Eigenvalues of the truncated Helmholtz solution operator under strong trapping

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January 22, 2021

Abstract

For the Helmholtz equation posed in the exterior of a Dirichlet obstacle, we prove that if there exists a family of quasimodes (as is the case when the exterior of the obstacle has stable trapped rays), then there exist near-zero eigenvalues of the standard variational formulation of the exterior Dirichlet problem (recall that this formulation involves truncating the exterior domain and applying the exterior Dirichlet-to-Neumann map on the truncation boundary).

The significance of this result is a) the finite-element method for computing approximations to solutions of the Helmholtz equation is based on the standard variational formulation, and b) the location of eigenvalues, and especially near-zero ones, plays a key role in understanding how iterative solvers such as the generalised minimum residual method (GMRES) behave when used to solve linear systems, in particular those arising from the finite-element method.

The result proved in this paper is thus the first step towards rigorously understanding how GMRES behaves when applied to discretisations of high-frequency Helmholtz problems under strong trapping (the subject of the companion paper [MGSS21]).

1 Introduction

1.1 Preliminary definitions

Let $\Omega_- \subset \mathbb{R}^d, d \geq 2$ be a bounded open set such that its open complement $\Omega_+ := \mathbb{R}^d \setminus \overline{\Omega_-}$ is connected. Let $\Gamma_D := \partial\Omega_-$, where the subscript D stands for "Dirichlet". Let Ω_1 be another bounded open set such that $\mathrm{conv}(\Omega_-) \Subset \Omega_1$, where conv denotes the convex hull and $\Subset$ denotes compact containment. Let $\Omega_{\rm tr} := \Omega_1 \setminus \Omega_-$, and $\Gamma_{\rm tr} := \partial\Omega_1$, where the subscript tr stands for "truncated". We assume throughout that Γ_D and $\Gamma_{\rm tr}$ are both C^∞. Let γ_0^D and $\gamma_0^{\rm tr}$ denote the Dirichlet traces on Γ_D and $\Gamma_{\rm tr}$ respectively, and let γ_1^D and $\gamma_1^{\rm tr}$ denote the respective Neumann traces, where the normal vector points out of $\Omega_{\rm tr}$ on both Γ_D and $\Gamma_{\rm tr}$. Let

$$H^1_{0,D}(\Omega_{\rm tr}) := \big\{ v \in H^1(\Omega_{\rm tr}) : \gamma_0^D v = 0 \big\}.$$

Let $\mathcal{D}(k) : H^{1/2}(\Gamma_{\rm tr}) \to H^{-1/2}(\Gamma_{\rm tr})$ be the Dirichlet-to-Neumann map for the equation $\Delta u + k^2 u = 0$ posed in the exterior of Ω_1 with the Sommerfeld radiation condition

$$\frac{\partial u}{\partial r}(x) - \mathrm{i}ku(x) = o\left(\frac{1}{r^{(d-1)/2}}\right) \tag{1.1}$$

as $r := |x| \to \infty$, uniformly in $\widehat{x} := x/r$. We say that a function satisfying (1.1) is *k-outgoing*. When $\Gamma_{\rm tr} = \partial B_R$, for some $R > 0$, the definition of $\mathcal{D}(k)$ in terms of Hankel functions and polar coordinates (when $d = 2$)/spherical polar coordinates (when $d = 3$) is given in, e.g., [MS10, Equations 3.7 and 3.10].

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Definition 1.1 (Eigenvalues of the truncated exterior Dirichlet problem) *We say μ_ℓ is an* eigenvalue of the truncated exterior Dirichlet problem at frequency $k_\ell > 0$, *with corresponding eigenfunction u_ℓ, if $u_\ell \in H^1_{0,D}(\Omega_{\rm tr}) \setminus \{0\}$ and $\mu_\ell \in \mathbb{C}$ satisfies*

$$(\Delta + k_\ell^2)u_\ell = \mu_\ell u_\ell \quad \text{in } \Omega_{\rm tr} \quad \text{and} \quad \gamma_1^{\rm tr} u_\ell = \mathcal{D}(k_\ell)(\gamma_0^{\rm tr} u_\ell). \tag{1.2}$$

Definition 1.2 (Quasimodes) *A* family of quasimodes of quality $\epsilon(k)$ *is a sequence $\{(u_\ell, k_\ell)\}_{\ell=1}^\infty \subset H^2(\Omega_{\rm tr}) \cap H^1_{0,D}(\Omega_{\rm tr}) \times \mathbb{R}$ such that the frequencies $k_\ell \to \infty$ as $\ell \to \infty$ and there is a compact subset $\mathcal{K} \Subset \Omega_1$ such that, for all ℓ,* supp $u_\ell \subset \mathcal{K}$,

$$\big\|(\Delta + k_\ell^2)u_\ell\big\|_{L^2(\Omega_{\rm tr})} \le \epsilon(k_\ell) \quad \text{and} \quad \|u_\ell\|_{L^2(\Omega_{\rm tr})} = 1.$$

Definition 1.3 (Quasimodes with multiplicity) *Let $\{(u_\ell, k_\ell)\}_{\ell=1}^\infty$ be a quasimode with quality $\epsilon(k)$ and let $\{(m_j, k_j^-, k_j^+)\}_{j=1}^\infty \subset \mathbb{N} \times \mathbb{R}^2$ be such that $k_j^- \to \infty$ and $k_j^- \le k_j^+$. Define*

$$\mathcal{W}_j := \big\{\ell \,:\, k_\ell \in [k_j^-, k_j^+]\big\}.$$

We say u_ℓ has multiplicity m_j in the window $[k_j^-, k_j^+]$ if

$$|\mathcal{W}_j| = m_j, \qquad |\langle u_{\ell_1}, u_{\ell_2}\rangle_{L^2(\Omega_{\rm tr})}| \le \epsilon(k_j^-) \quad \text{for } \ell_1 \neq \ell_2, \ \ell_1, \ell_2 \in \mathcal{W}_j.$$

We assume throughout that the quality, $\epsilon(k)$, of a quasimode is a decreasing function of k; this can always be arranged by replacing $\varepsilon(k)$ by $\tilde{\varepsilon}(k) := \sup_{\tilde{k} \ge k} \varepsilon(\tilde{k})$.

We use the notation that $A = \mathcal{O}(k^{-\infty})$ as $k \to \infty$ if, given $N > 0$, there exists C_N and k_0 such that $|A| \le C_N k^{-N}$ for all $k \ge k_0$, i.e. A decreases superalgebraically in k.

1.2 The main results

Theorem 1.4 (From quasimodes to eigenvalues) *Let $\alpha > 3(d+1)/2$. Suppose there exists a family of quasimodes of quality $\epsilon(k)$ with*

$$S_1 \exp(-S_2 k) \le \epsilon(k) \ll k^{1-\alpha}$$

for some $S_1, S_2 > 0$. Then there exists $k_0 > 0$ (depending on α) such that, if ℓ is such that $k_\ell \ge k_0$, then there exists an eigenvalue of the truncated exterior Dirichlet problem at frequency k_ℓ satisfying

$$|\mu_\ell| \le k_\ell^\alpha \epsilon(k_\ell),$$

We now give three specific cases when the assumptions of Theorem 1.4 hold. The first two cases are via the quasimode constructions of [BCWG+11a, Theorem 2.8, Equations 2.20 and 2.21] and [CP02, Theorem 1] for obstacles whose exteriors support elliptic-trapped rays. The third case is via the "resonances to quasimodes" result of [Ste00, Theorem 1]; recall that the resonances of the exterior Dirichlet problem are the poles of the meromorphic continuation of the solution operator from $\operatorname{Im} k \ge 0$ to $\operatorname{Im} k < 0$; see, e.g., [DZ19, Theorem 4.4. and Definition 4.6].

Lemma 1.5 (Specific cases when the assumptions of Theorem 1.4 hold)

(i) Let $d = 2$. Given $a_1 > a_2 > 0$, let

$$E := \left\{(x_1, x_2) \,:\, \left(\frac{x_1}{a_1}\right)^2 + \left(\frac{x_2}{a_2}\right)^2 < 1\right\}. \tag{1.3}$$

If Γ_D coincides with the boundary of E in the neighborhoods of the points $(0, \pm a_2)$, and if Ω_+ contains the convex hull of these neighbourhoods, then the assumptions of Theorem 1.4 hold with

$$\epsilon(k) = \exp(-C_1 k)$$

*for some $C_1 > 0$ (independent of k).**

* In [BCWG+11a, Theorem 2.8], Ω_+ is assumed to contain the whole ellipse E. However, inspecting the proof, we see that the result remains unchanged if E is replaced with the convex hull of the neighbourhoods of $(0, \pm a_2)$. Indeed, the idea of the proof is to consider a family of eigenfunctions of the ellipse localising around the periodic orbit $\{(0, x_2) : |x_2| \le a_2\}$.

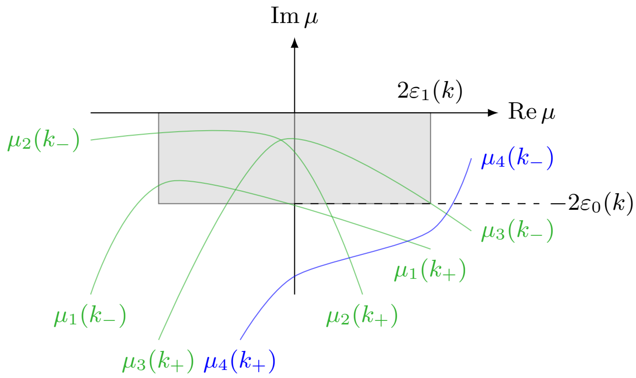


Figure 1.1: Paths of the eigenvalues, μ_j, of the truncated problem are shown as functions of $k \in [k_-, k_+]$. Those eigenvalues shown in green correspond to members of the box $\mathcal{E}$ defined by (1.5) (shaded), while the eigenvalue in blue is not in $\mathcal{E}$.

(ii) Suppose $d \geq 2$, $\Gamma_D \in C^\infty$, and Ω_+ contains an elliptic-trapped ray such that (a) Γ_D is analytic in a neighbourhood of the ray and (b) the ray satisfies the stability condition [CP02, (H1)]. If $q > 11/2$ when $d = 2$ and $q > 2d + 1$ when $d \geq 3$, then the assumptions of Theorem 1.4 hold with

$$\epsilon(k) = \exp(-C_2 k^{1/q})$$

for some $C_2 > 0$ (independent of k).

(iii) Suppose there exists a sequence of resonances $\{\lambda_j\}_{\ell=1}^\infty$ of the exterior Dirichlet problem with

$$0 \leq -\operatorname{Im}\lambda_j = \mathcal{O}\big(|\lambda_j|^{-\infty}\big) \quad \text{and} \quad \operatorname{Re}\lambda_\ell \to \infty \quad \text{as} \quad \ell \to \infty. \tag{1.4}$$

then there exists a family of quasimodes of quality $\epsilon(k) = \mathcal{O}(k^{-\infty})$ and thus the assumptions of Theorem 1.4 hold.

Remark 1.6 (Resonances $\iff$ quasimodes $\iff$ eigenvalues) *Part (iii) of Lemma 1.5 is the "resonances to quasimodes" result of [Ste00, Theorem 1]. The converse implication, i.e. that a family of quasimodes of quality $\epsilon(k) = \mathcal{O}(k^{-\infty})$ implies a sequence of resonances satisfying (1.4), was proved in [TZ98], [Ste99] (following [SV95, SV96]), see also [DZ19, Theorem 7.6]. Therefore the "quasimodes to eigenvalues" result of Theorem 1.4 is equivalent to a "resonances to eigenvalues" result. In fact, in Appendix A we show that the existence of $\mathcal{O}(k^{-\infty})$ eigenvalues implies the existence of quasimodes of quality $\mathcal{O}(k^{-\infty})$. We therefore have that resonances $\iff$ quasimodes $\iff$ eigenvalues.*

With $\{\mu_j(k)\}_j$ the set of eigenvalues, counting multiplicities, of the truncated exterior Dirichlet problem at frequency k, let

$$\mathcal{E}(\varepsilon_1, \varepsilon_0, k_-, k_+) := \Big\{\mu_j(k) \,:\, \mu_j(k) \in (-2\varepsilon_1, 2\varepsilon_1) - \mathrm{i}(0, 2\varepsilon_0) \text{ for some } k \in [k_-, k_+]\Big\}; \tag{1.5}$$

$\mathcal{E}$ is therefore the counting function of the eigenvalues, $\mu_j(k)$, that pass through a rectangle next to zero in μ as k varies in the interval $[k_-, k_+]$; see Figure 1.1. †

Theorem 1.7 (From quasimodes to eigenvalues, with multiplicities) *Let $k_j^-, k_j^+ \to \infty$ such that there is $C > 0$ satisfying $k_j^- \leq k_j^+ \leq C k_j^-$. Suppose there exists a family of quasimodes of quality $\epsilon(k) \ll k^{-(5d+3)/2}$ and multiplicity m_j in the window $[k_j^-, k_j^+]$ (in the sense of Definition 1.3). If $\epsilon_0(k)$ is such that, for some $S_3, S_4, S_5 > 0$,*

$$S_3 \exp(-S_4 k) \leq \epsilon_0(k) \leq S_5 k^{-(d+1)/2} \text{ for all } k \quad \text{and} \quad \epsilon_0(k) \gg k^{2d+1}\epsilon(k) \text{ as } k \to \infty,$$

† In Figure 1.1 we have drawn the paths of the eigenvalues as arbitrary curves. We see later in Figure 1.7 an example where the paths appear to be horizontal lines; this is consistent with the intuition that eigenvalues should be shifted resonances.

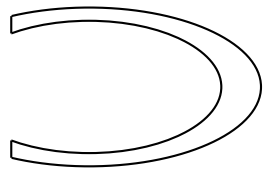


Figure 1.2: The two obstacles Ω_- considered in the numerical experiments

then there exists $k_0 > 0$ *such that if* $k_j^- \geq k_0$,

$$\left|\mathcal{E}\Big((k_j^-)^{(d+1)/2}\epsilon_0(k_j^-)\,,\,\epsilon_0(k_j^-)\,,\,k_j^-\,,\,k_j^+\Big)\right| \geq m_j.$$

Observe that if $k_j^+ = k_j^-$, then (up to algebraic powers of k) Theorem 1.7 reduces to Theorem 1.4, except that now multiplicities are counted; therefore the "quasimodes to eigenvalues" result holds with multiplicities (just as the "quasimodes to resonances" result of [Ste99] includes multiplicities).

The ideas used in the proof of Theorems 1.4 and 1.7 are discussed in §1.5 below.

1.3 Numerical experiments illustrating the main results

Description of the obstacles Ω_-. In this section, Ω_- is one of the two "horseshoe-shaped" 2-d domains shown in Figure 1.2. We define the *small cavity* as the region between the two elliptic arcs

$$(\cos(t), 0.5\sin(t)), \quad t \in [-\phi_0, \phi_0] \quad \text{and} \quad (1.3\cos(t), 0.6\sin(t)), \quad t \in [-\phi_1, \phi_1]$$
$$\text{with } \phi_0 = 7\pi/10 \quad \text{and} \quad \phi_1 = \arccos\left(\frac{1}{1.3}\cos(\phi_0)\right);$$

this corresponds to the interior of the solid lines in Figure 1.2. We define the *large cavity* as the region between the two arcs now with $\phi_0 = 9\pi/10$. (Note that our small cavity is the same as the cavity considered in the numerical experiments in [BCWG+11b, Section IV].)

In both cases, Γ_D coincides with the boundary of the ellipse E (1.3) with $a_1 = 1$ and $a_2 = 0.5$ in the neighbourhood of its minor axis. Part (i) of Lemma 1.5 (i.e., the results of [BCWG+11a]) then implies that there exist quasimodes with exponentially-small quality.

We choose these particular Ω_- because we can compute the frequencies k_ℓ in the quasimode. Indeed, the functions u_ℓ in the quasimode construction in [BCWG+11a] are based on the family of eigenfunctions of the ellipse localising around the periodic orbit $\{(0, x_2) : |x_2| \leq a_2\}$; when the eigenfunctions are sufficiently localised, the eigenfunctions multiplied by a suitable cut-off function form a quasimode, with frequencies k_ℓ equal to the square roots of eigenvalues of the ellipse. By separation of variables, k_ℓ can be expressed as the solution of a multiparametric spectral problem involving Mathieu functions; see see [BCWG+11a, Appendix A] and [MGSS21, Appendix C].

When giving specific values of k_ℓ below, we use the notation from [BCWG+11a, Appendix A] and [MGSS21, Appendix D] that $k^e_{m,n}$ and $k^o_{m,n}$ are the frequencies associated with the eigenfunctions of the ellipse that are even/odd, respectively, in the angular variable, with m zeros in the radial direction (other than at the centre or the boundary) and n zeros in the angular variable in the interval $[0, \pi)$.

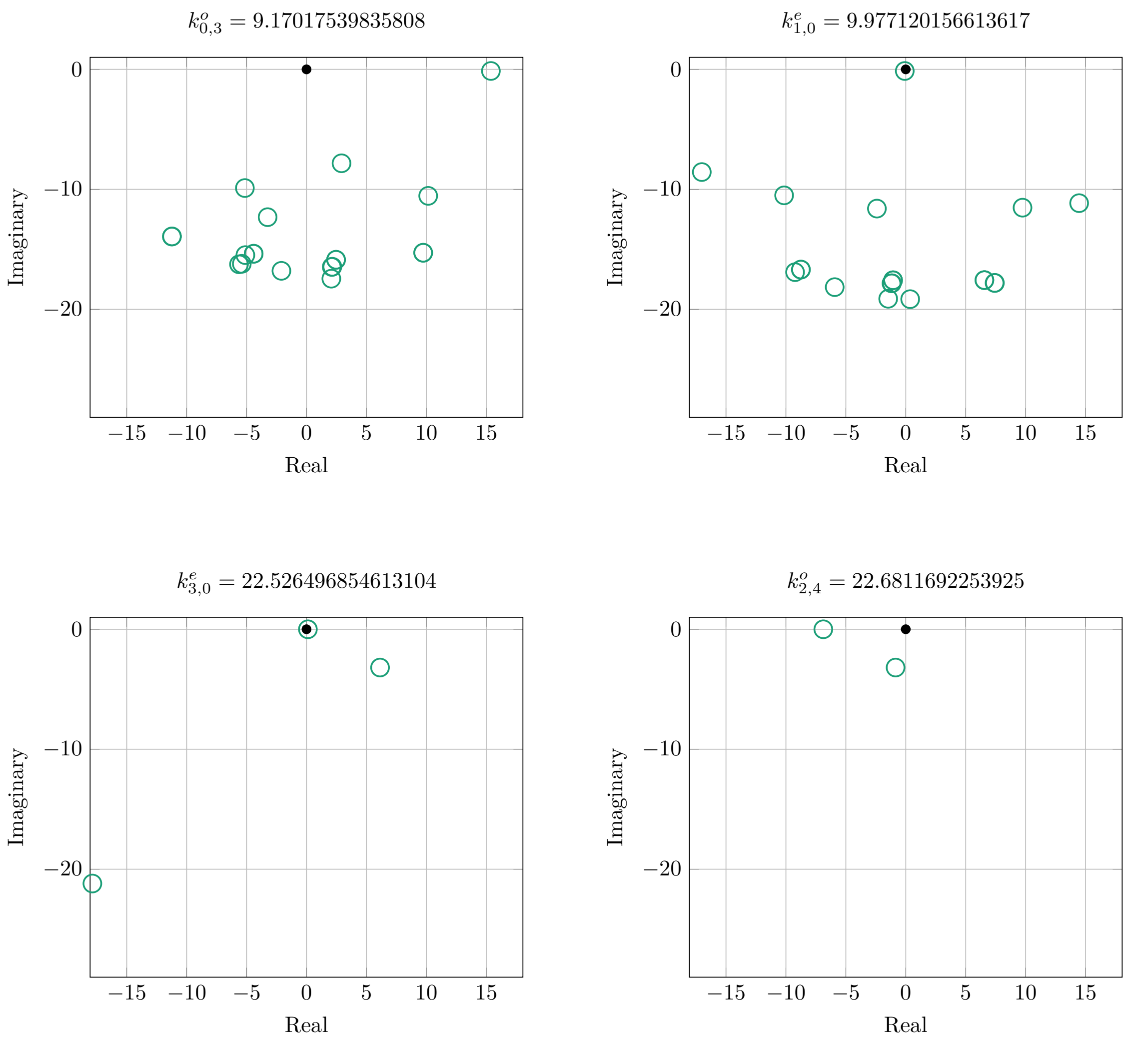


Figure 1.3: The eigenvalues of the truncated exterior Dirichlet problem (Definition 1.1) near the origin when Γ_D is equal to the small cavity. The eigenvalues are plotted at several frequencies, k, corresponding to eigenvalues of the ellipse. In each plot, the origin is marked with a black dot, and the eigenvalues are shown as green circles.

Plots of the eigenvalues and eigenfunctions Figures 1.3 and 1.4 plot the near-zero eigenvalues of the truncated exterior Dirichlet problem for the small and large cavity, respectively, at frequencies corresponding to eigenvalues of the ellipse. Figures 1.5 and 1.6 plot the corresponding eigenfunctions. In all these figures $\Gamma_{\mathrm{tr}} = \partial B(0, 2)$.

Figure 1.4 shows that the large cavity has an eigenvalue very close to zero at each of the four frequencies considered, qualitatively illustrating Theorem 1.4. In contrast, Figure 1.3 shows that the small cavity only has an eigenvalue very close to zero at the frequencies $k^e_{1,0}$ and $k^e_{3,0}$ (top right and bottom left in the figures) and not at $k^e_{0,3}$ and $k^o_{2,4}$ (top left and bottom right). The reason for this is clear from the plots of the eigenfunctions of the truncated exterior Dirichlet problem: looking at Figure 1.5, we see that at $k^e_{0,3}$ and $k^o_{2,4}$ the eigenfunctions are not well localised around the minor axis of the ellipse to be inside the small cavity – in the top left and bottom right of Figure 1.5 we see them "leaking out" of the small cavity. However, looking at Figure 1.6, we see that the corresponding eigenfunctions are localised sufficiently to be inside the large cavity, and thus generate an eigenvalue very close to zero. In these plots, the eigenfunctions are normalised so that their $L^2(\Omega_{\mathrm{tr}})$ norm equals one.

Figure 1.7 plots the trajectories of the near-zero eigenvalues as functions of k for both the small cavity (left plot) and large cavity (right plot) for $k \in (2.5, 12.5)$, with the spectra computed

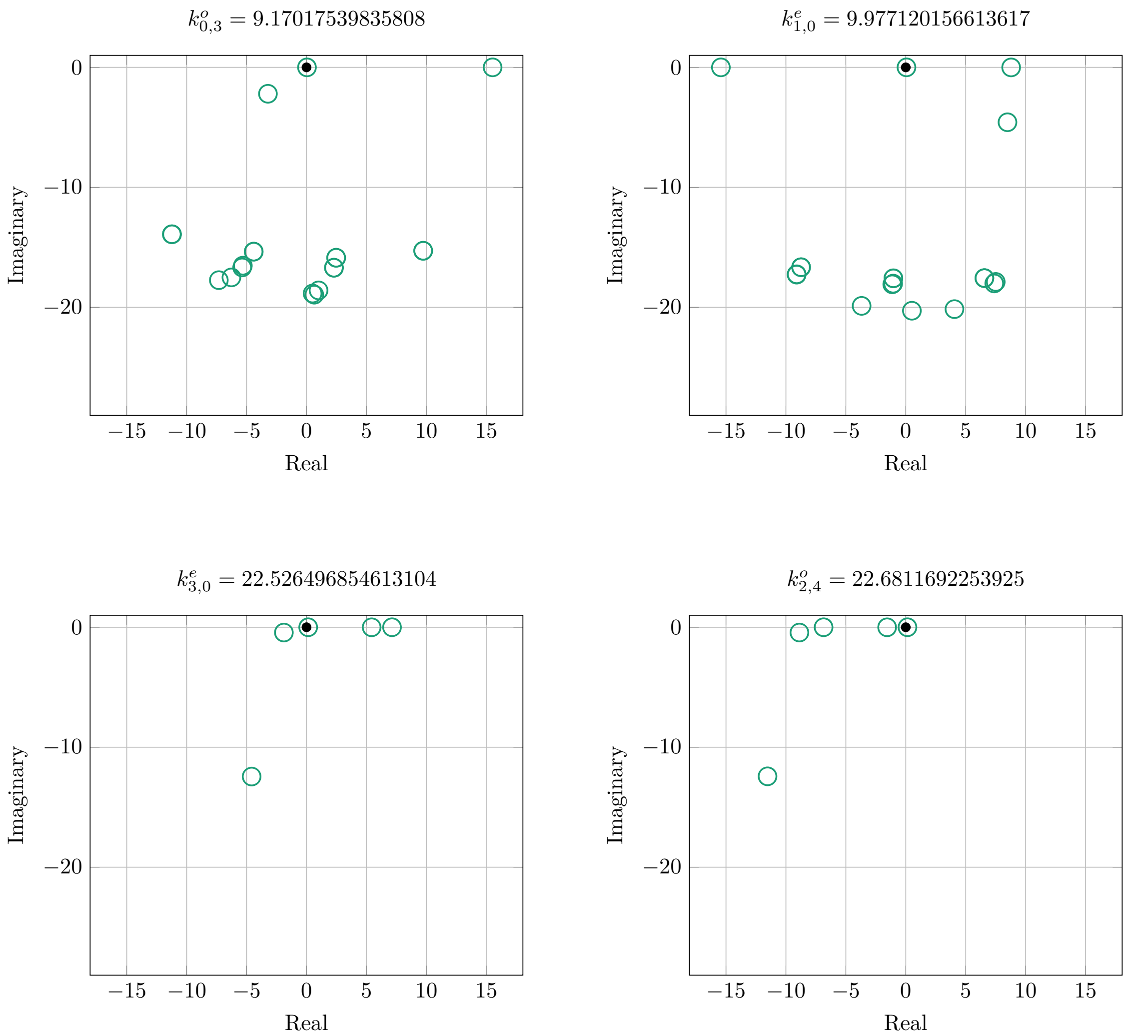


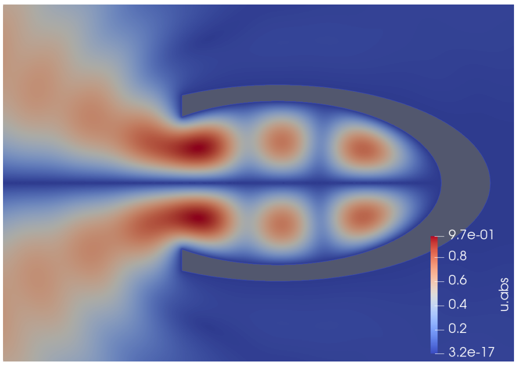
Figure 1.4: The eigenvalues of the truncated exterior Dirichlet problem (Definition 1.1) near the origin when Γ_D is equal to the large cavity. The eigenvalues are plotted at several frequencies, k, corresponding to eigenvalues of the ellipse. In each plot, the origin is marked with a black dot, and the eigenvalues are shown as green circles.

every 0.025. For Figure 1.7, $\Gamma_{\rm tr} = \partial B(0, 1.5)$; this change (compared to $\Gamma_{\rm tr} = \partial B(0, 2)$ for the earlier figures) is to reduce the cost of each eigenvalue solve, because each of the two plots in Figure 1.7 requires 400 such solves. Since we use the exact (up to discretisation error) Dirichlet-to-Neumann map on $\Gamma_{\rm tr}$, we expect there to be no difference between choosing $\Gamma_{\rm tr} = \partial B(0, 1.5)$ and $\Gamma_{\rm tr} = \partial B(0, 2)$ (in particular Figures 1.3 and 1.4 are unchanged when $\Gamma_{\rm tr}$ is changed from $\partial B(0, 2)$ to $\partial B(0, 1.5)$).

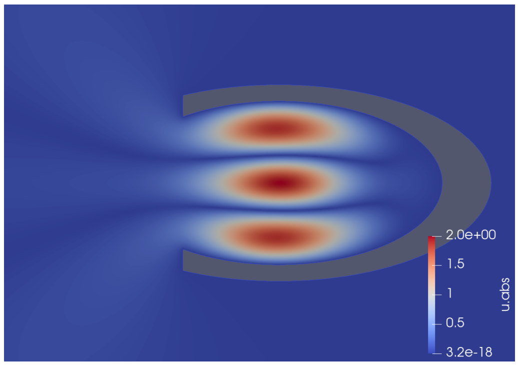
The eigenvalues that enter the red rectangle in Figure 1.7 are coloured green; these are members of $\mathcal{E}(0.2, 0.05, 2.5, 12.5)$, where $\mathcal{E}$ is defined by (1.5). Similar to the eigenvalues plots in Figures 1.3 and 1.4, Figure 1.7 shows that the large cavity has more near-zero eigenvalues for the range of k considered than the small cavity. This is expected since a larger number of the eigenfunctions of the ellipse are localized in the large cavity than in the small cavity.

How the eigenvalues and eigenfunctions were computed. Definition 1.1 (of the eigenvalues of the truncated Dirichlet problem) implies that if μ_ℓ is an eigenvalue at frequency k_ℓ, and with corresponding eigenfunction u_ℓ, then

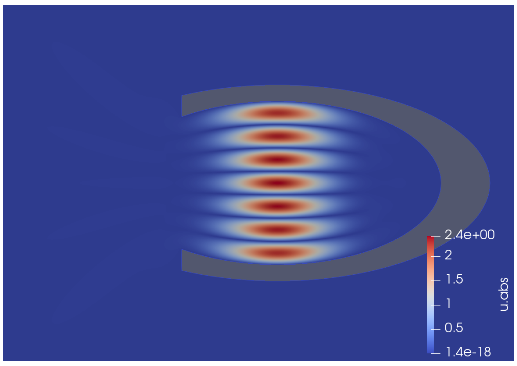
$$a(u_\ell, v) = \mu_\ell (u_\ell, v)_{L^2(\Omega_{\rm tr})} \quad \text{for all } v \in H^1_{0,D}(\Omega_{\rm tr}), \tag{1.6}$$



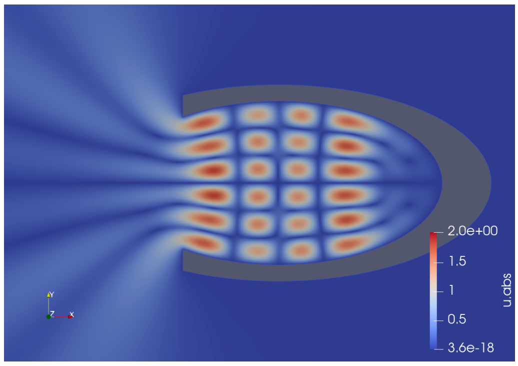
(a) $k^o_{0,3} = 9.17017539835808$



(b) $k^e_{1,0} = 9.977120156613617$



(c) $k^e_{3,0} = 22.526496854613104$



(d) $k^o_{2,4} = 22.6811692253925$

Figure 1.5: Absolute value of the eigenfunction of the truncated exterior Dirichlet problem associated with the smallest eigenvalue the small cavity.

where the sesquilinear form $a(\cdot,\cdot)$ is that appearing in the standard variational (i.e. weak) formulation of the Helmholtz exterior Dirichlet problem.

Definition 1.8 (Variational formulation of Helmholtz exterior Dirichlet problem) *Given $k > 0$, Ω_- as above, and $F \in (H^1_{0,D}(\Omega_{\rm tr}))^*$, let $u \in H^1_{0,D}(\Omega_{\rm tr})$ be the solution of the variational problem*

$$\text{find } u \in H^1_{0,D}(\Omega_{\rm tr}) \quad \text{such that} \quad a(u,v) = F(v) \quad \text{for all } v \in H^1_{0,D}(\Omega_{\rm tr}), \tag{1.7}$$

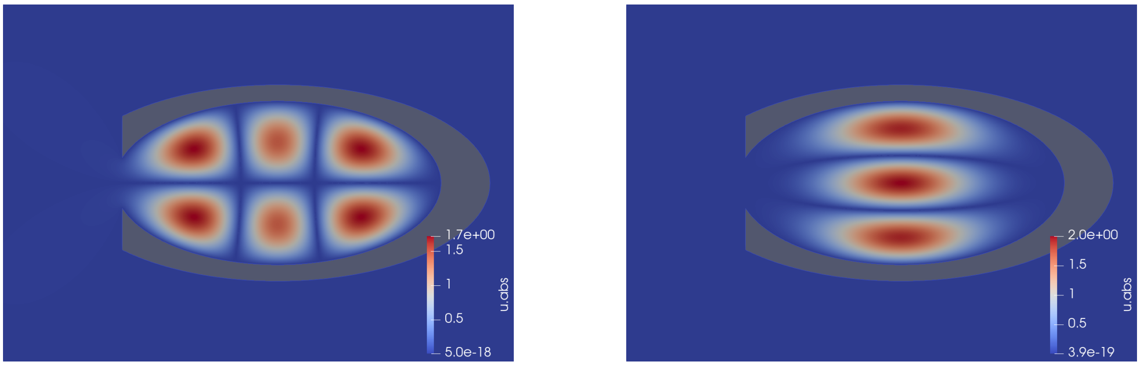
where

$$a(u,v) := \int_{\Omega_{\rm tr}} \Big(\nabla u \cdot \overline{\nabla v} - k^2 u\overline{v}\Big) - \big\langle \mathcal{D}(k)(\gamma_0^{\rm tr} u), \gamma_0^{\rm tr} v\big\rangle_{\Gamma_{\rm tr}}, \tag{1.8}$$

where $\langle\cdot,\cdot\rangle_{\Gamma_{\rm tr}}$ denotes the duality pairing on $\Gamma_{\rm tr}$ that is linear in the first argument and antilinear in the second.

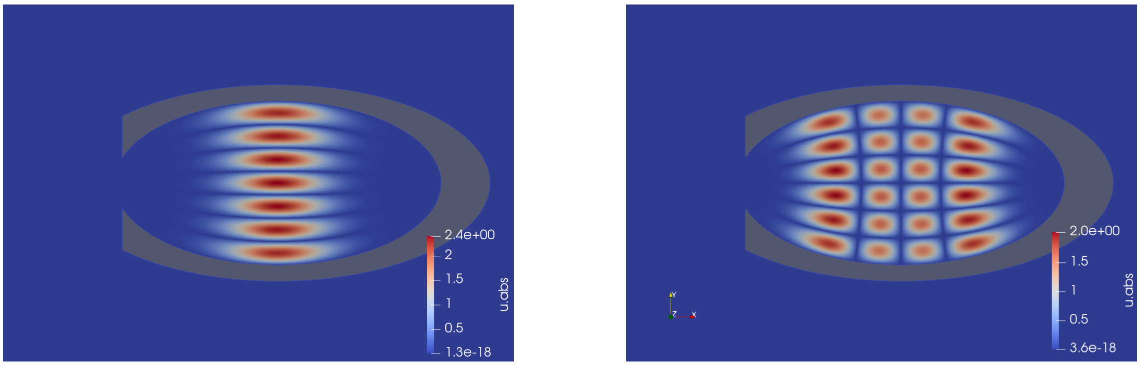
The figures above were created by solving the eigenvalue problem (1.6) using the finite-element method with piecewise-linear elements (i.e. the polynomial degree, p, equals one) and meshwidth h, equal $(2\pi/30)k^{-3/2}$. The Dirichlet-to-Neumann map, $\mathcal{D}(k)$, in $a(\cdot,\cdot)$ was computed using boundary integral equations – see Appendix B for details. The accuracy, uniform in frequency, of the finite-element applied the variational problem (1.7) with $p = 1$ and $hk^{3/2}$ sufficiently small has been known empirically for a long time, and was recently proved in [LSW19] for the case when the Dirichlet-to-Neumann map is realised exactly.

Since computing the Dirichlet-to-Neumann map is relatively expensive, in practice one often approximates it using a perfectly-matched layer (PML) or an absorbing boundary condition (such as the impedance boundary condition). The plots of the eigenfunctions and near-zero eigenvalues of the corresponding truncated exterior Dirichlet problems are very similar to those above; this too is expected since the quasimode is supported in a neighbourhood of the obstacle.



(a) $k^o_{0,3} = 9.17017539835808$

(b) $k^e_{1,0} = 9.977120156613617$



(c) $k^e_{3,0} = 22.526496854613104$

(d) $k^o_{2,4} = 22.6811692253925$

Figure 1.6: Absolute value of the eigenfunction of the truncated exterior Dirichlet problem associated with the smallest eigenvalue for the large cavity.

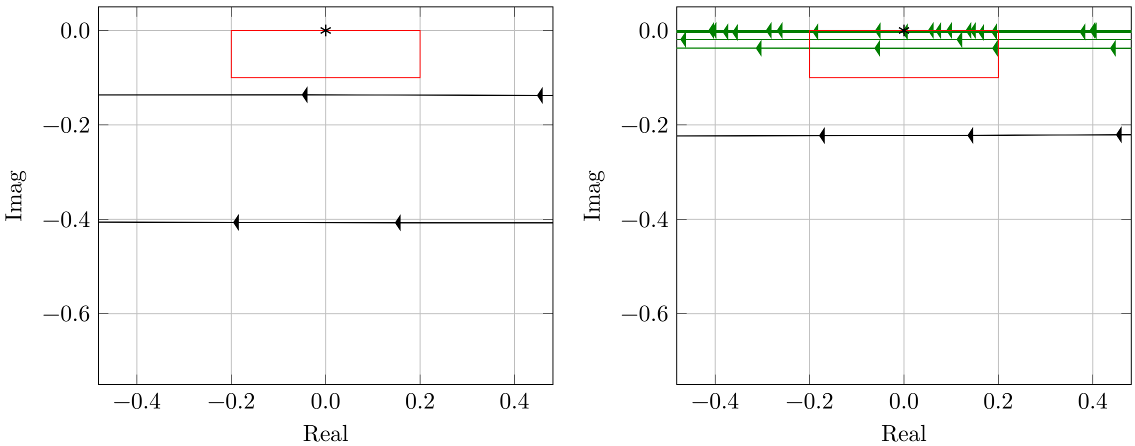


Figure 1.7: Paths of the eigenvalues for $k \in (2.5, 12.5)$ for the small cavity (left) and the large cavity (right). The eigenvalues that enter the red rectangle are coloured green.

1.4 Implications of the main results for numerical analysis of the Helmholtz exterior Dirichlet problem

Theorems 1.4 and 1.7 are the first step towards rigorously understanding how iterative solvers such as the generalised minimum residual method (GMRES) behave when applied to discretisations of high-frequency Helmholtz problems under strong trapping (the subject of the companion paper [MGSS21]). We now explain this in more detail.

As we saw in (1.6), the eigenvalues of truncated exterior Dirichlet problem (in sense of Definition

1.1) correspond to eigenvalues of sesquilinear form of standard variational formulation (Definition 1.8). The standard variational formulation is the basis of the finite-element method for computing approximations to the solution of the variational problem (1.7). Indeed, the finite-element method consists of choosing a piecewise-polynomial subspace of $H^1_{0,D}(\Omega_{\rm tr})$ and solving the variational problem (1.7) in this subspace.

A very popular way of solving the linear systems resulting from the finite-element method applied to the Helmholtz scattering problems is via iterative solvers such as GMRES [SS86]; this choice is made because the linear systems are (i) large and (ii) non-self-adjoint. Regarding (i): the systems are large since the number of degrees of freedom must be $\gg k^d$ to resolve the oscillations in the solution, see, e.g., the literature review in [LSW19, §1.1]. Regarding (ii): non-self-adjointness of the linear systems arises directly from the non-self-adjointness of the underlying Helmholtz scattering problem; GMRES is applicable to such systems, unlike the conjugate gradient method.

There is currently large research interest in understanding how iterative methods behave when applied to Helmholtz linear systems, and in designing good preconditioners for these linear systems; see the literature reviews [Erl08, EG12, GZ19], [GSZ20, §1.3].

The location of eigenvalues, especially near-zero ones, is crucial in understanding the behaviour of iterative methods. In the Helmholtz context, eigenvalue analyses of iterative methods applied to nontrapping problems include, for finite-element discretisations, [EO99, **?**, VGEV07, EG12, VG14, CG17, LXSdH20], and, for boundary-element discretisations, [CH01, DDL13, CDLL17].

The paper [MGSS21] analyses GMRES applied to discretisations of Helmholtz problems with strong trapping, using the "cluster plus outliers" GMRES convergence theory from [CIKM96] (with this idea arising in the context of the conjugate gradient method [Jen77] and used subsequently in, e.g., [ESW02]). The paper [MGSS21] obtains bounds on how the number of GMRES iterations depends on the frequency, under various assumptions about the eigenvalues. In particular, Theorem 1.4 proves [MGSS21, Assumption A2] for the standard variational formulation of the truncated exterior Dirichlet problem. We highlight that, although the results in [MGSS21] are about unpreconditioned systems, they give insight into the design of preconditioners. Indeed, a successful preconditioner for Helmholtz problems with strong trapping will need to specifically deal with the near-zero eigenvalues created by trapping. Theorem 1.4 and 1.7 give information about the location and multiplicities of these eigenvalues, and [MGSS21] shows how these locations and multiplicities affect GMRES.

1.5 The ideas behind the proof of Theorem 1.4

Semiclassical notation. Instead of working with the parameter k and being interested in the large-k limit, the semiclassical literature usually works with a parameter $h := k^{-1}$ and is interested in the small-h limit. So that we can easily recall results from this literature, we also work with the small parameter k^{-1}, but to avoid a notational clash with the meshwidth of the FEM, we let $\hbar := k^{-1}$ (the notation $\hbar$ comes from the fact that the semiclassical parameter is related to Planck's constant, which is written as $2\pi\hbar$; see, e.g., [Zwo12, §1.2]. Theorem 1.4 is then restated in semiclassical notation as Theorem 2.2 below.

The solution operator of the truncated problem. Let $R_{\Omega_{\rm tr}}(\lambda, z) : L^2(\Omega_{\rm tr}) \to L^2(\Omega_{\rm tr})$ be the solution operator for the truncated problem

$$\begin{cases} (-\hbar^2\Delta - \lambda^2 - z)u = f \quad \text{ in } \Omega_{\rm tr} \\ \gamma_0^D u = 0, \\ \gamma_1^{\rm tr} u = \mathcal{D}(\lambda/\hbar)\gamma_0^{\rm tr} u; \end{cases} \tag{1.9}$$

that is $R_{\Omega_{\rm tr}}(\lambda, z)$ satisfies

$$\begin{cases} (-\hbar^2\Delta - \lambda^2 - z)R_{\Omega_{\rm tr}}(\lambda, z)f = f \quad \text{ in } \Omega_{\rm tr} \\ \gamma_0^D R_{\Omega_{\rm tr}}(\lambda, z)f = 0 \\ \gamma_1^{\rm tr} R_{\Omega_{\rm tr}}(\lambda, z)f = \mathcal{D}(\lambda/\hbar)\gamma_0^D R_{\Omega_{\rm tr}}(\lambda, z)f. \end{cases} \tag{1.10}$$

Note that, at this point, it is not clear that the problem (1.9) is well posed and that the family of operators $R_{\Omega_{\rm tr}}(\lambda, z)$ is well defined. We address this in Lemma 1.9 below.

We study $R_{\Omega_{\rm tr}}(\lambda, z)$ by relating it to the solution operator of a more-standard scattering problem. Namely, let $V \in L^\infty(\Omega_+)$ with $\operatorname{supp} V \Subset \mathbb{R}^d$, and consider the problem

$$\begin{cases} (-\hbar^2\Delta - \lambda^2 + V)u = f & \text{on } \Omega_+, \\ \gamma_0^D u = 0, \\ u \text{ is } \lambda/\hbar \text{ outgoing.} \end{cases} \tag{1.11}$$

By, e.g., [DZ19, Chapter 4], the inverse of (1.11) is a meromorphic family of operators (for $\lambda \in \mathbb{C}$ when d is odd or λ in the logarithmic cover of $\mathbb{C} \setminus \{0\}$ when d is even) $R_V(\lambda) : L^2_{\rm comp}(\Omega_+) \to L^2_{\rm loc}(\Omega_+)$ with finite-rank poles satisfying

$$\begin{cases} (-\hbar^2\Delta - \lambda^2 + V)R_V(\lambda)f = f & \text{in } \Omega \\ \gamma_0^D R_V(\lambda)f = 0, \\ R_V(\lambda)f \text{ is } \lambda/\hbar \text{ outgoing.} \end{cases} \tag{1.12}$$

Observe that, although both $R_{\Omega_{\rm tr}}(\lambda, z)$ and $R_V(\lambda)$ depend on $\hbar$, we omit this dependence in the notation to keep expressions compact.

The following two lemmas (proved in §2.2) relate $R_{\Omega_{\rm tr}}(\lambda, z)$ and $R_V(\lambda)$ and then characterise the eigenvalues of the truncated exterior Dirichlet problem as poles of $R_{\Omega_{\rm tr}}(\lambda, z)$ as a function of z.

We use three indicator functions: $1_{\Omega_{\rm tr}}$ denotes the function in $L^\infty(\Omega_+)$ that is one on $\Omega_{\rm tr}$ and zero otherwise, $1^{\rm res}_{\Omega_{\rm tr}}$ denotes the restriction operator $L^2(\Omega_+) \to L^2(\Omega_{\rm tr})$, and $1^{\rm ext}_{\Omega_{\rm tr}}$ denotes the extension-by-zero operator $L^2(\Omega_{\rm tr}) \to L^2(\Omega_+)$.

Lemma 1.9 *Define*

$$R(\lambda, z) := R_V(\lambda) \qquad \textit{with } V(z) = -z 1_{\Omega_{\rm tr}}. \tag{1.13}$$

Then

$$R_{\Omega_{\rm tr}}(\lambda, z) = 1^{\rm res}_{\Omega_{\rm tr}} R(\lambda, z) 1^{\rm ext}_{\Omega_{\rm tr}}, \tag{1.14}$$

and thus $R_{\Omega_{\rm tr}}(\lambda, z)$ is a meromorphic family of operators in λ for $\lambda \in \mathbb{C}$ when d is odd and λ in the logarithmic cover of $\mathbb{C} \setminus \{0\}$ when d is even.

Lemma 1.10 *For $\lambda \in \mathbb{R} \setminus \{0\}$, $z \mapsto R(\lambda, z)$ is a meromorphic family of operators $L^2_{\rm comp}(\Omega_+) \to L^2_{\rm loc}(\Omega_+)$ with finite rank poles.*

Corollary 1.11 *If z_j is a pole of $z \mapsto R_{\Omega_{\rm tr}}(1, z)$, then $\mu_\ell := -\hbar_j^{-2} z_j$ is an eigenvalue of the truncated exterior Dirichlet problem (in the sense of Definition 1.1).*

The key point is that we are interested in $R_{\Omega_{\rm tr}}(\lambda, z)$ as a meromorphic family in the variable z, in contrast to the more-familiar study of $R_V(\lambda)$ as a meromorphic family in the variable λ.

Recap of "from quasimodes to resonances". Recall that resonances of $-\hbar^2\Delta + V$ are defined as poles of the meromorphic continuation of $R_V(w)$ into $\operatorname{Im} w < 0$, see [DZ19, §4.2, §7.2]. The "quasimodes to resonances" argument of [TZ98] (following [SV95, SV96]; see also [DZ19, Theorem 7.6]) shows that existence of quasimodes (as in Definition 1.2) implies existence of resonances close to the real axis; the additional arguments in [Ste99] then prove the corresponding result with multiplicities.

These arguments use the *semiclassical maximum principle* (a consequence of the maximum principle of complex analysis, see Theorem 2.7 below) combined with the bounds

$$\|\chi R_V(\lambda)\chi\|_{L^2 \to L^2} \leq C \exp\Big(C\hbar^{-d} \log \delta^{-1}\Big), \qquad \lambda^2 \in \Omega \setminus \bigcup_{w \in \operatorname{Res}(-\hbar^2\Delta + V)} B(w, \delta), \tag{1.15}$$

for $\Omega \Subset \{\operatorname{Re} w > 0\}$, and

$$\|R_V(\lambda)\|_{L^2 \to L^2} \leq \frac{1}{\operatorname{Im}(\lambda^2)} \quad \text{for } \operatorname{Im}(\lambda^2) > 0; \tag{1.16}$$

see [TZ98, Lemma 1], [TZ00, Proposition 4.3], [DZ19, Theorem 7.5].

From quasimodes to eigenvalues. Theorems 1.4 and 1.7 are proved using the same ideas as in the quasimodes to resonances arguments, except that now we work in the complex z-plane (with real λ) instead of the complex λ-plane. The analogue of the bounds (1.15) and (1.16) are given in the following lemma.

Lemma 1.12 (Bounds on $R_{\Omega_{\rm tr}}(\lambda, z)$) *Let $0 < a < b$ and let $z_j(\hbar, \lambda)$ be the poles of $R_{\Omega_{\rm tr}}(\lambda, z)$ (as a meromorphic function of z). Then there exist $C_1, \varepsilon_1 > 0$ such that for all $0 < \hbar < 1$, $\lambda^2 \in [a, b]$ and $\delta > 0$,*

$$\|R_{\Omega_{\rm tr}}(\lambda, z)\|_{L^2(\Omega_{\rm tr})\to L^2(\Omega_{\rm tr})} \leq \exp\Big(C_1\hbar^{-d}\log\delta^{-1}\Big) \quad \text{for } z \in B(0, \varepsilon_1\hbar)\Big\backslash\bigcup_j B(z_j(\hbar,\lambda), \delta). \quad (1.17)$$

Furthermore, there exists $C_2 > 0$ such that

$$\|R_{\Omega_{\rm tr}}(\lambda, z)\|_{L^2(\Omega_{\rm tr})\to L^2(\Omega_{\rm tr})} \leq C_2\frac{\langle z\rangle}{\operatorname{Im} z} \quad \text{for } \operatorname{Im} z > 0, \quad (1.18)$$

where $\langle z\rangle := (1 + |z|^2)^{1/2}$.

The bound (1.17) is proved by finding a parametrix for $-\hbar^2\Delta - \lambda^2 - z1_{\Omega_{\rm tr}}$ (i.e. an approximation to $R_{\Omega_{\rm tr}}(\lambda, z)$) via a boundary complex absorbing potential. While parametrices based on complex absorption are often used in scattering theory (see, e.g., [DZ16, DG17] [DZ19, Theorem 7.4]), parametrices based on boundary complex absorption appear to be new in the literature. One of the main features of the argument below is that it relies on a comparison of the (in principle, trapping) billiard flow with the non-trapping free flow to obtain estimates on the parametrix. A similar argument should work for boundaries in *any* non-trapping background.

1.6 Outline of the rest of the paper

In §2 we prove Lemmas 1.9 and 1.10 and then collect preliminary results about the generalized bicharacteristic flow (§2.4), the geometry of trapping (§2.5), complex scaling (§2.6), and defect measures (§2.7). In §3 we find a parametrix for $R_{\Omega_{\rm tr}}(\lambda, z)$ via a boundary complex absorbing potential. In §4 we prove Lemma 1.12. In §5 we prove Theorems 1.4 and 1.7 using Lemma 1.12 and the semiclassical maximum principle.

2 Preliminary results

2.1 Restatement of Theorems 1.4 and 1.7 in semiclassical notation

Definition 2.1 (Quasimodes in $\hbar$ notation) *A family of quasimodes of quality $\varepsilon(\hbar)$ is a sequence $\{(u_\ell, \hbar_\ell)\}_{\ell=1}^\infty \subset H^2(\Omega_{\rm tr}) \cap H^1_{0,D}(\Omega_{\rm tr}) \times \mathbb{R}$ such that $\hbar_\ell \to 0$ as $\ell \to \infty$ and there is a compact subset $\mathcal{K} \Subset \Omega_1$ such that, for all ℓ, $\operatorname{supp} u_\ell \subset \mathcal{K}$,*

$$\big\|(-\hbar^2\Delta - 1)u_\ell\big\|_{L^2(\Omega_{\rm tr})} \leq \varepsilon(\hbar_\ell) \quad \text{and} \quad \|u_\ell\|_{L^2(\Omega_{\rm tr})} = 1.$$

Let

$$\varepsilon(\hbar) := \hbar^2\epsilon(\hbar^{-1}).$$

Theorem 1.4 is then equivalent to the following result in the sense that the following result holds if and only if Theorem 1.4 holds with $\mu_\ell := \hbar_\ell^{-2} z_\ell$.

Theorem 2.2 (Analogue of Theorem 1.4 in $\hbar$ notation) *Let $\alpha > 3(d+1)/2$. Suppose there exists a family of quasimodes in the sense of Definition 2.1 and constants $\widetilde{S}_1, \widetilde{S}_2 > 0$ such that the quality $\varepsilon(\hbar)$ satisfies*

$$\widetilde{S}_1 \exp(-\widetilde{S}_2/\hbar) \leq \varepsilon(\hbar) \ll \hbar^{1+\alpha}. \quad (2.1)$$

Then there exists $\hbar_0 > 0$ (depending on α) such that, if ℓ is such that $\hbar_\ell \leq \hbar_0$ then there exists $z_\ell \in \mathbb{C}$ and $0 \neq u_\ell \in H^1_{0,D}(\Omega_{\rm tr})$ with

$$(-\hbar_\ell^2\Delta - 1 + z_\ell)u_\ell = 0 \text{ in } \Omega_{\rm tr}, \quad \gamma_1^{\rm tr} u_\ell = \mathcal{D}(\hbar_\ell^{-1})(\gamma_0^{\rm tr} u_\ell), \quad \text{and} \quad |z_\ell| \leq \hbar_\ell^{-\alpha}\varepsilon(\hbar_\ell). \quad (2.2)$$

Definition 2.3 (Quasimodes with multiplicity in $\hbar$ notation) *Let $0 \le a(\hbar) \le b(\hbar) < \infty$ be two functions of $\hbar$. A* family of quasimodes of quality $\varepsilon(\hbar)$ and multiplicity $m(\hbar)$ *in the window $[a(\hbar), b(\hbar)]$ is a sequence $\{\hbar_j\}_{j=1}^{\infty}$ such that $\hbar_j \to 0$ as $j \to \infty$ and for every j there exist $\{(u_{j,\ell}, E_{j,\ell})\}_{\ell=1}^{m(\hbar_j)} \subset H^2(\Omega_{\rm tr}) \cap H^1_{0,D}(\Omega_{\rm tr}) \times [a(\hbar_j), b(\hbar_j)]$ with*

$$\left\|(-\hbar_j^2\Delta - E_{j,\ell})u_{j,\ell}\right\|_{L^2(\Omega_{\rm tr})} = \varepsilon(\hbar_j),\ \ \|u_{j,\ell}\|_{L^2(\Omega_{\rm tr})} = 1,\ \ \left|\langle u_{j,\ell_1}, u_{j,\ell_2}\rangle_{L^2(\Omega_{\rm tr})}\right| \le \hbar_j^{-2}\varepsilon(\hbar_j) \text{ for } \ell_1 \neq \ell_2,$$

and supp $u_{j,\ell} \subset \mathcal{K}$ *for all j and ℓ, where $\mathcal{K} \Subset \Omega_1$.*

With $\{z_p(\hbar,\lambda)\}_p$ the set of poles of $z \mapsto R_{\Omega_{\rm tr}}(\lambda, z)$ counting multiplicities, let

$$\mathcal{Z}(\varepsilon_1, \varepsilon_0, a, b; \hbar) := \Big\{ z_p(\hbar,\lambda) \ :\ z_p(\hbar,\lambda) \in (-2\varepsilon_1, 2\varepsilon_1) - {\rm i}(0, 2\varepsilon_0) \text{ for some } \lambda^2 \in [a,b] \Big\}; \tag{2.3}$$

$\mathcal{Z}$ is therefore the counting function of the poles of $z \mapsto R_{\Omega_{\rm tr}}(\lambda, z)$ that enter a rectangle next to zero in z as λ^2 varies from a to b.

Theorem 2.4 (Analogue of Theorem 1.7 in $\hbar$ notation) *Let $0 < a_0 \le a(\hbar) \le b(\hbar) < b_0 < \infty$ and suppose there exists a family of quasimodes with quality*

$$\varepsilon(\hbar) \ll \hbar^{(5d+3)/2} \tag{2.4}$$

and multiplicity $m(\hbar)$ in the window $[a(\hbar), b(\hbar)]$ (in the sense of Definition 2.3). If $\varepsilon_0(\hbar)$ is such that, for some $\widetilde{S}_3, \widetilde{S}_4, \widetilde{S}_5 > 0$,

$$\widetilde{S}_3 \exp(-\widetilde{S}_4/\hbar) \le \varepsilon_0(\hbar) \le \widetilde{S}_5 \hbar^{(d+1)/2} \quad \text{for all } \hbar, \quad \text{and} \quad \varepsilon_0(\hbar) \gg \hbar^{-2d-1}\varepsilon(\hbar) \quad \text{as } \hbar \to 0, \tag{2.5}$$

then there exists $\hbar_0 > 0$ such that if $\hbar_j \le \hbar_0$, then

$$\left|\mathcal{Z}\Big(\hbar_j^{-(d+1)/2}\varepsilon_0(\hbar_j)\,,\,\varepsilon_0(\hbar_j)\,,\,a(\hbar_j)\,,\,b(\hbar_j)\,;\,\hbar_j\Big)\right| \ge m(\hbar_j).$$

Proof of Theorem 1.7 from Theorem 2.4. We first show that if there exists a family of quasimodes u_j with multiplicity m_ℓ in the window $[k_\ell^-, k_\ell^+]$ in k notation (i.e. in the sense of Definition 1.3), then there exists a family of quasimodes in $\hbar$ notation (in the sense of Definition 2.3).

Without loss of generality, each $k_\ell \in [k_j^-, k_j^+]$ for some j (if necessary by adding a window with $k_j^- = k_j^+ = k_\ell$), i.e. given ℓ in the index set of the quasimode, there exists j such that $\ell \in W_j$. We now index the quasimode with the index j describing the windows $[k_j^-, k_j^+]$. Let

$$\hbar_j := (k_j^-)^{-1}, \qquad m(\hbar_j) := m_j, \qquad a(\hbar_j) := 1, \qquad b(\hbar_j) := \frac{(k_j^+)^2}{(k_j^-)^2},$$
$$\varepsilon(\hbar_j) := \hbar_j^2 \epsilon(\hbar_j^{-1}), \qquad \text{and} \qquad E_{j,\ell} := \frac{(k_\ell)^2}{(k_j^-)^2} \text{ and } u_{j,\ell} := u_\ell \text{ for } \ell \in \mathcal{W}_j.$$

Then,

$$\left\|(\hbar_j^2\Delta + E_{j,\ell})u_{j,\ell}\right\|_{L^2(\Omega_{\rm tr})} = (k_j^-)^{-2}\left\|(\Delta + k_\ell^2)u_\ell\right\|_{L^2(\Omega_{\rm tr})} = (k_j^-)^{-2}\epsilon(k_\ell) \le (k_j^-)^{-2}\epsilon(k_j^-) = \varepsilon(h_j),$$

where we have used that $\epsilon(k)$ is a decreasing function of k. Therefore, we have shown that there exists a family of quasimodes with multiplicity $m(\hbar)$ in the window $[a(\hbar), b(\hbar)]$ in $\hbar$ notation (i.e. in the sense of Definition 2.3).

The result of Theorem 1.7 then follows from the result of Theorem 2.4 since (a) if $\lambda^2 \in [a(\hbar), b(\hbar)]$ and $\lambda/\hbar = k$, then $k \in [k_j^-, k_j^+]$, and (b) if

$$z \in \mathcal{Z}\Big(\hbar_j^{-(d+1)/2}\varepsilon_0(\hbar_j)\,,\,\varepsilon_0(\hbar_j)\,,\,a(\hbar_j)\,,\,b(\hbar_j)\,;\,\hbar_j\Big),$$

then

$$\mu := \hbar_j^{-2} z \in \mathcal{E}\Big((k_j^-)^{(d+1)/2}\epsilon_0(k_j^-)\,,\,\epsilon_0(k_j^-)\,,\,k_j^-\,,\,k_j^+\Big).$$

■

2.2 Results about meromorphic continuation

Proof of Lemma 1.9. Once we show (1.14), the meromorphicity of $R_{\Omega_{\rm tr}}(\lambda, z)$ in λ follows from the corresponding result for $R_V(\lambda)$ [DZ19, Theorem 4.4].

We first show that the appropriate extension of a solution of (1.9) is a solution of (1.11) with $V(z) = -z1_{\Omega_{\rm tr}}$. We then show that the appropriate restriction of the solution of (1.11) with $V(z) = -z1_{\Omega_{\rm tr}}$ is a solution of (1.9).

Given $f \in L^2(\Omega_{\rm tr})$, suppose that u solves (1.9). Then, by the definition of the operator $\mathcal{D}$, there exists a $\lambda/\hbar$-outgoing function $v \in H^2_{\rm loc}(\mathbb{R}^d \setminus \Omega_1)$ such that

$$(-\hbar^2\Delta - \lambda^2)v = 0 \quad \text{on } \mathbb{R}^d \setminus \overline{\Omega_1}, \qquad \text{and} \qquad \gamma_0^{\rm tr} v = \gamma_0^{\rm tr} u, \quad \gamma_1^{\rm tr} v = \gamma_1^{\rm tr} u.$$

Therefore,

$$\widetilde{v} := 1^{\rm ext}_{\Omega_{\rm tr}} u + 1^{\rm ext}_{\mathbb{R}^d\setminus\Omega_1} v$$

is in $H^2_{\rm loc}(\Omega_+)$ (since both its Dirichlet and Neumann traces match across $\partial\Omega_1$) and

$$(-\hbar^2\Delta - \lambda^2)\widetilde{v} = z1_{\Omega_{\rm tr}}\widetilde{v} + 1^{\rm ext}_{\Omega_{\rm tr}} f \quad \text{on } \Omega_+.$$

By the definition of $R(\lambda, z)$ as the solution of (1.12) with $V(z) = -z1_{\Omega_{\rm tr}}$,

$$\widetilde{v} = R(\lambda, z)1^{\rm ext}_{\Omega_{\rm tr}} f, \qquad \text{which implies that} \qquad u = 1^{\rm res}_{\Omega_{\rm tr}} R(\lambda, z)1^{\rm ext}_{\Omega_{\rm tr}} f.$$

Now suppose $\widetilde{f} \in L^2(\Omega_+)$. Then, by (1.13) and (1.11),

$$\begin{cases} (-\hbar^2\Delta - \lambda^2 - z1_{\Omega_{\rm tr}})R(\lambda, z)\widetilde{f} = \widetilde{f} & \text{in } \Omega, \\ R(\lambda, z)\widetilde{f} = 0 & \text{on } \Gamma_D, \\ R(\lambda, z)\widetilde{f} \text{ is } \lambda/\hbar\text{-outgoing.} \end{cases} \tag{2.6}$$

Therefore, if $\widetilde{f} = 1^{\rm ext}_{\Omega_{\rm tr}} f$ and $v := R(\lambda, z)\widetilde{f}$, then $(-\hbar^2\Delta - \lambda^2)R(\lambda, z)\widetilde{f} = 0$ in $\mathbb{R}^d \setminus \overline{\Omega_1}$ and v is $\lambda/\hbar$-outgoing. This last fact implies that

$$\gamma_1^{\rm tr}(1^{\rm res}_{\mathbb{R}^d\setminus\overline{\Omega_{\rm tr}}} v) = \mathcal{D}(\lambda/\hbar)\gamma_0^{\rm tr}(1^{\rm res}_{\mathbb{R}^d\setminus\overline{\Omega_{\rm tr}}} v). \tag{2.7}$$

Since $v = R(\lambda, z)\widetilde{f} \in H^2_{\rm loc}(\Omega_+)$, the Dirichlet and Neumann traces of v across $\Gamma_{\rm tr}$ do not have jumps, so that (2.7) implies that

$$\gamma_1^{\rm tr}(1^{\rm res}_{\Omega_{\rm tr}} v) = \mathcal{D}(\lambda/\hbar)\gamma_0^{\rm tr}(1^{\rm res}_{\Omega_{\rm tr}} v). \tag{2.8}$$

Then, by (2.6) and (2.8), $u := 1^{\rm res}_{\Omega_{\rm tr}} v$ solves (1.9) and the proof is complete. ■

Proof of Lemma 1.10. Since

$$(-\hbar^2\Delta - \lambda^2 - z1_{\Omega_{\rm tr}})R(\lambda, 0) = I - z1_{\Omega_{\rm tr}} R(\lambda, 0),$$

the definition of $R(\lambda, z)$ (1.13) implies that

$$R(\lambda, z) = R(\lambda, 0)\big(I - z1_{\Omega_{\rm tr}} R(\lambda, 0)\big)^{-1}. \tag{2.9}$$

We now claim that, for any $\rho \in C^\infty(\overline{\Omega_+})$ with $\operatorname{supp}\rho \Subset \mathbb{R}^d$ and $\rho \equiv 1$ on $\Omega_{\rm tr}$

$$\big(I - z1_{\Omega_{\rm tr}} R(\lambda, 0)\big)^{-1} = \big(I - z1_{\Omega_{\rm tr}} R(\lambda, 0)\rho\big)^{-1}\big(I + z1_{\Omega_{\rm tr}} R(\lambda, 0)(1-\rho)\big). \tag{2.10}$$

Indeed,

$$I - z1_{\Omega_{\rm tr}} R(\lambda, 0) = \Big(I - z1_{\Omega_{\rm tr}} R(\lambda, 0)(1-\rho)\big(I - z1_{\Omega_{\rm tr}} R(\lambda, 0)\rho\big)^{-1}\Big)\big(I - z1_{\Omega_{\rm tr}} R(\lambda, 0)\rho\big).$$

and thus

$$\big(I - z1_{\Omega_{\rm tr}} R(\lambda, 0)\big)^{-1} = \big(I - z1_{\Omega_{\rm tr}} R(\lambda, 0)\rho\big)^{-1}\Big(I - z1_{\Omega_{\rm tr}} R(\lambda, 0)(1-\rho)\big(I - z1_{\Omega_{\rm tr}} R(\lambda, 0)\rho\big)^{-1}\Big)^{-1}. \tag{2.11}$$

Observe that since $\rho R(\lambda,0)\rho : L^2(\Omega_+) \to L^2(\Omega_+)$ is compact, $1_{\Omega_{\rm tr}} R(\lambda,0)\rho : L^2(\Omega_+) \to L^2(\Omega_+)$ is compact, and the analytic Fredholm theorem [DZ19, Theorem C.8] implies that

$$z \mapsto (I - z 1_{\Omega_{\rm tr}} R(\lambda,0)\rho)^{-1} \text{ is a meromorphic family of operators for } z \in \mathbb{C} \tag{2.12}$$

with finite rank poles.

Now, since $(1-\rho)1_{\Omega_{\rm tr}} = 0$, for $|z|$ small enough,

$$(1-\rho)\big(I - z 1_{\Omega_{\rm tr}} R(\lambda,0)\rho\big)^{-1} = (1-\rho)\sum_{j=0}^{\infty}(z 1_{\Omega_{\rm tr}} R(\lambda,0)\rho)^k = (1-\rho). \tag{2.13}$$

However, by (2.12) both the left- and right-hand sides of (2.13) are meromorphic for $z \in \mathbb{C}$. Therefore, (2.13) holds for all $z \in \mathbb{C}$ and hence

$$\big(I - z 1_{\Omega_{\rm tr}} R(\lambda,0)(1-\rho)\big)^{-1} = I + z 1_{\Omega_{\rm tr}} R(\lambda,0)(1-\rho). \tag{2.14}$$

Using (2.13) and (2.14) in (2.11), we obtain (2.10). Therefore, for $\chi \equiv 1$ on $\Omega_{\rm tr}$ and $\rho \equiv 1$ on $\operatorname{supp}\chi$, (2.9), (2.10) and (2.13) imply that

$$\chi R(\lambda,z)\chi = \chi R(\lambda,0)\rho\big(I - z 1_{\Omega_{\rm tr}} R(\lambda,0)\rho\big)^{-1}\chi.$$

Using (2.12) again completes the proof. ∎

With $z_0(\hbar,\lambda)$ a pole of $R_{\Omega_{\rm tr}}(\lambda,z)$, let

$$\Pi_{z_0(\hbar,\lambda)} := -\frac{1}{2\pi {\rm i}}\oint_{z_0(\hbar,\lambda)} R_{\Omega_{\rm tr}}(\lambda,z)\,dz \quad \text{and} \quad m_R\big(z_0(\hbar,\lambda)\big) := \operatorname{rank}\Pi_{z_0(\hbar,\lambda)}, \tag{2.15}$$

where $\oint_{z_0(\hbar,\lambda)}$ denotes integration over a circle containing z_0 and no other pole of $R_{\Omega_{\rm tr}}(\lambda,z)$.

The following result then holds by, e.g., [DZ19, Theorem C.9].

Lemma 2.5 *For $\lambda \in \mathbb{R}\setminus\{0\}$, $\Pi_{z_0(\hbar,\lambda)} : L^2(\Omega_{\rm tr}) \to L^2(\Omega_{\rm tr})$ is a bounded projection with finite rank.*

The next result concerns the singular behaviour of $R_{\Omega_{\rm tr}}(\lambda,z)$ near its poles in z, and is analogous to (parts of) [DZ19, Theorem 4.7] concerning the singular behaviour of $R_V(\lambda)$ near its poles in λ.

Lemma 2.6 *For $\lambda \in \mathbb{R}\setminus\{0\}$, if $z_0 = z_0(\hbar,\lambda)$ and $m_R(z_0) > 0$, then there exists $M_{z_0} > 0$ such that*

$$R_{\Omega_{\rm tr}}(\lambda,z) = -\sum_{\ell=1}^{M_{z_0}} \Pi_{z_0}\frac{(-\hbar^2\Delta - \lambda^2 - z)^{\ell-1}}{(z-z_0)^\ell} + A(z,z_0,\lambda)$$

where $z \mapsto A(z,z_0,\lambda)$ is holomorphic near z_0.

Proof. By Lemma 1.10, for $\lambda \in \mathbb{R}\setminus\{0\}$, $z \mapsto R_{\Omega_{\rm tr}}(\lambda,z)$ is a meromorphic family of operators (in the sense of [DZ19, Definition C.7]) from $L^2(\Omega_{\rm tr}) \to L^2(\Omega_{\rm tr})$ and thus there exists $M_{z_0} > 0$, finite-rank operators $A_\ell(\lambda) : L^2(\Omega_{\rm tr}) \to L^2(\Omega_{\rm tr})$, $\ell = 1,\ldots,M_{z_0}$, and a family of operators $z \mapsto A(z,z_0,\lambda)$ from $L^2(\Omega_{\rm tr}) \to L^2(\Omega_{\rm tr})$, holomorphic near z_0, such that

$$R_{\Omega_{\rm tr}}(\lambda,z) = \sum_{\ell=1}^{M_{z_0}}\frac{A_\ell(\lambda)}{(z-z_0)^\ell} + A(z,z_0,\lambda).$$

By integrating around z_0 and using the residue theorem, we have $A_1 = -\Pi_{z_0}$. Then, with $\equiv$ denoting equality up to holomorphic operators,

$$\begin{aligned} R_{\Omega_{\rm tr}}(\lambda,z)(-\hbar^2\Delta - \lambda^2 - z) &\equiv \sum_{\ell=1}^{M_{z_0}}\left(\frac{A_\ell(-\hbar^2\Delta - \lambda^2 - z_0)}{(z-z_0)^\ell} - \frac{A_\ell}{(z-z_0)^{\ell-1}}\right), \\ &= \sum_{\ell=1}^{M_{z_0}}\frac{A_\ell(-\hbar^2\Delta - \lambda^2 - z_0) - A_{\ell+1}}{(z-z_0)^\ell}, \end{aligned}$$

where we define $A_{M_{z_0}+1} := 0$. Since $R_{\Omega_{\rm tr}}(\lambda,z)(-\hbar^2\Delta - \lambda^2 - z) = I$ on $H^2(\Omega_{\rm tr}) \cap H^1_0(\Omega_+)$, $A_{\ell+1} = A_\ell(-\hbar^2\Delta - \lambda^2 - z)$, $\ell = 1,\ldots,M_{z_0}$, and the result follows from density of $H^2(\Omega_{\rm tr}) \cap H^1_0(\Omega_+)$ in $L^2(\Omega_{\rm tr})$. ∎

2.3 The semiclassical maximum principle

The following result is the semiclassical maximum principle of [TZ98, Lemma 2], [TZ00, Lemma 4.2] (see also [DZ19, Lemma 7.7]).

Theorem 2.7 (Semiclassical maximum principle) *Let $\mathcal{H}$ be an Hilbert space and $z \mapsto Q(z, \hbar) \in \mathcal{L}(\mathcal{H})$ an holomorphic family of operators in a neighbourhood of*

$$\Omega(\hbar) := \big(w - 2\beta(\hbar), w + 2\beta(\hbar)\big) + \mathrm{i}\big(-\delta(\hbar)\hbar^{-L}, \delta(\hbar)\big), \tag{2.16}$$

where

$$0 < \delta(\hbar) < 1, \qquad \textit{and} \qquad \beta(\hbar)^2 \geq C h^{-3L} \delta(\hbar)^2 \tag{2.17}$$

for some $L > 0$ and $C > 0$. Suppose that

$$\|Q(z, \hbar)\|_{\mathcal{H}\to\mathcal{H}} \leq \exp(C\hbar^{-L}), \qquad z \in \Omega, \tag{2.18}$$

$$\|Q(z, \hbar)\|_{\mathcal{H}\to\mathcal{H}} \leq \frac{C}{\operatorname{Im} z}, \qquad \operatorname{Im} z > 0, \quad z \in \Omega. \tag{2.19}$$

Then

$$\|Q(z, \hbar)\|_{\mathcal{H}\to\mathcal{H}} \leq \frac{C}{\delta(\hbar)} \exp(C + 1) \quad \textit{for all } z \in \big[w - \beta(\hbar), w + \beta(\hbar)\big]. \tag{2.20}$$

References for proof. Let $f, g \in \mathcal{H}$ with $\|f\|_{\mathcal{H}} = \|g\|_{\mathcal{H}} = 1$, and let

$$F(z, \hbar) := \big\langle Q(z + w, h)g, f\big\rangle_{\mathcal{H}}.$$

The result (2.20) follows from the "three-line theorem in a rectangle" (a consequence of the maximum principle) stated as [DZ19, Lemma D.1] applied to the holomorphic family $(F(\cdot, h))_{0<h\ll 1}$ with

$$R = 2\beta(\hbar), \qquad \delta_+ = \delta(\hbar), \qquad \delta_- = \delta(\hbar)h^{-L},$$
$$M = M_- = \exp(C\hbar^{-L}), \qquad M_+ = C\,\delta(\hbar)^{-1}.$$

■

2.4 The generalized bicharacteristic flow

Recall that

$$T^*_{\overline{\Omega_+}}\mathbb{R}^d := \big\{(x, \xi) \in T^*\mathbb{R}^d, x \in \overline{\Omega_+}\big\} = \big\{x \in \overline{\Omega_+}, \xi \in \mathbb{R}^d\big\}$$

and

$$S^*_{\overline{\Omega_+}}\mathbb{R}^d := \big\{(x, \xi) \in S^*\mathbb{R}^d, x \in \overline{\Omega_+}\big\} = \big\{x \in \overline{\Omega_+}, \xi \in \mathbb{R}^d \text{ with } |\xi| = 1\big\}.$$

We write $\varphi_t : S^*_{\Omega_+}\mathbb{R}^d \to S^*_{\Omega_+}\mathbb{R}^d$ for the generalized bicharacteristic flow associated to a symbol p (see e.g. [Hör85, §24.3]). Since the flow over the interior is generated by the Hamilton vector field H_p, for any symbol $b \in C_c^\infty(T^*_{\Omega_+}\mathbb{R}^d)$,

$$\partial_t(b \circ \varphi_t) = H_p b = \{p, b\}, \tag{2.21}$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket; see [Zwo12, §2.4].

We primarily consider the case when p is the semiclassical principal symbol of the Helmholtz equation, namely $p = |\xi|^2 - 1$. By Hamilton's equations, away from the boundary of Ω_+, the corresponding flow satisfies $\dot{x}_i = 2\xi_i$ and $\dot{\xi}_i = 0$, and thus, for $\rho = (x, \xi)$ with x away from Γ_D, $\varphi_t(\rho) = x + 2t\xi$ for t sufficiently small; i.e., the flow has speed two.

We let $\pi_{\mathbb{R}}$ denote the projection operator onto the spatial variables; i.e.

$$\pi_{\mathbb{R}} : T^*_{\overline{\Omega_+}}\mathbb{R}^d \to \overline{\Omega_+}, \quad \pi_{\mathbb{R}}((x, \xi)) = x.$$

2.5 Geometry of trapping

Let $\chi \in C^\infty(\overline{\Omega_+};[0,1])$ with $\operatorname{supp}\chi \Subset \mathbb{R}^d$ and $\chi \equiv 1$ near Ω_- and define $r : T^*_{\Omega_+}\mathbb{R}^d \to \mathbb{R}$ by

$$\mathbf{r}(x,\xi) := (1-\chi(x))|x|$$

so that there is $c > 0$ such that for $r_0 > c$,

$$\{x\,:\,\mathbf{r} > r_0\} = \mathbb{R}^d \setminus B(0,r_0).$$

Moreover, note that $\{\mathbf{r} \le c\}$ is compact for every c. Next, define the *directly escaping sets*,

$$\mathcal{E}_\pm := \Big\{(x,\xi) \in S^*\mathbb{R}^d \mid \mathbf{r}(x,\xi) \ge r_0, \quad \pm\langle x,\xi\rangle_{\mathbb{R}^d} \ge 0\Big\},$$
$$\mathcal{E}^o_\pm := \Big\{(x,\xi) \in S^*\mathbb{R}^d \mid \mathbf{r}(x,\xi) \ge r_0, \quad \pm\langle x,\xi\rangle_{\mathbb{R}^d} > 0\Big\}.$$

Then,

$$\rho \in \mathcal{E}_\pm \quad \text{implies that} \quad \varphi_{\pm t}(\rho) \in \mathcal{E}_\pm \quad \text{and} \quad \mathbf{r}(\varphi_{\pm t}(\rho)) \ge \sqrt{\mathbf{r}(\rho)^2 + 4t^2}, \quad \text{for all } t \ge 0. \tag{2.22}$$

Therefore, $\mathbf{r}(\varphi_t(\rho)) \to \infty$ as $t \to \pm\infty$ and hence $\rho \in \mathcal{E}_\pm$ escapes forward/backward in time. This, in particular implies that

$$\mathbf{r}(\rho) \ge r_0,\, \mathbf{r}(\varphi_{\mp t_0}(\rho)) \le \mathbf{r}(\rho) \text{ for some } t_0 > 0 \quad \Rightarrow \quad \pm\langle x(\rho), \xi(\rho)\rangle > 0. \tag{2.23}$$

We now define the *outgoing tail* $\Gamma_+ \subset S^*_\Omega\mathbb{R}^d$, the *incoming tail* $\Gamma_- \subset S^*_\Omega\mathbb{R}^d$, and the *trapped set*, K by

$$\Gamma_\pm := \{q \in S^*_\Omega\mathbb{R}^d \mid \mathbf{r}(\varphi_t(q)) \not\to \infty,\, t \to \mp\infty\}, \qquad K := \Gamma_+ \cap \Gamma_-; \tag{2.24}$$

i.e. the outgoing tail is the set of trajectories that do not escape as $t \to -\infty$, the incoming tail is the set of trajectories that do not escape as $t \to \infty$, and the trapped set is the set of trajectories that do not escape in either time direction.

We now recall some basic properties of $\Gamma_\pm$ and K, with these proved in a more general setting in [DZ19, §6.1].

Lemma 2.8
(i) The sets $\Gamma_\pm, K$ are closed in $S^*_\Omega\mathbb{R}^d$ and $K \subset \{\mathbf{r} < r_0\}$.
(ii) Suppose that $\rho_n \in S^*_{\Omega_+}\mathbb{R}^d$ with $\rho_n \to \rho$ and there are $t_n \to \infty$ such that $\varphi_{\pm t_n}(\rho_n) \to \rho_\infty$. Then $\rho \in \Gamma_\mp$.

Proof. (i) We show that Γ_- is closed in $S^*_\Omega\mathbb{R}^d$. Suppose that $\rho_0 \in S^*_\Omega\mathbb{R}^d \setminus \Gamma_-$. Then $\mathbf{r}(\varphi_t(\rho_0)) \to \infty$ as $t \to \infty$. In particular, there are $0 < t_1 < t_2$ such that $\mathbf{r}(\varphi_{t_2}(\rho_0)) \ge r_0$ and $\mathbf{r}(\varphi_{t_1}(\rho_0)) \le \mathbf{r}(\varphi_{t_2}(\rho_0))$. So, applying (2.23) with $\rho = \varphi_{t_2}(\rho_0)$, we have $\varphi_{t_2}(\rho_0) \in \mathcal{E}^o_+$. Since $\mathcal{E}^o_+$ is open and φ_{t_2} is continuous we have $\varphi_{t_2}(\rho) \in \mathcal{E}^o_+$ for all ρ sufficiently close to ρ_0 and hence, by (2.22), $\rho \notin \Gamma_-$. Therefore Γ_- is closed. By an identical argument Γ_+ and hence $\Gamma_- \cap \Gamma_+$ are closed.

Now, we show that $K \subset \{\mathbf{r} < r_0\}$. Note that $S^*_\Omega\mathbb{R}^d \cap \{\mathbf{r} \ge r_0\} \subset \mathcal{E}_+ \cup \mathcal{E}_-$. But, $\mathcal{E}_+ \cap \Gamma_- = \emptyset$ and $\mathcal{E}_- \cap \Gamma_+ = \emptyset$ and hence $S^*_\Omega\mathbb{R}^d \cap \{\mathbf{r} \ge r_0\} \cap \Gamma_+ \cap \Gamma_- = \emptyset$ as claimed.

(ii) We prove the result for $t_n \to \infty$; the proof of the other case is similar. Seeking a contradiction, assume that $\rho \notin \Gamma_-$. Then there exists $T > 0$ such that $\mathbf{r}(\varphi_T(\rho)) \in \mathcal{E}^o_+$ and hence, since φ_T is continuous, and $\mathcal{E}^o_+$ is open, for n large enough, $\varphi_T(\rho_n) \in \mathcal{E}^o_+$. But then, by (2.22) and (2.23) for $t \ge T$ $\mathbf{r}(\varphi_t(\rho_n)) \ge \sqrt{r_0^2 + 4(t-T)^2}$. In particular, for n large enough,

$$\mathbf{r}(\varphi_{T_n}(\rho_n)) \ge \sqrt{r_0^2 + 4(T_n - T)^2} \to \infty$$

which contradicts the fact that $\mathbf{r}(\varphi_{T_n}(\rho_n)) \to \rho_\infty$. ∎

2.6 Complex scaling

We now review the method of complex scaling following [DZ19, §4.5]. We first fix a small angle of scaling, $\theta > 0$, and the radius, $r_1 > r_0$, where the scaling starts; without loss of generality, we assume that $\Omega_1 \Subset \{x : \mathbf{r} \le r_1\}$. Let $f_\theta \in C^\infty([0,\infty)$ satisfy

$$\begin{aligned} &f_\theta(r) \equiv 0, \quad r \le r_1; \qquad f_\theta(r) = r\tan\theta, \quad r \ge 2r_1; \\ &f'_\theta(r) \ge 0, \quad r \ge 0; \qquad \{f'_\theta(r) = 0\} = \{f_\theta(r) = 0\}. \end{aligned} \tag{2.25}$$

Then, consider the totally real submanifold (see [DZ19, Definition 4.28])

$$\Gamma_\theta := \left\{ x + \mathrm{i} f_\theta(|x|)\frac{x}{|x|} \;:\; x \in \mathbb{R}^d \right\} \subset \mathbb{C}^d$$

and note that we identify Ω_- with its image on Γ_θ. We define the complex scaled operator P_θ on Ω by the Dirichlet realization of

$$P_\theta := \Big(\frac{1}{1+\mathrm{i} f'_\theta(r)}\hbar D_r\Big)^2 - \frac{(d-1)\mathrm{i}}{(r+\mathrm{i} f_\theta(r))(1+\mathrm{i} f'_\theta(r))}\hbar^2 D_r - \frac{\hbar^2 \Delta_\phi}{(r+\mathrm{i} f_\theta(r))^2}, \qquad \{\mathbf{r} \ge r_0\}.$$

where Δ_ϕ denotes the Laplacian on the round sphere S^{d-1}. Note that P_θ is a semiclassical differential operator of second order such that on $r \le r_1$, $P_\theta = -\hbar^2\Delta$ with principal symbol, p_θ, satisfying $p_\theta(x,\xi) = |\xi|^2$ on $\{\mathbf{r} \le r_1\}$, and in polar coordinates $x = r\phi$,

$$p_\theta(r,\phi,\xi_r,\xi_\phi) = \frac{\xi_r^2}{(1+\mathrm{i} f'_\theta(r))^2} + \frac{|\xi_\phi|^2}{(r+\mathrm{i} f_\theta(r))^2}. \tag{2.26}$$

Now, by e.g. [DZ19, Theorems 4.36,4.38], for $\operatorname{Im}(e^{\mathrm{i}\theta}\lambda) > 0$,

$$P_\theta - \lambda^2 : H^2(\Omega_+) \cap H_0^1(\Omega_+) \to L^2(\Omega_+) \text{ is a Fredholm operator of index zero.} \tag{2.27}$$

In particular, for $V \in L^\infty(\mathbb{R}^d)$, $\operatorname{supp} V \subset \{r < r_1\}$, this implies that

$$P_\theta - \lambda^2 + V : H^2(\Omega_+) \cap H_0^1(\Omega_+) \to L^2(\Omega_+) \text{ is a Fredholm operator of index zero.} \tag{2.28}$$

Moreover, by [DZ19, Theorem 4.37], $(P_\theta - \lambda^2 + V)^{-1}$ has the same poles as $R_V(\lambda)$ and, for $\chi \in C_c^\infty(\{x : \mathbf{r} \le r_1\})$ with $\operatorname{supp}\chi \Subset \mathbb{R}^d$,

$$\chi(P_\theta - \lambda^2 + V)^{-1}\chi = \chi R_V(\lambda)\chi. \tag{2.29}$$

2.7 Defect measures

We say that a sequence $\{u_{\hbar_n}\}_{n=1}^\infty$ with $\|u_{\hbar_n}\|_{L^2(\mathbb{R}^d)} \le C$ for all n (with C independent of n) has *defect measure* μ if for all $a \in C_c^\infty(T^*\mathbb{R}^d)$,

$$\big\langle \operatorname{Op}_{\hbar_n}(a) u_{\hbar_n}, u_{\hbar_n} \big\rangle_{L^2(\mathbb{R}^d)} \to \int a\, d\mu,$$

where $\operatorname{Op}_\hbar(a)$ denotes the quantisation of the symbol a; see [DZ19, Equation E.1.18], [DZ19, Equation 4.1.2]. By, e.g., [Zwo12, Theorem 5.2], μ is a positive Radon measure on $T^*\mathbb{R}^d$. We say that $u_{\hbar_n}$ and $f_{\hbar_n}$ have joint defect measure μ^j if

$$\big\langle \operatorname{Op}_{\hbar_n}(a) u_{\hbar_n}, f_{\hbar_n} \big\rangle_{L^2(\mathbb{R}^d)} \to \int a\, d\mu^j. \tag{2.30}$$

We usually suppress the n in the notation and instead write that $u_\hbar$ has defect measure μ and $u_\hbar$ and $f_\hbar$ have joint defect measure μ^j.

Lemma 2.9 **([Zwo12, Theorem 5.3].)** *Let $\mathsf{P} \in \Psi^m(\mathbb{R}^d)$ and suppose that $u_\hbar$ has defect measure μ and satisfies*

$$\|\mathsf{P}u_\hbar\|_{L^2(\mathbb{R}^d)} = o(1).$$

Then, $\operatorname{supp}\mu \subset \{\sigma_\hbar(\mathsf{P}) = 0\}$, *where $\sigma_\hbar(\mathsf{P})$ is the semiclassical principal symbol of P.*

The following lemma is the defect-measure analogue of the propagation of singularities result [DZ19, Theorem E.47]. Before proving it, we recall the fundamental result that, for $a, b \in C_c^\infty(T^*\mathbb{R}^d)$,

$$\hbar^{-1}\sigma_\hbar\big(\big[\,\mathrm{Op}_\hbar(a), \mathrm{Op}_\hbar(b)\big]\big) = -\mathrm{i}\{a, b\}, \tag{2.31}$$

where (as in §2.4) $\{\cdot,\cdot\}$ denotes the Poisson bracket; see [DZ19, Equation E.1.44], [Zwo12, Page 68].

Lemma 2.10 *Let $\mathsf{P} \in \Psi^m(\mathbb{R}^d)$ with $\operatorname{Im}\mathsf{P} \le 0$ and suppose that $u_\hbar$ has defect measure μ and satisfies*

$$\mathsf{P}u_\hbar = \hbar f_\hbar,$$

where $\|f_\hbar\|_{L^2(\mathbb{R}^d)} \le C$ and $u_\hbar$ and $f_\hbar$ have joint defect measure μ^j. Then, for all real valued $a \in C_c^\infty(T^*\mathbb{R}^d)$,

$$\mu(H_{\operatorname{Re}\sigma_\hbar(\mathsf{P})}a^2) \ge -2\operatorname{Im}\mu^j(a^2).$$

Proof. Let $A = \mathrm{Op}_\hbar(a)$. Since $\sigma_\hbar(A^*) = a$ (by [DZ19, Equation E.1.45]) and thus $\sigma_\hbar(A^*A) = a^2$ (by [DZ19, Equation E.1.43]), by the definition of the joint measure (2.30),

$$2\hbar^{-1}\operatorname{Im}\big\langle A^*Au_\hbar, \mathsf{P}u_\hbar\big\rangle = 2\operatorname{Im}\mu^j(a^2) + o(1), \tag{2.32}$$

and, by (2.31) and (2.21),

$$\hbar^{-1}\operatorname{Im}\big\langle [A^*A, \operatorname{Re}\mathsf{P}]u_\hbar, u_\hbar\big\rangle = \mu(H_{\operatorname{Re}\sigma_\hbar(\mathsf{P})}a^2).$$

Since $2\operatorname{Im} z = \operatorname{Im}(z - \overline{z})$ and $\mathsf{P} = \operatorname{Re}\mathsf{P} + \mathrm{i}\operatorname{Im}\mathsf{P}$ with $\operatorname{Re}\mathsf{P}$ and $\operatorname{Im}\mathsf{P}$ both self-adjoint,

$$\begin{aligned}
&-2\hbar^{-1}\operatorname{Im}\big\langle A^*Au_\hbar, \mathsf{P}u_\hbar\big\rangle,\\
&\quad= \hbar^{-1}\operatorname{Im}\Big(\big\langle \mathsf{P}u_\hbar, A^*Au_\hbar\big\rangle - \big\langle A^*Au_\hbar, \mathsf{P}u_\hbar\big\rangle\Big),\\
&\quad= \hbar^{-1}\operatorname{Im}\Big(\big\langle (A^*A\operatorname{Re}\mathsf{P} - \operatorname{Re}\mathsf{P}A^*A)u_\hbar, u_\hbar\big\rangle + \mathrm{i}\big\langle (A^*A\operatorname{Im}\mathsf{P} + \operatorname{Im}\mathsf{P}A^*A)u_\hbar, u_\hbar\big\rangle\Big),\\
&\quad= \hbar^{-1}\operatorname{Im}\big\langle (A^*A\operatorname{Re}\mathsf{P} - \operatorname{Re}\mathsf{P}A^*A)u_\hbar, u_\hbar\big\rangle + 2\hbar^{-1}\operatorname{Re}\big\langle A^*A\operatorname{Im}\mathsf{P}u_\hbar, u_\hbar\big\rangle),\\
&\quad= \mu(H_{\operatorname{Re}\sigma_\hbar(\mathsf{P})}a^2) + o(1) + 2h^{-1}\operatorname{Re}\big\langle \operatorname{Im}\mathsf{P}Au_\hbar, Au_\hbar\big\rangle + 2h^{-1}\operatorname{Re}\big\langle A^*[A, \operatorname{Im}\mathsf{P}]u_\hbar, u_\hbar\big\rangle,\\
&\quad\le \mu(H_{\operatorname{Re}\sigma_\hbar(\mathsf{P})}a^2) + o(1) + 2h^{-1}\operatorname{Re}\big\langle A^*[A, \operatorname{Im}\mathsf{P}]u_\hbar, u_\hbar\big\rangle,
\end{aligned} \tag{2.33}$$

where the last line follows from the fact that $\operatorname{Im}\mathsf{P} \le 0$. By (2.31),

$$\operatorname{Re}\hbar^{-1}\sigma_\hbar\big(A^*[A, \operatorname{Im}\mathsf{P}]\big) = \operatorname{Re}\big(-\mathrm{i}a\{a, \operatorname{Im}\sigma_\hbar(\mathsf{P})\}\big) = 0,$$

and therefore, since the kernel of $\sigma_\hbar : \Psi^{-\infty} \to S^{-\infty}/hS^{-\infty}$ is $h\Psi^{-\infty}$, $h^{-1}\operatorname{Re}A^*[A, \operatorname{Im}\mathsf{P}] \in h\Psi^{-\infty}$ and, in particular,

$$\operatorname{Re}\langle A^*[A, \operatorname{Im}\mathsf{P}]u_\hbar, u_\hbar\rangle = \mathcal{O}(\hbar^2). \tag{2.34}$$

The lemma follows from combining (2.33) with (2.34) and (2.32), and sending $\hbar \to 0$. ∎

Corollary 2.11 *Suppose the assumptions of Lemma 2.10 hold and, in addition, $\mu^j = 0$. Then, with φ_t the bicharacteristic flow corresponding to the symbol $\operatorname{Re}\sigma_\hbar(\mathsf{P})$, for any $B \subset T^*\mathbb{R}^d$,*

$$\mu\big(\varphi_t(B)\big) \le \mu(B) \quad \textit{for } t \ge 0. \tag{2.35}$$

Corollary 2.11 shows that, under the assumptions of Lemma 2.10, we have information about the defect measures of sets moving forward under the flow.

Proof of Corollary 2.11. By (2.21),

$$\partial_t\left(\int (a^2 \circ \varphi_t)\, d\mu\right) = \int \partial_t(a^2 \circ \varphi_t)\, d\mu = \mu\big(H_{\operatorname{Re}\sigma_\hbar(\mathsf{P})}a^2\big) \ge 0,$$

and thus

$$\int a^2\, d\mu \ge \int (a^2 \circ \varphi_{-t})\, d\mu. \tag{2.36}$$

Let 1_B be the indicator function of $B \subset T^*\mathbb{R}^d$. By approximating 1_B by squares of smooth symbols, compactly supported symbols (2.36) holds with $a^2 = 1_B$. Since $1_B \circ \varphi_{-t} = 1_{\varphi_t(B)}$ the result (2.35) follows. More precisely, we first let B open and $K_n \Subset B$ compact with $K_n \uparrow B$ and choose $a_n \in C_c^\infty(T^*\mathbb{R}^d)$ with $a_n \equiv 1$ on K_n and $\operatorname{supp} a_n \subset B$. The result for B open follows by monotonicity of measure from below; the result for general B follows by outer regularity of μ. ∎

We now review some properties of defect measures when $u_\hbar$ satisfies the Helmholtz equation. Let $f_\hbar \in L^2_{\mathrm{comp}}(\mathbb{R}^d)$ be such that $\|f_\hbar\|_{L^2(\mathbb{R}^d)} \leq C$. Let $\psi \in C_c^\infty(\mathbb{R};[0,1])$ with $\psi \equiv 1$ on $[-E,E]$ and $\operatorname{supp}\psi \subset [-2E,2E]$ for some $E \geq 1$. Let $Q_b \in \Psi^{\mathrm{comp}}(\Gamma_D)$ with symbol

$$\sigma_\hbar(Q_b) = -\psi(|\xi'|_g).$$

Let $u \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ be a solution to

$$(-\hbar^2\Delta - 1)u_\hbar = \hbar f_\hbar \quad \text{in } \Omega_+, \qquad (Q_b\hbar D_\nu + 1)u_\hbar|_{\Gamma_D} = 0.$$

Suppose that $u_\hbar 1^{\mathrm{ext}}_{\Omega_+}$ has defect measure μ and $u_\hbar 1^{\mathrm{ext}}_{\Omega_+}$ and $f_\hbar$ have joint defect measure μ^j. On Γ_D, let ν_j be the joint measure associated with the Dirichlet and Neumann traces and ν_n be the measure associated with the Neumann trace; see [GLS21, Theorem 2.3]. In what follows, we only use the fact that there exists $\dot{n}^j$ such $\dot{n}^j\nu_n = \nu_j$; see [GLS21, Lemma 2.14].

With u as above, let $\mu^{\mathrm{in/out}}$ be the positive measures on $T^*\Gamma_D$, supported in the hyperbolic set $\mathcal{H}_{\Gamma_D}$, and defined in [GLS21, Lemma 2.9]/[Mil00, Proposition 1.7, Part (ii)].

In the following lemma, ${}^bT^*M$ denotes the b-cotangent bundle to Ω_+ and $\pi : T^*M \to {}^bT^*\Omega_+$ is defined by $\pi(x_1,x',\xi_1,\xi') := (x_1,x',x_1\xi_1,\xi')$ (for more details about ${}^bT^*M$, see, e.g., [Hör85, Section 18.3], [GSW20, Section 4B]).

Lemma 2.12 *With $u_\hbar$, μ, μ^j, μ^{in}, μ^{out}, and $\dot{n}^j$ as above,*

(i) $\operatorname{supp}\mu \subset S^*\Omega_+$

(ii) For all $\chi \in C_c^\infty(\mathbb{R}^d \setminus \Omega_-)$, $\lim_{\hbar\to 0}\|\chi u_\hbar\|^2_{L^2} = \mu(|\chi|^2)$

(iii) For all $a \in C_c^\infty({}^bT^*\Omega_+)$,

$$\pi_*\mu(a\circ\varphi_t) - \pi_*\mu(a) = \int_0^t \Big(-2\,\mathrm{Im}\,\pi_*\mu^j + \delta(x_1)\otimes(\mu^{\mathrm{in}} - \mu^{\mathrm{out}}) + \frac{1}{2}\,\mathrm{Re}\,\dot{n}^j H_p^2 x_1 \mu 1_{\mathcal{G}}\Big)(a\circ\varphi_s)\,ds$$

where $\mathcal{G} \subset S^*_{\Gamma_D}\mathbb{R}^d$ is the gliding set defined by

$$\mathcal{G} := \Big\{x_1 = H_p x_1 = 0, \qquad H_p^2 x_1 < 0\Big\}.$$

and the boundary Γ_D is given locally as $\{x_1 = 0\}$, with Ω_+ equal to $\{x_1 > 0\}$.

(iv) On $\mathcal{H}_{\Gamma_D}$, $\mu^{\mathrm{out}} = \alpha\,\mu^{\mathrm{in}}$, where

$$\alpha := \left|\frac{\sqrt{1-|\xi'|_g^2} - 1}{\sqrt{1-|\xi'|_g^2} + 1}\right|^2. \tag{2.37}$$

References for the proof. Parts (i) and (ii) are proved in [GSW20, Lemma 4.2]. Part (iii) is proved in [GLS21, Theorem 2.15] (following [GSW20, Lemma 4.8]), and Part (iv) is proved in [GLS21, Lemma 2.12] (following [Mil00, Proposition 1.10, Part (iii)]). ∎

3 Parametrix for $(P_\theta - \lambda^2)$ via boundary complex absorption

We now find a parametrix for $(P_\theta - \lambda^2)$ using a complex absorbing potential on the boundary Γ_D. We then obtain by perturbation a parametrix for $(P_\theta - \lambda^2 - z1_{\Omega_{\mathrm{tr}}})$ for z sufficiently small.

First, let

$$\mathcal{P}_\theta(\lambda) := \begin{pmatrix} P_\theta - \lambda^2 \\ \gamma_D \end{pmatrix} : H^2(\Omega_+) \to L^2(\Omega_+) \oplus H^{3/2}(\Gamma_D).$$

Then let $E : H^{3/2}(\Gamma_D) \to H^2(\Omega_+)$ be an extension operator satisfying

$$\gamma_D E g = g, \qquad g \in H^{\frac{3}{2}}(\partial\Omega).$$

Simple calculation then implies that

$$(\mathcal{P}_\theta(\lambda))^{-1} = \Big(\mathcal{R}_\theta(\lambda),\, E - \mathcal{R}_\theta(\lambda)(P_\theta - \lambda^2)E\Big), \tag{3.1}$$

where $\mathcal{R}_\theta(\lambda) := (P_\theta - \lambda^2)^{-1}$ is the inverse of (2.27).

Lemma 3.1 *The operator $\mathcal{P}_\theta(\lambda)$ is Fredholm with index zero.*

Proof. Recall that the map (2.27) is Fredholm with index zero. First, note that if $\mathcal{P}_\theta(\lambda)u = 0$, then $u \in H_0^1(\Omega_+) \cap H^2(\Omega_+)$ and in particular, $u \in \ker(P_\theta - \lambda^2)$. Therefore, since $P_\theta - \lambda^2 : H_0^1(\Omega_+) \cap H^2(\Omega_+) \to L^2(\Omega_+)$ is Fredholm, $\ker \mathcal{P}_\theta(\lambda)$ is finite dimensional. To see that the cokernel $L^2(\Omega_+) \oplus H^{3/2}(\Gamma_D) \big/ \mathcal{P}_\theta(\lambda) H^2(\Omega_+)$ is finite dimensional, define the map

$$\begin{gathered}\pi : L^2(\Omega_+) \oplus H^{3/2}(\Gamma_D) \Big/ \mathcal{P}_\theta(\lambda)H^2(\Omega_+) \to L^2(\Omega_+) \Big/ (P_\theta - \lambda^2)\big(H_0^1(\Omega_+) \cap H^2(\Omega_+)\big), \\ (f, g) + \mathcal{P}(\lambda)H^2(\Omega_+) \mapsto f - (P_\theta - \lambda^2)Eg + (P_\theta - \lambda^2)\big(H_0^1(\Omega_+) \cap H^2(\Omega_+)\big).\end{gathered}$$

First, observe that this map is well defined since if $(f_1, g_1) + \mathcal{P}_\theta(\lambda) = (f_2, g_2) + \mathcal{P}_\theta(\lambda)$ then there is $u \in H^2(\Omega_+)$ such that

$$(f_1 - f_2, g_1 - g_2) = \big((P_\theta - \lambda^2)u, \gamma_D u\big).$$

In particular,

$$(f_1 - f_2) - (P_\theta - \lambda^2)E(g_1 - g_2) = (P_\theta - \lambda^2)(u - E(g_1 - g_2)) \in (P_\theta - \lambda^2)\big(H_0^1(\Omega_+) \cap H^2(\Omega_+)\big),$$

so $\pi(f_1, g_1) = \pi(f_2, g_2)$.

Now, suppose that $\pi(f, g) = 0$. Then, there is $u \in H_0^1(\Omega_+) \cap H^2(\Omega_+)$ such that

$$f - (P_\theta - \lambda^2)Eg = (P_\theta - \lambda^2)u.$$

Therefore,

$$(f, g) - \mathcal{P}_\theta(\lambda)Eg = (f - (P_\theta - \lambda^2)Eg, 0) = ((P_\theta - \lambda^2)u, 0) \in \mathcal{P}_\theta(\lambda)H^2(\Omega_+),$$

and π is injective. For an injective operator, dim(domain) $\leq$ dim(range) $\leq$ dim(codomain); therefore

$$\dim\Big(L^2(\Omega_+) \oplus H^{3/2}(\Gamma_D) \Big/ \mathcal{P}_\theta(\lambda)H^2(\Omega_+)\Big) \leq \dim\Big(L^2(\Omega_+) \Big/ (P_\theta - \lambda^2)\big(H_0^1(\Omega_+) \cap H^2(\Omega_+)\big)\Big) < \infty$$

Since $P_\theta - \lambda^2 : H_0^1(\Omega_+) \cap H^2(\Omega_+) \to L^2(\Omega_+)$ is Fredholm, $\mathcal{P}_\theta(\lambda)$ is Fredholm. To see that $\mathcal{P}_\theta(\lambda)$ has index zero, recall that the index is constant in λ by, e.g., [DZ19, Theorem C.5], and observe that the formula (3.1) implies that the inverse exists for some λ. ■

We now define our complex absorbing operator. Let $\psi \in C_c^\infty(\mathbb{R}; [0, 1])$ with $\psi \equiv 1$ on $[-b, b]$ and $\operatorname{supp}\psi \subset [-2b, 2b]$. It will be convenient to have a specific notation for the Neumann trace with the standard derivative operator replaced by $D := -i\hbar\partial$. We therefore let $\gamma_{1,\hbar}^D := -i\hbar\gamma_1^D$. Let

$$\mathcal{P}_{\theta,Q}(\lambda) := \begin{pmatrix} P_\theta - \lambda^2 \\ Q_b\gamma_{1,\hbar}^D + \gamma_0^D \end{pmatrix} : H^2(\Omega_+) \to L^2(\Omega_+) \oplus H^{3/2}(\Gamma_D).$$

where $Q_b \in \Psi^{\text{comp}}(\Gamma_D)$ with symbol

$$\sigma_\hbar(Q_b) = -\psi(|\xi'|_g).$$

Note that

$$\mathcal{P}_Q(\lambda) = \mathcal{P}_\theta(\lambda) + \begin{pmatrix} 0 \\ Q_b\gamma_{1,\hbar}^D \end{pmatrix}$$

and hence is a compact perturbation of $\mathcal{P}_\theta(\lambda)$. Therefore, by Lemma 3.1, $\mathcal{P}_Q(\lambda)$ is Fredholm with index zero.

Lemma 3.2 *Let Q_b be as above and $0 < a < b$ and $C_1 > 0$. Then there exists $C > 0$ such that for all $\lambda \in [a, b] + \mathrm{i}[-C_1\hbar, C_1\hbar]$,*

$$\|\gamma^D_{1,\hbar} u\|_{L^2(\Gamma_D)} + \|u\|_{H^2_\hbar(\Omega_+)} \leq Ch^{-1} \left\|(P_\theta - \lambda^2)u\right\|_{L^2(\Omega_+)} + C \left\|(Q_b\gamma^D_{1,\hbar} + \gamma^D_0)u\right\|_{H^{3/2}_\hbar(\Gamma_D)}.$$

In particular, since $\mathcal{P}_{\theta,Q}(\lambda)$ is Fredholm with index zero,

$$\mathcal{R}_{\theta,Q}(\lambda) := (\mathcal{P}_{\theta,Q}(\lambda))^{-1}$$

exists and satisfies

$$\|\gamma^D_{1,\hbar}\mathcal{R}_{\theta,Q}(\lambda)(f, g)\|_{L^2(\Gamma_D)} + \|\mathcal{R}_{\theta,Q}(\lambda)(f, g)\|_{H^2_\hbar(\Omega_+)} \leq C\Big(h^{-1}\|f\|_{L^2(\Omega_+)} + \|g\|_{H^{3/2}_\hbar(\Gamma_D)}\Big). \tag{3.2}$$

Observe that the bound (3.2) has the same $\hbar$-dependence as the standard non-trapping resolvent estimate.

Before proving Lemma 3.2 we show how a parametrix for the operator $(P_\theta - \lambda^2 - z1_{\Omega_{\rm tr}})$ can be expressed in terms of $\mathcal{R}_{\theta,Q}(\lambda)$. Let

$$\mathcal{P}_{\theta,Q}(\lambda, z) := \begin{pmatrix} P_\theta - \lambda^2 - z1_{\Omega_{\rm tr}} \\ Q_b\gamma^D_{1,\hbar} + \gamma^D_0 \end{pmatrix} : H^2(\Omega_+) \to L^2(\Omega_+) \oplus H^{3/2}(\Gamma_D).$$

By Lemma 3.2, the bound (3.2), and inversion by Neumann series for $|z| \leq \frac{\hbar}{2C}$ (where C is the constant from Lemma 3.2)

$$\mathcal{R}_{\theta,Q}(\lambda, z) := (\mathcal{P}_{\theta,Q}(\lambda, z))^{-1}$$

exists and satisfies

$$\|\gamma^D_{1,\hbar}\mathcal{R}_{\theta,Q}(\lambda, z)(f, g)\|_{L^2(\Gamma_D)} + \|\mathcal{R}_{\theta,Q}(\lambda, z)(f, g)\|_{H^2_\hbar(\Omega_+)} \leq 2C\Big(h^{-1}\|f\|_{L^2(\Omega_+)} + \|g\|_{H^{3/2}_\hbar(\Gamma_D)}\Big). \tag{3.3}$$

Next, let

$$\mathcal{P}_\theta(\lambda, z) := \begin{pmatrix} P_\theta - \lambda^2 - z1_{\Omega_{\rm tr}} \\ \gamma^D_0 \end{pmatrix} : H^2(\Omega_+) \to L^2(\Omega_+) \oplus H^{3/2}_\hbar(\Gamma_D). \tag{3.4}$$

If $\mathcal{R}_{\theta,Q}(\lambda, z)$ exists, then

$$\mathcal{P}_\theta(\lambda, z) = \big(I + K(\lambda, z)\big)\mathcal{P}_{\theta,Q}(\lambda, z),$$

where

$$K(\lambda, z) := Q\mathcal{R}_{\theta,Q}(\lambda, z) \quad \text{ and } \quad Q := \begin{pmatrix} 0 \\ -Q_b\gamma^D_{1,\hbar} \end{pmatrix}. \tag{3.5}$$

Since $K(\lambda, z) : L^2(\Omega_+) \oplus H^{3/2}(\Gamma_D) \to L^2(\Omega_+) \oplus H^{3/2}(\Gamma_D)$ is compact, $(I + K(\lambda, z))^{-1}$ is a meromorphic family of operators by [DZ19, Theorem C.8]. Therefore, for $|z| \leq \frac{\hbar}{2C}$,

$$\mathcal{P}_\theta(\lambda, z)^{-1} = \mathcal{R}_{\theta,Q}(\lambda, z)\big(I + K(\lambda, z)\big)^{-1}. \tag{3.6}$$

Let $R_\theta(\lambda, z)$ be the inverse of the map (2.28) with $V = -z1_{\Omega_{\rm tr}}$, i.e.

$$R_\theta(\lambda, z) := (P_\theta - \lambda^2 - z1_{\Omega_{\rm tr}})^{-1}. \tag{3.7}$$

Then, for $|z| \leq \frac{\hbar}{2C}$,

$$R_\theta(\lambda, z) = \mathcal{P}_\theta(\lambda, z)^{-1}\begin{pmatrix} I \\ 0 \end{pmatrix} = \mathcal{R}_{\theta,Q}(\lambda, z)\big(I + K(\lambda, z)\big)^{-1}\begin{pmatrix} I \\ 0 \end{pmatrix}, \tag{3.8}$$

which is the required parametrix.

Proof of Lemma 3.2. Suppose that the estimate fails with $\|u\|_{H^2_\hbar(\Omega_+)}$ replaced by $\|u\|_{L^2(\Omega_+)}$, then there are $\hbar_n \to 0$, $\lambda_n \in [a, b] + \mathrm{i}[-C\hbar, C\hbar]$, and $(\tilde{u}_n, \tilde{f}_n, \tilde{g}_n) \in H^2(\Omega_+) \oplus L^2(\Omega_+) \oplus H^{3/2}_\hbar(\Gamma_D)$ with

$$\big\|\tilde{f}_n\big\|_{L^2(\Omega_+)} + \|\tilde{g}_n\|_{H^{3/2}_\hbar(\Gamma_D)} = 1, \qquad \|\tilde{u}_n\|_{L^2} = n,$$

and with

$$\mathcal{P}_{\theta,Q}(\tilde{u}_n) = (\hbar_n \tilde{f}_n, \tilde{g}_n).$$

In particular, renomarlizing $u_n := \tilde{u}_n/n$, $f_n := \tilde{f}_n/n$, and $g_n := \tilde{g}_n/n$,

$$\|f_n\|_{L^2(\Omega_+)} = \hbar^{-1}\left\|(P_\theta - \lambda_n^2)u_n\right\|_{L^2(\Omega_+)} \leq \frac{1}{n}$$

and

$$\|g_n\|_{L^2(\Gamma_D)} = \left\|(Q_b\gamma^D_{1,\hbar} + \gamma_D)u_n\right\|_{L^2(\Gamma_D)} \leq \frac{1}{n}.$$

Now, since $0 < a \leq \operatorname{Re}\lambda_n \leq b$, we may rescale $\hbar_n$ to $\tilde{\hbar} := \hbar_n/\operatorname{Re}\lambda_n$ and hence replace $\operatorname{Re}\lambda_n$ by 1. Note that this rescaling does not cause any issues since $b^{-1}\hbar_n \leq \tilde{\hbar}_n \leq a^{-1}\hbar_n$. Extracting a subsequence, we can assume that $u_n 1^{\rm ext}_{\Omega_+}$ has defect measure μ (see e.g. [Zwo12, Theorem 5.2]) and $\hbar_n^{-1}\operatorname{Im}\lambda_n \to \operatorname{Im}\beta_\infty$, and $\operatorname{Re}\lambda_n = 1$. Since $\|f_{\hbar_n}\|_{L^2} \to 0$, $\mu^j = 0$.

Let $\chi, \chi_0 \in C_c^\infty(\mathbb{R}^d;[0,1])$ with $\operatorname{supp}\chi \Subset \mathbb{R}^d$ and $\chi,\chi_0 \equiv 1$ in a neighborhood of $\{\mathbf{r} \leq 2r_1\}$ and $\operatorname{supp}\chi_0 \subset \{\chi \equiv 1\}$. We first show that

$$\|(1-\chi)u_n\|_{L^2(\Omega_+)} = \mathcal{O}(\hbar_n). \tag{3.9}$$

To do this, observe that, by (2.26),

$$|\sigma_\hbar(P_\theta - \lambda_n^2)(x,\xi)| = \left|\frac{|\xi|^2}{(1+\mathrm{i}\tan\theta)^2} - 1\right| \geq c(|\xi|^2+1), \qquad \mathbf{r}(x,\xi) \geq 2r_1. \tag{3.10}$$

Therefore, by ellipticity, for W a neighborhood of $\operatorname{supp}\partial\chi$,

$$\|u_n\|_{H^2_\hbar(W)} \leq C\big(\left\|(P_\theta - \lambda^2)u_n\right\|_{L^2(\Omega_+)} + \|u_n\|_{L^2(\Omega_+)}\big). \tag{3.11}$$

Now, by (3.10) and the definitions of χ and χ_0,

$$\left|\sigma\Big(\operatorname{Op}_\hbar\big((1+|\xi|^2)^{-1}\big)(1-\chi_0)(P_\theta - \lambda_n^2)(1-\chi_0) - \mathrm{i}\chi_0\Big)\right| \geq c.$$

Therefore, by [Zwo12, Theorem 4.29],

$$\begin{aligned}
&\|(1-\chi)u_n\|_{L^2(\Omega_+))}\\
&\quad\leq C\left\|\big[\operatorname{Op}_\hbar((1+|\xi|^2)^{-1})(1-\chi_0)(P_\theta-\lambda_n^2)(1-\chi_0) - \mathrm{i}\chi_0\big](1-\chi)u_n\right\|_{L^2(\mathbb{R}^d)}\\
&\quad= C\left\|\operatorname{Op}_\hbar((1+|\xi|^2)^{-1})(1-\chi_0)(P_\theta-\lambda_n^2)(1-\chi)u_n\right\|_{L^2(\mathbb{R}^d)}.
\end{aligned} \tag{3.12}$$

But,

$$\begin{aligned}
&\left\|\operatorname{Op}_\hbar((1+|\xi|^2)^{-1})(1-\chi_0)(P_\theta-\lambda_n^2)(1-\chi)u_n\right\|_{L^2(\mathbb{R}^d)},\\
&\qquad\leq C\,\|(1-\chi)\hbar_n f_n\|_{L^2(\Omega_+)} + \|[P_\theta,\chi]u_n\|_{H_\hbar^{-2}(\Omega_+)},\\
&\qquad\leq C\,\|(1-\chi)\hbar_n f_n\|_{L^2(\Omega_+)} + C\hbar_n\,\|u_n\|_{L^2(\Omega_+)} = \mathcal{O}(\hbar_n).
\end{aligned} \tag{3.13}$$

where we have used (3.11) in the second inequality; (3.9) then follows from combining (3.12) and (3.13).

We now show that $\mu(T^*\mathbb{R}^d) = 1$. First, observe that

$$(P_\theta - \lambda_n^2)\chi u_n = [P_\theta,\chi]u_n + o(\hbar_n)_{L^2}. \tag{3.14}$$

Indeed, using (3.11) in (3.14) we find that

$$(P_\theta - \lambda_n^2)\chi u_n = \mathcal{O}(\hbar_n)_{L^2}.$$

Since $(P_\theta - \lambda^2) = (-\hbar^2\Delta - \lambda^2)$ on $\operatorname{supp}\chi$, we can now apply Lemma 2.12 (with u in that lemma replaced by χu_n here) to find that

$$\mu(\chi^2) = \lim_{\hbar\to 0}\|\chi u_n\|^2_{L^2(\Omega_+)} = \lim_{\hbar\to 0}\|u_n\|^2_{L^2(\Omega_+)} = 1,$$

where we have used (3.9) in the second equality. Moreover,

$$\mu(T^*\mathbb{R}^d) \leq \lim_{\hbar\to 0} \|u_n\|^2_{L^2(\Omega_+)} = 1,$$

so that in fact $\mu(T^*\mathbb{R}^d) = 1$.

We now show that $\mu = 0$ which is a contradiction. To do this, we start by observing that (3.9) implies that $\mu(\{\mathbf{r} \geq 2r_1\}) = 0$. Now, Lemma 2.12, along with Lemma 2.10 together with the fact that $\operatorname{Im}(P_\theta) \leq 0$, allows us to propagate forward along the generalized bicharacteristic flow (in the sense of Corollary 2.11), but not backward. In particular, since $\mu(\{\mathbf{r} \geq 2r_1\} = 0)$, this implies that $\operatorname{supp}\mu \subset \Gamma_+$. Indeed, suppose that $A \subset S^*_{\overline{\Omega_+}}\mathbb{R}^d$ is compact and $A \cap \Gamma_+ = \emptyset$. Then, by the definition of Γ_+ (2.24), for each $\rho \in A$ there is $t_\rho > 0$ such that $\mathbf{r}(\varphi_{-t_\rho}(\rho)) > \max(2r_1, \mathbf{r}(\rho))$. Hence, by (2.22) for $t \geq t_\rho$, $\mathbf{r}(\varphi_{-t}(\rho)) > 2r_1$ and by continuity of φ_{-t_ρ}, there is a neighborhood U_ρ of ρ such that $\varphi_{-t}(U_\rho) \subset \{\mathbf{r} > 2r_1\}$ for $t \geq t_\rho$. In particular, by compactness of A, there is $T > 0$ such that $\varphi_{-T}(A) \subset \{\mathbf{r} > 2r_1\}$. Then by (2.35), $\mu(A) \leq \mu(\varphi_{-T}(A)) = 0$. Now, by Lemma 2.8, Γ_+ is closed and hence we may write $(\Gamma_+)^c = \cup_n A_n$ with A_n compact. In particular, $\mu((\Gamma_+)^c) = 0$ by monotonicity from below.

Next, note that since $\operatorname{Im}\sigma_\hbar(P_\theta - \lambda^2) < 0$ on $\{f_\theta \neq 0\}$,

$$\operatorname{supp}\mu \subset \{f_\theta = 0\}$$

by Lemma 2.9. In particular, by the definition of f_θ,

$$\operatorname{supp}\mu \subset \{\mathbf{r} < 2r_1\}.$$

To complete the proof, we need to show that in fact $\mu(\Gamma_+) = 0$. This is where the boundary term Q_b is used.

We claim there are $T, c > 0$ such that

$$\mu(\varphi_{-T}(A)) \geq \mathrm{e}^c \mu(A) \tag{3.15}$$

for all A. Once this is done, we have that $\mu \equiv 0$. To see this, observe that if $\mu(A) > 0$, then by induction $\mu(\varphi_{-nT}(A)) \geq \mathrm{e}^{nc}\mu(A)$. Taking $N > -(\log\mu(A))/c$, we have $\mu(\varphi_{-NT}(A)) > 1$, which is a contradiction to $\mu(T^*\mathbb{R}^d) = 1$.

We now prove (3.15). First, note that the statement is empty if $\mu(A) = 0$. Therefore, we can assume that $\mu(A) > 0$. Since $\operatorname{supp}\mu \subset \Gamma_+$, we assume that $A \subset \Gamma_+$; since Γ_+ is closed, we can assume that A is compact. Now, by (2.22), (2.23), and (2.24),

$$\Gamma_+ \cap \{r_0 \leq \mathbf{r} \leq 2r_1\} \subset \bigcup_{t=0}^{\sqrt{(2r_1)^2 - r_0^2}} \varphi_t(\Gamma_+ \cap \{\mathbf{r} \leq r_0\}).$$

Therefore, increasing T by $\sqrt{(2r_1)^2 - r_0^2}$, we may assume that $A \subset \{\mathbf{r} < r_0\} \cap \Gamma_+$.

We claim that there are $\varepsilon_1, T > 0$ such that for all $\rho \in \Gamma_+$ with $\mathbf{r}(\rho) < r_0$.

$$\int_0^T \Big(-\frac{1}{2}H_p^2 x_1 1_{\mathcal{G}}(\varphi_{-t}(\rho)) + |H_p x_1(\varphi_{-t}(\rho))|^{-1}\delta\big(x_1(\varphi_{-t}(\rho))\big)\log\alpha\big(\pi_{\Gamma_D}(\varphi_{-t}(\rho))\big)\Big)\, dt \leq -\varepsilon_1. \tag{3.16}$$

where $\pi_{\Gamma_D} : S^*_{\Gamma_D}\mathbb{R}^d \to T^*\Gamma_D$ is the orthogonal projection and α is given by (2.37).

Once (3.16) is proved, we claim that Lemma 2.12 implies (3.15) with $(c, T) = (\varepsilon_1, T)$. Indeed, suppose that (3.16) holds and that $\mu(A) > 0$, $A \subset \Gamma_+ \cap \{\mathbf{r} < r_0\}$ and A is closed. Then, let $0 \leq a \in C_c^\infty({}^bT^*\mathbb{R}^d \setminus \Omega_-)$ with $a \equiv 1$ on A and

$$\bigcup_{t\in[0,-T]} \varphi_t(\operatorname{supp} a) \subset \Big\{\mathbf{r} < \frac{r_0 + r_1}{2}\Big\}. \tag{3.17}$$

Now, let $\chi \equiv 1$ on $\{\mathbf{r} \leq \frac{r_0+r_1}{2}\}$ with $\operatorname{supp}\chi \subset \{\mathbf{r} < r_1\}$. Then,

$$(-\hbar^2\Delta - 1)\chi u = [-\hbar^2\Delta, \chi]u + \hbar f, \qquad \chi u|_{\Gamma_D} = 0$$

with $f = o(1)_{L^2}$ and hence by Lemma 2.12

$$\pi_*\mu(\chi^2(a\circ\varphi_t))-\pi_*\mu(\chi^2 a) = \int_0^t \Big(-4\langle \xi,\partial\chi\rangle\mu+\delta(x_1)\otimes(\mu^{\mathrm{in}}-\mu^{\mathrm{out}})+\frac{1}{2}\operatorname{Re}\dot{n}^j H_p^2 x_1\mu 1_{\mathcal{G}}\Big)(\chi^2(a\circ\varphi_s))\,ds.$$

But, by (3.17), $\chi^2\equiv 1$ on $\operatorname{supp} a\circ\varphi_t$ for $t\in[0,T]$ and on $\mathcal{G}$, $\dot{n}^j=1$. In particular, for $t\in[0,T]$

$$\pi_*\mu(a\circ\varphi_t)-\pi_*\mu(a) = \int_0^t \Big(\delta(x_1)\otimes(\mu^{\mathrm{in}}-\mu^{\mathrm{out}})+\frac{1}{2}H_p^2 x_1\mu 1_{\mathcal{G}}\Big)(a\circ\varphi_s)\,ds.$$

Finally, since A is closed we may approximate 1_A by smooth, compactly supported functions to obtain

$$\pi_*\mu(\varphi_{-t}(A))-\pi_*\mu(A) = \int_0^t \Big(\delta(x_1)\otimes(\mu^{\mathrm{in}}-\mu^{\mathrm{out}})+\frac{1}{2}H_p^2 x_1\mu 1_{\mathcal{G}}\Big)(1_A\circ\varphi_s)\,ds. \tag{3.18}$$

Now, to study (3.18), we first assume that A and t are such that for all $\rho\in A$ and $s\in[0,2t]$, $\varphi_{-s}(\rho)$ does not lie in the glancing region ($H_p x_1=0$) and each trajectory intersects $\partial\Omega_-$ exactly once and does so for $s\in(0,t)$. Shrinking the support of a further if necessary, we can find $\Sigma\subset^b T^*\mathbb{R}^d\setminus\Omega_-$ transverse to H_p such that

$$F:[-t,t]\times\Sigma\ni(s,\rho)\mapsto\varphi_{-s}(\rho)\in^b T^*_{\Gamma_D}\mathbb{R}^d$$

are smooth coordinates and $\varphi_{-s}(A)$ is in the image of F for all $s\in[0,t]$. Then, (3.18) reads

$$\begin{aligned}
&\pi_*\mu(\varphi_{-t}(A))-\pi_*\mu(A)\\
&=\int_0^t \Big(\delta(x_1)\otimes(\mu^{\mathrm{in}}-\mu^{\mathrm{out}})\Big)(1_A\circ\varphi_{t'})\,dt'\\
&=\int_0^t\int_{-t}^t\int_\Sigma (|H_p x_1|(s,\rho)\delta(s)\otimes(1_A(s-t',\rho))d(\mu^{\mathrm{in}}-\mu^{\mathrm{out}})(\rho)\,ds\,dt'\\
&=\int_\Sigma\int_0^t \Big(|H_p x_1|(0,\rho)(\alpha^{-1}(\rho)-1)1_A(-t',\rho)\Big)d\mu^{\mathrm{out}}(\rho)dt'
\end{aligned}$$

Now, arguing as in [GLS21, Lemma 2.16], we obtain that $\pi_*\mu=|H_p x_1|\mu^{\mathrm{out}}1_{t<0}dt+|H_p x_1|\mu^{\mathrm{in}}1_{t>0}dt$ and hence,

$$\mu(A)=\int_\Sigma\int_0^t |H_p x_1|(0,\rho)1_A(-t',\rho)d\mu^{\mathrm{out}}(\rho)dt'.$$

Therefore

$$\begin{aligned}
\pi_*\mu(\varphi_{-t}(A)) &\geq \inf_{F^{-1}(A)}(\alpha(\rho))^{-1}\pi_*\mu(A)\\
&=\inf_A \mathrm{e}^{-\int_0^t(|H_p x_1|(\varphi_{-t}(\rho))^{-1}\delta(x_1(\varphi_{-t}(\rho)))\log\alpha(\pi_{\Gamma_D}(\varphi_{-t}(\rho)))}\pi_*\mu(a).
\end{aligned}$$

Next, we assume that

$$A\subset\Big\{\varphi_s\big(\{x_1=H_p x_1=0\}\big)\setminus\{H_p x_1\neq 0,\, x_1=0\,:\, s\in[0,t]\Big\},$$

so that, in particular, trajectories from A do not intersect the hyperbolic set. In this case, (3.18) implies that

$$\partial_s\pi_*\mu(\varphi_{-s}(A))=\Big(\frac{1}{2}H_p^2 x_1 1_{\mathcal{G}}\pi_*\mu\Big)(\varphi_{-s}(A)), \tag{3.19}$$

In particular, shrinking A in necessary, we may choose $\Sigma\subset\{x_1=H_p x_1=0\}$ transverse to H_p and work in coordinates

$$[0,t]\times\Sigma\ni(s,\rho)\mapsto\varphi_{-s}(\rho)\in\Big\{\varphi_{-s}\big(\{x_1=H_p x_1=0\}\big)\,:\,s\in[0,t]\Big\}.$$

In these coordinates, (3.19) implies that $\pi_*\mu$ is absolutely continuous with respect to t in the sense that there is a family of measures, $t \mapsto \nu_t$ on Σ such that $\nu_t(\Sigma) \in L^1$ and $\mu = \nu_t dt$. Moreover,

$$\int_B d\nu_s(\rho) = \int_B \mathrm{e}^{\int \frac{1}{2} H_p^2 x_1 1_{\mathcal{G}}(\varphi_{-s}(\rho))ds} d\nu(\rho).$$

In particular,

$$\pi_*\mu(\varphi_{-t}(A)) \geq \inf_A \mathrm{e}^{\int \frac{1}{2} H_p^2 x_1 1_{\mathcal{G}}(\varphi_{-s}(\rho))ds} \pi_*\mu(A)$$

Putting everything together, we have for all A and $0 \leq t \leq T$,

$$\begin{aligned}
&\pi_*\mu(\varphi_{-t}(A)) \\
&\geq \inf_A \exp\Big(- \int_0^t (|H_p x_1|(\varphi_{-t}(\rho))^{-1} \delta(x_1(\varphi_{-t}(\rho))) \log \alpha(\pi_{\Gamma_D}(\varphi_t(\rho))) - \frac{1}{2} H_p^2 x_1 1_{\mathcal{G}}(\varphi_{-s}(\rho)))ds\Big) \pi_*\mu(A) \\
&\geq \mathrm{e}^{\varepsilon_1} \pi_*\mu(A)
\end{aligned}$$

as claimed.

Therefore, it is enough to prove (3.16). Seeking a contradiction, we assume that for every $\varepsilon_1 > 0$ and $T > 0$ there is $\rho \in \Gamma_+$ with $\mathbf{r}(\rho) < r_0$ such that

$$\int_0^T \left(-\frac{1}{2} H_p^2 x_1 1_{\mathcal{G}}(\varphi_{-t}(\rho)) + |H_p x_1(\varphi_{-t}(\rho))|^{-1} \delta(x_1(\varphi_{-t}(\rho))) \log \alpha(\pi_{\Gamma_D}(\varphi_{-t}(\rho))) \right) dt \geq -\varepsilon_1. \tag{3.20}$$

Note that since both terms are non-positive (since $\alpha \leq 1$), this implies that each term is $\geq -\varepsilon_1$.

Now, if $\varphi_{-t}(\rho) \in \mathcal{G}$ for $t \in [t_1, t_2]$, then, since the flow in $\mathcal{G}$ is given by the flow of the vector field

$$H_p^G := H_p + \frac{H_p^2 x_1}{H_{x_1}^2 p} H_{x_1}, \qquad p = |\xi|^2 - 1,$$

(see [Hör85, Def. 24.3.6]), we obtain

$$\varphi_{-t_2}(\rho) = \exp(-(t_2 - t_1) H_{|\xi|^2}(\rho)) + \mathcal{O}\left(\int_{t_1}^{t_2} H_p^2 x_1(\varphi_{-t}(\rho)) dt \right).$$

On the other hand, if $\varphi_{-t}(\rho) \notin \mathcal{G}$ for $t \in [t_1, t_2]$, and has exactly one intersection with Γ_D, then

$$\varphi_{-t_2}(\rho) = \exp(-(t_2 - t_1) H_{|\xi|^2}(\varphi_{-t_1}(\rho)) + \mathcal{O}\left(|t_2 - t_1| 2\sqrt{1 - |\xi_r'|_g^2} \right).$$

where $|\xi_r'|_g$ is measured at the point of reflection. All together, since

$$\log \alpha = -4\sqrt{1 - |\xi'|_g^2} + \mathcal{O}\big(1 - |\xi'|_g^2\big),$$

we obtain from (3.20) that

$$\varphi_{-T}(\rho) = \exp(-T H_{|\xi|^2}(\rho)) + \mathcal{O}(\varepsilon_1)$$

Therefore, choosing $T \gg r_0$, and ε_1 small enough, we obtain

$$\mathrm{dist}(\pi_{\mathbb{R}}(\varphi_{-T}(\rho)), \pi_{\mathbb{R}}(\rho)) > 3r_0$$

which is a contradiction to $\rho \in \Gamma_+ \cap \{\mathbf{r} \leq r_0\}$.

Thus, we have proved

$$\|u\|_{L^2(\Omega_+)} \leq C\hbar^{-1} \left\| (P_\theta - \lambda^2) u \right\|_{L^2(\Omega_+)} + C \left\| (Q_b \gamma_{1,\hbar}^D + \gamma_0^D) u \right\|_{H_\hbar^{3/2}(\Gamma_D)}. \tag{3.21}$$

where here, and in the rest of the proof, C denotes a constant, independent of $\hbar$, λ, and z, whose value may change from line to line. We now need to obtain a bound on the $H_\hbar^2$ norm of u, as opposed to just the L^2 norm in (3.21). By a standard elliptic parametrix construction, for $\chi_1 \in C^\infty(\overline{\Omega_+})$ supported away from Γ_D, we have

$$\begin{aligned}\|\chi_1 u\|_{H^2_\hbar(\Omega_+)} &\leq C \left\|(P_\theta - \lambda^2)u\right\|_{L^2(\Omega_+)} + C\,\|u\|_{L^2(\Omega_+)} \\ &\leq Ch^{-1} \left\|(P_\theta - \lambda^2)u\right\|_{L^2(\Omega_+)} + C\left\|(Q_b\gamma^D_{1,\hbar} + \gamma^D_0)u\right\|_{H^{3/2}_\hbar(\Gamma_D)},\end{aligned}$$

by (3.21). Finally, using the trace estimate from [GLS21, Corollary 4.2] we have for $\chi_2 \in C^\infty(\{x : \mathbf{r} \leq r_0\})$ with $\operatorname{supp}\chi_2 \Subset \mathbb{R}^d$,

$$\left\|\gamma^D_{1,\hbar}u\right\|_{L^2(\Gamma_D)} \leq C\,\|\chi_2 u\|_{L^2(\Omega_+)} + \left\|(-\hbar^2\Delta - 1)\chi_2 u\right\|_{L^2(\Omega_+)}.$$

Elliptic regularity for the Laplacian then implies that

$$\begin{aligned}\|\chi_2 u\|_{H^2_\hbar(\Omega_+)} &\leq C\left\|(-\hbar^2\Delta - \lambda^2)\chi_2 u\right\|_{L^2} + C\,\|\chi u\|_{L^2} + C\left\|\gamma^D_0 u\right\|_{H^{3/2}_\hbar(\Gamma_D)} \\ &\leq Ch^{-1}\left\|(P_\theta - \lambda^2)u\right\|_{L^2} + C\left\|(Q_b\gamma^D_{1,\hbar} + \gamma^D_0)u\right\|_{H^{3/2}_\hbar(\Gamma_D)},\end{aligned}$$

where we have used (3.21). Combining the bounds on $\|\chi_1 u\|_{H^2_\hbar(\Omega_+)}$, $\|\chi_2 u\|_{H^2_\hbar(\Omega_+)}$, and $\|\gamma^D_{1,\hbar}u\|_{L^2(\Gamma_D)}$, we obtain (3.2). ∎

4 Proof of Lemma 1.12

With $R(\lambda, z)$ defined by (1.13), $R_\theta(\lambda, z)$ defined by (3.7), and $\chi \in C^\infty$ with $\operatorname{supp}\chi \subset \{x : \mathbf{r} \leq r_1\}$ and $\operatorname{supp}\chi \Subset \mathbb{R}^d$, (2.29) implies that

$$\chi R_\theta(\lambda, z)\chi = \chi R(\lambda, z)\chi. \tag{4.1}$$

Recalling (1.14), we see that to prove the bounds (1.17), (1.18) it is sufficient to bound

$$\|R_\theta(\lambda, z)\|_{L^2(\Omega_{\rm tr}) \to L^2(\Omega_{\rm tr})}.$$

We first focus on proving the bound for $\operatorname{Im} z > 0$ (1.18). By the definitions of $\mathcal{P}_\theta(\lambda, z)$ (3.4) and $R_\theta(\lambda, z)$ (3.7), the bound (1.18) follows if we can prove the following.

Lemma 4.1 *There exists $C > 0$ such that if $\operatorname{Re}\lambda > 0$, $\operatorname{Im}\lambda = 0$,*

$$\|\mathcal{P}_\theta(\lambda, z)^{-1}\|_{L^2(\Omega_{\rm tr})\otimes H^{3/2}_\hbar(\Gamma_D) \to L^2(\Omega_{\rm tr})} \leq C\langle z\rangle(\operatorname{Im} z)^{-1} \quad \textit{for } \operatorname{Im} z > 0. \tag{4.2}$$

Moreover, there exists $\varepsilon > 0$ small enough such that if $\operatorname{Re}\lambda > 0$, $\operatorname{Im}\lambda = 0$,

$$\|\mathcal{P}_\theta(\lambda, z)^{-1}\|_{L^2(\Omega_+)\otimes H^{3/2}_\hbar(\Gamma_D) \to H^2_\hbar(\Omega_+)} \leq C(\operatorname{Im} z)^{-1} \quad \textit{for } \operatorname{Im} z > 0 \textit{ and } |z| \leq \varepsilon\hbar. \tag{4.3}$$

To prove Lemma 4.1, we need the following result about the sign of the Dirichlet-to-Neumann map.

Lemma 4.2 *For $\operatorname{Re}\lambda > 0$, and $\operatorname{Im}\lambda \geq 0$ we have $\operatorname{Im}\mathcal{D}(\lambda/\hbar) \geq 0$.*

Proof. Let $G(\lambda)$ be the meromorphic continuation from $\operatorname{Im}\lambda > 0$ of the solution operator satisfying

$$(-\hbar^2\Delta - \lambda^2)G(\lambda)g = 0 \text{ in } \mathbb{R}^d \setminus \overline{\Omega_1}, \qquad G(\lambda)g|_{\Gamma_{\rm tr}} = g,$$

and G is $\lambda/\hbar$-outgoing; then $\mathcal{D}(\lambda/\hbar) = \gamma_1^{\rm tr}G(\lambda)$. Note that for $\operatorname{Im}\lambda > 0$, $G(\lambda) : H^{1/2}(\Gamma_{\rm tr}) \to H^1(\mathbb{R}^d \setminus \Omega_1)$. Therefore, for $\operatorname{Re}\lambda > 0$ and $\operatorname{Im}\lambda > 0$, by integration by parts,

$$\begin{aligned}0 &= \left\langle(-\hbar^2\Delta - \lambda^2)G(\lambda)g, G(\lambda)g\right\rangle_{\mathbb{R}^d\setminus\Omega_1} \\ &= \|\hbar\nabla G(\lambda)g\|^2_{L^2(\mathbb{R}^d\setminus\Omega_1)} - \lambda^2\|G(\lambda)g\|^2_{L^2(\mathbb{R}^d\setminus\Omega_1)} + \hbar^2\left\langle\mathcal{D}(\lambda/\hbar)g, g\right\rangle_{\Gamma_{\rm tr}}.\end{aligned}$$

Therefore, taking imaginary parts

$$2\operatorname{Re}\lambda\operatorname{Im}\lambda\|G(\lambda)g\|_{L^2(\mathbb{R}^d\setminus\Omega_1)} = \hbar^2\operatorname{Im}\langle\mathcal{D}(\lambda/\hbar)g, g\rangle_{\Gamma_{\rm tr}}$$

and in particular, for $\operatorname{Re}\lambda > 0$, $\operatorname{Im}\lambda > 0$

$$0 \leq \operatorname{Im}\langle \mathcal{D}(\lambda/\hbar)g, g\rangle_{\Gamma_{\mathrm{tr}}}$$

Now, since the right hand side continues analytically from $\operatorname{Im}\lambda > 0$ to $\operatorname{Im}\lambda = 0$, we have

$$\operatorname{Im}\langle \mathcal{D}(\lambda/\hbar)g, g\rangle_{\Gamma_{\mathrm{tr}}} \geq 0$$

for $\operatorname{Re}\lambda > 0$ and $\operatorname{Im}\lambda = 0$. ∎

Proof of Lemma 4.1. Let $u \in H^2_{\mathrm{loc}}(\Omega_+)$. Then, let $v = u - E\gamma_D u \in H^2_{\mathrm{loc}}(\Omega_+) \cap H^1_{0,\mathrm{loc}}(\Omega_+)$. By integration by parts,

$$\begin{aligned}
-\operatorname{Im}\left\langle (P_\theta - \lambda^2 - z 1_{\Omega_{\mathrm{tr}}})v, v\right\rangle_{\Omega_{\mathrm{tr}}} &= -\operatorname{Im}\left\langle (-\hbar^2\Delta - \lambda^2 - z 1_{\Omega_{\mathrm{tr}}})v, v\right\rangle_{\Omega_{\mathrm{tr}}} \\
&= (\operatorname{Im} z)\|u\|^2_{L^2(\Omega_{\mathrm{tr}})} + \hbar^2 \operatorname{Im}\left\langle \mathcal{D}(\lambda/\hbar)v, v\right\rangle_{\Gamma_{\mathrm{tr}}} \geq (\operatorname{Im} z)\|v\|^2_{L^2(\Omega_{\mathrm{tr}})}.
\end{aligned}$$

Therefore, there exists $C, C_1, C_2 > 0$ such that for $\operatorname{Im} z > 0$,

$$\begin{aligned}
\|u\|_{L^2(\Omega_{\mathrm{tr}})} &\leq \|v\|_{L^2(\Omega_{\mathrm{tr}})} + \|E\gamma_0^D u\|_{L^2(\Omega_{\mathrm{tr}})} \\
&\leq (\operatorname{Im} z)^{-1}\|(-\hbar^2\Delta - \lambda^2 - z1_{\Omega_{\mathrm{tr}}})v\|_{L^2(\Omega_{\mathrm{tr}})} + C_1\|\gamma_0^D u\|_{H^{3/2}_\hbar(\Gamma_D)} \\
&\leq (\operatorname{Im} z)^{-1}\|(P_\theta - \lambda^2 - z1_{\Omega_{\mathrm{tr}}})u\|_{L^2(\Omega_{\mathrm{tr}})} + C_2\langle z\rangle(\operatorname{Im} z)^{-1}\|E\gamma_0^D u\|_{H^2_\hbar(\Omega_{\mathrm{tr}})} + C_1\|\gamma_0^D u\|_{H^{3/2}_\hbar(\Gamma_D)} \\
&\leq C\langle z\rangle(\operatorname{Im} z)^{-1}\left\|\mathcal{P}_\theta(\lambda, z)\right\|_{L^2(\Omega_{\mathrm{tr}})\oplus H^{3/2}_\hbar(\Gamma_D)},
\end{aligned}$$

by the definition of $\mathcal{P}_\theta(\lambda, z)$ (3.4). Having obtained the bound (4.2) on $\|u\|_{L^2(\Omega_{\mathrm{tr}})}$, we now prove the bound (4.3) on $\|u\|_{H^2_\hbar(\Omega_{\mathrm{tr}})}$. Using, e.g., the trace estimate from [GLS21, Corollary 4.2] (in a similar way to the end of the proof of Lemma 3.2), we have

$$\|\gamma^D_{1,\hbar}u\|_{L^2(\Gamma_D)} \leq C\hbar^{-1}\|(-\hbar^2\Delta - \lambda^2 - z1_{\Omega_{\mathrm{tr}}})u\|_{L^2(\Omega_{\mathrm{tr}})} + C\langle z\rangle\|u\|_{L^2(\Omega_{\mathrm{tr}})}. \tag{4.4}$$

Furthermore, by Lemma 3.2 and (3.3), there exists $\varepsilon > 0$ small enough such that for $\operatorname{Im} z > 0$ and $|z| \leq \varepsilon\hbar$,

$$\begin{aligned}
\|u\|_{H^2_\hbar(\Omega_+)} &\leq C\hbar^{-1}\left\|(P_\theta - \lambda^2 - z1_{\Omega_{\mathrm{tr}}})u\right\|_{L^2(\Omega_+)} + C\left\|(Q_b\gamma^D_{1,\hbar} + \gamma^D_0)u\right\|_{H^{3/2}_\hbar(\Gamma_D)} \\
&\leq C\big(\hbar^{-1} + \langle z\rangle^2(\operatorname{Im} z)^{-1}\big)\left\|(P_\theta - \lambda^2 - z1_{\Omega_{\mathrm{tr}}})u\right\|_{L^2} + C\left\|\gamma^D_0 u\right\|_{H^{3/2}_\hbar(\Gamma_D)},
\end{aligned}$$

by (4.4) and the fact that $P_\theta = -\hbar^2\Delta$ on Ω_{tr}; this implies (4.3) and the proof is complete. ∎

Having proved the bound (1.18), we now prove the bound (1.17). From (3.8),

$$R_\theta(\lambda, z) = \mathcal{R}_{\theta,Q}(\lambda, z)(I + K(\lambda, z))^{-1}\begin{pmatrix} I \\ 0 \end{pmatrix} \tag{4.5}$$

where $K(\lambda, z)$ is defined by (3.5). Since we have the bound (3.3) on $\mathcal{R}_{\theta,Q}(\lambda, z)$, to bound $R_\theta(\lambda, z)$ we only need to bound $(I + K(\lambda, z))^{-1}$.

Recalling the definition of trace class operators (see [DZ19, Definition B.17]), letting $\mathcal{H} := L^2(\Omega_+)\oplus H^{3/2}_\hbar(\Gamma_D)$, we see that $K(\lambda, z)$ (defined by (3.5)) is trace class and using similar reasoning to that in [DZ19, Page 434], together with (3.3)

$$\|K(\lambda, z)\|_{\mathcal{L}_1(\mathcal{H};\mathcal{H})} \leq C\left\|\langle hD\rangle^{3/2}Q_b\right\|_{\mathcal{L}_1(L^2(\Gamma_D))}\|\gamma^D_{1,\hbar}R_{\theta,Q}(\lambda, z)\|_{\mathcal{H}\to L^2(\Gamma_D)} \leq C\hbar^{1-d}\hbar^{-1} \leq C\hbar^{-d}. \tag{4.6}$$

Since $\mathcal{R}_{\theta,Q}(\lambda, z)$ exists for $|z| \leq \varepsilon\hbar$, by [DZ19, Equation B.4.7] and (3.5), $K(\lambda, z)$ is trace class for $|z| \leq \varepsilon\hbar$. Therefore, by the results [DZ19, Equation B.5.21] and [DZ19, Equation B.5.19] about trace-class operators,

$$\left\|(I + K(\lambda, z))^{-1}\right\|_{\mathcal{H}\to\mathcal{H}} \leq \det\big(I + K(\lambda, z)\big)^{-1}\det\big(I + [K(\lambda, z)^*K(\lambda, z)]^{1/2}\big),$$

$$\leq \det\left(I+K(\lambda,z)\right)^{-1}\exp\left(\|[K(\lambda,z)^*K(\lambda,z)]^{1/2}\|_{\mathcal{L}_1(\mathcal{H})}\right),$$
$$\leq \det\left(I+K(\lambda,z)\right)^{-1}\exp\left(\|[K(\lambda,z)\|_{\mathcal{L}_1(\mathcal{H})}\right), \tag{4.7}$$

where we have used the definition of the trace class norm $\|\cdot\|_{\mathcal{L}_1}$ in terms of singular values (see [DZ19, Equation B.4.2]) to write

$$\left\|[K(\lambda,z)^*K(\lambda,z)]^{1/2}\right\|_{\mathcal{L}_1(\mathcal{H})} = \|K(\lambda,z)\|_{\mathcal{L}_1(\mathcal{H})}.$$

and using this in (4.7) we find that

$$\left\|(I+K(\lambda,z))^{-1}\right\|_{L^2\to L^2} \leq \det\left(I+K(\lambda,z)\right)^{-1}\exp(C\hbar^{-d}) \quad \text{for } |z|\leq\varepsilon\hbar. \tag{4.8}$$

To estimate $\det(I+K(\lambda,z))^{-1}$ we use the same idea used to prove the bound (1.15), namely the following complex-analysis result

Lemma 4.3 ([DZ19, Equation D.1.13].) *Let $\Omega_0 \Subset \Omega_1 \Subset \mathbb{C}$, let f be holomorphic in a neighbourhood of Ω_1 with zeros $z_j, j=1,2,\ldots,$ and let $z_0\in\Omega_1$. There exists $C=C(\Omega_0,\Omega_1,z_0)$ such that for any $\delta>0$ sufficiently small*

$$\log|f(z)| \geq -C\log\left(\delta^{-1}\right)\left(\max_{z\in\Omega_1}\log|f(z)|-\log|f(z_0)|\right) \qquad \textit{for } z\in\Omega_0\setminus\bigcup_j B(z_j,\delta).$$

Applying this result with $f(z)=\det(I+K(\lambda,z))$, we see that to get an upper bound on $\log\det(I+K(\lambda,z))^{-1}$ we only need a lower bound on $\det(I+K(\lambda,z_0))$ for some $|z_0|\leq\varepsilon\hbar$ and an upper bound on $\det(I+K(\lambda,z))$ for all $|z|\leq\varepsilon\hbar$.

To obtain the upper bound for all $|z|\leq\varepsilon\hbar$, we again use [DZ19, Equation B.5.19] and (4.6) to obtain

$$|\det(I+K(\lambda,z))| \leq \exp(\|K(\lambda,z)\|_{\mathcal{L}_1}) \leq \exp(C\hbar^{-d}) \quad \text{for } |z|\leq\varepsilon\hbar. \tag{4.9}$$

To obtain the lower bound for some $|z_0|\leq\varepsilon\hbar$, we first observe that, from (3.6),

$$\left(I+K(\lambda,z)\right)^{-1} = \mathcal{P}_{\theta,Q}(\lambda,z)\mathcal{P}_\theta(\lambda,z)^{-1} = I - Q\mathcal{P}_\theta(\lambda,z)^{-1}.$$

so that

$$\left|\det\left(I+K(\lambda,z)\right)\right|^{-1} = \left|\det\left(I-Q\mathcal{P}_\theta(\lambda,z)^{-1}\right)\right|.$$

Since $Q\mathcal{P}_\theta(\lambda,z)$ is trace class, we use [DZ19, Equation B.5.19], [DZ19, Equation B.4.7], (4.6), and (4.3) to obtain

$$\log\left|\det(I+K(\lambda,z_0))\right|^{-1} \leq \|Q\|_{\mathcal{L}_1(H^2_\hbar(\Omega_+);\mathcal{H})}\left\|\mathcal{P}_\theta(\lambda,z_0)^{-1}\right\|_{\mathcal{H}\to H^2_\hbar(\Omega_+)} \leq C\hbar^{-d} \text{ for } z_0 = \mathrm{i}\varepsilon\hbar. \tag{4.10}$$

Therefore, combining Lemma 4.3, (4.9), and (4.10), we have

$$\log\left|\det\left(I+K(\lambda,z)\right)^{-1}\right| \leq C\hbar^{-d}\log\delta^{-1}, \qquad z\in B(0,\varepsilon_1\hbar)\Big\backslash\bigcup_{z_j}B(z_j,\delta)$$

where z_j are the poles of $(I+K(\lambda,z))^{-1}$. Therefore, combining this last bound with (4.5), (4.8), and (3.3), we have

$$\|R_\theta(\lambda,z)\|_{L^2(\Omega_+)\to L^2(\Omega_+)} \leq \exp\left(C\hbar^{-d}\log\delta^{-1}\right) \quad \text{for } z\in B(0,\varepsilon_1\hbar)\Big\backslash\bigcup_{z_j}B(z_j,\delta).$$

where z_j are the poles of $\mathcal{R}_\theta(\lambda,z)$. The bound (1.17) and the fact that z_j are the poles of $R_{\Omega_{\mathrm{tr}}}(\lambda,z)$ then follow from the relation (4.1) and Lemma 1.9.

5 Proofs of Theorems 2.2 and 2.4

5.1 Proof of Theorem 2.2

With Lemma 1.12 in hand, this proof is very similar to [DZ19, Proof of Theorem 7.6], except that now we work in the complex z plane as opposed to the complex λ plane. In addition, in this proof, the roles of ε_0 and ε are swapped compared to [DZ19, Proof of Theorem 7.6].

Let

$$\varepsilon_0(\hbar) := \hbar^{-\alpha}\varepsilon(\hbar), \tag{5.1}$$

with $\alpha > 0$ to be fixed later in the proof. The bound in (2.1) then implies that, given $\hbar_0$, there exists C' (depending on $\hbar_0$ and α) such that

$$\log\left(\frac{2}{\varepsilon_0(\hbar)}\right) \leq \frac{C'}{\hbar} \quad \text{for all } 0 < \hbar \leq \hbar_0. \tag{5.2}$$

Seeking a contradiction, we assume that there are no eigenvalues in $B(0, \varepsilon_0(\hbar))$. Since $\operatorname{supp} u_\ell \Subset \Omega_1$,

$$R(1,0)(-\hbar_j^2\Delta - 1)u_\ell = u_\ell.$$

Therefore, if we can show that the assumption that when $\hbar = \hbar_j$ there are no eigenvalues in $B(0, \varepsilon_0(\hbar_j))$ implies that

$$\|R(1,0)\|_{L^2(\Omega_{\rm tr})\to L^2(\Omega_{\rm tr})} < \frac{1}{2}\big(\varepsilon(\hbar_j)\big)^{-1}, \tag{5.3}$$

then we obtain a contradiction to $\|u_\ell\|_{L^2(\Omega_{\rm tr})} = 1$. We prove (5.3) by using Theorem 2.7 where $\Omega(\hbar)$ is a box (to be specified below) in $B(0, \varepsilon_0(\hbar)/2)$ with Lemma 1.12 providing the bounds (2.18) and (2.19).

We first use the bound (1.17) from Lemma 1.12. This bound is valid for $z \in B(0, \varepsilon_1\hbar)$ and away from the poles. The definition of $\varepsilon_0(\hbar)$ (5.1) and the upper bound in (2.1) implies that $B(0, \varepsilon_0(\hbar)/2) \subset B(0, \varepsilon_1\hbar)$ for $\hbar$ sufficiently small. We then choose δ in (1.17) to equal $\varepsilon_0(\hbar)/2$ and use (5.2) so that, for all $\hbar_j$ sufficiently small,

$$\|R(1,z)\|_{L^2(\Omega_{\rm tr})\to L^2(\Omega_{\rm tr})} \leq \exp\left(C_1C'\hbar_j^{-(d+1)}\right) \quad \text{for all } z \in B(0, \varepsilon_0(\hbar_j)/2), \tag{5.4}$$

and thus for all $z \in \Omega(\hbar_j)$ (since $\Omega(\hbar_j) \subset B(0, \varepsilon_0(\hbar_j)/2)$). We now let

$$Q(z,\hbar) := R_{\Omega_{\rm tr}}(1,z), \quad L := d+1, \quad \text{and } C := \max\big\{C_1C',\, C_2\, c\big\},$$

where $c = c(\hbar_0)$ is chosen large enough such that $\langle z\rangle \leq c$ for all $z \in B(0, \varepsilon_0(\hbar)/2)$ and $\hbar \leq h_0$; these choices ensures that the right-hand sides of the bounds (5.4) and (1.18) are bounded by the right-hand sides of (2.18) and (2.19) respectively. We then let

$$w = 0, \quad 2\beta(\hbar) = \frac{1}{4}\varepsilon_0(\hbar), \quad \text{and} \quad \delta(\hbar) = M\varepsilon(\hbar)$$

with M chosen (sufficiently large) later in the proof. For the assumptions of Theorem 2.7 to hold at $\hbar = \hbar_j$, we need that (i) the box $\Omega(\hbar_j)$ defined by (2.16) is inside $B(0, \varepsilon_0(\hbar_j)/2)$ (so that the bound (2.18) follows from (5.4)) and (ii) the second inequality in (2.17) is satisfied. The first requirement is ensured if

$$\delta(\hbar_j)\hbar_j^{-(d+1)} \ll \frac{1}{2}\varepsilon_0(\hbar_j), \quad \text{that is} \quad M\varepsilon(\hbar_j)\hbar_j^{-(d+1)} \ll \frac{1}{2}\hbar_j^{-\alpha}\varepsilon(\hbar_j),$$

which is satisfied if $\alpha > d+1$ and $\hbar_j$ is sufficiently small. The second requirement is

$$\frac{1}{8}h^{-2\alpha}\varepsilon(\hbar)^2 \geq C\hbar^{-3(d+1)}M\varepsilon(\hbar)^2;$$

given M, this inequality is satisfied if $\alpha > 3(d+1)/2$ and $\hbar$ is sufficiently small.

Therefore, if $\alpha > 3(d+1)/2$, the assumptions of Theorem 2.7 are all satisfied at $\hbar = \hbar_j$ (for $\hbar_j$ sufficiently small), and the result is that the bound (2.20) holds for all $z \in [-\beta(\hbar_j), \beta(\hbar_j)]$, and thus, in particular, at $z = 0$. Therefore, for all $\hbar_j$ sufficiently small,

$$\|R(1,0)\|_{L^2(\Omega_{\rm tr})\to L^2(\Omega_{\rm tr})} \leq \frac{C}{M\varepsilon(\hbar_j)}\exp(1+C).$$

We now choose

$$M := 2C\exp(1+C),$$

and obtain (5.3), i.e. the desired contradiction to there being no eigenvalues in $B(0, \varepsilon_0(\hbar_j))$.

5.2 Proof of Theorem 2.4

We first recall the following lemma proved in [Ste99, Lemma 4]; see also [Laz93, Lemma AII.20].

Lemma 5.1 *Let $f_1, \ldots, f_N$ be N vectors in a Hilbert space $\mathcal{H}$ with*

$$\big|\langle f_i, f_j\rangle_{\mathcal{H}} - \delta_{ij}\big| \leq \varepsilon \quad \text{for all } \; i,j = 1,\ldots,N.$$

If $\varepsilon < N^{-1}$, then $f_1, \ldots, f_N$ are linearly independent.

We use Lemma 5.1 both in the proof of Theorem 2.4 below, and in the proof of the following preparatory result.

Lemma 5.2 *Let $m(\hbar_j)$ and $\varepsilon(\hbar)$ be as in Theorem 2.4 (so that, in particular $\varepsilon(\hbar) \ll \hbar^{(5d+3)/2}$ as $\hbar \to 0$). Then there exists $C > 0$ (independent of $\hbar_j$) such that*

$$m(\hbar_j) \leq C\hbar_j^{-d}. \tag{5.5}$$

Proof. First observe that it is sufficient to prove the result for sufficiently small $\hbar_j$ (equivalently, sufficiently large j). Let $P(\hbar_j) = -\hbar_j^2\Delta$ with zero Dirichlet boundary conditions on Γ_D and $\Gamma_{\rm tr}$. $P(\hbar_j)$ is therefore self-adjoint with discrete spectrum and, since $\operatorname{supp} u_{j,\ell} \subset \mathcal{K} \Subset \Omega_1$,

$$\big\|\big(P(\hbar_j) - E_{j,\ell}\big)u_{j,\ell}\big\|_{L^2(\Omega_{\rm tr})} = \varepsilon(\hbar_j) \quad \text{ for all } j, \ell.$$

Let $\mu > c > 0$, let $\Pi(\hbar_j)$ be the orthogonal projection on to the eigenspaces corresponding to all eigenvalues of $P(\hbar_j)$ in $[a_0 - \mu, b_0 + \mu]$, and let $M(\hbar_j)$ be the number of these eigenvalues (counting multiplicities). By the Weyl law (with no remainder term) on manifolds with boundary (see e.g. [Hör85, Theorem 17.5.3]),

$$M(\hbar_j) \leq C\hbar_j^{-d}.$$

Furthermore, $\operatorname{rank}\Pi(\hbar_j) \leq M(\hbar_j)$ and thus to prove the result (5.5) it is sufficient to prove that $m(\hbar_j) \leq \operatorname{rank}\Pi(\hbar_j)$. To keep expressions compact, we now write P and Π instead of $P(\hbar_j)$ and $\Pi(\hbar_j)$.

Since Π commutes with $(P - E_{j,\ell})^{-1}$, and $(P - E_{j,\ell})$ is invertible on $(I - \Pi)L^2$,

$$\big(I - \Pi\big)u_{j,\ell} = \big(P - E_{j,\ell}\big)^{-1}\big(I - \Pi\big)\big(P - E_{j,\ell}\big)u_{j,\ell}. \tag{5.6}$$

Since P is self-adjoint, the spectral theorem (see, e.g., [DZ19, Theorem B.8]) implies that

$$\big\|\big(P - E_{j,\ell}\big)^{-1}\big(I - \Pi\big)\big\|_{L^2(\Omega_{\rm tr})\to L^2(\Omega_{\rm tr})} \leq \frac{1}{\mu}. \tag{5.7}$$

Therefore, combining (5.6) and (5.7), we have

$$\big\|\big(I - \Pi\big)u_{j,\ell}\big\|_{L^2(\Omega_{\rm tr})\to L^2(\Omega_{\rm tr})} \leq \frac{\varepsilon(\hbar_j)}{\mu}$$

(compare to [Laz93, Equation 32.2] and the first displayed equation in [Ste99, §3]). Then, for $\ell_1, \ell_2 \in \{1, \ldots, m(\hbar_j)\}$,

$$\big|\langle \Pi u_{j,\ell_1}, \Pi u_{j,\ell_2}\rangle_{L^2(\Omega_{\rm tr})} - \delta_{\ell_1\ell_2}\big| \leq \big|\langle u_{j,\ell_1}, u_{j,\ell_2}\rangle_{L^2(\Omega_{\rm tr})} - \delta_{\ell_1\ell_2}\big|$$

$$
\begin{aligned}
&+\left|\langle u_{j,\ell_1},(I-\Pi)u_{j,\ell_2}\rangle_{L^2(\Omega_{\rm tr})}\right|+\left|\langle (I-\Pi)u_{j,\ell_1},\Pi u_{j,\ell_2}\rangle_{L^2(\Omega_{\rm tr})}\right|,\\
&\le \hbar_j^{-2}\varepsilon(\hbar_j)+\frac{2}{\mu}\varepsilon(\hbar_j), \qquad (5.8)\\
&\ll \hbar_j^{(5d-1)/2}\quad \text{as } j\to\infty,
\end{aligned}
$$

where we have used that $\|\Pi\|_{L^2(\Omega_{\rm tr})\to L^2(\Omega_{\rm tr})}\le 1$ since Π is orthogonal. By Lemma 5.1, any subset of $\{\Pi u_{j,\ell}\}_{\ell=1}^{m(\hbar_j)}$ with cardinality $\ll \hbar_j^{-(5d-1)/2}$ is linearly independent. Seeking a contradiction assume that (5.5) does not hold, i.e. for all $C>0$ there exists j such that $m(\hbar_j)>C\hbar_j^{-d}$. Choose a subset of $\{\Pi u_{j,\ell}\}_{\ell=1}^{m(\hbar_j)}$ with cardinality $\lfloor C\hbar_j^{-d}+1\rfloor$. By the above argument, this subset is linearly independent, and thus $\lfloor C\hbar_j^{-d}+1\rfloor\le \operatorname{rank}\Pi(\hbar_j)=M(\hbar_j)\le C\hbar_j^{-d}$ which is the required contradiction. ∎

Proof of Theorem 2.4. The proof is similar to that of the corresponding "quasimodes to resonances" result [Ste99, Theorem 1] (see also [DZ19, §7.7, Exercise 1]), except that we use the semiclassical maximum principle in the z plane (as in the proof of Theorem 2.2), and now we also work in an interval in λ (as opposed to at $\lambda=1$ in the proof of Theorem 2.2). To keep the expressions compact, during the proof we drop the subscript j on $\hbar_j, E_{j,\ell}$, and $u_{j,\ell}$.

Let

$$
\mathcal{Z}:=\mathcal{Z}\big(\varepsilon_1(\hbar)\,,\,\varepsilon_0(\hbar)\,,\,a(\hbar),\,b(\hbar)\,;\,\hbar\big),
$$

where $\mathcal{Z}(\varepsilon_1,\varepsilon_0,a,b;\hbar)$ is defined by (2.3), $\varepsilon_0(\hbar)$ is as in the statement of the theorem, and $\varepsilon_1(\hbar)\ll\hbar$ will be fixed later. We assume throughout that $|\mathcal{Z}|<\infty$, since otherwise the proof is trivial. Let $\Pi(\hbar)$ denote the orthogonal projection onto

$$
\bigcup_{z_p\in\mathcal{Z}}\Pi_{z_p}(L^2(\Omega_{\rm tr})),
$$

where Π_{z_p} is defined in (2.15). Let $\widetilde{\mathcal{Z}}(\lambda)$ be the set of distinct values of $z_p(\hbar,\lambda)\in\mathcal{Z}$. (While $\mathcal{Z}$ is independent of λ, $\widetilde{\mathcal{Z}}$ in principle depends on λ, since the multiplicity of the poles of $z\mapsto R_{\Omega_{\rm tr}}(z,\lambda)$ in principle depends on λ.) Note that for $z_p\neq z_q$, $\operatorname{rank}(\Pi_{z_p}+\Pi_{z_q})=\operatorname{rank}\Pi_{z_p}+\operatorname{rank}\Pi_{z_q}$; therefore

$$
\operatorname{rank}\Pi(\hbar)=\sum_{z_p\in\widetilde{\mathcal{Z}}(\lambda)}\operatorname{rank}\Pi_{z_p(\hbar,\lambda)}=\sum_{z_p\in\widetilde{\mathcal{Z}}(\lambda)}m_R\big(z_p(\hbar,\lambda)\big)=|\mathcal{Z}|,
$$

where $m_R(z_0)$ is defined in (2.15). To prove the theorem, therefore, it is sufficient to show that $m(\hbar)\le\operatorname{rank}\Pi(\hbar)$.

Seeking a contradiction, we assume that $\operatorname{rank}\Pi(\hbar)<m(\hbar)$. By Lemma 2.6, near z_p, the singular part of $R_{\Omega_{\rm tr}}(\lambda,z)$ is in the range of $\Pi_{z_p}(\hbar,\lambda)$, and therefore $z\mapsto(I-\Pi(\hbar))R_{\Omega_{\rm tr}}(\lambda,z)$ is holomorphic on $\Omega(\hbar)$ for all $\lambda^2\in[a(\hbar),b(\hbar)]$. Let $\widetilde{\Omega}(\hbar)\subset\Omega(\hbar)$ be defined by

$$
\widetilde{\Omega}(\hbar):=\big(-\varepsilon_1(\hbar),\,\varepsilon_1(\hbar)\big)-{\rm i}\big(0,\varepsilon_0(\hbar)\big). \tag{5.9}
$$

Our goal is to apply the semiclassical maximum principle (Theorem 2.7) in subsets of $\widetilde{\Omega}(\hbar)$ with $Q(z,\hbar)=(I-\Pi(\hbar))R_{\Omega_{\rm tr}}(\lambda,z)$.

By Lemma 1.12, the fact that $\max(\varepsilon_0,\varepsilon_1)\ll\hbar$, and the fact that $\Pi(\hbar)$ is orthogonal (and so $\|I-\Pi(\hbar)\|_{L^2(\Omega_{\rm tr})\to L^2(\Omega_{\rm tr})}\le 1$),

$$
\big\|\big(I-\Pi(\hbar)\big)R_{\Omega_{\rm tr}}(\lambda,z)\big\|_{L^2(\Omega_{\rm tr})\to L^2(\Omega_{\rm tr})}\le\exp\Big(C_1\hbar^{-d}\log\delta^{-1}\Big)\qquad\text{for } z\in\widetilde{\Omega}(\hbar)\Big\backslash\bigcup_m B(z_m(\hbar,\lambda),\delta) \tag{5.10}
$$

and for $\lambda^2\in[a(\hbar),b(\hbar)]$, where the $z_m(\hbar,\lambda)$ are the poles of $R_{\Omega_{\rm tr}}(\lambda,z)$ such that $B(z_m(\hbar,\lambda),\delta)\cap\widetilde{\Omega}(\hbar)\neq\emptyset$. If $\delta>\min\{\varepsilon_0(\hbar),\varepsilon_1(\hbar)\}$, then these $z_m(\hbar,\lambda)$ might include poles that are not equal to $z_p(\hbar,\lambda)\in\mathcal{Z}$, but we restrict δ so that this is not the case. Indeed, we now choose $\delta>0$ so that the bound in (5.10) holds for all $z\in\widetilde{\Omega}(\hbar)$ and for all $\lambda^2\in[a(\hbar),b(\hbar)]$.

If δ and z_m are such that $B(z_m,\delta)\Subset\Omega(\hbar)$, then the bound in (5.10) holds on $\partial B(z_m,\delta)$, and then, since $z\mapsto(I-\Pi(\hbar))R_{\Omega_{\rm tr}}(\lambda,z)$ is holomorphic in $\Omega(\hbar)$, the maximum principle implies that the bound in (5.10) holds in $B(z_m,\delta)$. We now restrict δ so that there cannot be a connected union of $B(z_m,\delta)$ that intersects both $\widetilde{\Omega}(\hbar)$ and $\partial\Omega(\hbar)$. Once this is ruled out, the maximum principle and the fact that $z\mapsto(I-\Pi(\hbar))R_{\Omega_{\rm tr}}(\lambda,z)$ is holomorphic in $\Omega(\hbar)$ imply that the bound in (5.10) holds in $\widetilde{\Omega}(\hbar)$. Since we have assumed that $\operatorname{rank}\Pi(\hbar)<m(\hbar)$, and $m(\hbar)\le C\hbar^{-d}$ by (5.5), there exist a maximum of $C\hbar^{-d}$ of balls of radius δ. In particular, the maximum distance between any two points in such a connected union is bounded by $2C\delta(\hbar)\hbar^{-d}$ and hence, a connected union intersecting both $\partial\Omega(\hbar)$ and $\tilde{\Omega}(\hbar)$ is ruled out if

$$2\delta C\hbar^{-d}<\min\big\{\varepsilon_0(\hbar),\varepsilon_1(\hbar)\big\}. \tag{5.11}$$

We now assume that $\varepsilon_0(\hbar)\le\varepsilon_1(\hbar)$ and set

$$\delta:=\frac{\varepsilon_0(\hbar)\,\hbar^d}{4C}, \tag{5.12}$$

so that (5.11) holds. The first inequality in (2.5) implies that, given $\hbar_0$, there exists C' (depending on $\hbar_0$) such that

$$\log\delta^{-1}\le\frac{C'}{\hbar}\qquad\text{for all } 0<\hbar\le\hbar_0.$$

Therefore, the end result is that, if $\hbar$ is sufficiently small,

$$\big\|\big(I-\Pi(\hbar)\big)R_{\Omega_{\rm tr}}(\lambda,z)\big\|_{L^2(\Omega_{\rm tr})\to L^2(\Omega_{\rm tr})}\le\exp\Big(C\hbar^{-d-1}\Big)\text{ for } z\in\widetilde{\Omega}(\hbar)\text{ and }\lambda^2\in[a(\hbar),b(\hbar)], \tag{5.13}$$

where $C:=\max\{C_1C',cC_2\}$, where, as in the proof of Theorem 2.2, $c=c(\hbar_0)$ is chosen large enough such that $\langle z\rangle\le c$ for all $z\in\widetilde{\Omega}(\hbar)$ and $\hbar\le h_0$.

We apply the semiclassical maximum principle (Theorem 2.7) with

$$w=0,\quad \beta(\hbar)=\varepsilon_1(\hbar),\quad \delta(\hbar)=\hbar^{d+1}\varepsilon_0(\hbar),\quad\text{ and }\quad L=d+1,$$

and we now fix $\varepsilon_1(\hbar)$ as

$$\varepsilon_1(\hbar):=\frac{\hbar^{(d+1)/2}\varepsilon_0(\hbar)}{C};$$

observe that this definition of $\varepsilon(\hbar)$ satisfies both the second requirement in (2.17) and our previous assumption that $\varepsilon_0(\hbar)\le\varepsilon_1(\hbar)$. The result of Theorem 2.7 is that

$$\begin{gathered}\big\|\big(I-\Pi(\hbar)\big)R_{\Omega_{\rm tr}}(\lambda,z)\big\|_{L^2(\Omega_{\rm tr})\to L^2(\Omega_{\rm tr})}\le C\exp(C+1)\frac{\hbar^{-(d+1)}}{\varepsilon_0(\hbar)}\\ \text{for } z\in\big[-\varepsilon_1(\hbar),\varepsilon_1(\hbar)\big]\quad\text{and}\quad\lambda^2\in[a(\hbar),b(\hbar)].\end{gathered} \tag{5.14}$$

The definitions of E_ℓ and u_ℓ imply that

$$\big(I-\Pi(\hbar)\big)R_{\Omega_{\rm tr}}\big(\sqrt{E_\ell},0\big)\big(-\hbar^2\Delta-E_\ell\big)u_\ell=\big(I-\Pi(\hbar)\big)u_\ell,$$

for $\ell=1,\ldots,m(\hbar)$. Since $E_\ell\in[a(\hbar),b(\hbar)]$ for all ℓ, the fact that the bound (5.14) holds for all $\lambda^2\in[a(\hbar),b(\hbar)]$ implies that

$$\big\|(I-\Pi(\hbar))u_\ell\big\|_{L^2(\Omega_{\rm tr})\to L^2(\Omega_{\rm tr})}\le C\exp(C+1)\hbar^{-(d+1)}\frac{\varepsilon(\hbar)}{\varepsilon_0(\hbar)}$$

for $\ell=1,\ldots,m(\hbar)$. Therefore

$$\Big|\big\langle\Pi(\hbar)u_{\ell_1},\Pi(\hbar)u_{\ell_2}\big\rangle_{L^2(\Omega_{\rm tr})}-\delta_{\ell_1\ell_2}\Big|\le\varepsilon(\hbar)+2C\exp(C+1)\hbar^{-(d+1)}\frac{\varepsilon(\hbar)}{\varepsilon_0(\hbar)}$$

(compare to (5.8), but note that now the projection Π is different). Using the inequality (2.4) and the second inequality in (2.5), we have

$$\Big|\big\langle\Pi(\hbar)u_{\ell_1},\Pi(\hbar)u_{\ell_2}\big\rangle_{L^2(\Omega_{\rm tr})}-\delta_{\ell_1\ell_2}\Big|\ll\hbar^d\quad\text{ and thus }\quad\Big|\big\langle\Pi(\hbar)u_{\ell_1},\Pi(\hbar)u_{\ell_2}\big\rangle_{L^2(\Omega_{\rm tr})}-\delta_{\ell_1\ell_2}\Big|\le\frac{\hbar^d}{C},$$

where C is the constant in (5.5). By (5.5) and Lemma 5.1, $\{\Pi(h)u_\ell\}_{\ell=1}^{m(\hbar_j)}$ are linearly independent, and thus $\operatorname{rank}\Pi(\hbar)\ge m(\hbar)$, which is the desired contradiction to the assumption that $\operatorname{rank}\Pi(\hbar)<m(\hbar)$. ■

Acknowledgements

EAS gratefully acknowledges discussions with Alex Barnett (Flatiron Institute) that started his interest in eigenvalues of discretisations of the Helmholtz equation under strong trapping. JG thanks Maciej Zworski (UC Berkeley) for bringing to his attention the paper [Ste00]. PM thanks Pierre Jolivet (IRIT, CNRS) for his help with the software FreeFEM.

PM and EAS were supported by EPSRC grant EP/R005591/1.

A From eigenvalues to quasimodes

Lemma A.1 (From eigenvalues to quasimodes in $\hbar$ notation) *Suppose that there exist $z = \mathcal{O}(\hbar^\infty)$ and u satisfying (2.2) with $\|u\|_{L^2(\Omega_{\rm tr})} = 1$. Let $\chi \in C_c^\infty(\Omega_1)$ with $\chi \equiv 1$ in a neighborhood of $\pi_{\mathbb{R}}(K)$. Then χu is a quasimode (in the sense of Definition 2.1) of quality $\varepsilon(\hbar) = \mathcal{O}(\hbar^\infty)$ satisfying*

$$\|u - \chi u\|_{H^2_\hbar(\Omega_{\rm tr})} = \mathcal{O}(\hbar^\infty).$$

Proof. The proof is similar to the proof of the "resonances to quasimodes" result of [Ste00, Theorem 1], except that we avoid using results about $\mathcal{D}$ for strictly convex obstacles that are used in [Ste00] and instead use a commutator argument.

First observe that

$$(-\hbar^2\Delta - 1 - z)u = 0 \qquad \text{in } \Omega_{\rm tr},$$

so that

$$u = 1^{\rm res}_{\Omega_{\rm tr}} R(1,0) 1^{\rm ext}_{\Omega_{\rm tr}}\, z\, u.$$

Therefore,

$$u = 1^{\rm res}_{\Omega_{\rm tr}} R_\theta(1,0) 1^{\rm ext}_{\Omega_{\rm tr}}\, z\, u$$

by (2.29) and the definition of $R_\theta(\lambda, z)$ (3.7). Let

$$v = R_\theta(1,0) 1^{\rm ext}_{\Omega_{\rm tr}} z\, u, \tag{A.1}$$

and observe that $v = u$ on $\Omega_{\rm tr}$.

We now claim that, since $z = \mathcal{O}(\hbar^\infty)$ and $\Omega_{\rm tr} \Subset \mathbb{R}^d$, $\mathrm{WF}_\hbar(v) \subset \Gamma_+$ (defined by (2.24)). By the definition of the wavefront set [DZ19, Definition E.36], this is equivalent to $Av = \mathcal{O}(\hbar^\infty)$ for all A with $\mathrm{WF}_\hbar(A) \subset (\Gamma^+)^c$. This then follows by noting that $(P_\theta - 1)v = \mathcal{O}(\hbar^\infty)_{L^2_{\rm comp}}$ and applying [DZ19, Theorem E.47], [Hör85, Section 24.4], [Vas08, Theorem 8.1] ‡ (with, in the notation of [DZ19, Theorem E.47], $B_1 = I$, $B = \mathbf{P} = P_\theta - 1$), together with the facts that $\sigma_\hbar(\mathrm{Im}(P_\theta - 1)) \le 0$ and that $P_\theta - 1$ is elliptic on $\{\mathbf{r} \ge 2r_1\}$ (so that if $(x_0, \xi_0) \in \mathrm{WF}_\hbar(A)$ then there exists $T \ge 0$ such that $\varphi_{-T}(x_0, \xi_0) \in \mathrm{ell}_\hbar(P_\theta - 1)$).

Now let $\chi \in C_c^\infty(\Omega_1)$ with $\chi \equiv 1$ in a neighborhood of $\pi_{\mathbb{R}}(K)$. We claim that $\chi v = \chi u$ is a quasimode with quality $\varepsilon(\hbar) = \mathcal{O}(\hbar^\infty)$. To prove this, since

$$\|u - \chi u\|_{H^2_\hbar(\Omega_{\rm tr})} = \|(1-\chi)v\|_{H^2_\hbar(\Omega_{\rm tr})} = \|(1-\chi)v\|_{H^2_\hbar(\Omega_{\rm tr}\setminus\{\chi \equiv 1\})}, \tag{A.2}$$

it is sufficient to prove that v is $\mathcal{O}(\hbar^\infty)_{H^2_{\hbar,\rm loc}}$ outside a compact set.

Our first step is to prove that, with $r_0 < a < b < r_1$, for $\hbar$ sufficiently small,

$$\|v\|_{L^2(\mathbf{r} > a)} \le C h^{-1} \|(P_\theta - 1)v\|_{L^2(\Omega_+)} + C\|v\|_{L^2(a < \mathbf{r} < b)}, \tag{A.3}$$

where here, and in the rest of the proof, C denotes a constant, independent of $\hbar$ and z, whose value may change from line to line. To prove (A.3), first observe that, since $P_\theta - 1$ is elliptic on

‡Strictly speaking [DZ19, Theorem E.47] is used away from the boundary and [Vas08, Theorem 8.1] is written for the time dependent problem, but the semiclassical version can be easily recovered by applying the time dependent results to $e^{it/\hbar}v(x)$. It is then necessary to use the arguments in [Hör85, Section 24.4] to obtain the 'diffractive improvement' i.e. that singularities hitting a diffractive point follow only the flow of H_p rather than sticking to the boundary. A careful examination of [Hör85, Lemma 24.4.7] shows that the norm on the error term on $(P_\theta - 1)v$ is correct.

$\mathbf{r} \geq 2r_1$, by [DZ19, Theorem E.33] (more precisely its proof together with the calculus from [Zwo12, Chapter 4]),

$$\|v\|_{L^2(\mathbf{r}>3r_1)} \leq C\|(P_\theta - 1)v\|_{L^2(\Omega_+)} + C_N h^N \|v\|_{L^2(\mathbf{r}>2r_1)}$$

and hence

$$\|v\|_{L^2(\mathbf{r}>3r_1)} \leq C\|(P_\theta - 1)v\|_{L^2(\Omega_+)} + C_N h^N \|v\|_{L^2(2r_1<\mathbf{r}<4r_1)}. \tag{A.4}$$

Next, observe that there exists $T > 0$ such that for all $\rho \in \Gamma_+ \cap \{\frac{a+b}{2} < \mathbf{r} < 4r_1\}$, there exists $0 \leq t \leq T$ such that $a < \mathbf{r}(\varphi_{-t}(\rho)) < b$. In particular, using [DZ19, Theorem E.47] again, we have

$$\|v\|_{L^2(\frac{a+b}{2}<\mathbf{r}<4r_1)} \leq Ch^{-1}\|(P_\theta - 1)v\|_{L^2(\Omega_+)} + \|v\|_{L^2(a<\mathbf{r}<b)} + C_N h^N \|v\|_{L^2(\mathbf{r}>a)}.$$

Using this and (A.4) in

$$\|v\|_{L^2(\mathbf{r}>a)} \leq \|v\|_{L^2(a<\mathbf{r}<b)} + \|v\|_{L^2(\frac{a+b}{2}<\mathbf{r}<4r_1)} + \|v\|_{L^2(\mathbf{r}>3r_1)},$$

we obtain (A.3) for $\hbar$ sufficiently small.

The next part of the proof involves using a commutator argument to control (up to h^∞ errors) $\|v\|_{L^2(a<\mathbf{r}<b)}$ by the norm on a slightly bigger region and with a gain of $\hbar$ (see (A.5) below). Let $\psi \in C_c^\infty(-r_1, r_1)$ with $\psi \equiv 1$ on $\{|x| \leq r_0\}$, $x\psi'(x) \leq 0$, and $x\psi'(x) < 0$ on $a \leq |x| \leq b$. Then,

$$\begin{aligned}
&2\hbar^{-1}\operatorname{Im}\big\langle(-\hbar^2\Delta - 1)v, \psi(\mathbf{r})v\big\rangle_{L^2(\Omega_+)} \\
&\quad= -\mathrm{i}\hbar^{-1}\Big(\big\langle(-\hbar^2\Delta - 1)v, \psi(\mathbf{r})v\big\rangle_{L^2(\Omega_+)} - \big\langle\psi(\mathbf{r})v, (-\hbar^2\Delta - 1)v\big\rangle_{L^2(\Omega_+)}\Big) \\
&\quad= \mathrm{i}\hbar^{-1}\big\langle[-\hbar^2\Delta, \psi(\mathbf{r})]v, v\big\rangle_{L^2(\Omega_+)} \\
&\quad= \big\langle\big(2\psi'(\mathbf{r})\hbar D_r - \mathrm{i}\hbar[\Delta(\psi(\mathbf{r}))]\big)v, v\big\rangle_{L^2(\Omega_+)}
\end{aligned}$$

By the definition of Γ_+ (2.24), $\sigma_\hbar(\psi'(\mathbf{r})hD_r) = \psi'(\mathbf{r})\langle\xi, \frac{x}{|x|}\rangle < -c < 0$ on $\Gamma_+ \cap \{a \leq \mathbf{r} \leq b\}$. Therefore, since $\mathrm{WF}_\hbar(v) \subset \Gamma_+$, for $\psi_1 \in C_c^\infty(r_0 < \mathbf{r} < r_1)$ with $\psi_1 \equiv 1$ on $\operatorname{supp}\partial\psi(\mathbf{r})$

$$2\hbar^{-1}\operatorname{Im}\big\langle(-\hbar^2\Delta - 1)v, \psi(\mathbf{r})v\big\rangle_{L^2(\Omega_+)} \leq -c\|v\|^2_{L^2(a<\mathbf{r}<b)} + C\hbar\|\psi_1 v\|^2_{L^2(\Omega_+)} + C_N\hbar^N\|v\|_{L^2(\Omega_+)},$$

by the microlocal Garding inequality [DZ19, Prop. E.34]. Therefore,

$$\|v\|^2_{L^2(a<\mathbf{r}<b)} \leq C\hbar^{-N-1}\|(-\hbar^2\Delta - 1)v\|^2_{L^2(\mathbf{r}<r_1)} + C\hbar\|\psi_1 v\|^2_{L^2(\Omega_+)} + C_N\hbar^N\|v\|^2_{L^2(\Omega_+)} \tag{A.5}$$

We now use the propagation estimate again to control (up to $\hbar^\infty$ errors) $\|\psi_1 v\|^2_{L^2(\Omega_+)}$ by $\|v\|^2_{L^2(a<\mathbf{r}<b)}$. Suppose that $\rho \in \mathbf{r}^{-1}(\{\operatorname{supp}\psi_1\}) \cap \Gamma_+$. Then, there exists $|t| \leq \sqrt{r_1^2 - r_0^2}$ such that $\varphi_t(\rho) \in \{a < \mathbf{r} < b\}$. Therefore, by standard propagation estimates [DZ19, Theorem E.47], again using that $\mathrm{WF}_\hbar(v) \subset \Gamma_+$, we have

$$\|\psi_1 v\|^2_{L^2(\Omega_+)} \leq C\hbar^{-1}\|(-\hbar^2\Delta - 1)v\|^2_{L^2(\mathbf{r}\leq r_1)} + C\|v\|^2_{L^2(a<\mathbf{r}<b)} + C_N\hbar^N\|v\|^2_{L^2(\Omega_+)}. \tag{A.6}$$

We next use the propagation estimate again to control $\|v\|_{L^2(\{\mathbf{r}\leq r_1\}\setminus\{\chi\equiv 1\})}$ by $\|v\|^2_{L^2(a<\mathbf{r}<b)}$. To do this, we need that there exists $T > 0$ such that for all $\rho \in S^*_{\Omega_+\setminus\{\chi\equiv 1\}}\Omega_+$ with $\mathbf{r}(\rho) \leq r_1$ there is $|t| \leq T$ with $a < \mathbf{r}(\varphi_t(\rho)) < b$. Suppose not; then there exist $\rho_n \in S^*_{\Omega_+\setminus\{\chi\equiv 1\}}\Omega_+$ with $\mathbf{r}(\rho_n) \leq r_1$ and $T_n \to \infty$ such that

$$\bigcup_{|t|\leq T_n} \varphi_t(\rho_n) \cap \{a < \mathbf{r} < b\} = \emptyset.$$

By (2.22), we have $\mathbf{r}(\rho_n) \leq r_0$ and also $\mathbf{r}(\varphi_{\pm T_n}(\rho_n)) \leq r_0$. In particular, we may assume that $\rho_n \to \rho \in \{\mathbf{r} \leq r_0\} \setminus K$ (since $\pi_\mathbb{R}(K) \Subset \{\chi \equiv 1\}$) and $\varphi_{\pm T_n}(\rho_n) \to \rho_\pm$. Then, by Lemma 2.8, $\rho \in \Gamma_+ \cap \Gamma_- = K$, which is a contradiction. Applying the propagation estimate (using the existence of the uniform time T), we have

$$\|v\|^2_{L^2(\{\mathbf{r}\leq r_1\}\setminus\{\chi\equiv 1\})} \leq C\hbar^{-1}\|(-\hbar^2\Delta - 1)v\|^2_{L^2(\mathbf{r}\leq r_1)} + C\|v\|^2_{L^2(a<\mathbf{r}<b)} + C_N\hbar^N\|v\|^2_{L^2(\Omega_+)}. \tag{A.7}$$

Finally, we control $\|v\|_{L^2(\Omega_+\setminus\Omega_{\rm tr})}$. For this, note that, $v = u1_{\Omega_{\rm tr}} + v1_{(\Omega_{\rm tr})^c}$ and by (A.3) and (A.7) we have

$$\|v\|_{L^2(\Omega_+\setminus\Omega_{\rm tr})} \leq C\hbar^{-1}\|(P_\theta - 1)v\|_{L^2(\Omega_+)} + C\|v\|_{L^2(a<\mathbf{r}<b)} + C_N\hbar^N\|u\|_{L^2(\Omega_{\rm tr})}. \tag{A.8}$$

Now, using (A.6) and (A.7) in (A.5), and applying (A.8) taking $\hbar > 0$ small enough, and using the definition of v (A.1) and that $v = u$ on $\Omega_{\rm tr}$, we have

$$\begin{aligned}\|v\|^2_{L^2(a<\mathbf{r}<b)} &\leq C\hbar^{-N-1}\|(-\hbar^2\Delta - 1)v\|^2_{L^2(\mathbf{r}\leq r_1)} + C\hbar^N\|v\|^2_{L^2(\Omega_+)}\\ &= C\hbar^{-N-1}\|(P_\theta - 1)v\|^2_{L^2(\Omega_+)} + C\hbar^N\|u\|^2_{L^2(\Omega_{\rm tr})} + C\hbar^N\|v\|^2_{L^2(\Omega_+\setminus\Omega_{\rm tr})}.\end{aligned}$$

Then, using (A.8),

$$\|v\|^2_{L^2(a<\mathbf{r}<b)} \leq C\hbar^N\|u\|^2_{L^2(\Omega_{\rm tr})} + Ch^N\|v\|_{L^2(a<\mathbf{r}<b)},$$

and, taking $\hbar$ small enough, we obtain

$$\|v\|_{L^2(a<\mathbf{r}<b)} \leq C\hbar^N\|u\|_{L^2(\Omega_{\rm tr})}\leq C\hbar^N,$$

since $\|u\|_{L^2(\Omega_{\rm tr})} = 1$. Therefore, using (A.7), (A.8), and the definition of v (A.1), we have

$$\|\psi(\mathbf{r})v\|^2_{L^2(\Omega_+\setminus\{\chi\equiv1\})} = \mathcal{O}(\hbar^\infty).$$

so that, since $\mathrm{WF}_\hbar(v) \subset S^*\mathbb{R}^d$ (which is compact),

$$\|\psi(\mathbf{r})v\|^2_{H^2_\hbar(\Omega_+\setminus\{\chi\equiv1\})} = \mathcal{O}(\hbar^\infty);$$

the result then follows from (A.2). ■

B Details of how the eigenvalues/eigenfunctions were computed in §1.3

When discretising sesquilinear form $a(\cdot,\cdot)$ defined by (1.8), we need to calculate the Dirichlet-to-Neumann map $\mathcal{D}(k)$. Instead of approximating $\mathcal{D}(k)$ using either a perfectly-matched layer (PML) or an absorbing boundary condition, we use boundary integral operators to find $\mathcal{D}(k)$ "exactly" (i.e. up to the discretisation of these integral operators).

Recall that the single-layer potential on $\Gamma_{\rm tr}$ is defined for $\varphi \in L^1(\Gamma)$ by

$$\mathcal{S}_k\varphi(x) := \int_{\Gamma_{\rm tr}} \Phi_k(x,y)\varphi(y)\,ds(y) \qquad \text{for all } x \in \mathbb{R}^d \setminus \Gamma_{\rm tr},$$

where, in 2-d, $\Phi_k(x,y) := \mathrm{i}H_0^{(1)}(k|x-y|)/4$, where $H_0^{(1)}$ is the order zero Hankel function of the first kind. The single-layer and adjoint-double-layer operators are then defined, respectively, by $S_k := \gamma_0^{\rm tr}\mathcal{S}_k$ and $D_k' := \gamma_1^{\rm tr}\mathcal{S}_k - I/2$, where the traces are taken from inside $\Omega_{\rm tr}$. With these definitions, for values of k for which $S_k : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is invertible,

$$\mathcal{D}(k) = \left(-\frac{1}{2}I + D_k'\right)S_k^{-1} \tag{B.1}$$

see, e.g., [CWGLS12, Page 136].

To avoid the operator product in (B.1), we introduce the auxiliary variable $\varphi_\ell = S_k^{-1}(\gamma_0^{\rm tr}(u_\ell)) \in H^{-1/2}(\Gamma_{\rm tr})$. The eigenvalue problem (1.6) can therefore be rewritten as: find $u_\ell \in H^1_{0,D}(\Omega_{\rm tr})$ and $\varphi_\ell \in H^{1/2}(\Gamma_{\rm tr})$ such that

$$\begin{aligned}\big(\nabla u_\ell, \nabla v\big)_{L^2(\Omega_{\rm tr})} - k^2\big(u_\ell, v\big)_{L^2(\Omega_{\rm tr})} - \left\langle\left(-\frac{1}{2}I + D_k'\right)\varphi_\ell, \gamma_0^{\rm tr}v\right\rangle_{\Gamma_{\rm tr}} &= \mu_\ell\big(u_\ell, v\big)_{L^2(\Omega_{\rm tr})},\\ \text{and} \qquad \big\langle\gamma_0^{\rm tr}u_\ell, \psi\big\rangle_{\Gamma_{\rm tr}} - \big\langle S_k\varphi_\ell, \psi\big\rangle_{\Gamma_{\rm tr}} &= 0,\end{aligned} \tag{B.2}$$

for all $v \in H^1_{0,D}(\Omega_{\rm tr})$ and $\psi \in H^{-1/2}(\Gamma_{\rm tr})$. We note that this formulation is the transpose of the Johnson–Nédélec FEM-BEM coupling [JN80] applied to the eigenvalue problem (1.6); see, e.g., [GHS12, Equation 9].

We use piecewise-linear basis functions to discretise (B.2), and obtain the following generalised eigenvalue problem

$$\widetilde{\mathbf{A}} = \begin{pmatrix} \mathbf{A}_k & -\frac{1}{2}\mathbf{M}^{\rm tr} + \mathbf{D}'_k \\ \mathbf{M}^{\rm tr} & -\mathbf{S}_k \end{pmatrix} \mathbf{u}_\ell = \mu_\ell \begin{pmatrix} \mathbf{M} \\ 0 \end{pmatrix} \mathbf{u}_\ell =: \mu_\ell \mathbf{B} \mathbf{u}_\ell, \tag{B.3}$$

where $\mathbf{M}$ and $\mathbf{M}^{\rm tr}$ are the mass matrices on $\Omega_{\rm tr}$ and $\Gamma_{\rm tr}$, $\mathbf{S}_k$ and $\mathbf{D}_k$ are discretisations of the single- and adjoint-double layer operators, and $\mathbf{A}_k$ is the Galerkin matrix corresponding to the discretisation of $(\nabla u_\ell, \nabla v) - k^2(u_\ell, v)$.

To build the matrices in (B.3) and solve this problem, we use PETSc [BGMS97, BAA+19, BAA+20] and the eigensolver SLEPc [RCRT20, HRV05] via the software FreeFEM [Hec12]. Since we are interested in the eigenvalues near the origin, we use the shift-and-invert technique, i.e, we compute the largest eigenvalues of the problem $(\widetilde{\mathbf{A}})^{-1}\mathbf{B}\mathbf{u}_\ell = \nu_\ell \mathbf{u}_\ell$, and then set $\mu_\ell = 1/\nu_\ell$. To obtain the action of $(\widetilde{\mathbf{A}})^{-1}$, we use SuperLU [LD03] to compute the LU factorisation of $\widetilde{\mathbf{A}}$.

References

[BAA+19] S. Balay, S. Abhyankar, M. F. Adams, J. Brown, P. Brune, K. Buschelman, L. Dalcin, V. Eijkhout, W. D. Gropp, D. Kaushik, M. G. Knepley, L. Curfman McInnes, K. Rupp, B. F. Smith, S. Zampini, and H. Zhang. PETSc web page, 2019.

[BAA+20] S. Balay, S. Abhyankar, M. F. Adams, J. Brown, P. Brune, K. Buschelman, L. Dalcin, A. Dener, V. Eijkhout, W. D. Gropp, D. Karpeyev, D. Kaushik, M. G. Knepley, D. A. May, L. Curfman McInnes, R. Tran Mills, T. Munson, K. Rupp, P. Sanan, B. F. Smith, S. Zampini, H. Zhang, and H. Zhang. PETSc users manual. Technical Report ANL-95/11 - Revision 3.14, Argonne National Laboratory, 2020.

[BCWG+11a] T. Betcke, S. N. Chandler-Wilde, I. G. Graham, S. Langdon, and M. Lindner. Condition number estimates for combined potential boundary integral operators in acoustics and their boundary element discretisation. *Numer. Methods Partial Differential Eq.*, 27(1):31–69, 2011.

[BCWG+11b] T. Betcke, S. N. Chandler-Wilde, I. G. Graham, S. Langdon, and M. Lindner. Condition number estimates for combined potential boundary integral operators in acoustics and their boundary element discretisation. *Numer. Methods Partial Differential Eq.*, 27(1):31–69, 2011.

[BGMS97] S. Balay, W. D. Gropp, L. Curfman McInnes, and B. F. Smith. Efficient management of parallelism in object oriented numerical software libraries. In E. Arge, A. M. Bruaset, and H. P. Langtangen, editors, *Modern Software Tools in Scientific Computing*, pages 163–202. Birkhäuser Press, 1997.

[CDLL17] S. Chaillat, M. Darbas, and F. Le Louër. Fast iterative boundary element methods for high-frequency scattering problems in 3d elastodynamics. *Journal of Computational Physics*, 341:429–446, 2017.

[CG17] P.-H. Cocquet and M. J. Gander. How large a shift is needed in the shifted Helmholtz preconditioner for its effective inversion by multigrid? *SIAM J. Sci. Comp.*, 39(2):A438–A478, 2017.

[CH01] K. Chen and P. J. Harris. Efficient preconditioners for iterative solution of the boundary element equations for the three-dimensional Helmholtz equation. *Applied Numerical Mathematics*, 36(4):475–489, 2001.

[CIKM96] S. L. Campbell, I. C. F. Ipsen, C. T. Kelley, and C. D. Meyer. Gmres and the minimal polynomial. *BIT Numerical Mathematics*, 36(4):664–675, 1996.

[CP02] F. Cardoso and G. Popov. Quasimodes with exponentially small errors associated with elliptic periodic rays. *Asymptotic Analysis*, 30(3, 4):217–247, 2002.

[CWGLS12] S. N. Chandler-Wilde, I. G. Graham, S. Langdon, and E. A. Spence. Numerical-asymptotic boundary integral methods in high-frequency acoustic scattering. *Acta Numerica*, 21(1):89–305, 2012.

[DDL13] M. Darbas, E. Darrigrand, and Y. Lafranche. Combining analytic preconditioner and fast multipole method for the 3-D Helmholtz equation. *Journal of Computational Physics*, 236:289–316, 2013.

[DG17] S. Dyatlov and J. Galkowski. Fractal Weyl laws and wave decay for general trapping. *Nonlinearity*, 30(12):4301, 2017.

[DZ16] S. Dyatlov and M. Zworski. Dynamical zeta functions for Anosov flows via microlocal analysis. *Annales de l'ENS*, 49, 2016.

[DZ19] S. Dyatlov and M. Zworski. *Mathematical theory of scattering resonances*, volume 200 of *Graduate Studies in Mathematics*. American Mathematical Society, 2019.

[EG12] O. G. Ernst and M. J. Gander. Why it is difficult to solve Helmholtz problems with classical iterative methods. In I. G. Graham, T. Y. Hou, O. Lakkis, and R. Scheichl, editors, *Numerical Analysis of Multiscale Problems*, volume 83 of *Lecture Notes in Computational Science and Engineering*, pages 325–363. Springer, 2012.

[EO99] H. C. Elman and D. P. O'leary. Eigenanalysis of some preconditioned Helmholtz problems. *Numerische Mathematik*, 83(2):231–257, 1999.

[Erl08] Y.A. Erlangga. Advances in iterative methods and preconditioners for the Helmholtz equation. *Archives of Computational Methods in Engineering*, 15(1):37–66, 2008.

[ESW02] H. C. Elman, D. J. Silvester, and A. J. Wathen. Performance and analysis of saddle point preconditioners for the discrete steady-state Navier-Stokes equations. *Numerische Mathematik*, 90(4):665–688, 2002.

[GHS12] G. N. Gatica, G. C. Hsiao, and F.-J. Sayas. Relaxing the hypotheses of Bielak–MacCamyÕs BEM–FEM coupling. *Numerische Mathematik*, 120(3):465–487, 2012.

[GLS21] J. Galkowski, D. Lafontaine, and E. A. Spence. Local absorbing boundary conditions on fixed domains give order-one errors for high-frequency waves. *preprint*, 2021.

[GSW20] J. Galkowski, E. A. Spence, and J. Wunsch. Optimal constants in nontrapping resolvent estimates. *Pure and Applied Analysis*, 2(1):157–202, 2020.

[GSZ20] I. G. Graham, E. A. Spence, and J. Zou. Domain Decomposition with Local Impedance Conditions for the Helmholtz Equation with Absorption. *SIAM Journal on Numerical Analysis*, 58(5):2515–2543, 2020.

[GZ19] M.J. Gander and H. Zhang. A class of iterative solvers for the Helmholtz equation: factorizations, sweeping preconditioners, source transfer, single layer potentials, polarized traces, and optimized Schwarz methods. *SIAM Review*, 61(1):3–76, 2019.

[Hec12] F. Hecht. New development in FreeFem++. *Journal of numerical mathematics*, 20(3-4):251–266, 2012.

[Hör85] L. Hörmander. *The analysis of linear partial differential operators III: pseudo-differential operators*. Springer, 1985.

[HRV05] V. Hernandez, J. E. Roman, and V. Vidal. SLEPc: A scalable and flexible toolkit for the solution of eigenvalue problems. *ACM Trans. Math. Software*, 31(3):351–362, 2005.

[Jen77] A. Jennings. Influence of the eigenvalue spectrum on the convergence rate of the conjugate gradient method. *IMA Journal of Applied Mathematics*, 20(1):61–72, 1977.

[JN80] C. Johnson and J. C. Nédélec. On the coupling of boundary integral and finite element methods. *Mathematics of computation*, pages 1063–1079, 1980.

[Laz93] V. F. Lazutkin. *KAM theory and semiclassical approximations to eigenfunctions*. Springer Science & Business Media, 1993.

[LD03] X. S. Li and J. W. Demmel. SuperLU_DIST: A scalable distributed-memory sparse direct solver for unsymmetric linear systems. *ACM Trans. Mathematical Software*, 29(2):110–140, 2003.

[LSW19] D. Lafontaine, E. A. Spence, and J. Wunsch. A sharp relative-error bound for the Helmholtz h-FEM at high frequency. *arXiv preprint arXiv:1911.11093*, 2019.

[LXSdH20] X. Liu, Y. Xi, Y. Saad, and M. V. de Hoop. Solving the Three-Dimensional High-frequency Helmholtz Equation Using Contour Integration and Polynomial Preconditioning. *SIAM Journal on Matrix Analysis and Applications*, 41(1):58–82, 2020.

[MGSS21] P. Marchand, J. Galkowski, A. Spence, and E. A. Spence. Applying GMRES to the Helmholtz equation with strong trapping: how does the number of iterations depend on the frequency? *preprint*, 2021.

[Mil00] L. Miller. Refraction of high-frequency waves density by sharp interfaces and semiclassical measures at the boundary. *J. Math. Pures Appl. (9)*, 79(3):227–269, 2000.

[MS10] J. M. Melenk and S. Sauter. Convergence analysis for finite element discretizations of the Helmholtz equation with Dirichlet-to-Neumann boundary conditions. *Math. Comp*, 79(272):1871–1914, 2010.

[RCRT20] J. E. Roman, C. Campos, E. Romero, and A. Tomas. SLEPc users manual. Technical Report DSIC-II/24/02 - Revision 3.14, D. Sistemes Informàtics i Computació, Universitat Politècnica de València, 2020.

[SS86] Y. Saad and M. H. Schultz. GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems. *SIAM Journal on scientific and statistical computing*, 7(3):856–869, 1986.

[Ste99] P. Stefanov. Quasimodes and resonances: sharp lower bounds. *Duke mathematical journal*, 99(1):75–92, 1999.

[Ste00] P. Stefanov. Resonances near the real axis imply existence of quasimodes. *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics*, 330(2):105–108, 2000.

[SV95] P. Stefanov and G. Vodev. Distribution of resonances for the neumann problem in linear elasticity outside a strictly convex body. *Duke Mathematical Journal*, 78(3):677–714, 1995.

[SV96] P. Stefanov and G. Vodev. Neumann resonances in linear elasticity for an arbitrary body. *Communications in mathematical physics*, 176(3):645–659, 1996.

[TZ98] S.-H. Tang and M. Zworski. From quasimodes to resonances. *Mathematical Research Letters*, 5:261–272, 1998.

[TZ00] S.-H. Tang and M. Zworski. Resonance expansions of scattered waves. *Comm. Pure Appl. Math.*, 53(10):1305–1334, 2000.

[Vas08] A. Vasy. Propagation of singularities for the wave equation on manifolds with corners. *Annals of Mathematics*, 168:749–812, 2008.

[VG14] A. Vion and C. Geuzaine. Double sweep preconditioner for optimized Schwarz methods applied to the Helmholtz problem. *Journal of Computational Physics*, 266:171–190, 2014.

[VGEV07] M. B. Van Gijzen, Y. A. Erlangga, and C. Vuik. Spectral analysis of the discrete Helmholtz operator preconditioned with a shifted Laplacian. *SIAM J. Sci. Comp.*, 29(5):1942–1958, 2007.

[Zwo12] M. Zworski. *Semiclassical analysis*, volume 138 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.