\|\cdot\|_{H^s_\hbar(\Gamma)}$ is defined by (3.5). Let

$$\phi_{m,\hbar}(t) := \frac{\exp(imt)}{\sqrt{2\pi} \langle m\hbar \rangle^s}. \tag{6.4}$$

Then $\widehat{\phi}_m(n) = \delta_{mn} \langle m\hbar \rangle^{-s}$, $|||\phi_m|||^2_{H^s_\hbar(0,2\pi)} = 1$, and the Fourier expansion in (6.1) implies that

$$f(t) = \sum_{n=-\infty}^{\infty} \langle f, \phi_{m,\hbar} \rangle_{H^s_\hbar(0,2\pi)} \phi_{m,\hbar}(t).$$

6.2 The Galerkin projection.

For $v \in L^2(0, 2\pi)$, let

$$P^G_{\mathscr{T}_N} v(t) := \sum_{m=-N}^{N} \langle v, \phi_{m,\hbar} \rangle_{L^2(0,2\pi)} \phi_{m,\hbar}(t) = \sum_{m=-N}^{N} \widehat{v}(m) \frac{\exp(imt)}{\sqrt{2\pi}}. \tag{6.5}$$

Then $P^G_{V_N} : L^2(0, 2\pi) \to V_N$ is a projection, where V_N is defined by (2.20). and (6.5) is equivalent to (2.1) with the the norm (2.22).

Lemma 6.1 (Approximation property of Galerkin projection) *For all $t \geq q$ there is $C > 0$ such that*

$$\|I - P^G_{\mathscr{T}_N}\|_{H^t_\hbar(\Gamma) \to H^q_\hbar(\Gamma)} \leq \frac{C}{\langle (N + 1)\hbar \rangle^{t-q}}. \tag{6.6}$$

Proof. By (6.5) and (6.2), for $v \in H^t_\hbar(0, 2\pi)$,

$$\left|\!\left|\!\left| (I - P^G_{\mathscr{T}_N})v \right|\!\right|\!\right|^2_{H^q_\hbar(0,2\pi)} = \frac{1}{2\pi} \sum_{|m| \geq N+1} |\widehat{v}(n)|^2 \frac{\langle n\hbar \rangle^{2t}}{\langle n\hbar \rangle^{2(t-q)}} \leq \frac{1}{\langle (N + 1)\hbar \rangle^{2(t-q)}} |||v|||^2_{H^t_\hbar(0,2\pi)};$$

the result then follows from the norm equivalence (6.3). ∎

6.3 The collocation projection.

With V_N defined by (2.20) and t_j defined by (2.21), let

$$(P^C_{\mathscr{T}_N} v)(t_j) = v(t_j), \quad j = 0, 1, \ldots, 2N, \quad \text{and} \quad P^C_{\mathscr{T}_N} v \in V_N \tag{6.7}$$

(i.e., (2.3) with x_j replaced by t_j). The points t_j are unisolvent for V_N – and hence $P^C_{\mathscr{T}_N}$ is a well-defined projection – by, e.g., [72, §11.3], [5, §3.2.2]. We now recall the explicit expression for $P^C_{\mathscr{T}_N}$ in terms of the Lagrange basis functions (see, e.g., [5, §3.2.2, Page 62]). For $j = 0, 1, \ldots, 2N$, let

$$\ell_j(t) := \frac{2}{2N + 1} D_N(t - t_j), \quad \text{where} \quad D_N(t) := \frac{1}{2} + \sum_{j=1}^{N} \cos(jt) = \frac{\sin(N + 1/2)t}{2\sin(t/2)};$$

the definition of $D_N(t)$ implies that $\ell_j(t_m) = \delta_{jm}$. Then

$$P^C_{\mathscr{T}_N} v(t) = \sum_{j=0}^{2N} v(t_j)\ell_j(t). \tag{6.8}$$

The definition (6.7) implies that

$$\text{if } m = \ell(2N + 1) + \mu \text{ with } |\mu| \leq N \text{ then } \big(P^C_{\mathscr{T}_N}(\exp(im\cdot)\big)(t) = \exp(i\mu t). \tag{6.9}$$

Observe that the definition of $P^C_{V_N}$ (6.7) implies that for $v \in \mathscr{T}_N$, $P^C_{\mathscr{T}_N} v = v$, so $(I - P^C_{\mathscr{T}_N})P^G_{\mathscr{T}_N} = 0$, and thus

$$I - P^C_{\mathscr{T}_N} = (I - P^C_{\mathscr{T}_N})(I - P^G_{\mathscr{T}_N}). \tag{6.10}$$

Lemma 6.2 (Approximation property of collocation projection) *Given $t \geq q \geq 0$ with $t > 1/2$, then there exists $C > 0$ such that if $(N + 1) \geq k$ then*

$$\|I - P^C_{\mathscr{T}_N}\|_{H^t_\hbar(\Gamma) \to H^q_\hbar(\Gamma)} \leq C \left(\frac{k}{N}\right)^{t-q}. \tag{6.11}$$

Proof. The analogous bound when $\hbar = 1$ (i.e., the unweighted case) is proved in, e.g., [73, Theorem 2.1], [99, Lemma 4.1], [72, Theorem 11.8], [100, Theorem 8.3.1]. The bound (6.11) follows from repeating these proofs, now with $k \neq 1$; the requirement that $(N + 1)k^{-1} \geq 1$ is used in the inequality $\langle ax \rangle \leq |a|\langle x \rangle$ when $|a| \geq 1$. ∎

7 Best approximation by trigonometric polynomials in plane-wave scattering

7.1 Statement of the two main results in this section

The abstract results in §4 give conditions under which the projection-method error, $v - v_N$, is bounded in terms of the projection error $\|(I - P^G_{\mathscr{T}_N})v\|_{H^s_\hbar}$. In this section we bound the projection error $\|(I - P^G_{\mathscr{T}_N})v\|_{H^s_\hbar}$ in terms of $\|v\|_{H^s_\hbar}$ when v is the BIE solution corresponding to the plane-wave scattering problem (i.e., when the right-hand side f of $\mathcal{A}v = f$ is given by Theorem A.2). These bounds then lead to the bounds on the relative error for plane wave data (2.5), (2.9), (2.25), and (2.28)).

The two main results of this section are the following two theorems.

Theorem 7.1 (Galerkin and collocation projection error for trigonometric polynomials) *Suppose $\mathscr{T}_N$ is given by (2.20) and that $\mathcal{A}$ is one of A_k, A'_k, $B_{k,\mathrm{reg}}$, or $B'_{k,\mathrm{reg}}$ and the right-hand side f is as described in Theorem A.2.*

Then $v \in H^s(\Gamma)$ for all $s \geq 0$ and, given $\varepsilon > 0$ and $k_0 > 0$, if $N \geq (c_{\max} + \varepsilon)k$ and $k \geq k_0$, then

$$\frac{\left\|(I - P^G_{\mathscr{T}_N})v\right\|_{H^s_k(\Gamma)}}{\|v\|_{H^s_k(\Gamma)}} = O(k^{-\infty}), \tag{7.1}$$

where $P^G_{\mathscr{T}_N}$ is defined by (2.1)/(6.5).

Furthermore, if $s > 1/2$, then (7.1) holds with $P^G_{\mathscr{T}_N}$ replaced by $P^C_{\mathscr{T}_N}$ defined by Definition 2.3/Equation (6.7).

The result in Theorem 7.1 for $I - P^C_N$ follows immediately from the result for $I - P^G_N$ by using the identity (6.10) and the fact that $I - P^C_N$ is bounded on H^s_k (by Lemma 6.2).

Theorem 7.2 (Oscillatory behaviour of BIE solution under plane-wave scattering) *Suppose that $\mathcal{A}$ is one of A_k, A'_k, $B_{k,\mathrm{reg}}$, or $B'_{k,\mathrm{reg}}$ and the right-hand side f is as described in Theorem A.2. Then $v \in H^t(\Gamma)$ for all $t \geq 0$ and given $t \geq s \geq 0$ and $k_0 > 0$ there exists $C_{\rm osc}(t, s, k_0) > 0$ such that*

$$\|v\|_{H^t_k(\Gamma)} \leq C_{\rm osc} \|v\|_{H^s_k(\Gamma)} \qquad \textit{for all } k \geq k_0. \tag{7.2}$$

In §5.2, the bound in Theorem 7.2 is combined with the polynomial-approximation assumptions Assumptions 2.4 and 2.7 to prove bounds on $\|(I - P_N)v\|_{H^s_k(\Gamma)}$ (where P_N is either the Galerkin or collocation projection).

7.2 Bounds on the high-frequency components of f and v

In the next lemma we use the notation that γ^+ is the Dirichlet trace on Γ from Ω^+ (as used in §A).

Lemma 7.3 (The high-frequency components of f are superalgebraically small) *Suppose that $B \in \Psi^m_\hbar(\Gamma)$ and $\chi \in C^\infty_c(\mathbb{R}; [0, 1])$ with $\operatorname{supp}(1 - \chi) \cap [-R, R]$ for some $R > 0$. Then for all $N, \hbar_0 > 0$ there exists $C_N > 0$ such that for all $a \in \mathbb{R}^d$ with $|a| \leq R$ and for all $0 < \hbar \leq \hbar_0$,*

$$\|(I - \chi(|\hbar D'|^2_g))B\gamma^+ \mathrm{e}^{\mathrm{i}kx\cdot a}\|_{H^N_\hbar(\Gamma)} \leq C_N \hbar^N.$$

The idea behind the proof is that if $|a| \leq R$ then $B\gamma^+\mathrm{e}^{\mathrm{i}kx\cdot a}$ contains frequencies $\leq kR$, and $(I - \chi(|\hbar D'|^2_g)$ is a frequency cut-off to frequencies $> kR$; we now proof this rigorously using the quantisation definition (3.8) and integration by parts.

Proof. By a partition of unity, we can work in local coordinates where

$$\Gamma = \big\{(x', F(x')) \,:\, x' \in U \subset \mathbb{R}^{d-1}\big\}.$$

By the composition formula for symbols [112, Theorem 4.14], [35, Proposition E.8], $(I - \chi(|\hbar D'|_g^2))B = \mathrm{Op}_\hbar(b) + O(\hbar^\infty)_{\Psi^{-\infty}(\Gamma)}$ for some $b \in S^m$ satisfying $\operatorname{supp} b \cap \{|\xi'|_g \le R\} = \emptyset$. For such a b and $\psi \in C_c^\infty(\mathbb{R}^{d-1})$ supported in U, by (3.8),

$$\mathrm{Op}_\hbar(b)\psi(x')\gamma^+ \mathrm{e}^{\mathrm{i}kx\cdot a} = (2\pi\hbar)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(\langle x'-y',\xi'\rangle + \langle a',y'\rangle + F(y')a_1)} b(x',\xi')\psi(y')dy'd\xi'. \tag{7.3}$$

In these coordinates, the metric g on Γ is given by

$$g(\partial_{x^i},\partial_{x^j}) = \delta_{ij} + \partial_{x^i}F\,\partial_{x^j}F$$

and hence, in these coordinates,

$$|\xi'|_g^2(x') = \langle G(x')\xi',\xi'\rangle, \qquad G(x') = I - \frac{1}{1+|\partial_{x'}F(x')|^2}(\partial_{x'}F(x'))^t(\partial_{x'}F(x')).$$

The phase in the integrand in (7.3) is stationary in y' when $\xi'_* = a' + a_1\partial_{y'}F(y')$, and

$$|\xi'_*|_g^2 = \Big\langle G(x')\big(a' + a_1\partial_{y'}F(y')\big), \big(a' + a_1\partial_{y'}F(y')\big)\Big\rangle.$$

By direct computation, we find that $|\xi'_*|_g^2 \le |a|^2 \le R^2$. Therefore, since $|\xi'|_g > R$ on the support of the integrand in (7.3), the phase is non-stationary in y' and repeated integration by parts in y' (see, e.g., [112, Lemmas 3.10 and 3.14]) shows that

$$\big\|\mathrm{Op}_\hbar(b)\psi(x')\gamma^+\mathrm{e}^{\mathrm{i}kx\cdot a}\big\|_{H_\hbar^N(\Gamma)} \le C_N\hbar^N.$$

■

Lemma 7.4 (The high-frequency components of v are superalgebraically small) *Suppose that $\mathcal{A}$ is one of A_k, A'_k, $B_{k,\mathrm{reg}}$, or $B'_{k,\mathrm{reg}}$ and the right-hand side f is as described in Theorem A.2. For any $\widetilde{\chi} \in C_c^\infty(\mathbb{R}^d;[0,1])$ with $\operatorname{supp}(1-\widetilde{\chi}) \cap [-1,1] = \emptyset$ and any $s > 0$,*

$$\frac{\big\|(I - \widetilde{\chi}(|\hbar D'|_g^2))v\big\|_{H_\hbar^s(\Gamma)}}{\|v\|_{H_\hbar^s(\Gamma)}} = O(\hbar^\infty).$$

To prove Lemma 7.4 we need the following result.

Lemma 7.5 (Lower bound on $\|v\|_{H_\hbar^s(\Gamma)}$) *Suppose that $\mathcal{A}$ is one of A_k, A'_k, $B_{k,\mathrm{reg}}$, or $B'_{k,\mathrm{reg}}$ and the right-hand side f is as described in Theorem A.2. Given $s > 0$ there exists $M \in \mathbb{R}$ such that, given $\hbar_0 > 0$, there exists $C > 0$ such that*

$$\|v\|_{H_\hbar^s(\Gamma)} \ge C\hbar^M \quad \textit{for all } 0 < \hbar \le \hbar_0.$$

Proof. Since

$$\|v\|_{H_\hbar^s(\Gamma)} \ge \|f\|_{H_\hbar^s(\Gamma)}\big(\|\mathcal{A}\|_{H_\hbar^s(\Gamma)\to H_\hbar^s(\Gamma)}\big)^{-1},$$

we see that the required bound from below on $\|v\|_{L^2(\Gamma)}$ follows if we can show that (i) $\|f\|_{H_\hbar^s(\Gamma)}$ is bounded below algebraically in $\hbar$, and (ii) $\|\mathcal{A}\|_{H_\hbar^s(\Gamma)\to H_\hbar^s(\Gamma)}$ is bounded above algebraically in $\hbar^{-1}$.

Condition (ii) follows from Lemma 4.8, the bounds on $\|A'_k\|_{L^2(\Gamma)\to L^2(\Gamma)} = \|A_k\|_{L^2(\Gamma)}$ in [25], [47, Theorem 2], [58, Theorem A.1], [39, Theorem 4.5], and the bounds on $\|B_{k,\mathrm{reg}}\|_{L^2(\Gamma)\to L^2(\Gamma)} = \|B'_{k,\mathrm{reg}}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ in [45, Theorems 2.1].

Condition (i) follows from explicitly calculating the $\hbar$ dependence of $\|f\|_{H_\hbar^s(\Gamma)}$ for the four different fs given in Theorem A.2; for the f for $B_{k,\mathrm{reg}}$, we additionally have to use the bound $\|(S_{\mathrm{i}k})^{-1}\|_{H_\hbar^{s-1}(\Gamma)\to H_\hbar^s(\Gamma)} \lesssim \hbar^{-1}$ from [45, Corollary 3.7]. ■

Proof of Lemma 7.4. By §5.1, $\mathcal{A}$ satisfies Assumption 4.1 and thus the conclusions of Lemma 4.4 hold. Given $\widetilde{\chi}$, let $\chi \in C_c^\infty(\mathbb{R};[0,1])$ be such that $\operatorname{supp}(1-\chi)\cap[-1,1] = \emptyset$, and $\operatorname{supp}\chi \cap \operatorname{supp}(1-\widetilde{\chi}) = \emptyset$; this implies that $\{\widetilde{\chi} \not\equiv 1\} \Subset \{\chi \not\equiv 1\}$, and thus, by (4.11) and Lemma 4.4, (4.18) holds. The elliptic estimate (3.22) then implies that, given $s, M, N \in \mathbb{R}$, there exists $C, \hbar_0 > 0$ such that, for all $0 < \hbar \le \hbar_0$,

$$\begin{aligned}\big\|\big(1 - \widetilde{\chi}(|\hbar D'|_g^2)\big)v\big\|_{H_\hbar^s(\Gamma)} &\le C\Big(\big\|\big(1 - \chi(|\hbar D'|_g^2)\big)\mathcal{A}v\big\|_{H_\hbar^s(\Gamma)} + \hbar^M\|v\|_{H_\hbar^{s-N}(\Gamma)}\Big)\\ &= C\Big(\big\|\big(I - \chi(|\hbar D'|_g^2)\big)f\big\|_{H_\hbar^s(\Gamma)} + \hbar^M\|v\|_{H_\hbar^{s-N}(\Gamma)}\Big).\end{aligned} \tag{7.4}$$

The result then follows by choosing $N = 0$ and using Lemmas 7.3 and 7.5. ■

7.3 Proof of Theorem 7.2

First observe that it is sufficient to prove the bound (7.2) for k_0 sufficiently large. Let $\widetilde{\chi}$ be as in Lemma 7.4, and write

$$v = \big(1 - \widetilde{\chi}(|\hbar D'|_g^2)\big)v + \widetilde{\chi}(|\hbar D'|_g^2)v.$$

By (3.15), given $t \geq s$ and $\hbar_0 > 0$, there exists $C > 0$ such that, for all $0 < \hbar \leq \hbar_0$,

$$\left\|\widetilde{\chi}(|\hbar D'|_g^2)v\right\|_{H_\hbar^t(\Gamma)} \leq C \left\|v\right\|_{H_\hbar^s(\Gamma)}.$$

Combining this with the result of Lemma 7.4, we have

$$\left\|v\right\|_{H_\hbar^t(\Gamma)} \leq O(\hbar^\infty) \left\|v\right\|_{H_\hbar^t(\Gamma)} + C \left\|v\right\|_{H_\hbar^s(\Gamma)},$$

and the result follows.

7.4 Proof of Theorem 7.1

As noted after the statement of the theorem, the result in Theorem 7.1 for $I - P^C_{\mathscr{T}_N}$ follows immediately from the result for $I - P^G_{\mathscr{T}_N}$ by using the identity (6.10) and the fact that $I - P^C_{\mathscr{T}_N}$ is bounded on H^s_k (by Lemma 6.2).

To prove the result for $I - P^G_{\mathscr{T}_N}$, we first show that the operator $I - P^G_{\mathscr{T}_N}$ can be naturally expressed in terms of functions of ∂_t^2. Indeed, we claim that

$$P^G_{\mathscr{T}_N} = \mathbb{1}_{(-\infty,1]}\big(-N^{-2}\partial_t^2\big), \tag{7.5}$$

where ∂_t is understood as an operator on functions on Γ via $\partial_t u = (\partial_t(u \circ \gamma^{-1})) \circ \gamma$.

To see this, recall that for $f \in L^\infty(\mathbb{R})$ and $v \in L^2(\Gamma)$,

$$f(-\partial_t^2)v := \sum_{m=1}^{\infty} f(\lambda_m)(v, u_{\lambda_m})_{L^2(0,2\pi)} u_{\lambda_m}, \tag{7.6}$$

where $\{u_{\lambda_m}\}_{m=1}^\infty$ is an orthonormal basis for $L^2([0,2\pi])$ of eigenfunctions of $-\partial_t^2$; i.e.,

$$(-\partial_t^2 - \lambda_m)u_{\lambda_m} = 0 \quad \text{ and } \quad \|u_{\lambda_m}\|_{L^2(\Gamma)} = 1.$$

Thus

$$u_{\lambda_m}\big(\gamma(t)\big) = \frac{\exp(imt)}{\sqrt{2\pi}}, \quad \text{i.e.,} \quad u_{\lambda_m}(x) = \frac{\exp(im\gamma^{-1}(x))}{\sqrt{2\pi}},$$

where $\lambda_m = m^2$. Therefore, (7.6) implies that

$$\begin{aligned}
\mathbb{1}_{(-\infty,1]}\big(-N^{-2}\partial_t^2\big) &= \sum_{m=1}^{\infty} 1_{(-\infty,1]}\big(N^{-2}\lambda_m\big)\big(v, u_{\lambda_m}\big)_{L^2(0,2\pi)} u_{\lambda_m} \\
&= \sum_{\lambda_m \leq N^2} \widehat{v}(n)\frac{\exp(int)}{\sqrt{2\pi}} = \sum_{m=-N}^{N} \widehat{v}(n)\frac{\exp(int)}{\sqrt{2\pi}},
\end{aligned}$$

which is $P^G_{\mathscr{T}_N}$ given by (6.5), so that (7.5) holds, and also

$$(I - P^G_{\mathscr{T}_N}) = \mathbb{1}_{(1,\infty)}\big(-N^{-2}\partial_t^2\big) = \mathbb{1}_{\big((N\hbar)^2,\infty\big)}\big(-\hbar^2\partial_t^2\big). \tag{7.7}$$

Theorem 7.1 is a consequence of the following result.

Lemma 7.6 *Given $\Theta, \epsilon > 0$, let $\chi \in C_c^\infty(\mathbb{R}; [0,1])$ be such that* $\operatorname{supp}(1-\chi) \cap [-\Theta^2, \Theta^2] = \emptyset$ *and* $\operatorname{supp}\chi \subset [-(\Theta + \epsilon/(2c_{\max}))^2, (\Theta + \epsilon/(2c_{\max})^2)]$. *If $N \geq (\Theta c_{\max} + \epsilon)k$, then*

$$(I - P^G_{\mathscr{T}_N}) = (I - P^G_{\mathscr{T}_N})\big(I - \chi(|\hbar D'|_g^2)\big) + O(\hbar^\infty)_{\Psi_\hbar^{-\infty}(\Gamma)} = \big(I - \chi(|\hbar D'|_g^2)\big)(I - P^G_{\mathscr{T}_N}) + O(\hbar^\infty)_{\Psi_\hbar^{-\infty}(\Gamma)}.$$

The result (7.1) follows by applying Lemma 7.6 to v with $\Theta = 1$ and using Lemma 7.4 (noting that $\widetilde{\chi}$ in Lemma 7.4 can be taken to be χ via the choice $\Theta = 1$). It therefore remains to prove Lemma 7.6.

Proof of Lemma 7.6. Given $\varepsilon > 0$, let $\chi_0 \in C_c^\infty(\mathbb{R}; [0,1])$ be such that $\operatorname{supp}\chi_0 \subset (-(\Theta c_{\max} + \varepsilon)^2, (\Theta c_{\max} + \varepsilon)^2)$ and $\operatorname{supp}(1-\chi_0) \cap [-(\Theta c_{\max} + \varepsilon/2)^2, (\Theta c_{\max} + \varepsilon/2)^2] = \emptyset$.

If $N \geq (\Theta c_{\max} + \varepsilon)k$ then

$$(I - P^G_{\mathscr{T}_N})\chi_0(-\hbar^2\partial_t^2) = \chi_0(-\hbar^2\partial_t^2)(I - P^G_{\mathscr{T}_N}) = 0,$$

since the image of $\chi_0(-\hbar^2\partial_t^2)$ contains frequencies $< (\Theta c_{\max} + \varepsilon)k$, and $I - P^G_{\mathscr{T}_N}$ restricts to frequencies $\geq (\Theta c_{\max} + \varepsilon)k$ by (7.7). In particular,

$$(I - P^G_{\mathscr{T}_N}) = (I - P^G_{\mathscr{T}_N})\big(I - \chi_0(-\hbar^2\partial_t^2)\big) = \big(I - \chi_0(-\hbar^2\partial_t^2)\big)(I - P^G_{\mathscr{T}_N}).$$

Now, $-\partial_t^2 = -\Delta_{\Gamma, g_{\text{flat}}}$ where g_{flat} is the metric on Γ given by $|\dot{\gamma}(t)|_{g_{\text{flat}}(\gamma(t))} = 1$. Therefore, by Lemma 3.9,

$$\mathrm{WF}_\hbar(I - P^G_{\mathscr{T}_N}) \subset \Big\{|\xi'|^2_{g_{\text{flat}}} \geq (\Theta c_{\max} + \epsilon)^2\Big\}.$$

Since the arc length metric equals $|\dot{\gamma}(t)|^2 dt^2$,

$$\frac{|\xi'|_{g(\gamma(t))}}{|\xi'|_{g_{\text{flat}}(\gamma(t))}} = |\dot{\gamma}(t)|^{-1} \geq \inf_{t\in[0,2\pi]} |\dot{\gamma}(t)|^{-1} = c_{\max}^{-1}.$$

By the support properties of χ and (3.19),

$$\mathrm{WF}_\hbar\, \chi(|\hbar D'|_g^2) \subset \Big\{|\xi'|_g^2 \leq (\Theta + \epsilon/(2c_{\max}))^2\Big\} \subset \Big\{|\xi'|^2_{g_{\text{flat}}} \leq (\Theta c_{\max} + \epsilon/2)^2\Big\}$$

so that

$$\mathrm{WF}_\hbar\, \chi(|\hbar D'|_g^2) \cap \mathrm{WF}_\hbar(I - P^G_{\mathscr{T}_N}) = \emptyset;$$

the result then follows from (3.18). ∎

8 Proofs of the Galerkin and collocation results for trigonometric polynomials

Theorems 2.22 and 2.24 (on, respectively, the Galerkin and collocation method with trigonometric polynomials) are proved using the arguments in the proof of Theorem 4.2 with the additional structure of the Galerkin projection (in particular, (7.7)), and we prove these last.

8.1 Proof of Theorem 2.22 (Galerkin method with trigonometric polynomials)

First observe that the bound on the relative error (2.25) follows from the combination of (2.24) and (7.1). Therefore, we only need to prove the quasi-optimality bound (2.24)

Theorem 4.2 was based on the fact that the identity (4.29) holds under the condition (4.36). In the case of trigonometric polynomials, we instead use the simpler setup that (4.29) holds under the condition (4.30).

Let $\Xi \geq 1$, $\epsilon > 0$, and let χ be as in Lemma 7.6 with $\Theta := \Xi$; i.e. $\operatorname{supp}(1-\chi) \cap [-\Xi^2, \Xi^2] = \emptyset$ and $\operatorname{supp}\chi \subset [-(\Xi + \epsilon/(2c_{\max}))^2, (\Xi + \epsilon/(2c_{\max})^2)]$.

Then, if $N \geq (\Xi c_{\max} + \epsilon)k$, the combination of (4.29), Lemma 7.6, and Assumption 1.3 implies that

$$\begin{aligned}
&(I + P^G_{\mathscr{T}_N} L)^{-1}(I - P^G_{\mathscr{T}_N})\\
&= (I+L)^{-1}(I - \chi(|\hbar D'|_g^2))(I - P^G_{\mathscr{T}_N})\Big(I + (P^G_{\mathscr{T}_N} - I)(I - \chi(|\hbar D'|_g^2))L(I+L)^{-1}\Big)^{-1}(I - P^G_{\mathscr{T}_N})\\
&\qquad + O(\hbar^\infty)_{\Psi_\hbar^{-\infty}(\Gamma)}.
\end{aligned} \tag{8.1}$$

Now, by (4.22) and (4.24) from Lemma 4.9 and the fact that $L_{\max} = 1$ (by §5.1),

$$\big\|(I+L)^{-1}(I-\chi(|\hbar D'|_g^2))\big\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq (1+C\hbar) \tag{8.2}$$

(compare to (4.40)).

By the combination of Lemma 4.11, (8.1) and the bounds (8.2) and $\|(I-P^G_{\mathscr{T}_N})\|_{L^2\to L^2}\leq 1$, to prove (2.24), we only need to show that if $N \geq (\Xi R_{\min} c_{\max} + \epsilon)k$ then

$$\big\|(P^G_{\mathscr{T}_N} - I)(I-\chi(|\hbar D'|_g^2))L(I+L)^{-1}\big\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq C\big(\Xi^{-1}+\hbar\big)$$

(compare to (4.39)). Similarly, since $\|(I-P^G_{\mathscr{T}_N})\|_{L^2\to L^2}\leq 1$, it is sufficient to show that

$$\big\|(I-\chi(|\hbar D'|_g^2))L(I+L)^{-1}\big\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq C\big(\Xi^{-1}+\hbar\big). \tag{8.3}$$

To prove (8.3), we choose $\psi_i \in C_c^\infty(\mathbb{R};[0,1])$, $i=0,1$ with $\operatorname{supp}(1-\psi_i)\cap[-1,1]=\emptyset$, $\operatorname{supp}\psi_i\cap \operatorname{supp}(1-\chi)=\emptyset$, $\operatorname{supp}\psi_0\cap\operatorname{supp}(1-\psi_1)=\emptyset$ (i.e., ψ_0 is "smaller than" ψ_1 which is "smaller than" χ) – such a choice is possible since $\Xi\geq 1$, and thus $[-\Xi^2,\Xi^2]\supset[-1,1]$. Then, by Lemma 4.3, (4.23), (3.21), and Assumption 1.3,

$$\begin{aligned}
&\big(I-\chi(|\hbar D'|_g^2)\big)L(I+L)^{-1}\\
&\quad= \big(I-\chi(|\hbar D'|_g^2)\big)L\big(I-\psi_1(|\hbar D'|_g^2)\big)(I+L)^{-1} + O(\hbar^\infty)_{\Psi_\hbar^{-\infty}(\Gamma)}\\
&\quad= \big(I-\chi(|\hbar D'|_g^2)\big)L\big(I-\psi_1(|\hbar D'|_g^2)\big)\big(I+(1-\psi_0(|\hbar D'|_g^2))L\big)^{-1} + O(\hbar^\infty)_{\Psi_\hbar^{-\infty}(\Gamma)}\\
&\quad= \big(I-\chi(|\hbar D'|_g^2)\big)L\big(I+(1-\psi_0(|\hbar D'|_g^2))L\big)^{-1} + O(\hbar^\infty)_{\Psi_\hbar^{-\infty}(\Gamma)}\\
&\quad= \big(I-\chi(|\hbar D'|_g^2)\big)\big(1-\psi_0(|\hbar D'|_g^2)\big)L\big(I+\big(1-\psi_0(|\hbar D'|_g^2)\big)L\big)^{-1} + O(\hbar^\infty)_{\Psi_\hbar^{-\infty}(\Gamma)}.
\end{aligned} \tag{8.4}$$

By (4.24) from Lemma 4.9 and the fact that $L_{\max}=1$ (by §5.1),

$$\big\|\big(I+\big(1-\psi_0(|\hbar D'|_g^2)\big)L\big)^{-1}\big\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq 1+C\hbar,$$

and thus to prove (8.3) it is sufficient to prove that

$$\big\|\big(I-\chi(|\hbar D'|_g^2)\big)\big(1-\psi_0(|\hbar D'|_g^2)\big)L\big\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq C\big(\Xi^{-1}+\hbar\big). \tag{8.5}$$

Now, by Part (ii) of Assumption 4.1, $\big(1-\psi_0(|\hbar D'|_g^2)\big)L\in\Psi_\hbar^{-1}(\Gamma)$ so that, by the definition of S^m (3.7) and (3.14),

$$\big|\sigma_\hbar\big(\big(1-\psi_0(|\hbar D'|_g^2)\big)L\big)\big| \leq C\langle\xi'\rangle^{-1} \quad \text{for all } (x',\xi')\in T^*\Gamma.$$

Then, since $1-\chi(|\xi'|_g^2)=0$ for $|\xi'|_g\leq\Xi$,

$$\big|\sigma_\hbar\big(\big(I-\chi(|\hbar D'|_g^2)\big)\big(1-\psi_0(|\hbar D'|_g^2)\big)L\big)\big| \leq C\langle\xi'\rangle^{-1}\leq C'\Xi^{-1} \quad \text{for all } (x',\xi')\in T^*\Gamma.$$

The bound (8.5) then holds by Lemma 3.2, and the the proof is complete.

8.2 Proof of Theorem 2.24 (collocation method with trigonometric polynomials)

First observe that the bound (2.28) on the relative error then follows from combining (2.27) with Theorem 7.1, (6.10), and Assumption 1.3, since the superalgebraic decay in (7.1) (with $P^G_{\mathscr{T}_N}$ replaced by $P^C_{\mathscr{T}_N}$) absorbs the ρ on the right-hand side of (2.27). Therefore, we only need to prove the quasi-optimality bound (2.27).

As in the proof of Theorem 2.22, we use the identity (4.29) under the condition (4.30). Let $\chi\in C_c^\infty(\mathbb{R};[0,1])$ be such that Lemma 7.6 holds with $\Theta=1$. Then, by (6.10) and Lemma 7.6, if $N\geq(c_{\max}+\epsilon)k$,

$$\begin{aligned}
(P^C_{\mathscr{T}_N}-I)L(I+L)^{-1} &= (P^C_{\mathscr{T}_N}-I)(I-P^G_{\mathscr{T}_N})L(I+L)^{-1}\\
&= (P^C_{\mathscr{T}_N}-I)(I-P^G_{\mathscr{T}_N})(1-\chi(|\hbar D'|_g^2))L(I+L)^{-1} + O(\hbar^\infty)_{\Psi_\hbar^{-\infty}(\Gamma)}
\end{aligned}$$

$$= (P^C_{\mathscr{T}_N} - I)(1 - \chi(|\hbar D'|^2_g))L(I+L)^{-1} + O(\hbar^\infty)_{\Psi_\hbar^{-\infty}(\Gamma)}. \tag{8.6}$$

We now claim that if $N \geq (c_{\max} + \epsilon)k$ then

$$\big\|(P^C_N - I)(I - \chi(|\hbar D'|^2_g))L(I+L)^{-1}\big\|_{H^s_\hbar(\Gamma)\to H^s_\hbar(\Gamma)} \leq C\frac{k}{N}. \tag{8.7}$$

Indeed, since

$$\|I - P^C_{\mathscr{T}_N}\|_{H^{s+1}_\hbar(\Gamma)\to H^s_\hbar(\Gamma)} \leq C\frac{k}{N},$$

for $s > 1/2$ (by (6.11)),to prove (8.7) it is sufficient to prove that

$$\big\|(I - \chi(|\hbar D'|^2_g))L(I+L)^{-1}\big\|_{H^s_\hbar(\Gamma)\to H^{s+1}_\hbar(\Gamma)} \leq C. \tag{8.8}$$

Let $\widetilde{\chi} \in C^\infty_c(\mathbb{R};[0,1])$ be such that $\operatorname{supp}(1-\widetilde{\chi}) = \emptyset$ and $\operatorname{supp}(1-\chi)\cap\operatorname{supp}\widetilde{\chi} = \emptyset$. Then, by Lemma 4.3, (3.18), (3.17), and Assumption 1.3,

$$(I - \chi(|\hbar D'|^2_g))L(I+L)^{-1} = (I - \chi(|\hbar D'|^2_g))L(I - \widetilde{\chi}(|\hbar D'|^2_g))(I+L)^{-1} + O(\hbar^\infty)_{\Psi^{-\infty}}.$$

Part (ii) of Assumption 4.1 and Part (ii) of Theorem 3.1 imply that $\|(1 - \chi(|\hbar D'|^2_g))L\|_{H^s_\hbar(\Gamma)\to H^{s+1}_\hbar(\Gamma)} \leq C$, and then the bound (8.8) follows by combining this with (4.23), and (4.24).

The combination of (8.7) and (8.6) implies that if $N \geq (c_{\max} + \epsilon)k$ then

$$\|(I + (P^C_{\mathscr{T}_N} - I)L(I+L)^{-1})^{-1}\|_{H^s_\hbar(\Gamma)\to H^s_\hbar(\Gamma)} \leq 1 + C\frac{k}{N}. \tag{8.9}$$

Therefore, by (4.29), to prove Theorem 2.24 we only need to bound $(I+L)^{-1}(I - P^C_{\mathscr{T}_N})$. Now, by (4.43) and the fact that $L_{\max} = 1$,

$$\begin{aligned}
&\big\|(I+L)^{-1}(I - P^C_{\mathscr{T}_N})\big\|_{H^s_\hbar(\Gamma)\to H^s_\hbar(\Gamma)} \\
&\leq \big\|(I+L)^{-1}(1 - \chi(-\hbar^2\Delta_g))(I - P^C_{\mathscr{T}_N})\big\|_{H^s_\hbar(\Gamma)\to H^s_\hbar(\Gamma)} + \big\|(I+L)^{-1}\chi(-\hbar^2\Delta_g)(I - P^C_{\mathscr{T}_N})\big\|_{H^s_\hbar(\Gamma)\to H^s_\hbar(\Gamma)} \\
&\leq \|I - P^C_{\mathscr{T}_N}\|_{H^s_\hbar(\Gamma)\to H^s_\hbar(\Gamma)} + C\hbar + \big\|(I+L)^{-1}\chi(-\hbar^2\Delta_g)(I - P^C_{\mathscr{T}_N})\big\|_{H^s_\hbar(\Gamma)\to H^s_\hbar(\Gamma)}.
\end{aligned}$$

Finally, by the approximation property in Lemma 6.2 with $t = s$ and $q = 0$, the smoothing property of $\chi(-\hbar^2\Delta_g)$, and Lemma 4.6,

$$\big\|(I+L)^{-1}(I - P^C_{\mathscr{T}_N})\big\|_{H^s_\hbar(\Gamma)\to H^s_\hbar(\Gamma)} \leq \|I - P^C_{\mathscr{T}_N}\|_{H^s_\hbar(\Gamma)\to H^s_\hbar(\Gamma)} + C\hbar + C\left(\frac{k}{N}\right)^s\rho. \tag{8.10}$$

The combination of (4.29), (8.9), and (8.10) completes the proof of (2.27), and the proof of Theorem 2.24 is complete.

Before ending this section, we record the following corollary of our proof that we use in our study of the Nyström method.

Lemma 8.1 *Suppose that $\mathcal{A}$ satisfies Assumptions 4.1 and 1.3. Then for all $s \geq 0$, $k_0 > 0$ there there is $C > 0$ such that for all $k \geq k_0$, $k \notin \mathcal{J}$, and $N \geq Ck$,*

$$(I + P^C_{\mathscr{T}_N}L)^{-1} = (I+L)^{-1}\Big(I + (P^C_{\mathscr{T}_N} - I)L(I+L)^{-1}\Big)^{-1}. \tag{8.11}$$

Proof. The combination of (8.6) and (8.7) imply that

$$\|(P^C_{\mathscr{T}_N} - I)L(I+L)^{-1}\|_{H^s_\hbar\to H^s_\hbar} \leq C'k/N.$$

The result (8.11) then follows from (4.28). ∎

9 Description of the Nyström method

Assumption 9.1 (Class of weakly-singular integrals) *There exist $L_j(t,\tau)$, $j=1,2$, such that*

$$(Lv)(t) := \int_0^{2\pi} \log\left(4\sin^2\left(\frac{t-\tau}{2}\right)\right) L_1(t,\tau)\, v(\tau)\, d\tau + \int_0^{2\pi} L_2(t,\tau)\, v(\tau)\, d\tau. \tag{9.1}$$

Definition 9.2 (Approximation by quadrature) *With L defined by (9.1) and the collocation projection $P^C_{\mathscr{T}_N}$ defined by (6.7) with an odd number of evenly-spaced points (2.21),*

$$(\mathfrak{L}^N v)(t) := \int_0^{2\pi} \log\left(4\sin^2\left(\frac{t-\tau}{2}\right)\right) P^C_{\mathscr{T}_N,\tau}\big(L_1(t,\tau)\, v(\tau)\big)\, d\tau + \int_0^{2\pi} P^C_{\mathscr{T}_N,\tau}\big(L_2(t,\tau)\, v(\tau)\big)\, d\tau. \tag{9.2}$$

Using the expression (6.8) for $P^C_{\mathscr{T}_N,\tau}$ and the explicit expressions for integrals of trigonometric polynomials against the log factor (see, e.g., [72, Lemma 8.23]), one can write $(\mathfrak{L}^N v)(t)$ defined by (9.2) in terms of $L_j(t,t_j)v(t_j)$, $j=0,\ldots,2N$, multiplying trigonometric polynomials in t; see, e.g., [72, Equations 12.18-12.20]. The Nyström method we consider for computing approximations to $(I+L)v = c_0^{-1} f$ is then (1.5) with $L_V = \mathfrak{L}^N$ and $P_V = P^C_N$.

Our estimates for the Nyström method are given in terms the following quantities (see Theorem 10.1 below): for $j=1,2$, let

$$\widehat{L}_{j,m}(t) := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-i\tau m} L_j(t,\tau) d\tau, \tag{9.3}$$

and for $N\geq 0$, $0\leq n\leq N$, $s\in\mathbb{R}$, $0<\epsilon<1$, let

$$\begin{aligned} F^{s,\epsilon}_{\mathscr{L}}(N,n,L) &:= k^{-1} \sum_{N-n<|m|\leq(2-\epsilon)N-n} \|\widehat{L}_{1,m}\|_{H^s_k(\Gamma)} \langle m/k\rangle^s, \\ F^{s,\epsilon}_{\mathscr{H}}(N,n,L) &:= \sum_{|m|>(2-\epsilon)N-n} \Big(\|\widehat{L}_{1,m}\|_{H^s_{\hbar}(\Gamma)} + \|\widehat{L}_{2,m}\|_{H^s_k(\Gamma)}\Big)\langle m/k\rangle^s. \end{aligned} \tag{9.4}$$

We also write $F^{s,\epsilon}_{\mathscr{L}}(N,L) := F^{s,\epsilon}_{\mathscr{L}}(N,N,L)$ and $F^{s,\epsilon}_{\mathscr{H}}(N,L) := F^{s,\epsilon}_{\mathscr{H}}(N,N,L)$.

We now recall the standard method (a.k.a. "Kress quadrature") for writing the Dirichlet (1.2) and Neumann (1.3) BIEs with the perturbation L satisfying Assumption 9.1; this method is based on the splitting

$$\frac{\mathrm{i}}{4} H_0^{(1)}(\mu) = -\frac{1}{4\pi} J_0(\mu) \log\mu^2 + T(\mu,k), \tag{9.5}$$

where both $J_0(\mu)$ and $T(\mu,k)$ are analytic in μ.

Lemma 9.3 *The operators $\eta_D S_k$, K_k, and K'_k satisfy Assumption 9.1. In all three cases, there are functions $\widetilde{L}_j : \mathbb{R}^2\times\mathbb{T}_t\times\mathbb{T}_\tau$ and $N_0\in\mathbb{R}$ such that*

$$\begin{aligned} L_j(t,\tau) &= \widetilde{L}_j\big(k(\gamma(t)-\gamma(\tau)),t,\tau\big), \qquad j=1,2, \\ \widetilde{L}_1(\mu,t,\tau) &= k\int_{\mathbb{S}^1} e^{i\langle\mu,\omega\rangle} f(\omega,t,\tau) dS(\omega), \qquad |\partial^\alpha f|\leq C_\alpha, \\ \operatorname{supp}\mathcal{F}_{\mu\to\xi}\big(\partial_t^\alpha\partial_\tau^\beta \widetilde{L}_2(\cdot,t,\tau)\big)(\xi) &\subset \big\{|\xi|\leq 1\big\}, \qquad |\partial^\alpha \widetilde{L}_2(\mu,t,\tau)|\leq C_\alpha k^{N_0}. \end{aligned} \tag{9.6}$$

Proof. We first write

$$\begin{aligned} &\frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|) \\ &= -\frac{1}{2\pi} J_0\big(k|x-y|\big)\log(k|x-y|) + \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|) + \frac{1}{2\pi} J_0\big(k|x-y|\big)\log(k|x-y|) \\ &= -\frac{1}{4\pi} J_0\big(k|x-y|\big)\log|x-y|^2 + \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|) + \frac{1}{2\pi} J_0\big(k|x-y|\big)\log(k|x-y|) \end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{\log k}{2\pi} J_0\big(k|x-y|\big) \\
&=: -\frac{1}{4\pi} J_0\big(k|x-y|\big)\log|x-y|^2 + S_1\big(k|x-y|,k\big).
\end{aligned}
$$

By the asymptotics of $H_0^{(1)}(z)$ as $z \to 0$ (see, e.g., [94, §10.8]), $\frac{\mathrm{i}}{4}H_0^{(1)}(z) + \frac{1}{2\pi}J_0(z)\log(z)$ is analytic at $z=0$, and thus $S_1(\mu,k)$ is analytic in μ.

Next, observe that, since $\gamma : \mathbb{R}/2\pi\mathbb{Z} \to \Gamma$ is smooth, $|\gamma'| \geq c > 0$, and γ is bijectitve,

$$
e(t,\tau) := \frac{|\gamma(t)-\gamma(\tau)|^2}{4\sin^2\left(\frac{t-\tau}{2}\right)} \in C^\infty\big((\mathbb{R}/2\pi\mathbb{Z})^2\big), \qquad e(t,\tau) > c > 0.
$$

In particular, with $f(t,\tau) := \log e(t,\tau) \in C^\infty$,

$$
\log\left(|\gamma(t)-\gamma(\tau)|^2\right) =: \log\Big(4\sin^2\left(\tfrac{t-\tau}{2}\right)\Big) + f(t,\tau), \tag{9.7}
$$

so that

$$
\begin{aligned}
&\frac{\mathrm{i}}{4}H_0^{(1)}(k|\gamma(t)-\gamma(\tau)|) \\
&= -\frac{1}{4\pi}J_0\big(k|\gamma(t)-\gamma(\tau)|\big)\log\Big(4\sin^2\left(\tfrac{t-\tau}{2}\right)\Big) - \frac{1}{4\pi}J_0\big(k|\gamma(t)-\gamma(\tau)|\big)f(t,\tau) + S_1\big(k|\gamma(t)-\gamma(\tau)|,k\big).
\end{aligned}
$$

Thus, for $\eta_D S_k$ we set

$$
\begin{aligned}
L_1(t,\tau) &:= -\frac{\eta_D}{4\pi}J_0\big(k|\gamma(t)-\gamma(\tau)|\big), \\
L_2(t,\tau) &:= -\frac{\eta_D}{4\pi}J_0\big(k|\gamma(t)-\gamma(\tau)|\big)f(t,\tau) + \eta_D S_1\big(k|\gamma(t)-\gamma(\tau)|,k\big).
\end{aligned}
$$

The required properties for $L_1(t,\tau)$ now follow from a standard integral representation of $J_0(|x-y|)$ as the Fourier transform of the surface measure on $\mathbb{S}^1$ (see, e.g., [105, Page 154]). The required properties for $L_2(t,\tau)$ follow from the Payley–Wiener theorem (see, e.g., [105, Theorem 4.1]), the fact that $J_0(\mu)$ and $S_1(\mu,k)$ are analytic and the asymptotic growth

$$
|J_0(z)| + |H_0^{(1)}(z)| \leq C\log|z|e^{|\operatorname{Im} z|} \qquad \text{for all } z \in \mathbb{C}\setminus(-\infty,0]
$$

see, e.g., [94, §10.8 and §10.17]. The proofs for K_k and K'_k are nearly identical, using instead the same analyticity and growth properties for $\partial_\mu S_1(\mu,k)$. ∎

Lemma 9.4 *The operators $kS_{\mathrm{i}k}$, $K_{\mathrm{i}k}$, $K'_{\mathrm{i}k}$ satisfy Assumption 9.1 and in all three cases, there are functions $\widetilde{L}_j : \mathbb{R}^2 \times \mathbb{T}_t \times \mathbb{T}_\tau$ such that*

$$
L_j(t,\tau) = \widetilde{L}_j\big(k(\gamma(t)-\gamma(\tau)),t,\tau,k\big), \qquad j=1,2, \tag{9.8}
$$

and for all $j=1,2$, $\alpha \in \mathbb{N}$, $N>0$, there is $C_{\alpha N} > 0$ such that

$$
|\partial^\alpha \widetilde{L}_1(\mu,t,\tau)| \leq C_{\alpha N} k\log k\langle\mu\rangle^{-N}, \qquad |\partial^\alpha \widetilde{L}_2(\mu,t,\tau)| \leq C_{\alpha N} k\langle\mu\rangle^{-N}. \tag{9.9}
$$

Proof. The main difference here compared to Lemma 9.3 is that $J_0(ix)$ grows exponentially and $H_0^{(1)}(ix)$ decays exponentially as $x \to \infty$ (see, e.g., [94, §10.17]). We therefore use the upper bounds that for any $\alpha \in \mathbb{N}$ and $\epsilon > 0$ there is $C_{\alpha\epsilon} > 0$ such that

$$
\begin{aligned}
|\partial^\alpha J_0(\mathrm{i}x)| &\leq C_{\alpha\epsilon}e^{(1+\epsilon)|x|}, & x &\in \mathbb{R}, \\
|\partial^\alpha H_0^{(1)}(\mathrm{i}x)| &\leq C_{\alpha\epsilon}e^{-(1-\epsilon)|x|}\log|x|, & x &> 0.
\end{aligned} \tag{9.10}
$$

As in Lemma 9.3, with f as in (9.7),

$$
\frac{\mathrm{i}}{4}H_0^{(1)}(\mathrm{i}k|\gamma(t)-\gamma(\tau)|) = -\frac{1}{4\pi}J_0\big(\mathrm{i}k|\gamma(t)-\gamma(\tau)|\big)\log\Big(4\sin^2\left(\tfrac{t-\tau}{2}\right)\Big)
$$

$$-\frac{1}{4\pi}J_0\big(\mathrm{i}k|\gamma(t)-\gamma(\tau)|\big)f(t,\tau)+S_1(\mathrm{i}k|\gamma(t)-\gamma(\tau)|,\mathrm{i}k),$$

with

$$|\partial^\alpha S_1(\mu,\mathrm{i}k)|\le C_{\alpha,\epsilon}e^{(1+\epsilon)|\mu|}\log k$$

by (9.10). We let $\chi\in\mathcal{S}(\mathbb{R})$ with $\chi\equiv 1$ near 0, and

$$|\chi(x)|\le Ce^{-r|x|}\qquad\text{for some}\qquad r>1+\epsilon, \tag{9.11}$$

and write

$$\begin{aligned}\frac{\mathrm{i}}{4}H_0^{(1)}(\mathrm{i}k|\gamma(t)-\gamma(\tau)|)&=-\frac{1}{4\pi}J_0\big(\mathrm{i}k|\gamma(t)-\gamma(\tau)|\big)\chi\big(k|\gamma(t)-\gamma(\tau)|\big)\log\Big(4\sin^2\big(\tfrac{t-\tau}{2}\big)\Big)\\&\quad+\chi(k|\gamma(t)-\gamma(\tau)|)\Big(S_1(\mathrm{i}k|\gamma(t)-\gamma(\tau)|,\mathrm{i}k)-\frac{1}{4\pi}J_0\big(\mathrm{i}k|\gamma(t)-\gamma(\tau)|\big)f(t,\tau)\Big)\\&\quad+\Big(1-\chi\big(k|\gamma(t)-\gamma(\tau)|\big)\Big)\frac{\mathrm{i}}{4}H_0^{(1)}\big(\mathrm{i}k|\gamma(t)-\gamma(\tau)|\big).\end{aligned}$$

Therefore, if

$$\begin{aligned}L_1(t,\tau)&:=-\frac{k}{4\pi}J_0\big(\mathrm{i}k|\gamma(t)-\gamma(\tau)|\big)\chi\big(k|\gamma(t)-\gamma(\tau)|\big),\\L_2(t,\tau)&:=k\chi(k|\gamma(t)-\gamma(\tau)|)\Big(S_1(\mathrm{i}k|\gamma(t)-\gamma(\tau)|,\mathrm{i}k)-\frac{1}{4\pi}J_0\big(\mathrm{i}k|\gamma(t)-\gamma(\tau)|\big)f(t,\tau)\Big)\\&\qquad+k\Big(1-\chi\big(k|\gamma(t)-\gamma(\tau)|\big)\Big)\frac{\mathrm{i}}{4}H_0^{(1)}\big(\mathrm{i}k|\gamma(t)-\gamma(\tau)|\big),\end{aligned}$$

then the result for kS_{ik} follows from the bounds (9.10) and (9.11). For K_{ik} and K'_{ik}, the estimates follow similarly after taking appropriate derivatives of $H_0^{(1)}$, J_0, and S_1. ∎

Lemma 9.5 *The operator $k^{-1}(H_k-H_{\mathrm{ik}})$ satisfies Assumption 9.1 and there are functions $\widetilde{L}_{j,i}:\mathbb{R}^2\times\mathbb{T}_t\times\mathbb{T}_\tau$, $j=1,2$, $i=1,2$, and $N_0\in\mathbb{R}$ such that*

$$L_j(t,\tau)=\widetilde{L}_{j,1}\big(k(\gamma(t)-\gamma(\tau)),t,\tau,k\big)+\widetilde{L}_{j,2}\big(k(\gamma(t)-\gamma(\tau)),t,\tau,k\big),\qquad j=1,2,$$

and for $j=1,2$, all $\alpha\in\mathbb{N}$, $N>0$, there are C_α and $C_{\alpha N}>0$ such that

$$\widetilde{L}_{1,1}(\mu,t,\tau)=k\int_{\mathbb{S}^1}e^{i\langle\mu,\omega\rangle}f(\omega,t,\tau)dS(\omega),\qquad|\partial^\alpha f|\le C_\alpha,$$
$$\operatorname{supp}\mathcal{F}_{\mu\to\xi}\big(\partial_t^\alpha\partial_\tau^\beta\widetilde{L}_{2,1}(\cdot,t,\tau)\big)(\xi)\subset\{|\xi|\le 1\},\qquad|\partial^\alpha\widetilde{L}_{2,1}(\mu,t,\tau)|\le C_\alpha k^{N_0}$$

(compare to (9.6)) and

$$|\partial^\alpha\widetilde{L}_{1,2}(\mu,t,\tau)|\le C_{\alpha N}k\log k\langle\mu\rangle^{-N},\qquad|\partial^\alpha\widetilde{L}_{2,2}(\mu,t,\tau)|\le C_{\alpha N}k\langle\mu\rangle^{-N}$$

(compare to (9.9)).

Proof. Recall (from, e.g, [94, §10.8]) that

$$H_1^{(1)}(z)+i\frac{2}{\pi z}-i\frac{2}{\pi}J_1(z)\log\frac{z}{2}=:S_2(z)$$

is analytic. Bessel's equation and the fact that $H_0^{(1)'}(\mu)=-H_1^{(1)}(\mu)$ imply that

$$H_0^{(1)''}(\mu)=-\mu^{-1}H_0^{(1)'}(\mu)-H_0^{(1)}(\mu)=\mu^{-1}H_1^{(1)}(\mu)-H_0^{(1)}(\mu).$$

Therefore,

$$\begin{aligned}&H_0^{(1)''}(\mu)+H_0^{(1)''}(i\mu)\\&=\mu^{-1}\Big(-iH_1^{(1)}(i\mu)+H_1^{(1)}(\mu)\Big)-H_0^{(1)}(\mu)-H_0^{(1)}(i\mu)\end{aligned}$$

$$= \mu^{-1}\Big(- iS_2(i\mu) + S_2(\mu) + \frac{2}{\pi} J_1(i\mu)\log\frac{i\mu}{2} + i\frac{2}{\pi}J_1(\mu)\log\frac{\mu}{2}\Big) - H_0^{(1)}(\mu) - H_0^{(1)}(i\mu)$$
$$= \mu^{-1}\Big(iJ_1(i\mu) - iS_2(i\mu) + S_2(\mu) + \frac{2}{\pi}\big(J_1(i\mu) + iJ_1(\mu)\big)\log\frac{\mu}{2}\Big) - H_0^{(1)}(\mu) - H_0^{(1)}(i\mu).$$

Recall that J_1 is analytic, vanishes at 0, and for all $\alpha \in \mathbb{N}$, there are $C > 0$ and $N > 0$ such that

$$|J_1(z)| \leq C\langle z\rangle^N e^{|\operatorname{Im} z|} \quad \text{and} \quad |S_2(z)| \leq C\langle z\rangle^N e^{|\operatorname{Im} z|} \quad \text{for all } z \in \mathbb{C}.$$

The result then follows by introducing cutoffs on all terms where Bessel functions, J_0, J_1 are evaluated at $ik|\gamma(t) - \gamma(\tau)|$ as in Lemma 9.4 to obtain $\widetilde{L}_{j,2}$, and analyzing the remaining terms as in Lemma 9.3 to obtain $\widetilde{L}_{j,1}$. ■

10 Abstract result about the convergence of the Nyström method

For our abstract theorem on the Nyström method, we assume that $\mathcal{A} = I + L$ satisfies Assumption 4.1 with

$$L = \sum_{i=1}^{I} \widetilde{L}_i + \sum_{j=1}^{J} \widetilde{L}_{j,a}\widetilde{L}_{j,b}, \qquad L_N := \sum_{i=1}^{I} \widetilde{\mathfrak{L}}_i^N + \sum_{j=1}^{J} \widetilde{\mathfrak{L}}_{j,a}^N P^C_{\mathscr{T}_N} \widetilde{\mathfrak{L}}_{j,b}^N, \tag{10.1}$$

and $\widetilde{L}_i$, $\widetilde{L}_{j,\cdot}$ satisfying Assumption 9.1 and $\widetilde{\mathfrak{L}}_i^N$ and $\widetilde{\mathfrak{L}}_{j,\cdot}^N$ defined as in Definition 9.2. We further assume that for any $\chi \in C_c^\infty$ with $\operatorname{supp}(1-\chi) \cap [-1,1] = \emptyset$, $(1-\chi(|\hbar D'|_g^2))\widetilde{L}_{j,b}$, $\widetilde{L}_{j,b}(1-\chi(|\hbar D'|_g^2)) \in \Psi_\hbar^{-1}(\Gamma)$.

We define

$$\mathcal{F}^{s,\epsilon}_{\mathscr{L}}(N,n) := \max\Big(\sup_i F^{s,\epsilon}_{\mathscr{L}}(N,n,\widetilde{L}_i), \sup_j F^{s,\epsilon}_{\mathscr{L}}(N,n,\widetilde{L}_{j,a}), \sup_j F^{s,\epsilon}_{\mathscr{L}}(N,n,\widetilde{L}_{j,b})\Big),$$
$$\mathcal{F}^{s,\epsilon}_{\mathscr{H}}(N,n) := \max\Big(\sup_i F^{s,\epsilon}_{\mathscr{H}}(N,n,,\widetilde{L}_i), \sup_j F^{s,\epsilon}_{\mathscr{H}}(N,n,\widetilde{L}_{j,a}), \sup_j F^{s,\epsilon}_{\mathscr{H}}(N,n,\widetilde{L}_{j,b})\Big),$$
$$\|\widetilde{L}\|_{s,r} := \sum_j \|\widetilde{L}_{j,a}\|_{H^s_\hbar(\Gamma)\to H^s_\hbar(\Gamma)} + \|\widetilde{L}_{j,b}\|_{H^r_\hbar(\Gamma)\to H^r_\hbar(\Gamma)},$$
$$\mathcal{B}^{r,s}(\widetilde{L}) := \sum_j \|\widetilde{L}_{j,a}\|_{H^s_\hbar(\Gamma)\to H^s_\hbar(\Gamma)} \|\widetilde{L}_{j,b}\|_{H^r_\hbar(\Gamma)\to H^r_\hbar(\Gamma)}.$$

We also write $\mathcal{F}^{s,\epsilon}_{\mathscr{L}}(N) := \mathcal{F}^{s,\epsilon}_{\mathscr{L}}(N,N)$ and $\mathcal{F}^{s,\epsilon}_{\mathscr{H}}(N) := \mathcal{F}^{s,\epsilon}_{\mathscr{H}}(N,N)$.

Theorem 10.1 (New abstract result about the convergence of the Nyström method) *Suppose that $\mathcal{A}$ satisfies Assumptions 4.1 and 1.3, and L and L_N are as in (10.1). Given $k_0 > 0$, $M > 0$, $t^* \geq s \geq t > 1/2$, there exist $c, C > 0$ such that the following holds.*

Given $f \in H^s_\hbar(\Gamma)$, let $v \in H^s_\hbar(\Gamma)$ be the solution of

$$\mathcal{A}v := c_0(I + L)v = f. \tag{10.2}$$

If $k \geq k_0$,

$$\Big(\frac{k}{N}\Big)^{t^*-s} \rho \leq c, \tag{10.3}$$

$$\Big[1 + \rho\Big(\frac{k}{N}\Big)^{s-t}\Big]\Big(\frac{k}{N}\Big)\mathcal{F}^{s,\epsilon}_{\mathscr{L}}(N)\Big(1 + \|\widetilde{L}\|_{s,s} + \|\widetilde{L}\|_{t,t}\Big) \leq c \tag{10.4}$$

$$\Big[1 + \rho\Big(1 + \Big(\frac{k}{N}\Big)^{s+1}\Big)\Big]\mathcal{F}^{s,\epsilon}_{\mathscr{H}}(N)\Big(1 + \|\widetilde{L}\|_{s,s} + \|\widetilde{L}\|_{t,t}\Big) \leq c, \quad \textit{and} \tag{10.5}$$

$$\Big[1 + \rho\Big(\frac{k}{N}\Big)^{s+1}\Big]\Big(\frac{k}{N}\Big)\mathcal{B}^{s,s}(\widetilde{L}) + \rho\Big(\frac{k}{N}\Big)^{s-t+1}\mathcal{B}^{t,s}(\widetilde{L}) \leq c, \tag{10.6}$$

then the solution $v_N \in V_N$ to

$$(I + P^C_{\mathscr{T}_N} L_N) v_N = P^C_{\mathscr{T}_N} f \tag{10.7}$$

exists, is unique, and satisfies the error estimate

$$\begin{aligned}
&\|v - v_N\|_{H^s_\hbar(\Gamma)} \\
&\quad \le C\bigg[\Big(1 + \rho\Big(\frac{k}{N}\Big)^s\Big)\|(I - P^C_{\mathscr{T}_N})v\|_{H^s_\hbar(\Gamma)} + \Big(1 + \rho\Big(\frac{k}{N}\Big)^{s+1}\Big)\big\|P^C_{\mathscr{T}_N}(L_N - L)P^C_{\mathscr{T}_N} v\big\|_{H^s_\hbar(\Gamma)} \\
&\qquad\qquad + \rho\big\|P^C_{\mathscr{T}_N}(L_N - L)P^C_{\mathscr{T}_N} v\big\|_{L^2(\Gamma)}\bigg], \qquad (10.8)\\
&\le C\bigg[\Big(1 + \rho\Big(\frac{k}{N}\Big)^s\Big)\Big(\big\|(I - P^C_{\mathscr{T}_N})v\big\|_{H^s_\hbar(\Gamma)} + \big\|P^C_{\mathscr{T}_N}(L_N - L)P^C_{\mathscr{T}_N} v\big\|_{H^s_\hbar(\Gamma)}\Big) \\
&\qquad + \rho\bigg(\Big[\Big(\frac{k}{N}\Big)^{s+1}\mathcal{F}^{s,\epsilon}_{\mathscr{L}}(N) + \mathcal{F}^{0,\epsilon}_{\mathscr{H}}(N)\Big]\big(1 + \|\widetilde{L}\|_{0,0} + \|\widetilde{L}\|_{s,s}\big) + \Big(\frac{k}{N}\Big)^{s+1}\mathcal{B}^{0,s}(\widetilde{L})\bigg)\|v\|_{H^s_\hbar(\Gamma)}\bigg].
\end{aligned} \tag{10.9}$$

Before proving Theorem 10.1, we need a few technical lemmas; the first is an analogue of Lemma 4.9 for $(I + P^C_{\mathscr{T}_N} L)^{-1}$.

Lemma 10.2 *Suppose that $\mathcal{A}$ satisfies Assumptions 4.1 and 1.3. Given $t^* > 1/2$ and $\chi \in C^\infty_c$ with* $\operatorname{supp}(1-\chi) \cap [-1,1] = \emptyset$, *there exist $C, c > 0$ such that if $(k/N)^{t^*-s}\rho \le c$ then*

$$\big\|(I + P^C_{\mathscr{T}_N} L)^{-1}\big(I - \chi(-\hbar^2\Delta_\Gamma)\big)\big\|_{H^s_\hbar(\Gamma)\to H^s_\hbar(\Gamma)} \le C\Big(1 + \rho\Big(\frac{k}{N}\Big)^{s+1}\Big) \tag{10.10}$$

and

$$\big\|(I + P^C_{\mathscr{T}_N} L)^{-1}\chi(-\hbar^2\Delta_\Gamma)\big\|_{L^2(\Gamma)\to H^s_\hbar(\Gamma)} \le C\rho. \tag{10.11}$$

Proof. By Lemma 8.1, if $N \ge Ck$ then

$$(I + P^C_{\mathscr{T}_N} L)^{-1} = (I + L)^{-1}\Big(I + (P^C_{\mathscr{T}_N} - I)L(I + L)^{-1}\Big)^{-1}. \tag{10.12}$$

We prove the bounds (10.10) and (10.11) using (4.45). Observe that $P^C_{\mathscr{T}_N}$ satisfies (4.1) for $0 \le q \le t \le \infty$ and $t > 1/2$ by (6.11); i.e., $t_{\max}$ in the arguments in §4.4 can be taken to be arbitrarily large. We therefore apply (4.45) with $q_{\min} = 0$ and $t_{\max} = t^*$, so that the condition (4.3) under which (4.45) holds becomes $(k/N)^{t^*}\rho \le c$.

Given χ as in the statement of the lemma, choose $\epsilon > 0$ in the definitions of $\Pi_{\mathscr{L}}$ and $\Pi_{\mathscr{H}}$ (4.31) so that

$$1 - \chi(-\hbar^2\Delta_\Gamma) = \Pi_{\mathscr{H}}(1 - \chi(-\hbar^2\Delta_\Gamma)).$$

Then choose $\chi^< \in C^\infty_c$ with $\operatorname{supp}(1 - \chi^<) \cap [-1,1] = \emptyset$ such that

$$\chi^<(-\hbar^2\Delta_\Gamma) = \chi^<(-\hbar^2\Delta_\Gamma)\Pi_{\mathscr{L}}$$

(this requires, in particular, that $\operatorname{supp}\chi^< \cap \operatorname{supp}(1-\chi) = \emptyset$). Finally, choose $\chi^\ll \in C^\infty_c$ with $\operatorname{supp}(1-\chi^<) \cap [-1,1] = \emptyset$ such that $\operatorname{supp}\chi^\ll \cap \operatorname{supp}(1-\chi^<) = \emptyset$. Now, by (10.12), Lemma 4.6, Lemma 4.9, and Corollary 3.10,

$$\begin{aligned}
&\big\|(I + P^C_{\mathscr{T}_N} L)^{-1}\big(I - \chi(-\hbar^2\Delta_\Gamma)\big)\big\|_{H^s_\hbar\to H^s_\hbar} \\
&= \big\|(I+L)^{-1}\big(I + (P^C_{\mathscr{T}_N} - I)L(I+L)^{-1}\big)^{-1}\big(I - \chi(-\hbar^2\Delta_\Gamma)\big)\big\|_{H^s_\hbar(\Gamma)\to H^s_\hbar(\Gamma)} \\
&\le \big\|(I+L)^{-1}\chi^<(-\hbar^2\Delta_\Gamma)\big(I + (P^C_{\mathscr{T}_N} - I)L(I+L)^{-1}\big)^{-1}\big(I - \chi(-\hbar^2\Delta_\Gamma)\big)\big\|_{H^s_\hbar\to H^s_\hbar} \\
&\quad + \big\|(I+L)^{-1}(I - \chi^\ll(-\hbar^2\Delta_\Gamma))\big(I - \chi^<(-\hbar^2\Delta_\Gamma)\big)\big(I + (P^C_{\mathscr{T}_N} - I)L(I+L)^{-1}\big)^{-1}\big(I - \chi(-\hbar^2\Delta_\Gamma)\big)\big\|_{H^s_\hbar\to H^s_\hbar} \\
&\le C\rho\big\|\chi^<(-\hbar^2\Delta_\Gamma)\big(I + (P^C_{\mathscr{T}_N} - I)L(I+L)^{-1}\big)^{-1}\big(I - \chi(-\hbar^2\Delta_\Gamma)\big)\big\|_{H^s_\hbar\to H^s_\hbar} \\
&\qquad + C\big\|\big(I - \chi^<(-\hbar^2\Delta_\Gamma)\big)\big(I + (P^C_{\mathscr{T}_N} - I)L(I+L)^{-1}\big)^{-1}\big(I - \chi(-\hbar^2\Delta_\Gamma)\big)\big\|_{H^s_\hbar\to H^s_\hbar}
\end{aligned}$$

$$\le C\rho\big\|\Pi_{\mathscr{L}}\big(I+(P^C_{\mathscr{T}_N}-I)L(I+L)^{-1}\big)^{-1}\Pi_{\mathscr{H}}\big\|_{H^s_\hbar\to H^s_\hbar}$$
$$+C\big\|\big(I+(P^C_{\mathscr{T}_N}-I)L(I+L)^{-1}\big)^{-1}\Pi_{\mathscr{H}}\big\|_{H^s_\hbar\to H^s_\hbar}.$$

Therefore, by (4.45), if $(k/N)^{t^*}\rho\le c$ then

$$\big\|(I+P^C_{\mathscr{T}_N}L)^{-1}\big(I-\chi(-\hbar^2\Delta_\Gamma)\big)\big\|_{H^s_\hbar\to H^s_\hbar}\le C\rho(k/N)^{s+1}+C(k/N)^{s+1}+1+Ck/N,$$

which implies (10.10).

For (10.11), we apply again (4.45) (with $q_{\min}=0$ and $t_{\max}=t^*$), except this time, given χ as in the statement of the lemma, we choose $\epsilon>0$ in the definitions of $\Pi_{\mathscr{L}}$ and $\Pi_{\mathscr{H}}$ (4.31) so that

$$\chi(-\hbar^2\Delta_\Gamma)=\Pi_{\mathscr{L}}\chi(-\hbar^2\Delta_\Gamma).$$

Then, by (10.12), (4.45), and Lemma 4.6, if $(k/N)^{t^*-s}\rho\le c$ then

$$\begin{aligned}\big\|(I+P^C_{\mathscr{T}_N}L)^{-1}\chi(-\hbar^2\Delta_\Gamma)\big\|_{H^s_\hbar\to H^s_\hbar}&\le C\rho\big\|\big(I+(P^C_{\mathscr{T}_N}-I)L(I+L)^{-1}\big)^{-1}\chi(-\hbar^2\Delta_\Gamma)\big\|_{H^s_\hbar\to H^s_\hbar}\\&\le C\rho\big\|\big(I+(P^C_{\mathscr{T}_N}-I)L(I+L)^{-1}\big)^{-1}\Pi_{\mathscr{L}}\big\|_{H^s_\hbar\to H^s_\hbar}\\&\le C\rho\end{aligned}\tag{10.13}$$

Finally, let $\chi^>\in C^\infty_c$ with $\operatorname{supp}(1-\chi^>)\cap[-1,1]=\emptyset$, and $\operatorname{supp}\chi\cap\operatorname{supp}(1-\chi^>)=\emptyset$. By the definition of $\|\cdot\|_{H^s_\hbar(\Gamma)}$ (3.5) and the fact that $\widetilde{\chi}$ has compact support, $\chi^>(-\hbar^2\Delta_\Gamma):L^2(\Gamma)\to H^s_\hbar(\Gamma)$ with bounded norm. The result (10.11) then follows from the bound (10.13), the fact that $\chi=\chi\chi^>$, and the smoothing property of $\chi^>(-\hbar^2\Delta_\Gamma)$. ■

Lemma 10.3 (Error estimate in terms of the discrete inverse) *If $P_N:H^s_\hbar(\Gamma)\to V_N$ is a projection, $L_N:H^s_\hbar(\Gamma)\to H^s_\hbar(\Gamma)$, and $I+P_NL_NP_N:H^s_\hbar(\Gamma)\to H^s_\hbar(\Gamma)$ is invertible, then the solution $v_N\in V_N$ to (10.7) exists, is unique, and satisfies*

$$v-v_N=\big(I+P_NL_NP_N\big)^{-1}\Big((I-P_NL)(I-P_N)v+P_N(L_N-L)P_Nv\Big).\tag{10.14}$$

Proof. First observe that if $v_N\in H^s_\hbar(\Gamma)$ and $(I+P_NL_N)v_N=P_Nf$, then $v_N\in V_N$, and hence $(I+P_NL_NP_N)v_N=P_Nf$. Furthermore, if $(I+P_NL_NP_N):H^s_\hbar(\Gamma)\to H^s_\hbar(\Gamma)$ is invertible, then there is $v_N\in H^s_\hbar(\Gamma)$ such that $(I+P_NL_NP_N)v_N=P_Nf$. This v_N satisfies $(I-P_N)v_N=0$ so that $v_N\in V_N$ and thus $(I+P_NL_N)v_N=P_Nf$. Therefore, if $(I+P_NL_NP_N):H^s_\hbar(\Gamma)\to H^s_\hbar(\Gamma)$ is invertible, then the equation (10.7) has a unique solution in V_N and it, in addition, satisfies $(I+P_NL_NP_N)v_N=P_Nf$. By this and the fact that $(I+L)v=f$,

$$\begin{aligned}(I+P_NL_NP_N)(v-v_N)=(I+P_NL_NP_N)v-P_Nf&=(I-P_N)v+P_NL_NP_Nv-P_NLv\\&=(I-P_NL)(I-P_N)v+P_N(L_N-L)P_Nv\end{aligned}$$

and the result follows. ■

Lemma 10.4 *Suppose that L satisfies Assumption 9.1, L_N is defined by Definition 9.2. Then, given $r\ge s\ge0$, $0<\epsilon<1$, there exist $C>0$ such that for all $0\le n\le n'\le N$,*

$$\begin{aligned}&\big\|(L-L_N)P^C_{\mathscr{T}_n}\big\|_{H^r_\hbar(\Gamma)\to H^s_\hbar(\Gamma)}\\&\le C\Bigg(\Big(\Big(\frac{k}{N}\Big)^{r-s+1}\mathcal{F}^{r,\epsilon}_{\mathscr{L}}(N,n')+\mathcal{F}^{s,\epsilon}_{\mathscr{H}}(N,n')\Big)(1+\|\widetilde{L}\|_{s,s}+\|\widetilde{L}\|_{r,r})\\&+\Big(\Big(\frac{k}{N}\Big)\mathcal{F}^{s,\epsilon}_{\mathscr{L}}(N)+\mathcal{F}^{s,\epsilon}_{\mathscr{H}}(N)\Big)\Big(\Big(\frac{k}{N}\Big)^{r-s+1}\mathcal{F}^{r,\epsilon}_{\mathscr{L}}(N,n')+\mathcal{F}^{s,\epsilon}_{\mathscr{H}}(N,n')\Big)+\Big(\frac{k}{N}\Big)^{r-s+1}\mathcal{B}^{s,r}(\widetilde{L})\\&+\max_j\Big(\Big(\frac{k}{N}\Big)^{r-s+1}\mathcal{F}^{r,\epsilon}_{\mathscr{L}}(N)+\mathcal{F}^{s,\epsilon}_{\mathscr{H}}(N)\Big)\cdot\|P^C_{\mathscr{T}_N}(I-P^C_{\mathscr{T}_{n'}})\|_{H^r_\hbar\to H^r_\hbar}\|(I-P^G_{\mathscr{T}_{n'}})\widetilde{L}_{j,b}P^G_{\mathscr{T}_n}\|_{H^r_\hbar\to H^r_\hbar}\Bigg).\end{aligned}\tag{10.15}$$

In particular, if $n = n' = N$,

$$
\begin{aligned}
&\big\|(L - L_N)P^C_{\mathscr{T}_N}\big\|_{H^r_\hbar(\Gamma)\to H^s_\hbar(\Gamma)}\\
&\quad\le C\bigg[\Big(\Big(\frac{k}{N}\Big)^{r-s+1}\mathcal{F}^{r,\epsilon}_{\mathscr{L}}(N) + \mathcal{F}^{s,\epsilon}_{\mathscr{H}}(N)\Big)(1 + \|\widetilde{L}\|_{s,s} + \|\widetilde{L}\|_{r,r})\\
&\qquad + \Big(\Big(\frac{k}{N}\Big)\mathcal{F}^{s,\epsilon}_{\mathscr{L}}(N) + \mathcal{F}^{s,\epsilon}_{\mathscr{H}}(N)\Big)\Big(\Big(\frac{k}{N}\Big)^{r-s+1}\mathcal{F}^{r,\epsilon}_{\mathscr{L}}(N) + \mathcal{F}^{s,\epsilon}_{\mathscr{H}}(N)\Big) + \Big(\frac{k}{N}\Big)^{r-s+1}\mathcal{B}^{s,r}(\widetilde{L})\bigg]. \quad (10.16)
\end{aligned}
$$

Furthermore, if $n' \ge n + \epsilon k$, and $n = \Xi k$ with $\Xi \ge c_{\max} + \epsilon$, then for all $M > 0$ there is $C_M > 0$ such that

$$
\begin{aligned}
&\big\|(L - L_N)P^C_{\mathscr{T}_n}\big\|_{H^r_\hbar(\Gamma)\to H^s_\hbar(\Gamma)}\\
&\quad\le C\bigg[\Big(\Big(\frac{k}{N}\Big)^{r-s+1}\mathcal{F}^{r,\epsilon}_{\mathscr{L}}(N,n') + \mathcal{F}^{s,\epsilon}_{\mathscr{H}}(N,n')\Big)(1 + \|\widetilde{L}\|_{s,s} + \|\widetilde{L}\|_{r,r})\\
&\qquad + \Big(\Big(\frac{k}{N}\Big)\mathcal{F}^{s,\epsilon}_{\mathscr{L}}(N) + \mathcal{F}^{s,\epsilon}_{\mathscr{H}}(N)\Big)\Big(\Big(\frac{k}{N}\Big)^{r-s+1}\mathcal{F}^{r,\epsilon}_{\mathscr{L}}(N,n') + \mathcal{F}^{s,\epsilon}_{\mathscr{H}}(N,n')\Big) + \Big(\frac{k}{N}\Big)^{r-s+1}\mathcal{B}^{s,r}(\widetilde{L})\bigg]\\
&\qquad + C_M\Big(\Big(\frac{k}{N}\Big)^{r-s+1}\mathcal{F}^{r,\epsilon}_{\mathscr{L}}(N) + \mathcal{F}^{s,\epsilon}_{\mathscr{H}}(N)\Big)\|\widetilde{L}\|_{0,0}k^{-M}. \quad (10.17)
\end{aligned}
$$

We postpone the proof of Lemma 10.4 and first prove Theorem 10.1.

Proof of Theorem 10.1. In a similar way to (4.28), we write

$$
\begin{aligned}
I + P^C_{\mathscr{T}_N}L_NP^C_{\mathscr{T}_N} &= I + P^C_{\mathscr{T}_N}L + P^C_{\mathscr{T}_N}(L_N - L)P^C_{\mathscr{T}_N} + P^C_{\mathscr{T}_N}L(P^C_{\mathscr{T}_N} - I)\\
&= (I + P^C_{\mathscr{T}_N}L)\Big(I + (I + P^C_{\mathscr{T}_N}L)^{-1}P^C_{\mathscr{T}_N}L(P^C_{\mathscr{T}_N} - I) + (I + P^C_{\mathscr{T}_N}L)^{-1}P^C_{\mathscr{T}_N}(L_N - L)P^C_{\mathscr{T}_N}\Big).
\end{aligned} \tag{10.18}
$$

By decreasing c if necessary, we see that the condition (10.3) implies that $N \ge Ck$ for C as in Lemma 8.1; thus $I + P^C_{\mathscr{T}_N}L$ is invertible. Then, by (10.18) and Neumann series, if

$$\big\|(I + P^C_{\mathscr{T}_N}L)^{-1}P^C_{\mathscr{T}_N}L(P^C_{\mathscr{T}_N} - I)\big\|_{H^s_\hbar(\Gamma)\to H^s_\hbar(\Gamma)} \le 1/4 \tag{10.19}$$

and

$$\big\|(I + P^C_{\mathscr{T}_N}L)^{-1}P^C_{\mathscr{T}_N}(L_N - L)P^C_{\mathscr{T}_N}\big\|_{H^s_\hbar(\Gamma)\to H^s_\hbar(\Gamma)} \le 1/4, \tag{10.20}$$

then $I + P^C_{\mathscr{T}_N}L_NP^C_{\mathscr{T}_N}$ is invertible with

$$\big\|(I + P^C_{\mathscr{T}_N}L_NP^C_{\mathscr{T}_N})^{-1}v\big\|_{H^s_\hbar(\Gamma)} \le 2\big\|\big(I + P^C_{\mathscr{T}_N}L\big)^{-1}v\big\|_{H^s_\hbar(\Gamma)} \quad \text{for all } v \in H^s_\hbar(\Gamma). \tag{10.21}$$

We first prove that (10.19) holds, by establishing that

$$\big\|(I + P^C_{\mathscr{T}_N}L)^{-1}P^C_{\mathscr{T}_N}L(P^C_{\mathscr{T}_N} - I)\big\|_{H^s_\hbar\to H^s_\hbar} \le C\bigg(1 + \rho\Big(\frac{k}{N}\Big)^{s+1}\bigg)\frac{k}{N} + C\rho\Big(\frac{k}{N}\Big)^{s}, \tag{10.22}$$

We then prove the result (10.8) under the assumption that (10.20) holds, and come back and establish (10.20) at the end; we proceed in this order because we use (10.22) in the proof of (10.8).

Let $\chi \in C^\infty_c(-1-\epsilon, 1+\epsilon)$ with $\operatorname{supp}(1-\chi) \cap [-1,1] = \emptyset$. Then,

$$
\begin{aligned}
&(I + P^C_{\mathscr{T}_N}L)^{-1}P^C_{\mathscr{T}_N}L\big(1 - \chi(-\hbar^2\Delta_\Gamma)\big)(P^C_{\mathscr{T}_N} - I)\\
&\qquad = (I + P^C_{\mathscr{T}_N}L)^{-1}\Big(\chi(-\hbar^2\Delta_\Gamma) + \big(1 - \chi(-\hbar^2\Delta_\Gamma)\big)\Big)P^C_{\mathscr{T}_N}L\big(1 - \chi(-\hbar^2\Delta_\Gamma)\big)(P^C_{\mathscr{T}_N} - I).
\end{aligned}
$$

Now, by (10.10), (6.11) with $t = q = s > 1/2$, Part (ii) of Assumption 4.1, Corollary 3.10, and (6.11) with $t = s$ and $q = s - 1$,

$$\big\|(I + P^C_{\mathscr{T}_N}L)^{-1}\big(1 - \chi(-\hbar^2\Delta_\Gamma)\big)P^C_{\mathscr{T}_N}L\big(1 - \chi(-\hbar^2\Delta_\Gamma)\big)(P^C_{\mathscr{T}_N} - I)\big\|_{H^s_\hbar\to H^s_\hbar}$$

$$\leq C\Big(1+\rho\Big(\frac{k}{N}\Big)^{s+1}\Big)\big\|L(1-\chi(-\hbar^2\Delta_\Gamma))(P^C_{\mathscr{T}_N}-I)\big\|_{H^s_\hbar\to H^s_\hbar} \leq C\Big(1+\rho\Big(\frac{k}{N}\Big)^{s+1}\Big)\frac{k}{N}. \quad (10.23)$$

Next, by (10.11), (6.11) with $t=q=1$, Part (ii) of Assumption 4.1 and Corollary 3.10, and (6.11) with $t=s, q=0$,

$$\begin{aligned}
&\big\|(I+P^C_{\mathscr{T}_N}L)^{-1}\chi(-\hbar^2\Delta_\Gamma)P^C_{\mathscr{T}_N}L\big(1-\chi(-\hbar^2\Delta_\Gamma)\big)(P^C_{\mathscr{T}_N}-I)\big\|_{H^s_\hbar\to H^s_\hbar}\\
&\leq \big\|(I+P^C_{\mathscr{T}_N}L)^{-1}\chi(-\hbar^2\Delta_\Gamma)\|_{H^1_\hbar\to H^s_\hbar}\big\|P^C_{\mathscr{T}_N}\big\|_{H^1_\hbar\to H^1_\hbar}\big\|L\big(1-\chi(-\hbar^2\Delta_\Gamma)\big)\big\|_{L^2\to H^1_\hbar}\big\|(P^C_{\mathscr{T}_N}-I)\big\|_{H^s_\hbar\to L^2}\\
&\leq C\rho\Big(\frac{k}{N}\Big)^s. \qquad (10.24)
\end{aligned}$$

Finally, by (10.11) again, Lemma 4.6, the fact that $\chi(-\hbar^2\Delta_\Gamma): L^2\to H^s_\hbar$ with bounded norm, and (6.11) with $t=s$ and $q=0$,

$$\begin{aligned}
&\big\|(I+P^C_{\mathscr{T}_N}L)^{-1}P^C_{\mathscr{T}_N}L\chi(-\hbar^2\Delta_\Gamma)(P^C_{\mathscr{T}_N}-I)\big\|_{H^s_\hbar\to H^s_\hbar}\\
&=\big\|(I-(I+P^C_{\mathscr{T}_N}L)^{-1})\chi(-\hbar^2\Delta_\Gamma)(P^C_{\mathscr{T}_N}-I)\big\|_{H^s_\hbar\to H^s_\hbar}\leq C\rho\big\|(P^C_{\mathscr{T}_N}-I)\big\|_{H^s_\hbar\to L^2}\leq C\rho\Big(\frac{k}{N}\Big)^s. \qquad (10.25)
\end{aligned}$$

Combining (10.23), (10.24), and (10.25), we obtain (10.22), and thus (10.19) is ensured if (10.3) holds.

We now prove (10.8) under the assumption that (10.20) holds. By (10.21), the right-hand side of (10.14) can be bounded as follows:

$$\begin{aligned}
&\Big\|\big(I+P^C_{\mathscr{T}_N}L_NP^C_{\mathscr{T}_N}\big)^{-1}\Big[(I-P^C_{\mathscr{T}_N}L)(I-P^C_{\mathscr{T}_N})v+P^C_{\mathscr{T}_N}(L_N-L)P^C_{\mathscr{T}_N}v\Big]\Big\|_{H^s_\hbar(\Gamma)}\\
&\leq 2\Big\|\big(I+P^C_{\mathscr{T}_N}L\big)^{-1}\Big[(I-P^C_{\mathscr{T}_N}L)(I-P^C_{\mathscr{T}_N})v+P^C_{\mathscr{T}_N}(L_N-L)P^C_{\mathscr{T}_N}v\Big]\Big\|_{H^s_\hbar(\Gamma)}\\
&=2\Big\|\Big[I-2(I+P^C_{\mathscr{T}_N}L)^{-1}P^C_{\mathscr{T}_N}L\Big](I-P^C_{\mathscr{T}_N})v+(I+P^C_{\mathscr{T}_N}L)^{-1}P^C_{\mathscr{T}_N}(L_N-L)P^C_{\mathscr{T}_N}v\Big\|_{H^s_\hbar(\Gamma)}\\
&\leq 2\big\|(I-P^C_{\mathscr{T}_N})v\big\|_{H^s_\hbar(\Gamma)}+4\big\|(I+P^C_{\mathscr{T}_N}L)^{-1}P^C_{\mathscr{T}_N}L(I-P^C_{\mathscr{T}_N})\big\|_{H^s_\hbar(\Gamma)\to H^s_\hbar(\Gamma)}\big\|(I-P^C_{\mathscr{T}_N})v\big\|_{H^s_\hbar(\Gamma)}\\
&\qquad+2\big\|(I+P^C_{\mathscr{T}_N}L)^{-1}P^C_{\mathscr{T}_N}(L_N-L)P^C_{\mathscr{T}_N}v\big\|_{H^s_\hbar(\Gamma)}. \qquad (10.26)
\end{aligned}$$

The second term on the right-hand side of (10.26) can be estimated via (10.22). To bound the third term on the right-hand side of (10.26), we observe that, by Lemma 10.2,

$$\begin{aligned}
&\big\|(I+P^C_{\mathscr{T}_N}L)^{-1}P^C_{\mathscr{T}_N}(L_N-L)P^C_{\mathscr{T}_N}v\big\|_{H^s_\hbar(\Gamma)}\\
&\qquad\leq \big\|(I+P^C_{\mathscr{T}_N}L)^{-1}\big(I-\chi(-\hbar^2\Delta_g)\big)\big\|_{H^s_\hbar(\Gamma)\to H^s_\hbar(\Gamma)}\big\|P^C_{\mathscr{T}_N}(L_N-L)P^C_{\mathscr{T}_N}v\big\|_{H^s_\hbar(\Gamma)}\\
&\qquad\qquad+\|(I+P^C_{\mathscr{T}_N}L)^{-1}\chi(-\hbar^2\Delta_g)\|_{L^2(\Gamma)\to H^s_\hbar(\Gamma)}\big\|P^C_{\mathscr{T}_N}(L_N-L)P^C_{\mathscr{T}_N}v\big\|_{L^2(\Gamma)}\\
&\qquad\leq C\Big(1+\rho\Big(\frac{k}{N}\Big)^{s+1}\Big)\big\|P^C_{\mathscr{T}_N}(L_N-L)P^C_{\mathscr{T}_N}v\big\|_{H^s_\hbar(\Gamma)}+C\rho\big\|P^C_{\mathscr{T}_N}(L_N-L)P^C_{\mathscr{T}_N}v\big\|_{L^2(\Gamma)}. \quad (10.27)
\end{aligned}$$

Therefore, inserting the bounds (10.22) and (10.27) into (10.26), we obtain that

$$\begin{aligned}
&\Big\|\big(I+P^C_{\mathscr{T}_N}L_NP^C_{\mathscr{T}_N}\big)^{-1}\Big[(I-P^C_{\mathscr{T}_N}L)(I-P^C_{\mathscr{T}_N})v+P^C_{\mathscr{T}_N}(L_N-L)P^C_{\mathscr{T}_N}v\Big]\Big\|_{H^s_\hbar(\Gamma)}\\
&\leq C\Big(1+\rho\Big(\frac{k}{N}\Big)^{s}\Big)\big\|(I-P^C_{\mathscr{T}_N})v\big\|_{H^s_\hbar(\Gamma)}+C\Big(1+\rho\Big(\frac{k}{N}\Big)^{s+1}\Big)\big\|P^C_{\mathscr{T}_N}(L_N-L)P^C_{\mathscr{T}_N}v\big\|_{H^s_\hbar(\Gamma)}\\
&\quad+C\rho\big\|P^C_{\mathscr{T}_N}(L_N-L)P^C_{\mathscr{T}_N}v\big\|_{L^2(\Gamma)},
\end{aligned}$$

which combined with (10.14) implies the bound (10.8).

To complete the proof of (10.8), we now need to check that (10.20) holds. By (10.11), (10.10), and the fact that $P^C_{\mathscr{T}_N}: H^s_\hbar(\Gamma)\to H^s_\hbar(\Gamma)$ is uniformly bounded (for $s>1/2$) by (6.11), for $s\geq t>1/2$,

$$\big\|(I+P^C_{\mathscr{T}_N}L)^{-1}P^C_{\mathscr{T}_N}(L_N-L)P^C_{\mathscr{T}_N}\big\|_{H^s_\hbar\to H^s_\hbar}$$

$$
\begin{aligned}
&\leq \left\|(I+P^C_{\mathscr{T}_N}L)^{-1}\chi(-\hbar^2\Delta_\Gamma)P^C_{\mathscr{T}_N}(L_N-L)P^C_{\mathscr{T}_N}\right\|_{H^s_\hbar\to H^s_\hbar}\\
&\qquad\qquad\qquad + \left\|(I+P^C_{\mathscr{T}_N}L)^{-1}(1-\chi(-\hbar^2\Delta_\Gamma))P^C_{\mathscr{T}_N}(L_N-L)P^C_{\mathscr{T}_N}\right\|_{H^s_\hbar\to H^s_\hbar}\\
&\leq C\rho\left\|(L_N-L)P^C_{\mathscr{T}_N}\right\|_{H^s_\hbar\to H^t_\hbar}+C\Big(1+\rho\Big(\frac{k}{N}\Big)^{s+1}\Big)\left\|(L_N-L)P^C_{\mathscr{T}_N}\right\|_{H^s_\hbar\to H^s_\hbar}.
\end{aligned}
$$

Now, by the bound (10.15) from Lemma 10.4, first applied with $r=t$ and then with $r=s$,

$$
\begin{aligned}
&\left\|(I+P^C_{\mathscr{T}_N}L)^{-1}P^C_{\mathscr{T}_N}(L_N-L)P^C_{\mathscr{T}_N}\right\|_{H^s_\hbar\to H^s_\hbar}\\
&\leq C\rho\bigg(\Big(\frac{k}{N}\Big)^{s-t+1}\mathcal{B}^{t,s}(\widetilde{L})+\Big[\Big(\frac{k}{N}\Big)^{s-t+1}\mathcal{F}^{s,\epsilon}_{\mathscr{L}}(N)+\mathcal{F}^{t,\epsilon}_{\mathscr{H}}(N)\Big]\\
&\qquad\qquad\qquad\times\Big[1+\|\widetilde{L}\|_{s,s}+\|\widetilde{L}\|_{t,t}+\Big(\frac{k}{N}\Big)\mathcal{F}^{s,\epsilon}_{\mathscr{L}}(N)+\mathcal{F}^{s,\epsilon}_{\mathscr{H}}(N)\Big]\bigg)\\
&\quad+C\Big(1+\rho\Big(\frac{k}{N}\Big)^{s+1}\Big)\bigg(\Big(\frac{k}{N}\Big)\mathcal{B}^{s,s}(\widetilde{L})+\Big[\Big(\frac{k}{N}\Big)\mathcal{F}^{s,\epsilon}_{\mathscr{L}}(N)+\mathcal{F}^{s,\epsilon}_{\mathscr{H}}(N)\Big]\\
&\qquad\qquad\qquad\times\Big[1+\|\widetilde{L}\|_{s,s}+\Big(\frac{k}{N}\Big)\mathcal{F}^{s,\epsilon}_{\mathscr{L}}(N)+\mathcal{F}^{s,\epsilon}_{\mathscr{H}}(N)\Big]\bigg)\\
&\leq C\Big(\frac{k}{N}+\rho\Big(\frac{k}{N}\Big)^{s-t+1}\Big)\mathcal{F}^{s,\epsilon}_{\mathscr{L}}(N)\Big[1+\|\widetilde{L}\|_{s,s}+\|\widetilde{L}\|_{t,t}\Big]\\
&\quad+C\bigg(1+\rho\Big(1+\Big(\frac{k}{N}\Big)^{s+1}\Big)\bigg)\mathcal{F}^{s,\epsilon}_{\mathscr{H}}(N)\Big[1+\|\widetilde{L}\|_{s,s}+\|\widetilde{L}\|_{t,t}\Big]\\
&\quad+C\Big[1+\rho\Big(\frac{k}{N}\Big)^{s+1}\Big]\Big(\frac{k}{N}\Big)\mathcal{B}^{s,s}(\widetilde{L})+\rho\Big(\frac{k}{N}\Big)^{s-t+1}\mathcal{B}^{t,s}(\widetilde{L})
\end{aligned}
$$

where in the last line we have used (10.5) and (10.6) to see that

$$
\left|\Big(\frac{k}{N}\Big)\mathcal{F}^{s,\epsilon}_{\mathscr{L}}(N)+\mathcal{F}^{s,\epsilon}_{\mathscr{H}}(N)\right|\leq C.
$$

Therefore, the conditions (10.4), (10.5), and (10.6) ensure that (10.20) holds, and the proof of (10.8) is complete.

Finally, to obtain (10.9), we bound $\left\|P^C_{\mathscr{T}_N}(L_N-L)P^C_{\mathscr{T}_N}v\right\|_{L^2(\Gamma)}$ in (10.8) by (10.16). Indeed, by (10.16),

$$
\begin{aligned}
&\left\|P^C_{\mathscr{T}_N}(L_N-L)P^C_{\mathscr{T}_N}v\right\|_{L^2(\Gamma)}\\
&\leq C\bigg[\Big(\Big(\frac{k}{N}\Big)^{s+1}\mathcal{F}^{s,\epsilon}_{\mathscr{L}}(N)+\mathcal{F}^{0,\epsilon}_{\mathscr{H}}(N)\Big)(1+\|\widetilde{L}\|_{0,0}+\|\widetilde{L}\|_{s,s})\\
&\qquad+\Big(\Big(\frac{k}{N}\Big)\mathcal{F}^{0,\epsilon}_{\mathscr{L}}(N)+\mathcal{F}^{0,\epsilon}_{\mathscr{H}}(N)\Big)\Big(\Big(\frac{k}{N}\Big)^{s+1}\mathcal{F}^{s,\epsilon}_{\mathscr{L}}(N)+\mathcal{F}^{0,\epsilon}_{\mathscr{H}}(N)\Big)+\Big(\frac{k}{N}\Big)^{s+1}\mathcal{B}^{0,s}(\widetilde{L})\bigg]\|v\|_{H^s_\hbar(\Gamma)},\\
&\leq C\bigg[\Big(\Big(\frac{k}{N}\Big)^{s+1}\mathcal{F}^{s,\epsilon}_{\mathscr{L}}(N)+\mathcal{F}^{0,\epsilon}_{\mathscr{H}}(N)\Big](1+\|\widetilde{L}\|_{0,0}+\|\widetilde{L}\|_{s,s})+\Big(\frac{k}{N}\Big)^{s+1}\mathcal{B}^{0,s}(\widetilde{L})\bigg]\|v\|_{H^s_\hbar(\Gamma)}.
\end{aligned}
$$

■

Before proving Lemma 10.4, we record the following two lemmas. In these lemmas and the proof of Lemma 10.4, we use the notation that

$$
(\varphi_m\circ\gamma)(t):=\exp(imt) \tag{10.28}
$$

(compare to (6.4)).

Lemma 10.5 (Multiplication by trigonometric polynomials) *Given $s\in\mathbb{R}$ there exists $C>0$ such that, for all $v\in H^s_\hbar(\Gamma)$ and all $m\in\mathbb{R}$,*

$$
\|\varphi_m v\|_{H^s_\hbar(\Gamma)}\leq C\langle m\hbar\rangle^s\|v\|_{H^s_\hbar(\Gamma)}. \tag{10.29}
$$

Proof. This result is well-known (see, e.g., [98, Prop. 3]), but since the proof is very short, we give it here. By (6.2) and (6.3),

$$\begin{aligned}\|\varphi_m v\|_{H^s_\hbar(\Gamma)} &\leq C_2 \sum_{n=-\infty}^{\infty} \left|\widehat{(v\circ\gamma)}(n-m)\right|^2 \langle n\hbar\rangle^{2s} \\ &= C_2 \sum_{n=-\infty}^{\infty} \left|\widehat{(v\circ\gamma)}(n)\right|^2 \langle (n+m)\hbar\rangle^{2s} \leq C' \sum_{n=-\infty}^{\infty} \left|\widehat{(v\circ\gamma)}(n)\right|^2 \langle n\hbar\rangle^{2s}\langle m\hbar\rangle^{2s},\end{aligned}$$

by Peetre's inequality. The result (10.29) then follows from using (6.2) and (6.3) again. ■

Lemma 10.6 *Let $n \leq N$. If $n+|m|\leq N$ then*

$$(I-P^C_{\mathscr{T}_N})\varphi_m P^C_{\mathscr{T}_n} = 0. \tag{10.30}$$

Furthermore, if $n+|m|\leq 2N$ then

$$P^G_{\mathscr{T}_{2N-n-|m|}}(I-P^C_{\mathscr{T}_N})\varphi_m P^C_{\mathscr{T}_n} = 0. \tag{10.31}$$

Proof. For $(I-P^C_{\mathscr{T}_N})\varphi_m P^C_{\mathscr{T}_n}$ not to be zero, there must exist $\xi\in\mathbb{Z}$ such that

$$N+1 \leq |\xi| \leq n+|m|, \tag{10.32}$$

so that if $n+|m|\leq N$ then (10.30) holds. Then for $P^G_{\mathscr{T}_{2N-n-|m|}}(I-P^C_{\mathscr{T}_N})\varphi_m P^C_{\mathscr{T}_n}$ not to be zero, we need, in addition to (10.32), that, by (6.9), there exists $\ell\in 0,1,2,\ldots$ such that

$$\big||\xi|-\ell(2N+1)\big| \leq 2N-n-|m|. \tag{10.33}$$

If $n+|m|\leq N$, then (10.32) cannot hold, so we can restrict attention to the case $n+|m|>N$. In this case, (10.33) implies that $||\xi|-\ell(2N+1)|\leq N$, which cannot hold with $\ell=0$ by the lower bound in (10.32). Finally, (10.33) implies that

$$|\xi|-\ell(2N+1) \geq |m|+n-2N \quad \text{so that} \quad |\xi| \geq |m|+n+\ell+2N(\ell-1),$$

which contradicts (10.32) if $\ell\geq 1$; i.e., we have proved (10.31). ■

We adopt the notation of [72, §12.4], [73] and define

$$(A_0 w)(t) := \int_0^{2\pi} \log\left(4\sin^2\left(\frac{t-\tau}{2}\right)\right) w(\tau)d\tau. \tag{10.34}$$

Up to a factor of π^{-1}, A_0 is the Laplace single-layer operator on the unit circle with arc-length parametrisation (since there $|x(t)-x(\tau)|^2 = 4\sin^2((t-\tau)/2)$). Therefore, $A_0 : H^s(0,2\pi)\to H^{s+1}(0,2\pi)$ for all $s\in\mathbb{R}$ (by, e.g., [93, Theorem 4.4.1]).

Lemma 10.7 *If L satisfies Assumption 9.1 then for all $c,\epsilon>0$ there exists $C>0$ such that for $N\geq ck$, $0\leq n\leq N$, $r\geq s>1/2$, and $\mathfrak{L}^N$ as in (9.2),*

$$\|(L-\mathfrak{L}^N)P^C_{\mathscr{T}_n}\|_{H^r_\hbar(\Gamma)\to H^s_\hbar(\Gamma)} \leq C\left[\left(\frac{k}{N}\right)^{r-s+1} F^{r,\epsilon}_{\mathscr{L}}(N,n,L) + \left(\frac{k}{N}\right)^{\min\{s,1\}} F^{s,\epsilon}_{\mathscr{H}}(N,n,L)\right]. \tag{10.35}$$

Proof. By the definitions of L in (9.1) and $\mathfrak{L}^N$ in (9.2),

$$(L-\mathfrak{L}^N)P^C_{\mathscr{T}_n} = I_1 + I_2,$$

where

$$(I_1 v)(t) := \int_0^{2\pi} \log\left(4\sin^2\left(\frac{t-\tau}{2}\right)\right) \left(I-P^C_{\mathscr{T}_N,\tau}\right)\left(L_1(t,\tau)\,P^C_{\mathscr{T}_n}v(\tau)\right)d\tau$$

and

$$(I_2 v)(t) := \int_0^{2\pi} \big(I - P^C_{\mathscr{T}_N,\tau}\big)\big(L_2(t,\tau)\, P^C_{\mathscr{T}_n} v(\tau)\big)\, d\tau.$$

We prove (10.35) by proving that

$$\begin{aligned}\|I_1\|_{H^r_\hbar(\Gamma)\to H^s_\hbar(\Gamma)} &\leq C\Big(\frac{k}{N}\Big)^{r-s+1} k^{-1} \sum_{N-n\leq|m|\leq(2-\epsilon)N-n} \big\|\widehat{L}_{1,m}(t)\big\|_{H^s_\hbar(\Gamma)} \langle m/k\rangle^r \\ &\quad + C\Big(\frac{k}{N}\Big)^{\min\{s,1\}} \sum_{|m|>(2-\epsilon)N-n} \big\|\widehat{L}_{1,m}(t)\big\|_{H^s_\hbar(\Gamma)} \langle m/k\rangle^s \end{aligned} \tag{10.36}$$

and

$$\|I_2\|_{H^r_\hbar(\Gamma)\to H^s_\hbar(\Gamma)} \leq C\Big(\frac{k}{N}\Big)^s \sum_{|m|>2N-n} \|\widehat{L}_{2,m}(t)\|_{H^s_\hbar(\Gamma)} \langle m/k\rangle^s. \tag{10.37}$$

By the definition of $\widehat{L}_{j,m}(t)$ from (9.3) and (6.1),

$$L_j(t,\tau) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \widehat{L}_{j,m}(t) \mathrm{e}^{\mathrm{i}m\tau}.$$

Then

$$(I_1 v)(t) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \widehat{L}_{1,m}(t)\, A_0\Big((I - P^C_{\mathscr{T}_N})(\varphi_m P^C_{\mathscr{T}_n} v)\Big)(t), \tag{10.38}$$

where A_0 is defined by (10.34) and

$$\begin{aligned}(I_2 v)(t) &= \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \widehat{L}_{2,m}(t) \int_0^{2\pi} (I - P^C_{\mathscr{T}_N,\tau})\big(\varphi_m(\tau) P^C_{\mathscr{T}_n} v(\tau)\big) d\tau \\ &= \sum_{m=-\infty}^{\infty} \widehat{L}_{2,m}(t)\ P^G_{\mathscr{T}_0}(I - P^C_{\mathscr{T}_N})\big(\varphi_m P^C_{\mathscr{T}_n} v\big).\end{aligned} \tag{10.39}$$

We first prove the bound on I_2 in (10.36) (since this is slightly easier than proving the bound on I_1). By, e.g., [98, Prop. 2], [100, §5.13], for all $v \in H^s_\hbar(\Gamma)$ and $s > 1/2$,

$$\big\|\widehat{L}_{2,m} v\big\|_{H^s_\hbar(\Gamma)\to H^s_\hbar(\Gamma)} \leq C\big\|\widehat{L}_{2,m}\big\|_{H^s_\hbar(\Gamma)} \|v\|_{H^s_\hbar(\Gamma)}\,. \tag{10.40}$$

By (10.40), (10.31), (10.29), (6.5), (6.11),

$$\begin{aligned}\|I_2 v\|_{H^s_\hbar(\Gamma)} &\leq C \sum_{|m|\geq 2N-n} \big\|\widehat{L}_{2,m}\big\|_{H^s_\hbar(\Gamma)} \big\|P^G_{\mathscr{T}_0}(I - P^C_{\mathscr{T}_N})(\varphi_m P^C_{\mathscr{T}_n} v)\big\|_{H^s_\hbar(\Gamma)} \\ &\leq C\|v\|_{H^s_\hbar(\Gamma)} \sum_{|m|\geq 2N-n} \|\widehat{L}_{2,m}\|_{H^s_\hbar(\Gamma)} \langle mk^{-1}\rangle^s (k/N)^s, \\ &\leq C\|v\|_{H^r_\hbar(\Gamma)} \sum_{|m|\geq 2N-n} \|\widehat{L}_{2,m}\|_{H^s_\hbar(\Gamma)} \langle mk^{-1}\rangle^s (k/N)^s,\end{aligned}$$

which is (10.37). We now bound I_1. Using in the expression (10.38) the properties (10.40), (10.30), (10.31), and (5.5), we find that

$$\begin{aligned}\|I_1 v\|_{H^s_\hbar(\Gamma)} &\leq C \sum_{m=-\infty}^{\infty} \big\|\widehat{L}_{1,m}(t)\big\|_{H^s_\hbar(\Gamma)} \|A_0(I - P^C_{\mathscr{T}_N})\varphi_m P^C_{\mathscr{T}_n} v\|_{H^s_\hbar(\Gamma)} \\ &= C \sum_{|m|>N-n} \big\|\widehat{L}_{1,m}(t)\big\|_{H^s_\hbar(\Gamma)} \|A_0(I - P^C_{\mathscr{T}_N})\varphi_m P^C_{\mathscr{T}_n} v\|_{H^s_\hbar(\Gamma)} \\ &\leq C \sum_{N-n<|m|\leq(2-\epsilon)N-n} \big\|\widehat{L}_{1,m}(t)\big\|_{H^s_\hbar(\Gamma)} \|A_0(I - P^G_{\mathscr{T}_{2N-|m|-n}})(I - P^C_{\mathscr{T}_N})\varphi_m P^C_{\mathscr{T}_n} v\|_{H^s_\hbar(\Gamma)}\end{aligned}$$

$$+ C \sum_{|m|>(2-\epsilon)N-n} \big\|\widehat{L}_{1,m}(t)\big\|_{H^s_\hbar(\Gamma)} \|A_0(I - P^C_{\mathscr{T}_N})\varphi_m P^C_{\mathscr{T}_n} v\|_{H^s_\hbar(\Gamma)}.$$

We now claim that $\|A_0\|_{H^{s-1}_\hbar(\Gamma)\to H^s_\hbar(\Gamma)} \le C$. Indeed, for $s \ge 1$, since $A_0 : H^{s-1}(\Gamma) \to H^s(\Gamma)$,

$$\big\|A_0 u\big\|_{L^2(\Gamma)} \le C\|u\|_{L^2(\Gamma)} \quad \text{and} \quad \big\|A_0 u\big\|_{H^s(\Gamma)} \le C\|u\|_{H^{s-1}(\Gamma)}$$

so that

$$\big\|A_0 u\big\|_{H^s_\hbar(\Gamma)} \le C\Big(\big\|A_0 u\big\|_{L^2(\Gamma)} + \hbar^s\big\|A_0 u\big\|_{H^s_\hbar(\Gamma)}\Big) \le C\big(\|u\|_{L^2(\Gamma)} + \hbar^s\|u\|_{H^{s-1}(\Gamma)}\big) \le C\|u\|_{H^{s-1}_\hbar(\Gamma)}.$$

The result for $s < 1$ follows by duality and interpolation. Therefore,

$$\begin{aligned}
\|I_1 v\|_{H^s_\hbar(\Gamma)} \le C &\sum_{N-n<|m|\le(2-\epsilon)N-n} \big\|\widehat{L}_{1,m}(t)\big\|_{H^s_\hbar(\Gamma)} \|A_0(I - P^G_{\mathscr{T}_{2N-|m|-n}})(I - P^C_{\mathscr{T}_N})\varphi_m P^C_{\mathscr{T}_n} v\|_{H^s_\hbar(\Gamma)} \\
&+ \sum_{|m|>(2-\epsilon)N-n} \big\|\widehat{L}_{1,m}(t)\big\|_{H^s_\hbar(\Gamma)} \|(I - P^C_{\mathscr{T}_N})\varphi_m P^C_{\mathscr{T}_n} v\|_{H^{s-1}_\hbar(\Gamma)} \\
\le C &\sum_{N-n<|m|\le(2-\epsilon)N-n} \big\|\widehat{L}_{1,m}(t)\big\|_{H^s_\hbar(\Gamma)} \|A_0(I - P^G_{\mathscr{T}_{2N-|m|-n}})(I - P^C_{\mathscr{T}_N})\varphi_m P^C_{\mathscr{T}_n} v\|_{H^s_\hbar(\Gamma)} \\
&+ \sum_{|m|>(2-\epsilon)N-n} \big\|\widehat{L}_{1,m}(t)\big\|_{H^s_\hbar(\Gamma)} \Big(\frac{k}{N}\Big)^{s-\max\{s-1,0\}} \langle m/k\rangle^s \|v\|_{H^s_\hbar(\Gamma)}. \qquad (10.41)
\end{aligned}$$

where we have used (6.11) and (10.29) in the last step. Now, since A_0 is a (non-semiclassical) pseudodifferential operator of order -1, for any M

$$\|P^G_{\mathscr{T}_{N-\frac{|m|-n}{2}}} A_0(I - P^G_{\mathscr{T}_{2N-|m|-n}})\|_{H^{-M}\to H^M} \le C_M(2N - |m| - n)^{-M}.$$

Next, observe that for any $s \in \mathbb{R}$, since $2N - |m| - n \ge \epsilon N \ge c\epsilon k$, then

$$\begin{aligned}
\|(I - P^G_{\mathscr{T}_{N-\frac{|m|-n}{2}}})u\|_{H^s_\hbar(\Gamma)} &\le C\hbar^s\|(I - P^G_{\mathscr{T}_{N-\frac{|m|-n}{2}}})u\|_{H^s(\Gamma)}, \\
\|(I - P^G_{\mathscr{T}_{2N-|m|-n}})u\|_{H^s(\Gamma)} &\le C\hbar^{-s}\|(I - P^G_{\mathscr{T}_{2N-|m|-n}})u\|_{H^s_\hbar(\Gamma)}, \\
\|(I - P^G_{\mathscr{T}_{2N-|m|-n}})u\|_{H^{s-1}(\Gamma)} &\le C(2N - |m| - n + 1)^{-1}\|(I - P^G_{\mathscr{T}_{2N-|m|-n}})u\|_{H^s(\Gamma)}.
\end{aligned}$$

By (6.6) and the fact that $2N - |m| - n \ge \epsilon N \ge c\epsilon k$,

$$\|I - P^G_{\mathscr{T}_{N-\frac{|m|-n}{2}}}\|_{H^s_\hbar(\Gamma)\to H^s_\hbar(\Gamma)} \le C.$$

Combining these last five displayed bounds, we obtain that

$$\begin{aligned}
&\|A_0(I - P^G_{\mathscr{T}_{2N-|m|-n}})w\|_{H^s_\hbar(\Gamma)} \\
&\quad\le \big\|\big(I - P^G_{\mathscr{T}_{N-\frac{|m|-n}{2}}}\big)A_0(I - P^G_{\mathscr{T}_{2N-|m|-n}})w\big\|_{H^s_\hbar(\Gamma)} + C_M(2N - |m| - n)^{-M}\,\|w\|_{H^s_\hbar(\Gamma)}\,, \\
&\quad\le C\hbar^s\big\|\big(I - P^G_{\mathscr{T}_{N-\frac{|m|-n}{2}}}\big)A_0(I - P^G_{\mathscr{T}_{2N-|m|-n}})w\big\|_{H^s(\Gamma)} + C_M(2N - |m| - n)^{-M}\,\|w\|_{H^s_\hbar(\Gamma)}\,, \\
&\quad\le C\hbar^s\big\|(I - P^G_{\mathscr{T}_{2N-|m|-n}})w\big\|_{H^{s-1}(\Gamma)} + C_M(2N - |m| - n)^{-M}\,\|w\|_{H^s_\hbar(\Gamma)}\,, \\
&\quad\le C(2N - |m| - n + 1)^{-1}\hbar^s\big\|(I - P^G_{\mathscr{T}_{2N-|m|-n}})w\big\|_{H^s(\Gamma)} + C_M(2N - |m| - n)^{-M}\,\|w\|_{H^s_\hbar(\Gamma)}\,, \\
&\quad\le C(2N - |m| - n + 1)^{-1}\big\|(I - P^G_{\mathscr{T}_{2N-|m|-n}})w\big\|_{H^s_\hbar(\Gamma)} + C_M(2N - |m| - n)^{-M}\,\|w\|_{H^s_\hbar(\Gamma)}\,.
\end{aligned}$$

Using this along with (6.11) and (10.29) in the estimate (10.41), we obtain

$$\|I_1 v\|_{H^s_\hbar(\Gamma)} \le C \sum_{N-n<|m|\le(2-\epsilon)N-n} \big\|\widehat{L}_{1,m}(t)\big\|_{H^s_\hbar(\Gamma)} (2N - n + 1 - |m|)^{-1}\langle m/k\rangle^r \Big(\frac{k}{N}\Big)^{r-s} \|v\|_{H^r_\hbar(\Gamma)}$$

$$
\begin{aligned}
&+ \sum_{|m|>(2-\epsilon)N-n} \big\|\widehat{L}_{1,m}(t)\big\|_{H_\hbar^s(\Gamma)} \Big(\frac{k}{N}\Big)^{s-\max\{s-1,0\}} \langle m/k\rangle^s \|v\|_{H_\hbar^s(\Gamma)}\\
&\le C\|v\|_{H_\hbar^r(\Gamma)} \Big(\frac{k}{N}\Big)^{r-s+1} k^{-1} \sum_{N-n<|m|\le(2-\epsilon)N-n} \big\|\widehat{L}_{1,m}(t)\big\|_{H_\hbar^s(\Gamma)} \langle m/k\rangle^r\\
&\quad + C\|v\|_{H_\hbar^r(\Gamma)} \Big(\frac{k}{N}\Big)^{\min\{s,1\}} \sum_{|m|>(2-\epsilon)N-n} \big\|\widehat{L}_{1,m}(t)\big\|_{H_\hbar^s(\Gamma)} \langle m/k\rangle^s,
\end{aligned}
$$

where we have used (6.11) and the fact that $N \ge ck$ in the last step. ∎

The only thing the remains to do in the proof of Lemma 10.4 is to estimate the difference

$$
(\widetilde{L}_{j,a}\widetilde{L}_{j,b} - \widetilde{\mathfrak{L}}^N_{j,a} P^C_{\mathscr{T}_N} \widetilde{\mathfrak{L}}^N_{j,b}) P^C_{\mathscr{T}_N}.
$$

Lemma 10.8 *Let T_a and T_b Assumptions (9.1). Then, for all $r \ge s \ge 0$ and $0<\epsilon<1$, there is $C>0$ such that for $0 \le n \le n' \le N$, and $\mathfrak{T}_a^N$, $\mathfrak{T}_b^N$ as in (9.2),*

$$
\begin{aligned}
&\|(T_aT_b - \mathfrak{T}_a^N P^C_{\mathscr{T}_N} \mathfrak{T}_b^N) P^C_{\mathscr{T}_n}\|_{H_\hbar^r(\Gamma)\to H_\hbar^s(\Gamma)}\\
&\quad\le C\Bigg(\Big(\Big(\frac{k}{N}\Big)^{r-s+1} F^{r,\epsilon}_{\mathscr{L}}(N,n,T_b) + F^{s,\epsilon}_{\mathscr{H}}(N,n,T_b)\Big) \|T_a\|_{H_\hbar^s\to H_\hbar^s}\\
&\quad + \Big(\Big(\frac{k}{N}\Big)^{r-s+1} F^{r,\epsilon}_{\mathscr{L}}(N,n',T_a) + F^{s,\epsilon}_{\mathscr{H}}(N,n',T_a)\Big) \|T_b\|_{H_\hbar^r\to H_\hbar^r}\\
&\quad + \Big(\Big(\frac{k}{N}\Big) F^{s,\epsilon}_{\mathscr{L}}(N,T_a) + F^{s,\epsilon}_{\mathscr{H}}(N,T_a)\Big)\Big(\Big(\frac{k}{N}\Big)^{r-s+1} F^{r,\epsilon}_{\mathscr{L}}(N,n,T_b) + F^{s,\epsilon}_{\mathscr{H}}(N,n,T_b)\Big)\\
&\quad + \Big(\frac{k}{N}\Big)^{r-s+1} \|T_a\|_{H_\hbar^s\to H_\hbar^s} \|T_b\|_{H_\hbar^r\to H_\hbar^r}\\
&\quad + \Big(\Big(\frac{k}{N}\Big)^{r-s+1} F^{r,\epsilon}_{\mathscr{L}}(N,T_a) + F^{s,\epsilon}_{\mathscr{H}}(N,T_a)\Big) \cdot \|P^C_{\mathscr{T}_N}(I - P^C_{\mathscr{T}_{n'}})\|_{H_\hbar^r\to H_\hbar^r} \|(I - P^G_{\mathscr{T}_{n'}}) T_b P^G_{\mathscr{T}_n}\|_{H_\hbar^r\to H_\hbar^r}\Bigg).
\end{aligned}
$$

In particular, if $n = n' = N$,

$$
\begin{aligned}
&\|(T_aT_b - \mathfrak{T}_a^N P^C_{\mathscr{T}_N} \mathfrak{T}_b^N) P^C_{\mathscr{T}_N}\|_{H_\hbar^r(\Gamma)\to H_\hbar^s(\Gamma)}\\
&\quad\le C\Bigg(\Big(\Big(\frac{k}{N}\Big)^{r-s+1} F^{r,\epsilon}_{\mathscr{L}}(N,T_b) + F^{s,\epsilon}_{\mathscr{H}}(N,T_b)\Big) \|T_a\|_{H_\hbar^s\to H_\hbar^s}\\
&\quad + \Big(\Big(\frac{k}{N}\Big)^{r-s+1} F^{r,\epsilon}_{\mathscr{L}}(N,T_a) + F^{s,\epsilon}_{\mathscr{H}}(N,T_a)\Big) \|T_b\|_{H_\hbar^r\to H_\hbar^r}\\
&\quad + \Big(\Big(\frac{k}{N}\Big) F^{s,\epsilon}_{\mathscr{L}}(N,T_a) + F^{s,\epsilon}_{\mathscr{H}}(N,T_a)\Big)\Big(\Big(\frac{k}{N}\Big)^{r-s+1} F^{r,\epsilon}_{\mathscr{L}}(N,T_b) + F^{s,\epsilon}_{\mathscr{H}}(N,T_b)\Big)\\
&\quad + \Big(\frac{k}{N}\Big)^{r-s+1} \|T_a\|_{H_\hbar^s\to H_\hbar^s} \|T_b\|_{H_\hbar^r\to H_\hbar^r}\Bigg).
\end{aligned}
$$

Suppose, in addition, that for all $\chi \in C_c^\infty(\mathbb{R})$ with $\chi \equiv 1$ near $[-1,1]$, $(1-\chi(|\hbar D'|_g^2))T_b \in \Psi_\hbar^{-1}(\Gamma)$. Then, for $n' \ge n + \epsilon k$ and $n = \Xi k$ with $\Xi \ge c_{\max} + \epsilon$, then for all $M>0$ there is $C_M > 0$ such *that*

$$
\begin{aligned}
&\|(T_aT_b - \mathfrak{T}_a^N P^C_{\mathscr{T}_N}\mathfrak{T}_b^N)P^C_{\mathscr{T}_n}\|_{H^r_\hbar(\Gamma)\to H^s_\hbar(\Gamma)}\\
&\quad\le C\Bigg(\Big(\Big(\frac{k}{N}\Big)^{r-s+1}F^{r,\epsilon}_{\mathscr{L}}(N,n,T_b)+F^{s,\epsilon}_{\mathscr{H}}(N,n,T_b)\Big)\|T_a\|_{H^s_\hbar\to H^s_\hbar}\\
&\qquad+\Big(\Big(\frac{k}{N}\Big)^{r-s+2}F^{r,\epsilon}_{\mathscr{L}}(N,n',T_a)+F^{s,\epsilon}_{\mathscr{H}}(N,n',T_a)\Big)\|T_b\|_{H^r_\hbar\to H^{r+1}_\hbar}\\
&\qquad+\Big(\Big(\frac{k}{N}\Big)F^{s,\epsilon}_{\mathscr{L}}(N,T_a)+F^{s,\epsilon}_{\mathscr{H}}(N,T_a)\Big)\Big(\Big(\frac{k}{N}\Big)^{r-s+1}F^{r,\epsilon}_{\mathscr{L}}(N,n,T_b)+F^{s,\epsilon}_{\mathscr{H}}(N,n,T_b)\Big)\\
&\qquad+\Big(\frac{k}{N}\Big)^{r-s+1}\|T_a\|_{H^s_\hbar\to H^s_\hbar}\|T_b\|_{H^r_\hbar\to H^r_\hbar}\\
&\qquad+C_M\Big(\Big(\frac{k}{N}\Big)^{r-s+1}F^{r,\epsilon}_{\mathscr{L}}(N,T_a)+F^{s,\epsilon}_{\mathscr{H}}(N,T_a)\Big)\|T_b\|_{L^2(\Gamma)\to L^2(\Gamma)}k^{-M}\Bigg).
\end{aligned}
$$

Proof. We write

$$
\begin{aligned}
(T_aT_b - \mathfrak{T}_a^N P^C_{\mathscr{T}_N}\mathfrak{T}_b^N)P^C_{\mathscr{T}_n} &= T_aP^C_{\mathscr{T}_N}T_bP^C_{\mathscr{T}_n} - \mathfrak{T}_a^N P^C_{\mathscr{T}_N}\mathfrak{T}_b^N P^C_{\mathscr{T}_n} + T_a(I-P^C_{\mathscr{T}_N})T_bP^C_{\mathscr{T}_n}\\
&= T_aP^C_{\mathscr{T}_N}(T_b-\mathfrak{T}_b^N)P^C_{\mathscr{T}_n} + (T_a-\mathfrak{T}_a^N)P^C_{\mathscr{T}_N}\mathfrak{T}_b^N P^C_{\mathscr{T}_n} + T_a(I-P^C_{\mathscr{T}_N})T_bP^C_{\mathscr{T}_n}\\
&= T_a(T_b-\mathfrak{T}_b^N)P^C_{\mathscr{T}_n} + (T_a-\mathfrak{T}_a^N)P^C_{\mathscr{T}_{n'}}T_bP^C_{\mathscr{T}_n} + (T_a-\mathfrak{T}_a^N)P^C_{\mathscr{T}_N}(\mathfrak{T}_b^N-T_b)P^C_{\mathscr{T}_n}\\
&\qquad + T_a(I-P^C_{\mathscr{T}_N})T_bP^C_{\mathscr{T}_n} + (T_a-\mathfrak{T}_a^N)P^C_{\mathscr{T}_N}(I-P^C_{\mathscr{T}_{n'}})T_bP^C_{\mathscr{T}_n}
\end{aligned}
$$

So,

$$
\begin{aligned}
&\|(T_aT_b - \mathfrak{T}_a^N P^C_{\mathscr{T}_N}\mathfrak{T}_b^N)P^C_{\mathscr{T}_n}\|_{H^r_\hbar\to H^s_\hbar}\\
&\quad\le C\Big(\|T_a\|_{H^s_\hbar\to H^s_\hbar(\Gamma)}\|(T_b-\mathfrak{T}_b^N)P^C_{\mathscr{T}_n}\|_{H^r_\hbar\to H^s_\hbar} + \|(T_a-\mathfrak{T}_a^N)P^C_{\mathscr{T}_{n'}}\|_{H^r_\hbar(\Gamma)\to H^s_\hbar}\|T_b\|_{H^r_\hbar(\Gamma)\to H^r_\hbar}\\
&\qquad+\|(T_a-\mathfrak{T}_a^N)P^C_{\mathscr{T}_N}\|_{H^s_\hbar\to H^s_\hbar}\|(\mathfrak{T}_b^N-T_b)P^C_{\mathscr{T}_n}\|_{H^r_\hbar\to H^s_\hbar}\\
&\qquad+\|T_a\|_{H^s_\hbar\to H^s_\hbar}\|I-P^C_{\mathscr{T}_N}\|_{H^r_\hbar\to H^s_\hbar}\|T_b\|_{H^r_\hbar\to H^r_\hbar}\\
&\qquad+\|(T_a-\mathfrak{T}_a^N)P^C_{\mathscr{T}_N}\|_{H^s_\hbar\to H^s_\hbar}\|P^C_{\mathscr{T}_N}(I-P^C_{\mathscr{T}_{n'}})\|_{H^s_\hbar\to H^s_\hbar}\|(I-P^G_{\mathscr{T}_{n'}})T_bP^G_{\mathscr{T}_n}\|_{H^r_\hbar(\Gamma)\to H^s_\hbar}\Big)\\
&\quad\le C\Bigg(\Big(\Big(\frac{k}{N}\Big)^{r-s+1}F^{r,\epsilon}_{\mathscr{L}}(N,n,T_b)+F^{s,\epsilon}_{\mathscr{H}}(N,n,T_b)\Big)\|T_a\|_{H^s_\hbar\to H^s_\hbar}\\
&\qquad+\Big(\Big(\frac{k}{N}\Big)^{r-s+1}F^{r,\epsilon}_{\mathscr{L}}(N,n',T_a)+F^{s,\epsilon}_{\mathscr{H}}(N,n',T_a)\Big)\|T_b\|_{H^r_\hbar\to H^r_\hbar}\\
&\qquad+\Big(\Big(\frac{k}{N}\Big)F^{s,\epsilon}_{\mathscr{L}}(N,T_a)+F^{s,\epsilon}_{\mathscr{H}}(N,T_a)\Big)\Big(\Big(\frac{k}{N}\Big)^{r-s+1}F^{r,\epsilon}_{\mathscr{L}}(N,n,T_b)+F^{s,\epsilon}_{\mathscr{H}}(N,n,T_b)\Big)\\
&\qquad+\Big(\frac{k}{N}\Big)^{r-s+1}\|T_a\|_{H^s_\hbar\to H^s_\hbar}\|T_b\|_{H^r_\hbar\to H^r_\hbar}\\
&\qquad+\Big(\Big(\frac{k}{N}\Big)^{r-s+2}F^{r,\epsilon}_{\mathscr{L}}(N,T_a)+F^{s,\epsilon}_{\mathscr{H}}(N,T_a)\Big)\cdot\|P^C_{\mathscr{T}_N}(I-P^C_{\mathscr{T}_{n'}})\|_{H^r_\hbar\to H^r_\hbar}\|(I-P^G_{\mathscr{T}_{n'}})T_bP^G_{\mathscr{T}_n}\|_{H^r_\hbar\to H^r_\hbar}\Bigg).
\end{aligned}
$$

The implication when $n=n'=N$ follows immediately since $P^C_{\mathscr{T}_N}(I-P^C_{\mathscr{T}_N})=0$.

Now, suppose that $(1-\chi(|\hbar D'|^2_g))T_b\in\Psi^{-1}_\hbar(\Gamma)$ for all $\chi\in C_c^\infty$ with $\chi\equiv 1$ near $[-1,1]$, $n'\ge n+\epsilon k$, and $n=\Xi k$ with $\Xi\ge(1+\epsilon)c_{\max}$. Then let $\chi_0,\chi_1\in C_c^\infty(\mathbb{R})$ with $\chi_0\equiv 1$ near $[-\Xi^2,\Xi^2]$, $\operatorname{supp}\chi_0\subset\{\chi_1\equiv 1\}$, and $\operatorname{supp}\chi_1\subset(-(\Xi+\epsilon)^2,(\Xi+\epsilon)^2)$ we observe

$$
P^G_{\mathscr{T}_n}=\chi_0(k^{-2}\partial_t^2)P^G_{\mathscr{T}_n},\qquad (I-P^G_{\mathscr{T}_{n'}})=(I-P^G_{\mathscr{T}_{n'}})(1-\chi_1(k^{-2}\partial_t^2))
$$

By Lemma 7.6 (with $\Xi=1$) together with the fact that $n'\ge(c_{\max}+\epsilon)k$, there is $\chi\in C_c^\infty$ with $\chi\equiv 1$ near $[-1,1]$ such that

$$
(I-P^G_{\mathscr{T}_{n'}})=(I-P^G_{\mathscr{T}_{n'}})(1-\chi(|\hbar D'|^2_g))+O(\hbar^\infty)_{\Psi^{-\infty}_\hbar(\Gamma)}.
$$

Therefore,

$$\begin{aligned}
&(I-P^G_{\mathscr{T}_{n'}})T_bP^G_{\mathscr{T}_n}\\
&=(I-P^G_{\mathscr{T}_{n'}})(1-\chi(|\hbar D'|^2_g))T_bP^G_{\mathscr{T}_n}+O(\hbar^\infty\|T_b\|_{L^2(\Gamma)\to L^2(\Gamma)})_{\Psi_\hbar^{-\infty}(\Gamma)}\\
&=(I-P^G_{\mathscr{T}_{n'}})(1-\chi_1(|k^{-2}\partial_t|^2)(1-\chi(|\hbar D'|^2_g))T_b\chi_0(k^{-2}\partial_t^2))P^G_{\mathscr{T}_n}+O(\hbar^\infty\|T_b\|_{L^2(\Gamma)\to L^2(\Gamma)})_{\Psi_\hbar^{-\infty}(\Gamma)}\\
&=O(\hbar^\infty\|T_b\|_{L^2(\Gamma)\to L^2(\Gamma)})_{\Psi_\hbar^{-\infty}(\Gamma)},
\end{aligned}$$

where the last line follows from the fact that $(1-\chi(|\hbar D'|^2_g))T_b\in\Psi^{-1}_\hbar$, that both $(1-\chi_1(|k^{-1}\partial_t|^2))\in\Psi^0_\hbar$ and $\chi_0(|k^{-1}\partial_t|^2))\in\Psi^0_\hbar(\Gamma)$, and

$$\mathrm{WF}(\chi_0(k^{-2}\partial_t^2))\cap\mathrm{WF}(1-\chi_1(k^{-2}\partial_t^2))=\emptyset.$$

■

The next lemma is used to prove the results for plane-wave data.

Lemma 10.9 *Suppose that there is $\Xi>0$ such that for all $\psi\in C^\infty_c(\mathbb{R})$ with $\operatorname{supp}(1-\psi)\cap[-\Xi,\Xi]=\emptyset$ and all $M>0$, there is $C_M>0$ such that*

$$\|(1-\psi(|\hbar D'|^2_g))v\|_{H^M_\hbar(\Gamma)}\le C_M\hbar^M, \tag{10.42}$$

and for all $\chi\in C^\infty_c(\mathbb{R})$ with $\chi\equiv1$ near $[-\Xi,\Xi]$, $(1-\chi(|\hbar D'|^2_g))\widetilde{L}_b\in\Psi^{-1}_\hbar(\Gamma)$. Then for all $r\ge s\ge0$, $M>0$, and $s\in\mathbb{R}$, there are $C_1,C_2>0$ such that for all $N\ge C_1k$

$$\begin{aligned}
&\|P^C_{\mathscr{T}_N}(L_N-L)P^C_{\mathscr{T}_N}v\|_{H^s_\hbar(\Gamma)}\\
&\le C\Bigg(\Big(\Big(\frac{k}{N}\Big)^{r-s+1}\mathcal{F}^{r,\epsilon}_{\mathscr{L}}(N,C_2k)+\mathcal{F}^{s,\epsilon}_{\mathscr{H}}(N,C_2k)\Big)\big(1+\|\widetilde{L}\|_{s,s}+\|\widetilde{L}\|_{r,r}\big)\\
&\quad+\Big(\Big(\frac{k}{N}\Big)\mathcal{F}^{s,\epsilon}_{\mathscr{L}}(N)+\mathcal{F}^{s,\epsilon}_{\mathscr{H}}(N)\Big)\Big(\Big(\frac{k}{N}\Big)^{r-s+1}\mathcal{F}^{r,\epsilon}_{\mathscr{L}}(N,C_2k)+\mathcal{F}^{s,\epsilon}_{\mathscr{H}}(N,C_2k)\Big)+\Big(\frac{k}{N}\Big)^{r-s+1}\mathcal{B}^{s,r}(\widetilde{L})\\
&\quad+C_M\Big(\Big(\frac{k}{N}\Big)^{r-s+1}\mathcal{F}^{r,\epsilon}_{\mathscr{L}}(N)+\mathcal{F}^{s,\epsilon}_{\mathscr{H}}(N)\Big)\|\widetilde{L}\|_{0,0}k^{-M}\Bigg)\|v\|_{H^r_\hbar(\Gamma)}.
\end{aligned}$$

Proof. First observe that, if $N\ge n$,

$$\begin{aligned}
(L-L_N)P^C_{\mathscr{T}_N}v&=(L-L_N)P^C_{\mathscr{T}_n}v-(L-L_N)(P^C_{\mathscr{T}_n}-P^C_{\mathscr{T}_N})v\\
&=(L-L_N)P^C_{\mathscr{T}_n}v-(L-L_N)P^C_{\mathscr{T}_n}(I-P^C_{\mathscr{T}_N})v.
\end{aligned}\tag{10.43}$$

We now claim that there exists $C_3>0$ such that

$$\|(I-P^C_{\mathscr{T}_{C_3c_{\max}k}})v\|_{H^r_\hbar(\Gamma)}=O(\hbar^\infty). \tag{10.44}$$

Assuming this claim, the result follows from the combination of (10.43), (10.44), and (10.17), by setting n and n' as in (10.44) and then setting $C_1=\max\{C_3,1\}c_{\max}+2\epsilon$.

To see (10.44), recall that from (6.10) and Lemma 7.6 that there is $\chi_1\in C^\infty_c(\mathbb{R})$ with $\chi_1\equiv1$ on $[-C_3,C_3]$ such that

$$\begin{aligned}
(I-P^C_{\mathscr{T}_{C_3c_{\max}k}})&=(I-P^C_{\mathscr{T}_{C_3c_{\max}k}})(I-P^G_{C_3c_{\max}k})\\
&=(I-P^C_{\mathscr{T}_{C_3c_{\max}k}})(I-P^G_{C_3c_{\max}k})(1-\chi_1(|\hbar D'|^2_g))+O(\hbar^\infty)_{\Psi_\hbar^{-\infty}(\Gamma)}.
\end{aligned}\tag{10.45}$$

Let $C_3>\Xi$. Then, for $\psi\in C^\infty_c((-C_3,C_3))$ with $\psi\equiv1$ near $[-\Xi,\Xi]$,

$$(1-\chi_1(|\hbar D'|^2_g))\psi(|\hbar D'|^2_g)=O(\hbar^\infty)_{\Psi_\hbar^{-\infty}(\Gamma)}. \tag{10.46}$$

Combining (10.45) and (10.46), we see that

$$I-P^C_{\mathscr{T}_{C_3c_{\max}k}}=(I-P^C_{\mathscr{T}_{C_3c_{\max}k}})\big(1-\psi(|\hbar D'|^2_g)\big)+O(\hbar^\infty)_{\Psi_\hbar^{-\infty}(\Gamma)},$$

and then (10.44) holds by (10.42). ■

11 Application of the abstract Nyström results to the second-kind Helmholtz BIEs

11.1 Technical bounds on the Kress splittings

The next few lemmas bound $F_{\mathscr{L}}$ and $F_{\mathscr{H}}$ (defined by (9.4)) for the splittings discussed in Lemmas 9.3 to 9.5 for any curve Γ.

Lemma 11.1 *For L_1 and L_2 satisfying (9.6), $0 < \epsilon < 1$, $k_0 > 0$, $s \in \mathbb{R}$, $M > 0$, there is $C > 0$ such that for all $k > k_0$, $N > (1+\epsilon)(1-\epsilon)^{-1}c_{\max}k$*

$$F^{s,\epsilon}_{\mathscr{L}}(N, L) \le C\sqrt{\log k}, \qquad F^{s,\epsilon}_{\mathscr{H}}(N, L) \le Ck^{-M}. \tag{11.1}$$

Furthermore, if $N - n \ge (1+\epsilon)c_{\max}k$, then

$$\max\big\{F^{s,\epsilon}_{\mathscr{L}}(N, n, L), F^{s,\epsilon}_{\mathscr{H}}(N, n, L)\big\} \le Ck^{-M}. \tag{11.2}$$

Proof. By (9.3) and (9.6),

$$D_t^\ell \widehat{L}_{1,m} = \frac{k}{\sqrt{2\pi}}\int_0^{2\pi}\int_{\mathbb{S}^1} D_t^\ell\Big(e^{ik\big(\langle\gamma(t)-\gamma(\tau),\omega\rangle - m\tau/k\big)} f(\omega, t, \tau)\Big) dS(\omega)d\tau.$$

When $m > (1+\epsilon)c_{\max}k$,

$$\big| - \langle\gamma'(\tau), \omega\rangle - m/k\big| > c_\epsilon |m/k|,$$

and thus we can integrate by parts in τ to obtain

$$|D_t^\ell \widehat{L}_{1,m}| \le C_M k^{-M}\langle m/k\rangle^{-M}, \qquad |m| > (1+\epsilon)c_{\max}k. \tag{11.3}$$

Next, by the Cauchy–Schwarz inequality,

$$k^{-1}\sum_{|m|\le(1+\epsilon)c_{\max}k} \|\widehat{L}_{1,m}\|_{H^s_\hbar(\Gamma)}\langle m/k\rangle^s \le C_s k^{-1/2}\sqrt{\sum_{|m|\le(1+\epsilon)c_{\max}k}\|\widehat{L}_{1,m}\|^2_{H^s_\hbar(\Gamma)}}, \tag{11.4}$$

where we have used that

$$\sqrt{\sum_{|m|\le(1+\epsilon)c_{\max}k}\langle m/k\rangle^{2s}} \le C_s k^{1/2}.$$

Let L_1 be the operator with kernel $L_1(t, \tau)$. By (9.3), it is enough to estimate for any $\ell \ge 0$

$$\begin{aligned}\sum_{|m|\le(1+\epsilon)c_{\max}k}\big\|(k^{-1}D_t)^\ell \widehat{L}_{1,m}\big\|^2_{L^2(\Gamma)} &= \sum_{|m|\le(1+\epsilon)c_{\max}k}\big\|(k^{-1}D_t)^\ell L_1(e^{-im\tau})\big\|^2_{L^2(\Gamma)}\\ &\le C\sum_{|m|\le(1+\epsilon)c_{\max}k}\big\|(k^{-1}D_t)^\ell L_1(e^{-im\tau})\big\|^2_{L^2(0,2\pi)}\\ &\le C\big\|(k^{-1}D_t)^\ell L_1\big\|^2_{\mathrm{HS}},\end{aligned} \tag{11.5}$$

where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert–Schmidt norm (see, e.g., [72, Page 32]).

Now, the kernel of $(k^{-1}D_t)^\ell L_1$ is given by

$$K(\tau, s) := k\int_{\mathbb{S}^1} e^{ik\big(\langle\gamma(t)-\gamma(\tau),\omega\rangle\big)} f_\ell(\omega, t, \tau)dS(\omega),$$

for some smooth f_ℓ with all derivatives bounded uniformly in k. Therefore, since the Hilbert–Schmidt norm is equal to the L^2 norm of the kernel (see, e.g., [5, Equation 1.2.33]),

$$\begin{aligned}&\|(k^{-1}D_t)^\ell L_1\|^2_{\mathrm{HS}}\\ &= k^2\int_0^{2\pi}\int_0^{2\pi}\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} e^{ik(\langle\gamma(t)-\gamma(\tau),\omega\rangle - \langle\gamma(t)-\gamma(\tau),\zeta\rangle)}\widetilde{f}_\ell(\omega, t, \tau, \zeta)dS(\omega)dS(\zeta)dtd\tau\end{aligned}$$

$$= k^2 \int_0^{2\pi}\int_0^{2\pi}\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} e^{ik|\gamma(t)-\gamma(\tau)|\left(\left\langle \frac{\gamma(t)-\gamma(\tau)}{|\gamma(t)-\gamma(\tau)|},\omega\right\rangle - \left\langle \frac{\gamma(t)-\gamma(\tau)}{|\gamma(t)-\gamma(\tau)|},\zeta\right\rangle\right)} \widetilde{f}_\ell(\omega,t,\tau,\zeta)dS(\omega)dS(\zeta)dtd\tau$$
$$\le Ck + k^2\int_0^{2\pi}\int_{|t-\tau|\ge Ck^{-1}}\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} e^{ik|\gamma(t)-\gamma(\tau)|\left(\left\langle \frac{\gamma(t)-\gamma(\tau)}{|\gamma(t)-\gamma(\tau)|},\omega\right\rangle - \left\langle \frac{\gamma(t)-\gamma(\tau)}{|\gamma(t)-\gamma(\tau)|},\zeta\right\rangle\right)} \widetilde{f}_\ell(\omega,t,\tau,\zeta)dS(\omega)dS(\zeta)dtd\tau.$$

We now apply stationary phase in ζ and ω. Let

$$\Phi := \langle \gamma(t)-\gamma(\tau),\omega\rangle - \langle\gamma(t)-\gamma(\tau),\zeta\rangle.$$

When $\omega\in\mathbb{S}^1$, $\omega' = \omega^\perp$ and $(\omega^\perp)' = -\omega$ so that

$$\partial_\omega\Phi = \big\langle \gamma(t)-\gamma(\tau),\omega^\perp\big\rangle \quad \text{ and } \quad \partial_\zeta\Phi = -\big\langle\gamma(t)-\gamma(\tau),\zeta^\perp\big\rangle$$

so that there are stationary points when both ζ and ω equal $\pm(\gamma(t)-\gamma(\tau))$. Since

$$\partial^2_{\tau\omega}\Phi = \begin{pmatrix} -\big\langle\gamma(t)-\gamma(\tau),\omega\big\rangle & 0\\ 0 & \big\langle\gamma(t)-\gamma(\tau),\zeta\big\rangle\end{pmatrix}$$

and $|t-\tau|\ge k^{-1}$, these stationary phase are nondegenerate, and the principle of stationary phase (see, e.g., [112, Theorem 3.16], [61, Theorem 7.7.5]) implies that

$$\|(k^{-1}D_t)^\ell L_1\|_{\rm HS} \le Ck + Ck\int_0^{2\pi}\int_{|t-\tau|\ge Ck^{-1}} |\gamma(t)-\gamma(\tau)|^{-1}dtd\tau \le Ck\log k.$$

Combining this with (11.4) and (11.5), we obtain

$$k^{-1}\sum_{|m|\le(1+\epsilon)c_{\max}k}\|\widehat{L}_{1,m}\|_{H^s_\hbar(\Gamma)}\langle m/k\rangle^s \le C_s\sqrt{\log k}.$$

The first estimate in (11.1) then follows from this last inequality combined the definition of $F^{s,\epsilon}_L(N,L)$ (9.4), the choice $N>(1+\epsilon)(1-\epsilon)^{-1}c_{\max}k$, and (11.3).

For the second estimate in (11.1), by (9.3) and (9.6) (including, in particular, the support property of the Fourier transform of $\widetilde{L}$),

$$D_t^\ell\widehat{L}_{2,m} = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi}\int_{|\xi|\le 1} e^{ik(\langle\gamma(t)-\gamma(\tau),\xi\rangle - m\tau/k)}\widehat{\widetilde{L}}_2(\xi,t,\tau)d\xi d\tau.$$

As before, we integrate by parts in τ when $|m|>(1+\epsilon)c_{\max}k$ to obtain that

$$|D_t^\ell\widehat{L}_{2,m}| \le C_M k^{-M}\langle m/k\rangle^{-M}, \qquad |m|>(1+\epsilon)c_{\max}k, \tag{11.6}$$

which, together with (11.3) immediately implies the second estimate in (11.1).

Finally, the result (11.2) follows from (11.3) and (11.6). ∎

Lemma 11.2 *Let L_1 and L_2 satisfy (9.9). Then, for all $M>0$, $s\in\mathbb{R}$, and $0<\epsilon<1$, there is $C>0$ such that*

$$F^{s,\epsilon}_{\mathscr{L}}(N,L)\le C, \qquad F^{s,\epsilon}_{\mathscr{H}}(N,L)\le C\langle N/k\rangle^{-M}k\log k. \tag{11.7}$$

Proof. By the definitions of $F_{\mathscr{L}}$ and $F_{\mathscr{H}}$ (9.4), it is sufficient to prove that

$$\|\widehat{L}_{1,m}\|_{H^s_k(\Gamma)}\le C_{M,s}\langle m/k\rangle^{-M}, \qquad \|\widehat{L}_{2,m}\|_{H^s_k(\Gamma)}\le C_{M,s}\log k\langle m/k\rangle^{-M}.$$

By (9.8) and (9.9),

$$D_t^\ell\widehat{L}_{1,m}(t) = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} e^{-i\tau m}D_t^\ell\big(\widetilde{L}_1\big(k(\gamma(t)-\gamma(\tau)),t,\tau\big)\big)d\tau.$$

By integration by parts in τ and the property (9.9) of $\widetilde{L}_1$,

$$\begin{aligned}|D_t^\ell \widehat{L}_{1,m}(t)| &= \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} e^{-i\tau m} D_t^\ell \frac{(1+k^{-2}mD_\tau)^N}{\langle m/k\rangle^{2N}}\big(\widetilde{L}_1\big(k(\gamma(t)-\gamma(\tau)),t,\tau\big)\big)d\tau\\ &\le C\int_0^{2\pi} k^{\ell+1}\langle m/k\rangle^{-N}\langle k(\gamma(t)-\gamma(\tau))\rangle^{-2}d\tau\\ &\le k^\ell\langle m/k\rangle^{-N}.\end{aligned}$$

The proof of the estimate on $\widehat{L}_{2,m}$ is identical. ∎

Lemma 11.3 *Let L_1 and L_2 satisfy (9.6). If Γ is convex with non-vanishing curvature and unit parametrized, then for all $k_0 > 0$ and $s \in \mathbb{R}$ there is $C > 0$ such that for all $k > k_0$, $0 < \epsilon < 1$ and $N \in \mathbb{R}$,*

$$F^{s,\epsilon}_{\mathscr{L}}(N, L) \le C. \tag{11.8}$$

Proof. We decompose L_1 into two pieces. Let $\chi \in C_c^\infty(\mathbb{R})$ with $\mathrm{supp}(1-\chi)\cap[-1,1] = \emptyset$ and set $\chi_\epsilon(x) := \chi(\epsilon^{-1}x)$. Then

$$(k^{-1}D_t)^\ell L_1(t,\tau) = (k^{-1}D_t)^\ell L_1(t,\tau)\chi_\epsilon(t-\tau)+(k^{-1}D_t)^\ell L_1(t,\tau)\big(1-\chi_\epsilon(t-\tau)\big) =: I_1(t,\tau)+I_2(t,\tau). \tag{11.9}$$

Let $\widehat{I}_{j,m}(t)$ be the Fourier transform in τ of $I_j(t,\tau)$ (compare to (9.3)), so that, by the definition of $F_{\mathscr{L}}$ (9.4), it is sufficient to prove that

$$k^{-1}\sum_m \|\widehat{I}_{1,m}\|_{L^2(\Gamma)}\langle m/k\rangle^s + k^{-1}\sum_m \|\widehat{I}_{2,m}\|_{L^2(\Gamma)}\langle m/k\rangle^s \le C. \tag{11.10}$$

Now, by (9.6),

$$\widehat{I}_{1,m}(t) = k\int\int_{\mathbb{S}^1} e^{ik\big(\langle\gamma(\tau)-\gamma(t),\omega\rangle-\tau m/k\big)} f_\ell(\omega,t,\tau)\chi_\epsilon(t-\tau)dS(\omega)d\tau.$$

For general m, we may perform stationary phase in ω alone to find

$$\widehat{I}_{1,m}(t) = k\int\sum_\pm e^{\pm ik|\gamma(t)-\gamma(\tau)|-i\tau m}\langle k|\gamma(t)-\gamma(\tau)|\rangle^{-1/2} f_{\pm,\ell}(t,\tau)\chi_\epsilon(t-\tau)d\tau,$$

where

$$|f_{\pm,\ell}(t,\tau)| \le C.$$

Integrating in τ and using that $|\dot\gamma| > c > 0$ and $\gamma : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^2$ is a bijection, we then obtain for any m that

$$|\widehat{I}_{1,m}(t)| \le C_\ell k^{1/2}. \tag{11.11}$$

Recalling the goal (11.10), we see it is therefore enough to obtain estimates when $|m-k| \ge M'k^{1/2}$ and $|m+k| \ge M'k^{1/2}$, i.e., when $Mk^{-1/2} < |1-(m/k)^2|$. We consider the three cases

$$(m/k)^2-1 \le -\delta,\quad -\delta < (m/k)^2-1 < -Mk^{-1/2},\quad \text{and}\quad (m/k)^2-1 > Mk^{-1/2} \tag{11.12}$$

separately. Let

$$\Phi := \langle\gamma(\tau)-\gamma(t),\omega\rangle - \tau m/k$$

be the phase function. Then,

$$\partial_\tau\Phi = \langle\gamma'(\tau),\omega\rangle - m/k, \qquad \partial_\omega\Phi = \langle\gamma(\tau)-\gamma(t),\omega^\perp\rangle. \tag{11.13}$$

Since ω is a variable on the unit circle and $|\gamma'(t)| = 1$ for all t, at a critical point, (τ_c, ω_c), where $\partial_\tau\Phi = \partial_\omega\Phi = 0$,

$$\gamma(t)-\gamma(\tau_c) = \pm|\gamma(t)-\gamma(\tau_c)|\omega_c, \qquad \gamma'(\tau_c) = (m/k)\omega_c \pm \sqrt{1-(m/k)^2}\,\omega_c^\perp, \tag{11.14}$$

and

$$\partial^2_{\tau\omega}\Phi = \begin{pmatrix} \langle \gamma''(\tau_c), \omega_c \rangle & \langle \gamma'(\tau_c), \omega_c^\perp \rangle \\ \langle \gamma'(\tau_c), \omega_c^\perp \rangle & -\langle \gamma(\tau_c) - \gamma(t), \omega_c \rangle \end{pmatrix} = \begin{pmatrix} \langle \gamma''(\tau_c), \omega_c \rangle & \pm\sqrt{1-(m/k)^2} \\ \pm\sqrt{1-(m/k)^2} & -\langle \gamma(\tau_c) - \gamma(t), \omega_c \rangle \end{pmatrix}. \tag{11.15}$$

When $1-(m/k)^2 \geq \delta$, the Hessian is non-degenerate, and there exists (τ_c, ω_c) such that $\Phi(\tau_c, \omega_c) = 0$; we then apply the principle of stationary phase in (τ, ω). In this case, for $\epsilon > 0$ chosen small enough depending on $\delta > 0$, the only critical point occurs when $\tau_c = t$ and so

$$\partial^2_{\tau\omega}\Phi = \begin{pmatrix} \langle \gamma''(\tau_c), \omega_c \rangle & \mp\sqrt{1-(m/k)^2} \\ \mp\sqrt{1-(m/k)^2} & 0 \end{pmatrix}.$$

Thus $|\det \partial^2_{\tau\omega}\Phi| > c > 0$ and the principle of stationary phase (see, e.g., [112, Theorem 3.16]) implies that

$$|\widehat{I}_{1,m}(t)| \leq C_{\ell\delta} \qquad \text{for all } 1-(m/k)^2 \geq \delta. \tag{11.16}$$

We next consider $1-(m/k)^2 < -Mk^{-1/2}$. In this case, we integrate by parts with $\frac{k^{-1}\langle D\Phi, D_{\tau,\omega}\rangle}{|D\Phi|^2}$, using that

$$|\partial_\tau \Phi| + |\partial_\omega \Phi| \geq |m/k| - 1 + 1 - |\langle \gamma'(\tau), \omega \rangle| + c|\tau - t|^2,$$

to obtain

$$|\widehat{I}_{1,m}(t)| \leq Ck^{1-N'} \int \Big(|m/k| - 1 + 1 - |\langle \gamma'(\tau), \omega \rangle| + c|\tau - t|^2\Big)^{-2N'} dS(\omega) d\tau \tag{11.17}$$

$$\leq Ck^{1-N'} \int_0^{2\pi} \int_0^{\pi} \Big(|m/k| - 1 + 1 - \sin\theta + c|t-\tau|^2\Big)^{-2N'} d\theta d\tau. \tag{11.18}$$

Now

$$\int_{-\pi/2-\epsilon}^{\pi/2+\epsilon} \big(A + 1 - \sin\theta\big)^{-2N'} d\theta \leq \int_{-\epsilon}^{\epsilon} \big(A + C\phi^2\big)^{-2N'} d\phi = A^{1/2-2N'} \int_{-A^{1/2}\epsilon}^{A^{1/2}\epsilon} \big(1 + Cs^2\big)^{-2N'} d\phi$$
$$\leq CA^{1/2-2N'}$$

and

$$\int_{|\theta-\pi/2|>\epsilon} \big(A + 1 - \sin\theta\big)^{-2N'} d\theta \leq CA^{-2N'},$$

so that

$$|\widehat{I}_{1,m}(t)| \leq Ck^{1-N'} \int_0^{2\pi} \Big(|m/k| - 1 + c|t-\tau|^2\Big)^{1/2-2N'} d\tau \leq Ck^{1+\ell-N'} \big(|m/k| - 1\big)^{1-2N'}. \tag{11.19}$$

Finally, we consider $M/k^{1/2} < 1-(m/k)^2 < \delta$. For this, we again perform stationary phase in (ω, τ). Recall that $\gamma''(\tau_c) = -n(\tau_c)\kappa(\tau_c)$, where n is the outward-pointing unit normal vector to Ω^- and κ is the (signed) curvature. Since n is perpendicular to γ', by both equations in (11.14),

$$-\langle \gamma''(\tau_c), \omega_c \rangle \langle \gamma(\tau_c) - \gamma(t), \omega_c \rangle = |\kappa(\tau_c)| \sqrt{1-(m/k)^2} |\gamma(t) - \gamma(\tau_c)|.$$

Therefore, by (11.15),

$$\det \partial^2_{\tau\omega}\Phi = \sqrt{1-(m/k)^2}\Big(|\kappa(\tau_c)||\gamma(t) - \gamma(\tau_c)| - \sqrt{1-(m/k)^2}\Big). \tag{11.20}$$

We now seek to show that $|\det \partial^2_{\tau\omega}\Phi| > 0$ when $|\kappa(\tau)| \geq c$ for all τ; i.e., when Γ is convex with non-vanishing curvature. By the first equation in (11.14), with coordinates chosen so that $\omega_c = (1, 0)$, for $r = \pm 1$,

$$\pm\big(|\gamma(t) - \gamma(\tau_c)|, 0\big)$$

$$
\begin{aligned}
&= \gamma(t) - \gamma(\tau_c) \\
&= \gamma'(\tau_c)(t-\tau_c) + \frac{1}{2}\gamma''(\tau_c)(t-\tau_c^2) + O((t-\tau_c)^3) \\
&= \big(m/k, r\sqrt{1-(m/k)^2}\big)(t-\tau_c) \pm \frac{1}{2}\kappa(\tau_c)\big(-r\sqrt{1-(m/k)^2}, m/k\big)(t-\tau_c)^2 + O\big((t-\tau_c)^3\big).
\end{aligned}
$$

The two components of this last equation imply that

$$
\begin{aligned}
(t-\tau_c)\Big(r\sqrt{1-(m/k)^2} \pm \frac{1}{2}\kappa(\tau_c)(m/k)(t-\tau_c) + O\big((t-\tau_c)^2\big)\Big) = 0, \quad \text{and} \\
|\gamma(t)-\gamma(\tau_c)| = \Big|m/k \pm \frac{1}{2}\kappa(\tau_c)(-r\sqrt{1-(m/k)^2})(t-\tau_c) + O((t-\tau_c)^2\Big||t-\tau_c|
\end{aligned}
$$

Therefore, either $t = \tau_c$, in which case $|\det \partial^2_{\tau\omega}\Phi| = -\sqrt{1-(m/k)^2}$ by (11.15), or

$$
\frac{1}{2}\kappa(\tau_c)(t-\tau_c) = \mp r(k/m)\sqrt{1-(m/k)^2} + O((t-\tau_c)^2).
$$

In this latter case, since $\kappa(\tau_c) > 0$,

$$
|t-\tau_c| = O(\sqrt{1-(m/k)^2})
$$

and

$$
\frac{|\gamma(t)-\gamma(\tau_c)|}{|t-\tau_c|} = |m/k| + O\big(1-(m/k)^2\big).
$$

Thus,

$$
\begin{aligned}
|\kappa(\tau_c)||\gamma(t)-\gamma(\tau_c)| &= |\kappa(\tau_c)||t-\tau_c|(|m/k| + O\big(1-(m/k)^2)\big)) \\
&= \big|(2k/m\sqrt{1-(m/k)^2} + O\big((1-(m/k)^2)\big)\big|\big(|m/k| + O\big((1-(m/k)^2)\big)\big) \\
&= 2\sqrt{1-(m/k)^2} + O\big(1-(m/k)^2\big),
\end{aligned}
$$

Thus, by (11.20), at either critical point

$$
|\det \partial^2_{\tau\omega}\Phi| \geq c\big(1-(m/k)^2\big).
$$

Performing stationary phase, (using [61, Theorem 7.7.5] with $k = 1$) we then obtain

$$
|\widehat{I}_{1,m}(t)| \leq C_\ell(1-(m/k)^2)^{-1/2} \quad \text{for } M/k^{1/2} < 1-(m/k)^2 < \delta. \tag{11.21}
$$

We now combine (11.11), (11.16), (11.19), and (11.21), to see that the bound on $\widehat{I}_{1,m}$ in (11.10) holds (i.e., the contribution to $F_{\mathscr{L}}$ from I_1 is uniformly bounded). Indeed, splitting the sum into the four regions (three as in (11.12) and the fourth equal to $Mk^{-1/2} < |1-(m/k)^2|$) and inputting (11.11), (11.16), (11.19), and (11.21), we obtain that

$$
\begin{aligned}
\sum_m \|\widehat{I}_{1,m}\|_{L^2(\Gamma)}\langle m/k\rangle^s \leq C &\sum_{Mk^{-1/2}<|1-(m/k)^2|} \langle m/k\rangle^s k^{1/2} + C \sum_{(m/k)^2-1\leq -\delta} \langle m/k\rangle^s \\
&+ C \sum_{(m/k)^2-1>Mk^{-1/2}} \langle m/k\rangle^s k^{1-N'}\big(|m/k|-1\big)^{1-2N'} \\
&+ C \sum_{Mk^{-1/2}<1-(m/k)^2<\delta} \langle m/k\rangle^s \big(1-(m/k)^2\big)^{-1/2} \\
&\leq Ck + C\int_{Mk^{-1/2}<1-(m/k)^2<\delta} \big(1-(m/k)^2\big)^{-1/2} dm \\
&\leq Ck + Ck\int_{Mk^{-1/2}<1-x^2<\delta} \big(1-x^2\big)^{-1/2} dx,
\end{aligned}
$$

so that the bound on $\widehat{I}_{1,m}$ in (11.10) holds.

Next, we consider I_2. For this, we use a partition of unity, $\{\psi_j\}_{j=1}^J$, with $\sup_j \operatorname{diam}(\operatorname{supp}\psi_j) < \frac{\epsilon}{2}$. To write

$$I_2(t,\tau) = \sum_{j,k} \psi_j(t) I_2(t,\tau)\psi_k(\tau).$$

Since $\chi_\epsilon \equiv 1$ on $[-\epsilon,\epsilon]$, we can assume that $\operatorname{supp}\psi_j \cap \operatorname{supp}\psi_k = \emptyset$.

Since Γ is convex with non-vanishing curvature,

$$\frac{\gamma(t)-\gamma(\tau)}{|\gamma(t)-\gamma(\tau)|} \neq \pm\gamma'(t) \qquad \text{for } t \in \operatorname{supp}\psi_j, \tau \in \operatorname{supp}\psi_k; \tag{11.22}$$

i.e., the ray from $\gamma(\tau)$ to $\gamma(t)$ is not tangent to Γ at $\gamma(t)$. We now write

$$\begin{aligned}&\int \psi_j(t) I_2(t,\tau)\psi_k(\tau)\overline{\psi_j(t) I_2(t,s)\psi_k(s)}dt \\ &= k^{2+2\ell}\int\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} e^{ik(\langle\gamma(\tau)-\gamma(t),\omega\rangle - \langle\gamma(s)-\gamma(t),\zeta\rangle)} \\ &\qquad \psi_j^2(t)\psi_k(\tau)\psi_k(s)(1-\chi_\epsilon(t-s))(1-\chi_\epsilon(t-\tau))f_\ell(\omega,t,\tau)\overline{f_\ell(\zeta,t,s)}dS(\omega)dS(\zeta)dt.\end{aligned}$$

Denote the phase function by

$$\Psi := \big\langle \gamma(\tau)-\gamma(t),\omega\big\rangle - \big\langle\gamma(s)-\gamma(t),\zeta\big\rangle.$$

First, by integrating by parts in ζ, we may assume that

$$\big|\big\langle\gamma(t)-\gamma(s),\zeta^\perp\big\rangle\big| \ll 1,$$

at the cost of an $O(k^{-\infty})$ error term. Next, we perform stationary phase in (t,ω);

$$\partial_t\Psi = \langle\gamma'(t),\zeta-\omega\rangle, \qquad \partial_\omega\Psi = \langle\gamma(t)-\gamma(\tau),\omega^\perp\rangle,$$

and

$$\partial^2_{t\omega}\Psi = \begin{pmatrix}\langle\gamma''(t),\zeta-\omega\rangle & \langle\gamma'(t),\omega^\perp\rangle\\ \langle\gamma'(t),\omega^\perp\rangle & \langle\gamma(t)-\gamma(\tau),\omega\rangle\end{pmatrix}.$$

By our assumption on $\operatorname{supp}\psi_j$ and $\operatorname{supp}\psi_k$, we have $\omega_c = \zeta$ and $\gamma(t_c)-\gamma(\tau) = \pm|\gamma(t_c)-\gamma(\tau)|\omega$. Thus, by (11.22), $|\langle\gamma'(t_c),\zeta^\perp\rangle| > c > 0$. In particular, $|\det\partial^2\Psi| > c > 0$ and we can perform stationary phase to obtain

$$\int \psi_j(t)I_2(t,\tau)\psi_k(\tau)\overline{\psi_j(t)I_2(t,s)\psi_k(s)}dt = k^{1+2\ell}\int_{\mathbb{S}^1} e^{ik(\langle\gamma(\tau)-\gamma(s),\zeta\rangle)}\widetilde{f}_\ell(\zeta,s,\tau)\widetilde{\chi}_\epsilon(s-\tau)dS(\zeta),$$

for some $\widetilde{\chi} \in C_c^\infty(\mathbb{R})$ with $\widetilde{\chi} \equiv 1$ in a small neighborhood of 0. In particular, this has exactly the same form as I_1 and hence

$$\|\widehat{I}_{2,m}\|_{L^2} = \sqrt{\Big|\sum_{j,k}\langle\psi_j I_2\psi_k e^{-im\tau},\psi_j I_2\psi_k e^{-im\tau}\rangle\Big|} \leq \sum_{j,k}\sqrt{\|(\psi_j I_2\psi_k)^*\psi_j I_2\psi_k e^{-im\tau}\|_{L^2}}.$$

In particular, by (11.11), (11.16), (11.19), and (11.21),

$$k^{-1}\sum_m \|\widehat{I}_{2,m}\|_{L^2} \leq Ck^\ell.$$

■

Although, at the moment, we are only able to prove Lemma 11.3 in the case of Γ convex with non-vanishing curvature, and in other cases lose a $\sqrt{\log k}$ as in Lemma 11.1; we conjecture that this loss is technical.

Conjecture 11.4 *Let Γ be smooth and parametrized by γ where $0 < |\gamma'| < c_{\max}$. Then for all $k_0 > 0$ and $s \in \mathbb{R}$ there is $C > 0$ such that for all $N > 0$, $k > k_0$, $0 < \epsilon < 1$,*

$$F^{s,\epsilon}_{\mathscr{L}}(N,L) \leq C.$$

If this corollary holds, then the conclusion of Point (ii) of Theorem 2.26 holds with $N \geq C_1 k$; i.e., we need not consider the case of convex obstacles separately in Theorem 2.26.

11.2 Application of Theorem 10.1 to Dirichlet and Neumann BIEs

Proof of Theorem 2.26 Parts (i)-(iii) of Theorem 2.26 follow from Theorem 10.1 combined with

- the fact $\mathcal{B}^{r,s}(\widetilde{L}) = 0 = \|\widetilde{L}\|_{s,r}$ for all s, r (since the sums over j in (10.1) are empty),
- for Part (i), Lemmas 9.3 and 11.3 (applied to $\eta_D S_k$ K_k, and K'_k),
- for Parts (ii) and (iii), Lemmas 9.3 and 11.1.

For the result about plane wave data (2.30), we use, in addition, Lemma 10.9 – together with Lemmas 7.4 and Lemma 7.5 – and the bounds (11.2).

Proof of Theorem 2.27 Theorem 2.27 follow from Theorem 10.1 after using Lemmas 9.3 and 11.1 on K_k and K'_k, Lemmas 9.4 and 11.2 on S_{ik}, and the combination of Lemmas 9.5, 11.1, and 11.2 on $k^{-1}(H_k - H_{ik})$.

As in the Dirichlet case, the result about plane wave data (2.30) uses, in addition, Lemma 10.9 together with Lemmas 7.4 and Lemma 7.5.

12 From quasimodes to pollution for projection methods

In this section, we first prove a version of Theorem 2.10 for an abstract projection method under an appropriate assumption on the approximation space (see Lemma 12.6). We then specialize to the case of piecewise polynomial spaces, proving Theorem 2.10.

12.1 Abstract assumptions on the projection

We start by stating some abstract assumptions on the approximation space and projection. Throughout, we let $\{V_h\}_{h>0} \subset L^2(\Gamma)$ be a family of finite dimensional approximation spaces with $P^G_{V_h} : L^2(\Gamma) \to V_h$ orthogonal projection.

Assumption 12.1 *Let $t_{\max} \geq 0$. For any $k_0 > 0$, there is $C > 0$ such that for all $0 < h < 1$, and $k_0 < k$,*

$$\|I - P^G_{V_h}\|_{H^{t_{\max}}(\Gamma)\to L^2(\Gamma)} \leq C h^{t_{\max}}.$$

It will also be convenient to assume that V_h contains the constants.

Assumption 12.2 *The subspace V_h contains the constants.*

To state our next assumption, let $-\Delta_g$ denote the Laplace–Beltrami operator on Γ and $\{\phi_{\lambda_j}\}_{j=1}^{\infty}$ be an orthonormal basis satisfying

$$(-\Delta_g - \lambda_j^2)\phi_{\lambda_j} = 0,$$

and define the set of functions *oscillating with frequency between ka and kb* as

$$\mathcal{E}_k(a, b) := \operatorname{span}\big\{\phi_{\lambda_j} \,:\, ak < \lambda_j \leq bk\big\}. \tag{12.1}$$

We say that a function $u = u(k)$ is *k-oscillating* if there are $0 < a < b$ and $k_0 > 0$ such that $u(k)$ is oscillating with frequency between ka and kb for all $k > k_0$.

We denote by

$$\Pi_j(a) : L^2(\Gamma) \to \mathcal{E}_k(2^{-j-1}a, 2^{-j}a),$$

the orthogonal projection onto $\mathcal{E}_k(2^{-j-1}a, 2^{-j}a)$.

We then make an assumption on $P^G_{V_h}$ that quantifies the angle between $(V_h)^{\perp}$ and oscillating functions.

Assumption 12.3 *Let $\{V_h\}_{h>0} \subset L^2(\Gamma)$ and $P^G_{V_h} : L^2(\Gamma) \to V_h$ a corresponding family of projections. Let $\Xi_0 > 1$, $k_0 > 0$ and $J = J(k) : [k_0, \infty) \to (0, \infty)$. Then, for $k \geq k_0$ there exists $\mathrm{Tan}(V_h, P^G_{V_h}, k) < \infty$ such that*

$$\sum_{j=0}^{J-1} 2^{jt_{\max}} \Pi_j(\Xi_0)(I - P^G_{V_h}) \sum_{\ell=0}^{J-1} 2^{\ell t_{\max}} \Pi_\ell(\Xi_0) : H_k^{t_{\max}}(\Gamma) \to \mathcal{E}_k(2^{-J}\Xi_0, \Xi_0)$$

is surjective and has a right inverse $\mathcal{R}(k,h) : \mathcal{E}_k(2^{-J}\Xi_0, \Xi_0) \to \mathcal{E}(2^{-J}\Xi_0, \Xi_0)$ with

$$\|\mathcal{R}(k,h)\|_{L^2(\Gamma) \to H_k^{t_{\max}}(\Gamma)} \leq C\,\mathrm{Tan}(V_h, P^G_{V_h}, k).$$

Remark 12.4 *The notation* Tan *comes from the fact that if $J = 1$, then this number estimates the tangent of the angle between $(V_h)^\perp$ and the range of $1_{[2^{-2}\Xi_0^2, \Xi_0^2]}(-k^{-2}\Delta_g)$.*

12.2 Pollution under Assumptions 12.1 and 12.3

We now prove the analogue of Theorem 2.10 under Assumptions 12.1 and 12.3. Before proceeding, we need a technical lemma. For $a > 0$, $J \geq 1$, define

$$\mathcal{W}_J(a) := \sum_{j=0}^{J-1} 2^{jt_{\max}} \Pi_j(a).$$

Lemma 12.5 *Let $m \geq 0$, $a > 0$. Then there is $C > 0$ such that for all $k_0 > 0$, $k > k_0$, defining $J = J(k) := \log_2 k - \log_2 k_0 + 1$ we have*

$$\|\mathcal{W}_J u\|_{H^m} \leq C \max(k^{t_{\max}} k_0^{-t_{\max}} \langle k_0 \rangle^m, \langle k \rangle^m) \Big\| \sum_{j=0}^{J-1} \Pi_j(a) u \Big\|_{L^2}.$$

Proof. Notice that

$$\begin{aligned}
\|\mathcal{W}_J(a)u\|_{H^m}^2 &\leq C\|(-\Delta_g + 1)^{m/2} \mathcal{W}_J u\|_{L^2}^2 = C \sum_{j=0}^{J-1} \|(-\Delta_g + 1)^{m/2} 2^{jt_{\max}} \Pi_j(a) u\|_{L^2}^2 \\
&\leq C \sum_{j=0}^{J-1} (k^2 2^{-2j-2} a^2 + 1)^m 2^{2jt_{\max})} \|\Pi_j(a)u\|_{L^2}^2 \\
&\leq C \sum_{j=0}^{J-1} 2^{2jt_{\max}} \langle 2^{-j} k \rangle^{2m} \|\Pi_j(a)u\|_{L^2}^2 \\
&\leq C \max(k^{2t_{\max}} k_0^{-2t_{\max}} \langle k_0 \rangle^{2m}, \langle k \rangle^{2m}) \sum_{j=0}^{J-1} \|\Pi_j(a)u\|_{L^2}^2 \\
&= C \max(k^{2t_{\max}} k_0^{-2t_{\max}} \langle k_0 \rangle^{2m}, \langle k \rangle^{2m}) \Big\| \sum_{j=0}^{J-1} \Pi_j(a) u \Big\|_{L^2}^2.
\end{aligned}$$

■

Lemma 12.6 *Let $C > 0$, $t_{\max} \geq 0$ $\Xi_0 > 1$, $0 < \epsilon < \frac{1}{10}$ $k_0 > 0$, $0 < \epsilon_0 < \Xi_0$, $\chi \in C_c^\infty((-(1+2\epsilon)^2, (1+2\epsilon)^2))$ with $\chi \equiv 1$ on $[-1-\epsilon, 1+\epsilon]$, $\mathcal{A} = I + L$, and $\mathcal{J} \subset (0, \infty)$ such that Assumptions 1.3 and 4.1 hold. Let V_h, P_h satisfies Assumptions 12.1 and 12.3 with*

$$\mathrm{Tan}(\mathcal{V}_h, P_h, k) \leq C(hk)^{-\gamma}, \qquad \gamma \geq t_{\max}, \qquad 1 \leq J(k) \leq \log_2 k + C.$$

Then there is $c > 0$ such that the following holds. For any $k_n \to \infty$, $k_0 < k_n \notin \mathcal{J}$, $u_n, f_n \in L^2(\Gamma)$, $\beta_n, \alpha_n \in (0, 1]$ satisfying

$$\beta_n \|(I + L(k_n))^{-1}\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq c, \tag{12.2}$$

and

$$(I+L(k_n))u_n = f_n, \qquad \|f_n\|_{L^2(\Gamma)} \le \alpha_n, \qquad \|u_n\|_{L^2(\Gamma)} = 1, \tag{12.3}$$

$$\|(1-\chi(\Xi_0^{-2}k_n^{-2}\Delta_g))f_n\|_{L^2(\Gamma)} \le \beta_n \qquad \|\chi(\epsilon_0^{-2}k_n^{-2}\Delta_g)f_n\|_{L^2(\Gamma)} \le \beta_n, \tag{12.4}$$

$$2^{-J(k_n)t_{\max}}(hk_n)^{2t_{\max}-\gamma}\alpha_n\|1_{(0,2^{-2J+4}\Xi_0^2]}(-k^{-2}\Delta_g)(I+L^*)^{-1}u_n\|_{L^2} \le c, \tag{12.5}$$

then for any $0<h<1$ *and* n *such that* $(I+P^G_{V_h}L(k_n))$ *has an inverse,*

$$\big\|(I+P^G_{V_h}L(k_n))^{-1}(I-P^G_{V_h})\big\|_{L^2(\Gamma)\to L^2(\Gamma)} \ge c\begin{cases}(hk_n)^{-t_{\max}}, & \alpha_n \le (hk_n)^{\gamma}\\ (hk_n)^{\gamma-t_{\max}}\alpha_n^{-1}, & (hk_n)^{\gamma}\le \alpha_n \le c(hk_n)^{\gamma-t_{\max}}.\end{cases}$$

Proof. To ease the notation, we omit the subscript n in the proof. By (12.4), there is $j_0>0$ such that for all $k>k_0$,

$$f-\sum_{j=0}^{j_0}\Pi_j(2\Xi_0)f = O(\beta(\hbar))_{L^2(\Gamma)}. \tag{12.6}$$

By Assumption 12.3

$$g := \mathcal{R}(k,h)\sum_{j=0}^{j_0}2^{jt_{\max}}\Pi_j(2\Xi_0)f$$

satisfies

$$\sum_{j=0}^{J-1}2^{jt_{\max}}\Pi_j(2\Xi_0)(I-P^G_{V_h})\sum_{\ell=0}^{J-1}2^{\ell t_{\max}}\Pi_\ell(2\Xi_0)g = \sum_{j=0}^{j_0}2^{jt_{\max}}\Pi_j(2\Xi_0)f, \tag{12.7}$$

and, using Lemma 12.5,

$$\|\tilde f\|_{H^{t_{\max}}(\Gamma)} \le Ck^{t_{\max}}\operatorname{Tan}(\mathcal{V}_h,P_h,\hbar)\alpha, \quad \text{where} \quad \tilde f := \mathcal{W}_J(2\Xi_0)g = \sum_{\ell=0}^{J-1}2^{\ell t_{\max}}\Pi_j(2\Xi_0)g. \tag{12.8}$$

Applying $\Pi_\ell(2\Xi_0)$ to (12.7), we see that

$$\Pi_\ell(2\Xi_0)(I-P^G_{V_h}))\tilde f = \begin{cases}\Pi_\ell(2\Xi_0)f, & \ell=0,\dots,j_0,\\ 0, & \ell = j_0+1,\dots,J-1,\end{cases}$$

and combining this with (12.6) we find that

$$\sum_{j=0}^{J-1}\Pi_j(2\Xi_0)(I-P^G_{V_h})\tilde f = \sum_{j=0}^{j_0}\Pi_j(2\Xi_0)f = f+O(\beta(\hbar))_{L^2(\Gamma)}. \tag{12.9}$$

In particular, since $-\Delta_g\phi_0 = 0$ implies that ϕ_0 is a constant, Assumption 12.2 implies that $1_{\{0\}}(-k^{-2}\Delta_g)(I-P^G_{V_h})=0$ and then (12.9) implies that

$$(I-P^G_{V_h})\tilde f = f+O(\beta)_{L^2(\Gamma)}+1_{(0,2^{-2J+4}\Xi_0^2]}(-k^{-2}\Delta_g)(I-P^G_{V_h})\tilde f+1_{(4\Xi_0^2,\infty)}(-k^{-2}\Delta_g)(I-P^G_{V_h})\tilde f. \tag{12.10}$$

Define

$$\begin{aligned}v &:= (I+L)^{-1}(I-P^G_{V_h})\tilde f\\ &= u+O(\beta\|(I+L)^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)})_{L^2(\Gamma)}\\ &\quad+(I+L)^{-1}\Big(1_{(0,2^{-2J+4}\Xi_0^2]}(-k^{-2}\Delta_g)(I-P^G_{V_h})\tilde f+1_{(4\Xi_0^2,\infty)}(-k^{-2}\Delta_g)(I-P^G_{V_h})\tilde f\Big).\end{aligned} \tag{12.11}$$

Then, $(I+L)v = (I-P^G_{V_h})\tilde f$, and hence

$$\begin{aligned}(I+P^G_{V_h}L)v &= (I+L)v-(I-P^G_{V_h})Lv = (I-P^G_{V_h})\tilde f-(I-P^G_{V_h})\big[(I-P^G_{V_h})\tilde f-v\big]\\ &= (I-P^G_{V_h})v.\end{aligned}$$

Moreover, by Lemma 4.9, the fact that $P^G_{V_h}$ is self-adjoint, Assumption 12.1, and (12.8),

$$\begin{aligned}\|v\|^2_{L^2(\Gamma)} &= 1 - O(\beta\|(I+L)^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}) - O(\|(I-P^G_{V_h})\tilde{f}\|_{L^2(\Gamma)})\\ &\quad - 2\langle u, (I+L)^{-1}1_{(0,2^{-2J+4}\Xi_0^2]}(-k^{-2}\Delta_g)(I-P^G_{V_h})\tilde{f}\rangle\\ &\quad + \|(I+L)^{-1}1_{(0,2^{-2J+4}\Xi_0^2]}(-k^{-2}\Delta)(I-P^G_{V_h})\tilde{f}\|^2_{L^2(\Gamma)}\\ &\geq 1 - O(\beta\|(I+L)^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}) - C(hk)^{t_{\max}-\gamma}\alpha\\ &\quad - 2\langle (I-P^G_{V_h})1_{(0,2^{-2J+4}\Xi_0^2]}(-k^{-2}\Delta_g)(I+L^*)^{-1}u, (I-P^G_{V_h})\tilde{f}\rangle\\ &\geq 1 - O(\beta\|(I+L)^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}) - C(hk)^{t_{\max}-\gamma}\alpha\\ &\quad - \|(I-P^G_{V_h})1_{(0,2^{-2J+4}\Xi_0^2]}(-k^{-2}\Delta_g)(I+L^*)^{-1}u\|_{L^2(\Gamma)}\|(I-P^G_{V_h})\tilde{f}\|_{L^2(\Gamma)}\\ &\geq 1 - O(\beta\|(I+L)^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}) - C(hk)^{t_{\max}-\gamma}\alpha\\ &\quad - C2^{-Jt_{\max}}(hk)^{2t_{\max}-\gamma}\alpha\|1_{(0,2^{-2J+4}\Xi_0^2]}(-k^{-2}\Delta_g)(I+L^*)^{-1}u\|_{L^2(\Gamma)}.\end{aligned}$$

Therefore, if $\alpha \ll (hk)^{\gamma-t_{\max}}$, then, by (12.2) and (12.5), $\|v\|_{L^2(\Gamma)} \geq c > 0$. Now, by (12.11), Lemma 4.9, and (12.8),

$$\begin{aligned}\|(I-P^G_{V_h})v\|_{L^2(\Gamma)} &\leq \|(I-P^G_{V_h})(1-\chi(-\hbar^2\Delta_g))(I+L)^{-1}(I-P^G_{V_h})\tilde{f}\|_{L^2(\Gamma)} + \|(I-P^G_{V_h})\chi(-\hbar^2\Delta_g)v\|_{L^2(\Gamma)}\\ &\leq C\|(I-P^G_{V_h})\tilde{f}\|_{L^2(\Gamma)} + C(hk)^{t_{\max}}\|v\|_{L^2(\Gamma)}\\ &\leq C(hk)^{t_{\max}}\operatorname{Tan}(V_h,P_h,k)\alpha + C(hk)^{t_{\max}}\|v\|_{L^2(\Gamma)}.\end{aligned}$$

Provided the inverse $(I+P^G_{V_h}L)^{-1}$ exists, we have $(I+P^G_{V_h}L)^{-1}(I-P^G_{V_h})v = v$. Thus, for $\alpha \ll (hk)^{\gamma-t_{\max}}$, we have $\alpha \geq \|(I+L)^{-1}\|^{-1} \geq \beta$, $(hk)^{t_{\max}}\operatorname{Tan}(V_h,P_h,k) \geq 1$, and

$$\begin{aligned}\frac{\|(I+P^G_{V_h}L)^{-1}(I-P^G_{V_h})v\|_{L^2(\Gamma)}}{\|(I-P^G_{V_h})v\|_{L^2(\Gamma)}} &\geq \frac{\|v\|_{L^2(\Gamma)}}{C(hk)^{t_{\max}}\operatorname{Tan}(V_h,P_h,k)\alpha + C(hk)^{t_{\max}}\|v\|_{L^2(\Gamma)}}\\ &\geq \frac{1}{C(hk)^{t_{\max}}\operatorname{Tan}(V_h,P_h,k)\alpha + C(hk)^{t_{\max}}}\\ &\geq \frac{1}{C(hk)^{t_{\max}-\gamma}\alpha + C(hk)^{t_{\max}}}\\ &\geq c\begin{cases}(hk)^{-t_{\max}}, & \alpha \leq (hk)^{\gamma}\\ (hk)^{\gamma-t_{\max}}\alpha^{-1}, & (h\hbar)^{\gamma} \leq \alpha \ll (hk)^{\gamma-t_{\max}}.\end{cases}\end{aligned}$$

■

12.3 Galerkin projection onto piecewise polynomials satisfies Assumption 12.3: proof of Theorem 2.10

First observe that Assumption 2.4 implies Assumption 12.1. Therefore, we only need to check Assumption 12.3.

Lemma 12.7 *Let $p \geq 0$, $a > 0$ $C > 0$. Then there are $k_1 \geq 0$, $c > 0$ such that for any $k_0 > k_1$, defining $J := \log_2 k - \log_2 k_0 + 1$, for all $hk < C$,*

$$c(hk)^{p+1}\Big\|\sum_{j=0}^{J-1}\Pi_j(a)u\Big\|_{L^2} \leq \Big\|(I-P^G_{V_h})\mathcal{W}_J(a)u\Big\|_{L^2}.$$

Moreover, if $p = 0$, then $k_1 = 0$.

Proof. First, observe that

$$\Big\langle \Delta_g^{p+1}\sum_{j=0}^{J-1}2^{j(p+1)}\Pi_j(a)u,\sum_{j=0}^{J-1}2^{j(p+1)}\Pi_j(a)u\Big\rangle = \sum_{j=0}^{J-1}\Big\langle (2^{2j}\Delta_g)^{p+1}\Pi_j(a)u,\Pi_j(a)u\Big\rangle$$
$$\geq 2^{-2(p+1)}a^{2(p+1)}k^{2(p+1)}\sum_{j=0}^{J-1}\|\Pi_j(a)u\|_{L^2}^2 \tag{12.12}$$
$$\geq 2^{-2(p+1)}a^{2(p+1)}k^{2(p+1)}\Big\|\sum_{j=0}^{J-1}\Pi_j(a)u\Big\|_{L^2}^2.$$

Set

$$v := \sum_{j=0}^{J-1}2^{j(p+1)}\Pi_j(a)u. \tag{12.13}$$

Then, using Lemma 12.5 in the arguments in [40, Lemma 3.2], we obtain (for $k_0>0$ large enough when $p\geq 1$ or $k_0>0$ when $p=0$)

$$\sum_{T\in\mathcal{T}_h}\sum_{|\alpha|=p+1}\|\partial_x^\alpha(v\circ\gamma_T)\|^2_{L^2(\Omega_h,dx)} \leq C\epsilon k^{2(p+1)}\Big\|\sum_{j=0}^{J-1}\Pi_j(a)u\Big\|_{L^2}^2+\epsilon^{-1}Ch^{-2(p+1)}\|(I-P^G_{V_h})v\|^2_{L^2}. \tag{12.14}$$

Arguing as in [40, Lemma 3.1], we also obtain that for $k>k_1$

$$ck^{2(p+1)}\Big\|\sum_{j=0}^{J-1}\Pi_j(a)u\Big\|_{L^2}^2 \leq \sum_{T\in\mathcal{T}_h}\sum_{|\alpha|=p+1}\|\partial_x^\alpha(v\circ\gamma_T)\|^2_{L^2(\Omega_h,dx)} \tag{12.15}$$

and $k_1=0$ if $p=0$. (Note that the arguments of [40, Lemmas 3.1 and 3.2] use only estimates on the Sobolev norms of v, and the analogue of (12.12), and hence can be easily adapted to the case when v is given by (12.13).)

Combining (12.14), (12.15), and taking $\epsilon>0$ small enough proves the lemma. ∎

Corollary 12.8 *Let $p\geq 0$, $a>0$, $C>0$. Then there is $k_1\geq 0$ such that given $k_0>k_1$, there is $C_1>0$ such that for all $k>k_0$, $hk\leq C$, setting $J:=\log_2 k-\log_2 k_0+1$, the operator $\mathcal{W}_J(a)(I-P^G_{V_h})\mathcal{W}_J(a):\mathcal{E}_k(2^{-J}a,a)\to\mathcal{E}_k(2^{-J}a,a)$ is invertible and satisfies*

$$\Big\|\big[\mathcal{W}_J(a)(I-P^G_{V_h})\mathcal{W}_J(a)\big]^{-1}\Big\|_{H_k^{-p-1}\to H_k^{p+1}}\leq C_1(hk)^{2(p+1)}.$$

Moreover, if $p=0$, then $k_1=0$.

Proof. Observe that since $P^G_{V_h}$ and $\mathcal{W}_J$ are self-adjoint and $(I-P^G_{V_h})^2=(I-P^G_{V_h})$, by Lemma 12.7

$$c(hk)^{2(p+1)}\Big\|\sum_{j=0}^{J-1}\Pi_j(a)u\Big\|_{L^2}^2\leq\|(I-P^G_{V_h})\mathcal{W}_Ju\|^2_{L^2}=\Big\langle\mathcal{W}_J(I-P^G_{V_h})\mathcal{W}_Ju,\sum_{j=0}^{J-1}\Pi_j(a)u\Big\rangle.$$

Hence,

$$c(hk)^{2(p+1)}\Big\|\sum_{j=0}^{J-1}\Pi_j(a)u\Big\|_{L^2}\leq\|\mathcal{W}_J(I-P^G_{V_h})\mathcal{W}_Ju\|_{L^2},$$

and the lemma follows since $\mathcal{E}_k(2^{-J}a,a)$ is finite dimensional and there is $C>0$ such that

$$C^{-1}\|u\|_{H_k^{p+1}}\leq\|u\|_{L^2}\leq C\|u\|_{H_k^{-p-1}},\qquad\text{for all } u\in\mathcal{E}_k(2^{-J}a,a).$$

∎

We can now prove Theorem 2.10.

Proof of Theorem 2.10. By Corollary 12.8, there is $k_1 \geq 0$ such that for all $k_0 > k_1$, $(V_h, P^G_{V_h})$ satisfies Assumptions 12.3 with $\mathrm{Tan}(V_h, P^G_{V_h}, k) \leq C(hk)^{-2(p+1)}$, and $J = \log_2 k - \log_2 k_0 + 1$. Morevover, if $p = 0$, then $k_1 = 0$. By Assumption 2.4 part (iii), $(P^G_{V_h}, V_h)$ satisfies Assumption 12.1 with $t_{\max} = p + 1$ and, since a constant is a piecewise polynomial, V_h contains the constants. Therefore, the assumptions of Lemma 12.6 hold with $t_{\max} = p + 1$ and $\gamma = 2(p + 1)$. Finally, we need to show that the condition (12.5) (i.e., the second condition in (2.13)) is not required when $p = 0$. To do this, observe that there is $k_* > 0$ such that $1_{(0,k_*^2 k^{-2})}(k^{-2}\Delta_g) = 0$ and hence, choosing $k_0 > 0$ small enough when $p = 0$ completes the proof. ∎

13 Construction of quasimodes implying pollution

In this section, we present the construction of quasimodes that lead to Theorems 2.12 and 2.15. In fact, we prove results that allow for the passage from quasimodes for the scattering problem i.e. outgoing u such that $(-\hbar^2\Delta - 1)u \in L^2_{\mathrm{comp}}(\Omega^+)$ and

$$\|(-\hbar^2\Delta - 1)u\|_{L^2(\Omega^+)} \ll \hbar\|\chi u\|_{L^2(\Omega^+)}, \qquad u|_\Gamma = 0, \qquad u \text{ outgoing},$$

where $\chi \in C^\infty_c$ with $\chi \equiv 1$ in a neighborhood of the convex hull of Ω^-, with a few additional properties to quasimodes usable in Theorem 2.10.

Before proceeding, we recall the following formulas for the inverses of $A_{k,\eta}$, $A'_{k,\eta}$, $B_{k,\mathrm{reg}}$, and $B'_{k,\mathrm{reg}}$ [26, Theorem 2.33] [45, Lemma 7.4]

$$\begin{aligned}
A_k^{-1} &= I - P^{-,\eta_D}_{\mathrm{ItD}}(P^+_{\mathrm{DtN}} - ik^{-1}\eta_D)\\
(A'_k)^{-1} &= I - (P^+_{\mathrm{DtN}} - ik^{-1}\eta_D)P^{-,\eta_D}_{\mathrm{ItD}}\\
B^{-1}_{k,\mathrm{reg}} &= P^+_{\mathrm{NtD}}k^{-1}S^{-1}_{ik} - (I - i\eta_N P^+_{\mathrm{NtD}}k^{-1}S^{-1}_{ik})P^{-,\eta_N,S_{ik}}_{\mathrm{ItD}}\\
(B'_{k,\mathrm{reg}})^{-1} &= S^{-1}_{ik}k^{-1}P^+_{\mathrm{NtD}} - k^{-1}S^{-1}_{ik}P^{-,\eta_N,S_{ik}}_{\mathrm{ItD}}(kS_{ik} - i\eta_N P^+_{\mathrm{NtD}})
\end{aligned} \tag{13.1}$$

Here, we define $P^\pm_{\mathrm{DtN}}f := k^{-1}\partial_\nu u_1$, where

$$(-\Delta - k^2)u_1 = 0 \text{ in } \Omega^+, \quad u_1|_\Gamma = f, \qquad u_1 \text{ is outgoing } (+)/ \text{ incoming } (-),$$

$P^\pm_{\mathrm{NtD}}f := u_2|_\Gamma$, where

$$(-\Delta - k^2)u_2 = 0 \text{ in } \Omega^+, \quad k^{-1}\partial_\nu u_2|_\Gamma = f, \qquad u_2 \text{ is outgoing}(+)/ \text{ incoming } (-),$$

$P^{\pm,\eta}_{\mathrm{ItD}}f := u_3|_\Gamma$, where

$$(-\Delta - k^2)u_3 = 0 \text{ in } \Omega^-, \quad (k^{-1}\partial_\nu u_3 \pm ik^{-1}\eta u_3)|_\Gamma = f,$$

and $P^{\pm,\eta,S_{ik}}_{\mathrm{ItD}}f := u_4|_\Gamma$, where

$$(-\Delta - k^2)u_4 = 0 \text{ in } \Omega^-, \quad (S_{ik}\partial_\nu u_4 \pm i\eta u_4)|_\Gamma = f.$$

13.1 Dynamical Preliminaries

Let $\Omega \subset \mathbb{R}^d$ be open with smooth, compact, boundary, Γ. In order to define the relevant dynamics, we recall the notion of the b-contangent bundle of Ω. We have $T^*\Omega \rightharpoonup {}^bT^*\Omega$ via the canonical projection map $\pi_b : T^*\Omega \to {}^bT^*\Omega$ given, in local coordinates (x_1, x') where $\partial\Omega = \{x_1 = 0\}$ and $\Omega = \{x_1 > 0\}$ by

$$\pi_b(x_1, x', \xi_1, \xi') = (x_1, x', x_1\xi_1, \xi').$$

Let g be a metric on Ω and assume throughout this section that for every point $\mathcal{Z} := \rho \in \pi_b(S^*M)$, there is exactly one GBB through ρ. This, for instance, is the case of $\partial\Omega$ is nowhere tangent to the geodesics in Ω to infinite order or $\partial\Omega$ has negative semidefinite second fundamental form with respect to the inward normal. We denote the generalized broken bicharacteristic flow on $\mathcal{Z}$, by

$$\varphi^\Omega_t : \mathcal{Z} \to \mathcal{Z}.$$

It will sometimes be convenient below to denote $x(\varphi_t^\Omega(\rho)) := \pi_{\mathbb{R}^d}(\varphi_t^\Omega(\rho))$, where $\pi_{\mathbb{R}^d} : \mathcal{Z} \to \mathbb{R}^d$ denote projection to the base.

Let $f : \mathcal{Z} \to \mathbb{R}$ be a boundary defining function for Ω and g_Γ denote the metric induced on $T^*\Gamma$ by g.. We write

$$\mathcal{H} := \{(x', \xi') \in T^*\Gamma \,:\, |\xi'|_{g_\Gamma} < 1\}, \quad \mathcal{E} := \{(x', \xi') \in T^*\Gamma \,:\, |\xi'|_{g_\Gamma} > 1\}$$
$$\mathcal{G} := \{(x', \xi') \in T^*\Gamma \,:\, |\xi'|_{g_\Gamma} = 1\}, \quad \mathcal{G}_d := \{(x', \xi') \in \mathcal{G} \,:\, H^2_{|\xi|^2_g} f > 0\}, \qquad \mathcal{G}_g := \mathcal{G} \setminus \mathcal{G}_d.$$

We then define the incoming (Γ_-) and outgoing (Γ_+) sets,

$$\Gamma_\pm := \{\rho \in \mathcal{Z} \,:\, \limsup_{t\to\infty} |x(\varphi_{\pm t}(\rho))| < \infty\},$$

and the trapped set

$$K := \Gamma_+ \cap \Gamma_-.$$

We say (Ω, g) is trapping if $K \neq \emptyset$. We denote by $\mathrm{ch}(U)$, the convex hull of a set U.

Basic properties of the trapped set

Lemma 13.1 *Suppose that $\Omega^- \Subset \mathbb{R}^d$ has connected complement and $\Omega = \Omega^+ := \mathbb{R}^d \setminus \overline{\Omega}$ and $\mathrm{supp}(g - I)$ is compact. The trapped set is closed and satisfies*

$$K \subset \mathrm{ch}(\overline{\Omega^-} \cup \mathrm{supp}(g - I)).$$

Proof. We start by proving the inclusion of K. Suppose that $\rho \in \mathcal{Z}$ with $x(\rho) \notin \mathrm{ch}(\overline{\Omega^-} \cup \mathrm{supp}(g-I))$. Then, for

$$|t| \leq d\big(x(\rho), \mathrm{ch}(\overline{\Omega^-} \cup \mathrm{supp}(g - I)\big)$$

we have

$$x(\varphi_t^{\Omega^+}) = x(\rho) \pm t\xi(\rho).$$

Now, by convexity, for one choice of $\pm$,

$$\{x(\rho) \pm [0, \infty)\xi(\rho)\} \cap \mathrm{ch}(\overline{\Omega^-} \cup \mathrm{supp}(g - I)) = \emptyset$$

In particular, ρ not trapped either forward or backward in time.

Since K is invariant under $\varphi_t^{\Omega^+}$ and $K \subset \mathrm{ch}(\overline{\Omega^-} \cup \mathrm{supp}(g - I))$, there is $C > 0$ such that for any $\{\rho_n\}_{n=1}^\infty \in K$ with $\rho_n \to \rho$, and any $t \in \mathbb{R}$, $|x(\varphi_t^{\Omega^+}(\rho_n))| \leq C$. In particular, since $\varphi_t^{\Omega^+}$ is continuous, $|x(\varphi_t^{\Omega^+}(\rho))| \leq C$. Since t is arbitrary this implies $\rho \in K$ and hence K is closed. ∎

Lemma 13.2 *Suppose that $\Omega^- \Subset \mathbb{R}^d$ has connected complement and $\Omega = \Omega^+ := \mathbb{R}^d \setminus \overline{\Omega}$ and $\mathrm{supp}(g - I)$ is compact and Ω^+ is trapping. Then $\Gamma_- \setminus K \neq \emptyset$.*

Proof. Since $|\xi|_g^2$ is homogeneous degree 2, Γ_- is nonempty if and only if Γ_+ is nonempty. Therefore, it is enough to show that $\Gamma_+ \cup \Gamma_- \neq \emptyset$.

Let $\{q_n\}_{n=1}^\infty \subset \mathcal{Z} \setminus K$ with $d(q_n, K) \to 0$. Such a sequence exists since K is closed and K is not equal to $\mathcal{Z}$. Let

$$C_1 > \mathrm{diam}(\mathrm{ch}(\overline{\Omega^-} \cup \mathrm{supp}(g - I))).$$

Without loss of generality, we may assume $|x(q_n)| \leq C_1$. Then, since the $\varphi_t^{\Omega^+}$ is continuous and for all $q' \in K$, $t \in \mathbb{R}$, $|x(\varphi_t^{\Omega^+}(q'))| \leq C_1$, for any $T > 0$, there is n large enough such that $|x(\varphi_t^+(q_n))| \leq 2C_1$ for $|t| \leq T$. Extracting subsequences, we may assume that

$$|x(\varphi_t^{\Omega^+}(q_n))| \leq 2C_1, \quad 0 \leq t \leq n.$$

Now, since $q_n \notin K$, there is $s_n > n$ and $\mu \in \{1, -1\}$ such that $2C_1 < |x(\varphi^+_{\mu s_n}(q_n))| < 3C_1$. Then, since $|x(q_n)| \leq C_1$ we have $x(\varphi^+_{\mu t}(q_n)) \leq 3C_1$ for $0 \leq t \leq s_n$. Taking a subsequences, we may assume that $\varphi^+_{s_n}(q_n) \to q$ with

$$2C_1 \leq |x(q)| \leq 3C_1$$

Now, fix $T > 0$ then,

$$d(\varphi^{\Omega^+}_{-\mu T}(q), \varphi^{\Omega^+}_{\mu(s_n-T)}(q_n)) \to 0,$$

and for $n > T$

$$|x(\varphi^{\Omega^+}_{\mu(s_n-T)}(q_n)))| \le 3C_1$$

In particular,

$$|x(\varphi^{\Omega^+}_{-\mu T}(q))| \le 3C_1$$

for all $T \ge 0$. In particular, $q \in (\Gamma_- \cup \Gamma_+) \setminus K$.

∎

Lemma 13.3 *Suppose that* $\Omega^- \Subset \mathbb{R}^d$ *has connected complement and smooth boundary,* $\Omega = \Omega^+ := \mathbb{R}^d \setminus \overline{\Omega}$, $\operatorname{supp}(g - I)$ *is compact, and* $q \in (\Gamma_- \setminus K) \cap T^*\Gamma$. *Then* $\xi'(q) \neq 0$.

Proof. Suppose that $q \in (\Gamma_- \setminus K) \cap T^*\Gamma$ and $\xi'(q) = 0$. Then $x(\varphi^{\Omega^+}_t(q)) = x(\varphi^+_{-t}(q))$, and hence, since $q \in \Gamma_- \setminus K$, $|x(\varphi^+_{-t}(q))| \to \infty$. Therefore, $|x(\varphi^{\Omega^+}_t(q))| \to \infty$. In particular $q \notin \Gamma_-$. ∎

Distinguished neighborhoods of a GBB segment and perturbations near the boundary Our construction of approximate solutions of Helmholtz problems for given data with precise knowledge of the approximate solution's wavefront set uses perturbations of the Hamiltonian $p = |\xi|^2 - 1$ near the boundary and neighborhoods U of GBB segments such that i) the operator is unchanged in U and ii) every trajectory either escapes backward in time, enters the interior of the domain outside of U, or hits a distinguished subset of $T^*\Gamma$.

For our next lemma, we need some additional notation. First, for $U, V \subset \mathcal{H}$, $\mu \in \{1, -1\}$, and $q \in \mathcal{Z}$ define

$$T^\mu_U(q) := \inf\{t > 0 : \varphi^\Omega_{\mu t}(q) \in U\}, \qquad T^\mu_{U<V}(q) := \inf\{0 < t < T^\mu_V(q) \,:\, \varphi^\Omega_{\mu t}(q) \in U\}.$$

Next, we write $a \in S^{\mathrm{comp}}_{\mathsf{T}}(\Omega)$ if in a Fermi coordinate (x_1, x', ξ_1, ξ') neighborhood of the boundary,

$$a = \tilde{a}(x_1, x', \xi'), \qquad \tilde{a} \in C^\infty_c(\mathbb{R} \times T^*\Gamma).$$

Lemma 13.4 *Let* $\mu \in \{1, -1\}$, $\mathcal{O} \subset \mathcal{H}$ *open and* $\mathcal{B} \subset \mathcal{H}$ *closed. Suppose that* $q \in \mathcal{Z}$ *and* $0 < T_q < T_{\mathcal{B}\cup\{q\}}$ *such that* $\varphi_{T_q}(q) \in \mathcal{O}$. *Then there is* $\delta_0 > 0$, *a closed neighborhood,* $V \subset T^*\Gamma \setminus (S^*\Gamma \cup \mathcal{B} \cup \{q\})$ *of* $\varphi^\Omega_{T_q}(q)$ *and for all* $0 < \delta < \delta_0$ *a closed neighborhood* $U \subset \mathcal{Z}$ *of* q, *and a continuous function* $T : U \to [T_q - \delta, T_q + \delta]$ *such that*

$$\varphi^\Omega_{T(q')}(q') \in V, \quad \inf\{d(\varphi^{\Omega^+}_t(q'), U \cup \mathcal{B}) : \delta < t < T(q'), \varphi_t(q') \in T^*\Gamma, q' \in U\} > 0.$$

Moreover, for any $R > 0$ *and any open* $V_- \subset V^o$ *with* $q \in V_-$, *there are* $T > 0$ *and* $a \in S^{\mathrm{comp}}_{\mathsf{T}}(\Omega)$, $0 \le a \le \frac{1}{2}|\xi'|^2_{g_\Gamma}$, $a|_\Gamma = 0$ *with*

$$\operatorname{supp} a \cap \Sigma_q = \emptyset, \qquad \Sigma_q := \bigcup_{0 \le t \le T_q} \varphi^{\Omega^+}_{\mu t}(q)$$

such that, putting $\mathcal{Z}_a := {}^b\pi(\{|\xi|^2_g - a - 1 = 0\})$, *for any GBB,* γ, *for* $|\xi|^2_g - a - 1$, *there is* $0 \le t \le T$ *such that*

$$\gamma(-\mu t) \in V_- \cup \Big(\{|x| > R\} \cup \{d(x, \Gamma) > 0\}\Big) \setminus \Sigma_q. \tag{13.2}$$

Proof. The existence of V and U as described follows from the facts that 1) φ^Ω_t is continuous, 2) since $\mathcal{O} \subset \mathcal{H}$, $\partial_t \varphi^\Omega_t|_{T_q - 0}$ is transverse to Γ, and 3) $T_q < T_{\mathcal{B}\cup\{q\}}$ and $\mathcal{B} \cup \{q\}$ is closed.

For use below, let $t_0, c_0 > 0$ small enough that

$$d(\varphi^\Omega_{-\mu t_0}(q), q) > c_0.$$

To construct the desired a, we work in Fermi normal coordinates $[0, \epsilon)_{x_1} \times \Gamma_{x'}$ near Γ. In these coordinates,

$$|\xi|^2_g - 1 = \xi_1^2 - r(x, \xi'), \qquad r(0, x', \xi') = 1 - |\xi'|^2_{g_\Gamma}.$$

Let $\chi \in C_c^\infty((-\frac{1}{2},\frac{1}{2});[0,1])$ with $0 \notin \operatorname{supp}(1-\chi)$. Let also $\psi_\epsilon \in C_c^\infty(T^*\Gamma;[0,1])$ with $\operatorname{supp}(1-\psi_\epsilon)\cap\Sigma_q=\emptyset$, $\operatorname{supp}\psi_\epsilon \subset \{d((x',\xi'),\Sigma_q)<\epsilon\}$. Then, for some $0<\epsilon<c_0/2$ to be determined, define

$$a := \epsilon^{-1}x_1|\xi'|^2_{g_\Gamma}\chi(\epsilon^{-1}x_1)\chi(\epsilon^{-1}(r(0,x',\xi')))(1-\psi_\epsilon(x',\xi')).$$

Without loss of generality, we may assume that $\Gamma \subset \{|x| \le R\}$. Therefore, to prove (13.2), suppose that there is no such T. Then for all n there is γ_n such that

$$\gamma_n(-\mu[0,n]) \subset [V_-]^c \cap \Big(T^*\Gamma \cup \Sigma_q\Big). \tag{13.3}$$

Suppose first that $\gamma_n(0) \in \Sigma_q \cap [V^o]^c$. Then, since $\operatorname{supp} a \cap \Sigma_q = \emptyset$, there is $T_q+\delta > S > \inf\{0 \le t \,:\, \varphi^\Omega_{-\mu t}(\gamma_n(0)) \notin \Sigma_q\} =: S_0$ such that $\gamma_n(-\mu t) = \varphi^\Omega_{-\mu t}(\gamma_n(0))$ for $0 \le t \le S$. In particular, this implies that

$$\varphi^\Omega_{-\mu t}(\gamma_n(0)) \notin \Sigma_q, \qquad S_0 < t < S.$$

Since $\varphi^\Omega_{T_q}(q) \in \mathcal{H}$,

$$\inf\{t > S_0 \,:\, \gamma_n(-\mu t) \in \Sigma_q\} < \inf\{t > S_0 \,:\, \gamma_n(-\mu t) \notin T^*\Gamma\}.$$

In particular, for $\epsilon > 0$ small enough (depending only on $\|\partial_{x_1} r|_{x_1=0=r=0}\|_{L^\infty}$), $a|_{x_1=0}=0$, this implies

$$\cup_{S_0<t<n}\gamma_n(-\mu t) \subset \{r(0,x',\xi') = \partial_{x_n} r + \partial_{x_n} a = 0\} \subset \{d((x',\xi'),\Sigma_q)<\epsilon\},$$

and $\gamma_n(-\mu t) = \varphi^\Omega_{-\mu t}(\gamma_n(0))$ for $0 \le t \le n$. Now, let $t = S_0 + t_0$. Then,

$$c_0 < d(\varphi^\Omega_{-\mu t_0}(q),\Sigma_q) = d(\varphi^\Omega_{-\mu t}(\gamma_n(0)),\Sigma_q) < \epsilon < c_0/2,$$

which is a contradiction for n large enough.

Next, suppose that $\gamma_n(0) \in (T^*\Gamma \cap [V_-]^c)\setminus \Sigma_q$ and there is $0 \le t \le n$ such that $\gamma_n(-\mu t) \in \Sigma_q$. Then, since $V \subset \mathcal{H}$, $q \notin V$, and $\Sigma_q \cap \operatorname{supp} a = \emptyset$, there is $0<s<t$ such that $\gamma_n(-\mu s) \notin \Sigma_q \cup T^*\Gamma$, which contradicts (13.3).

Finally, we consider the case $\gamma_n(-\mu t) \subset (T^*\Gamma \cap [V_-]^c \cap \Sigma_q^c)$, $0 \le t \le n$. For $\epsilon > 0$ small enough (depending only on $\|\partial_{x_1} r|_{x_1=0=r=0}\|_{L^\infty}$), this implies that $\gamma_n(-\mu t) \in \{r(0,x',\xi') = \partial_{x_n} r + \partial_{x_n} a = 0\} \subset \{d((x',\xi'),\Sigma_q)<\epsilon\}$ and, since $a|_{x_1=0}=0$, $\gamma_n(-\mu t) = \varphi^\Omega_{-\mu t}(\gamma_n(0))$. The continuity of φ^Ω_t and the fact that Σ_q is a flow-line of length T_q implies that for any $\beta>0$, there are $\epsilon>0$ small enough (depending only on the continuity properties of φ^Ω_t) and $0<t<T_q+\beta$ such that $d(\varphi^\Omega_{-\mu t}(\gamma_n(0)),q) < \beta$ and hence, choosing $\beta>0$ small enough (again depending only on the continuity properties of φ^Ω_t),

$$d(\varphi^\Omega_{-\mu(t+t_0)}(\gamma_n(0)),\varphi^\Omega_{-\mu t_0}(q)) < c_0/2,$$

which implies

$$d(\varphi^\Omega_{-\mu(t+t_0)}(\gamma_n(0)),\Sigma_q) > c_0/2 > \epsilon,$$

a contradiction. Thus, we have proved (13.2). ∎

Lemma 13.5 *Let $\Omega^- \Subset \mathbb{R}^d$ open with smooth boundary and connected complement $\Omega := \mathbb{R}^d \setminus \overline{\Omega^-}$ and suppose $q \in \mathcal{Z}\setminus\Gamma_-$. Then for any $R>0$, there are $T>0$ and $a \in S^{\mathrm{comp}}_{\mathsf{T}}(\Omega)$ with $a \ge 0$, $a|_\Gamma = 0$ such that*

$$\gamma_q \cap \operatorname{supp} a = \emptyset, \qquad \gamma_q := \bigcup_{t\ge 0}\varphi^\Omega_t(q),$$

and, putting $\mathcal{Z}_a := {}^b\pi(\{|\xi|^2_g - a - 1 = 0\})$, for any GBB, γ, for $|\xi|^2_g - a - 1$, there is $0 \le t \le T$ such that

$$\gamma(-t) \in \Big(\{|x|>R\} \cup \{d(x,\Gamma)>0\}\Big)\setminus\gamma_q. \tag{13.4}$$

Proof. The proof follows the same lines as that of Lemma 13.4 but is simpler because we need not consider any special subsets of $T^*\Gamma$. ∎

13.2 Semiclassical b and tangential-pseudodifferential operators

b-pseudodifferential operators The class of b-pseudodifferential operators that we work with is the natural class of operators quantizing differential operators that are tangential to the boundary of Ω_+. Away from $\partial\Omega_+$ they are pseudodifferential operators in the sense of §3, but near $\partial\Omega_+$ they have a different form. In particular, in coordinates (x_1, x') with $\partial\Omega_+ = \{x_1 = 0\}$, their symbols are functions on the b-cotangent bundle, ${}^bT^*\Omega_+$, whose sections are of the form

$$\sigma\frac{dx_1}{x_1} + \xi' dx'.$$

Notice that ${}^bT^*\Omega_+$ is the dual to sections of $T^*\Omega_+$ that are tangent to $\partial\Omega_+$. We also write ${}^b\overline{T^*\Omega_+}$ for the fiber radially compactified b-contangent bundle; i.e., ${}^bT^*\Omega_+$ with the sphere at infinity in (σ, ξ') attached.

In coordinates, b-pseudodifferential operators are of the form

$$\mathrm{Op}_b(a)(u)(x) = \frac{1}{(2\pi\hbar)^d}\int e^{\frac{i}{\hbar}((x_1-y_1)\xi_1+(x'-y'),\xi')}\phi(x_1/y_1)a(x_1, x', x_1\xi_1, \xi')u(y)dyd\xi,$$

where $\phi \in C_c^\infty(1/2, 2)$ with $\phi \equiv 1$ near 1 and for some m

$$|D_x^\alpha D_\sigma^j D_{\xi'}^\beta a(x_1, x', \sigma, \xi')| \le C_{j\alpha\beta}\langle(\sigma, \xi')\rangle^{m-j-|\beta|}.$$

In this case, we write $\mathrm{Op}_b(a) \in \Psi_b^m(\Omega_+)$ and $a \in S^m({}^bT^*\Omega_+)$. When $m = 0$ we write $S({}^bT^*\Omega_+)$ and $\Psi_b(\Omega_+)$ respectively. We also write $\Psi_b^{-\infty} = \cap_m \Psi_b^m$.

The class comes equipped with principal symbol map ${}^b\sigma : {}^b\Psi^m(\Omega_+) \to S^m({}^bT^*\Omega_+)/h^bS^{m-1}(T^*\Omega_+)$ such that if $A \in \Psi_b(\Omega_+)$ and $\sigma(A) = 0$ then $A \in h\Psi_b^{m-1}(\Omega_+)$. We now introduce two important sets for b-pseudodifferential operators. For $A \in \Psi_b^m(\Omega_+)$ and $q \in {}^b\overline{T^*\Omega_+}$, we say $q \in {}^b\operatorname{Ell}(A)$ if there is a neighbourhood, U of q such that

$$|\sigma(A)(q')|\langle(\sigma, \xi')\rangle^{-m} > c > 0, \qquad q' \in U \cap {}^bT^*\Omega_+.$$

Next, we say $q \notin {}^b\mathrm{WF}(A)$ if there is $E \in {}^b\Psi(\Omega_+)$ with $q \in {}^b\operatorname{Ell}(E)$ such that

$$EA \in \hbar^\infty\Psi_b^{-\infty}.$$

For a more complete treatment of these operators, we refer the reader to [59, Appendix A] and the references therein.

Tangential pseudodifferential operators In addition to the class of b-pseudodifferential operators, it is useful to have a subclass consisting of tangential pseudodifferential operators. For this, let g be a Riemannian metric on Ω and $(x_1, x') \in \mathbb{R} \times \Gamma$ be Fermi normal coordinates near Γ. That is,

$$\Omega = \{x_1 > 0\},\ x_1 = d(x, \partial\Omega).$$

We then use the canonical coordinates (x_1, x', ξ_1, ξ') on $T^*\Omega$ and define

$$S_{\mathsf{T}}^m := \{a(x, \xi') \,:\, |\partial_x^\alpha \partial_{\xi'}^\beta a(x, \xi')| \le C_{\alpha\beta}\langle\xi'\rangle^{m-|\alpha|},$$

with $S_{\mathsf{T}}^{\mathrm{comp}}$, S_{T}^∞, and $S_{\mathsf{T}}^{-\infty}$ defined as for the usual symbol classes. We then set

$$\Psi_{\mathsf{T}}^m(\Omega) := \{\mathrm{Op}(a) \,:\, a \in S_{\mathsf{T}}^m(\Omega)\},$$

again, with $\Psi_{\mathsf{T}}^{\mathrm{comp}}$, Ψ_{T}^∞, and $\Psi_{\mathsf{T}}^{-\infty}$.

Wavefront sets We have already defined wavefront sets of pseudodifferential operators, but it is useful, in addition, to have the notion of a wavefront set of certain classes of distributions.

Definition 13.6 *Let $s \in \mathbb{R}$ and M be a smooth manifold (possibly with boundary). We say that $u \in H_{\hbar,\mathrm{loc}}^s(M)$ is $\hbar$-tempered if for all $\chi \in C_c^\infty(\overline{M})$, there are $N > 0$ and $C > 0$ such that*

$$\|\chi u\|_{H_\hbar^s} \le Ch^{-N}.$$

We can now define the wavefront set and b-wavefront set of a distribution.

Definition 13.7 *Let M be a smooth manifold without boundary and $s,t\in\mathbb{R}$ and suppose that $u\in H^s_{\hbar,\mathrm{loc}}(M)$ is $\hbar$-tempered. Then, we say that $(x_0,\xi_0)\in\overline{T^*M}$ is not in $\mathrm{WF}^t_{H^s_\hbar}(u)$ if there are $C>0$ and $A\in\Psi^0(M)$ with A elliptic at (x_0,ξ_0) such that*

$$\|Au\|_{H^s_\hbar}\leq C\hbar^t.$$

We write $\mathrm{WF}_{H^s_\hbar}(u)=\cup_t\mathrm{WF}^t_{H^s_\hbar}(u)$, $\mathrm{WF}^t(u)=\cup_s\mathrm{WF}^t_{H^s_\hbar}(u)$, and $\mathrm{WF}(u)=\cup_s\mathrm{WF}_{H^s_\hbar}(u)$.

Definition 13.8 *Let Ω be a smooth manifold with boundary and $s,t\in\mathbb{R}$ and suppose that $u\in H^s_{\hbar,\mathrm{loc}}(M)$ is $\hbar$-tempered. Then, we say that $(x_0,\xi_0)\in\overline{{}^bT^*M}$ is not in ${}^b\mathrm{WF}^t_{H^s_\hbar}(u)$ if there are $C>0$ and $A\in{}^b\Psi^0(M)$ with A elliptic at (x_0,ξ_0) such that*

$$\|Au\|_{H^s_\hbar}\leq C\hbar^t.$$

We write ${}^b\mathrm{WF}_{H^s_\hbar}(u)=\cup_t{}^b\mathrm{WF}^t_{H^s_\hbar}(u)$. Since $H^1_\hbar$ will be our default space on a manifold with boundary, we sometimes abuse notation and write ${}^b\mathrm{WF}(u)$ for ${}^b\mathrm{WF}_{H^1_\hbar}(u)$.

We record the following lemma which is a consequence of [62, Theorem 18.3.32] and allows us to use tangential pseudodifferential operators as test operators rather than the full b calculus.

Lemma 13.9 *Let $t,s\in\mathbb{R}$ and suppose that $u\in H^s_{\hbar,\mathrm{loc}}$ is h tempered and $b\in S^0_{\mathrm{T}}(M)$. Then,*

$${}^b\mathrm{WF}^t_{H^s_\hbar}(u)|_{\partial M}\subset{}^b\mathrm{WF}^t_{H^s_\hbar}(\mathrm{Op}(b)u)\cup\{(x',\xi')\in\overline{T^*\Gamma}\,:\,\sigma(b)(0,x',\xi')=0\}.$$

Defect measures In the most general settings below, we prove qualitative pollution bounds; i.e. we show that there is pollution, but do not determine the rate. For this, it is convenient to use the notion of defect measures.

Definition 13.10 *Suppose that $\hbar_n\to0^+$ and $\{u_{\hbar_n}\}_{n=1}^\infty$ satisfies $\sup_n\|\chi u_{\hbar_n}\|_{L^2(\Omega^+)}<\infty$ for all $\chi\in C^\infty_c(\overline{\Omega}_+)$. Then there is a subsequence $h_{n_k}\to0$ and a positive radon measure μ on $T^*\mathbb{R}^d$ such that for any $a\in\Psi^{\mathrm{comp}}(\mathbb{R}^d)$ we have*

$$\langle\mathrm{Op}(a)u_{h_{n_k}}1_{\Omega^+},u_{h_{n_k}}1_{\Omega^+}\rangle_{L^2(\mathbb{R}^d)}\to\int\sigma(a)d\mu.$$

We record the following fact about defect measures for Helmholtz solutions. For its proof, we refer the reader to e.g. [44, Lemma 2.12], [50, Lemma 4.2,4.8], and [22].

Lemma 13.11 *Suppose that $\chi\in C^\infty_c(\overline{\Omega}_+)$, $f\in L^2(\Omega^+)$, with $\|f\|_{L^2}\leq C$,*

$$(-\hbar^2\Delta-1)u=\hbar\chi f\ \text{ in }\Omega^+,\qquad u|_\Gamma=0,\qquad u\text{ is outgoing},$$

and u has defect measure μ. Then, for any $\psi\in C^\infty_c(\overline{\Omega})$, $\mu(\psi^2)=\lim_{h\to0^+}\|\psi u\|_{L^2}$, $\operatorname{supp}{}^b\pi_*\mu\subset\mathcal{Z}$, and ${}^b\pi_*\mu$ is invariant under $\varphi_t^{\Omega^+}$ away from ${}^b\mathrm{WF}(f)$ and $\mathcal{G}^\infty$ and, for R large enough,

$$\mu(\mathcal{I}_-(R)\setminus\bigcup_{t>0}\varphi_t^{\Omega^+}({}^b\mathrm{WF}(f))=0,$$

where

$$\mathcal{I}_-(R):=\{(x,\xi)\in T^*\Omega^+\,:\,|x|>R,\langle\tfrac{x}{|x|},\xi\rangle\leq0\}.$$

13.3 Propagation of singularities and solvability with boundary damping

Let $\Omega \subset \mathbb{R}^d$ be an open with smooth, compact boundary Γ, g be a Riemannian metric on Ω with $g_{ij}(x) \equiv \delta_{ij}$ for $|x| \gg 1$ and ν be the outward unit normal to Ω with respect to g. Let $Q \in \Psi^{-1}(\Gamma)$, and $W \in \Psi^{\mathrm{comp}}(\Omega)$ and $A \in \Psi^{\mathrm{comp}}_{\mathsf{T}}(\Omega)$, We study the problem

$$\begin{cases} \mathsf{P}u := (-\hbar^2\Delta_g - iW - 1 - A)u = \hbar f_1 & \text{in } \Omega \\ \mathsf{B}u := Q\hbar D_\nu u - u = f_2 & \text{on } \Gamma \\ u \text{ is outgoing.} \end{cases} \tag{13.5}$$

We prove the following propagation of singularities theorem in Appendix B.

Theorem 13.12 (Propagation of singularities) *Let $Q \in \Psi^{-1}(\Gamma)$ with $\sigma(Q) \geq 0$, $A \in \Psi^{\mathrm{comp}}_{\mathsf{T}}$ with $A|_\Gamma = 0$ and $\sigma(A) \in \mathbb{R}$, $W \in \Psi^{\mathrm{comp}}(\Omega)$ with $\sigma(W) \geq 0$, $N > 0$, $s \in \mathbb{R}$, and $u \in H^1_\hbar$ be $\hbar$-tempered. Then, letting $S^*_a\Omega := \{(x,\xi) : |\xi|^2_g - a = 1\}$, $\mathcal{Z} := \pi_b(S^*_a\Omega)$,*

$${}^b\mathrm{WF}^s_{H^2_\hbar}(u) \subset \big(\mathcal{Z} \cup {}^b\mathrm{WF}^s_{L^2}(\mathsf{P}u) \cup \mathrm{WF}^{s-\frac{1}{2}}_{H^{3/2}_\hbar}(\mathsf{B}u)\big) \setminus \{\sigma(W) > 0\}$$

and for all $\beta > 0$,

$${}^b\mathrm{WF}^s_{H^1_\hbar}(u) \cap (\mathcal{Z} \setminus \{\sigma(W) > 0\})$$

is a union of maximally extended backward GBBs for the Hamiltonian $|\xi|^2_g - a - 1$ in

$$\mathcal{Z} \setminus \Big({}^b\mathrm{WF}^{s+1}_{L^2}(\mathsf{P}u) \cup \big(\mathrm{WF}^s_{L^2}(\mathsf{B}u) \cap (\mathcal{G}_d \cup \mathcal{H})\big) \cup \big(\mathrm{WF}^{s+\beta}_{L^2}(\mathsf{B}u) \cap \mathcal{G}_g\big)\Big).$$

Remark 13.13 *The differences between Theorem 13.12 and existing results in the literature are that Theorem 13.12 (i) covers semiclassical boundary damping (via Q) and (ii) quantifies the dependence of the microlocalised $H^1_\hbar$ norm of u on the microlocalised L^2 norm of $\mathsf{B}u$ along the GBBs. Surprisingly, even for the Dirichlet problem, (ii) appears not to be in the literature. The most subtle part of the analysis is in the glancing region, where we follow the proof in [62, Chapter 24].*

We also need the following lemma, which is also proved in Appendix B.

Lemma 13.14 *Let $V \Subset \mathcal{H}$. Then there is $\Lambda^{-1} \in \Psi^{-1}(\Gamma)$ with $\sigma(\Lambda^{-1}) \geq c\langle\xi\rangle^{-1} > 0$ such that for all $q \in V$, $s \in \mathbb{R}$, and all $\hbar$-tempered $u \in H^1_\hbar$ with*

$$q \notin {}^b\mathrm{WF}^{s+1}_{L^2}(\mathsf{P}u) \cap \mathrm{WF}^s(\Lambda^{-1}\hbar D_\nu u - u),$$

there is $\epsilon > 0$ such that

$$\bigcup_{0<s\leq\epsilon} \varphi_s(q) \notin \mathrm{WF}^s_{H^1_\hbar}(u).$$

Finally, we require a lemma that gives conditions under which the problem (13.5) is uniquely solvable.

Lemma 13.15 *Let $\Omega \subset \mathbb{R}^d$ open with smooth boundary, $\Gamma \Subset \mathbb{R}^d$ and g a Riemannian metric on Ω with $\operatorname{supp}(g - I) \Subset \mathbb{R}^d$. Let $\hbar_0 > 0$, $Q \in \Psi^{-1}(\Gamma)$ with $\sigma(Q) \geq 0$, $A \in \Psi^{\mathrm{comp}}_{\mathsf{T}}$ with $A|_\Gamma = 0$, $\sigma(A) \in \mathbb{R}$, $W \in \Psi^{\mathrm{comp}}(\Omega)$ with $\sigma(W) \geq 0$. Suppose that for all $R > 0$ there are $\delta > 0$, $T > 0$ such that for all $\rho \in \mathcal{Z} \cap (B(0,R))$, either there is $0 \leq t \leq T$ such that*

$$\varphi^\Omega_{-t}(\rho) \in \{\sigma(W) > 0\} \cup \big(\mathcal{H} \cap \{\sigma(Q) > 0\}\big) \cup T^*\mathbb{R}^d \setminus B(0,R) \tag{13.6}$$

or

$$\bigcup_{t-\delta}^{t} \varphi^\Omega_s(\rho) \subset \mathcal{G} \cap \{\sigma(Q) > 0\}. \tag{13.7}$$

Then for every $R > 0$ and $\chi \in C^\infty_c(\overline{\Omega})$, there is $C > 0$ such that for all $f_1 \in L^2(\Omega)$, $\operatorname{supp} f_1 \subset B(0,R)$, $f_2 \in H^{3/2}_\hbar(\Gamma)$, the solution, u to (13.5) exists, is unique and for all $0 < \hbar < \hbar_0$,

$$\|\chi u\|_{H^2_\hbar(\Omega)} + \|u\|_{H^{3/2}_\hbar(\Gamma)} + \|\hbar\partial_\nu u\|_{H^{1/2}_\hbar(\Gamma)} \leq C(\|f_1\|_{L^2(\Omega)} + \|f_2\|_{H^{3/2}_\hbar(\Gamma)}).$$

Proof. This lemma follows directly from the analysis in [44, Lemma 3.2]. ■

With these lemmas in hand, we are able to construct approximate solutions to (13.5) that have prescribed wavefront set properties.

Lemma 13.16 (Construction of approximate solutions to (13.5)) *Let $\Omega \subset \mathbb{R}^d$ open with smooth boundary, $\Gamma \Subset \mathbb{R}^d$ and g a Riemannian metric on Ω with $\operatorname{supp}(g-I) \Subset \mathbb{R}^d$. Let $V \Subset \mathcal{H}$, $\hbar_0 > 0$, $Q \in \Psi^{-1}(\Gamma)$ with $\sigma(Q) \geq 0$, $U \subset \mathcal{Z}$ and suppose that for any $R > 0$ and $\rho \in U$ there is $0 \leq t < \infty$ such that*

$$\varphi_t^{\Omega}(\rho) \in (\mathrm{WF}(Q - \Lambda^{-1}))^c \cup \{|x| > R\}$$

Then for all $R > 0$, there is $C > 0$ such that for all $\hbar$-tempered,$f_1 \in L^2(\Omega)$, $\operatorname{supp} f_1 \subset B(0,R)$, $f_2 \in H_\hbar^{3/2}(\Gamma)$, with

$$({}^b\mathrm{WF}(f_1) \cup \mathrm{WF}(f_2)) \cap \mathcal{Z} \subset U,$$

there is $u \in H^2_{\hbar,\mathrm{loc}}(\Omega_+)$ such that, u satisfies

$$\begin{cases} (-\hbar^2\Delta_g - 1)u = \hbar f_1 + O(\hbar^\infty)_{L^2_{\mathrm{comp}}} & \text{in } \Omega \\ Q\hbar D_\nu u - u = f_2 & \text{on } \Gamma \\ u & \text{is outgoing,} \end{cases} \tag{13.8}$$

$$\|u\|_{H_\hbar^2(\Omega)} + \|u\|_{H_\hbar^{3/2}(\Gamma)} + \|\hbar\partial_\nu u\|_{H_\hbar^{1/2}(\Gamma)} \leq C(\|f_1\|_{L^2(\Omega)} + \|f_2\|_{H_\hbar^{3/2}(\Gamma)}), \tag{13.9}$$

and

$${}^b\mathrm{WF}(u) \subset {}^b\mathrm{WF}(f_1) \cup \mathrm{WF}(f_2) \cup \bigcup_{q \in {}^b\mathrm{WF}(f_1)\cup \mathrm{WF}(f_2)} \overline{\bigcup_{0\leq t < T_Q(q)} \varphi_t^{\Omega}(q)},$$

where

$$T_Q(q) := \inf\{t \geq 0 \,:\, \varphi_t^{\Omega}(q) \in (\mathrm{WF}(Q - \Lambda_+^{-1}))^c\}.$$

Proof. Let $q \in {}^b\mathrm{WF}(f_1) \cup \mathrm{WF}(f_2) \cap \mathcal{Z}$. Then, there is $0 \leq t \leq T$ such that

$$\varphi_t^{\Omega}(\rho) \in (\mathrm{WF}(Q - \Lambda^{-1}))^c \cup \{|x| > R\}.$$

In particular, either Lemma 13.4 or 13.5 applies. Let a_q as constructed there so that either (13.2) or (13.4) holds. Then, since $\mathcal{Z}$ is compact, there is a neighborhood U_q of q and $V_q \Subset V_q^+ \Subset T^*\Omega$ such that

$$\bigcup_{q' \in U \cap \mathcal{Z}} \overline{\bigcup_{0 \leq t < T_Q(q')} \varphi_t^{\Omega}(q')} \cap (\operatorname{supp} a \cup V_q^+) = \emptyset.$$

and such that (13.2) or (13.4) holds with $\{d(x,\Gamma) > 0\}$ replaced by V_q.

Now, since ${}^b\mathrm{WF}(f_1) \cup \mathrm{WF}(f_2) \cap \mathcal{Z}$ is compact, there are $q_1, \dots, q_N$ such that

$${}^b\mathrm{WF}(f_1) \cup \mathrm{WF}(f_2) \cap \mathcal{Z} \subset \bigcup_{i=1}^N U_{q_i}.$$

Let $\psi_i \in C_c^\infty(U_{q_i})$ be a partition of unity near ${}^b\mathrm{WF}(f_1)$ and $\chi_i \in C_c^\infty(U_i \cap T^*\Gamma)$ a partition of unity near $\mathrm{WF}(f_2) \cap \mathcal{Z}$ and set $\psi_0 := 1 - \sum_i \psi_i$, $\chi_0 := 1 - \sum_i \chi_i$. Let also $W_i \in \Psi^{\mathrm{comp}}(\Omega)$ with $\sigma(W_i) = 1$ on V_{q_i} and $\mathrm{WF}(W_i) \Subset V_{q_i}^+$, and $A_i = \mathrm{Op}(a_{q_i})$, $i = 1, \dots, N$, and set $W_0 = W_1$, $A_0 = A_1$.

Then, by Lemma 13.15, for $i = 0, 1, \dots, N$, there is u_i satisfying

$$\begin{cases} (-\hbar^2\Delta_g - A_i - iW_i^*W_i - 1)u = \hbar^b\mathrm{Op}(\psi_i)f_1 & \text{in } \Omega \\ Q\hbar D_\nu u_i - u_i = \mathrm{Op}(\chi_i)f_2 & \text{on } \Gamma \\ u_i & \text{is outgoing,} \end{cases}$$

with u_i satisfying (13.9). Moreover, by Theorem 13.12, Lemma 13.14, and the outgoing property,

$${}^b\mathrm{WF}(u_i) \subset {}^b\mathrm{WF}(f_1) \cup \mathrm{WF}(f_2) \cup \bigcup_{q \in ({}^b\mathrm{WF}(f_1)\cup\mathrm{WF}(f_2))\cap U_{q_i}} \overline{\bigcup_{0 \leq t < T_Q(q)} \varphi_t^{\Omega}(q)}$$

In particular, ${}^b\mathrm{WF}(u_i) \cap ({}^b\mathrm{WF}(A_i) \cap {}^b\mathrm{WF}(W_i)) = \emptyset$, and hence

$$\begin{cases} (-\hbar^2\Delta_g - 1)u = \hbar^b\mathrm{Op}(\psi_i)f_1 + O(\hbar^\infty)_{L^2_{\mathrm{comp}}} & \text{in } \Omega \\ Q\hbar D_\nu u_i - u_i = \mathrm{Op}(\chi_i)f_2 & \text{on } \Gamma \\ u_i & \text{is outgoing.} \end{cases}$$

Setting $u := \sum_{i=0}^N u_i$ completes the proof. ■

13.4 Review of resolvents and layer potentials

In this section, we recall some facts about the free, outgoing resolvent and boundary layer potentials; we use the latter to control the boundary traces of solutions to $(-\hbar^2\Delta - 1)u = 0$. Recall that the outgoing resolvent $R_0(\hbar) : L^2_{\mathrm{comp}} \to L^2_{\mathrm{loc}}$ is the unique solution to

$$(-\hbar^2\Delta - 1)R_0(\hbar)f = f \text{ in } \mathbb{R}^d, \qquad R_0(\hbar)f \text{ is outgoing.}$$

It is well known (see e.g. [50]) that for all $\chi \in C_c^\infty(\mathbb{R}^d)$, $\hbar_0 > 0$, and $s \in \mathbb{R}$, there is $C > 0$ such that for $0 < \hbar < \hbar_0$,

$$\|\chi R_0(\hbar)\chi\|_{H^s_\hbar \to H^{s+2}_\hbar} \le C\hbar^{-1}. \tag{13.10}$$

The single and double layer potentials associated to Γ are then given, respectively, for $f \in L^2(\Gamma)$ by

$$\mathcal{S}\ell f(x) := \hbar^2 \int_\Gamma R_0(\hbar)(x,y)f(y)dS(y), \qquad \mathcal{D}\ell f(x) := \hbar^2 \int_\Gamma \partial_{\nu_y} R_0(\hbar)(x,y)f(y)dS(y),$$

where $R_0(\hbar)(x,y)$ is the kernel of $R_0(\hbar)$.

Lemma 13.17 *Let $\chi \in C_c^\infty(\mathbb{R}^d)$, and $B \in \Psi^{\mathrm{comp}}(\mathbb{R}^d)$ with $\mathrm{WF}(B) \cap \bigcup_{t\ge 0} \varphi_t^{\mathbb{R}^d}(S^*\Gamma) = \emptyset$. Then, for all $\hbar_0 > 0$ there is $C > 0$ such that for $0 < \hbar < \hbar_0$,*

$$\|B\mathcal{S}\ell\|_{L^2(\Gamma)\to L^2(\mathbb{R}^d)} \le C\hbar, \qquad \|\chi\mathcal{D}\ell\|_{L^2(\Gamma)\to L^2(\mathbb{R}^d)} \le C. \tag{13.11}$$

Proof. Let L be a smooth vectorfield with $L|_\Gamma = \partial_\nu$. We start by recalling that

$$\mathcal{S}\ell = R_0(\hbar)\hbar^2\delta_\Gamma, \qquad \mathcal{D}\ell = R_0(\hbar)L^*\hbar^2\delta_\Gamma.$$

Let $u \in L^2(\Gamma)$ and $v \in L^2(\mathbb{R}^d)$. Then,

$$\langle B\mathcal{S}\ell u, v\rangle_{L^2(\mathbb{R}^d)} = \langle BR_0(\hbar)\hbar^2\delta_\Gamma u, v\rangle_{L^2(\mathbb{R}^d)} = \langle u, \gamma R_0^*(\hbar)\hbar^2 B^* v\rangle_{L^2(\Gamma)}$$

Since R_0^* is the *incoming* resolvent,

$$(-\hbar^2\Delta - 1)R_0^*(\hbar)\hbar^2 B^* v = \hbar^2 B^* v, \qquad \mathrm{WF}(R_0^*(\hbar)\hbar^2 B^* v) \subset \mathrm{WF}(B^* v) \cup \bigcup_{t\le 0} \varphi_t^{\mathbb{R}^d}\big(\mathrm{WF}(B^*) \cap S^*\mathbb{R}^d\big),$$

In particular, $\mathrm{WF}(R_0^*(\hbar)\hbar^2 B^* v) \cap S^*\Gamma = \emptyset$. Thus, for $\chi \in C_c^\infty(\mathbb{R}^d)$ with $\chi \equiv 1$ near Γ, standard trace estimates (see e.g. [43, Proposition 3.1])

$$\|\gamma R_0^*(\hbar)\hbar^2 B^* v\|_{L^2(\Gamma)} \le C(\|\chi R_0^*(\hbar)\hbar^2 B^* v\|_{L^2(\mathbb{R}^d)} + \hbar\|B^* v\|_{L^2(\mathbb{R}^d)}) \le C\hbar\|B^* v\|_{L^2(\mathbb{R}^d)} \le C\hbar\|v\|_{L^2(\mathbb{R}^d)},$$

and hence

$$|\langle B\mathcal{S}\ell u, v\rangle_{L^2(\mathbb{R}^d)}| \le C\hbar\|u\|_{L^2(\Gamma)}\|v\|_{L^2(\mathbb{R}^d)},$$

which implies the first estimate in (13.11) by duality.

For the second, we proceed similarly.

$$\langle \mathcal{D}\ell u, v\rangle_{L^2(\mathbb{R}^d)} = \langle R_0(\hbar)\hbar^2 L^*\delta_\Gamma u, v\rangle_{L^2(\mathbb{R}^d)}. = \langle u, \gamma\hbar L R_0^*(\hbar)\hbar v\rangle_{L^2(\Gamma)}$$

Then, we have

$$(-\hbar^2\Delta - 1)R_0^*(\hbar)\hbar v = \hbar v,$$

and hence, using e.g. [107, Corollary 0.6],

$$\|\hbar\gamma L R_0^*(\hbar)hv\|_{L^2(\Gamma)} \le C(\|R_0^*(\hbar)\hbar v\|_{L^2(\mathbb{R}^d)} + \|v\|_{L^2(\mathbb{R}^d)}) \le C\|v\|_{L^2(\mathbb{R}^d)},$$

which, arguing as above, implies the second estimate in (13.11) by duality. ■

Next, we consider the Dirichlet resolvent $R_D : L^2_{\mathrm{comp}}(\Omega^+) \to L^2_{\mathrm{loc}}(\Omega^+)$ defined by

$$(-\hbar^2\Delta - 1)R_D f = f, \qquad R_D f|_\Gamma = 0, \qquad R_D f \text{ is outgoing,}$$

and the Poisson operator $G_D : H^1_\hbar(\Gamma) \to L^2_{\mathrm{loc}}(\Omega^+)$, defined by

$$(-\hbar^2\Delta - 1)G_D g = 0, \qquad G_D g|_\Gamma = g, \qquad G_D g \text{ is outgoing.}$$

We need the following assumption on R_D.

Assumption 13.18 *There is $P_{\mathrm{inv}} \ge 0$ and $\tilde{\mathcal{J}} \subset (0,\infty)$ such that for all $\chi \in C_c^\infty(\Omega^+)$, there is $C > 0$ such that*

$$\|\chi R_D \chi\|_{L^2(\Omega^+)\to L^2(\Omega^+)} \le C\hbar^{-1-P_{\mathrm{inv}}}, \qquad \hbar \notin \tilde{\mathcal{J}}$$

Note that by [77], for any $\Omega^- \Subset \mathbb{R}^d$ with smooth boundary and connected complement, and any $\delta > 0$ there is $\mathcal{J}$ with $|\mathcal{J}| < \delta$ such that Assumption 13.18 holds. Moreover, for any $N > 0$, one can find $\mathcal{J}$ and $C > 0$ such that $|\mathcal{J} \cap (k,\infty)| \le Ck^{-N}$ for all $k > 1$.

We now show that, provided the data is located away from Γ_-, G_D and R_D behave as in the nontrapping case.

Lemma 13.19 *Suppose that Assumption (13.18) holds. Then for all $B \in \Psi^0(\Gamma)$ with $\mathrm{WF}(B) \cap \Gamma_- = \emptyset$ and all $\chi \in C_c^\infty(\overline{\Omega^+})$, there is $C > 0$ such that*

$$\|\chi G_D B\|_{H_\hbar^{3/2}(\Gamma)\to H^2_\hbar(\Omega^+)} \le C, \qquad \hbar \notin \mathcal{J},$$

and, if $u \in H_\hbar^{3/2}$ is $\hbar$-tempered

$${}^b\mathrm{WF}(G_D B u) \subset \mathrm{WF}(B) \cup \bigcup_{t\ge 0} \varphi_t^\Omega(\mathrm{WF}(B) \cap \mathcal{Z}).$$

Proof. Since $\mathrm{WF}(B) \cap \Gamma_- = \emptyset$, Lemma 13.16 implies that for any $g \in H_\hbar^{3/2}$ there is $u \in H^2_{\hbar,\mathrm{loc}}$ such that for all $\chi \in C_c^\infty(\Omega^+)$,

$$\|\chi u\|_{H^2_\hbar} \le C\|f\|_{H_\hbar^{3/2}}$$

and

$${}^b\mathrm{WF}(u) \subset \mathrm{WF}(B) \cup \bigcup_{t\ge 0} \varphi_t^{\Omega+}({}^b\mathrm{WF}(B) \cap \mathcal{Z}),$$

and

$$\begin{cases} (-\hbar^2\Delta - 1)u = f = O(\hbar^\infty \|g\|_{H_\hbar^{3/2}})_{L^2_{\mathrm{comp}}} & \text{in } \Omega^+ \\ u = Bg & \text{on } \Gamma, \\ u \text{ is outgoing.} \end{cases}$$

In particular, by Assumption 13.18,

$$G_D B g = u - R_D f = u + O(\hbar^\infty)_{H^2_{\hbar,\mathrm{loc}}},$$

and the lemma follows. ■

The next lemma is proved using an almost identical argument

Lemma 13.20 *Suppose that Assumption 13.18 holds. Then for all $B \in \Psi^{\mathrm{comp}}(\Omega^+)$ with $\mathrm{WF}(B) \cap \Gamma_- = \emptyset$ and all $\chi \in C_c^\infty(\overline{\Omega^+})$, there is $C > 0$ such that*

$$\|\chi R_D B\|_{L^2 \to L^2(\Omega^+)} \le C\hbar^{-1}, \qquad \hbar \notin \tilde{\mathcal{J}},$$

and, if $u \in L^2$ is $\hbar$-tempered,

$${}^b\mathrm{WF}(R_D B u) \subset {}^b\mathrm{WF}(B) \cup \bigcup_{t\ge 0} \varphi_t^\Omega({}^b\mathrm{WF}(B) \cap \mathcal{Z}).$$

We also require the following estimate on the wavefront set properties of R_D which follows directly from Theorem 13.12, the outgoing condition, and Assumption 13.18

Lemma 13.21 *Suppose that Assumption 13.18 holds. Then, for all $B \in {}^b\Psi^{\mathrm{comp}}(\Omega^+)$, $\chi \in C_c^\infty(\overline{\Omega^+})$, and $u \in L^2$ $\hbar$-tempered,*

$$ {}^b\mathrm{WF}(R_D Bu) \subset {}^b\mathrm{WF}(B) \cup \bigcup_{t \geq 0} \varphi_t^{\Omega^+}({}^b\mathrm{WF}(B) \cap \mathcal{Z}) \cup \Gamma_+. $$

13.5 Non-interference of the Impedance-to-Dirichlet maps

In order to pass from quasimodes in Ω^+ to quasimodes for A_k and A_k', we need two lemmas that give conditions guaranteeing that

$$ \|P_{\mathrm{ItD}}^{-;\eta} P_{\mathrm{DtN}}^+ f\|_{L^2} \sim \|P_{\mathrm{DtN}}^+ f\|_{L^2}, \qquad \|P_{\mathrm{ItD}}^{+;\eta} P_{\mathrm{DtN}}^- f\|_{L^2} \sim \|P_{\mathrm{DtN}}^- f\|_{L^2}. $$

Lemma 13.22 *Let $C > 0$ and suppose that $f \in L^2(\Gamma)$, with*

$$ \|f\|_{L^2(\Gamma)} = o(1), \qquad \|P_{\mathrm{DtN}}^+ f\|_{L^2} = 1, $$

and there is $B \in \Psi^{\mathrm{comp}}(\mathbb{R}^d)$ with $\mathrm{WF}(B) \cap \bigcup_{t>0} \varphi_t^{\mathbb{R}^d}(S^*\Gamma) = \emptyset$ such that

$$ \|BG_D f\|_{L^2(\Omega^+)} \geq c > 0. \tag{13.12} $$

Then, for all $C^{-1} \leq \hbar\eta \leq C$,

$$ \|P_{\mathrm{ItD}}^{-;\eta} P_{\mathrm{DtN}}^+ f\| \geq c > 0. $$

Proof. Suppose there is a sequence $\hbar_n \to 0$ with $\|P_{\mathrm{ItD}}^{-;\eta} P_{\mathrm{DtN}}^+ f\| \to 0$. Then, there are solutions u_1, u_2 to

$$ \begin{cases} (-\hbar^2\Delta - 1)u_1 = 0 & \text{in } \Omega \\ (\hbar\partial_\nu - i\hbar\eta)u_1 = \hbar\partial_\nu u_2, \end{cases} \qquad \begin{cases} (-\hbar^2\Delta - 1)u_2 = 0 & \text{in } \mathbb{R}^d \setminus \Omega \\ u_2|_{\partial\Omega} = f \\ u_2 \text{ outgoing} \end{cases} $$

with $\|u_1|_{\partial\Omega}\|_{L^2} = o(1)$ and $\|f\|_{L^2} = o(1)$. Thus,

$$ \begin{aligned} (u_1 1_\Omega + u_2 1_{\mathbb{R}^d\setminus\Omega}) &= R_0(\hbar)(\hbar^2(u_1 - u_2)|_\Gamma \otimes \partial_\nu \delta_\Gamma + hR_0(\hbar)(\hbar\partial_\nu u_1 - \hbar\partial_\nu u_2) \otimes \delta_\Gamma \\ &= \mathcal{D}\ell(u_1 - u_2)|_\Gamma) + \hbar^{-1}\mathcal{S}\ell(\hbar\partial_\nu u_1 - \hbar\partial_\nu u_2) \\ &= \mathcal{D}\ell(u_1 - f) + \hbar^{-1}\mathcal{S}\ell i\hbar\eta u_1 \\ &= \mathcal{D}\ell o(1)_{L^2} + \hbar^{-1}\mathcal{S}\ell o(1)_{L^2}. \end{aligned} $$

In particular, for any $B \in \Psi^c(\mathbb{R}^d)$ with

$$ \mathrm{WF}(B) \cap \bigcup_{t \geq 0} \varphi_t^{\mathbb{R}^d}(S^*\Gamma) = \emptyset, $$

we have

$$ B((u_1 1_\Omega + u_2 1_{\mathbb{R}^d\setminus\Omega})) = o(1)_{L^2}. $$

This contradicts (13.12). ∎

Lemma 13.23 *Let $C, c > 0$ and suppose that $f \in L^2(\Gamma)$, with*

$$ \|f\|_{L^2(\Gamma)} = o(1), \qquad \|P_{\mathrm{DtN}}^- f\|_{L^2} = 1, $$

and there is $B \in \Psi^{\mathrm{comp}}(\mathbb{R}^d)$ with $\mathrm{WF}(B) \cap \bigcup_{t \leq 0} \varphi_t^{\mathbb{R}^d}(S^*\Gamma \cup N^*\Gamma) = \emptyset$ such that

$$ \|BG_D^+(h) f\|_{L^2(\Omega^+)} \geq c > 0. \tag{13.13} $$

Then, there is $A \in \Psi(\Gamma)$ with $\mathrm{WF}(A) \cap \{\xi' = 0\} = \emptyset$ such that for all $C^{-1} \leq \hbar\eta \leq C$,

$$ \|AP_{\mathrm{ItD}}^{+;\eta} P_{\mathrm{DtN}}^- f\| \geq c > 0. $$

Proof. Suppose there is a sequence $\hbar_n \to 0$ with $\|AP^{+,\eta}_{\mathrm{ItD}}P^{-}_{\mathrm{DtN}}f\| \to 0$. Then, there are solutions u_1, u_2 to

$$\begin{cases}(-\hbar^2\Delta - 1)u_1 = 0 & \text{in } \Omega\\ (\hbar\partial_\nu + i\hbar\eta)u_1 = \hbar\partial_\nu u_2, & \end{cases} \qquad \begin{cases}(-\hbar^2\Delta - 1)u_2 = 0 & \text{in } \mathbb{R}^d \setminus \Omega\\ u_2|_{\partial\Omega} = f & \\ u_2 \text{ incoming} & \end{cases}$$

with $\|Au_1|_{\partial\Omega}\|_{L^2} = o(1)$ and $\|f\|_{L^2} = o(1)$. Thus,

$$\begin{aligned}(u_1 1_\Omega + u_2 1_{\mathbb{R}^d\setminus\Omega}) &= R_0^*(\hbar)(\hbar^2(u_1 - u_2)|_\Gamma \otimes \partial_\nu\delta_\Gamma + hR_0^*(\hbar)(\hbar\partial_\nu u_1 - \hbar\partial_\nu u_2) \otimes \delta_\Gamma\\ &= \widetilde{\mathcal{D}\ell}(u_1 - u_2)|_\Gamma) + \hbar^{-1}\widetilde{\mathcal{S}\ell}(\hbar\partial_\nu u_1 - \hbar\partial_\nu u_2)\\ &= \widetilde{\mathcal{D}\ell}(u_1 - f) + \hbar^{-1}\widetilde{\mathcal{S}\ell}i\hbar\eta u_1\\ &= \widetilde{\mathcal{D}\ell}(u_1) - \widetilde{\mathcal{D}\ell}o(1)_{L^2} + \hbar^{-1}\widetilde{\mathcal{S}\ell}i\hbar\eta u_1.\end{aligned}$$

In particular, for any $B \in \Psi^c(\mathbb{R}^d)$ with

$$\mathrm{WF}(B) \cap \bigcup_{t\geq 0} \varphi_t^{\mathbb{R}^d}(S^*\Gamma \cup \mathrm{WF}(I - A)) = \emptyset,$$

we have

$$B((u_1 1_\Omega + u_2 1_{\mathbb{R}^d\setminus\Omega})) = B\widetilde{\mathcal{D}\ell}(u_1) + \hbar^{-1}\widetilde{\mathcal{S}\ell}i\hbar\eta u_1 + o(1)_{L^2} = B\widetilde{\mathcal{D}\ell}(Au_1)\hbar^{-1}\widetilde{\mathcal{S}\ell}i\hbar\eta Au_1 + o(1)_{L^2} = o(1)_{L^2}.$$

This contradicts (13.13). ∎

13.6 From interior quasimodes to quasimodes for BIEs

We are now in a position to show that appropriate quasimodes in the bulk produce quasimodes for the BIE operators that are effective for demonstrating pollution. Before constructing these quasimodes, we need two technical lemmas that reduce (12.4) to a wavefront set statement.

Lemma 13.24 *Suppose that Assumption 1.1 holds Assumption 13.18 holds for some $\tilde{\mathcal{J}}$. Then Assumption 1.3 holds with $\mathcal{A} = A_k$ or A'_k with $\mathcal{J} := \{k : k^{-1} \in \tilde{\mathcal{J}}\}$.*

Proof. Notice that Lemma 13.15 implies

$$\|P^{-,\eta}_{\mathrm{ItD}}\|_{H_\hbar^{1/2} \to H_\hbar^{3/2}} \leq C.$$

Next, we show that that there are $\chi \in C_c^\infty$ and $N_0 > 0$ such that

$$\|P^{+}_{\mathrm{DtN}}\|_{H_\hbar^{3/2} \to H_\hbar^{1/2}} \leq C\|\chi R_D\chi\|_{L^2\to L^2}.$$

For this, observe that there is $E : H_\hbar^{3/2}(\Gamma) \to H^2_{\hbar,\mathrm{comp}}(\Omega^+)$, such that $\|E\|_{H_\hbar^{3/2}\to H_\hbar^2} \leq C\hbar^{-1/2}$. Therefore,

$$P^{+}_{\mathrm{DtN}} = \hbar\partial_\nu R_D(-\hbar^2\Delta - 1)E,$$

and we use the estimate: for any $\chi \in C_c^\infty(\overline{\Omega}^+)$ with $\chi \equiv 1$ near Γ,

$$\|\hbar\partial_\nu u\|_{H_\hbar^{1/2}(\Gamma)} \leq C\hbar^{-1/2}\|\chi u\|_{H_\hbar^2},$$

to conclude that

$$\|P^{+}_{\mathrm{DtN}}\|_{H_\hbar^{3/2}\to H_\hbar^{1/2}} \leq C\hbar^{-1}\|\chi R_D\chi\|_{L^2\to H_\hbar^2} \leq C\hbar^{-1}\|\chi R_D\chi\|_{L^2\to L^2}.$$

Together with (13.1), this implies that for $\Psi \in \Psi^{\mathrm{comp}}(\Gamma)$,

$$\|A_k^{-1}\Psi\|_{L^2(\Gamma)\to L^2(\Gamma)} + \|(A'_k)^{-1}\Psi\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq C\hbar^{-1}\|\chi R_D\chi\|_{L^2\to L^2}.$$

Lemma 4.9, or more precisely its proof, then completes the proof of the lemma. ∎

Lemma 13.25 *Suppose that $f \in L^2(\Gamma)$ is $\hbar$-tempered and*

$$\mathrm{WF}(f) \subset \{(x', \xi') \in \mathcal{H} : 0 < |\xi'|_{g_\Gamma}\}. \tag{13.14}$$

Then for any $k_0 > 0$, $\epsilon > 0$, $\chi \in C_c^\infty((-(1+2\epsilon)^2, (1+2\epsilon)^2))$ with $\chi \equiv 1$ on $[-1-\epsilon, 1+\epsilon]$, there is $\epsilon_0 > 0$ such that for any $N > 0$ there is $C > 0$ such that

$$\|(1-\chi(k^{-2}\Delta_g))f\|_{L^2(\Gamma)} \le Ck^{-N}, \qquad \|\chi(\epsilon_0^{-2}k^{-2}\Delta_g)f\|_{L^2(\Gamma)} \le Ck^{-N}$$

Proof. The proof follows from the definition of wavefront set and Corollary 3.10. ∎

We now give a few abstract lemmas that pass from quasimodes in Ω^+ to data, f on Γ satisfying (13.14) and (12.3).

Lemma 13.26 (From Helmholtz quasimode data to growing P^+_{DtN}) *Let $\Omega^- \Subset \mathbb{R}^d$ open with smooth boundary and connected complement $\Omega^+ := \mathbb{R}^d \setminus \overline{\Omega}_-$ such that Assumption 13.18 holds. Let $\mu \in \{1, -1\}$, $\mathcal{O} \subset \mathcal{H}$ open, $\mathcal{B} \subset \mathcal{H}$ closed and $B \in \Psi^{\mathrm{comp}}(\Omega^+)$ such that*

$$\bigcup_{\mu t \ge 0} \varphi^{\mathbb{R}^d}_{\mu t}(S^*\Gamma) \cap \mathrm{WF}(B) = \emptyset. \tag{13.15}$$

Suppose that there are $u \in H^2_\hbar(\Omega^+)$, $\tilde{g} \in L^2_{\mathrm{comp}}(\Omega^+)$ such that

$$\sup_{q \in {}^b\mathrm{WF}(\tilde{g} \cap \mathcal{Z})} T^\mu_{\mathcal{O} < \mathcal{B} \cup \{q\}}(q) < \infty, \qquad \|Bu\|_{L^2(\Omega^+)} \ge c > 0 \tag{13.16}$$

$$(-\hbar^2\Delta - 1)u = \hbar\tilde{g}, \qquad u|_\Gamma = 0, \qquad \|\tilde{g}\|_{L^2} = o(1), \quad u \text{ is } \begin{cases} \text{outgoing} & \mu = 1 \\ \text{incoming} & \mu = -1 \end{cases}. \tag{13.17}$$

Then for any function $T : {}^b\mathrm{WF}(\tilde{g}) \to [0, \infty)$ such that $T(q) < T_{\mathcal{B} \cup \{q\}}(q)$, $\varphi^{\Omega^+}_{\mu T(q)}(q) \in \mathcal{O} \setminus (\mathcal{B} \cup \{q\})$, and any $\epsilon > 0$ there is $c_1 > 0$ such that for any $\hbar \notin \mathcal{J}$, $q_0 \in {}^b\mathrm{WF}(\tilde{g})$ and $g_1 \in L^2(\Gamma)$ such that

$$\|P^\mu_{\mathrm{DtN}} g_1\|_{L^2} \ge c_1(\|Bu\|_{L^2}/\|\tilde{g}\|_{L^2})\|g_1\|_{L^2}, \tag{13.18}$$

$$\mathrm{WF}(g_1) \subset B(\varphi^{\Omega_+}_{T(q_0)}(q_0), \epsilon) \subset \mathcal{O} \setminus (\mathcal{B} \cup \{q_0\}), \tag{13.19}$$

and

$$\|BG_D g_1\|_{L^2} \ge c_1.$$

Proof. We consider the case of $\mu = 1$, the other being almost identical but with the role of R_D played by R_D^*.

Let $q \in {}^b\mathrm{WF}(\tilde{g}) \cap \mathcal{Z}$ and $0 < T_q < T_{\mathcal{B} \cup \{q\}}$ such that $\varphi^{\Omega^+}_{T^1_q}(q) \in \mathcal{O} \setminus (\mathcal{B} \cup \{q\})$. Let $V^1_q \Subset V^2_q \subset \mathcal{O} \cap (B(\varphi^{\Omega_+}_{T_q}(q), \epsilon) \setminus (\mathcal{B} \cup \{q\}))$ be a neighborhood of $\varphi^{\Omega^+}_{T_q}(q)$ and U_q a neighborhood of q such that

$$\inf_{q' \in U_q} T^1_{V^1_q < (\mathcal{B} \cup \{q'\}}(q') < \infty.$$

Such a neighborhood exists because $\varphi^{\Omega^+}_t$ is continuous and $\varphi^{\Omega^+}_{\mu T(q)}(q) \in \mathcal{H} \setminus (\mathcal{B} \cup \{q\})$.

Since ${}^b\mathrm{WF}(\tilde{g}) \cap \mathcal{Z}$ is compact, there are $\{q_1, \ldots q_N\}$ such that

$${}^b\mathrm{WF}(\tilde{g}) \cap \mathcal{Z} \subset \bigcup_{i=1}^N U_{q_i}.$$

Let $\{\psi_j\}_{j=1}^N \in C_c^\infty({}^bT^*\Omega^+)$ with $\sum_j \psi_j \equiv 1$ on ${}^b\mathrm{WF}(\tilde{g}) \cap \mathcal{Z}$ and $\operatorname{supp} \psi_j \subset U_{q_j}$. Then, by Lemma 13.20 and hence, using Assumption 13.18, for any $\chi \in C_c^\infty$,

$$\|\chi u - \sum_j \chi R_D {}^b\mathrm{Op}(\psi_j)\hbar\tilde{g}\|_{H^2_\hbar} \le C\|\tilde{g}\|_{L^2}.$$

In particular, there is $i \in \{1, \dots, N\}$ such that

$$\|BR_D{}^b\mathrm{Op}(\psi_i)\tilde{g}\|_{L^2} \geq \frac{1}{N}\|BR_D\tilde{g}\|_{L^2} \geq \frac{c}{N}, \qquad \|{}^b\mathrm{Op}(\psi_i)\tilde{g}\|_{L^2} \leq C\|\tilde{g}\|_{L^2} = o(1). \tag{13.20}$$

Let $g := {}^b\mathrm{Op}(\psi_i)\tilde{g}$. then, by Lemma 13.16 there is u_1 satisfying

$$\begin{cases} (-\hbar^2\Delta - 1)u_1 = \hbar g + O(\hbar^\infty)_{L^2_{\mathrm{comp}}} & \text{in } \Omega^+ \\ u_1 \text{ outgoing} \\ Q\Lambda^{-1}\hbar D_\nu u_1 - u_1 = 0 & \text{on } \Gamma, \end{cases}$$

where $Q \in \Psi^{\mathrm{comp}}(\Gamma)$ with $\sigma(Q) \geq 0$, $\mathrm{WF}(Q) \subset V^2_{q_i}$ and $\mathrm{WF}(Q - \Lambda^{-1}) \cap V^1_{q_i} = \emptyset$, where $\Lambda^{-1} \in \Psi^{-1}$ is from Lemma 13.16. Moreover, for any $\chi \in C^\infty_c(\overline{\Omega^+})$,

$${}^b\mathrm{WF}(u_1) \subset {}^b\mathrm{WF}(g) \cup \bigcup_{q \in {}^b\mathrm{WF}(g)} \overline{\bigcup_{0 \leq t < T_{V^1_{q_i}}(q)} \varphi^\Omega_t(q)}, \qquad \mathrm{WF}(u_1|_\Gamma) \subset V^2_q,$$

and

$$\|\chi u_1\|_{H^2_\hbar} \leq c\|g\|_{L^2}. \tag{13.21}$$

Put $u_2 := R_D g - u_1 + R_D((-\hbar^2\Delta - 1)u_1 - g)$. Then

$$(-\hbar^2\Delta - 1)u_2 = 0 \text{ in } \Omega^+, \qquad u_2|_\Gamma = -u_1|_\Gamma, \qquad u_2(\hbar) \text{ is outgoing,}$$

and, using Assumption 13.18 and that $(-\hbar^2\Delta - 1)u_1 - g = O(\hbar^\infty)_{L^2_{\mathrm{comp}}}$,

$$u_2 = R_D g - u_1 + O(\hbar^\infty)_{H^2_{\hbar,\mathrm{loc}}}.$$

In particular, by (13.20) and (13.21)

$$\|Bu_2\|_{L^2} \geq \|BR_D g\| - \|Bu_1\|_{L^2} - O(\hbar^\infty) \geq c - C\|\tilde{g}\|_{L^2} + O(\hbar^\infty). \tag{13.22}$$

Furthermore, since $\mathrm{WF}(u_1|_\Gamma) \subset \mathcal{H}$, for $\chi \in C^\infty_c(\overline{\Omega^+})$ with $\mathrm{supp}(1 - \chi) \cap \Gamma = \emptyset$,

$$\|u_2|_\Gamma\|_{L^2} = \|u_1|_\Gamma\|_{L^2} \leq \|\chi u_1\|_{L^2} + C\|g\|_{L^2} + O(\hbar^\infty)\|g\|_{L^2} \leq C\|\tilde{g}\|_{L^2} + O(\hbar^\infty)\|\tilde{g}\|_{L^2}. \tag{13.23}$$

Now,

$$u_2 = \mathcal{S}\ell\partial_\nu u_2|_\Gamma - \mathcal{D}\ell u_2|_\Gamma.$$

Therefore, using

$$c - C\|\tilde{g}\|_{L^2} - O(\hbar^\infty) \leq \|Bu_2\|_{L^2} \leq \|B\mathcal{S}\ell\partial_\nu u_2|_\Gamma\| + \|B\mathcal{D}\ell u_2|_\Gamma\| \leq C\|\hbar\partial_\nu u_2\|_{L^2} + C\|\tilde{g}(h)\|_{L^2}. \tag{13.24}$$

In particular, since $\|\tilde{g}\|_{L^2} = o(1)$, using (13.22), (13.23), and (13.24), for h small enough

$$c\frac{\|u_2|_\Gamma\|_{L^2}}{\|\tilde{g}\|_{L^2}} \leq c \leq \|\hbar\partial_\nu u_2\|_{L^2} = \|P^+_{\mathrm{DtN}}u_2|_\Gamma\|_{L^2}$$

■

Lemma 13.27 (From Helmholtz quasimode data to BIE quasimode data for A_k) *Let $\Omega^- \Subset \mathbb{R}^d$ open with smooth boundary and connected complement $\Omega^+ := \mathbb{R}^d \setminus \overline{\Omega}_-$ and suppose that Assumptions 1.1 and 13.18 hold. Let $\mu = 1$, $B \in \Psi^{\mathrm{comp}}(\Omega^+)$ satisfy (13.15) and suppose there are $u \in H^2_{\hbar,\mathrm{loc}}(\Omega^+)$, $g \in L^2_{\mathrm{comp}}(\Omega^+)$ satisfying (13.16) and (13.17) with*

$$\mathcal{O} = \mathcal{H}_0 := \{(x', \xi') \in \mathcal{H}, :\ |\xi'| \neq 0\}, \qquad \mathcal{B} = \emptyset.$$

Then for all $\hbar \notin \tilde{\mathcal{J}}$, there are $v, f \in L^2(\Gamma)$ such that f satisfies (13.14) and

$$A_k v = f, \qquad \|f\|_{L^2(\Gamma)} \leq C\|\tilde{g}\|_{L^2(\Omega^+)}\|v\|_{L^2(\Gamma)}.$$

Proof. By Lemma 13.26, there is $g_1 \in L^2(\Gamma)$ with $\mathrm{WF}(g_1) \subset \mathcal{H}_0$ such that

$$\|P^+_{\mathrm{DtN}} g_1\|_{L^2} \geq \|Bu\|_{L^2}/\|\tilde{g}\|_{L^2}\|g_1\|_{L^2},$$

and

$$\|BG_D g_1\|_{L^2} \geq c > 0.$$

Thus, Lemma 13.22 implies that

$$\|P^{-,\eta}_{\mathrm{ItD}} P^+_{\mathrm{DtN}} g_1\|_{L^2} \geq c\|\tilde{g}_1\|^{-1}_{L^2}\|g_1\|_{L^2}.$$

In particular,

$$\|A_k^{-1} u_2|_\Gamma\|_{L^2} = \|(I - P^{-,\eta}_{\mathrm{ItD}}(P^+_{\mathrm{DtN}} - i\hbar\eta))u_2|_\Gamma\|_{L^2} \geq (c\|\tilde{g}\|^{-1}_{L^2} - C)\|g_1\|_{L^2},$$

which completes the proof. ∎

We give two versions of Lemma 13.27 for $A_{k,\eta}$. The first is simpler but has more restrictive assumptions.

Lemma 13.28 (From Helmholtz quasimode data to BIE quasimode data for A'_k:I) *Let $\Omega^- \Subset \mathbb{R}^d$ open with smooth boundary and connected complement $\Omega^+ := \mathbb{R}^d \setminus \overline{\Omega}_-$ and suppose that Assumptios 1.1 and 13.18 hold. Let $\mu = -1$, $B \in \Psi^{\mathrm{comp}}(\Omega^+)$ satisfy (13.15) and*

$$\mathrm{WF}(B) \cap \bigcup_{t\geq 0} \varphi^{\mathbb{R}^d}_{-t}(N^*\Gamma), \tag{13.25}$$

and suppose there are $u \in H^2_{\hbar,\mathrm{loc}}(\Omega^+)$, $g \in L^2_{\mathrm{comp}}(\Omega^+)$ satisfying (13.16) and (13.17) with $\mathcal{O} = \mathcal{H}$ Then for all $\hbar \notin \tilde{\mathcal{J}}$, there are $v, f \in L^2(\Gamma)$ such that f satisfies (13.14) and

$$A'_k v = f, \qquad \|f\|_{L^2(\Gamma)} \leq C\|\tilde{g}\|_{L^2(\Omega^+)}\|v\|_{L^2(\Gamma)}.$$

Proof. By Lemma 13.26 with $\mu = -1$, there is $g_1 \in L^2(\Gamma)$ with $\mathrm{WF}(g_1) \subset \mathcal{H}$ such that

$$\|P^-_{\mathrm{DtN}} g_1\|_{L^2} \geq \|Bu\|_{L^2}/\|\tilde{g}\|_{L^2}\|g_1\|_{L^2},$$

Thus, Lemma 13.23 implies that there is $\tilde{B} \in \Psi(\Gamma)$ with $\mathrm{WF}(\tilde{B}) \cap \{\xi' = 0\} = \emptyset$ such that

$$\|\tilde{B} P^{+,\eta}_{\mathrm{ItD}} P^-_{\mathrm{DtN}} g_1\|_{L^2} \geq c\|\tilde{g}_1\|^{-1}_{L^2}\|g_1\|_{L^2}.$$

In particular, since $(P^{+,\eta}_{\mathrm{ItD}} P^-_{\mathrm{DtN}})^* = P^+_{\mathrm{DtN}} P^{-,\eta}_{\mathrm{ItD}}$, we have

$$\|P^+_{\mathrm{DtN}} P^-_{\mathrm{ItD}} \tilde{B}\|_{L^2 \to L^2} = \|\tilde{B} P^+_{\mathrm{ItD}} P^-_{\mathrm{DtN}}\|_{L^2 \to L^2} \geq \frac{c}{\|\tilde{g}\|_{L^2}}.$$

Since

$$\|(A'_{k,\eta})^{-1}\tilde{B}\|_{L^2(\Gamma)\to L^2(\Gamma)} \geq \|(P^+_{\mathrm{DtN}} P^{-,\eta}_{\mathrm{ItD}})\tilde{B}\|_{L^2(\Gamma)\to L^2(\Gamma)} - C,$$

this completes the proof. ∎

In some settings it is useful to have a more refined estimate than Lemma 13.28 provides. In this case the dynamics in the interior of Ω^- also play a role.

Lemma 13.29 (From Helmholtz quasimode data to BIE quasimode data for A'_k:II) *Let $\Omega^- \Subset \mathbb{R}^d$ open with smooth boundary and connected complement $\Omega^+ := \mathbb{R}^d \setminus \overline{\Omega}_-$ and suppose that Assumptions 1.1 and 13.18 hold. Let Γ_- be the incoming set for Ω^+ and define*

$$\mathcal{O} := \{(x', \xi') \in \mathcal{H} \setminus \Gamma_- \,:\, |\xi'| > 0\}, \qquad \mathcal{B} := \Gamma_- \cap \{\xi' = 0\}.$$

Suppose that there is $\tilde{g} \in H^{3/2}_\hbar(\Gamma)$ satisfying, $\mathrm{WF}(\tilde{g}) \subset \mathcal{H}_0$, (13.16) with $\mathcal{O}$ and $\mathcal{B}$ as above, and

$$\|\tilde{g}\|_{H^{3/2}_\hbar} = o(1), \qquad \|\hbar P^+_{\mathrm{DtN}}\tilde{g}\|_{L^2(\Gamma)} = 1.$$

Then for all $\hbar \notin \mathcal{J}$, there are $v, f \in L^2(\Gamma)$ such that f satisfies (13.14) and

$$A'_k v = f, \qquad \|f\|_{L^2(\Gamma)} \leq C\frac{\|\tilde{g}\|_{L^2(\Gamma)}}{\|\hbar P^+_{\mathrm{DtN}}\tilde{g}\|_{L^2(\Gamma)}}, \qquad \|v\|_{L^2(\Gamma)} = 1.$$

Proof. Let $q \in \mathrm{WF}(\tilde{g})$ and $0 < T_q < T_{\mathcal{B}\cup\{q\}}$ such that $\varphi^{\Omega^-}_{T_q}(q) \in \mathcal{O} \setminus (\mathcal{B} \cup \{q\})$.

Let $V_q^1 \Subset V_q^2 \subset \mathcal{O} \cap (B(\varphi^{\Omega+}_{T_q}(q), \epsilon) \setminus (\mathcal{B} \cup \{q\}))$ be a neighborhood of $\varphi^{\Omega^+}_{T_q}(q)$ and U_q neighborhoods of q such that

$$\inf_{q' \in U_q} T^{\mu}_{V_q^1 < (\mathcal{B}\cup\{q'\})}(q') < \infty.$$

Such a neighborhood exists because $\varphi^{\Omega^+}_t$ is continuous and $\varphi^{\Omega^+}_{\mu T(q)}(q) \in \mathcal{H} \setminus (\mathcal{B} \cup \{q\})$.

Since $\mathrm{WF}(\tilde{g}) \cap \mathcal{Z}$ is compact, there are $\{q_1, \dots q_N\}$ such that

$$\mathrm{WF}(\tilde{g}) \subset \bigcup_{i=1}^{N} U_{q_i}.$$

Let $\{\psi_j\}_{j=1}^N \in C_c^\infty(T^*\Gamma)$ with $\sum_j \psi_j \equiv 1$ on $\mathrm{WF}(\tilde{g}) \cap \mathcal{Z}$ and $\operatorname{supp} \psi_j \subset U^1_{q_j}$. Then, by Lemma 13.19,

$$\|P^+_{\mathrm{DtN}} u - \sum_j P^+_{\mathrm{DtN}} Op(\psi_j)\tilde{g}\|_{H_\hbar^{1/2}} \leq C\|\tilde{g}\|_{H_\hbar^{3/2}}.$$

In particular, there is $i \in \{1, \dots, N\}$ such that

$$\|P^+_{\mathrm{DtN}} \mathrm{Op}(\psi_i)\tilde{g}\|_{L^2} \geq \frac{1}{N}\|P^+_{\mathrm{DtN}}\tilde{g}\|_{L^2} \geq \frac{c}{N}, \qquad \|\mathrm{Op}(\psi_i)\tilde{g}\|_{L^2} \leq C\|\tilde{g}\|_{L^2} = o(1).$$

Let $Q \in \Psi^{\mathrm{comp}}(\Gamma)$ with $\sigma(Q) \geq 0$

$$\mathrm{WF}(Q) \subset V^1_{q_i}, \qquad \mathrm{WF}(Q - I) \cap V^2_{q_i} = \emptyset,$$

and $Q_0 \in \Psi(\Gamma)$ with $0 \leq \sigma(Q_0) \leq 1$,

$$\mathrm{WF}(Q_0) \cap \bigcup_{q \in U_{q_i}} \bigcup_{0 \leq t \leq T_{V^1_{q_i}}(q)} \varphi^{\Omega_-}_t(q) \cap \Gamma_- = \emptyset, \tag{13.26}$$

and $\mathrm{WF}(Q_0 - I) \cap \{\xi' = 0\} = \emptyset$. Then, by Lemma 13.16, there is u satisfying

$$\begin{cases} (-\hbar^2\Delta - 1)u = O(\hbar^\infty \|g\|_{L^2})_{L^2} & \text{in } \Omega^- \\ ((Q_0 - I)Q\tilde{\Lambda}^{-1} + Q_0\eta^{-1})\hbar D_\nu u - u = g & \text{on } \tilde{\Gamma}, \end{cases}$$

where $\tilde{\Lambda} \in \Psi^{-1}$ with Λ^{-1} as in Lemma 13.16

$$\mathrm{WF}(\tilde{\Lambda}^{-1} - \Lambda^{-1}) \cap \mathrm{WF}(Q) = \emptyset, \qquad \sigma(\tilde{\Lambda}^{-1}) \geq c\langle\xi'\rangle^{-1} > 0.$$

Moreover,

$$\begin{aligned} &\|u\|_{L^2(\tilde{\Omega}_-)} + \|u|_\Gamma\|_{H_\hbar^{3/2}} + \|\hbar\partial_\nu u|_\Gamma\|_{H_\hbar^{1/2}} \leq C\|g\|_{H_\hbar^{3/2}}, \\ &{}^b\mathrm{WF}(u) \subset U_{q_i} \cup \bigcup_{q' \in U_{q_i}} \bigcup_{0 \leq t \leq T^1_{V^1_{q_i}}(q')} \varphi^{\Omega_-}_t(U_{q_i} \cap \mathcal{Z}), \end{aligned}$$

and using $\mathrm{WF}(Q_0) \cap \mathrm{WF}(g) = \emptyset$ and (13.26)

$$\mathrm{WF}((\hbar D_\nu - \eta)u|_\Gamma) \cap \{\xi' = 0\} = \emptyset, \qquad \mathrm{WF}(u|_{\tilde{\Gamma}} - g) \cap \Gamma_- = \emptyset$$

Using Lemma 13.19 this implies

$$\|P^+_{\mathrm{DtN}} P^{-,\eta}_{\mathrm{ItD}}(\hbar D_\nu - \eta)u|_\Gamma\| \geq \|\hbar P^+_{\mathrm{DtN}} g\|_{L^2(\Gamma)} - C\|g\|_{L^2(\Gamma)},$$

and

$$\|(\hbar D_\nu - \eta)u\|_{L^2(\Gamma)} \leq C\|g\|_{L^2(\Gamma)}.$$

Thus, putting $f = (\hbar D_\nu - \eta)u$, we have

$$\begin{aligned} \|(A'_{k,\eta})^{-1} f\|_{L^2(\Gamma)} &\geq \|(P^+_{\mathrm{DtN}} P^{-,\eta}_{\mathrm{ItD}}) f\|_{L^2(\Gamma)} - C\|f\|_{L^2(\Gamma)} \\ &\geq c\|\hbar P^+_{\mathrm{DtN}}\tilde{g}\|_{L^2(\Gamma)} / \|\tilde{g}\|_{L^2(\Gamma)} \|f\|_{L^2(\Gamma)} - C\|f\|_{L^2(\Gamma)}, \end{aligned}$$

which proves the claim. ■

13.7 Pollution in concrete situations

13.7.1 A good quasimode on the diamond domain: Proof of Theorem 2.12

In this section, we construct a quasimode that yields quantitative pollution. Let $0 < \epsilon < \pi/2$ and $\Omega_0 \subset (-\pi+\epsilon, \pi-\epsilon) \times (-\pi+\epsilon, \pi-\epsilon)$ be convex with smooth boundary and

$$\partial\Omega_0 \supset \cup_{\pm}\big(\{\pm(\pi-\epsilon)\} \times (-\pi+2\epsilon, \pi-2\epsilon) \cup (-\pi+2\epsilon, \pi-2\epsilon) \times \{\pm(\pi-\epsilon)\}.$$

Then, define

$$\Omega^- := \Omega_0 + \{(-2\pi+\epsilon, 0), (0, 2\pi-\epsilon), (0, -2\pi+\epsilon), (0, 2\pi-\epsilon)\}. \tag{13.27}$$

Observe that Ω^- has connected complement and smooth boundary, Γ, such that

$$\Omega^- \cap \{(x,y) \in [-\pi,\pi] \times [-\pi,\pi]\} = \emptyset,$$
$$[-\pi+2\epsilon, \pi-2\epsilon] \times \{-\pi,\pi\} \cup \{-\pi,\pi\} \times [-\pi+2\epsilon, \pi-2\epsilon] \subset \Gamma.$$

Now, let $0 < \delta < \epsilon$, $\chi \in C_c^\infty((2\delta, 2\pi-2\delta); [0,1])$, and define

$$u := \Big[\chi(y-x)e^{in(x+y)} - \chi(2\pi-y-x)e^{in(x-y)} + \chi(x-y)e^{in(-x-y)} - \chi(2\pi+y+x)e^{in(-x+y)}\Big]1_{[-\pi,\pi]}(x)1_{[-\pi,\pi]}(y)$$

Notice that

$$\begin{aligned}
\operatorname{supp}\chi(y-x) \cap [-\pi,\pi] \times [-\pi,\pi] &\subset [-\pi, \pi-2\delta) \times (-\pi+2\delta, \pi] \\
\operatorname{supp}\chi(2\pi-y-x) \cap [-\pi,\pi] \times [-\pi,\pi] &\subset (-\pi+2\delta, \pi] \times (-\pi+2\delta, \pi] \\
\operatorname{supp}\chi(x-y) \cap [-\pi,\pi] \times [-\pi,\pi] &\subset (-\pi+2\delta, \pi] \times [-\pi, \pi-2\delta) \\
\operatorname{supp}\chi(2\pi+y+x) \cap [-\pi,\pi] \times [-\pi,\pi] &\subset [-\pi, \pi-2\delta) \times [-\pi, \pi-2\delta),
\end{aligned}$$

and $u|_\Gamma = 0$. Hence, $u \in H_0^1(\Omega^+) \cap C_c^\infty(\overline{\Omega^+})$.

Set $\hbar_n = \frac{1}{\sqrt{2}n}$,

$$\begin{aligned}
\tilde{g} := \frac{\hbar_n}{2}\Big[&\chi''(y-x)e^{in(x+y)} - \chi''(2\pi-y-x)e^{in(x-y)} \\
&+ \chi''(x-y)e^{in(-x-y)} - \chi''(2\pi+y+x)e^{in(-x+y)}\Big]1_{[-\pi,\pi]}(x)1_{[-\pi,\pi]}(y),
\end{aligned}$$

so that for $|\epsilon| \leq C$,

$$(-\hbar_n^2\Delta - 1 + \epsilon\hbar_n^2)u = \hbar_n^2(\tilde{g} + \epsilon)u.$$

Observe that

$$\begin{aligned}
({}^b\mathrm{WF}(u) \cup {}^b\mathrm{WF}(g)) \cap T^*\{x = -\pi\} &\subset \{(y, \tfrac{1}{\sqrt{2}}) \,:\, y \in \operatorname{supp}\chi(y+\pi)\} \\
({}^b\mathrm{WF}(u) \cup {}^b\mathrm{WF}(g)) \cap T^*\{y = \pi\} &\subset \{(x, \tfrac{1}{\sqrt{2}}) \,:\, x \in \operatorname{supp}\chi(\pi-x)\} \\
({}^b\mathrm{WF}(u) \cup {}^b\mathrm{WF}(g)) \cap T^*\{x = \pi\} &\subset \{(y, -\tfrac{1}{\sqrt{2}}) \,:\, y \in \operatorname{supp}\chi(\pi-y)\} \\
({}^b\mathrm{WF}(u) \cup {}^b\mathrm{WF}(g)) \cap T^*\{y = -\pi\} &\subset \{(x, -\tfrac{1}{\sqrt{2}}) \,:\, x \in \operatorname{supp}\chi(\pi+x)\} \\
({}^b\mathrm{WF}(u) \cup {}^b\mathrm{WF}(g)) \cap T^*(-\pi,\pi) \times (-\pi,\pi) &\subset \bigcup_{\pm}\{(\xi,\eta) = \pm(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}), \pm(x-y) \in \operatorname{supp}\chi\} \\
&\quad \cup \bigcup_{\pm}\{(\xi,\eta) = \pm(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}), 2\pi \pm (-x-y) \in \operatorname{supp}\chi\},
\end{aligned}$$

and hence

$$\sup_{q \in {}^b\mathrm{WF}(u)} T_{\mathcal{H}_0 < \{q\}} < \infty. \tag{13.28}$$

Moreover, for $\delta > 0$ small enough, since Ω_0 is convex,

$$^b\mathrm{WF}(u) \cap \bigcup_{t \geq 0}(S^*\Gamma \cup N^*\Gamma) = \emptyset. \tag{13.29}$$

Hence, there is B satisfying (13.15) and (13.25) such that (13.16) and (13.17) hold. Thus, u and g satisfy the hypotheses of both Lemma 13.27 and 13.28. This implies that for $\hbar = \hbar_n/\sqrt{1+\epsilon\hbar_n^2}$, there are $f_1, f_2 \in L^2(\Gamma)$ with f_i satisfying (13.14) such that

$$\|A_k^{-1}g_1\|_{L^2(\Gamma)} \geq c\hbar^{-1}\|g_1\|_{L^2(\Gamma)}, \qquad \|(A_k')^{-1}g_2\|_{L^2(\Gamma)} \geq c\hbar^{-1}\|g_2\|_{L^2(\Gamma)}.$$

Hence, Theorems 2.12 and 2.13 follow from Theorem 2.10.

13.7.2 Non-quantitative pollution

To prove the non-quantitative pollution results, we first find appropriate data g with wavefront set near a point in $\Gamma_- \setminus K$ such that $R_D\hbar g$ is "large". Using the fact that $\Gamma_- \setminus K$ avoids the normal bundle (Lemma 13.3), we then apply Lemmas 13.27 and 13.29 to produce quasimodes for A_k and A_k', respectively.

Lemma 13.30 (Growing Dirichlet resolvents with data near a point in Γ_-) *Let $\Omega^- \Subset \mathbb{R}^d$ with smooth boundary, Γ and connected complement $\Omega^+ := \mathbb{R}^d \setminus \overline{\Omega^-}$ and suppose that $K \neq \emptyset$ and Assumption 13.18 holds. Then, for any $q \in \Gamma_- \setminus T^*\Gamma$ and $\delta > 0$, there are $g \in L^2_{\mathrm{comp}}(\Omega^+)$ and $\chi \in C_c^\infty(\overline{\Omega^+})$ such that*

$$\mathrm{WF}(g) \subset \{\varphi_t^{\Omega^+}(q) \,:\, -2\delta < t < -\delta\}, \qquad \|g\|_{L^2} = 1, \tag{13.30}$$

and

$$\lim_{\hbar\to 0^+, \hbar\notin\tilde{\mathcal{J}}} \|\chi R_D \hbar g\|_{L^2} = \infty. \tag{13.31}$$

Proof. Let $(x_0,\xi_0) = q \in \Gamma_- \setminus (K \cup T^*\Gamma)$. Without loss of generality, we may assume $x_0 = 0$ and $\xi_0 = (1,0,\dots,0)$.

We start by finding $\delta > 0$ and $v \in H^2_{\hbar,\mathrm{comp}}(\Omega^+)$, $g \in L^2_{\mathrm{comp}}$ such that $B(0,\delta) \subset \Omega^+$, and for $\chi \in C_c^\infty(B(0,\delta))$ with $0 \notin \mathrm{supp}(1-\chi)$,

$$\begin{aligned} (-\hbar^2\Delta - 1)v = \hbar g, \qquad \|g\|_{L^2} \leq C, \qquad \|\chi v\|_{L^2} \geq c \\ \mathrm{WF}(g) \subset \{(x_1,0,\dots,0),(1,0,\dots,0) \,:\, \delta < |x_1| < 2\delta\}, \\ \mathrm{WF}(v) \subset \{(x_1,0,\dots,0),(1,0,\dots,0) \,:< |x_1| < 2\delta\}. \end{aligned} \tag{13.32}$$

To do this, recall that by [112, Theorem 12.3], there is a neighborhood U of $(0,0)$, a symplectomorphism $\kappa : U \to T^*\mathbb{R}^d$ such that $\kappa^*(|\xi|^2 - 1) = \xi_1$, operators T_1, T_2 and such that for any u with $\mathrm{WF}(u) \subset U$, and v with $\mathrm{WF}(v) \subset \kappa(U)$,

$$\begin{gathered} T_1(-\hbar^2\Delta - 1)T_2 u = 2\hbar D_{x_1}u + O(\hbar^\infty)_{\Psi^{-\infty}}u, \\ T_1T_2u = u + O(\hbar^\infty)_{\Psi^{-\infty}}u, \qquad T_2T_1v = v + O(\hbar^\infty)_{\Psi^{-\infty}}v. \\ \mathrm{WF}(T_2u) \subset \kappa(\mathrm{WF}(u)). \end{gathered}$$

Let $\chi \in C_c^\infty((-2,2))$ with $\mathrm{supp}(1-\chi) \cap [-1,1] = \emptyset$ near 0. Define

$$u := \hbar^{-\frac{d-1}{4}} e^{-|x'|^2/2\hbar}\chi(\delta^{-1}x_1).$$

Then,

$$2\hbar D_{x_1}u = 2\hbar\delta^{-1}\hbar^{-\frac{d-1}{4}}e^{-|x'|^2/2\hbar}\chi'(\delta^{-1}x_1),$$

so that

$$\mathrm{WF}(u) \subset \{|x_1| < 2\delta, x' = 0, \xi = 0\}, \qquad \mathrm{WF}(\hbar D_{x_1}u) \subset \{\delta < |x_1| < 2\delta, x' = 0, \xi = 0\}.$$

Putting $v := T_2 u$, we have

$$\begin{gathered} \mathrm{WF}(v) = \kappa(\{|x_1| < 2\delta, x' = 0, \xi = 0\}) \subset \{|x_1| < 2\delta, x' = 0, \xi = (1,0,\dots,0)\}, \\ \mathrm{WF}((-\hbar^2\Delta - 1)v) \subset \kappa(\{\delta < |x_1| < 2\delta, x' = 0, \xi = 0\}) \subset \{\delta < |x_1| < 2\delta, x' = 0, \xi = (1,0,\dots,0)\}, \end{gathered}$$

and for any $\psi \in C_c^\infty(B(0,\delta))$, with $\operatorname{supp}(1-\psi)\cap B(0,\delta/2)=\emptyset$,

$$\|(-\hbar^2\Delta-1)v\|_{L^2}\leq C\hbar, \qquad \|\psi v\|_{L^2}\geq c>0.$$

By inserting appropriate cutoffs equal to one near 0, we may assume that v and hence also $g := \hbar^{-1}(-\hbar^2\Delta-1)v$ have compact support. Hence, we have achieved (13.32).

Let $\psi \in C_c^\infty(0,1)$ with $\operatorname{supp}(1-\psi)\cap(\delta,2\delta)=\emptyset$. We claim that there is $\chi \in C_C^\infty(\overline{\Omega^+})$ such that

$$\frac{\|g\|_{L^2}}{\|\chi R_D\hbar\psi(x_1)g\|_{L^2}}=o(1). \tag{13.33}$$

Suppose by contradiction that for any $\psi\in C_c^\infty(\overline{\Omega^+})$, there are $C>0$ and $\hbar_n\to 0$ such that

$$\|\psi R_D(\hbar_n)\hbar_n\psi(x_1)g(\hbar_n)\|_{L^2}\leq C\|g(\hbar_n)\|_{L^2}.$$

Then, without loss of generality we may assume $\|g\|_{L^2}=1$ and, setting $w:=v-R_Dh\psi g$, up to extracting a further subsequence, we may assume that w has defect measure μ. By Lemma 13.21,

$$(0,(1,0,\dots,0))\notin {}^b\mathrm{WF}(\psi R_D(\hbar_n)\hbar_n\psi(x_1)g(\hbar_n))$$

and hence, by (13.32) and, using Lemma 13.21 again

$$\mu(\{((x_1,0,\dots,0),(1,0,\dots,0)\,:\,|x_1|<\delta\})>c>0.$$

Now,

$$(-\hbar^2\Delta_g-1)w=(1-\psi)\hbar g$$

and hence by Lemma 13.11, μ is invariant under $\varphi_t^{\Omega^+}$ away from $\{((x_1,0,\dots,0),(1,0,\dots,0))\,:\,-2\delta<x_1<-\delta\}$. Since $q\in\Gamma_-$, this implies that there is $R>0$ such that

$$\mu(|x|<R)=\infty,$$

a contradiction. Hence (13.33) holds, completing the proof. ■

Lemma 13.31 (Quasimodes for A_k under geometric assumptions) *Let $\Omega^-\Subset\mathbb{R}^d$ with smooth boundary, Γ, and connected complement $\Omega^+:=\mathbb{R}^d\setminus\overline{\Omega^-}$ and suppose that $K\neq\emptyset$, Assumption 13.18 holds and*

$$K\cap\bigcup_{t\geq 0}\varphi_t^{\mathbb{R}^d}(S^*\Gamma)=\emptyset. \tag{13.34}$$

Then, if Assumption 1.1 holds there are $v,f\in L^2(\Gamma)$ such that f satisfies (13.14),

$$A_kv=f,\qquad \|v\|_{L^2}=1,\qquad \lim_{h\to 0^+,h\notin\mathcal{J}}\|f\|_{L^2}=0.$$

Moreover, there is $B\in\Psi^{\mathrm{comp}}(\Omega_+)$ satisfying (13.15) such that for all $q\in\Gamma_-\setminus(K\cup T^*\Gamma)$, there is g satisfying (13.30) and

$$\lim_{\hbar\to 0^+,\hbar\notin\tilde{\mathcal{J}}}\|BR_D\hbar g\|_{L^2}=\infty.$$

Proof. By Lemma 13.2 there is $q\in\Gamma_-\setminus K$ and, since $|\varphi_{-t}^{\Omega_+}(q)|\to\infty$ as $t\to\infty$, we may assume without loss of generality that $q\notin T^*\Gamma$. Then, by Lemma 13.30 there is g satisfying (13.30) and (13.31). Since $q\in\Gamma_-$, there is $T>0$ such that $\varphi_t^{\Omega_+}(q)\in T^*\Gamma\cap\Gamma_-\setminus K$ and hence, by Lemma 13.3 $\varphi_t^{\Omega^+}(q)\notin\{\xi'=0\}$. Using that $K\cap S^*\Gamma=\emptyset$ and increasing t if necessary, there is $t>0$ such that $\varphi_t^{\Omega^+}(q)\in\mathcal{H}\cap\{\xi'\neq 0\}$.

In particular,

$$\sup_{q\in{}^b\mathrm{WF}(g)}T^1_{\mathcal{H}_0<\{q\}}(q)<\infty.$$

Now, by (13.34) for any $R>0$ there are $U\Subset U_1\Subset T^*\Omega^+$ open with

$$U_1\cap\bigcup_{t\geq 0}\varphi_t^{\mathbb{R}^d}(S^*\Gamma)=\emptyset$$

and $T>0$ such that for all $q\in\mathcal{Z}\cap B(0,R)$, there is $0\leq s\leq T$ such that $\varphi_{-s}^{\Omega^+}(q)\in U\cup\{|x|>R\}$. In particular, by Theorem 13.12 letting $B\in\Psi^{\mathrm{comp}}(\Omega^+)$ with $\mathrm{WF}(I-B)\cap U=\emptyset$ and $\mathrm{WF}(B)\subset U_1$,

$$\|\chi R_D\hbar g\|_{L^2}\leq C\|BR_D\hbar g\|_{L^2}+\|g\|_{L^2}.$$

In particular, using (13.31), we have

$$\lim_{\hbar\to0^+,\hbar\notin\tilde{\mathcal{J}}}\|BR_D\hbar g\|_{L^2}=\infty.$$

Thus, we have verified the hypotheses of Lemma 13.27 which completes the proof. ∎

Lemma 13.32 (Quasimodes for A_k' under geometric assumptions) *Let $\Omega^-\Subset\mathbb{R}^d$ with smooth boundary, Γ and connected complement $\Omega^+:=\mathbb{R}^d\setminus\overline{\Omega^-}$, that $K\neq\emptyset$, and Assumption 13.18 and (13.34) hold. Suppose in addition that there is $q_0\in\Gamma_-\cap\mathcal{H}\setminus K$ such that*

$$T^1_{\mathcal{H}\setminus(\Gamma_-\cup\{\xi'=0\})<\Gamma_-\cap\{\xi'=0\}}(q_0)<\infty. \tag{13.35}$$

Then, if Assumption 1.1 holds, there are $v,f\in L^2(\Gamma)$ such that f satisfies (13.14)

$$A_k'v=f,\qquad \|v\|_{L^2}=1,\qquad \lim_{\hbar\to0^+,\hbar\notin\tilde{\mathcal{J}}}\|f\|_{L^2}=0.$$

Proof. Since $q_0\in\mathcal{H}$, there is $t>0$ such that $\varphi_{-t}^{\Omega_+}(q_0)\in\Gamma_-\setminus(K\cup T^*\Gamma)$. Hence, by Lemma 13.31, there are B satisfying (13.15) and g satisfying (13.30) with $q=\varphi_{-t}^{\Omega^+}(q_0)$ such that

$$\lim_{\hbar\to0^+,\hbar\notin\tilde{\mathcal{J}}}\|BR_D\hbar g\|_{L^2}=\infty.$$

Now, by construction

$$\sup_{q\in{}^b\mathrm{WF}(g)}T^1_{\mathcal{H}_0<\{q\}}(q)<\infty,$$

and hence by Lemma 13.26 for any $\epsilon>0$, there is $g_1\in L^2(\Gamma)$ with

$$\|P^+_{\mathrm{DtN}}g_1\|_{L^2}=1,\qquad \|g_1\|_{L^2}=o(1),\qquad \mathrm{WF}(g_1)\subset B(q_0,\epsilon).$$

By (13.35), and upper semicontinuity of $T^1_{\mathcal{H}\setminus(\Gamma_-\cup\{\xi'=0\})<\Gamma_-\cap\{\xi'=0\}}$ there is $\epsilon>0$ small enough such that

$$\sup_{q'\in\mathrm{WF}(g_1)}T^1_{\mathcal{H}\setminus(\Gamma_-\cup\{\xi'=0\})<\Gamma_-\cap\{\xi'=0\}}(q')<\infty$$

and hence the Lemma follows from Lemma 13.29. ∎

The following two Theorems follow directly from Lemmas 13.31, 13.32, and 13.25, and Theorem 2.10.

Theorem 13.33 *Let $\Omega^-\Subset\mathbb{R}^d$ with smooth boundary, Γ, and connected complement $\Omega^+:=\mathbb{R}^d\setminus\overline{\Omega^-}$ and suppose that $K\neq\emptyset$, Assumptions 1.1 and 13.18, and (13.34) hold and*

$$\textit{either}\qquad p=0\qquad \textit{or}\qquad P_{\mathrm{inv}}\leq p+1.$$

Then the assumptions of Theorem 2.10 with $\mathcal{A}=A_k$ hold for some $\epsilon_0>0$, $\Xi_0=1$, all $\hbar\notin\tilde{\mathcal{J}}$, and some $\alpha=o(1)$, $\beta=O(\hbar^\infty)$.

Theorem 13.33 implies Theorem 2.15.

Theorem 13.34 *Let $\Omega^-\Subset\mathbb{R}^d$ with smooth boundary, Γ, and connected complement $\Omega^+:=\mathbb{R}^d\setminus\overline{\Omega^-}$, suppose that $K\neq\emptyset$ and Assumptions 1.1 and 13.18 hold, (13.34), there is $q_0\in\Gamma_-\cap\mathcal{H}\setminus K$ such that (13.35) holds, and*

$$\textit{either}\qquad p=0\qquad \textit{or}\qquad P_{\mathrm{inv}}\leq p+1.$$

Then the assumptions of Theorem 2.10 with $\mathcal{A}=A_{k,\eta}'$ hold for some $\epsilon_0>0$, $\Xi_0=1$, all $\hbar\notin\mathcal{J}$, and some $\alpha=o(1)$, $\beta=O(\hbar^\infty)$.

14 Pollution for the BIEs on the disk

We now turn to the study of pollution for BIEs on the unit disk, $B(0,1)\subset\mathbb{R}^2$ and prove Theorems 2.16, and 2.18. We parametrize use $\mathbb{R}/2\pi\mathbb{Z}$ as coordinates on $\partial B(0,1)$ with the coordinate map $\gamma(t)=(\cos(t),\sin(t))$. We record the following description of the single layer, double layer, and hypersingular operators as Fourier multipliers (see, e.g., [69]).

Lemma 14.1 *Let $\Omega^-=B(0,1)$, $\Omega^+:=\mathbb{R}^2\setminus\overline{B(0,1)}$. Then, for any $\tau\neq 0$,*

$$\begin{aligned} S_\tau e^{imt}&=\frac{\pi i}{2}H^{(1)}_{|m|}(\tau)J_{|m|}(\tau)e^{imt}\\ \big(\tfrac{1}{2}I+K'_\tau\big)e^{imt}=\big(\tfrac{1}{2}I+K_\tau\big)e^{imt}&=\frac{\pi i}{2}\tau H^{(1)}_{|m|}(\tau)J'_{|m|}(\tau)e^{imt}\\ H_\tau e^{imt}&=i\tau^2(H^{(1)}_{|m|})'(\tau)J'_{|m|}(\tau)e^{imt}. \end{aligned}\tag{14.1}$$

In particular, $A_{k,\eta}e^{imt}=A'_{k,\eta}e^{imt}=\lambda_m e^{imt}$ and $B_{k,\mathrm{reg}}e^{imt}=B'_{k,\mathrm{reg}}e^{imt}=\mu_m e^{imt}$ with

$$\begin{aligned} \lambda_m&:=\frac{\pi}{2}\Big(ikH^{(1)}_{|m|}(k)J'_{|m|}(k)+\eta H^{(1)}_{|m|}(k)J_{|m|}(k)\Big)\\ \mu_m&:=i\eta(1-\frac{\pi i}{2}kH^{(1)}_{|m|}(k)J'_{|m|}(k))+\frac{\pi i}{2}H^{(1)}_{|m|}(ik)J_{|m|}(ik)ik^2(H^{(1)}_{|m|})'(k)J'_{|m|}(k). \end{aligned}$$

We require asymptotic expansions for Bessel functions uniformly for large order [94, §10.20]. Define the decreasing, smooth bijection $\zeta:(0,\infty)\to(-\infty,\infty)$ by

$$\zeta(z):=\begin{cases}\Big(\frac{3}{2}\int_z^1\frac{\sqrt{1-t^2}}{t}dt\Big)^{2/3} & 0<z\le 1\\ -\Big(\frac{3}{2}\int_1^z\frac{\sqrt{t^2-1}}{t}dt\Big)^{2/3} & 1\le z<\infty,\end{cases}$$

and the Airy function by

$$Ai(x):=\frac{1}{\pi}\int_0^\infty\cos\Big(\frac{1}{3}t^3+xt\Big)dt.$$

Then, for any $I\Subset(0,\infty)$, and all $z\in I$,

$$\begin{aligned} J_m(mz)&=\Big(\frac{4\zeta}{1-z^2}\Big)^{\frac14}\Big(m^{-1/3}Ai(m^{2/3}\zeta)(1+O_I(m^{-2}))+O_I(m^{-5/3}Ai'(m^{2/3}\zeta))\Big)\\ H^{(1)}_m(mz)&=2e^{-\pi i/3}\Big(\frac{4\zeta}{1-z^2}\Big)^{\frac14}\Big(m^{-1/3}Ai(e^{2\pi i/3}m^{2/3}\zeta)(1+O_I(m^{-2}))+O_I(m^{-5/3}Ai'(e^{2\pi i/3}m^{2/3}\zeta))\Big)\\ J'_m(mz)&=-\frac{2}{z}\Big(\frac{4\zeta}{1-z^2}\Big)^{-\frac14}\Big(m^{-2/3}Ai'(m^{2/3}\zeta)(1+O_I(m^{-2}))+O_I(m^{-4/3}Ai(m^{2/3}\zeta))\Big)\\ (H^{(1)}_m)'(mz)&=\frac{4e^{-2\pi i/3}}{z}\Big(\frac{4\zeta}{1-z^2}\Big)^{-\frac14}\Big(m^{-2/3}Ai'(e^{2\pi i/3}m^{2/3}\zeta)(1+O_I(m^{-2}))+O_I(m^{-4/3}Ai(e^{2\pi i/3}m^{2/3}\zeta))\Big). \end{aligned}$$

We also recall the following estimates for the Airy function [94, §9.8]

$$|Ai(x)|\le C\langle x\rangle^{-1/4},\qquad |Ai'(x)|\le C\langle x\rangle^{1/4}.$$

Lemma 14.2 *Let $0<\zeta_1<\zeta_2<\dots$ such that $-\zeta_j$ are the zeros of Ai. Then, for all $k_0>0$ and $j\in\mathbb{N}$ there is $C>0$ such that for all $0<\epsilon<1$, $k>k_0$, and $m\in\mathbb{Z}\setminus\{0\}$ satisfying,*

$$|\zeta(k/|m|)+\zeta_j|m|^{-2/3}|<\epsilon k^{-2/3},$$

we have

$$|\mu_m|\le C(k^{-1/3}+(\epsilon+k^{-2/3})|\eta|),\qquad C^{-1}k\le|m|\le Ck.$$

Proof. We first observe that since

$$\|kS_{ik}\|_{L^2\to L^2}\le C,$$

we have

$$|kH^{(1)}_{|m|}(ik)J_{|m|}(ik)| \le C.$$

Therefore, it is enough to check that

$$|k(H^{(1)}_{|m|})'(k)J'_{|m|}(k)| \le Ck^{-1/3}, \qquad |(1 - \frac{\pi i}{2} kH^{(1)}_{|m|}(k)J'_{|m|}(k))| \le C(\epsilon + k^{-2/3}). \tag{14.2}$$

To do this, we first note that ζ^{-1}, is smooth, $|\zeta'| > c > 0$, and $\zeta^{-1}(0) = 1$. Hence,

$$\left|\frac{k}{|m|} - 1\right| \le Ck^{-2/3}, \qquad c < \left|\frac{1 - (\frac{k}{|m|})^2}{4\zeta(k/|m|)}\right| \le C. \tag{14.3}$$

To obtain the first inequality (14.2), we observe that

$$\begin{aligned}
&k(H^{(1)}_{|m|})'(k)J'_{|m|}(k) \\
&= -\frac{8ke^{-2\pi i/3}}{(k/|m|)^2}\Big(\frac{1-(k/|m|)^2}{4\zeta(k/|m|)}\Big)^{1/2}(|m|^{-4/3}Ai'(|m|^{2/3}\zeta(k/|m|))Ai'(e^{2\pi i/3}|m|^{2/3}\zeta(k/|m|)) + O(k|m|^{-2}) \\
&= O(k^{-1/3})
\end{aligned}$$

and to obtain the second inequality in (14.2), we use the fact that [94, §9.2]

$$Ai(e^{2\pi i/3}x)Ai'(x) - (Ai(e^{2\pi i/3}x))'Ai(x) = -\frac{e^{-\pi i/6}}{2\pi}$$

and hence, since $Ai(\zeta_j) = 0$ and Ai is smooth,

$$\begin{aligned}
&(1 - \frac{\pi i}{2} kH^{(1)}_{|m|}(k)J'_{|m|}(k)) \\
&= (1 + 2\pi i e^{-\pi i/3}Ai(e^{2\pi i/3}|m|^{2/3}\zeta(k/|m|))Ai'(|m|^{2/3}\zeta(k/|m|)) + O(k^{-2/3})) \\
&= O(\epsilon + k^{-2/3}).
\end{aligned}$$

■

Lemma 14.3 *Let $0 < \zeta'_1 < \zeta'_2 < \dots$ such that $-\zeta'_j$ are the zeros of Ai'. Then, for all $k_0 > 0$, $c > 0$ there is $C > 0$ such that for all $0 < \epsilon < 1$, $k_0 < k$, $m \in \mathbb{Z} \setminus \{0\}$, and $j \in \mathbb{N}$ satisfying,*

$$ck^{2/3} \le |\zeta'_j| \le 2c^{-1}k^{2/3}, \qquad |\zeta(k/|m|) + \zeta'_j|m|^{-2/3}| < \epsilon k^{-1},$$

we have

$$|\lambda_m| \le C(\epsilon + k^{-1} + k^{-1}|\eta|), \quad C^{-1}k \le |m| \le Ck.$$

Proof. To prove the lemma, we show that

$$|kH^{(1)}_{|m|}(k)J'_{|m|}(k)| \le C(\epsilon + k^{-2/3}), \qquad |H^{(1)}_{|m|}(k)J_{|m|}(k)| \le Ck^{-1}. \tag{14.4}$$

Since $\zeta : (0, \infty) \to (-\infty, \infty)$ is a smooth decreasing bijection, with smooth inverse, and $\zeta(0) = 1$, there are $c, C > 0$ such that

$$1 + c < \frac{k}{|m|} < C, \qquad -2\delta < \zeta(k/|m|) < -\delta.$$

To obtain the first inequality in (14.4), we observe that

$$\begin{aligned}
&kH^{(1)}_{|m|}(k)J'_{|m|}(k) \\
&= -4e^{-\pi i/3}Ai(e^{2\pi i/3}|m|^{2/3}\zeta(k/|m|))Ai'(|m|^{2/3}\zeta(k/|m|)) + O(k|m|^{-2}) \\
&= O(\epsilon + k^{-1}).
\end{aligned}$$

For the second inequality,

$$\begin{aligned}&H^{(1)}_{|m|}(k)J_{|m|}(k)\\&=2e^{-\pi i/3}\Big(\frac{4\zeta(k/|m|)}{1-(k/|m|)^2}\Big)^{\frac12}|m|^{-2/3}Ai(|m|^{2/3}\zeta(k/|m|))Ai(e^{2\pi i/3}|m|^{2/3}\zeta(k/|m|))+O(m^{-2})\\&\leq C|m|^{-1}\leq Ck^{-1}.\end{aligned}$$

■

As an easy Corollary of Lemmas 14.2 and 14.3 we obtain Theorems 2.16 and 2.18.

Proof of Theorem 2.16. By [45, Theorem 2.3], for $|\eta|\sim 1$,

$$\|B_{k,\mathrm{reg}}^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}\leq Ck^{1/3}.$$

Hence, $B_{k,\mathrm{reg}}$ satisfies Assumption 1.3 with $\mathcal{J}=\emptyset$. Now, given $k>k_0$, we must find m such that the hypotheses of Lemma 14.2 hold. Let $|j|\leq C$, and $z_j:=\zeta^{-1}(-k^{-2/3}\zeta_j)$. Then, $z_j=1+O(k^{-2/3})$ and let $m\in\mathbb{Z}$ such that $||m|-k/z_j|\leq\frac12$. Then, since $\frac14 k\leq|m|\leq\frac32 k$,

$$\Big|\frac{k}{|m|}-z_j\Big|\leq\frac{|z_j|}{2|m|}=\frac{1}{2|m|}(1+O(k^{-5/3}))=O(k^{-1}),$$

and hence

$$\zeta(k/|m|)+k^{-2/3}\zeta_j=O(k^{-1}).\tag{14.5}$$

Next,

$$||m|^{-2/3}\zeta_j-k^{-2/3}\zeta_j|=|\zeta_j||(k+O(1))^{-2/3}-k^{-2/3})|=O(k^{-5/3}).\tag{14.6}$$

Combining (14.5) and (14.6) yields

$$|\zeta(k/|m|)+m^{-2/3}\zeta_j|\leq Ck^{-1}.$$

In particular, by Lemma 14.2,

$$|\mu_m|\leq Ck^{-\frac13}.$$

Choosing $\epsilon_0=\frac18$, $\Xi_0=2$, and using that $\frac14 k\leq|m|\leq\frac32 k$, and $-\Delta_\Gamma e^{imt}=m^2e^{imt}$, we have

$$\chi(-\epsilon_0^{-2}k^{-2}\Delta_\Gamma)e^{imt}=0,\qquad(1-\chi(-\Xi_0^{-2}k^{-2}\Delta_\Gamma))e^{imt}=0,$$

and

$$\chi(-\epsilon_0^{-2}k^{-2}\Delta_\Gamma)(B^*_{k,\mathrm{reg}})^{-1}\mu_me^{imt}=\chi(-\epsilon_0^{-2}k^{-2}\Delta_\Gamma)\overline{\mu_m}^{-1}\mu_me^{imt}=0.$$

Hence, there is $C>0$ such that for any seqence $k_n\to\infty$, the hypotheses of Theorem 2.10 with $\alpha_n\leq Ck_n^{-1/3}$ and $\beta_n=0$. This completes the proof. ■

Proof of Theorem 2.18. By the bound on $P^{-,\eta}_{\mathrm{ItD}}$ from [27, Theorem 4.3] and the bound on P^+_{DtN} coming from the fact that $B(0,1)$ is nontrapping, for any k_0 there is $C>0$ such that for $k>k_0$,

$$\|A_k^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}\leq C|\eta|^{-1}.$$

Hence, A_k satisfies Assumption 1.3 with $\mathcal{J}=\emptyset$.

Now, given $m\in\mathbb{Z}\setminus\{0\}$ large enough, we must find $k>0$ such that the hypotheses of Lemma 14.3 hold. Let $\delta>0$ and $C_0>0$ such that

$$\zeta(x)\leq-3\delta x^{2/3},\qquad x>C_0.\tag{14.7}$$

Such a $\delta>0$ exists since [94, §10.20]

$$\zeta(x)=\Big(\frac32(\sqrt{z^2-1}-\operatorname{arcsec}z\Big)^{2/3},\quad z>1.$$

Now fix j such that

$$\delta|m|^{2/3}\leq\zeta_j'\leq2\delta|m|^{2/3},$$

and define $f:(0,\infty)\to(-\infty,\infty)$ by $f(x):=|m|^{2/3}\zeta(x/|m|)$. Then, f is smooth, $f(|m|)=0$ and, by (14.7)

$$f(x)\leq -3\delta k^{2/3}, \qquad x>C_0|m|$$

and hence for there is $|m|<k_j\leq C_0|m|$ such that $f(k_j)=\zeta_j'$. In particular, by Lemma 14.2,

$$|\lambda_m(k_j)|\leq C(1+|\eta|)k^{-1}.$$

Since $-\Delta_\Gamma e^{imt}=m^2e^{imt}$, and there is C_0 such that $C_0^{-1}k_j\leq|m|\leq C_0k$, choosing $\epsilon_0<\frac{1}{2}C_0^{-1}$ and $\Xi_0\geq 2C$ this implies the there are $k_m\to\infty$ and $\alpha_m\leq C|\eta_m|k_m^{-1}$ such that the hypotheses of Theorem 2.10 hold with $\beta_m=0$. This completes the proof. ∎

15 Details of the numerical experiments in §2

We now describe the parametrizations of the trapping and nontrapping domains considered in §2 and provide a brief description of the numerical solver used for the Galerkin examples (the Nyström method is discussed in Section 9).

Geometry parametrization: We solve the scattering problem for two nontrapping domains: the unit disk and a star-shaped domain whose parametrization is given by

$$\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}=(1+0.3\cos(t))\begin{bmatrix}\cos(t)\\ \sin(t)\end{bmatrix}, \qquad t\in[0,2\pi). \tag{15.1}$$

We consider two trapping domains. The first consists of 4 rounded and tilted squares, which we refer to as the "four diamonds" geometry. Let Γ_0 be the boundary of the square with vertices $(\pm4\sqrt{2}\pi/5,0),(0,\pm4\sqrt{2}\pi/5)$, and let $\widetilde{\Gamma}_0$ be Γ_0 but whose vertices are rounded using a Gaussian filter as described in [36]. Furthermore, let $\Gamma_j=(\pm\sqrt{2}\pi,\pm\sqrt{2}\pi)+\widetilde{\Gamma}_0$, $j=1,2,3,4$. Then the boundary of the four diamonds geometry is given by $\Gamma=\cup_{j=1}^4\Gamma_j$. The second trapping domain is a crescent shaped boundary that we refer to as the "cavity" geometry. Let $s\in[-\pi/2,\pi/2]$, $a=0.2$, and $b=\pi/12$. Let

$$\begin{aligned}\theta(s)&=b-a+2\left(1-\frac{(b-a)}{\pi}\right)\left(\frac{a}{\sqrt{\pi}}e^{-(s/a)^2}+s\cdot\operatorname{erf}(s/a)\right),\\ r(s)&=1-a\cdot\operatorname{erf}(s/a).\end{aligned} \tag{15.2}$$

Finally, let Γ be the union of the curve $(r(s)\sin(\theta(s)),r(s)\cos(\theta(s)))$ and it's reflection about the y axis. This results in a cavity whose opening is approximately a sector of b radians, and whose width is approximately $2a$. Since this curve is not smooth for the specific choices of a and b, we sample the curve at $M=400$ equispaced points in the following manner. Let $s_j=-\frac{\pi}{2}+(j-1/2)\pi$, $j=1,2,\dots M/2$, and consider

$$\begin{pmatrix}x_j\\ y_j\end{pmatrix}=\begin{cases}\begin{pmatrix}r(s_j)\sin(\theta(s_j))\\ r(s_j)\cos(\theta(s_j))\end{pmatrix}, & j=1,2,\dots M/2\\ \begin{pmatrix}-r(s_{M-j+1})\sin(\theta(s_{M-j+1}))\\ r(s_{M-j+1})\cos(\theta(s_{M-j+1}))\end{pmatrix}, & j=M/2+1,2,\dots M\end{cases}. \tag{15.3}$$

The boundary of the cavity domain is then defined to be the curve corresponding to the discrete Fourier series of the sampled curve above.

Galerkin Discretization: The geometries are discretized with N_{pan} equispaced panels in parameter space, denoted by Γ_j, $j=1,2,\dots N_{\text{pan}}$, sampled at 16th order Gauss-Legendre nodes on each panel. Let $\gamma_j(t):[-1,1]\to\Gamma_j$ denote the parametrization of panel Γ_j. For an integral operator with kernel K, the Galerkin discretization requires accurate evaluation of the integrals

$$I_{i,j,k,\ell}=\int_{-1}^{1}\int_{-1}^{1}K(\gamma_i(t),\gamma_j(s))P_k(t)P_\ell(s)|\gamma_i'(t)||\gamma_j'(s)|ds\,dt, \tag{15.4}$$

$i, j = 1, 2, \ldots N_{\text{pan}}$, $k, \ell = 0, \ldots, p$, where p is the order of the Galerkin discretization, and $P_\ell(t)$ is the Legendre polynomial of degree ℓ. $I_{i,j,k,\ell}$ corresponds to the contribution from basis function ℓ on Γ_j to basis function k on Γ_i. In all the integral representations considered, the kernel $K(x, y)$ has at most a log-singularity as $|x - y| \to 0$.

For the geometries considered and the equispaced discretization above, when panels Γ_i, and Γ_j do not share a vertex, the integrand is smooth and a high-order Gauss-Legendre quadrature rule suffices to approximate $I_{i,j,k,\ell}$. In particular, we use a 24th order Gauss-Legendre rule in both t and s to compute it.

Suppose now that Γ_i is adjacent to Γ_j with $\gamma_i(1) = \gamma_j(-1)$. The integrand in s has a near singularity close to $s = -1$. After computing the integral in s, the integrand in t has a log-singularity at $t = 1$. To handle the log-singularity at the end point, we use a custom quadrature rule that accurately computes all integrals of the form

$$\int_{-1}^{1} \Big(\log|1 - s| P_\ell(s) + \log|1 + s| P_m(s) + P_n(s) \Big)\, ds, \tag{15.5}$$

with $0 \le \ell, m, n \le q - 1$. Let $Q_{\log} = \{t_j, w_j\}$, $j = 1, 2, \ldots, N_q$, denote such a rule computed using Generalized Gaussian quadratures, see, e.g., [17]. We use a rule with $q = 20$ which results in a $N_q = 24$ point rule. To handle the integrand in s, we subdivide the $[-1, 1]$ into three equispaced panels and the panels at the ends are dyadically subdivided 4 times. A 32 point Gauss-Legendre quadrature rule is used on each of these panels resulting in a total 352 quadrature nodes.

Finally, when $\Gamma_i = \Gamma_j$, after computing the integral in s, the integrand in t now has a log-singularity at $t = \pm 1$. We use the quadrature rule $Q_{\log}$ above to compute the integral in t. For any given t_j, in order to compute the integral in s, we use a mapped version of $Q_{\log}$ on $[-1, t_j]$, and $[t_j, 1]$ to handle the log-singularity in the kernel.

Remark 15.1 *Far fewer nodes would suffice to compute the integral in s, but the quadrature rule above guarantees that error in the solution computed using the Galerkin discretization above will not be dominated by the quadrature error of computing $I_{i,j,k,\ell}$.*

Fast solver: The quadrature method described above can be easily coupled to fast multipole methods. The discretized linear system is solved using GMRES until the relative residual drops below 5×10^{-8}. The matrix vector product in each GMRES iteration is computed using `fmm2d` [4], a wideband fast-multipole method which uses far-field signatures to accelerate the translation operators at high frequencies [32]. The computational complexity of applying an $N \times N$ matrix using a wideband FMM for high-frequency problems is $O(N \log N)$.

A Definition of the scattering problems and the standard boundary-integral operators

In this section we show how scattering of a plane-wave by an obstacle with zero Dirichlet or Neumann boundary conditions can be reformulated as a BIE involving the operators A_k, A'_k (1.2) (for the Dirichlet problem) and $B_{k,\text{reg}}, B'_{k,\text{reg}}$ (1.3) (for the Neumann problem). The proof that the general Dirichlet and Neumann problems (i.e., with arbitrary boundary data) can be reformulated via these BIEs is very similar; see, e.g., [26, §2.6].

Let $\Omega^- \subset \mathbb{R}^d$, $d \ge 2$ be a bounded open set such that its open complement $\Omega^+ := \mathbb{R}^d \setminus \overline{\Omega^-}$ is connected. Let $\Gamma := \partial\Omega^-$. The results in the main body of the paper require that Γ is C^∞, but the results in this appendix hold when Γ is Lipschitz. Let ν be the outward-pointing unit normal vector to Ω^-, and let $\gamma^\pm$ and $\partial_\nu^\pm$ denote the Dirichlet and Neumann traces, respectively, on Γ from $\Omega^\pm$.

Definition A.1 (Plane-wave sound-soft/-hard scattering problems) *Given $k > 0$ and the incident plane wave $u^I(x) := \exp(\mathrm{i}kx \cdot \hat{a})$ for $\hat{a} \in \mathbb{R}^d$ with $|\hat{a}|_2 = 1$ find the total field $u \in H^1_{\text{loc}}(\Omega^+)$ satisfying $\Delta u + k^2 u = 0$ in Ω^+,*

$$\text{either } \gamma^+ u = 0 \text{ (sound-soft)} \quad \text{or} \quad \partial_\nu^+ u = 0 \text{ (sound-hard)} \quad \text{on } \Gamma,$$

and, where $u^S := u - u^I$ is the scattered field,

$$\frac{\partial u^S}{\partial r} - \mathrm{i}ku^S = o\left(\frac{1}{r^{(d-1)/2}}\right) \text{ as } r := |x| \to \infty, \text{ uniformly in } x/r. \tag{A.1}$$

The solutions of the sound-soft and sound-hard plane-wave scattering problems exist and are unique; see, e.g., [26, Theorem 2.12 and Corollary 2.13].

Let $\Phi_k(x, y)$ be the fundamental solution of the Helmholtz equation defined by

$$\Phi_k(x,y) := \frac{\mathrm{i}}{4}\left(\frac{k}{2\pi|x-y|}\right)^{(d-2)/2} H^{(1)}_{(d-2)/2}\big(k|x-y|\big) = \begin{cases} \dfrac{\mathrm{i}}{4} H^{(1)}_0\big(k|x-y|\big), & d = 2, \\ \dfrac{\mathrm{e}^{\mathrm{i}k|x-y|}}{4\pi|x-y|}, & d = 3, \end{cases} \tag{A.2}$$

where $H^{(1)}_\nu$ denotes the Hankel function of the first kind of order ν. The single- and double-layer potentials, $\mathcal{S}\ell$ and $\mathcal{D}\ell$ respectively, are defined for $k \in \mathbb{C}$, $\phi \in L^1(\Gamma)$, and $x \in \mathbb{R}^d \setminus \Gamma$ by

$$\mathcal{S}\ell\varphi(x) = \int_\Gamma \Phi_k(x,y)\varphi(y)\,\mathrm{d}s(y) \quad \text{and} \quad \mathcal{D}\ell\varphi(x) = \int_\Gamma \frac{\partial\Phi_k(x,y)}{\partial\nu(y)}\varphi(y)\,\mathrm{d}s(y). \tag{A.3}$$

The standard single-layer, adjoint-double-layer, double-layer, and hypersingular operators are defined for $k \in \mathbb{C}$, $\phi \in L^2(\Gamma)$, $\psi \in H^1(\Gamma)$, and $x \in \Gamma$ by

$$S_k\phi(x) := \int_\Gamma \Phi_k(x,y)\phi(y)\,ds(y), \qquad K'_k\phi(x) := \int_\Gamma \frac{\partial\Phi_k(x,y)}{\partial\nu(x)}\phi(y)\,ds(y), \tag{A.4}$$

$$K_k\phi(x) := \int_\Gamma \frac{\partial\Phi_k(x,y)}{\partial\nu(y)}\phi(y)\,ds(y), \quad H_k\psi(x) := \frac{\partial}{\partial\nu(x)}\int_\Gamma \frac{\partial\Phi_k(x,y)}{\partial\nu(y)}\psi(y)\,ds(y). \tag{A.5}$$

(We use the notation K_k, K'_k for the double-layer and its adjoint, instead of D_k, D'_k, to avoid a notational clash with the operator $D := -\mathrm{i}\partial$ used in the rest of the paper.)

Theorem A.2 *(i) If u is solution of the sound-soft scattering problem of Definition A.1, then*

$$A'_k\partial^+_\nu u = \partial^+_\nu u^I - \mathrm{i}\eta_D u^I \quad \text{and} \quad u = u^I - \mathcal{S}\ell(\partial^+_\nu u). \tag{A.6}$$

(ii) If $v \in L^2(\Gamma)$ is the solution to

$$A_k v = -\gamma^+ u^I, \quad \text{then} \quad u := u^I + (\mathcal{D}\ell - \mathrm{i}\eta_D\mathcal{S}\ell)v \tag{A.7}$$

is the solution of the sound-soft scattering problem of Definition A.1.

(iii) If u is solution of the sound-hard scattering problem of Definition A.1, then

$$B_{k,\mathrm{reg}}\gamma^+ u = \mathrm{i}\eta_N\gamma^+u^I - S_{\mathrm{i}k}\partial^+_\nu u^I \quad \text{and} \quad u = u^I + \mathcal{D}\ell(\gamma^+u). \tag{A.8}$$

(iv) If $v \in L^2(\Gamma)$ is the solution to

$$B'_{k,\mathrm{reg}} v = -\partial^+_\nu u^I, \quad \text{then} \quad u := u^I + (\mathcal{D}\ell S_{\mathrm{i}k} - \mathrm{i}\eta_N\mathcal{S}\ell)v$$

is the solution of the sound-hard scattering problem of Definition A.1.

References for the proof. Part (i) is proved in, e.g., [26, Theorem 2.46]. Part (ii) is proved in, e.g., [26, Equations 2.70-2.72]. Part (iii) is proved in, e.g., [45, Equation 1.6]. Part (iv) is proved in, e.g., [45, Equation 1.8]. ∎

Lemma A.3 *If Γ is C^1 then*

$$\|A'_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \geq 1/2, \qquad \|(A'_k)^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \geq 2,$$

$$\|B_{k,\mathrm{reg}}\|_{L^2(\Gamma)\to L^2(\Gamma)} \geq \left(\frac{\eta_N}{2} + \frac{1}{4}\right), \qquad \|(B_{k,\mathrm{reg}})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \geq \left(\frac{\eta_N}{2} + \frac{1}{4}\right)^{-1}.$$

Proof. The results for A'_k are proved in [25, Lemma 4.1] using that $A'_k - (1/2)I$ is compact when Γ is C^1. The results for $B_{k,\mathrm{reg}}$ are proved in an analogous way, using that $B_{k,\mathrm{reg}} - (\mathrm{i}\eta_N/2 - 1/4)I$ is compact by, e.g., [45, Proof of Theorem 2.2]. ∎

B Propagation of singularities

The goal of this section is to prove Theorem 13.12. Since propagation of singularities away from Γ follow from standard results (see e.g. [35, Appendix E]), we may work in a small neighborhood of a point on Γ. In particular, we work in Fermi normal coordinates (x_1, x') with $\Gamma = \{x_1 = 0\}$. Then, since $W \in \Psi^{\mathrm{comp}}(\Omega)$ it is, in particular, supported away from Γ, in these coordinates the operator $-\hbar^2\Delta - 1 - iW$ is given by

$$\hbar^2 D_{x_1}^2 + \hbar a(x)\hbar D_{x_1} - \tilde{R}(x, \hbar D_{x'})$$

for some $a \in C^\infty$, where $\tilde{R} \in \mathrm{Diff}_\hbar^2$ i.e. is a semiclassical differential operator of order 2. Moreover, $r(x,\xi) := \sigma(\tilde{R}) = 1 - |\xi'|_g$. Now, setting $\psi(x_1) = \frac{1}{2}\int_0^{x_1} a(s,x')ds$, we have

$$(-\hbar^2\Delta - 1)e^{-i\psi} = \hbar^2 D_{x_1}^2 - \tilde{R}(x, \hbar D_{x'}) + \hbar^2|\partial_{x_1}\psi|^2 + i\hbar^2\partial_{x_1}^2\psi.$$

Throughout this section, we use the notation $\Psi_{\mathsf{T}}^m := C^\infty((-\infty,\infty)_{x_1}; \Psi^m(\Gamma))$ and $S_{\mathsf{T}}^m :$ $C^\infty((-\infty,\infty); S^m(T^*\Gamma))$ and study operators of the form

$$P := \hbar^2 D_{x_1}^2 - R(x, \hbar D_{x'}), \tag{B.1}$$

with $R \in \Psi_{\mathsf{T}}^2$ having real principal symbol, r and $\{r = 0\} \cap \{\partial_{(x',\xi')} r = 0\} = \emptyset$.

For the boundary conditions, observe that since $\psi|_{x_1=0} = 1$, the boundary conditions are then changed to

$$(-Q\hbar D_{x_1}u - u) = e^{i\psi}(-Q\hbar D_{x_1})e^{-i\psi}e^{i\psi}u - e^{i\psi}u = (-Q\hbar D_{x_1} + Q\hbar\partial_{x_1}\psi u - u)$$

Notice that, for $\hbar$ small enough, $E := (1 + Q\hbar\partial_{x_1}\psi)^{-1} \in \Psi^0$ exists with $\sigma(E) = 1$. Therefore, $\tilde{Q} := EQ \in \Psi^{-1}$ with $\sigma(EQ) = \sigma(Q)$ and it is enough to study propagation of singularities for

$$Pu = f \text{ in } \{x_1 > 0\}, \qquad Q\hbar D_{x_1}u + u = g \text{ on } x_1 = 0, \tag{B.2}$$

where $Q \in \Psi^{-1}(\Gamma)$.

We divide our analysis into three regions

$$\mathcal{E} := \{(x',\xi') : r(0,x',\xi') < 0\}, \qquad \mathcal{H} := \{(x',\xi') : r(0,x',\xi') > 0\}, \qquad \mathcal{G} := \{(x',\xi') : r(0,x',\xi') = 0\},$$

respectively the elliptic, hyperbolic, and glancing regions.

B.1 Elliptic Estimates

We start by recalling a standard factorization together with elliptic estimates.

Lemma B.1 (Lemma 4.2 [38]) *Let P as in (B.1). Then for all $\chi \in S_{\mathsf{T}}^0$ with $\operatorname{supp}\chi \subset \{|r| > 0\}$, there are $\Lambda_\pm^j \in \Psi_{\mathsf{T}}^1$, $j = 0,1$ such that, on $\operatorname{supp}\chi$, $\sigma(\Lambda_\pm) = \pm\sqrt{r}$ on $\{r > 0\}$ and $\sigma(\Lambda_\pm) = i\sqrt{|r|}$ on $\{r < 0\}$ and*

$$\begin{aligned} \chi(x,\hbar D_{x'})P &= \chi(x,\hbar D_{x'})(\hbar D_{x_1} + \Lambda_-)(\hbar D_{x_1} - \Lambda_-) + O(\hbar^\infty)_{\Psi_{\mathsf{T}}^{-\infty}} \\ &= \chi(x,\hbar D_{x'})(\hbar D_{x_1} + \Lambda_+)(\hbar D_{x_1} - \Lambda_+) + O(\hbar^\infty)_{\Psi_{\mathsf{T}}^{-\infty}} \end{aligned} \tag{B.3}$$

Using energy estimates, one then obtains the following elliptic estimates.

Lemma B.2 *Let $Q \in \Psi^{-1}(\Gamma)$, $k \in \mathbb{N}$, $k \geq 2$, $s, N \in \mathbb{R}$, $0 < t_1 < t_2$, $X, \tilde{X} \in \Psi_{\mathsf{T}}^0$ such that $\mathrm{WF}(X) \subset \mathrm{Ell}(\tilde{X})$ and $\mathrm{WF}(\tilde{X}) \subset \{r < 0\}$ and there is $c > 0$ such that,*

$$|\sigma(Q)i\sqrt{|r|} + 1| > c > 0, \qquad \text{on } \mathrm{WF}(X).$$

Then there is $C > 0$ such that

$$\begin{aligned} \sum_{j=0}^{2} \|Xu\|_{H_\hbar^{k-j}((0,t_1);H_\hbar^{s+j})} \leq{}& C\sum_{j=0}^{k-2} \|\tilde{X}Pu\|_{H_\hbar^{k-2-j}((0,t_2);H_\hbar^{s+j})} + C\hbar^{\frac{1}{2}}\|\tilde{X}(Q\hbar D_{x_1}u - u)(0)\|_{H_\hbar^{s+k-\frac{1}{2}}} \\ &+ C\hbar^N(\|u\|_{H_\hbar^1((0,t_2);H_\hbar^{-N})} + \|Pu\|_{L^2((0,t_2);H_\hbar^{-N})} + \|u(0)\|_{H_\hbar^{-N}}). \end{aligned}$$

Proof. Let $X' \in \Psi^0(\Gamma)$ with $\mathrm{WF}(X) \subset \mathrm{Ell}(X')$ and $\mathrm{WF}(X') \subset \mathrm{Ell}(\tilde{X}) \cap \{|\sigma(Q)i\sqrt{|r|}+1| > c/2\}$. Then, since, on $\mathrm{WF}(X')$, $|\sigma(Q\Lambda_- + 1)| = |\sigma(Q)i\sqrt{|r|}+1| \geq c > 0$,

$$\begin{aligned}\|X'u(0)\|_{H_\hbar^{s+k-\frac{1}{2}}} &\leq C\|\tilde{X}(-Q\Lambda_- - 1)u(0)\|_{H_\hbar^{s+k-\frac{1}{2}}} + C\hbar^N\|u(0)\|_{H_\hbar^{-N}} \\ &\leq C\|\tilde{X}(-Q\hbar D_{x_1}u - u)(0)\|_{H_\hbar^{s+k-\frac{1}{2}}} + C\|\tilde{X}Q(\hbar D_{x_1} - \Lambda_-^0)u(0)\|_{H_\hbar^{s+k-\frac{1}{2}}} + C\hbar^N\|u(0)\|_{H_\hbar^{-N}}.\end{aligned}$$

Therefore, the lemma follows from [38, (4.28), Lemma 4.8]], where iE_- is replaced by Λ_-^0 in our notation. ■

B.2 Propagation in the hyperbolic region

We next proceed to the hyperbolic region. In this region, singularities follow broken bicharacteristics.

Lemma B.3 *Let $N > 0$, $Q \in \Psi^{-1}(\Gamma)$, $A \in \Psi_{\mathsf{T}}$ with $\mathrm{WF}(A) \cap T^*\Gamma \subset \{r > 0\} \cap \{|\sigma(Q)\sqrt{r}+1| > 0\}$ and $B_1 \in {}^b\Psi^0$ with $\mathrm{WF}(A) \subset {}^b\mathrm{Ell}(B_1)$. Then, there is $\epsilon_0 > 0$ small enough such that for all $B \in {}^b\Psi^{\mathrm{comp}}$, $B' \in \Psi^{\mathrm{comp}}(\Gamma)$ and $0 < T < \epsilon_0$ such that $\mathrm{WF}(A) \cap T^*\Gamma \subset \mathrm{Ell}(B')$, for all $\rho \in \mathrm{WF}(A)$, there is $0 \leq t \leq T$ such that*

$$\varphi_{-t}(\rho) \in \mathrm{Ell}(B), \qquad \bigcup_{0 \leq t \leq T} \varphi_{-t}(\rho) \subset {}^b\mathrm{Ell}(B_1),$$

there is $C > 0$ such that

$$\begin{aligned}&\|A\hbar D_{x_1}u(0)\|_{L^2} + \|Au(0)\|_{L^2} + \|A\hbar D_{x_1}u\|_{L^2} + \|Au\|_{L^2} \\ &\leq C(\hbar^{-1}\|B_1 Pu\|_{L^2} + \|B'(-Q\hbar D_{x_1} - u)(0)\|_{L^2} + \|Bu\|_{L^2}) \\ &\quad + C\hbar^N(\|(-Q\hbar D_{x_1}u - u)(0)\|_{H_\hbar^{-N}} + \|\hbar D_{x_1}u(0)\|_{H_\hbar^{-N}} + \|u\|_{L^2(0,2\epsilon)} + \|\hbar D_{x_1}u\|_{L^2(0,2\epsilon)} + \|Pu\|_{L^2(0,2\epsilon)})\end{aligned}$$

In order to prove Lemma B.3, we need a pseudodifferential cutoff that nearly commutes with one factor in (B.3).

Lemma B.4 *Let $a \in S^{\mathrm{comp}}(T^*\Gamma)$ with $\operatorname{supp} a \subset \{r > 0\}$. Then, there are $\epsilon > 0$ and $E_\pm \in \Psi_{\mathsf{T}}^{\mathrm{comp}}$ such that $E_\pm|_{x_1=0} = a(x', \hbar D_{x'})$, for any $\theta \in C_c^\infty(-\epsilon, \epsilon)$,*

$$\theta[\hbar D_{x_1} + \Lambda_\pm, E_\pm] = O(\hbar^\infty)_{\Psi_{\mathsf{T}}^{-\infty}},$$

and

$$\mathrm{WF}(E_\pm) \subset \bigcup_{0 \leq x_1 \leq 2\epsilon} \{(x_1, \exp(x_1 H_{\mp\sqrt{r}})(\operatorname{supp}(a)))\}.$$

Proof. We start by solving

$$\sigma([(\hbar D_{x_1} + \Lambda_\pm), \tilde{e}_0^\pm(x, \hbar D_{x'})]) = -i\hbar\partial_{x_1}\tilde{e}_0^\pm - i\hbar\{\pm\sqrt{r}, \tilde{e}_0^\pm\} = 0 \quad 0 < x_1 < 2\epsilon, \qquad \tilde{e}_0^\pm|_{x_1=0} = a.$$

This is possible for ϵ small enough since $\operatorname{supp} a \subset \{r > 0\}$ and hence $\{x_1 = 0\}$ is non-characteristic for the vector-field $\partial_{x_1} \pm H_{\sqrt{r}}$. Let $\psi, \psi_0, \psi_1 \ldots, \in C_c^\infty((-2\epsilon, 2\epsilon))$ with $x_1\psi'(x_1) \leq 0$, $\operatorname{supp}(1-\psi) \cap [-\epsilon, \epsilon] = \emptyset$, $\operatorname{supp}\psi \cap \operatorname{supp}(1-\psi_j) = \emptyset$, $j \geq 0$ and $\operatorname{supp}\psi_{j+1} \cap \operatorname{supp}(1-\psi_j) = \emptyset$, $j \geq 0$. Then, set $e_0^\pm := \psi\tilde{e}_0^\pm$ and $E_0^\pm := e_0^\pm(x, \hbar D_{x'})$. Then,

$$[(\hbar D_{x_1} + \Lambda_\pm), E_0^\pm(x, \hbar D_{x'})] = \hbar^2 b_0(x, \hbar D_{x'}), \text{ for } x_1 \notin \operatorname{supp}(1-\psi_0)$$

for some $b_0 \in S_{\mathsf{T}}^{\mathrm{comp}}$ with $\operatorname{supp} b_0 \subset \operatorname{supp}\psi e_0^\pm \subset \{r > 0\}$.

Suppose by induction that form some $N \geq 1$ we have found $e_0^\pm, \ldots, e_{N-1}^\pm \in S_{\mathsf{T}}^{\mathrm{comp}}$ with $\operatorname{supp} e_j^\pm \subset \operatorname{supp}\psi_0\tilde{e}_0^\pm \subset \{r > 0\}$ such that, with $E_{N-1}^\pm := \sum_{j=0}^{N-1}\hbar^j\psi_{N-1}e_j^\pm(x, \hbar D_{x'})$,

$$[(\hbar D_{x_1} + \Lambda_\pm), E_{N-1}^\pm(x, \hbar D_{x'})] = \hbar^{N+1}b_N(x, \hbar D_{x'}), \qquad \text{for } x_1 \notin \operatorname{supp}(1-\psi_{N-1}). \tag{B.4}$$

for some $b_N \in S^{\mathrm{comp}}_{\mathsf{T}}$ with $\operatorname{supp} b_N \subset \operatorname{supp} \psi_0 \tilde{e}_0^{\pm} \subset \{r > 0\}$. Then, let $e_N^{\pm} \in S^{\mathrm{comp}}_{\mathsf{T}}$ satisfy

$$\partial_{x_1} e_N^{\pm} - \{\mp\sqrt{r}, e_N^{\pm}\} = -b_N, \qquad e_N^{\pm}|_{x_1=0} = 0, \qquad x_1 \notin \operatorname{supp}(1 - \psi_{N-1}).$$

Defining $E_N^{\pm} := \sum_{j=0}^{N} \hbar^j \psi_N e_j^{\pm}(x, \hbar D_{x'})$, we have (B.4) with N replaced by $N+1$. Putting $E_{\pm} \sim \sum_{j=0}^{\infty} \hbar^j \psi e_j^{\pm}(x, \hbar D_{x'})$, completes the proof of the lemma. ∎

We can now prove Lemma B.3.

Proof of Lemma B.3. By a partition of unity argument, we may assume that $\varphi_{-T}(\rho) \in \mathrm{Ell}(B)$ for all $\rho \in \mathrm{WF}(A)$. In addition, since $\mathrm{WF}(A) \subset \{r > 0\}$, for any $\delta > 0$, may further assume that $\mathrm{WF}(B) \subset \{\xi_1 < 0,\ \epsilon/2 < x_1 < \epsilon\}$.

Let $a_i \in C_c^\infty(T^*\Gamma)$, $i = 1, 2$ with $\operatorname{supp} a_i \subset \{r > 0\}$, $\operatorname{supp} a_1 \cap \operatorname{supp}(1 - a_2) = \emptyset$,

$$\mathrm{WF}(A) \cap \bigcup_{-2\epsilon \le x_1 \le 2\epsilon} \{(x_1, \exp(x_1 H_{\sqrt{r}} \operatorname{supp}(1 - a_1)\} = \emptyset,$$

and

$$\bigcup_{-2\epsilon \le x_1 \le 2\epsilon} \{(x_1, \exp(x_1 H_{\sqrt{r}})(\operatorname{supp} a_2))\} \subset {}^b\mathrm{Ell}(B_1).$$

Let $E^i_{\pm}$ from Lemma B.4 with a replaced by a_i. Then, using the factorizations (B.3), we have

$$(\hbar D_{x_1} + \Lambda_{\pm}) E^i_{\pm} (\hbar D_{x_1} - \Lambda_{\pm}) = E^i_{\pm} P + O(\hbar^\infty)_{\Psi_{\mathsf{T}}^{-\infty}}. \tag{B.5}$$

Since $\operatorname{supp} a_i \subset {}^b\mathrm{Ell}(B_1)$, there is $\epsilon > 0$ small enough such that for $X \in \Psi(\{x_1 > 0\})$ with $\mathrm{WF}(X) \subset \{\pm\xi_1 \ge -\epsilon, 0 < x_1 < \epsilon\}$,

$$\|X E^i_{\pm}(\hbar D_{x_1} - \Lambda_{\pm}) u\|_{L^2} \le C \|B_1 P u\|_{L^2} + C\hbar^N \|u\|_{L^2},$$

In particular, there is $\chi \in C_c^\infty((\epsilon/2, \epsilon))$ with $\chi \equiv 1$ near $\frac{2\epsilon}{3}$ and X as above such that

$$\mathrm{WF}(E^i_{\pm}\chi) \subset (\{\pm\xi_1 > 0\} \cup \{p^2 + b^2 > c > 0\}) \cap {}^b\mathrm{Ell}(B_1).$$

Therefore,

$$\|\chi E^i_{\pm}(\hbar D_{x_1} - \Lambda_{\pm}) u\|_{L^2} \le C(\|B_1 P u\|_{L^2} + \|Bu\|_{L^2} + \hbar^N \|u\|_{L^2} + \hbar^N \|Pu\|_{L^2}). \tag{B.6}$$

Basic energy estimates together with the fact that on $\operatorname{supp} e_0$, $\operatorname{Im} \sigma(\Lambda) = 0$ (see e.g. [38, Lemma 4.4]) imply that for $0 \le t_1 \le \frac{\epsilon}{2}$ and $0 \le t \le \epsilon$,

$$\begin{aligned} &\|E^i_{\pm}(t_1)(\hbar D_{x_1} - \Lambda_{\pm}) u(t_1)\|_{L^2} \le C\hbar^{-1}(\|E^i_{\pm} P u\|_{L^2(0,\epsilon)} + \|\chi E^i_{\pm}(\hbar D_{x_1} - \Lambda_{\pm}) u\|_{L^2}), \\ &\qquad \|E^i_{\pm}(\hbar D_{x_1} - \Lambda_{\pm}) u(t)\|_{L^2} \le C\hbar^{-1}(\|E^i_{\pm} P u\|_{L^2(0,\epsilon)} + \|a_i(x', \hbar D_{x'})(\hbar D_{x_1} - \Lambda_{\pm}) u(0)\|_{L^2}). \end{aligned} \tag{B.7}$$

In particular, combining the first estimate with $t_1 = 0$, (B.6) and the fact that $\mathrm{WF}(E^i_{\pm}) \cap \{x_1 < \epsilon\} \subset {}^b\mathrm{Ell}(B_1)$, we obtain

$$\begin{aligned} &\|a_2(x', \hbar D_{x'})(I + \Lambda_+ Q)\hbar D_{x_1} u(0)\|_{L^2} + \|E^2_{\pm}(t_1)(\hbar D_{x_1} - \Lambda_{\pm}) u(t_1)\|_{L^2} \\ &\le C\|a_2(x', \hbar D_{x'})\Lambda_+(-Q\hbar D_{x_1} u - u)(0)\|_{L^2} + C\hbar^{-1}(\|B_1 P u\|_{L^2} + \|Bu\|_{L^2}) + C\hbar^N(\|u\|_{L^2} + \|Pu\|_{L^2}) \\ &\le C(\hbar^{-1}\|B_1 P u\|_{L^2} + \|B'(-Q\hbar D_{x_1} - u)(0)\|_{L^2} + \|Bu\|_{L^2}) + C\hbar^N(\|(-Q\hbar D_{x_1} u - u)(0)\|_{H_\hbar^{-N}} + \|u\|_{L^2} + \|Pu\|_{L^2}) \end{aligned}$$

It remains to observe that $\sigma(I + \Lambda_+ Q) > 0$ on $\operatorname{supp} a_2$ and hence

$$\begin{aligned} &\|a_2(x', \hbar D_{x'})\hbar D_{x_1} u(0)\| + \|a_2(x', \hbar D_{x'}) u(0)\|_{L^2} \\ &\le \|a_2(x', \hbar D_{x'})\hbar D_{x_1} u(0)\|_{L^2} + \|a_2(x, \hbar D_{x'})(-Q\hbar D_{x_1} u - u)(0)\|_{L^2} \\ &\le C(\hbar^{-1}\|B_1 P u\|_{L^2} + \|B'(-Q\hbar D_{x_1} - u)(0)\|_{L^2} + \|Bu\|_{L^2}) \\ &\qquad + C\hbar^N(\|(-Q\hbar D_{x_1} u - u)(0)\|_{H_\hbar^{-N}} + \|\hbar D_{x_1} u(0)\|_{H_\hbar^{-N}} + \|u\|_{L^2} + \|Pu\|_{L^2}) \end{aligned}$$

Now that we have control of the boundary traces, we can use the second inequality (B.7) to finish the proof. Indeed, since $\operatorname{supp} a_1 \cap \operatorname{supp}(1-a_2) = \emptyset$,

$$
\begin{aligned}
&\|E^1_\pm(\hbar D_{x_1} - \Lambda_-)u(t_1)\|_{L^2} + \|E^2_\pm(t_1)(\hbar D_{x_1} - \Lambda_+)u(t_1)\|_{L^2} \\
&\le C(\hbar^{-1}\|B_1 Pu\|_{L^2} + \|B'(-Q\hbar D_{x_1} - u)(0)\|_{L^2} + \|Bu\|_{L^2}) + C\|a_1(x', \hbar D_{x'})(\hbar D_{x_1} - \Lambda_\pm)u(0)\|_{L^2}) \\
&\quad + C\hbar^N(\|(-Q\hbar D_{x_1}u - u)(0)\|_{H_\hbar^{-N}} + \|\hbar D_{x_1}u(0)\|_{H_\hbar^{-N}} + \|u\|_{L^2} + \|Pu\|_{L^2}) \\
&\le C(\hbar^{-1}\|B_1 Pu\|_{L^2} + \|B'(-Q\hbar D_{x_1} - u)(0)\|_{L^2} + \|Bu\|_{L^2}) \\
&\quad + C\hbar^N(\|(-Q\hbar D_{x_1}u - u)(0)\|_{H_\hbar^{-N}} + \|\hbar D_{x_1}u(0)\|_{H_\hbar^{-N}} + \|u\|_{L^2} + \|Pu\|_{L^2})
\end{aligned}
$$

and hence, using that $\mathrm{WF}(A) \subset \mathrm{Ell}(E^i_\pm) \cap \{x_1 < \epsilon/2\}$ and $|\sigma(\Lambda_+ - \Lambda_-)| > c > 0$ on $\mathrm{WF}(A)$, we have

$$
\begin{aligned}
&\|A\hbar D_{x_1}u\|_{L^2} + \|Au\|_{L^2} \\
&\le C(\|E^1_\pm(\hbar D_{x_1} - \Lambda_-)u\|_{L^2} + \|E^2_\pm(\hbar D_{x_1} - \Lambda_+)u\|_{L^2} + \hbar^N(\|\hbar D_{x_1}u\|_{L^2} + \|u\|_{L^2}) \\
&\le C(\hbar^{-1}\|B_1 Pu\|_{L^2} + \|B'(-Q\hbar D_{x_1} - u)(0)\|_{L^2} + \|Bu\|_{L^2}) \\
&\quad + C\hbar^N(\|(-Q\hbar D_{x_1}u - u)(0)\|_{H_\hbar^{-N}} + \|\hbar D_{x_1}u(0)\|_{H_\hbar^{-N}} + \|u\|_{L^2} + \|\hbar D_{x_1}u\|_{L^2} + \|Pu\|_{L^2})
\end{aligned}
$$

■

As a corollary of (B.7), and Lemma B.1, we obtain Lemma 13.14

Proof of Lemma 13.14. Let $\chi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1})$ with $\chi(q) = 1$ such that

$$\|\chi(x, \hbar D_{x'})Pu\|_{L^2} \le C\hbar^{s+1}, \qquad \|\chi(0, x', \hbar D_{x'})(\hbar D_{x_1} - \Lambda_-)u\|_{L^2} \le C\hbar^s.$$

Let $a \in C_c^\infty(T^*\Gamma)$ with $\operatorname{supp} a \subset \{\chi > 0\}$ and $E_\pm$ as constructed in Lemma B.4, and $\epsilon > 0$ small enough that $\operatorname{supp} E_- \cap \{x_1 < \epsilon\} \subset \{\chi > 0\}$. Then, by (B.7),

$$\|E_-(\hbar D_{x_1} - \Lambda_-)u(t)\|_{L^2} \le C\hbar^{-1}\|E_+ Pu\|_{L^2(0,\epsilon)} + \|a(x, \hbar D_{x;})(\hbar D_{x_1} - \Lambda_-)u\|_{L^2} \le C\hbar^s.$$

In particular,

$${}^b\mathrm{WF}^s_{H^1_\hbar}(u) \cap \{x_1 > 0, \xi_1 > 0\} \cap \{\sigma(E_-) > 0\} = \emptyset,$$

and, since

$$\bigcup_{0<s<\epsilon} \varphi_s(q) \subset \{\sigma(E_-) > 0\},$$

the lemma follows. ■

B.3 Propagation in the glancing region

The most subtle part of the analysis is in the glancing region, where we follow the proof in [62, Chapter 24]. Throughout this section, $B_i \in \Psi^0_\Gamma$, $i = 0, 1$, satisfies

$$B(x, hD) := B_1(x, hD')\hbar D_{x_1} + B_0(x, \hbar D_{x'}), \qquad B_1^* = B_1, \qquad B_0^* = B_0 - [\hbar D_{x_1}, B_1]. \tag{B.8}$$

Then, integration by parts yields the following lemma.

Lemma B.5 (Lemma 24.4.2 [62]) *Let P as in (B.1) and B as in (B.8). Then,*

$$2\hbar^{-1}\operatorname{Im}\langle Pu, Bu\rangle_\Omega = \sum_{j,k=0}^{1} \operatorname{Re}\langle E_{jk}(x', \hbar D_{x'})u_k, u_j\rangle_\Gamma + \sum_{j,k=0}^{1} \operatorname{Re}\langle C_{jk}u_k, u_j\rangle_\Omega,$$

where $u_j = h^j D^j_{x_1}$ and $E_{11} = B_1$, $E^*_{01} = E_{10} = B_0$, $E_{00} = B_1(R + R^*)/2$, and the symbol, c_{jk} of C_{jk} is real and satisfies $c_{01} = c_{10}$,

$$\sum_{jk} c_{jk}(x, \xi')\xi_1^{j+k} = \{p, b\} + 2b\operatorname{Im} p_{-1},$$

where p_{-1} is the subprincipal symbol of P.

We use Lemma B.5 repeatedly in positive commutator type arguments. To do this, we first record a technical lemma similar to [62, Lemma 24.4.5].

Lemma B.6 *Let $A_{jk} \in \Psi^{\mathrm{comp}}$ have real principal symbols a_{jk} compactly supported in x, $a_{01} = a_{10}$ and such that*

$$\sum_{j,k=0}^{1} a_{jk}(x,\xi')\xi_1^{j+k} = -\psi(x,\xi)^2, \quad \text{when } \xi_1^2 = r(x,\xi'),$$

where $\psi \in C_c^\infty$. Furthermore, suppose that $\partial_{x_1} r > 0$ or $\partial_{(x',\xi')} r \neq 0$ on $\cup \operatorname{supp} a_{jk}$. Then, there are Ψ_i $i = 0,1$ with principal symbols ψ and $\partial_{\xi_1}\psi$ when $\xi_1 = r(x,\xi') = 0$ such that for any N

$$\begin{aligned}\operatorname{Re} \sum \langle A_{jk}(hD_1)^k u, (hD_1)^j u\rangle_\Omega &+ \|\Psi_0(x,\hbar D_{x'})u + \Psi_1(x,\hbar D_{x'})\hbar D_{x_1} u\|^2_{L^2(\Omega)} \\ &\le C\hbar^{-1}\|Pu\|^2_{L^2} + C\hbar(\|u\|^2_{L^2} + \|\hbar D_{x_1} u\|^2_{L^2} + \|\hbar D_{x_1} u(0)\|^2_{L^2} + \|(Q\hbar D_{x_1} + u)(0)\|^2_{L^2}).\end{aligned}$$

Proof. Let ψ_0, $\psi_1 \in C_c^\infty$ such that $\psi_j = \partial_{\xi_1}^j \psi$ when $\xi_1 = r(x,\xi') = 0$. and such that

$$\sum a_{jk}(x,\xi')\xi_1^{j+k} + (\psi_0(x,\xi') + \psi_1(x,\xi')\xi_1)^2 \le g(x,\xi')(\xi_1^2 - r(x,\xi')),$$

for some $g \in C_c^\infty$. This is possible by [62, Lemma 24.4.3].

Let Ψ_j and G with principal symbols ψ_j and g respectively. Then, by the Sharp Gårding inequality for systems,

$$\begin{aligned}\operatorname{Re} \Big(\sum \langle A_{jk}(x,\hbar D_{x'})u_k, u_j\rangle_\Omega &+ \sum \langle \Psi_j^*(x,\hbar D_{x'})\Psi_k(x,\hbar D_{x'})u_k, u_j\rangle_\Omega \\ &- \langle G(x,\hbar D_{x'})u_1, u_1\rangle_\Omega + \langle G(x,\hbar D_{x'})R(x,\hbar D_{x'})u_0, u_0\rangle_\Omega \Big) \\ \le C\hbar(\|u_0\|^2 + \|\hbar D_{x_1} u\|^2)&\end{aligned}$$

Next,

$$\begin{aligned}&\operatorname{Re} \Big(\langle G(x,\hbar D_{x'})u_1, u_1\rangle_\Omega - \langle G(x,\hbar D_{x'})R(x,\hbar D_{x'})u_0, u_0\rangle_\Omega \Big) \\ &= \operatorname{Re} \Big(\langle GPu, u\rangle_\Omega - i\hbar \langle G(0,x',\hbar D_{x'})\hbar D_{x_1} u, u\rangle|_\Gamma + \langle [\hbar D_{x_1}, G(x,\hbar D_{x'})]\hbar D_{x_1} u, u\rangle_\Omega \Big) \\ &\le C\hbar^{-1}\|Pu\|^2_{L^2} + C\hbar\|u\|_{L^2} + \hbar \operatorname{Im}\langle G(0,x',\hbar D_{x'})\hbar D_{x_1} u, u\rangle_\Gamma + C\hbar\|\hbar D_{x_1} u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}.\end{aligned}$$

■

We also need a microlocal estimate on the normal derivative. We start with a microlocal estimate on the normal derivative.

Lemma B.7 *There is $\delta_0 > 0$ such that for all $a \in S^{\mathrm{comp}}(T^*\Gamma)$, $\operatorname{supp} a \subset \{|r| \le \delta\}$, $A := a(x',\hbar D_{x'})$, $\psi \in C_c^\infty(-1,1)$ with $0 \notin \operatorname{supp}(1-\psi)$ and $\chi, \chi_1 \in C_c^\infty(\mathbb{R}^\times \mathbb{R}^{n-1})$ with $\operatorname{supp} a(x',\xi')\psi(x_1) \cap \operatorname{supp}(1-\chi) = \emptyset$, $\operatorname{supp}(1-\chi_1) \cap \operatorname{supp}\chi = \emptyset$. Then,*

$$\begin{aligned}\|A\hbar D_{x_1} u(0)\|_{L^2} &\le C\hbar^{-1}\|XPu\|_{L^2} + C\|Xu\|_{L^2} + C\|X(Q\hbar D_{x_1} u + u)(0)\|_{L^2} \\ &\quad + C\hbar^N(\|(Q\hbar D_{x_1} u + u)(0)\|_{L^2} + \|\hbar D_{x_1} u(0)\|_{L^2}). \\ \|X\hbar D_{x_1} u\|_{L^2} &\le C(\hbar^{-1}\|X_1 Pu\|_{L^2} + \|X_1 u\|_{L^2} + C\|X_1(Q\hbar D_{x_1} u + u)(0)\|_{L^2} \\ &\quad + C\hbar^N(\|(Q\hbar D_{x_1} u + u)(0)\|_{L^2} + \|\hbar D_{x_1} u(0)\|_{L^2})\end{aligned}$$

Proof. Let $\psi \in C_c^\infty((-1,1);[0,1])$ with $0 \notin \operatorname{supp}(1-\psi)$ and define

$$B_1 := A^*A\psi(x_1), \qquad B_0 := \frac{-i\hbar}{2} B_1 \psi'(x_1),$$

Then, Lemma B.5 yields, with $B = B_1 \hbar D_{x_1} + B_0$,

$$2\hbar^{-1}\operatorname{Im}\langle PXu, BXu\rangle_\Omega = \operatorname{Re}\sum_{j,k=0}^{1}\langle E_{jk}(\hbar D_{x_1})^k Xu, (\hbar D_{x_1})^j Xu\rangle_\Gamma + \operatorname{Re}\sum_{j,k=0}^{1}\langle C_{jk}(\hbar D_{x_1})^k Xu, (\hbar D_{x_1})^j Xu_j\rangle_\Omega,$$

where

$$E_{01}^* = E_{10} = 0, \qquad E_{11} = A^*A, \qquad E_{00} = A^*A(R+R^*)/2.$$

Hence,

$$\begin{aligned}
&\|A\hbar D_{x_1} Xu(0)\|_{L^2}^2 \\
&= 2\hbar^{-1}\operatorname{Im}\langle PXu, BXu\rangle_\Omega - \operatorname{Re}\sum_{j,k=0}^{1}\langle C_{jk}(\hbar D_{x_1})^k Xu, (\hbar D_{x_1})^j Xu\rangle_\Omega - \tfrac{1}{2}\langle A(R+R^*)Xu, AXu\rangle_\Gamma \\
&= 2\hbar^{-1}\operatorname{Im}\langle PXu, BXu\rangle_\Omega - \operatorname{Re}\sum_{j,k=0}^{1}\langle C_{jk}(\hbar D_{x_1})^k Xu, (\hbar D_{x_1})^j Xu\rangle_\Omega \\
&\quad - \tfrac{1}{2}\langle A(R+R^*)(X(Q\hbar D_{x_1}u+u), AX(Q\hbar D_{x_1}u+u)\rangle_\Gamma \\
&\quad + \tfrac{1}{2}\langle A(R+R^*)X(Q\hbar D_{x_1}u+u), AXQ\hbar D_{x_1}u\rangle_\Gamma + \tfrac{1}{2}\langle A(R+R^*)XQ\hbar D_{x_1}u, AX(Q\hbar D_{x_1}u+u)\rangle_\Gamma \\
&\quad - \tfrac{1}{2}\langle A(R+R^*)XQ\hbar D_{x_1}u, AXQ\hbar D_{x_1}u\rangle_\Gamma \\
&\le 2\hbar^{-1}\operatorname{Im}\langle PXu, BXu\rangle_\Omega + C\|X\hbar D_{x_1}u\|_{L^2} + C\|Xu\|_{L^2} \\
&\quad + C\|X(Q\hbar D_{x_1}u+u)(0)\|_{L^2}^2 + C\delta\|A\hbar D_{x_1}u(0)\|_{L^2}^2 + C\hbar\|X\hbar D_{x_1}u(0)\|_{L^2}^2 \\
&\quad + O(\hbar^\infty)(\|(Q\hbar D_{x_1}u+u)(0)\|_{L^2}^2 + \|\hbar D_{x_1}u(0)\|_{L^2}^2) \\
&\le C\hbar^{-2}\|XPu\|_{L^2}^2 + C\|X\hbar D_{x_1}u\|_{L^2} + C\|Xu\|_{L^2} \\
&\quad + C\|X(Q\hbar D_{x_1}u+u)(0)\|_{L^2}^2 + C\delta\|A\hbar D_{x_1}u(0)\|_{L^2}^2 + C\hbar\|X\hbar D_{x_1}u(0)\|_{L^2}^2 \\
&\quad + O(\hbar^\infty)(\|(Q\hbar D_{x_1}u+u)(0)\|_{L^2}^2 + \|\hbar D_{x_1}u(0)\|_{L^2}^2).
\end{aligned}$$

In particular, for $\delta > 0$ small,

$$\begin{aligned}
\|A\hbar D_{x_1}u(0)\|_{L^2} \le C\hbar^{-1}&\|XPu\|_{L^2} + C\|X\hbar D_{x_1}u\|_{L^2} + C\|Xu\|_{L^2} \\
&+ C\|X(Q\hbar D_{x_1}u+u)(0)\|_{L^2} + C\hbar^{1/2}\|X\hbar D_{x_1}u(0)\|_{L^2} \\
&+ O(\hbar^\infty)(\|(Q\hbar D_{x_1}u+u)(0)\|_{L^2} + \|\hbar D_{x_1}u(0)\|_{L^2}).
\end{aligned}$$

Now, we need to estimate $\|X\hbar D_{x_1}u\|_{L^2}$ by Pu and u. For this, self-adjoint.

$$\begin{aligned}
\langle Pu, X^*Xu\rangle &= \langle((\hbar D_{x_1})^2 - r(x,\hbar D_{x'}))u, X^*Xu\rangle \\
&= -\langle r(x,\hbar D_{x'})u, X^*Xu\rangle + \|X\hbar D_{x_1}u\|_{L^2}^2 \\
&\quad \langle(\hbar D_{x_1})u, [\hbar D_{x_1}, X^*X]u\rangle + i\hbar\langle\hbar D_{x_1}u, X^*Xu\rangle_\Gamma.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|X\hbar D_{x_1}u\|_{L^2}^2 &\le \langle Pu, X^*Xu\rangle + \langle r(x,\hbar D_{x'})Xu, Xu\rangle \\
&\quad + \langle[X,r]u, Xu\rangle - \langle(\hbar D_{x_1})u, [\hbar D_{x_1}, X^*]X + X^*[\hbar D_{x_1}, X]u\rangle \\
&\quad - \operatorname{Re} hi\langle X\hbar D_{x_1}u, X(Q\hbar D_{x_1}u+u)\rangle_\Gamma + \operatorname{Re} hi\langle X\hbar D_{x_1}u, XQ\hbar D_{x_1}u\rangle_\Gamma \\
&\le C\|XPu\|^2 + C\|Xu\|^2 + C\hbar(\|\tilde{X}u\|_{L^2}^2 + \|\tilde{X}\hbar D_{x_1}u\|_{L^2}^2) + C\hbar\|\tilde{X}\hbar D_{x_1}u(0)\|_{L^2}^2 \\
&\quad + C\hbar\|\tilde{X}(Q\hbar D_{x_1}+u)(0)\|_{L^2}^2 \\
&\quad + O(\hbar^\infty)(\|u\|_{L^2}^2 + \|\hbar D_{x_1}u\|_{L^2}^2 + \|(Q\hbar D_{x_1}u+u)(0)\|_{L^2}^2 + \|\hbar D_{x_1}u(0)\|_{L^2}^2).
\end{aligned}$$

The proof is completed by iteration. ■

B.4 Diffractive points

We start by considering diffractive points. The main technical estimate is provided in the next lemma.

Lemma B.8 *Let $Q \in \Psi^{-1}(\Gamma)$ with $\operatorname{Re}\sigma(Q) \geq 0$ near $r = 0$. Then there are $\delta_0 > 0$ and $M_0 > 0$ such that for all $M > M_0$, $f \in C_c^\infty((0,\infty) \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$, $b_1, b_0, v, \chi, \chi_+ \rho \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1}; \mathbb{R})$, $t, t_0 \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathbb{R})$, and $\psi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R})$ satisfying*

$$b_0|_{x_1=0} \geq 0, \qquad \operatorname{supp} b_i \cap \operatorname{supp}(1-\chi) = \emptyset, \qquad \operatorname{supp}\chi \cap \operatorname{supp}(1-\chi_+) = \emptyset$$
$$\operatorname{supp}\chi_+ \subset \{|r| \leq \delta_0\} \cap \{\partial_{x_1} r > 0\},$$
$$\operatorname{supp}\rho \subset \{r > 0\}, \qquad \operatorname{supp}\rho \cap \operatorname{supp}(1-\chi) = \emptyset, \qquad b_1|_{x_1=0} = -t^2, \qquad b_0|_{x_1=0} = tt_0.$$

setting $b := b_1\xi_1 + b_0$,

$$\{p, b\} + bM|\xi'| = -\psi^2 + \rho(\xi_1 - r^{1/2}), \quad b = v^2, \quad \textit{when } p = 0,$$

and for all $(x', \xi') \in \operatorname{supp}\chi$ either $r(x,\xi') < 0$ or there are $0 \leq s_\pm(x,\xi') \leq 1$ such that

$$\varphi_{-s_\pm}(x, \pm\sqrt{r(x,\xi')}, \xi') \in \{f > 0\}, \;\; \textit{and} \; \Big(\bigcup_{0 \leq s \leq s_\pm} \varphi_{-s}(x, \pm\sqrt{r(x,\xi')}, \xi')\Big) \cap \operatorname{supp}(1-\chi) = \emptyset.$$

Then there are $C > 0$, $\Psi_j \in C^\infty(\mathbb{R}_{x_1}; \Psi^{\mathrm{comp}}(\mathbb{R}^{n-1}))$, $j = 0, 1$ such that for all $0 < h < 1$, defining $X := \chi(x, \hbar D_{x'})$, $B_i := b_i(x, \hbar D_{x'})$, $T := t(x', \hbar D_{x'})$,

$$\begin{aligned} &\|\Psi_0 Xu + \Psi_1 \hbar D_{x_1} Xu\|_{L^2(\Omega)} + \|T\hbar D_{x_1} Xu(0)\|_{L^2} + \|BXu\|_{L^2} \\ &\leq C\hbar^{-1}\|XPu\| + C(1+\epsilon^{-1})\|X(Q\hbar D_{x_1}u + u)(0)\|_{L^2} + \|Fu\|_{L^2} \\ &\quad + C\hbar^{\frac{1}{2}}(\|X_+u\|_{L^2} + \|X_+\hbar D_{x_1}u\|_{L^2} + \|\hbar D_{x_1}Xu(0)\|_{L^2} + \|(Xu + Q\hbar D_{x_1}Xu)(0)\|_{L^2}) \\ &\quad + C\hbar^N(\|u\|_{L^2} + \|\hbar D_{x_1}u\|_{L^2} + \|Pu\|_{L^2} + \|u(0)\|_{L^2} + \|\hbar D_{x_1}u(0)\|_{L^2}), \end{aligned} \tag{B.9}$$

and $\sigma(\Psi_j)|_{r=0}(0, x', 0, \xi') = \partial_{\xi_1}^j \psi(0, x', 0, \xi')|_{r=0}$.

Proof. By Lemma B.5

$$\begin{aligned} 2\hbar^{-1}\operatorname{Im}\langle PXu, BXu\rangle = &\sum_{j,k=0}^{1} \langle E_{jk}(x', \hbar D_{x'})(\hbar D_{x_1})^k Xu, (\hbar D_{x_1})^j (Xu)\rangle_\Gamma \\ &+ \sum_{j,k=0}^{1} \langle C_{jk}(\hbar D_{x_1})^k Xu, (\hbar D_{x_1})^j (Xu)\rangle_\Omega, \end{aligned} \tag{B.10}$$

where

$$\sum c_{jk}\xi_1^{j+k} = \{p, b\} + 2b\operatorname{Im} p_{-1} = -bM - \psi^2, \qquad \text{when } \xi_1^2 = r.$$

Hence, by Lemma B.6,

$$\begin{aligned} &\operatorname{Re}\sum_{j,k=0}^{1} \langle C_{jk}(\hbar D_{x_1})^k Xu, (\hbar D_{x_1})^j (Xu)\rangle_\Omega \\ &\quad + \operatorname{Re}\langle (M - 2\operatorname{Im} p_{-1}(x, \hbar D_{x'}))Xu, BXu\rangle - \operatorname{Re}\langle (\hbar D_{x_1} - \Lambda(x, \hbar D_{x'}))Xu, \rho(x, \hbar D_{x'})Xu\rangle \\ &\leq -\|\Psi_0 Xu + \Psi_1 \hbar D_{x_1} Xu\|^2 + C\hbar^{-1}\|PXu\|_{L^2}^2 \\ &\quad + C\hbar(\|Xu\|_{L^2}^2 + \|\hbar D_{x_1}Xu\|_{L^2}^2 + \|(Q\hbar D_{x_1}Xu + Xu)(0)\|_{L^2}^2 + \|\hbar D_{x_1}Xu(0)\|_{L^2}^2). \end{aligned} \tag{B.11}$$

Next, observe that for M large enough,

$$-Mb + 2\operatorname{Im} p_{-1} b + \epsilon|b|^2 = -(M - 2\operatorname{Im} p_{-1} - v^2)v^2, \text{ when } \xi_1^2 = r.$$

Therefore, apply Lemma B.6 again we obtain

$$\begin{aligned} &-\operatorname{Re}\langle (M - 2\operatorname{Im} p_{-1}(x, \hbar D_{x'}))Xu, BXu\rangle + \|BXu\|_{L^2}^2 \\ &\leq C\hbar^{-1}\|PXu\|_{L^2}^2 \\ &\quad + C\hbar(\|Xu\|_{L^2}^2 + \|\hbar D_{x_1}Xu\|_{L^2}^2 + \|(Q\hbar D_{x_1}Xu + Xu)(0)\|_{L^2}^2 + \|\hbar D_{x_1}Xu(0)\|_{L^2}^2) \end{aligned} \tag{B.12}$$

Since $\operatorname{supp}\rho\cap\operatorname{supp}(1-\chi)=\emptyset$, there is $\tilde\chi\in C_c^\infty(\mathbb{R}^n\times\mathbb{R}^{n-1})$ such that $\operatorname{supp}\rho\cap\operatorname{supp}(1-\tilde\chi)=\emptyset$, $\operatorname{supp}\tilde\chi\cap\operatorname{supp}(1-\chi)=\emptyset$. Hence, using Lemma B.3 with $A=\tilde\chi(x,\hbar D_{x'})X$, $B'=X|_{x_1=0}$, $B_1=X$, and $B=F$, by Lemma B.3,

$$\begin{aligned}
&\langle(\hbar D_{x_1}-\Lambda(x,\hbar D_{x'}))Xu,\rho(x,\hbar D_{x'})Xu\rangle\\
&\le\|(\hbar D_{x_1}-\Lambda(x,\hbar D_{x'}))\tilde\chi(x,\hbar D_{x'})Xu\|_{L^2}^2+\|\rho(x,\hbar D_{x'})\tilde\chi(x,\hbar D_{x'})Xu\|_{L^2}^2\\
&\qquad+O(\hbar^\infty)(\|\hbar D_{x_1}u\|_{L^2}^2+\|u\|_{L^2}^2)\\
&\le C\hbar^{-2}\|XPu\|_{L^2}^2+\|X(Q\hbar D_{x_1}u+u)(0)\|_{L^2}^2+\|Fu\|_{L^2}^2\\
&\qquad+O(\hbar^\infty)(\|\hbar D_{x_1}u\|_{L^2}^2+\|u\|_{L^2}^2+\|Pu\|_{L^2}^2+\|(Q\hbar D_{x_1}u+u)(0)\|_{L^2}^2+\|\hbar D_{x_1}u(0)\|_{L^2}^2).
\end{aligned}\tag{B.13}$$

Next, we consider

$$\sum_{j,k=0}^{1}\langle E_{jk}(x',\hbar D_{x'})(\hbar D_{x_1})^kXu,(\hbar D_{x_1})^j(Xu)\rangle_\Gamma.$$

First, notice that, since $\sigma(E_{11})=-t^2$,

$$\langle E_{11}\hbar D_{x_1}Xu,\hbar D_{x_1}Xu\rangle_\Gamma+\|T\hbar D_{x_1}Xu(0)\|^2\le C\hbar\|\hbar D_{x_1}Xu(0)\|_{L^2}^2,\tag{B.14}$$

Next, since $\operatorname{Re}Q\ge 0$ on $\operatorname{WF}(B_0|_{x_1=0})$ and $\sigma(B_0)\ge 0$,

$$\begin{aligned}
&\operatorname{Re}(\langle E_{10}\hbar D_{x_1}Xu,Xu\rangle_\Gamma+\langle E_{01}Xu,\hbar D_{x_1}Xu\rangle_\Gamma)\\
&=2\operatorname{Re}\langle B_0\hbar D_{x_1}Xu,(Xu+Q\hbar D_{x_1}Xu)\rangle_\Gamma-2\operatorname{Re}\langle B_0\hbar D_{x_1}Xu,Q\hbar D_{x_1}Xu\rangle_\Gamma\\
&\le 2\operatorname{Re}\langle t_0(x',\hbar D_{x'})T\hbar D_{x_1}Xu,(Xu+Q\hbar D_{x_1}Xu)\rangle_\Gamma-2\operatorname{Re}\langle B_0\hbar D_{x_1}Xu,Q\hbar D_{x_1}Xu\rangle_\Gamma\\
&\qquad+C\hbar\|\hbar D_{x_1}Xu(0)\|_{L^2}^2+C\hbar\|X(Q\hbar D_{x_1}u+u)(0)\|_{L^2}^2\\
&\qquad+O(\hbar^\infty)(\|u(0)\|_{L^2}^2+\|\hbar D_{x_1}u(0)\|_{L^2}^2).\\
&\le\epsilon\|T\hbar D_{x_1}Xu\|_{L^2}^2+C\hbar\|\hbar D_{x_1}Xu(0)\|_{L^2}^2+C\epsilon^{-1}\|X(u(0)+Q\hbar D_{x_1}u(0))\|_{L^2}^2\\
&\qquad+O(\hbar^\infty)(\|u(0)\|_{L^2}^2+\|\hbar D_{x_1}u(0)\|_{L^2}^2).
\end{aligned}\tag{B.15}$$

Finally,

$$\begin{aligned}
&2\langle E_{00}Xu,Xu\rangle_\Gamma=\langle B_1(R+R^*)Xu,Xu\rangle_\Gamma\\
&=\langle B_1(R+R^*)Q\hbar D_{x_1}Xu,Q\hbar D_{x_1}Xu\rangle_\Gamma-\langle B_1(R+R^*)Q\hbar D_{x_1}Xu,Xu+Q\hbar D_{x_1}Xu\rangle_\Gamma\\
&\qquad+\langle B_1(R+R^*)(Xu+Q\hbar D_{x_1}Xu),(Xu+Q\hbar D_{x_1}Xu)\rangle_\Gamma\\
&\qquad-\langle B_1(R+R^*)(Xu+Q\hbar D_{x_1}Xu),Q\hbar D_{x_1}Xu\rangle_\Gamma\\
&\le\langle T^*T(R+R^*)Q\hbar D_{x_1}Xu,Q\hbar D_{x_1}Xu\rangle-\langle T^*T(R+R^*)Q\hbar D_{x_1}Xu,Xu+Q\hbar D_{x_1}Xu\rangle\\
&\qquad+\langle T^*T(R+R^*)(Xu+Q\hbar D_{x_1}Xu),(Xu+Q\hbar D_{x_1}Xu)\rangle\\
&\qquad-\langle T^*T(R+R^*)(Xu+Q\hbar D_{x_1}Xu),Q\hbar D_{x_1}Xu\rangle\\
&\qquad C\hbar(\|\hbar D_{x_1}Xu\|_{L^2}^2+\|X(u+Q\hbar D_{x_1}u)\|_{L^2}^2+O(\hbar^\infty)\|\hbar D_{x_1}u(0)\|_{L^2}+\|u(0)\|_{L^2})\\
&\le C\delta\|T\hbar D_{x_1}Xu\|_{L^2}^2+C\|X(Q\hbar D_{x_1}u+u)(0)\|_{L^2}^2\\
&\qquad C\hbar(\|\hbar D_{x_1}Xu\|_{L^2}^2+\|X(u+Q\hbar D_{x_1}u)\|_{L^2}^2+O(\hbar^\infty)\|\hbar D_{x_1}u(0)\|_{L^2}+\|u(0)\|_{L^2})
\end{aligned}\tag{B.16}$$

Combining (B.10), (B.11) (B.12), (B.13), (B.14), (B.15), and (B.16), and choosing $\epsilon>0$ small enough,

$$\begin{aligned}
&2\hbar^{-1}\operatorname{Im}\langle PXu,BXu\rangle+\|\Psi_0Xu+\Psi_1\hbar D_{x_1}Xu\|_{L^2}^2+\|BXu\|_{L^2}^2+\|T\hbar D_{x_1}Xu(0)\|_{L^2}^2\\
&\le C\hbar^{-1}\|PXu\|_{L^2}^2+C\hbar^{-2}\|XPu\|_{L^2}^2+C(1+\epsilon^{-1})\|X(Q\hbar D_{x_1}u+u)(0)\|_{L^2}^2\\
&\qquad+C\hbar(\|Xu\|_{L^2}^2+\|\hbar D_{x_1}Xu\|_{L^2}^2+\|(Q\hbar D_{x_1}Xu+Xu)(0)\|_{L^2}^2+\|\hbar D_{x_1}Xu\|_{L^2}^2)\\
&\qquad+C\hbar^N(\|\hbar D_{x_1}u\|_{L^2}^2+\|u\|_{L^2}^2+\|Pu\|_{L^2}^2+\|(Q\hbar D_{x_1}u+u)(0)\|_{L^2}^2+\|\hbar D_{x_1}u(0)\|_{L^2}^2)
\end{aligned}$$

Finally,

$$2\hbar^{-1}\operatorname{Im}\langle PXu,BXu\rangle=2\hbar^{-1}\operatorname{Im}\langle XP+[P,X]u,BXu\rangle$$

$$\geq 2\hbar^{-1}\langle XPu, BXu\rangle + O(\hbar^\infty)(\|u\|_{L^2}^2 + \|\hbar D_{x_1}u\|_{L^2}^2)$$
$$\geq -C\hbar^{-2}\|PXu\|_{L^2}^2 - \frac{1}{2}\|BXu\|_{L^2}^2 + O(\hbar^\infty)(\|u\|_{L^2}^2 + \|\hbar D_{x_1}u\|_{L^2}^2)$$

and hence the estimate follows after we notice that

$$\begin{aligned}\|PXu\|_{L^2} &= \|XPu\|_{L^2} + \|[P,X]u\|_{L^2}\\ &\leq \|XP\|_{L^2} + C\hbar(\|X_+u\|_{L^2} + \|X_+\hbar D_{x_1}u\|_{L^2}) + C\hbar^N(\|u\|_{L^2} + \|\hbar D_{x_1}u\|_{L^2})\end{aligned}$$

■

We now construct the functions required to make use of Lemma B.8.

Proposition B.9 *Suppose that* $r(0,y',\eta') = 0$ *and* $\partial_{x_1}r(0,y',\eta') > 0$ *and let* $\gamma(t) = \exp(tH_p)(0,y',\eta')$. *Then for all* $N>0$, $\chi \in C_c^\infty(\mathbb{R}^n\times\mathbb{R}^{n-1})$ *with* $\chi(0,y',\eta') = 1$ *there is* $0<s<1$ *such that for all* $f \in C_c^\infty(\mathbb{R}_+\times\mathbb{R}^{n-1}\times\mathbb{R}^n)$ *with* $f(\gamma(-s)) = 1$ *there are* $C>0$, $a\in C_c^\infty(\mathbb{R}^n\times\mathbb{R}^{n-1})$ *with* $a(0,y',\eta') = 1$ *such that for all* $0<h<1$

$$\begin{aligned}&\|A(x,\hbar D_{x'})u\|_{L^2} + \|A(0,x',\hbar D_{x'})\hbar D_{x_1}u\|_{L^2}\\ &\leq C(\hbar^{-1}\|XPu\|_{L^2} + \|Fu\|_{L^2} + \|X(Q\hbar D_{x_1}u+u)(0)\|_{L^2})\\ &\quad + C\hbar^N(\|Pu\|_{L^2} + \|u\|_{L^2} + \|(Q\hbar D_{x_1}u+u)(0)\|_{L^2}).\end{aligned}$$

Proof. To prove the proposition, we show that given a neighborhood, V of $(0,y',\eta')$ there are $a, a_1 \in C_c^\infty(\mathbb{R}^n\times\mathbb{R}^{n-1})$ with $a(0,y',\eta') = 1$ and $\operatorname{supp} a_1 \subset V$ such that

$$\begin{aligned}&\|a(x,\hbar D_{x'})u\|_{L^2} + \|a(x,\hbar D_{x'})\hbar D_{x_1}u\|_{L^2} + \|a(0,x',\hbar D_{x'})\hbar D_{x_1}u(0)\|_{L^2}\\ &\leq C(\hbar^{-1}\|XPu\|_{L^2} + \|Fu\|_{L^2} + \|X(Q\hbar D_{x_1}u+u)(0)\|_{L^2})\\ &\quad + C\hbar^{1/2}(\|A_1u\|_{L^2} + \|A_1\hbar D_{x_1}u\|_{L^2} + \|A_1\hbar D_{x_1}u(0)\|_{L^2})\\ &\quad + C\hbar^N(\|Pu\|_{L^2} + \|u\|_{L^2} + \|(Q\hbar D_{x_1}u+u)(0)\|_{L^2}).\end{aligned} \tag{B.17}$$

Then, iterating this estimate and using propagation of singularities away from the boundary implies the proposition.

Define

$$\phi := \xi_1 + \phi_0(x,\xi'), \qquad \phi_0 = x_1^2 + |x'-y'|^2 + |\xi'-\eta'|^2.$$

Then,

$$H_p\phi = \{\xi_1^2 - r, \phi\} = \partial_{x_1}r + 2\xi_1\partial_{x_1}\phi - H_r\phi.$$

There is a neighborhood, U, of (y',η') and $c>0$ such that

$$\partial_{x_1}r > 4c > 0, \quad |H_r\phi_0| + |\partial_{x_1}\phi| \leq c.$$

Hence,

$$H_p\phi > c > 0, \qquad \text{on } \{(x,\xi_1,\xi') : |\xi_1|\leq 1,\ (x,\xi')\in U\}.$$

Let $\delta>0$ small enough such that

$$\{\phi_0(x',\xi') \leq 3\delta\} \subset U\cap V\cap\{\chi>0\}.$$

We start by finding a function $\tilde{b}$ that has all the required properties except that it is not linear in ξ_1. Set

$$\tilde{b} := \chi_2(\phi_0/\delta)^2\chi_0(1-\phi/\delta),$$

where

$$\chi_0(t) := \begin{cases}\exp(-1/t) & t>0\\ 0 & t\leq 0\end{cases}, \quad \chi_2 \in C_c^\infty(-3,3;[0,1]),\ \operatorname{supp}(1-\chi_2)\cap[-2,2] = \emptyset.$$

For later use, we also let $\chi_1 \in C_c^\infty((-2,2);[0,1])$ with $\operatorname{supp}(1-\chi_1)\cap[-1,1] = \emptyset$.

We claim that

$$
\begin{aligned}
H_p\tilde{b} + M\tilde{b} &= -\psi^2 + \rho(\xi_1 - r^{1/2}), \quad \text{when } p = 0 \\
\psi &:= \chi_1(\delta^{-1}\xi_1)N^{1/2}, \\
-2r^{1/2}\rho &:= \begin{cases} -(1-\chi_1(\delta^{-1}\xi_1)^2)N + \chi_0(1-\delta^{-1}\phi)H_p\chi_2(\delta^{-1}\phi_0)^2|_{\xi_1=-\sqrt{r}} & r \geq 0, \\ 0 & r < 0, \end{cases} \\
N &:= \chi_2(\delta^{-1}\phi_0)^2(\chi_0'(1-\delta^{-1}\phi)\delta^{-1}H_p\phi - \chi_0(1-\delta^{-1}\phi)M).
\end{aligned}
\tag{B.18}
$$

To see this, we observe that

$$
\begin{aligned}
\operatorname{supp}\tilde{b} &\subset \{\phi \leq \delta\} \subset \{\xi_1 \leq \delta\}, \\
\operatorname{supp}\tilde{b} \cap \operatorname{supp}(\partial(\chi_2(\delta^{-1}\phi_0)) &\subset \{\phi_0 + \xi_1 \leq \delta,\, \phi_0 \geq 2\delta\} \subset \{\xi_1 \leq -\delta\}.
\end{aligned}
$$

In particular,

$$
\operatorname{supp}\tilde{b} \cap \operatorname{supp}(\partial(\chi_2(\delta^{-1}\phi_0)) \cap \{p = 0\} \subset \{r \geq \delta^2\},
$$

and (B.18) follows by a direct calculation.

Now, we need to check that ψ and ρ are smooth function. To see that $\psi \in C^\infty$, observe that on $\operatorname{supp}\psi$, $|\xi_1| \leq 2\delta$ and $|\phi_0| \leq 3\delta$. Therefore $(1 - \delta^{-1}\phi) \leq 3$. To use this, observe that

$$
\begin{aligned}
N^{1/2} &= \chi_2(\delta^{-1}\phi_0)[\chi_0'(1-\delta^{-1}\phi_0)\delta^{-1}H_p\phi]^{1/2}(1 - M\delta\chi_0(t)/\chi_0'(t)|_{t=1-\delta^{-1}\phi})^{1/2} \\
&= \chi_2(\delta^{-1}\phi_0)[\chi_0'(1-\delta^{-1}\phi_0)\delta^{-1}H_p\phi]^{1/2}(1 - M\delta(1-\delta^{-1}\phi)^2)^{1/2}.
\end{aligned}
$$

Therefore, on $\operatorname{supp}\psi$, $1 - M\delta(1-\delta^{-1}\phi)^2 \geq 1 - 9M\delta$ and for $\delta < \frac{1}{10M}$, using that $(\chi_0')^{1/2}$ is smooth, we see that $N^{1/2}$ is smooth.

Next, to see that $\rho \in C^\infty$, observe that

$$
\operatorname{supp}(1-\chi_1^2)N \cup \operatorname{supp}\chi_0(1-\delta^{-1}\phi)H_p\chi_2(\delta^{-1}\phi_0)^2 \subset \{\xi_1 \leq -\delta\}.
$$

Hence, $\operatorname{supp}\rho \subset r \geq \delta^2$ which implies that $\rho \in C^\infty$.

Finally, we observe that $\tilde{b}|_{p=0} = v^2$ with $v = \chi_1(\xi_1)\chi_2(\delta^{-1}\phi_0)[\chi_0(1-\delta^{-1}\phi)]^{1/2}$, provided that $\delta \leq 1$.

With $\tilde{b}$ in hand, we construct the required function b. By the Malgrange preparation theorem, there are $\tilde{b}_1, b_0 \in C^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1})$ such that

$$
\tilde{b}(x,\xi) = p(x,\xi)g(x,\xi) + \tilde{b}_1(x,\xi')\xi_1 + \bar{b}_0(x,\xi')
$$

Define $\bar{b} := \bar{b}_1(x,\xi')\xi_1 + \bar{b}_0(x,\xi')$.

Observe that on $p = 0$, $H_p\bar{b} = H_p\tilde{b}$ and $b = \bar{b}$. Hence,

$$
H_p\bar{b} + M\bar{b} = -\psi^2 + \rho(\xi_1 - r^{1/2}), \qquad \bar{b} = v^2 \qquad \text{on } p = 0. \tag{B.19}
$$

In addition,

$$
\begin{aligned}
\bar{b}_1(x,\xi') &= \frac{\tilde{q}(x,\sqrt{r(x,\xi')},\xi') - \tilde{q}(x,-\sqrt{r(x,\xi')},\xi')}{2\sqrt{r(x,\xi')}}, \text{ on } r(x,\xi') > 0, \\
\bar{b}_0(x,\xi') &= \frac{\tilde{q}(x,\sqrt{r(x,\xi')},\xi') + \tilde{q}(x,-\sqrt{r(x,\xi')})}{2}, \text{ on } r(x,\xi') \geq 0.
\end{aligned}
$$

We first modify $\bar{b}_1$ and find t. Notice that $-\partial_{\xi_1}\tilde{b} = \chi_2(\delta^{-1}\phi_0)^2\chi_0'(1-\phi/\delta)\delta^{-1}\phi_1$. Hence by [62, Lemma 24.4.9] there is $W_1 \in C^\infty(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n-1})$ such that $\bar{b}_1 = -W_1^2(r,x,\xi')$ when $r > 0$. Setting $b_1 := -W_1^2(r(x,\xi'),x,\xi')$ modifies $\bar{b}_1$ on $r < 0$ and hence does not affect (B.19). Therefore, we set $t(x,\xi') := W(r(x,\xi'),0,x,\xi')$.

Now, observe that there is $\tilde{t} \in C^\infty$ with $\tilde{t}(0,x,\xi') > 0$ such that

$$
t = \chi_2(\phi_0/\delta)\tilde{t}(r(x',\xi'),x,\xi').
$$

On the other hand, by [62, Theorem C.4.4], there is $W_2 \in C^\infty(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n-1})$ such that

$$\bar{b}_0(x,\xi') = \chi_2^2(\phi_0/\delta)W_2(r(x,\xi'),x,\xi'), \qquad r>0.$$

Setting $b_0(x,\xi') = \chi_2^2(\phi_0/\delta)W_2(r(x,\xi'),x,\xi')$ modifies b_0 only on $r<0$ and hence does not affect (B.19). Moreover, we have $b_0 = tt_0$ with $t_0 := \chi_2(\phi_0/\delta)W_2(r(x,\xi'),x,\xi')/\tilde{t}(r(x',\xi'),x,\xi') \in C^\infty$ since $\tilde{t}$ is non-vanishing near $r=0$.

Defining $b := b_1\xi_1 + b_0$ and letting $\tilde{\chi},\tilde{\chi}_+ \in C_c^\infty(\mathbb{R}^n\times\mathbb{R}^{n-1})$ with $\operatorname{supp}\tilde{\chi}_+ \subset U\cap\{\chi>0\}$, $\operatorname{supp}(1-\chi_+)\cap\operatorname{supp}\chi=\emptyset$, and $\operatorname{supp}(1-\chi)\cap\operatorname{supp}b_i=\emptyset$. we have verified all the hypotheses of Lemma B.8 and hence the estimate (B.9) holds for any $\delta>0$ small enough. Now, observe that

$$t(y',\eta') = (\chi_0'(1)\delta^{-1})^{1/2}, \qquad \psi_0^2(0,y',\eta') = \chi_0'(1)\partial_{x_1}r(0,y',\eta')\delta^{-1} - M\chi_0(1)$$
$$2\psi_0\psi_1(0,y',\eta') = -\chi_0''(1)\partial_{x_1}r(0,y',\eta')\delta^{-2} + \chi_0'(1)M\delta^{-1}.$$

In particular $t(y',\eta')>c$ and $\psi_1/\psi_0 = -c\delta^{-1}+O(1)$. hence, choosing, for instance δ and $\delta/2$, there is $a\in C_c^\infty(\mathbb{R}^n\times\mathbb{R}^{n-1})$ such that $a(0,y',\eta')=1$,

$$\begin{aligned}\|a(x,\hbar D_{x'})u\|_{L^2} + \|a(x,\hbar D_{x'})\hbar D_{x_1}u\|_{L^2} &\le C(\|\Psi_0^\delta u + \Psi_1^\delta \hbar D_{x_1}u\|_{L^2} + \|\Psi_0^{\delta/2}u + \Psi_1^{\delta/2}\hbar D_{x_1}u\|_{L^2})\\ &\quad + C\hbar^N(\|u\|_{L^2} + \|\hbar D_{x_1}u\|_{L^2})\\ \|a(0,x',\hbar D_{x'})\hbar D_{x_1}u(0)\|_{L^2} &\le C\|T^\delta \hbar D_{x_1}u(0)\|_{L^2} + C\hbar^N\|\hbar D_{x_1}u\|_{L^2}.\end{aligned}$$

Hence, using that $\operatorname{supp}\tilde{\chi}_+\subset V\cap\{\chi>0\}$ and letting s small enough that $\exp(-tH_p)(0,y',\eta')\subset\{\chi>0\}$, for $0\le t\le s$, we have proved (B.17) and hence the proposition. ■

B.5 Non-diffractive points

Finally, we consider rays tangent to the boundary that are non-diffractive. Once again, we proceed by using a positive commutator type estimate followed by a careful construction of an escape function.

Lemma B.10 *Let $Q\in\Psi^{-1}(\Gamma)$ with $\operatorname{Re}\sigma(Q)|_{r=0}\ge 0$. Then there are $\delta_0, M_0>0$ such that for all $N>0$, $M\ge M_0$, $b,v,a,f,\chi,\chi_+\in C_c^\infty(\mathbb{R}^n\times\mathbb{R}^{n-1};\mathbb{R})$, satisfying*

$$b|_{x_1=0}\ge 0, \qquad \operatorname{supp}b\cap\operatorname{supp}(1-\chi)=\emptyset, \qquad \operatorname{supp}\chi\cap\operatorname{supp}(1-\chi_+)=\emptyset, \qquad \operatorname{supp}\chi_+\subset\{|r|\le\delta_0\},$$
$$\{p,b\}+bM = -\psi^2 - a^2 + f^2, \quad b=v^2, \qquad \text{when } \xi_1^2 = r,$$

with $\psi\in C_c^\infty(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R})$ therehere is $C>0$ such that, defining $A:=a(x,\hbar D_{x'})$ $B:=b(x,\hbar D_{x'})$, $F=f(x,\hbar D_{x'})$, $X:=\chi(x,\hbar D_{x'})$, for all and $0<\mu<1$, $0<h<1$,

$$\begin{aligned}&\|\Psi_0 u + \Psi_1\hbar D_{x_1}u\|_{L^2} + \|AXu\|_{L^2}\\ &\le C\hbar^{-1}\|X_+Pu\| + C\mu^{-1}\|X_+(u+Q\hbar D_{x_1}u)(0)\|_{L^2} + \|FXu\|_{L^2} + (\mu+\hbar^{1/2})\|X_+u\|_{L^2}\\ &+ C\hbar^N(\|u(0)\|_{L^2} + \|\hbar D_{x_1}u(0)\|_{L^2}),\end{aligned}$$

where $\sigma(\Psi_j)(0,x',0,\xi')|_{r=0} = \partial_{\xi_1}^j\psi_j(0,x',0,\xi')|_{r=0}$.

Proof. The proof is similar to that of Lemma B.8. The changes are that we estimate

$$\begin{aligned}&\operatorname{Re}\sum\langle C_{jk}(x,\hbar D_{x'})(\hbar D_{x_1})^k Xu, (\hbar D_{x_1})^j Xu\rangle + \operatorname{Re}\langle(M-2\operatorname{Im}p_{-1}(x,\hbar D_{x'}))Xu, BXu\rangle\\ &\qquad + \|AXu\|_{L^2}^2 - \|FXu\|_{L^2}^2\\ &\le -\|\Psi_0 Xu + \Psi_1\hbar D_{x_1}Xu\|^2 + C\hbar^{-1}\|PXu\|_{L^2}^2 + C\hbar(\|Xu\|_{L^2}^2 + \|\hbar D_{x_1}Xu\|_{L^2}^2)\\ &\qquad + C\hbar\|\hbar D_{x_1}Xu\|^2 + C\hbar\|(uX + Q\hbar D_{x_1}Xu)\|_{L^2}^2,\end{aligned}$$

that $E_{00}=E_{11}=0$, and instead of (B.15) we use that $\operatorname{Re}Q\ge 0$ on $\mathrm{WF}(B|_{x_1=0})$ and $\sigma(B)\ge 0$, to obtain

$$\begin{aligned}&\operatorname{Re}(\langle E_{10}\hbar D_{x_1}Xu, Xu\rangle_\Gamma + \langle E_{01}Xu, \hbar D_{x_1}Xu\rangle_\Gamma)\\ &= 2\operatorname{Re}\langle B\hbar D_{x_1}Xu, (Xu+Q\hbar D_{x_1}Xu)\rangle_\Gamma - 2\operatorname{Re}\langle B\hbar D_{x_1}Xu, Q\hbar D_{x_1}Xu\rangle_\Gamma\\ &\le \mu^2\|B\hbar D_{x_1}Xu(0)\|_{L^2}^2 + C\hbar\|\hbar D_{x_1}Xu(0)\|_{L^2}^2 + C\mu^{-2}\|X(u(0)+Q\hbar D_{x_1}u(0))\|_{L^2}^2\\ &\quad + O(\hbar^\infty)(\|u(0)\|_{L^2}^2 + \|\hbar D_{x_1}u(0)\|_{L^2}^2).\end{aligned} \tag{B.20}$$

We then use Lemma B.7 to estimate $\|B\hbar D_{x_1}Xu(0)\|$ and $\|\hbar D_{x_1}Xu\|_{L^2}$ completing the proof of the lemma. ∎

Lemma B.11 *Let $N > 0$ $(y', \eta') \in \mathbb{R}^{2n-2}$, $r_0(y', \eta') = 0$ and $\chi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1})$ with $\chi(0, y', \eta') = 1$. Then there are $\epsilon_0 > 0, C_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, there is $\delta_\epsilon > 0$ such that for all $\delta \in (0, \delta_\epsilon)$ and $f \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1})$ with*

$$\{(x_1, x', \xi') : x_1 < C_0\epsilon\delta, (x', \xi') - (y', \eta') - \delta H_{r_0}(y', \eta')| < C_0\epsilon\delta\} \subset \{f > 0\},$$

there is $a \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1})$ with $a(0, y', \eta') = 1$ such that for any $\beta > 0$, there is $C > 0$ such that for all $0 < h < 1$,

$$\begin{aligned}\|Au\|_{L^2} \le C(\hbar^{-1}\|XPu\|_{L^2} + \|Fu\|_{L^2} + h^{-\beta}\|X(Q\hbar D_{x_1}u + u)(0)\|_{L^2})\\ + C\hbar^N(\|u\|_{L^2} + \|Pu\|_{L^2} + \|(Q\hbar D_{x_1}u + u)(0)\|_{L^2}).\end{aligned}$$

Proof. In order to start the proof, we need to define a hypersurface transversal to the flow of H_{r_0}. For this, we let $N \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ with $H_{r_0}N(x', \xi') = 1$ in a neighborhoood of (y', η') and $N(y', \eta') = 0$. We then define

$$\Sigma_{(y', \eta')} := \{(x', \xi') : N(x', \xi') = 0\}.$$

Let $\omega \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ solve

$$H_{r_0}\omega = 0 \text{ in neighborhood of } (y', \eta'), \qquad \omega = |x' - y'|^2 + |\xi' - \eta'|^2, \qquad (x', \xi') \in \Sigma_{(y', \eta')}. \tag{B.21}$$

Since $H_{r_0}N(y', \eta') = 1$, this uniquely defines ω in a neighborhood of (y', η').

Let $\gamma := \{\exp(tH_{r_0})(y', \eta) : |t| \le 1\}$. Then, $\omega|_\gamma = 0$, $\partial\omega|_\gamma = 0$, and there is $c > 0$ such that $\omega(x', \xi') \ge cd((x', \xi'), \gamma)^2$. In particular, since $r_0|_{\gamma=0}$ and $\partial\omega$ vanish on γ,

$$|r_0| + |\partial\omega| \le C\omega^{1/2}.$$

Now, set

$$\phi(x, \xi') := -N(x', \xi') + \frac{x_1}{\epsilon} + \frac{\omega}{\epsilon^2\delta}.$$

Then,

$$H_p\phi = -H_rN + \frac{2\xi_1}{\epsilon} - \frac{H_r\omega}{\delta\epsilon^2}.$$

Notice that

$$H_r\omega = H_{r_0}\omega + (H_r - H_{r_0})\omega = O(|\partial(r(x_1, x', \xi') - r(0, x', \xi'))||\partial\omega|) = O(x_1\omega^{1/2}),$$

and when $\xi_1^2 = r$,

$$\xi_1^2 = r \le r_0 + O(x_1) = O(\omega^{1/2} + x_1).$$

Next, when $N(x', \xi') \le 2\delta$ and $\phi(x, \xi') \le 2\delta$, we have

$$4\delta\epsilon \ge N\epsilon + 2\delta\epsilon \ge \epsilon\phi + N\epsilon \ge x_1 + \frac{\omega}{\delta\epsilon}.$$

Hence, since $x_1, \omega \ge 0$,

$$x_1 \le 4\delta\epsilon, \qquad \omega \le 4\delta^2\epsilon^2.$$

Finally, observe that

$$H_rN = 1 + (H_r - H_{r_0})N = 1 + O(x_1)$$

Therefore, for $\delta > 0$ small enough and $\epsilon \le 1$,

$$\begin{aligned}H_p\phi = H_rN + O(\epsilon^{-1}(\omega^{1/2} + x_1)^{1/2}) + O(\delta^{-1}\epsilon^{-2}x_1\omega^{1/2}) = 1 + O(\delta^{1/2}\epsilon^{-1/2})\\ \text{on } \{p = 0\} \cap \{N \le 2\delta\} \cap \{\phi \le 2\delta\}.\end{aligned}$$

Now, define

$$\chi_0(t) := \begin{cases} \exp(-1/t) & t > 0 \\ 0 & t \leq 0, \end{cases}$$

and let $\chi_1(t) \in C^\infty(\mathbb{R})$ with $\operatorname{supp} \chi_1 \subset (0, \infty)$, $\operatorname{supp}(1-\chi_1) \subset (-\infty, 1)$ and such that for all $\beta < 1$, there is $C_{\alpha\beta}$ such that

$$|D^\alpha \chi_1'| \leq C_{\alpha\beta} (\chi_1')^\beta.$$

Let $0 \leq t \leq 1$ and set

$$b_t(x', \xi') := \chi_0(1 + t - \delta^{-1}\phi)\chi_1(\epsilon^{-1}\delta^{-1}(-N(x', \xi') + \delta) + t).$$

Then,

$$H_p b_t = \chi_0' \delta^{-1} H_p \phi \chi_1 + \chi_0 \chi_1' \epsilon^{-1} \delta^{-1} H_r N$$

Observe that for $t_1 < t_2$,

$$\operatorname{supp} b_t \subset \{\phi \leq \delta(1+t)\} \cap \{N \leq \delta(1 + \epsilon t)\} \subset \{x_1 \leq 4\delta\epsilon,\ \omega \leq 4\delta^2\epsilon^2\} \cap \{N \leq 2\delta\}$$
$$\operatorname{supp} b_{t_1} \subset \{b_{t_2} > 0\},$$

In particular, for ϵ, δ small enough, $\operatorname{supp} b_t \subset \{\chi > 0\}$.

Now, define

$$f_t := (\chi_0(1 + t - \delta^{-1}\phi)\chi_1'(\epsilon^{-1}\delta^{-1}(-N + \delta) + t)\epsilon^{-1}\delta^{-1} H_r N)^{1/2}.$$

We claim that f_t is smooth and, setting $(y'(\delta), \eta'(\delta)) := \exp(\delta H_{r_0})(y', \eta')$,

$$\operatorname{supp} f_t \subset \{x_1^2 + |x' - y'(\delta)|^2 + |\xi' - \eta'(\delta)|^2 \leq C(\epsilon\delta)^2\} \subset \{f > 0\}. \tag{B.22}$$

To prove the claim first observe that

$$\operatorname{supp} f_t \subset \{|\delta - N| \leq \epsilon\delta\} \cap \{\phi \leq 2\delta\} \subset \{x_1 \leq 4\epsilon\delta, \omega \leq 4\epsilon^2\delta^2\}.$$

Therefore, since $(\chi_1')^{1/2}$, $\chi_0^{1/2}$ are smooth functions, and $H_r N > c > 0$ on $\operatorname{supp} f_t$, f_t is smooth. Moreover, on $\operatorname{supp} f_t$,

$$|N(y'(\delta), \eta'(\delta)) - N(x', \xi')| \leq \epsilon\delta + O(\delta^2).$$

Now, since $\omega \sim d((x', \xi'), \gamma)^2$ and $H_{r_0} N = 1$,

$$|N(x', \xi') - N(y'(\delta), \eta'(\delta))^2 + \omega \sim |x' - y'(\delta)|^2 + |\xi' - \eta'(\delta)|^2$$

and hence, for $\delta > 0$ small enough, we have proved (B.22).

Next, we find a_t and ψ_t such that

$$\chi_1(\chi_0'(1 + t - \delta^{-1}\phi)\delta^{-1} H_p \phi - M\chi_0(1 + t - \delta^{-1})\phi) = \psi_t^2 + a_t^2.$$

Define

$$a_t := (\chi_1 \chi_0'(1 + t - \delta^{-1}\phi)/(2\delta))^{1/2},$$
$$\psi_t := \chi_1^{1/2}(\chi_0'(1 + t - \delta^{-1}\phi)(H_p\phi - 1/2)\delta^{-1} - \chi_0(1 + t - \delta^{-1}\phi)M)^{1/2}.$$

Now, $a_t \in C^\infty$ since $\chi_1^{1/2} \in C^\infty$ and $(\chi_0')^{1/2} \in C^\infty$. Furthermore, for $0 \leq t \leq 1$ and $\epsilon < 1$,

$$a_t(0, y', \eta') = (\chi_0'(1+t)\chi_1(\epsilon^{-1} + t))^{1/2}(2\delta)^{-1/2} > c\delta^{-1/2}$$

To see that ψ_t is smooth in a neighborhood of $p = 0$, observe that $\chi_0(s)/\chi_0'(s) = s^2$ and hence,

$$\psi_t = \chi_1^{1/2}(\chi_0')^{1/2}((H_p\phi - 1/2)\delta^{-1} - (1 + t - \delta^{-1}\phi)^2 M)^{1/2}.$$

Now, $H_p\phi \geq \frac{3}{4}$ on $\operatorname{supp}\chi_1 \cap \operatorname{supp}\chi_0 \cap \{p=0\}$ and $(1+t-\delta^{-1}\phi) \leq 4$. Hence

$$(H_p\phi - 1/2)\delta^{-1} - (1+t-\delta^{-1}\phi)^2 M \geq \frac{1}{4\delta} - 16M \geq 1$$

for $\delta > 0$ small enough. Cutting off to an appropriate neighborhood of the characteristic set, we see that $\psi_t \in C^\infty$.

Now, let $0 \leq t_< < \cdots < t_1 < t_0 \leq 1$, $i = 1, \ldots, N$, $\operatorname{supp} b_{t_i} \subset \{a_{t_{i-1}} > 0\}$ and hence, letting $\chi_{t_i}, \chi_{+,t_i} \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1})$ with $\operatorname{supp}(1-\chi_{t_i}) \cap \operatorname{supp} b_{t_i} = \emptyset$, $\operatorname{supp}\chi_{t_i} \cap \operatorname{supp}(1-\chi_{+,t_i}) = \emptyset$, and $\operatorname{supp}\chi_{+,t_i} \subset \{a_{t_{i-1}} > 0\}$, we obtain from Lemma B.10 that for any $\mu > 0$,

$$\begin{aligned}
&\|A_{t_i} X u\|_{L^2} \\
&\leq C\hbar^{-1}\|X_{t_i} P u\| + C\mu^{-1}\|X_{t_i}(u + Q\hbar D_{x_1} u)(0)\|_{L^2} + C\|F_{t_i} X u\|_{L^2} + (\mu + \hbar^{1/2})\|X_{+,t_i} u\|_{L^2} \\
&\qquad + C\hbar^N(\|u(0)\|_{L^2} + \|\hbar D_{x_1} u(0)\|_{L^2} + \|u\|_{L^2} + \|Pu\|_{L^2}) \\
&\leq C\hbar^{-1}\|XPu\| + C\mu^{-1}\|X(u + Q\hbar D_{x_1} u)(0)\|_{L^2} + C\|Fu\|_{L^2} + (\mu + \hbar^{1/2})\|X_{+,t_i} u(0)\|_{L^2} \\
&\qquad + C\hbar^N(\|u(0)\|_{L^2} + \|\hbar D_{x_1} u(0)\|_{L^2} + \|u\|_{L^2} + \|Pu\|_{L^2})
\end{aligned}$$

Using this for $t_M < t_{M-1} < \cdots < t_0$, we have and letting

$$\begin{aligned}
&\|A_{t_M} X u\|_{L^2} \\
&\leq C\hbar^{-1}\|X_P u\| + C\mu^{-1}\|X(u + Q\hbar D_{x_1} u)\|_{L^2} + \|Fu\|_{L^2} + (\mu + \hbar^{1/2})^M \|X_{+,t_0} \hbar D_{x_1} X u(0)\|_{L^2} \\
&\qquad + C\hbar^N(\|u(0)\|_{L^2} + \|\hbar D_{x_1} u(0)\|_{L^2} + \|u\|_{L^2} + \|Pu\|_{L^2}).
\end{aligned}$$

Taking $\mu = \hbar^\beta$ and $M \geq N/\beta$ completes the proof. ∎

Acknowledgements

The authors thank Ivan Graham (University of Bath) and Pierre Marchand (INRIA Paris) for useful discussions, Alex Barnett (Flatiron Institute) for the parametrization of the cavity domain, and Jeremy Hoskins (University of Chicago) for the generalized Gaussian quadrature rule used in the Galerkin method, and Andras Vasy for a helpful discussion about the proof of propagation of singularities for boundary value problems.

JG was supported by EPSRC grants EP/V001760/1 and EP/V051636/1, Leverhulme Research Project Grant RPG-2023-325, and ERC Synergy Grant PSINumScat - 101167139, and EAS was supported by EPSRC grant EP/R005591/1 and ERC Synergy Grant PSINumScat - 101167139.

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