SURFACE PLASMONS IN METAMATERIAL CAVITIES: SCATTERING BY OBSTACLES WITH NEGATIVE WAVE SPEED

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Abstract. We study scattering by metamaterials with negative indices of refraction, which are known to support *surface plasmons* – long-lived states that are highly localized at the boundary of the cavity. This type of states has found uses in a variety of modern technologies. In this article, we study surface plasmons in the setting of non-trapping cavities; i.e. when all billiard trajectories outside the cavity escape to infinity. We characterize the indices of refraction which support surface plasmons, show that the corresponding resonances lie super-polynomially close to the real axis, describe the localization properties of the corresponding resonant states, and give an asymptotic formula for their number.

1. Introduction

We consider resonance phenomena for metamaterial cavities which exhibit a negative index of refraction or negative wave speed. These structures are known, in some contexts, to support surface plasmons – long lived states that are highly localized to the surface of the metamaterial [Mai07]. These surface plasmons offer strong light enhancement and are central to a range of modern technologies [SHV08]. Although negative index of refraction metamaterials have attracted some mathematical interest (see [CM23, CdV25, DBCM24, BBDCC12, BBDCC13, BBDCCJ14, BHM21]) their asymptotic behavior has remained largely unexplored. Under a relatively mild assumption on the metamaterial scatterer, we study plasmon resonances in a scalar model (i.e. in the Transverse Electric or Transverse Magnetic polarization). In this article, we give an accurate description of the asymptotic behavior of surface plasmons. We characterize the existence and absence of surface plasmons, accurately describe their localization properties, and provide an asymptotic formula for their number.

Let $d \geq 2$ and $\Omega_{\mathcal{I}} \subset \mathbb{R}^d$ be a bounded open domain with smooth boundary and connected complement. Define $\Omega_{\mathcal{O}} := \mathbb{R}^d \backslash \overline{\Omega_{\mathcal{I}}}$. We denote the shared boundary of $\Omega_{\mathcal{I}}$ and $\Omega_{\mathcal{O}}$ by $\partial\Omega$ and the outward pointing normal of $\Omega_{\mathcal{I}}$ by ν. Although we work in the more general setting of negative wave speeds below (see section 1.1), we state first a simple consequence of our main theorem. Let the *index of refraction* $n \in C^\infty(\overline{\Omega_{\mathcal{I}}}; (0,\infty))$ with $|n|_{\partial\Omega} - 1| > 0$. We call $\lambda \in \mathbb{C} \setminus \mathrm{i}(-\infty,\infty)$ a *resonance* if there is a non-zero solution $(u_{\mathcal{I}}, u_{\mathcal{O}}) \in H^2(\Omega_{\mathcal{I}}) \oplus H^2_{\mathrm{loc}}(\Omega_{\mathcal{O}})$ to

$$\begin{cases} (\operatorname{div} n^{-1}\nabla - \lambda^2)u_{\mathcal{I}} = 0 & \text{in } \Omega_{\mathcal{I}}, \\ (-\Delta - \lambda^2)u_{\mathcal{O}} = 0 & \text{in } \Omega_{\mathcal{O}}, \\ u_{\mathcal{O}} = u_{\mathcal{I}} & \text{on } \partial\Omega, \\ \partial_\nu u_{\mathcal{O}} = -n^{-1}\partial_\nu u_{\mathcal{I}} & \text{on } \partial\Omega, \\ u_{\mathcal{O}} \text{ is } \lambda\text{-outgoing.} \end{cases} \tag{1.1}$$

Here, we say that u is λ-outgoing if there is $g \in L^2_{\text{comp}}(\mathbb{R}^d)$ such that $u(x) = [R_0(\lambda)g](x)$ for $|x| \gg 1$, where $R_0(\lambda)$ is the free, outgoing resolvent – when λ is real, this outgoing condition becomes the usual Sommerfel radiation condition and $R_0(\lambda)$ is L^2-bounded for $\text{Im}(\lambda) > 0$.

The system (1.1) was studied for $\lambda \in \mathbb{R}$ in [CM23] and is a special case of our more general setting below (see (1.5) and Remark 1.7). We call $u_\lambda = u_{\mathcal{I}} 1_{\Omega_{\mathcal{I}}} + u_{\mathcal{O}} 1_{\Omega_{\mathcal{O}}}$ a resonant state for λ and write $\mathcal{R}(n, \Omega_{\mathcal{I}})$ for the set of resonances. We call a sequence of resonances $\{\lambda_j\}_{j=1}^\infty \subset \mathcal{R}$ with $|\lambda_j| \to \infty$ a *plasmon resonance* if for any $\psi \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp} \psi \cap \partial\Omega = \emptyset$, any $N > 0$, and any sequence of resonant states u_{λ_j}, we have

$$\lim_{j \to \infty} \frac{|\lambda_j|^N \|\psi u_{\lambda_j}\|_{L^2(\mathbb{R}^d)}}{\|u_{\lambda_j}\|_{L^2(\partial\Omega)}} = 0. \tag{1.2}$$

That is, any sequence of resonant states associated to λ_j concentrates asymptotically at $\partial\Omega$.

Throughout the text, we will assume that $\Omega_{\mathcal{I}}$ is *non-trapping* (See Figure 1). That is, all billiard trajectories (or more precisely generalized broken bicharacteristics; see e.g. [HÖ7, Section 24.3] for a definition) escape any compact set in finite time. This condition guarantees that resonant states corresponding to propagating modes cannot approach the real axis.

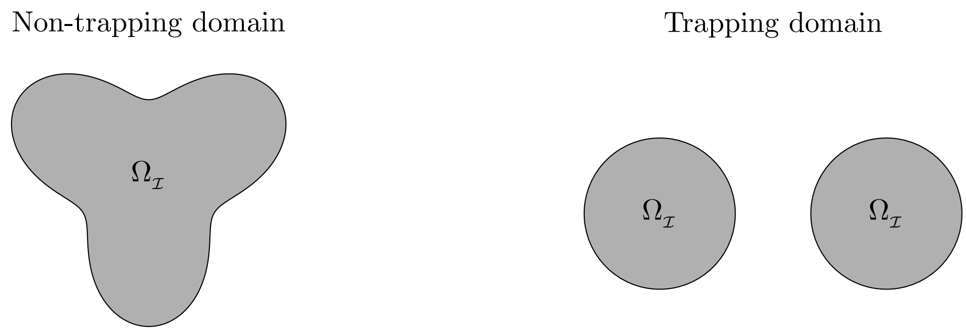


Figure 1. Examples of trapping and non-trapping domains.

We first determine conditions on the index of refraction, n, such that there are no resonances close to the real axis.

Theorem 1.1. *Suppose that $\Omega_{\mathcal{I}}$ is non-trapping and $n \in C^\infty(\overline{\Omega}_{\mathcal{I}}; (0, \infty))$ satisfies $n|_{\partial\Omega} < 1$. Then for all $M > 0$ there is $C > 0$ such that*

$$\mathcal{R}(n, \Omega_{\mathcal{I}}) \cap \{|\operatorname{Re} \lambda| > C\} \subset \{\operatorname{Im} \lambda < -M\}.$$

Next, in the complementary case, we describe the region in which resonances may lie and show that any such resonances are plasmonic.

Theorem 1.2. *Suppose that $\Omega_{\mathcal{I}}$ is non-trapping and $n \in C^\infty(\overline{\Omega}_{\mathcal{I}}; (0, \infty))$ satisfies $n|_{\partial\Omega} > 1$. Then for all $M > 0$, $N > 0$ there is C such that*

$$\mathcal{R}(n, \Omega_{\mathcal{I}}) \cap \{|\operatorname{Re} \lambda| > C\} \subset \{\operatorname{Im} \lambda < -M \text{ or } -|\operatorname{Re} \lambda|^{-N} < \operatorname{Im} \lambda < 0\}.$$

Moreover, any sequence $\{\lambda_j\}_{j=1}^\infty \subset \mathcal{R}(n, \Omega_{\mathcal{I}})$ with $|\operatorname{Re} \lambda_j| \to \infty$ and $|\operatorname{Im} \lambda_j|$ bounded is a plasmon resonance.

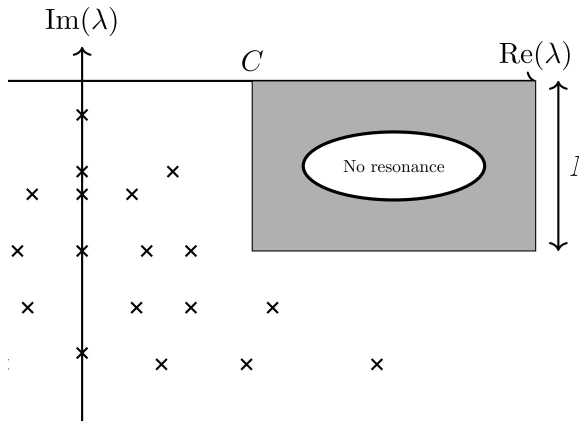
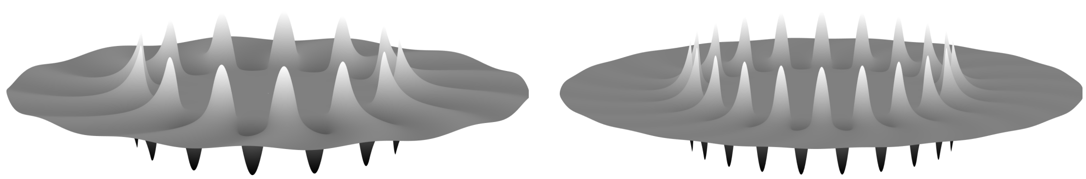


Figure 2. The figure shows the resonances for (1.1) with $n|_{\partial\Omega} < 1$ as x's. The resonance free region is determined by Theorem 1.1 or 1.8.



$\lambda_L \approx 8.4647 - 1.0396 \times 10^{-2}\mathrm{i}$ $\lambda_R \approx 13.145 - 8.5412 \times 10^{-4}\mathrm{i}$

Figure 3. Lemma 5.9 in fact shows that, modulo $O(|\lambda_j|^{-\infty})$, all surface plasmons are as pictured here (with $\Omega_{\mathcal{I}} = B(0,1)$). These plasmons concentrate in a $|\lambda_j|^{-1}$ neighborhood of the boundary, $\partial\Omega$ and oscillate at frequency $\sim |\lambda_j|$ in $\partial\Omega$. The functions plotted here are the real parts of the resonant state corresponding to $n|_{B(0,1)} \equiv 3$ with resonance $\lambda_L \approx 8.4647 \times 10^0 - 1.0396 \times 10^{-2}\mathrm{i}$ (on the left) and $\lambda_R \approx 13.145 - 8.5412 \times 10^{-4}\mathrm{i}$ (on the right).

Remark 1.3. *We prove much more than that the resonant states with* $|\operatorname{Im}\lambda_j|$ *bounded are plasmonic in the sense of* (1.2) *(see Lemma 5.9 and Figure 3). We are, in fact, able to describe their localization properties modulo* $|\lambda_j|^{-\infty}$*. For instance, one can see that for any* $\chi \in C_c^\infty(\mathbb{R}^d)$*,* $\|\chi\partial_x^\alpha u_{\lambda_j}(x)\|_{L^2(\mathbb{R}^d)} \le C|\lambda_j|^{-\frac{1}{2}+|\alpha|}$*,* $|\alpha| \le 2$*.*

The resonance free regions of Theorems 1.1 and 1.2 are pictured in Figures 2 and 4 respectively.

Finally, we determine the asymptotic number of plasmonic resonances, counted with multiplicity.

Theorem 1.4. *Suppose that $\Omega_{\mathcal{I}}$ is non-trapping and $n \in C^\infty(\overline{\Omega}_{\mathcal{I}}; (0,\infty))$ satisfies $n|_{\partial\Omega} > 1$. Then for all $M > 0$,*

$$\begin{aligned}\#\{\lambda_j \in \mathcal{R}(n,\Omega_{\mathcal{I}}) \,:\, 0 < \operatorname{Re}\lambda_j \le \lambda,\, \operatorname{Im}\lambda \ge -M\}\\ = \frac{\lambda^{d-1}}{(2\pi)^{d-1}}\operatorname{vol}_{T^*\partial\Omega}\Big(\Big\{(x',\xi') \in T^*\partial\Omega \,:\, |\xi'|^2_{g_{euc}} \le 1 + \frac{1}{n(x')-1}\Big\}\Big) + o(\lambda^{d-1}),\end{aligned}$$

where g_{euc} is the metric induced on $\partial\Omega$ by the Euclidean metric on $\mathbb{R}^d$.

Remark 1.5. *Note that Theorem 1.1 1.2, and 1.4 are the special cases of Theorem 1.8, Theorem 1.9, Theorem 1.10, and Theorem 1.11 below.*

Theorems 1.1 to 1.4 (and their analogs below) give a precise description of resonances near the real axis for a wide class of negative index of refraction scattering problems. They determine when such resonances exist, how many there are, and describe the asymptotic properties of the corresponding resonant states. While it is often possible to obtain asymptotic upper bounds on the number of resonances near the real axis (see e.g. [DD13, DG17, Dya19, SZ07] and references therein), it is very rare to be able to give an asymptotic count of these resonances – celebrated examples include scattering in one dimension [Zwo87], by convex obstacles [SZ99], by convex transparent obstacles [CPV01], and with normally hyperbolic trapping [Dya15]. Theorem 1.4 and its more general analog Theorem 1.11 provide another such example.

Relation with previous work on negative index of refraction metamaterials:

To the best of the authors' knowledge, the first mathematical paper considering negative index of refraction scattering is [CS85], where the authors study materials with constant index of refraction (i.e. $n|_{\Omega_{\mathcal{I}}} \equiv c_{\mathcal{I}}$) and show that the problem (1.1) is Fredholm under certain conditions on λ and $c_{\mathcal{I}}$. The works [BBDCC12, BBDCC13, BBDCCJ14] build on this theory, allowing n to be variable, and study the problem in lower regularity. In Appendix A, we give a different proof inspired by [CdV25] to show that (1.1) (or indeed the more general problem (1.5)) is Fredholm when $n|_{\partial\Omega}$ avoids 1.

We study scattering resonances in the context of negative index of refraction metamaterials. As far as the authors are aware, the only previous works in this context are [CdV25, CM23, DBCM24]. In [CM23, DBCM24], the authors study the case of $d = 2$ with $\Omega_{\mathcal{I}}$ having smooth boundary and show, without further assumptions on the geometry, that there are many resonances near the real axis in the case $n|_{\partial\Omega} > 1$ and many negative eigenvalues in the case $n|_{\partial\Omega} < 1$. In fact, the authors construct a sequence of quasimodes, u_j, with quasi-eigenvalue $\lambda_j \to \infty$ associated to (1.1) so that u_j are highly localised near $\partial\Omega$. However, they do not show that the true resonant states are highly localized. When $n|_{\partial\Omega} > 1$ is constant,

$$\lambda_j = \frac{2\pi j(n-1)}{n|\partial\Omega|},$$

and, if $\Omega_{\mathcal{I}}$ is non-trapping, we confirm from Theorem 1.4 that this sequence of quasi-eigenvalues captures most resonances near the real axis. Moreover, Theorem 1.2 shows that all corresponding resonant states are highly localized. Very recently, and independently from our work, [CdV25] considers the higher dimensional analog of [CM23].

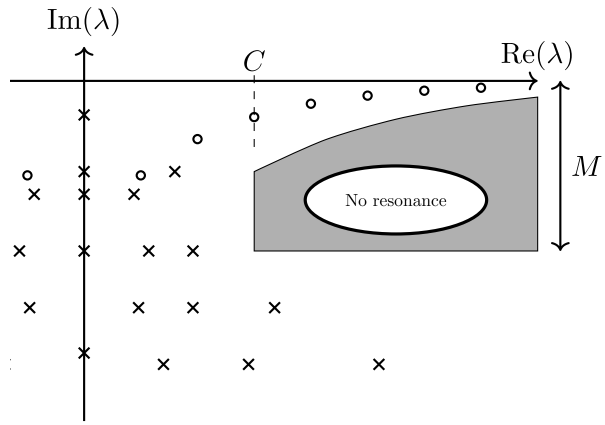


Figure 4. The figure shows the resonances for (1.1) with $n|_{\partial\Omega} > 1$. Non-plasmonic resonances are denoted with x's and plasmonic resonances with o's. The resonance free regions are those determined by Theorem 1.2 or 1.9. Theorems 1.4 and 1.11 determine the asymptotic number of plasmonic resonances.

The present article differs from and strengthens these earlier works in two substantial ways. First, under an additional natural dynamical assumption, we describe the location of all possible resonances near the real axis and show that if there are any, they must correspond to highly localized resonant states, and second, we determine how many such resonances there are.

Remark 1.6. *There are variety transmission problems, including by positive index of refraction materials, which are much better developed in the mathematical literature (See e.g.* [PV99b, CPV99, PV99a, CPV01, Gal19a, Gal19b, MS19]*).*

1.1. **Inhomogeneous metrics.** In this article, we study the general situation of a cavity with a negative definite Laplacian. This corresponds to a metamaterial where the material properties are not homogoneous and exhibit an effective negative wave-speed. To this end, let $g_{\mathcal{O}}$ be a smooth Riemannian metric on $\overline{\Omega_{\mathcal{O}}}$ and $g_{\mathcal{I}}$ a smooth Riemannian metric on $\overline{\Omega_{\mathcal{I}}}$. Let also $\rho_{\mathcal{I}} \in C^\infty(\overline{\Omega_{\mathcal{I}}};(0,\infty))$ and $\rho_{\mathcal{O}} \in C^\infty(\overline{\Omega_{\mathcal{O}}};(0,\infty))$. We assume that the geometry is Euclidean near infinity. That is

$$g^{ij}_{\mathcal{O}}(x) = \delta^{ij}, \qquad \rho_{\mathcal{O}}(x) = 1, \qquad \text{for } |x| \gg 1.$$

For a metric g and positive function ρ, we define the operator

$$\Delta_{g,\rho} u := \frac{1}{\rho\sqrt{|g|}}\partial_{x^i}(g^{ij}\sqrt{|g|}\rho\partial_j u(x)), \quad |g| := |\det g_{ij}|,$$

and note that $\Delta_{g,\rho}$ is symmetric on $L^2(\rho d\mathrm{vol}_g)$. We then define the unbounded operator $P : L^2(\Omega_{\mathcal{I}}, \rho_{\mathcal{I}} d\mathrm{vol}_{g_{\mathcal{I}}}) \oplus L^2(\Omega_{\mathcal{O}}, \rho_{\mathcal{O}} d\mathrm{vol}_{g_{\mathcal{O}}}) \to L^2(\Omega_{\mathcal{I}}, \rho_{\mathcal{I}} d\mathrm{vol}_{g_{\mathcal{I}}}) \oplus L^2(\Omega_{\mathcal{O}}, \rho_{\mathcal{O}} d\mathrm{vol}_{g_{\mathcal{O}}})$ given by

$$P(u_{\mathcal{I}}, u_{\mathcal{O}}) = (\Delta_{g_{\mathcal{I}},\rho_{\mathcal{I}}}, -\Delta_{g_{\mathcal{O}},\rho_{\mathcal{O}}})$$

with domain

$$\mathcal{D}(P) := \left\{ \begin{aligned} u = (u_{\mathcal{I}}, u_{\mathcal{O}}) \in H^1(\mathbb{R}^d) \cap \big(H^2(\Omega_{\mathcal{I}}) \oplus H^2(\Omega_{\mathcal{O}})\big) \ : \\ \rho_{\mathcal{O}} \partial_{\nu_{\mathcal{O}}} u_{\mathcal{O}} \mathrm{dvol}_{g_{\mathcal{O}}, \partial\Omega} = \rho_{\mathcal{I}} \partial_{\nu_{\mathcal{I}}} \mathrm{dvol}_{g_{\mathcal{I}}, \partial\Omega} \end{aligned} \right\}, \tag{1.3}$$

where $\partial_{\nu_{\mathcal{I}}}$ and $\partial_{\nu_{\mathcal{O}}}$ denote respectively the outward unit normal to $\Omega_{\mathcal{I}}$ with respect to $g_{\mathcal{I}}$ and the outward unit normal to $\Omega_{\mathcal{O}}$ with respect to $g_{\mathcal{O}}$.

The induced boundary volume forms from the exterior and interior metrics are not in general the same, i.e. $\mathrm{dvol}_{g_{\mathcal{O}}}$ is not necessarily the same as $\mathrm{dvol}_{g_{\mathcal{I}}}$. It is thus natural to define a function on $\partial\Omega$ to measure their difference. Hence, we define $\tau \in C^\infty(\partial\Omega)$

$$\tau \mathrm{dvol}_{g_{\mathcal{O}}, \partial\Omega} := \mathrm{dvol}_{g_{\mathcal{I}}, \partial\Omega}.$$

Notice that τ is positive on $\partial\Omega$.

We assume throughout the text that

$$\tau^2 \rho_{\mathcal{I}}^2 |\xi'|_{g_{\mathcal{I}}}^2 - \rho_{\mathcal{O}}^2 |\xi'|_{g_{\mathcal{O}}}^2 \neq 0, \qquad (x, \xi') \in T^*\partial\Omega. \tag{1.4}$$

Under assumption (1.4), the operator P is self-adjoint and is a black box Hamiltonian in the sense of [DZ19, Section 4.1] (See Appendix A).

We then let $R_P(\lambda) := (P - \lambda^2)^{-1} : L^2(\Omega_{\mathcal{I}}) \oplus L^2(\Omega_{\mathcal{O}}) \to L^2(\Omega_{\mathcal{I}}) \oplus L^2(\Omega_{\mathcal{O}})$. By [DZ19, Theorem 4.4], $R_P(\lambda) : L^2(\Omega_{\mathcal{I}}) \oplus L^2(\Omega_{\mathcal{O}}) \to \mathcal{D}(P)$ is meromorphic in $\operatorname{Im}\lambda > 0$ and has a meromorphic continuation to $\mathbb{C}$ for d odd and the logarithmic cover of $\mathbb{C} \setminus \{0\}$ for d even as an operator $R_P(\lambda) : L^2(\Omega_{\mathcal{I}}) \oplus L^2_{\mathrm{comp}}(\Omega_{\mathcal{O}}) \to \mathcal{D}_{\mathrm{loc}}(P)$, where

$$\mathcal{D}_{\mathrm{loc}}(P) := \Big\{ u = (u_{\mathcal{I}}, u_{\mathcal{O}}) \in H^1_{\mathrm{loc}}(\mathbb{R}^d) \cap \big(H^2(\Omega_{\mathcal{I}}) \oplus H^2_{\mathrm{loc}}(\Omega_{\mathcal{O}})\big) \ : \ \rho_{\mathcal{O}} \partial_{\nu_{\mathcal{O}}} u_{\mathcal{O}} = \tau \rho_{\mathcal{I}} \partial_{\nu_{\mathcal{I}}} u_{\mathcal{I}} \Big\}.$$

Defining $(u_{\mathcal{I}}, u_{\mathcal{O}}) := R_P(\lambda)(f_{\mathcal{I}}, f_{\mathcal{O}})$, $(u_{\mathcal{I}}, u_{\mathcal{O}})$ satisfies

$$\begin{cases} (\Delta_{g_{\mathcal{I}}, \rho_{\mathcal{I}}} - \lambda^2) u_{\mathcal{I}} = f_{\mathcal{I}} & \text{in } \Omega_{\mathcal{I}}, \\ (-\Delta_{g_{\mathcal{O}}, \rho_{\mathcal{O}}} - \lambda^2) u_{\mathcal{O}} = f_{\mathcal{O}} & \text{in } \Omega_{\mathcal{O}}, \\ u_{\mathcal{I}} = u_{\mathcal{O}} & \text{on } \partial\Omega, \\ \rho_{\mathcal{O}} \partial_{\nu_{\mathcal{O}}} u_{\mathcal{O}} - \tau \rho_{\mathcal{I}} \partial_{\nu_{\mathcal{I}}} u_{\mathcal{I}} = 0 & \text{on } \partial\Omega, \\ u_{\mathcal{O}} \text{ is } \lambda\text{-outgoing.} \end{cases} \tag{1.5}$$

Define the set of *resonances of* P by

$$\mathcal{R}(P) := \{\lambda \, : \, \lambda \text{ is a pole of } R_P(\lambda)\}.$$

Remark 1.7. *Note that* (1.1) *is a special case of our general setting* (1.5) *with* $g_{\mathcal{O}}^{ij} = \delta^{ij}$, $g_{\mathcal{I}}^{ij} = n^{-1}\delta^{ij}$, $\rho_{\mathcal{O}} = 1$, $\rho_{\mathcal{I}} = n^{-\frac{d}{2}}$, *and* $\tau = n^{\frac{d-1}{2}}$.

We now state the analogs of Theorems 1.1 to 1.4 in the more general setting of a negative wave speed. We begin in the case

$$0 > \rho_{\mathcal{O}}^2 |\xi'|_{g_{\mathcal{O}}}^2 - \tau^2 \rho_{\mathcal{I}}^2 |\xi'|_{g_{\mathcal{I}}}^2, \qquad (x', \xi') \in T^*\partial\Omega, \tag{1.6}$$

where we show that there are no resonances close to the real axis.

Theorem 1.8. *Suppose that* $(\Omega_{\mathcal{O}}, g_{\mathcal{O}})$ *is a non-trapping domain and* (1.6) *holds. Then for all* $M > 0$*, there is* $C > 0$ *such that*

$$\mathcal{R}(P) \cap \{|\operatorname{Re}\lambda| > C\} \subset \{\operatorname{Im}\lambda \leq -M\}.$$

Moreover, for $\chi \in C_c^\infty(\mathbb{R}^d)$*,*

$$\|\chi R_P(\lambda)\chi\|_{L^2 \to L^2} \leq C|\lambda|^{-1}, \qquad \lambda \in \{\operatorname{Re}\lambda > C, \operatorname{Im}\lambda > -M\}.$$

Our next three theorems consider the opposite case:

$$0 < \rho_{\mathcal{O}}^2 |\xi'|^2_{g_{\mathcal{O}}} - \tau^2 \rho_{\mathcal{I}}^2 |\xi'|^2_{g_{\mathcal{I}}}, \qquad (x', \xi') \in T^*\partial\Omega. \tag{1.7}$$

The combination of the next three theorems shows that there are many resonances resonances superpolynomially close to the real axis, all of which are plasmonic and, moreover, any sequence of resonances that is not superpolynomially close to the real axis must have imaginary part whose absolute value tends to infinity.

The first theorem provides a resonances free region.

Theorem 1.9. *Suppose that* $(\Omega_{\mathcal{O}}, g_{\mathcal{O}})$ *is a non-trapping domain and* (1.7) *holds. Then for all* $M > 0$*,* $N > 0$ *there is* $C > 0$ *such that*

$$\mathcal{R}(P) \cap \{|\operatorname{Re}\lambda| > C\} \subset \{\operatorname{Im}\lambda \leq -M\} \cup \{-|\lambda|^{-N} < \operatorname{Im}\lambda < 0\}.$$

Moreover,

$$\|\chi R_P(\lambda)\chi\|_{L^2 \to L^2} \leq C|\operatorname{Im}\lambda|^{-1}|\lambda|^{-1}, \qquad \lambda \in \{\operatorname{Re}\lambda > C, -M < \operatorname{Im}\lambda < (\operatorname{Re}\lambda)^{-N}\}.$$

Next, we show that any resonances with bounded imaginary parts are necessarily plasmonic.

Theorem 1.10. *Suppose that* $(\Omega_{\mathcal{O}}, g_{\mathcal{O}})$ *is a non-trapping domain and* (1.7) *holds. Then for any* $\{\lambda_j\}_{j=1}^\infty \subset \mathcal{R}(P)$ *with* $|\operatorname{Re}\lambda_j| \to \infty$ *and* $\sup|\operatorname{Im}\lambda_j| < \infty$*, any* $0 \neq u_{\lambda_j} \in \mathcal{D}_{\text{loc}}(P)$ *satisfying* (1.5) *with* $(f_{\mathcal{I}}, f_{\mathcal{O}}) = 0$*,* $N > 0$*, and any* $\psi \in C_c^\infty(\mathbb{R}^d)$ *with* $\operatorname{supp}\psi \cap \partial\Omega = \emptyset$*, we have*

$$\frac{|\lambda_j|^N \|\psi u_{\lambda_j}\|_{L^2(\mathbb{R}^d)}}{\|u_{\lambda_j}\|_{L^2(\partial\Omega)}} \to 0.$$

Finally, we give an asymptotic formula for the number of plasmonic resonances.

Theorem 1.11. *Suppose that* $(\Omega_{\mathcal{O}}, g_{\mathcal{O}})$ *is a non-trapping domain and* (1.7) *holds. Then for all* $M > 0$*,*

$$\#\{\lambda_j \in \mathcal{R}(P) \,:\, 0 < \operatorname{Re}\lambda_j \leq \lambda \,:\operatorname{Im}\lambda_j \geq -M\} = \frac{\lambda^{d-1}}{(2\pi)^{d-1}} \operatorname{vol}_{T^*\partial\Omega}(\mathcal{V}) + o(\lambda^{d-1}),$$

where

$$\mathcal{V} := \Big\{(x', \xi') \in T^*\partial\Omega \,:\, \rho_{\mathcal{O}}^2(x')|\xi'|^2_{g_{\mathcal{O}}} - \rho_{\mathcal{I}}^2(x')^2\tau^2(x')|\xi'|^2_{g_{\mathcal{I}}} \leq \rho_{\mathcal{O}}(x')^2 + \rho_{\mathcal{I}}^2(x')\tau^2(x')\Big\}.$$

1.2. **Outline and ideas from the proof.** We start by making a semiclassical rescaling, setting $\lambda = h^{-1}(1+z)$, with $|z| \le Mh$ and $0 < h < 1$. The goal of this article can then be rephrased in the following way.

Let $\chi \in C_c^\infty(\mathbb{R}^d)$ with $\chi \equiv 1$ on $\Omega_{\mathcal{I}}$, $L^2(\mathbb{R}^d) \ni f = (f_i, f_{\mathcal{O}}) \in L^2(\Omega_{\mathcal{I}}) \oplus L^2(\Omega_{\mathcal{O}})$. Our main goal will be to prove estimates for the solution, $(u_{\mathcal{I}}, u_{\mathcal{O}})$ to

$$\begin{cases} (P_{\mathcal{I}} - z^2)u_{\mathcal{I}} := (h^2\Delta_{g_{\mathcal{I}},\rho_{\mathcal{I}}} - z^2)u_{\mathcal{I}} = hf_i & \text{in } \Omega_{\mathcal{I}}, \\ (P_{\mathcal{O}} - z^2)u_{\mathcal{O}} := (-h^2\Delta_{g_{\mathcal{O}},\rho_{\mathcal{O}}} - z^2)u_{\mathcal{O}} = h\chi f_{\mathcal{O}} & \text{in } \Omega_{\mathcal{O}}, \\ u_{\mathcal{O}} = u_{\mathcal{I}} & \text{on } \partial\Omega, \\ \rho_{\mathcal{O}} h\partial_{\nu_{\mathcal{O}}} u_{\mathcal{O}} - \tau\rho_{\mathcal{I}} h\partial_{\nu_{\mathcal{I}}} u_{\mathcal{I}} = 0 & \text{on } \partial\Omega, \\ u_{\mathcal{O}} \text{ is } z/h\text{-outgoing}. \end{cases} \tag{1.8}$$

In section 3, using the solution of the Dirichlet problem in $\Omega_{\mathcal{I}}$ and the outgoing Dirichlet problem in $\Omega_{\mathcal{O}}$, we reduce these estimates to the study of

$$\begin{cases} (P_{\mathcal{O}} - z^2)v_{\mathcal{O}} = 0 & \text{in } \Omega_{\mathcal{O}}, \\ \rho_{\mathcal{O}} h\partial_{\nu_{\mathcal{O}}} v_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}}(z)v_{\mathcal{O}} =: g \in H_h^{\frac{1}{2}} & \text{on } \partial\Omega, \\ v_{\mathcal{O}} \text{ is } z/h\text{-outgoing}, \end{cases} \tag{1.9}$$

where $\Lambda_{\mathcal{I}}(z)$ is a certain Dirichlet-to-Neumann map associated with the inner problem. More precisely, we define by $\Lambda_{\mathcal{I}/\mathcal{O}}(z)w = \rho_{\mathcal{I}/\mathcal{O}} h\partial_{\nu_{\mathcal{I}/\mathcal{O}}} u_{\mathcal{I}/\mathcal{O}}$, where $u_{\mathcal{I}/\mathcal{O}} \in H^2_{\text{loc}}(\overline{\Omega}_{\mathcal{I}/\mathcal{O}})$ solves

$$\begin{cases} (P_{\mathcal{I}/\mathcal{O}} - z^2)u_{\mathcal{I}} = 0 & \text{in } \Omega_{\mathcal{I}/\mathcal{O}}, \\ u_{\mathcal{I}/\mathcal{O}} = w & \text{on } \partial\Omega, \\ u_{\mathcal{O}} \text{ is } z/h \text{ outgoing}. \end{cases}$$

After the reduction to (1.9), it becomes natural to study the Dirichlet-to-Neumann map for operators of the form

$$P(\omega; g, L) := -h^2\Delta_g + hL - \omega,$$

where $L = \sum_{i=1}^d L^i(x)hD_{x^i}$, and $|\omega - \omega_0| \le Ch$.

Our next theorem yields a parametrix for the Dirichlet-to-Neumann map in the elliptic region.

Theorem 1.12. *Let $\omega_0 \in \mathbb{R}$, L be a smooth vector field, and (M, g) a Riemannian manifold with boundary. Then for all $\varepsilon > 0$, $C > 0$, and $|\omega - \omega_0| \le Ch$ there is $E_\omega \in \Psi^1(\partial\Omega)$ with $\sigma(E_\omega) = \sqrt{|\xi'|_g^2 - \omega_0}$ such that for any $s \ge \frac{1}{2}$, $X \in \Psi^0(\partial M)$ with $\text{WF}_h(X) \subset \{|\xi'|_g > \omega_0\}$, $\delta > 0$, $N > 0$, there is $C_1 > 0$ such that for all $0 < h < 1$,*

$$\|X(h\partial_{\nu_g} u - E_\omega(u|_{\partial M}))\|_{H_h^s(\partial M)} \le C_1(h^{-\frac{1}{2}}\|P(\omega; g, L)u\|_{H_h^{s-\frac{1}{2}}(\partial M_\delta)} + h^N\|u\|_{H_h^1(\partial M_\delta)}),$$

where

$$\partial M_\delta := \{x \in M \,:\, d(x, \partial M) < \delta\}.$$

Notice that Theorem 1.12 implies that $\Lambda_{\mathcal{I}}(z) \in \Psi^1(\partial\Omega)$ with principal symbol

$$\sigma(\Lambda_{\mathcal{I}}(z)) = \rho_{\mathcal{I}}\sqrt{|\xi'|^2_{g_{\mathcal{I}}} + 1} \tag{1.10}$$

and, moreover, for $X \in \Psi^0(\partial\Omega)$ with $\mathrm{WF}_h(X) \subset \{|\xi'|_{g_{\mathcal{O}}} > 1\}$, $X\Lambda_{\mathcal{O}}(z) \in \Psi^1(\partial\Omega)$ with

$$\sigma(X\Lambda_{\mathcal{O}}(z)) = \rho_i \sigma(X)\sqrt{|\xi'|^2_{g_{\mathcal{O}}} - 1}. \tag{1.11}$$

The distinction between (1.6) and (1.7) can be seen from (1.10) and (1.11). Indeed,

$$\sigma(\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}}) = \rho_i \sigma(X)\sqrt{|\xi'|^2_{g_{\mathcal{O}}} - 1} - \tau\rho_{\mathcal{I}}\sqrt{|\xi'|^2_{g_{\mathcal{I}}} + 1}, \qquad |\xi'|_{g_{\mathcal{O}}} > 1,$$

and this symbol does not vanish in the case of (1.6), while it does in the case of (1.7). One can also see why the existence of $(x', \xi') \in T^*\partial\Omega$ such that

$$\tau^2\rho_{\mathcal{I}}^2|\xi'|^2_{g_{\mathcal{I}}} = \rho_{\mathcal{O}}^2|\xi'|^2_{g_{\mathcal{O}}}$$

may cause problems with self-adjointness. Indeed, in this case, the symbol is not uniformly elliptic as $|\xi'| \to \infty$ and hence, standard elliptic regularity results will fail.

In the case of (1.6), the knowledge of the symbol of $\Lambda_{\mathcal{I}}$ and that of $\Lambda_{\mathcal{O}}$ at high frequency is sufficient. However, in the case of (1.7) we need one more subtle piece of information about the Dirichlet-to-Neumann maps.

Theorem 1.13. *Let $M > 0$. Then there is $c > 0$ such that for all $0 < h < 1$, $|1 - z| \leq Mh$, and $u \in H_h^{\frac{1}{2}}(\partial\Omega)$,*

$$-\operatorname{sgn}(\operatorname{Im} z^2)\operatorname{Im}\langle \tau\Lambda_{\mathcal{I}}u, u\rangle_{L^2(\partial\Omega, \mathrm{dvol}_{g_{\mathcal{O}}})} \leq -c|\operatorname{Im} z^2|\|u\|^2_{L^2(\partial\Omega)}.$$

Furthermore, for all $X \in \Psi^{\mathrm{comp}}$ with $\mathrm{WF}_h(X) \subset \{|\xi'|_{g_{\mathcal{O}}} > 1\}$, and $N > 0$, there are $c > 0$ and $C_N > 0$ such that for all $0 < h < 1$, $|1 - z| < Mh$, and $u \in L^2(\partial\Omega)$, we have

$$\operatorname{sgn}(\operatorname{Im} z^2)\operatorname{Im}\langle \Lambda_{\mathcal{O}}u, u\rangle_{L^2(\partial\Omega, \mathrm{dvol}_{g_{\mathcal{O}}})} \leq -(c|\operatorname{Im} z^2| - C_N h^N)\|u\|^2_{L^2(\partial\Omega)}.$$

Provided that $|\operatorname{Im} z| > h^N$ for some N, Theorem 1.13 allows us to obtain estimates where $\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}}$ fails to be elliptic.

In order to finish the proofs of Theorem 1.8 and 1.9, we need to obtain estimates on $|\xi'|_{g_{\mathcal{O}}} \leq 1$. For this, we employ defect measure arguments similar to those in [GMS21, GSW20, Bur02].

The proof of Theorem 1.11 relies on Theorem 1.9 and a contour integration. Let $V(h) := [1-2\varepsilon, 1+2\varepsilon] \times \mathrm{i}[-h, h]$. First, using a complex absorbing potential to reduce to operators of trace class, one can find a compactly microlocalized pseudodifferential operator, X, with $\mathrm{WF}_h(X) \subset \{|\xi'|_{g_{\mathcal{O}}} > 1 + 2\varepsilon\}$ such that

$$\sum_{\lambda_j \in V(h)} \tilde{\psi}(\lambda_j) = \frac{1}{2\pi i}\int_{\partial V(h)} \tilde{\psi}(z)(\Lambda_{\mathcal{O}}(z) - \tau\Lambda_{\mathcal{I}}(z))^{-1}\partial_z(\Lambda_{\mathcal{O}}(z) - \Lambda_{\mathcal{I}}(z))X dz + O(h^\infty),$$

where $\psi \in C_c^\infty((1 - 2\varepsilon, 1 + 2\varepsilon))$ and $\tilde{\psi}$ is an almost analytic extension of ψ.

Then, since on $\mathrm{WF}_h(X)$, $\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}}$ is a pseudodifferential operator with symbol

$$\rho_{\mathcal{O}}\sqrt{|\xi'|^2_{g_{\mathcal{O}}} - z^2} - \tau\rho_{\mathcal{I}}\sqrt{|\xi'|_{g_{\mathcal{I}}} + z^2},$$

we will argue in this sketch as though the whole operator was such a pseudodifferential operator. In particular, we can find E an elliptic pseudodifferential operator such that

$$\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}} = E(B(z) - z^2),$$

where $B(z) \in \Psi_h^2$ with

$$B(z) = B_0 + hB_1(z), \qquad B_i \in \Psi_h^{2-i},$$

and

$$\sigma(B_0) = \frac{\rho_{\mathcal{O}}^2 |\xi'|_{g_{\mathcal{O}}}^2 - \rho_{\mathcal{I}}^2 \tau^2 |\xi'|_{g_{\mathcal{I}}}^2}{\rho_{\mathcal{O}}^2 + \rho_{\mathcal{I}}^2 \tau^2}.$$

Next, we closely follow the proof of the Weyl law for self-adjoint pseudodifferential operators. In particular, for $\mp \operatorname{Im} z > 0$, we write

$$(B - z^2)^{-1} = \frac{i}{h} \int_0^{\pm\infty} U(t) e^{\frac{i}{h} t z^2} dt,$$

where

$$(hD_t - B(z))U(t) = 0, \qquad U(0) = I.$$

Then,

$$\sum_{\lambda_j \in V(h)} \tilde{\psi}(\lambda_j) = \sum_{\pm} \pm \frac{1}{2\pi h} \int_0^{\pm\infty} \int_{\partial V(h)} \tilde{\psi}(z) U(t) e^{-\frac{i}{h} t z^2} E^{-1} \partial_z (\Lambda_{\mathcal{O}}(z) - \Lambda_{\mathcal{I}}(z)) W dz + O(h^\infty),$$

and, integrating by parts in z, we are able to replace the integral to time infinity by a finite integral; i.e. for $\chi \in C_c^\infty(-1,1)$ with $\chi \equiv 1$ near 0, we have

$$\sum_{\lambda_j \in V(h)} \tilde{\psi}(\lambda_j) = \frac{1}{2\pi h} \int_{-\infty}^{\infty} \int_{\partial V(h)} \tilde{\psi}(z) \chi(t) U(t) e^{-\frac{i}{h} t z^2} E^{-1} \partial_z (\Lambda_{\mathcal{O}}(z) - \Lambda_{\mathcal{I}}(z)) W dz + O(h^\infty).$$

At this point we can use an oscillatory integral approximation of $U(t)$ to compute the integrals and then approximate $1_{[1-\varepsilon, 1+\varepsilon]}$ by cutoff functions ψ, thereby finishing the proof of theorem.

1.3. **Structure of the paper.** Section 2 contains a review of some preliminary material including basic notation for semiclassical operators, and defect measures as well as propagation of defect measure results. In Section 3 we reformulate the problem as a scattering problem in the exterior of the obstacle $\Omega_{\mathcal{I}}$ with a non-standard boundary condition. Next, in Section 4, we prove Theorem 1.12 by implementing a factorization scheme for the Laplace-Beltrami operator near the boundary. We apply these methods specifically to $P_{\mathcal{O}} - z^2$ and $P_{\mathcal{I}} - z^2$ and prove Theorem 1.13 in Sections 4.4 and 4.5. In Section 5, we prove Theorems 1.8, 1.9, and 1.10. Finally, in Section 6, we prove Theorem 1.11. Appendix A shows that P is a black-box Hamiltonian.

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2. Preliminaries

2.1. **Semiclassical rescaling and pseudodifferential operators.** In order to prove our estimates, we reformulate our problem in semiclassical language; i.e. let $0 < h < 1$, $z = z(h) \in \mathbb{C}$ with $|1 - z| \leq Mh$ and set $\lambda = h^{-1}z$. We will also need semiclassical Sobolev spaces defined on a Riemannian manifold (M, g), for $k \in \mathbb{N}$ by the norm

$$\|u\|^2_{H^k_h(M)} := \sum_{|\alpha| \leq k} \|(hD)^\alpha u\|^2_{L^2(M)}.$$

We then define H^s_h for $s \geq 0$ by interpolation and H^{-s}_h by duality (Notice that when M has a boundary H^{-s}_h is the space of supported distributions).

We then write $f \in H^s_{h,\mathrm{loc}}(M)$ if for all $\chi \in C^\infty_c(M)$, $\chi f \in H^s_h(M)$. We write $f \in H^s_{h,\mathrm{comp}}$ if $f \in H^s_h(M)$ and f is compactly supported.

We use the language of semiclassical pseudodifferential operators frequently in this paper. We now briefly recall the concepts and notation (see [Zwo12] and[DZ19, Appendix E] for a complete treatment). We will define pseudodifferential operators on $\mathbb{R}^d$, the definitions on manifolds being similar and refer the reader to [DZ19, Appendix E] for the precise definitions on a manifold.

Semiclassical Pseudodifferential Operators on $\mathbb{R}^d$ We say $a \in C^\infty(T^*\mathbb{R}^d)$ is a symbol of order m and write $a \in S^m(T^*\mathbb{R}^d)$ if for all $\alpha, \beta \in \mathbb{N}^d$, there is $C_{\alpha\beta} > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m-|\beta|}, \qquad \langle \xi \rangle := (1 + |\xi|^2)^{1/2}.$$

We then quantize $a \in S^m(T^*\mathbb{R}^d)$ using the quantization

$$[\mathrm{Op}(a)u](x) := \frac{1}{(2\pi h)^d} \int e^{\frac{i}{h}\langle x-y,\xi\rangle} a(x,\xi) u(y) \mathrm{d}y \mathrm{d}\xi,$$

and define the class of semiclassical pseudodifferential operators of order m,

$$\Psi^m_h(\mathbb{R}^d) := \{\mathrm{Op}(a) \,:\, a \in S^m(T^*\mathbb{R}^d)\}.$$

We write $\Psi^\infty_h(\mathbb{R}^d) := \cup_m \Psi^m_h(\mathbb{R}^d)$ and $\Psi^{-\infty}_h(\mathbb{R}^d) := \cap_m \Psi^m_h(\mathbb{R}^d)$.

We next recall a few technical lemmas and definitions from the calculus of semiclassical pseudodifferential operators. The first gives the basic elements of the calculus.

Lemma 2.1 (Theorem 9.5 [Zwo12])**.** *Let $a \in S^{m_1}(\mathbb{R}^d)$ and $b \in S^{m_2}(\mathbb{R}^d)$. Then,*

$$h^{-1}(\mathrm{Op}(a)\,\mathrm{Op}(b) - \mathrm{Op}(ab)) \in \Psi^{m_1+m_2-1}_h(\mathbb{R}^d),$$
$$h^{-1}(\mathrm{Op}(a)^* - \mathrm{Op}(\bar{a})) \in \Psi^{m_1-1}_h(\mathbb{R}^d),$$
$$h^{-2}([\mathrm{Op}(a), \mathrm{Op}(b)] + hi\,\mathrm{Op}(\{a,b\})) \in \Psi^{m_1+m_2-2}_h(\mathbb{R}^d).$$

The next defines the principal symbol.

Lemma 2.2 (Principal Symbol Map, Proposition E.14 [DZ19])**.** *There is a map $\sigma_m : \Psi^m_h(\mathbb{R}^d) \to S^m(T^*\mathbb{R}^d)$ so that*

$$A - \mathrm{Op}(\sigma(A)) \in h\Psi^{m-1}(\mathbb{R}^d).$$

We write $\overline{T^*\mathbb{R}^d} = T^*\mathbb{R}^d \sqcup \mathbb{R}^d \times S^{d-1}$ for the fiber radially compactified cotangent bundle i.e. the cotangent bundle with the sphere at infinity in ξ attached.

We can now define the notion of the elliptic set.

Definition 2.3. *Let $A \in \Psi_h^m(\mathbb{R}^d)$. For $(x_0, \xi_0) \in \overline{T^*M}$, we say that A is elliptic at (x_0, ξ_0) and write $(x_0, \xi_0) \in \operatorname{ell}_h(A)$ if there is a neighborhood, U of (x_0, ξ_0) and $c > 0$ such that*

$$|\sigma(A)(x, \xi)| \geq c\langle \xi \rangle^m, \qquad (x, \xi) \in U.$$

Next, we define the wavefront set of a pseudodifferential operator.

Definition 2.4. *Let $A \in \Psi_h^m(\mathbb{R}^d)$. For $(x_0, \xi_0) \in \overline{T^*M}$ we say that (x_0, ξ_0) is not in the wavefront set of A and write $(x_0, \xi_0) \notin \operatorname{WF}_h(A)$ if there is $a \in S^m$ such that*

$$A = \operatorname{Op}(a) + O(h^\infty)_{\Psi_h^{-\infty}}$$

and $(x_0, \xi_0) \notin \operatorname{supp} a$.

The next lemma gives the so-called elliptic parametrix construction.

Lemma 2.5 (Proposiion E.32 [DZ19])**.** *Suppose that $A \in \Psi_h^{m_1}(\mathbb{R}^d)$ and $B \in \Psi_h^{m_2}(\mathbb{R}^d)$ and $\operatorname{WF}_h(A) \subset \operatorname{ell}_h(B)$. Then there is $E \in \Psi_h^{m_1 - m_2}(\mathbb{R}^d)$ with $\operatorname{WF}_h(E) \subset \operatorname{WF}_h(A)$ such that*

$$A = EB + O(h^\infty)_{\Psi_h^{-\infty}}.$$

The final lemma concerns boundedness of pseudodifferential operators.

Lemma 2.6 (Proposition E.19 [DZ19])**.** *Let $A \in \Psi_h^m(\mathbb{R}^d)$. Then, for all $s \in \mathbb{R}$, there is $C > 0$ such that for all $u \in H_h^{s+m}(\mathbb{R}^d)$ and $0 < h < 1$,*

$$\|Au\|_{H_h^s(\mathbb{R}^d)} \leq C\|u\|_{H_h^{s+m}(\mathbb{R}^d)}.$$

Tangential Pseudodifferential operators We will also have occasion to use tangential pseudodifferential operators on a domain $\Omega \subset \mathbb{R}^d$ with smooth boundary. Once again, we make these definition in local coordinates $\mathbb{R}_{x^1} \times \mathbb{R}^{d-1}_{x'}$, we $\Omega = \{x^1 > 0\}$.

We say that $a \in C^\infty(\mathbb{R} \times T^*\mathbb{R}^{d-1})$ is a tangential symbol of order m and write $a \in S^m_{\mathrm{t}}$ if $a \in C^\infty(\mathbb{R}_{x^1}; S^m(\mathbb{R}^{d-1}))$. We then define the class of tangential pseudodifferential operators of order m by

$$\Psi^m_{\mathrm{t},h} := \{\operatorname{Op}(a) \,:\, a \in S^m_{\tan,h}\}.$$

We also write $\Psi^\infty_{\mathrm{t},h} := \cup_m \Psi^m_{\mathrm{t},h}(\mathbb{R}^d)$ and $\Psi^{-\infty}_{\mathrm{t},h}(\mathbb{R}^d) := \cap_m \Psi^m_{\mathrm{t},h}(\mathbb{R}^d)$.

Notice that for any $A \in \Psi^m_{\mathrm{t},h}$ and $y \in \mathbb{R}$ A can be restricted to an operator on $\{x^1 = y\}$ and this operator lies in $\Psi_h^m(\{x^1 = y\})$.

2.2. **The operator in Fermi Normal Coordinates.** In Fermi normal coordinates $x = (x^1, x')$, where x^1 is the signed distance to the boundary, $\overline{\Omega}$ is given by $x^1 \geq 0$ and the metric is of the form

$$g = (\mathrm{d}x^1)^2 + \sum_{\alpha,\beta=2}^{d} g_{\alpha\beta}(x)\mathrm{d}x^\alpha \mathrm{d}x^\beta.$$

Then,

$$-h^2\Delta_{g,\rho} - z^2 = (hD_{x^1})^2 + ha(x)hD_{x^1} - R(x^1, x', hD_{x'}). \tag{2.1}$$

Here, a is a smooth function given by $a = (\sqrt{|g|}\rho)^{-1}D_{x^1}\sqrt{|g|}\rho$ with $\sqrt{|g|}$ being the Riemannian density function. Moreover, R is a tangential differential operator of order 2. The semiclassical principal symbol of R is given by $\sigma(R) = r(x^1, x', \xi')$ with $r(0, x', \xi') = 1 - |\xi'|_g^2$.

2.3. **Semiclassical defect measures.** Semiclassical defect measures are measures associated with a sequence (possibly subsequence) of functions $\{u(h)\}_{0<h<h_0}$. Some well-known existence theorem of semiclassical defect measures can be found in [DZ19, Appendix E.3] or [Zwo12, Chapter 5]. We will summarise them in the following.

- If $u_j := u(h_j)$ satisfies
$$\|\chi u_j\|_{L^2(\Omega)} \leq C_\chi \tag{2.2}$$
for some constant C_χ depending on $\chi \in C_c^\infty(\overline{\Omega})$ but not j, then there is a subsequence j_n and a non-negative Radon measure μ on $T^*\Omega$ such that
$$\langle \mathrm{Op}_h(a)(x, h_{j_n}D)u_{j_n}, u_{j_n}\rangle \to \int_{T^*\Omega} a(x,\xi)\mathrm{d}\mu \quad \text{for} \quad \forall a \in C_c^\infty(T^*\Omega). \tag{2.3}$$
- If u_j satisfies
$$\|u_j\|_{L^2(\partial\Omega)} \leq C_D, \tag{2.4}$$
then there is a subsequence j_n and a non-negative Radon measure υ_D on $T^*\partial\Omega$ such that
$$\langle \mathrm{Op}_h(b)(x', h_jD')u_j, u_j\rangle \to \int_{T^*\partial\Omega} b(x',\xi')\mathrm{d}\upsilon_D \quad \text{for} \quad \forall b \in C_c^\infty(T^*\partial\Omega).$$
- If u_j satisfies
$$\|hD_\nu u_j\|_{L^2(\partial\Omega)} \leq C_N, \tag{2.5}$$
then there exists a non-negative Radon measure υ_N on $T^*\partial\Omega$ such that
$$\langle \mathrm{Op}_h(b)(x', h_jD')hD_\nu u_j, hD_\nu u_j\rangle \to \int_{T^*\partial\Omega} b(x',\xi')\mathrm{d}\upsilon_N \quad \text{for} \quad \forall b \in C_c^\infty(T^*\partial\Omega).$$
- If u_j satisfies (2.4) and (2.5) then in addition to measures υ_D and υ_N, there exists another Radon measure υ_{DN} on $T^*\partial\Omega$ such that
$$\langle \mathrm{Op}_h(b)(x', h_jD')u_j, hD_\nu u_j\rangle \to \int_{T^*\partial\Omega} b(x',\xi')\mathrm{d}\upsilon_{DN} \quad \text{for} \quad \forall b \in C_c^\infty(T^*\partial\Omega).$$
- Let u_j satisfy
$$\begin{cases} (P_{\mathcal{O}} - z^2)u_j = h_jf_j & \text{in } \Omega, \\ (h_jD_{\nu_{\mathcal{O}}} + \Lambda)u_j = g_j & \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

for some $f_j \in L^2_{\text{comp}}(\Omega)$, $\|f_j\|_{L^2} \le C$ and $g_j \in H_h^{\frac{1}{2}}(\partial\Omega)$, $\|g_j\|_{H_h^{\frac{1}{2}}} \le C$, and $\Lambda \in \Psi_h^1(\partial\Omega)$. Then there exists Radon measures μ_f on $T^*\Omega$ and σ_g on $T^*\partial\Omega$ such that

$$\begin{cases} \langle \mathrm{Op}_h(a)(x, h_j D)u_j, f\rangle \to \int_{T^*\Omega} a(x,\xi)\mathrm{d}\mu_f \quad \text{for} \quad \forall a \in C_c^\infty(T^*\Omega), \\ \langle \mathrm{Op}_h(b)(x', h_j D')u_j, g\rangle \to \int_{T^*\partial\Omega} b(x',\xi')\mathrm{d}\upsilon_g \quad \text{for} \quad \forall b \in C_c^\infty(T^*\partial\Omega). \end{cases}$$

If u_j further satisfies (2.2), then $\mathrm{supp}(\mu) \cap T^*\Omega \subset \Sigma_p := \{p = 0\}$.

Notice that if u_j satisfies (2.4), (2.5) and (2.6) then

$$\upsilon_g = \upsilon_{DN} + \overline{\sigma(\Lambda)}\upsilon_D. \tag{2.7}$$

To obtain relationships between the interior defect measures and boundary defect measures one uses the following integration by parts formula.

Lemma 2.7 (Integration by parts)**.** *Suppose that*

$$P = (hD_{x^1})^2 + ha(x)hD_{x^1} - R(x, hD_{x'})$$

is formally self adjoint with respect to the density $\rho\mathrm{d}x$*. Let* $B = B_0 + B_1 hD_{x^1}$ *with* $B_i \in C^\infty_{\text{comp}}((-2\delta, 2\delta)_{x^1}; \Psi_h^{\ell_i}(\mathbb{R}^{d-1}))$ *for* $i = 1, 2$*. Moreover,* Ω *is defined for* $x^1 > 0$*. Then we have*

$$\begin{aligned} \frac{\mathrm{i}}{h}\langle [P,B]u,u\rangle_{L^2(\Omega)} &= -\frac{2}{h}\,\mathrm{Im}\left(\langle Bu, Pu\rangle_{L^2(\Omega)}\right) + \frac{\mathrm{i}}{h}\langle Pu, (B-B^*)u\rangle_{L^2(\Omega)} \\ &- \Big(\langle B_0 u, hD_{x^1}u\rangle_{L^2(\partial\Omega)} + \langle B_1 hD_{x^1}u, hD_{x^1}u\rangle_{L^2(\partial\Omega)} + \langle B_1 Ru, u\rangle_{L^2(\partial\Omega)} \\ &+\langle hD_{x^1}B_0u,u\rangle_{L^2(\partial\Omega)} + \langle [hD_{x^1}, B_1]hD_{x^1}u, u\rangle_{L^2(\partial\Omega)} + h\langle [a, B_1]hD_{x^1}u,u\rangle_{L^2(\partial\Omega)} + h\langle aB_0u,u\rangle_{L^2(\partial\Omega)}\Big), \end{aligned}$$

where $a = \rho^{-1}D_{x^1}\rho$*.*

Proof. Using the measures $\rho\mathrm{d}x$ in $\{x^1 > 0\} = \Omega$ and $\rho\mathrm{d}x'$ on $\{x^1 = 0\} = \partial\Omega$, from expression (2.1), we have

$$\begin{aligned} \langle PBu, u\rangle_{L^2(\Omega,\rho dx)} &= \langle Bu, Pu\rangle_{L^2(\Omega)} + \mathrm{i}h\left(\langle hD_{x^1}Bu,u\rangle_{L^2(\partial\Omega)} + \langle Bu, hD_{x^1}u\rangle_{L^2(\partial\Omega)}\right), \\ \langle BPu,u\rangle_{L^2(\Omega)} &= \langle Pu, Bu\rangle_{L^2(\Omega)} + \langle Pu, (B^*-B)u\rangle_{L^2(\Omega)} + \mathrm{i}h\langle Pu, B_1^*u\rangle_{L^2(\partial\Omega)}. \end{aligned}$$

One also has

$$\begin{aligned} \langle hD_{x^1}Bu, u\rangle_{L^2(\partial\Omega)} &= \langle (hD_{x^1})^2u, B_1^*u\rangle_{L^2(\partial\Omega)} \\ &\quad + \langle hD_{x^1}B_0u,u\rangle_{L^2(\partial\Omega)} + \langle [hD_{x^1}, B_1]hD_{x^1}u, u\rangle_{L^2(\partial\Omega)}. \end{aligned}$$

Therefore, the boundary contributions of $\langle [P,B]u,u\rangle_{L^2(\Omega)}$ is given by ih multiplying with

$$\begin{aligned}\langle Bu, hD_{x^1}u\rangle_{L^2(\partial\Omega)} + \langle hD_{x^1}Bu,u\rangle_{L^2(\partial\Omega)} - \langle Pu, B_1^* u\rangle_{L^2(\partial\Omega)}\\ = \langle Bu, hD_{x^1}u\rangle_{L^2(\partial\Omega)} + \langle (R - ah^2 D_{x^1})u, B_1^* u\rangle_{L^2(\partial\Omega)}\\ + \langle hD_{x^1}B_0 u,u\rangle_{L^2(\partial\Omega)} + \langle [hD_{x^1}, B_1]hD_{x^1}u,u\rangle_{L^2(\partial\Omega)}\\ = \langle Bu, hD_{x^1}u\rangle_{L^2(\partial\Omega)} + \langle B_1 Ru,u\rangle_{L^2(\partial\Omega)} + \langle hD_{x^1}B_0u,u\rangle_{L^2(\partial\Omega)}\\ + \langle [hD_{x^1}, B_1]hD_{x^1}u,u\rangle_{L^2(\partial\Omega)} + h\langle [a,B_1]hD_{x^1}u,u\rangle_{L^2(\partial\Omega)} + h\langle aB_0u,u\rangle_{L^2(\partial\Omega)}.\end{aligned}$$

This completes the proof. □

When $B_0 = 0$, we have

$$\begin{aligned}\frac{\mathrm{i}}{h}\langle [P, B_1 hD_{x^1}]u,u\rangle_{L^2(\Omega)} =\\ -\frac{2}{h}\operatorname{Im}\left(\langle B_1 hD_{x^1}u, Pu\rangle_{L^2(\Omega)}\right) + \frac{\mathrm{i}}{h}\langle Pu, (B_1 hD_{x^1} - (hD_{x^1})^* B_1^*)u\rangle_{L^2(\Omega)}\\ - \Big(\langle B_1 hD_{x^1}u, hD_{x^1}u\rangle_{L^2(\partial\Omega)} + \langle B_1 Ru,u\rangle_{L^2(\partial\Omega)}\\ + h\langle [a,B_1]hD_{x^1}u,u\rangle_{L^2(\partial\Omega)} + \langle [hD_{x^1},B_1]hD_{x^1}u,u\rangle_{L^2(\partial\Omega)}\Big).\end{aligned}$$

When $B_1 = 0$, we have

$$\begin{aligned}\frac{\mathrm{i}}{h}\langle [P,B_0]u,u\rangle_{L^2(\Omega)} = -\frac{2}{h}\operatorname{Im}\left(\langle B_0u, Pu\rangle_{L^2(\Omega)}\right) + \frac{\mathrm{i}}{h}\langle Pu, (B_0 - B_0^*)u\rangle_{L^2(\Omega)}\\ - \left(\langle B_0 u, hD_{x^1}u\rangle_{L^2(\partial\Omega)} + \langle hD_{x^1}B_0u,u\rangle_{L^2(\partial\Omega)} + h\langle aB_0u,u\rangle_{L^2(\partial\Omega)}\right).\end{aligned}$$

Using Lemma 2.7, one can obtain the results of [Mil00] (see also [GLS24]) on boundary defect measures.

Theorem 2.8. *Let u_j satisfy* (2.2)*,* (2.4)*,* (2.5)*, and* (2.6)*, then* $\operatorname{supp}\mu \subset \{\sigma(P-1) = 0\} \cap T^*\Omega$ *and we have*

$$\begin{aligned}\mu(H_p a) = -2\operatorname{Im}\mu_f(\operatorname{Re} a) - 2\mathrm{i}\operatorname{Im}\mu_f(\operatorname{Im} a)\\ - 2\operatorname{Re}\upsilon_g(a_{\mathrm{even}}) - 2|\sigma(\Lambda)|\operatorname{Im}\upsilon_D(a_{\mathrm{even}}) - \upsilon_N(a_{\mathrm{odd}}) - \upsilon_D(ra_{\mathrm{odd}}), \quad (2.8)\end{aligned}$$

where $a_{\mathrm{even}} = \frac{1}{2}(a(x, r^{\frac12}, \xi') + a(x, -r^{\frac12}, \xi'))$*,* $a_{\mathrm{odd}} = \frac{1}{2r^{\frac12}}(a(x, r^{\frac12}, \xi') - a(x, -r^{\frac12}, \xi'))$ *and* $r = \sigma(R)$ *in* (2.1)*. Let* $\pi : T^*\Omega \to {}^bT^*\Omega$ *define as* $\pi(x^1, x', \xi_1, \xi') = (x^1, x', x^1\xi_1, \xi')$*. If* $\partial\Omega$ *is nowhere tangent to* H_p *to infinite order. Then, for* $q \in C_c^\infty({}^bT^*\Omega; \mathbb{R})$*, we have*

$$\pi_*\mu(q\circ\varphi_t) - \pi_*\mu(q) = \int_0^t \left(-2\operatorname{Im}\pi_*\mu_f - 2\delta_{\partial\Omega}\otimes(\operatorname{Re}\upsilon_{DN}) + \upsilon_{\mathcal{G}}\right)(q\circ\varphi_s)\mathrm{d}s, \quad (2.9)$$

where φ_t *is the bicharacteristic flow and the measure* $\upsilon_{\mathcal{G}}$ *is only supported on the glancing set. Moreover,* $\mu_f = 0$ *if* $f = o(1)$ *and, similarly,* $\operatorname{Re}\upsilon_{DN} = \upsilon_{\mathcal{G}} = 0$ *if* $g = o(1)$*.*

Proof. We will briefly mention the proof. Identity (2.8) follows from identity (2.7), Lemma 2.7 and the fact that $(H_p a)|_{S^*\mathbb{R}^d} = H_p(a|_{S^*\mathbb{R}^d})$. Here, H_p is the Hamiltonian vector field generated by $p = \sigma(P)$. The proof of (2.9) can be found in [GLS24, Section 2]. It is clear that $f = o(1)$

implies $\mu_f = 0$. For $g = o(1)$, we know from (2.7) that $\upsilon_{DN} = -\overline{\sigma(\mathrm{i}\tau\Lambda_{\mathcal{I}})}\upsilon_D$, whose real part is zero as $\overline{\sigma(\mathrm{i}\tau\Lambda_{\mathcal{I}})}$ is purely imaginary (see Proposition 4.20). Finally, $\upsilon_{\mathcal{G}} = 0$ follows immediately from $\operatorname{Re}\upsilon_{DN} = 0$. □

2.4. **An estimate for the Neumann Trace.**

Lemma 2.9. *Suppose that*

$$P = (hD_{x^1})^2 + ha(x)hD_{x^1} - R(x, hD_{x'})$$

is formally self adjoint with respect to the density $\rho \mathrm{d}x$. *Then, for* $s \le \frac{1}{2}$ *and* $\varepsilon > 0$, *there is* $C > 0$ *such that for* $u \in H_h^1$ *with* $Pu \in L^2$,

$$\|hD_{x^1}u|_{x^1=0}\|_{H_h^s(\partial\Omega)} \le Ch^{-1}\|Pu\|_{L^2(0,\varepsilon)} + C\|u\|_{H_h^1(0,\varepsilon)} + C\|u|_{x^1=0}\|_{H_h^{s+1}(\partial\Omega)}.$$

Proof. Let $E \in \Psi_h^s(\partial\Omega)$ elliptic, $\chi \in C_c^\infty([0,\varepsilon))$ with $\chi \equiv 1$ near 0, set $B_1 := \chi(x^1)E$, $B_0 = 0$, $B := E^*E\chi(x^1)hD_{x^1}$. Then, by Lemma 2.7,

$$\begin{aligned}
\|hD_{x^1}u|_{x^1=0}\|^2_{H_h^s(\partial\Omega)} &\le \langle EhD_{x^1}u, EhD_{x^1}u\rangle_{L^2(\partial\Omega)} \\
&= \frac{\mathrm{i}}{h}\langle [P,B]u,u\rangle_{L^2(\Omega)} - \frac{2}{h}\operatorname{Im}\langle Bu, Pu\rangle_{L^2(\Omega)} - \langle ERu, Eu\rangle_{L^2(\partial\Omega)} \\
&\quad + h\langle [a,\chi(x^1)E^*E]hD_{x^1}u,u\rangle_{L^2(\partial\Omega)} \\
&\le C\|u\|^2_{H_h^1(0,\varepsilon)} + Ch^{-1}\|u\|_{H_h^1(0,\varepsilon)}\|Pu\|_{L^2(0,\varepsilon)} + \|u|_{x^1=0}\|^2_{H_h^{s+1}(\partial\Omega)} \\
&\quad + Ch^2\|hD_{x^1}u|_{x^1=0}\|_{H_h^s(\partial\Omega)}\|u|_{x^1=0}\|_{H_h^{s-1}(\partial\Omega)} \\
&\le C\|u\|^2_{H_h^1(0,\varepsilon)} + \delta\|u\|^2_{H_h^1(0,\varepsilon)} + Ch^{-2}\delta^{-1}\|Pu\|_{L^2(0,\varepsilon)} \\
&\quad + C(1+\delta^{-1})\|u|_{x^1=0}\|^2_{H_h^{s+1}(\partial\Omega)} + h^2\delta\|hD_{x^1}u|_{x^1=0}\|^2_{H_h^s(\partial\Omega)}.
\end{aligned}$$

Taking $\delta > 0$ small enough completes the proof of the lemma. □

3. Reformulation of negative index of refraction problem as an exterior problem

In this section, we reformulate the estimates for $R_P(z)$ in terms of an exterior scattering problem. To do this, we first review estimates for more classical resolvents.

3.1. **Review of estimates for the Dirichlet resolvent in $\Omega_{\mathcal{O}}$.**

Since $\Omega_{\mathcal{O}}$ is connected, $P_{\mathcal{O}}$ is self-adjoint with domain $H_0^1 \cap H^2$, and $g_{\mathcal{O}}^{ij}(x) = \delta^{ij}$, $\rho(x) \equiv 1$ for $|x|$ large enough, the theory of black-box scattering [DZ19, Chapter 4] implies that there is a meromorphic family of operators $R_{\mathcal{O}}(z) : L^2_{\mathrm{comp}}(\Omega_{\mathcal{O}}) \to H^2_{h,\mathrm{loc}} \cap H^1_{0,\mathrm{loc}}(\Omega_{\mathcal{O}})$ satisfying

$$(P_{\mathcal{O}} - z^2)R_{\mathcal{O}}(z)f = f \text{ in } \Omega_{\mathcal{O}},\ R_{\mathcal{O}}(z/h)f \text{ is } z/h\text{-outgoing}.$$

Moreover, since $g_{\mathcal{O}}$ is non-trapping on $\Omega_{\mathcal{O}}$, combining [Bur02, Theorem 1.3] with elliptic regularity, we have for any $M > 0$, there is $h_0 > 0$ such that $R_{\mathcal{O}}(z)$ is analytic in, $|1 - z| \le Mh$ and for all $\chi \in C_c^\infty(\overline{\Omega_{\mathcal{O}}})$ there is $C > 0$ such that

$$\|\chi R_{\mathcal{O}}(z)\chi\|_{L^2(\Omega_{\mathcal{O}}) \to H_h^2(\Omega_{\mathcal{O}})} \le Ch^{-1},\ 0 < h < h_0,\ |1 - z| < Mh. \tag{3.1}$$

By Lemma 2.9, this implies and hence also

$$\|h\partial_\nu R_{\mathcal{O}}(z)\chi\|_{L^2(\Omega_{\mathcal{O}})\to H_h^{1/2}(\partial\Omega)} \le Ch^{-1} + \|\chi R_{\mathcal{O}}(z)\chi\|_{L^2(\partial\Omega)\to H_h^1(\Omega_{\mathcal{O}})} \le Ch^{-1}. \tag{3.2}$$

Moreover, letting $E_{\mathcal{O}} : H_h^{3/2}(\partial\Omega) \to H^2_{h,\mathrm{comp}}(\Omega_{\mathcal{O}})$, be an extension operator, the operator $G_{\mathcal{O}}(z) : H_h^{3/2}(\partial\Omega) \to H^2_{h,\mathrm{loc}}(\Omega_{\mathcal{O}})$ defined by

$$G_{\mathcal{O}}(z)v := E_{\mathcal{O}}v - R_{\mathcal{O}}(z)E_{\mathcal{O}}v,$$

is a meromorphic family of operators satisfying

$$(P_{\mathcal{O}} - z^2)G_{\mathcal{O}}(z)v = 0 \text{ in } \Omega_{\mathcal{O}},\ G_{\mathcal{O}}(z)g|_{\partial\Omega} = v,\ G_{\mathcal{O}}(z)v \text{ is } z/h\text{-outgoing}.$$

We now obtain an improved version of [BSW16, Theorem 3.5]. The following proposition is an improvement of [BSW16, Theorem 3.5], where we combine the method used in the proof of [BSW16, Theorem 3.5] and Lemma 4.14.

Proposition 3.1. *Let $G_{\mathcal{O},h}$ be the Dirichlet map for $(P_{\mathcal{O}} - z^2)$ satisfying z/h outgoing condition. Then for all $M > 0$, $\chi \in C_c^\infty(\overline{\Omega_{\mathcal{O}}})$, and $j = 0, 1$ there are $C, h_0 > 0$ such that for $0 < h < h_0$,*

$$\|\chi G_{\mathcal{O},h}(z)\|_{H_h^{\frac{1}{2}+j}(\partial\Omega)\to H_h^{1+j}(\Omega_{\mathcal{O}})} \le C, \quad \text{for} \quad |1 - z| \le Mh, \tag{3.3}$$

and

$$\|h\partial_{\nu_{\mathcal{O}}} G_{\mathcal{O}}(z)\chi\|_{H_h^{\frac{1}{2}+j}(\partial\Omega)\to H_h^{-\frac{1}{2}+j}(\partial\Omega)} \le C. \tag{3.4}$$

Proof. Let $g \in H_h^{1/2}(\partial\Omega)$ and w be a solution to

$$\begin{cases} -h^2\Delta_{g_{\mathcal{O}},\rho_{\mathcal{O}}} w_+ - (1 + \mathrm{i}h)^2 w_+ = 0 & \text{in} \quad \Omega_{\mathcal{O}}, \\ w_+ = g & \text{on} \quad \partial\Omega. \end{cases}$$

Then Green's identity implies

$$\begin{aligned} &h\|w_+\|^2_{L^2(\Omega_{\mathcal{O}})} \le Ch|\langle h\partial_\nu w_+, g\rangle_{\partial\Omega}|, \\ &\|w_+\|^2_{H_h^1(\Omega_{\mathcal{O}})} \le C(\|w_+\|^2_{L^2(\Omega_{\mathcal{O}})} + h|\langle h\partial_\nu w_+, g\rangle_{\partial\Omega}|), \end{aligned} \tag{3.5}$$

where the boundary contribution at infinity is zero since $h > 0$ implies $w_+ \in H^1$.

Applying Lemma 2.9, one has for $s \le \frac{3}{2}$,

$$\|h\partial_{\nu_{\mathcal{O}}} w_+\|_{H_h^{s-1}(\partial\Omega)} \le \|w_+\|_{L^2(\Omega)}\|w_+\|_{H_h^1(\Omega)} + \|g\|_{H_h^s(\partial\Omega)}. \tag{3.6}$$

Hence, using (3.6) with $s = \frac{1}{2}$ and (3.5), we obtain

$$\|w_+\|_{H_h^1(\Omega_{\mathcal{O}})} \le C\varepsilon\|w_+\|_{H_h^1(\Omega_{\mathcal{O}})} + C(\varepsilon + \varepsilon^{-1})\|g\|_{H_h^{\frac{1}{2}}(\partial\Omega)}.$$

In particular, taking $\varepsilon = \frac{1}{2C}$, we have

$$\|w_+\|_{H_h^1(\Omega_{\mathcal{O}})} \le C\|g\|_{H_h^{\frac{1}{2}}(\partial\Omega)}. \tag{3.7}$$

Note also that if $g \in H_h^{3/2}$, then by elliptic regularity for $-h^2\Delta_{g_{\mathcal{O}},\rho_{\mathcal{O}}} + 1$,

$$\|w_+\|_{H_h^2(\Omega_{\mathcal{O}})} \le C(\|w_+\|_{H_h^1(\Omega)} + \|g\|_{H_h^{3/2}(\partial\Omega)}) \le C\|g\|_{H_h^{3/2}(\partial\Omega)}. \tag{3.8}$$

Let $\chi_1, \chi_2 \in C_0^\infty(\overline{\Omega_{\mathcal{O}}})$ $\chi_1 \equiv 1$ near $\partial\Omega$, and $\operatorname{supp}(1-\chi_2) \cap \operatorname{supp}\chi_1 = \emptyset$. Then, notice that

$$\begin{aligned} G_{\mathcal{O}}(z)g &= \chi_1 w_+ - R_{\mathcal{O}}(z)(P_{\mathcal{O}} - z^2)\chi_1 w_+ \\ &= \chi_1 w_+ - R_{\mathcal{O}}(z)\chi_2[-h^2\Delta_{g_{\mathcal{O}},\rho_{\mathcal{O}}}, \chi_1]w_+ - ((1+ih)^2 - z^2)R_{\mathcal{O}}(z)\chi_2\chi_1 w_+. \end{aligned}$$

The estimate (3.3) now follows from (3.7), (3.8), and the estimates

$$\|\chi R_{\mathcal{O}}\chi_2\|_{L^2(\Omega_{\mathcal{O}})\to H_h^2(\Omega_{\mathcal{O}})} \le Ch^{-1}, \qquad \|[-h^2\Delta_{g_{\mathcal{O}},\rho_{\mathcal{O}}}, \chi_1]\|_{H_h^1\to L^2} \le Ch.$$

The estimate (3.4) now follows from Lemma 2.9. □

3.2. **Review of estimates for the Dirichlet resolvent in $\Omega_{\mathcal{I}}$.** Since $P_{\mathcal{I}}$ is self-adjoint with domain $H_0^1(\Omega_{\mathcal{I}}) \cap H^2(\Omega_{\mathcal{I}})$ and $P_{\mathcal{I}} \le Ch$, for $|1-z| \le C_0 h$, there is a holomorphic family of operators $R_{\mathcal{I}}(z) := (P_{\mathcal{I}} - z^2)^{-1} : L^2(\Omega_{\mathcal{I}}) \to H_h^2(\Omega_{\mathcal{I}}) \cap H_0^1(\Omega_{\mathcal{I}})$ satisfying

$$(P_{\mathcal{I}} - z^2)R_{\mathcal{I}}(z)f = f \text{ in } \Omega_{\mathcal{I}}.$$

Moreover, $\|R_{\mathcal{I}}(z)\|_{L^2(\Omega_{\mathcal{I}})\to L^2(\Omega_{\mathcal{I}})} \le C$. Hence by elliptic regularity estimates (see e.g. [McL00, Theorem 4.18]),

$$\|R_{\mathcal{I}}(z)\|_{L^2(\Omega_{\mathcal{I}})\to H_h^2(\Omega_{\mathcal{I}})} \le C \tag{3.9}$$

and by Lemma 2.9

$$\|h\partial_\nu R_{\mathcal{I}}(z)\|_{L^2(\Omega_{\mathcal{I}})\to H_h^{\frac12}(\partial\Omega)} \le C. \tag{3.10}$$

In addition, letting $E_{\mathcal{I}} : H_h^{3/2}(\partial\Omega) \to H_h^2(\Omega_{\mathcal{I}})$ be an extension operator with

$$\|E_{\mathcal{I}}\|_{H_h^{3/2}(\partial\Omega)\to H_h^2(\Omega_{\mathcal{I}})} \le Ch^{1/2},$$

and defining $G_{\mathcal{I}}(z) : H_h^{3/2}(\partial\Omega) \to H_h^2(\Omega_{\mathcal{I}})$ by

$$G_{\mathcal{I}}(z)v := E_{\mathcal{I}}v - R_{\mathcal{I}}(z)E_{\mathcal{I}}v.$$

$G_{\mathcal{I}}(z)$ satisfies

$$(P_{\mathcal{I}} - z^2)G_{\mathcal{I}}(z)g = 0 \text{ in } \Omega_{\mathcal{I}},\ G_{\mathcal{I}}(z)g|_{\partial\Omega} = g,$$

and for any $M > 0$ there is $h_0 > 0$ such that for $0 < h < h_0$, $j = 0, 1$

$$\|G_{\mathcal{I}}(z)g\|_{H_h^{\frac12+j}\to H_h^{1+j}} + \|h\partial_\nu G_{\mathcal{I}}(z)\chi\|_{H_h^{\frac12+j}\to H_h^{-\frac12+j}} \le C, \quad 0 < h < h_0,\ |1-z| \le Mh. \tag{3.11}$$

3.3. **Reformulation of the negative index of refraction problem.** It is convenient to reduce our problem to studying the case of $f \equiv 0$ at the cost of placing inhomogeneous data in the boundary condition. To do this, define $v_{\mathcal{O}} := u_{\mathcal{O}} - R_{\mathcal{O}}(z)h\chi f_{\mathcal{O}}$ and $v_{\mathcal{I}} := u_{\mathcal{I}} - R_{\mathcal{I}}(z)hf_{\mathcal{I}}$.

Then, using (1.8) we obtain

$$\begin{cases} (P_{\mathcal{I}} - z^2)v_{\mathcal{I}} = 0 & \text{in } \Omega_{\mathcal{I}}, \\ (P_{\mathcal{O}} - z^2)v_{\mathcal{O}} = 0 & \text{in } \Omega_{\mathcal{O}}, \\ v_{\mathcal{O}} = v_{\mathcal{I}} & \text{on } \partial\Omega, \\ \rho_{\mathcal{O}}h\partial_{\nu_{\mathcal{O}}}v_{\mathcal{O}} - \tau\rho_{\mathcal{I}}h\partial_{\nu_{\mathcal{I}}}v_{\mathcal{I}} =: g & \text{on } \partial\Omega, \\ v_{\mathcal{O}} \text{ is } z/h\text{-outgoing.} \end{cases} \tag{3.12}$$

Notice that by (3.1),

$$\|\chi(v_{\mathcal{O}} - u_{\mathcal{O}})\|_{H_h^2(\Omega_{\mathcal{O}})} \leq Ch^{-1}\|f_{\mathcal{O}}\|_{L^2}, \tag{3.13}$$

by (3.9)

$$\|v_{\mathcal{I}} - u_{\mathcal{I}}\|_{H_h^2(\Omega_{\mathcal{I}})} \leq C\|f_{\mathcal{I}}\|_{L^2}. \tag{3.14}$$

Next, by (3.2) and (3.10),

$$\|g\|_{H_h^{1/2}} \leq C(h^{-1}\|f_{\mathcal{O}}\|_{L^2} + \|f_{\mathcal{I}}\|_{L^2}). \tag{3.15}$$

Finally, observe that

$$\|v_{\mathcal{I}}\|_{H_h^2} \leq C\|v_{\mathcal{O}}\|_{H_h^{3/2}(\partial\Omega)}. \tag{3.16}$$

Using (3.13), (3.14), (3.15), and (3.16), Theorem 1.8 will follow from the estimate

$$\|\chi v_{\mathcal{O}}\|_{H_h^2(\Omega_{\mathcal{O}})} + \|v_{\mathcal{O}}\|_{H_h^{\frac{3}{2}}(\partial\Omega)} \leq C\|g\|_{H_h^{1/2}(\partial\Omega)}, \quad |\operatorname{Im} z| < Mh, \tag{3.17}$$

and Theorem 1.9 will follow from the estimate

$$\|\chi v_{\mathcal{O}}\|_{H_h^2(\Omega_{\mathcal{O}})} + \|v_{\mathcal{O}}\|_{H_h^{\frac{3}{2}}(\partial\Omega)} \leq C|\operatorname{Im} z|^{-1}\|g\|_{H_h^{1/2}(\partial\Omega)}, \quad -Mh < \operatorname{Im} z < -h^N. \tag{3.18}$$

Since our goal is now to prove (3.17) and (3.18), we can now reduce (3.12) to an exterior scattering problem with a non-standard Robin-type boundary condition. For this, let $\Lambda_{\mathcal{I}}(z) : H_h^s(\partial\Omega) \to H_h^{s-1}(\partial\Omega)$ be the Dirichlet-to-Neumann (DtN) map defined as follows. $\Lambda_{\mathcal{I}}(z)u_0 := h\rho_{\mathcal{I}}\partial_{\nu_{\mathcal{I}}}G_{\mathcal{I}}u_0$, where $\nu_{\mathcal{I}}$ is outward normal with respect to the metric $g_{\mathcal{I}}$. We then rewrite (3.12) as

$$\begin{cases} (P_{\mathcal{O}} - z^2)v_{\mathcal{O}} = 0 & \text{in } \Omega_{\mathcal{O}}, \\ \rho_{\mathcal{O}} h\partial_{\nu_{\mathcal{O}}} v_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}}(z)v_{\mathcal{O}}|_{\partial\Omega} = g & \text{on } \partial\Omega, \\ v_{\mathcal{O}} \text{ is } z/h\text{-outgoing.} \end{cases} \tag{3.19}$$

The proof of Theorems 1.8 and 1.9 will consist of proving estimates on the solution to (3.19). Since we have eliminated $v_{\mathcal{I}}$ from (3.19), we will abuse notation slightly and simply write $v_{\mathcal{O}} = v$ from now on.

4. Microlocal description of the Dirichlet-to-Neumann map

In addition to the DtN map $\Lambda_{\mathcal{I}}(z)$, we will use the outgoing DtN map, $\Lambda_o(z) : H_h^s(\partial\Omega) \to H_h^{s-1}(\partial\Omega)$ defined as $h\rho_{\mathcal{O}}\partial_{\nu_{\mathcal{O}}}G_{\mathcal{O}}(z)$. In this section we provide a full description of $\Lambda_{\mathcal{I}}(z)$ as a pseudodifferential operator and a microlocal description of $\Lambda_o(z)$ on $|\xi'|_{g_{\mathcal{O}}} > 1$. In particular, we prove Theorem 1.12.

In fact, we generalize our situation slightly, defining for $L := \sum_{i=1}^d L^i(x)hD_{x^i}$ and $\omega = \omega_0 + h\omega_1 + o(h^2)$,

$$P(\omega; g, L) := -h^2\Delta_g + hL - \omega. \tag{4.1}$$

Notice that

$$\begin{aligned} P_{\mathcal{O}} - \omega &= -\frac{h^2}{\rho\sqrt{|g|}}\partial_i g_{\mathcal{O}}^{ij}\sqrt{|g|}\rho\partial_j - \omega \\ &= -h^2\Delta_{g_{\mathcal{O}}} - h\rho^{-1}(\partial_i\rho)g_{\mathcal{O}}^{ij}h\partial_j - \omega = P(\omega; g_{\mathcal{O}}, L_{\mathcal{O}}), \end{aligned} \tag{4.2}$$

with

$$L_{\mathcal{O}} := -\rho^{-1}(\partial_i \rho) g_{\mathcal{O}}^{ij} h \partial_j.$$

Similarly, there is $L_{\mathcal{I}}$ such that

$$-P_{\mathcal{I}} + \omega = P(-\omega; g_{\mathcal{I}}, L_{\mathcal{I}}). \tag{4.3}$$

4.1. **Semiclassical Lee-Uhlmann constructions.** To the best of authors' knowledge, the earliest paper showing DtN map as a classical (i.e. non-semiclassical) pseudodifferential operator and providing a way of calculating the full classical symbol expression of DtN map can be found in the work of Sylvester and Uhlmann [SU88]. The method of Sylvester and Uhlmann is based on the study of Calderón projector. A more direct approach to DtN map via factorization modulo smoothing operators can be found in the work of Lee–Uhlmann [LU89], and their method allows one to calculate its full classical asymptotic expansions in a simpler and more intuitive way. In this section, we give a semiclassical version of Lee–Uhlmann approach. While the results in this sections are considered folk-lore knowledge, we were unable to find a reference in the literature.

Remark 4.1. *In the simplest form of factorization problem, say solving the equation*

$$p(x;\lambda) := x^2 - \lambda^2 = 0 \tag{4.4}$$

for some unknown number x, we can factorize $p(x;\lambda)$ as $(x-\lambda)(x+\lambda)$ and obtain $x = \lambda$ or $x = -\lambda$ as solutions to this toy problem. To further determine which solution to be valid, we would need more information about x. Lee–Uhlmann's idea is essentially solving an analogue of (4.4) *for $p(x;\lambda)$ being the Laplace-Beltrami operator, x being an unknown operator with some given classical pseudodifferential operator λ and, furthermore, the right-hand-side of* (4.4) *is replaced by some smoothing operator. To determine which solution of x to be the right candidate boils down to choosing which $\pm\lambda$ that gives the well-posedness of the heat equation. In fact, it is due to the nature that the heat operator $e^{\pm\lambda t}$ is only well-posed in positive time t for $-\lambda$ if we assume $\lambda > 0$ (See* [LU89, Proposition 1.2]*).*

Our approach is essentially a semiclassical version of Lee-Uhlmann's method, i.e. $p(x;\lambda)$, x and λ are now semiclassical pseudodifferential operators. Note that Lemmas 4.5 and 4.6 to be proved in this section imply that depending on the sign of principal symbol of λ, the operator $x-\lambda$ enjoys different microlocal estimates. In other words, the operator $x+\lambda$ and $x-\lambda$ satisfy different microlocal estimates if we fix λ. In this way, we replaces the fulfillment of the well-posedness of the heat equation, as in Lee–Uhlmann's construction, by microlocal estimates. See Remark 4.9 for further details.

The following lemma gives a semiclassical factorisation of the semiclassical Laplace-Beltrami with potentials.

Lemma 4.2. *Let $\omega_0 \in \mathbb{R}$ and suppose that $|\omega_0 - \omega| < Ch$. Then, for $X \in \Psi^0_{\mathrm{t},h}(\Omega)$ such that $\mathrm{WF}_h(X) \subset \{(x,\xi) : \big||\xi'|_g^2 - \omega_0\big| > 0\}$, we have, in the boundary normal coordinates,*

$$XP(\omega; g, L) = X\left(hD_{x^1} + h\tilde{a} - \mathrm{i}E_{\pm}(x, hD_{x'})\right)\left(hD_{x^1} + \mathrm{i}E_{\pm}(x, hD_{x'})\right) + O(h^\infty)_{\Psi^{-\infty}_{\mathrm{t},h}}$$

in the boundary normal coordinates, where $\tilde{a} = a + L^1$, $E_\pm \in \Psi^1_{\mathrm{t},h}(\Omega)$ *and its principal symbols is given by*

$$\sigma(E_\pm) = e_1 := \begin{cases} \pm \mathrm{i}\sqrt{\omega_0 - |\xi'|_g^2} & |\xi'|_g^2 - \omega_0 < 0, \\ -\sqrt{|\xi'|_g^2 - \omega_0} & |\xi'|_g^2 - \omega_0 > 0. \end{cases} \tag{4.5}$$

Proof. Our strategy is to show $E_\pm = \sum_{j\le 1} h^{1-j}\operatorname{Op}_h(e_j)$ for some $e_j \in S^j_{1,0}(T^*\partial\Omega)$. First, set $E_\pm = \operatorname{Op}_h(e_1)$ with e_1 as in (4.5). Then it follows from equation (2.1) and definition of $P(\omega, g, L)$ that

$$X_+\left(hD_{x^1} + h\tilde{a} - \mathrm{i}E_\pm(x, hD_{x'})\right)\left(hD_{x^1} + \mathrm{i}E_\pm(x, hD_{x'})\right) = X_+P(\omega; g, L) + hX_+F_1,$$

where $F_1 \in \Psi^0_{\mathrm{t},h}$. This proves our first induction step. Suppose that we have proved the k-th induction step, i.e.

$$X_+\left(hD_{x^1} + h\tilde{a} - \mathrm{i}E_{+,k}(x, hD_{x'})\right)\left(hD_{x^1} + \mathrm{i}E_{+,k}(x, hD_{x'})\right) = X_+P(\omega; g) + h^kX_+F_k,$$

where $E_{+,k} = \sum_{-(k-1)<j\le 1} h^{1-j}\operatorname{Op}_h(e_j)$ and $F_k \in \Psi^{1-k}_{\mathrm{t},h}$. Then we set $e_{-(k-1)} = -\frac{1}{2}f_k/e_1$ with $f_k = \sigma(F_k)$.

$$\begin{aligned} &X_+\left(hD_{x^1} + h\tilde{a} - \mathrm{i}E_{+,k+1}\right)\left(hD_{x^1} + \mathrm{i}E_{+,k+1}\right) \\ &\quad = X_+\Big(hD_{x^1} + h\tilde{a} - \mathrm{i}E_{+,k} - \mathrm{i}h^k\operatorname{Op}_h(e_{-(k-1)})\Big)\Big(hD_{x^1} + \mathrm{i}E_{+,k} + \mathrm{i}h^k\operatorname{Op}_h(e_{-(k-1)})\Big) \\ &\qquad = X_+\Big(P(\omega; g) + h^kX_+F_k + +h^k(E_{+,k}\operatorname{Op}_h(e_{-(k-1)}) + \operatorname{Op}_h(e_{-(k-1)})E_{+,k}) \\ &\qquad\qquad + \mathrm{i}h^k[hD_{x^1}, \operatorname{Op}_h(e_{-(k-1)})]\Big) + O(h^{k+1})_{\Psi^{-k-1}_{\mathrm{t},h}} = X_+P(\omega; g) + h^{k+1}X_+F_{k+1}, \end{aligned}$$

which proves the $(k+1)$-th induction step and hence completes the proof. □

When $\omega_0 < 0$, Lemma 4.2 gives a full factorization for $P(\omega; g)$.

Corollary 4.3. *Let* $\omega_0 < 0$ *and suppose that* $|\omega - \omega_0| < Ch$. *Consider operator* $P(\omega; g, L)$. *Then* P *is strongly elliptic and we have, in the boundary normal coordinates,*

$$P(\omega; g) = \left(hD_{x^1} + h\tilde{a} - \mathrm{i}E_\pm(x, hD_{x'})\right)\left(hD_{x^1} + \mathrm{i}E_\pm(x, hD_{x'})\right) + O(h^\infty)_{\Psi^{-\infty}_{\mathrm{t},h}}, \tag{4.6}$$

where $E_\pm$ *is a first order semiclassical operator whose principal symbol is chosen to be* $\sigma(E_\pm) = \pm\sqrt{|\xi'|_g^2 - \omega_0}$.

Proof. The proof follows immediately from Proposition 4.2 with $X = I$. □

4.2. **Energy estimates for first order operators.** The first estimate is the following basic energy estimate, which can be applied to $E_\pm$.

Lemma 4.4. *Let* $\Lambda \in \Psi^1_{\mathrm{t},h}(\Omega)$ *with* $\operatorname{Im}\sigma(\Lambda) \le 0$. *Then, for all* $s \in \mathbb{R}$, *and* $t_0 < t_1$, *we have*

$$\|v|_{x^1=t_0}\|_{H^s_h(\partial\Omega)} \le C\left(h^{-1}\|(hD_{x^1} - \Lambda)v\|_{L^2((t_0,t_1);H^s_h(\partial\Omega))} + \|v\|_{L^2((t_0,t_1);H^s_h(\partial\Omega))}\right).$$

Proof. Let $A = \langle hD' \rangle^s$. We work in the boundary normal coordinates as before and start with the derivative of $\|v(x^1, \cdot)\|_{L^2(\mathbb{R}^{d-1})}$ in x^1. Omitting arguments x^1 and $\mathbb{R}^{d-1}$, we have

$$\begin{aligned}\frac{h}{2}\partial_{x^1}\|Av\|_{L^2}^2 &= -\operatorname{Im}\langle hD_{x^1}Av, Av\rangle = -\operatorname{Im}\langle (hD_{x^1} - \Lambda)Av, Av\rangle - \operatorname{Im}\langle \Lambda Av, Av\rangle \\ &= -\operatorname{Im}\langle A(hD_{x^1} - \Lambda)v, Av\rangle - \operatorname{Im}\left\langle \left(\Lambda + [\Lambda, A]A^{-1}\right)Av, Av\right\rangle \\ &\geq -\|A(hD_{x^1} - \Lambda)v\|_{L^2}\|AXv\|_{L^2} - Ch\|Av\|_{L^2}^2 \\ &\geq -h^{-1}\|A(hD_{x^1} + \Lambda)v\|_{L^2}^2 - Ch\|Av\|_{L^2}^2, \quad (4.7)\end{aligned}$$

where Gårding's inequality is used for $\operatorname{Im}\left\langle \left(\Lambda + [\Lambda, A]A^{-1}\right)Av, Av\right\rangle$. In other words,

$$\partial_{x^1}\|v\|_{H_h^s}^2 \geq -C\left(h^{-2}\|(hD_{x^1} + \Lambda)v\|_{H_h^s}^2 + \|v\|_{H_h^s}^2\right).$$

Let $t_- < t_0$ and $\varphi \in C_c^\infty(t_-, t_2)$ with $\varphi \geq 0$ and $\varphi = 1$ in a neighbourhood of $x^1 = t_0$. Then we have

$$\begin{aligned}\|v(t_0)\|_{H_h^s}^2 &= -\int_{t_0}^\infty \partial_s\left(\varphi(s)\|v(s)\|_{H_h^s}^2\right)\mathrm{d}s \\ &\leq C\int_{t_0}^{t_1}\|v(s)\|_{H_h^s}^2\mathrm{d}s - \int_{t_0}^\infty \varphi(s)\partial_s\|v(s)\|_{H_h^s}^2\mathrm{d}s \\ &\leq C\left(h^{-2}\|(hD_{x^1} - \Lambda)v(s)\|_{L^2((t_0,t_1);H_h^s(\partial\Omega))}^2 + \|v(s)\|_{L^2((t_0,t_1);H_h^s(\partial\Omega))}^2\right).\end{aligned}$$

□

Our next estimate allows us to both microlocalize and work in higher regularity than Lemma 4.4. The method for microlocalization was communicated from [GL25].

Lemma 4.5. *Let $\Lambda \in \Psi_{\mathrm{t},h}^1$, $\varepsilon > 0$, $X, \tilde{X} \in \Psi_{\mathrm{t},h}^0(\Omega)$ such that $\mathrm{WF}_h(X) \subset \mathrm{ell}_h(\tilde{X})$ and $\mathrm{WF}_h(X) \subset \{(x,\xi) : \operatorname{Im}\sigma(\Lambda)(x,\xi) < -\varepsilon\langle\xi\rangle\}$. Also, let $s \in \mathbb{R}$ and $0 \leq t_0 < t_1 < t_2$. Then, there exists $h_0, c, C > 0$ such that*

$$\begin{aligned}&\|Xv(t_0)\|_{H_h^s(\partial\Omega)} + ch^{-\frac{1}{2}}\|Xv\|_{L^2\left((t_0,t_1);H_h^{s+\frac{1}{2}}(\partial\Omega)\right)} \\ &\leq Ch^{-\frac{1}{2}}\|\tilde{X}(hD_{x^1} - \Lambda)v\|_{L^2\left((t_0,t_2);H_h^{s-\frac{1}{2}}(\partial\Omega)\right)} + h^N\|v\|_{L^2\left((t_0,t_2);H_h^{-N}(\partial\Omega)\right)} \\ &+ h^N\|(hD_{x^1} - \Lambda)v\|_{L^2\left((t_0,t_2);H_h^{-N}(\partial\Omega)\right)}\end{aligned}$$

for all $0 < h < h_0$.

Proof. Let $A = \langle hD' \rangle^s$. Exactly as in (4.7), one has

$$\frac{h}{2}\partial_{x^1}\|AXv\|_{L^2(\partial\Omega)}^2 = -\operatorname{Im}\langle A(hD_{x^1} - \Lambda)Xv, AXv\rangle - \operatorname{Im}\langle\left(\Lambda + [\Lambda, A]A^{-1}\right)AXv, AXv\rangle. \quad (4.8)$$

By the microlocal Gårding inequality [DZ19, Proposition E.34],

$$\operatorname{Im}\langle \Lambda AXv, AXv\rangle \leq -\varepsilon\|Xv\|_{H_h^{s+\frac{1}{2}}(\partial\Omega)}^2 + h^N\|v\|_{H_h^{-N}(\partial\Omega)}^2,$$

Therefore, plugging into (4.8), we arrive at

$$\frac{h}{2}\partial_{x^1}\|AXv\|^2_{L^2(\partial\Omega)} \geq -\|(hD_{x^1}-\Lambda)Xv\|^2_{H_h^{s-\frac{1}{2}}(\partial\Omega)} + c\|Xv\|^2_{H_h^{s+\frac{1}{2}}(\partial\Omega)} - h^N\|v\|^2_{H_h^{-N}(\partial\Omega)}. \tag{4.9}$$

We now claim that if $t_0 < t_1 < t_2$, and $X' \in \Psi^0_{h,\mathrm{t}}(\Omega)$ with $\mathrm{WF}_h(X) \subset \mathrm{ell}_h(X')$, then,

$$\begin{aligned}
&\|Xv(t_0)\|^2_{H_h^s(\partial\Omega)} + ch^{-1}\|Xv\|^2_{L^2\left((t_0,t_1);H_h^{s+\frac{1}{2}}(\partial\Omega)\right)} \\
&\leq Ch^{-1}\|X(hD_{x^1}-\Lambda)v\|^2_{L^2((t_0,t_2);H_h^{s-\frac{1}{2}}(\partial\Omega))} + C\|X'v\|^2_{L^2((t_0,t_2);H_h^{s-\frac{1}{2}}(\partial\Omega))} \\
&\quad + h^N\|v\|_{L^2((t_0,t_2);H_h^{-N}(\partial\Omega))}.
\end{aligned} \tag{4.10}$$

To prove this, $t_- < t_0$ and $\varphi \in C^\infty((t_-,t_2);[0,1])$ with $\varphi \equiv 1$ on $[t_0,t_1]$. Then by (4.9) with v, we have

$$\begin{aligned}
\|Xv(t_0)\|^2_{H_h^s(\partial\Omega)} &= -\int_{t_0}^{\infty}\partial_\tau\left(\varphi(\tau)\|Xv(\tau)\|^2_{H_h^s(\partial\Omega)}\right)\mathrm{d}\tau \\
&\leq Ch^{-1}\left(\|X(hD_{x^1}-\Lambda)v\|^2_{L^2((t_0,t_2);H_h^{s-\frac{1}{2}}(\partial\Omega))} - c\|Xv\|^2_{L^2((t_0,t_2);H_h^{s+\frac{1}{2}}(\partial\Omega))}\right) \\
&\quad + C\|Xv\|^2_{L^2((t_1,t_2);H_h^s(\partial\Omega))} \\
&\quad + Ch^{-1}\|[hD_{x^1}-\Lambda,X]v\|^2_{L^2((t_0,t_2);H_h^{s-\frac{1}{2}}(\partial\Omega)} + Ch^N\|v\|_{L^2((t_0,t_2);H_h^{-N}(\partial\Omega))},
\end{aligned}$$

which implies (4.10) since $\|Xv\|^2_{L^2((t_1,t_2);H_h^s(\partial\Omega))}$ can be absorbed by $\|Xv\|^2_{L^2((t_0,t_2);H_h^{s+\frac{1}{2}}(\partial\Omega))}$ for sufficiently small h and

$$\begin{aligned}
Ch^{-1}\|[hD_{x^1}-\Lambda,X]v\|^2_{L^2((t_0,t_2);H_h^{s-\frac{1}{2}}(\partial\Omega)} &+ \|Xv\|^2_{L^2((t_0,t_2);H_h^{s-\frac{1}{2}}(\partial\Omega))} \\
&\leq C\|X'v\|^2_{L^2((t_0,t_2);H_h^{s-\frac{1}{2}}(\partial\Omega))} + Ch^N\|v\|_{L^2((t_0,t_2);H_h^{-N}(\partial\Omega))}.
\end{aligned}$$

Now, we will prove by induction that for $t_0 < t_1 < t_2 < t_3$, and $X' \in \Psi^0_{h,\mathrm{t}}(\Omega)$ with $\mathrm{WF}_h(X) \subset \mathrm{ell}_h(X')$,

$$\begin{aligned}
&\|Xv(t_0)\|^2_{H_h^s(\partial\Omega)} + ch^{-1}\|Xv\|^2_{L^2\left((t_0,t_1);H_h^{s+\frac{1}{2}}(\partial\Omega)\right)} \\
&\leq Ch^{-1}\|X'(hD_{x^1}-\Lambda)v\|^2_{L^2((t_0,t_3);H_h^{s-\frac{1}{2}}(\partial\Omega))} + Ch^j\|X'v\|^2_{L^2((t_0,t_2);H_h^{s-j-\frac{1}{2}}(\partial\Omega))} \\
&\quad + h^N\|v\|_{L^2((t_0,t_3);H_h^{-N}(\partial\Omega))} + h^N\|(hD_{x^1}-\Lambda)v\|_{L^2((t_0,t_3);H_h^{-N}(\partial\Omega))}.
\end{aligned} \tag{4.11}$$

By (4.10) we have (4.11) with $j = 0$. Suppose that (4.11) holds for some $j \geq 0$. Let $t_0 < t_1 < t_2' < t_2 < t_3$ and $X'', X' \in \Psi^0_{h,\mathrm{t}}(\Omega)$ with $\mathrm{WF}_h(X) \subset \mathrm{ell}_h(X'')$ and $\mathrm{WF}_h(x'') \subset \mathrm{ell}_h(X')$. Then, by the induction hypothesis (4.11) holds.

$$\begin{aligned}&\|Xv(t_0)\|^2_{H_h^s(\partial\Omega)}+ch^{-1}\|Xv\|^2_{L^2\left((t_0,t_1);H_h^{s+\frac12}(\partial\Omega)\right)}\\&\le Ch^{-1}\|X''(hD_{x^1}-\Lambda)v\|^2_{L^2((t_0,t_3);H_h^{s-\frac12}(\partial\Omega))}+Ch^j\|X''v\|^2_{L^2((t_0,t_2');H_h^{s-j-\frac12}(\partial\Omega))}\\&\quad+h^N\|v\|_{L^2((t_0,t_3);H_h^{-N}(\partial\Omega))}+h^N\|(hD_{x^1}-\Lambda)v\|_{L^2((t_0,t_3);H_h^{-N}(\partial\Omega))}.\end{aligned}\tag{4.12}$$

By (4.10) with s replaced by $s-j$ and (t_0,t_1,t_2) replaced by (t_0,t_2',t_2), and (X,X') replaced by (X'',X'), we have

$$\begin{aligned}&\|X''v\|^2_{L^2((t_0,t_2');H_h^{s-j-\frac12}(\partial\Omega))}\\&\le C\|X''(hD_{x^1}-\Lambda)v\|^2_{L^2((t_0,t_2);H_h^{s-j-\frac12}(\partial\Omega))}+Ch\|X'v\|^2_{L^2((t_0,t_2);H_h^{s-j-\frac12}(\partial\Omega))}\\&\quad+h^N\|v\|_{L^2((t_0,t_2);H_h^{-N}(\partial\Omega))}.\end{aligned}$$

Using this in (4.12) then implies (4.11) with j replaced by $j+1$ and hence completes the proof of the lemma. □

We also need an estimate analogous to Lemma 4.5 for estimating v in the interior.

Lemma 4.6. *Let $\Lambda\in\Psi^1_{\mathrm{t},h}$ $\varepsilon>0$ $X,\tilde X\in\Psi^0_{\mathrm{t},h}(\Omega)$ such that $\mathrm{WF}_h(X)\subset\mathrm{ell}_h(\tilde X)$ and $\mathrm{WF}_h(X)\subset\{(x,\xi)\,:\,\operatorname{Im}\sigma(\Lambda)(x,\xi)>\varepsilon\langle\xi\rangle\}$. Also, let $s\in\mathbb{R}$ and $0\le t_0<t_1<t_2$. Then, there exists $h_0,c,C>0$ such that*

$$\begin{aligned}&\|Xv\|_{L^2\left((t_0,t_1);H_h^{s+\frac12}(\partial\Omega)\right)}\\&\le C\|\tilde X(hD_{x^1}-\Lambda)v\|_{L^2\left((t_0,t_2);H_h^{s-\frac12}(\partial\Omega)\right)}+Ch^{\frac12}\|\tilde Xv(t_0)\|_{H_h^s(\partial\Omega)}\\&\quad+h^N\|v\|_{L^2\left((t_0,t_2);H_h^{-N}(\partial\Omega)\right)}+h^N\|(hD_{x^1}-\Lambda)v\|_{L^2\left((t_0,t_2);H_h^{-N}(\partial\Omega)\right)}\\&\quad+h^N\|v(t_0)\|_{H_h^{-N}(\partial\Omega)},\end{aligned}\tag{4.13}$$

for all $0<h<h_0$.

Proof. Let $A=\langle hD'\rangle^s$. We start again from (4.8) and use the microlocal Gårding inequality [DZ19, Proposition E.34] to obtain

$$\operatorname{Im}\langle\Lambda AXv,AXv\rangle\ge\varepsilon\|Xv\|^2_{H_h^{s+\frac12}(\partial\Omega)}-h^N\|v\|^2_{H_h^{-N}(\partial\Omega)}.$$

Therefore, plugging into (4.8), we arrive at

$$\frac{h}{2}\partial_{x^1}\|AXv\|^2_{L^2(\partial\Omega)}\le\|(hD_{x^1}-\Lambda)Xv\|^2_{H_h^{s-\frac12}(\partial\Omega)}-c\|Xv\|^2_{H_h^{s+\frac12}(\partial\Omega)}+h^N\|v\|^2_{H_h^{-N}(\partial\Omega)}.\tag{4.14}$$

We now claim that if $t_0 < t_1 < t_2$, and $X' \in \Psi^0_{h,\mathrm{t}}(\Omega)$ with $\mathrm{WF}_h(X) \subset \mathrm{ell}_h(X')$, then,

$$\begin{aligned}&\|Xv\|^2_{L^2\left((t_0,t_1);H_h^{s+\frac12}(\partial\Omega)\right)}\\&\leq C\|X(hD_{x^1}-\Lambda)v\|^2_{L^2((t_0,t_2);H_h^{s-\frac12}(\partial\Omega))}+h\|Xv(t_0)\|^2_{H_h^s(\partial\Omega)}\\&\quad+Ch\|X'v\|^2_{L^2((t_0,t_2);H_h^{s-\frac12}(\partial\Omega))}+h^N\|v\|_{L^2((t_0,t_2);H_h^{-N}(\partial\Omega))}.\end{aligned}\tag{4.15}$$

To prove this, let $t_- < t_0$ and $\varphi \in C^\infty((t_-, t_2); [0, 1])$ with $\varphi \equiv 1$ on $[t_0, t_1]$. Then by (4.14) with v, we have

$$\begin{aligned}\|Xv(t_0)\|^2_{H_h^s(\partial\Omega)}&=-\int_{t_0}^{\infty}\partial_\tau\left(\varphi(\tau)\|Xv(\tau)\|^2_{H_h^s(\partial\Omega)}\right)\mathrm{d}\tau\\&\geq -Ch^{-1}\Big(\|X(hD_{x^1}-\Lambda)v\|^2_{L^2((t_0,t_2);H_h^{s-\frac12}(\partial\Omega))}-c\|Xv\|^2_{L^2((t_0,t_2);H_h^{s+\frac12}(\partial\Omega))}\Big)\\&\quad-C\|Xv\|^2_{L^2((t_1,t_2);H_h^s(\partial\Omega))}\\&\quad-Ch^{-1}\|[hD_{x^1}-\Lambda,X]v\|^2_{L^2((t_0,t_2);H_h^{s-\frac12}(\partial\Omega)}-Ch^N\|v\|_{L^2((t_0,t_2);H_h^{-N}(\partial\Omega))},\end{aligned}$$

which implies (4.15) since $\|Xv\|^2_{L^2((t_1,t_2);H_h^s(\partial\Omega))}$ can be absorbed by $\|Xv\|^2_{L^2\left((t_0,t_1);H_h^{s+\frac12}(\partial\Omega)\right)}$ for sufficiently small h and

$$\begin{aligned}&Ch^{-1}\|[hD_{x^1}-\Lambda,X]v\|^2_{L^2((t_0,t_2);H_h^{s-\frac12}(\partial\Omega)}+\|Xv\|^2_{L^2((t_0,t_2);H_h^{s-\frac12}(\partial\Omega))}\\&\qquad\leq C\|X'v\|^2_{L^2((t_0,t_2);H_h^{s-\frac12}(\partial\Omega))}+Ch^N\|v\|_{L^2((t_0,t_2);H_h^{-N}(\partial\Omega))}.\end{aligned}$$

Now, we will prove by induction that for $t_0 < t_1 < t_2 < t_3$, and $X' \in \Psi^0_{h,\mathrm{t}}(\Omega)$ with $\mathrm{WF}_h(X) \subset \mathrm{ell}_h(X')$,

$$\begin{aligned}\|Xv\|^2_{L^2\left((t_0,t_1);H_h^{s+\frac12}(\partial\Omega)\right)}&\leq C\|X'(hD_{x^1}-\Lambda)v\|^2_{L^2((t_0,t_3);H_h^{s-\frac12}(\partial\Omega))}\\&\quad+Ch\|X'v(t_0)\|^2_{H_h^s(\partial\Omega)}+Ch^{j+1}\|X'v\|^2_{L^2((t_0,t_2);H_h^{s-j-\frac12}(\partial\Omega))}\\&\quad+h^N\|v\|_{L^2((t_0,t_3);H_h^{-N}(\partial\Omega))}+h^N\|(hD_{x^1}-\Lambda)v\|_{L^2((t_0,t_3);H_h^{-N}(\partial\Omega))}\\&\quad+h^N\|v(t_0)\|_{H_h^{-N}(\partial\Omega)}.\end{aligned}\tag{4.16}$$

By (4.15) we have (4.16) with $j = 0$. Suppose that (4.16) holds for some $j \geq 0$. Let $t_0 < t_1 < t_2' < t_2 < t_3$ and $X'', X' \in \Psi^0_{h,\mathrm{t}}(\Omega)$ with $\mathrm{WF}_h(X) \subset \mathrm{ell}_h(X'')$ and $\mathrm{WF}_h(X'') \subset \mathrm{ell}_h(X')$. Then, by the induction hypothesis (4.16) holds.

$$\begin{aligned}\|Xv\|^2_{L^2\left((t_0,t_1);H_h^{s+\frac12}(\partial\Omega)\right)} \leq{}& C\|X''(hD_{x^1}-\Lambda)v\|^2_{L^2((t_0,t_3);H_h^{s-\frac12}(\partial\Omega))}\\ &+h\|X''v(t_0)\|^2_{H_h^s(\partial\Omega)}+Ch^{j+1}\|X''v\|^2_{L^2((t_0,t_2');H_h^{s-j-\frac12}(\partial\Omega))}\\ &+h^N\|v\|_{L^2((t_0,t_3);H_h^{-N}(\partial\Omega))}+h^N\|(hD_{x^1}-\Lambda)v\|_{L^2((t_0,t_3);H_h^{-N}(\partial\Omega))}\\ &+h^N\|v(t_0)\|_{H_h^{-N}(\partial\Omega)}.\end{aligned}\tag{4.17}$$

By (4.15) with s replaced by $s-j-1$ and (t_0,t_1,t_2) replaced by (t_0,t_2',t_2), and (X,X') replaced by (X'',X'), we have

$$\begin{aligned}&\|X''v\|^2_{L^2((t_0,t_2');H_h^{s-j-\frac12}(\partial\Omega))}\\ &\leq C\|X''(hD_{x^1}-\Lambda)v\|^2_{L^2((t_0,t_2);H_h^{s-j-\frac32}(\partial\Omega))}+h\|X''v(t_0)\|^2_{H_h^{s-j-1}(\partial\Omega)}\\ &\qquad+Ch\|X'v\|^2_{L^2((t_0,t_2);H_h^{s-j-\frac32}(\partial\Omega))}+h^N\|v\|_{L^2((t_0,t_2);H_h^{-N}(\partial\Omega))}.\end{aligned}$$

Using this in (4.17) then implies (4.16) with j replaced by $j+1$ and hence completes the proof of the lemma. □

4.3. **Estimates for the operator** P**.** In this section, we use the factorisation, Lemma 4.2 together with the estimates from the previous subsection to prove estimates on solutions to $Pu=f$.

Let $E_\pm$ be the factorisation operators of P defined in Lemma 4.2. We have the following microlocal estimates.

Lemma 4.7. *Let $X,\tilde X\in\Psi^0_{\mathrm{t},h}(\Omega)$ such that $\mathrm{WF}_h(X)\subset\mathrm{ell}_h(\tilde X)$, then for $t_0<t_1$, we have*

$$\begin{aligned}&\|X(hD_{x^1}+\mathrm{i}E_\pm)u|_{x^1=t_0}\|_{H_h^s(\partial\Omega)}\\ &\quad\leq C\left(h^{-1}\|XPu\|_{L^2((t_0,t_1);H_h^s(\partial\Omega))}+\|\tilde Xu\|_{H_h^1((t_0,t_1);H_h^s(\partial\Omega))}\right)+h^N\|\tilde Xu\|_{L^2\left((t_0,t_1);H_h^{-N}(\partial\Omega)\right)}.\end{aligned}$$

Proof. First note that

$$\begin{aligned}&(hD_{x^1}+ha-\mathrm{i}E_\pm)X(hD_{x^1}+\mathrm{i}E_\pm)u\\ &\qquad=XPu+([hD_{x^1}+ha-\mathrm{i}E_\pm,X])(hD_{x^1}+\mathrm{i}E_\pm)u+O(h^\infty)_{\Psi^{-\infty}_{\mathrm{t},h}}(\tilde Xu).\end{aligned}\tag{4.18}$$

Let $\Lambda=\mathrm{i}E_\pm-ha$, which satisfies the criteria in Lemma 4.4. Then by setting $v=X(hD_{x^1}+\mathrm{i}E_\pm)u$, we have

$$\|v|_{x^1=t_0}\|_{H_h^s(\partial\Omega)}\leq C\left(h^{-1}\|(hD_{x^1}-\Lambda)v\|_{L^2((t_0,t_1);H_h^s(\partial\Omega))}+\|v\|_{L^2((t_0,t_1);H_h^s(\partial\Omega))}\right).$$

That is

$$
\begin{aligned}
&\|X(hD_{x^1}+\mathrm{i}E_\pm)u|_{x^1=t_0}\|_{H^s_h(\partial\Omega)}\\
&\le C\Big(h^{-1}\|XPu\|_{L^2((t_0,t_1);H^s_h(\partial\Omega))}+h^{-1}\|[hD_{x^1}+ha-\mathrm{i}E_\pm,X](hD_{x^1}+\mathrm{i}E_\pm)u\|_{L^2((t_0,t_1);H^s_h(\partial\Omega))}\\
&\qquad+\|X(hD_{x^1}+\mathrm{i}E_\pm)u\|_{L^2((t_0,t_1);H^s_h(\partial\Omega))}\Big)+h^N\|\tilde{X}u\|_{L^2\left((t_0,t_1);H^{-N}_h(\partial\Omega)\right)},
\end{aligned}
$$

which proves the lemma. □

Lemma 4.8. *Let $\varepsilon>0$, $X,\tilde{X}\in\Psi^0_{\mathrm{t},h}(\Omega)$ such that $\mathrm{WF}_h(X)\subset\mathrm{ell}_h(\tilde{X})\subset\mathrm{WF}_h(\tilde{X})\subset\{(x,\xi):\mathrm{Re}\,\sigma(E_-)(x,\xi)<-\varepsilon\}$. Also, let $s\in\mathbb{R}$ and $t_0<t_1<t_2$. Then, there exists $h_0,c,C>0$ such that*

$$
\begin{aligned}
&\|X(hD_{x^1}+\mathrm{i}E_-)u(t_0)\|_{H^s_h(\partial\Omega)}+ch^{-\frac12}\|X(hD_{x^1}+\mathrm{i}E_-)u\|_{L^2\left((t_0,t_1);H^{s+\frac12}_h(\partial\Omega)\right)}\\
&\le Ch^{-\frac12}\|\tilde{X}Pu\|_{L^2((t_0,t_2);H^{s-\frac12}_h(\partial\Omega))}\\
&\quad+Ch^N\|u\|_{H^1_h((t_0,t_2);H^{-N}_h(\partial\Omega))}+Ch^N\|Pu\|_{L^2((t_0,t_2);H^{-N}_h(\partial\Omega))}
\end{aligned}
\tag{4.19}
$$

for all $0<h<h_0$. If $\mathrm{Re}\,\sigma(E_-)(x,\xi')<0$ on $[t_0,t_2]\times T^*\partial\Omega$, then we have $X=\tilde{X}=I$ and a better estimate

$$
\begin{aligned}
&\|(hD_{x^1}+\mathrm{i}E_-)u(t_0)\|_{H^s_h(\partial\Omega)}+ch^{-\frac12}\|(hD_{x^1}+\mathrm{i}E_-)u\|_{L^2\left((t_0,t_1);H^{s+\frac12}_h(\partial\Omega)\right)}\\
&\le Ch^{-\frac12}\|Pu\|_{L^2((t_0,t_2);H^{s-\frac12}_h(\partial\Omega))}+Ch^N\|u\|_{L^2((t_0,t_2);H^{-N}_h(\partial\Omega))}
\end{aligned}
\tag{4.20}
$$

for all $0<h<h_0$.

Remark 4.9. *Note that Lemma 4.7 says that both E_+ and E_- are good approximation to $h\partial_{x^1}$ if we allow some H^1_h-error of u. Lemma 4.8 says E_- is a better approximation than E_+ as the H^1_h-error of u is reduced to h^∞ small for E_-.*

An immediate corollary of Lemma 4.8 is the following elliptic estimate.

Corollary 4.10. *Let $\omega_0\in\mathbb{R}$ and suppose that $|\omega_0-\omega|<Ch$ and P be given as in (4.1), Then, there exists $h_0,C,\omega'>0$ such that for $0<h<h_0$,*

$$
\begin{aligned}
\|hD_{x^1}u(t_0)\|_{H^s_h(\partial\Omega)}&\le Ch^{-\frac12}\|(P+\omega')u\|_{L^2((t_0,t_2);H^{s-\frac12}_h(\partial\Omega))}\\
&\quad+\|u(t_0)\|_{H^{s+1}_h(\partial\Omega)}+Ch^N\|u\|_{L^2((t_0,t_2);H^{-N}_h(\partial\Omega))},
\end{aligned}
\tag{4.21}
$$

and the following estimate holds,

$$
\begin{aligned}
\|hD_{x^1}u\|_{L^2\left((t_0,t_1);H^{s+\frac12}_h(\partial\Omega)\right)}&\le C\|(P+\omega')u\|_{L^2((t_0,t_2);H^{s-\frac12}_h(\partial\Omega))}\\
&\quad+\|u\|_{L^2\left((t_0,t_1);H^{s+\frac32}_h(\partial\Omega)\right)}+Ch^N\|u\|_{L^2((t_0,t_2);H^{-N}_h(\partial\Omega))}.
\end{aligned}
$$

Proof of Corollary 4.10. We only need to choose ω' large enough such that $P+\omega'$ makes

$$\operatorname{Re}\sigma(E_-)(x,\xi')<0 \quad \text{on} \quad [t_0,t_2]\times T^*\partial\Omega,$$

then estimate (4.20) holds for $E_-(\omega')$ and $P+\omega'$. $\square$

Proof of Lemma 4.8. Let $t_0<t_1<t_2'<t_2$. We consider the E_- factorisation of (4.18) in Lemma 4.7. In other words, we set $\Lambda=\mathrm{i}E_- - ha$ and substitute $v=(hD_{x^1}+\mathrm{i}E_-)u$. Let $X'\in\Psi^0_{h,\mathrm{t}}(\Omega)$ with $\mathrm{WF}_h(X)\subset \mathrm{ell}_h(X_1)$ and $\mathrm{WF}_h(X_1)\subset\mathrm{ell}_h(\tilde X)$. Then replacing $(X,\tilde X)$ by (X,X_1) in Lemma 4.5, we obtain

$$\begin{aligned}
&\|X(hD_{x^1}+\mathrm{i}E_-)u(t_0)\|_{H_h^s(\partial\Omega)}+ch^{-\frac12}\|X(hD_{x^1}+\mathrm{i}E_-)u\|_{L^2\left((t_0,t_1);H_h^{s+\frac12}(\partial\Omega)\right)}\\
&\le Ch^{-\frac12}\|X_1Pu\|_{L^2((t_0,t_2');H_h^{s-\frac12}(\partial\Omega))}+Ch^N\|X_1(hD_{x^1}+\mathrm{i}E_-)u\|_{L^2((t_0,t_2');H_h^{-N}(\partial\Omega))}\\
&\quad + Ch^N\|Pu\|_{L^2((t_0,t_2');H_h^{-N}(\partial\Omega))}.
\end{aligned}\tag{4.22}$$

To obtain (4.19) we simply estimate

$$\begin{aligned}
&h^{-\frac12}\|X_1Pu\|_{L^2((t_0,t_2');H_h^{s-\frac12}(\partial\Omega))}+Ch^N\|X_1(hD_{x^1}+\mathrm{i}E_-)u\|_{L^2((t_0,t_2');H_h^{-N}(\partial\Omega))}\\
&\le Ch^{-\frac12}\|\tilde XPu\|_{L^2((t_0,t_2');H_h^{s-\frac12}(\partial\Omega))}+Ch^N\|\tilde Xu\|_{H_h^1((t_0,t_2');H_h^{-N}(\partial\Omega))}\\
&\quad + Ch^N\|Pu\|_{L^2((t_0,t_2');H_h^{-N}(\partial\Omega))}.
\end{aligned}$$

Now, we are left with the case $\operatorname{Re}\sigma(E_-)(x,\xi')<0$ on $T^*\Omega$. Note that by setting $X=X_1=I$ in (4.22), one has

$$\begin{aligned}
&\|(hD_{x^1}+\mathrm{i}E_-)u(t_0)\|_{H_h^s(\partial\Omega)}+ch^{-\frac12}\|(hD_{x^1}+\mathrm{i}E_-)u\|_{L^2\left((t_0,t_1);H_h^{s+\frac12}(\partial\Omega)\right)}\\
&\qquad\le Ch^{-\frac12}\|Pu\|_{L^2((t_0,t_2');H_h^{s-\frac12}(\partial\Omega))}+h^N\|u\|_{L^2((t_0,t_2');H_h^{-N}(\partial\Omega))}\\
&\qquad\quad + h^N\|(hD_{x^1}+\mathrm{i}E_-)u\|_{L^2((t_0,t_2');H_h^{-N}(\partial\Omega))}.
\end{aligned}\tag{4.23}$$

Since $\operatorname{Re}\sigma(E_-)(x,\xi')<0$ on $[t_0,t_2]\times T^*\partial\Omega$ implies P is elliptic in the neighbourhood of $[t_1,t_2']\times\partial\Omega$, this means

$$\|(hD_{x^1}+\mathrm{i}E_-)u\|_{H_h^1((t_1,t_2');H_h^{-N}(\partial\Omega))}\le C\|Pu\|_{L^2((t_0,t_2);H_h^{-N}(\partial\Omega))}+\|u\|_{L^2((t_0,t_2);H_h^{-N}(\partial\Omega))},$$

which can be applied to (4.23) to complete the proof. $\square$

The application of Lemma 4.6 to E_+ is given by the following.

Lemma 4.11. *Let* $\varepsilon > 0$*,* $X, \tilde{X} \in \Psi^0_{\mathrm{t},h}(\Omega)$ *such that* $\mathrm{WF}_h(X) \subset \mathrm{ell}_h(\tilde{X}) \subset \mathrm{WF}_h(\tilde{X}) \subset \{(x,\xi) : \operatorname{Re}\sigma(E_+)(x,\xi) > \varepsilon\}$*. Also, let* $s \in \mathbb{R}$ *and* $t_0 < t_1 < t_2$*. Then, there exists* $h_0, c, C > 0$ *such that*

$$\begin{aligned}
&\|X(hD_{x^1} + \mathrm{i}E_+)u\|_{L^2\left((t_0,t_1);H_h^{s+\frac{1}{2}}(\partial\Omega)\right)} \\
&\leq C\|\tilde{X}Pu\|_{L^2\left((t_0,t_2);H_h^{s-\frac{1}{2}}(\partial\Omega)\right)} + Ch^{\frac{1}{2}}\|\tilde{X}(hD_{x^1} + \mathrm{i}E_+)u(t_0)\|_{H_h^s(\partial\Omega)} \\
&\quad + h^N\|(hD_{x^1} + \mathrm{i}E_+)u\|_{L^2\left((t_0,t_2);H_h^{-N}(\partial\Omega)\right)} + h^N\|Pu\|_{L^2\left((t_0,t_2);H_h^{-N}(\partial\Omega)\right)} \\
&\quad + h^N\|(hD_{x^1} + \mathrm{i}E_+)u(t_0)\|_{H_h^{-N}(\partial\Omega)}.
\end{aligned} \tag{4.24}$$

for all $0 < h < h_0$*. If* $\operatorname{Re}\sigma(E_+)(x,\xi') > 0$ *on* $[t_0,t_2] \times T^*\partial\Omega$*, then we have* $X = \tilde{X} = I$ *and a better estimate*

$$\begin{aligned}
&\|(hD_{x^1} + \mathrm{i}E_+)u\|_{L^2\left((t_0,t_1);H_h^{s+\frac{1}{2}}(\partial\Omega)\right)} \\
&\qquad \leq C\|Pu\|_{L^2\left((t_0,t_2);H_h^{s-\frac{1}{2}}(\partial\Omega)\right)} + Ch^{\frac{1}{2}}\|(hD_{x^1} + \mathrm{i}E_+)u(t_0)\|_{H_h^s(\partial\Omega)}
\end{aligned} \tag{4.25}$$

for all $0 < h < h_0$*.*

Proof. By setting $\Lambda = \mathrm{i}E_+ - ha$ and $v = (hD_{x^1} + \mathrm{i}E_+)u$ in equation (4.13), which gives equation (4.24). Equation (4.25) follows immediately from equation (4.24). □

Combining Lemma 4.8 and Lemma 4.11 yields the following estimate.

Lemma 4.12. *Let* $\varepsilon > 0$*,* $B \in \Psi^1_h(\partial\Omega)$*,* $X, \tilde{X} \in \Psi^0_{\mathrm{t},h}(\Omega)$ *such that* $\mathrm{WF}_h(X) \subset \mathrm{ell}_h(\tilde{X}) \subset \mathrm{WF}_h(\tilde{X}) \subset \{(x,\xi) : \operatorname{Re}\sigma(E_-)(x,\xi) < -\varepsilon\}$*. Also, let* $s \in \mathbb{R}$ *and* $t_0 < t_1 < t_2$*. If* $\operatorname{Re}\sigma(E_-)(x,\xi) < -\varepsilon$*, then there exists* $h_0, C > 0$ *such that*

$$\begin{aligned}
&\|Xu\|_{H_h^1\left((t_0,t_1);H_h^s(\partial\Omega)\right)} + \|Xu\|_{L^2\left((t_0,t_1);H_h^{s+1}(\partial\Omega)\right)} \\
&\leq C\|\tilde{X}Pu\|_{L^2((t_0,t_2);H_h^{s-1}(\partial\Omega))} + Ch^{\frac{1}{2}}\|\tilde{X}u(t_0)\|_{H_h^{s+\frac{1}{2}}(\partial\Omega)} \\
&\quad + Ch^N\|u\|_{H_h^1((t_0,t_2);H_h^{-N}(\partial\Omega))} + Ch^N\|Pu\|_{L^2((t_0,t_2);H_h^{-N}(\partial\Omega))} + Ch^N\|u(t_0)\|_{H_h^{-N}(\partial\Omega)}
\end{aligned} \tag{4.26}$$

for all $0 < h < h_0$*.*

Proof. Let $\mathrm{WF}_h(X) \subset \mathrm{ell}_h(X') \subset \mathrm{WF}_h(X') \subset \mathrm{ell}_h(\tilde{X}) \subset \mathrm{WF}_h(\tilde{X})$. Combining (4.19) and (4.24) yields

$$\begin{aligned}
&h^{\frac{1}{2}}\|X(hD_{x^1} + \mathrm{i}E_-)u(t_0)\|_{H_h^s(\partial\Omega)} + c\|X(hD_{x^1} + \mathrm{i}E_-)u\|_{L^2\left((t_0,t_1);H_h^{s+\frac{1}{2}}(\partial\Omega)\right)} \\
&\leq C\|X'Pu\|_{L^2((t_0,t_2);H_h^{s-\frac{1}{2}}(\partial\Omega))} \\
&\qquad + Ch^N\|u\|_{H_h^1((t_0,t_2);H_h^{-N}(\partial\Omega))} + Ch^N\|Pu\|_{L^2((t_0,t_2);H_h^{-N}(\partial\Omega))}
\end{aligned}$$

and

$$\begin{aligned}
&\|X(hD_{x^1}+\mathrm{i}E_+)u\|_{L^2\left((t_0,t_1);H_h^{s+\frac12}(\partial\Omega)\right)}\\
&\le C\|X'Pu\|_{L^2\left((t_0,t_2);H_h^{s-\frac12}(\partial\Omega)\right)}+Ch^{\frac12}\|X'(hD_{x^1}+\mathrm{i}E_+)u(t_0)\|_{H_h^s(\partial\Omega)}\\
&\qquad+h^N\|(hD_{x^1}+\mathrm{i}E_+)u\|_{L^2\left((t_0,t_2);H_h^{-N}(\partial\Omega)\right)}+h^N\|Pu\|_{L^2\left((t_0,t_2);H_h^{-N}(\partial\Omega)\right)}\\
&\qquad+h^N\|(hD_{x^1}+\mathrm{i}E_+)u(t_0)\|_{H_h^{-N}(\partial\Omega)}.
\end{aligned}$$

With parallelogram law on Hilbert spaces, we have

$$\begin{aligned}
&\|XhD_{x^1}u\|_{L^2\left((t_0,t_1);H_h^{s+\frac12}(\partial\Omega)\right)}+\|X(E_+-E_-)u\|_{L^2\left((t_0,t_1);H_h^{s+\frac12}(\partial\Omega)\right)}\\
&\le C\|X'Pu\|_{L^2((t_0,t_2);H_h^{s-\frac12}(\partial\Omega))}+Ch^{\frac12}\|X'(hD_{x^1}+\mathrm{i}E_-)u(t_0)\|_{H_h^s(\partial\Omega)}\\
&\quad+Ch^{\frac12}\|X'(E_+-E_-)u(t_0)\|_{H_h^s(\partial\Omega)}+Ch^N\|u\|_{H_h^1((t_0,t_2);H_h^{-N}(\partial\Omega))}\\
&\quad+Ch^N\|Pu\|_{L^2((t_0,t_2);H_h^{-N}(\partial\Omega))}+Ch^N\|(hD_{x^1}+\mathrm{i}E_+)u(t_0)\|_{H_h^{-N}(\partial\Omega)}.
\end{aligned}$$

This shows

$$\begin{aligned}
&\|hD_{x^1}Xu\|_{L^2\left((t_0,t_1);H_h^{s+\frac12}(\partial\Omega)\right)}+\|Xu\|_{L^2\left((t_0,t_1);H_h^{s+\frac32}(\partial\Omega)\right)}\\
&\le C\|\tilde XPu\|_{L^2((t_0,t_2);H_h^{s-\frac12}(\partial\Omega))}+Ch^{\frac12}\|\tilde Xu(t_0)\|_{H_h^{s+1}(\partial\Omega)}\\
&\quad+Ch^N\|u\|_{H_h^1((t_0,t_2);H_h^{-N}(\partial\Omega))}+Ch^N\|Pu\|_{L^2((t_0,t_2);H_h^{-N}(\partial\Omega))}\\
&\quad+Ch^N\|hD_{x^1}u(t_0)\|_{H_h^{-N}(\partial\Omega)}+Ch^N\|u(t_0)\|_{H_h^{-N}(\partial\Omega)}+h\|\tilde Xu\|_{L^2\left((t_0,t_1);H_h^{s+\frac12}(\partial\Omega)\right)}.
\end{aligned}$$

A similar bootstrap argument as used in the proof of Lemma 4.5 will allow us to absorb the term $h\|\tilde Xu\|_{L^2\left((t_0,t_1);H_h^{s+\frac12}(\partial\Omega)\right)}$. On the other hand, estimate (4.21) allows us to kill the term $Ch^N\|hD_{x^1}u(t_0)\|_{H_h^{-N}(\partial\Omega)}$. This proves the Lemma by replacing s with $s-\frac12$. □

A simple elliptic parametrix estimate then yields

Lemma 4.13. *Let $\varepsilon>0$, $B\in\Psi_h^1(\partial\Omega)$, $X,\tilde X\in\Psi_{\mathrm{t},h}^0(\Omega)$ such that $\mathrm{WF}_h(X)\subset\mathrm{ell}_h(\tilde X)\subset\mathrm{WF}_h(\tilde X)\subset\{(x,\xi):\operatorname{Re}\sigma(E)(x,\xi)<-\varepsilon\}\cap\{(x,\xi):|\sigma(B)-\mathrm{i}\sigma(E)|(x,\xi)>0\}$. Also, let $s\in\mathbb R$ and $t_0<t_1<t_2$. Then, there exists $h_0,C>0$ such that*

$$\begin{aligned}
&\|Xu\|_{L^2\left((t_0,t_1);H_h^{s+1}(\partial\Omega)\right)}\\
&\le C\|\tilde XPu\|_{L^2\left((t_0,t_2);H_h^{s-1}(\partial\Omega)\right)}+Ch^{\frac12}\|\tilde X(hD_{x^1}+B)u(t_0)\|_{H_h^{s-\frac12}(\partial\Omega)}\\
&\qquad+Ch^N\|u\|_{H_h^1\left((t_0,t_2);H_h^{-N}(\partial\Omega)\right)}+Ch^N\|Pu\|_{L^2((t_0,t_2);H_h^{-N}(\partial\Omega))}+Ch^N\|u(t_0)\|_{H_h^{-N}(\partial\Omega)}
\end{aligned}\tag{4.27}$$

for all $0<h<h_0$.

Proof. Microlocal elliptic estimate says

$$\begin{aligned}\|Xu(t_0)\|_{H_h^{s+\frac{1}{2}}(\partial\Omega)} &\le C\|X(B-\mathrm{i}E_-)u(t_0)\|_{H_h^{s-\frac{1}{2}}(\partial\Omega)} \\ &\quad + Ch\|X'u(t_0)\|_{H_h^{s-\frac{1}{2}}(\partial\Omega)} + Ch^N\|u(t_0)\|_{H_h^{-N}(\partial\Omega)} \\ &\le C\|X(hD_{x^1}+B)u(t_0)\|_{H_h^{s-\frac{1}{2}}(\partial\Omega)} + C\|X(hD_{x^1}+\mathrm{i}E_-)u(t_0)\|_{H_h^{s-\frac{1}{2}}(\partial\Omega)} \\ &\quad + Ch\|\tilde{X}u(t_0)\|_{H_h^{s-\frac{1}{2}}(\partial\Omega)} + Ch^N\|u(t_0)\|_{H_h^{-N}(\partial\Omega)} \\ &\le Ch^{-\frac{1}{2}}\|\tilde{X}Pu\|_{L^2((t_0,t_2);H_h^{s-1}(\partial\Omega))} + C\|X(hD_{x^1}+B)u(t_0)\|_{H_h^{s-\frac{1}{2}}(\partial\Omega)} \\ &\quad + Ch^N\|u\|_{H_h^1((t_0,t_2);H_h^{-N}(\partial\Omega))} + Ch^N\|Pu\|_{L^2((t_0,t_2);H_h^{-N}(\partial\Omega))} \\ &\quad + Ch\|\tilde{X}u(t_0)\|_{H_h^{s-\frac{1}{2}}(\partial\Omega)} + Ch^N\|u(t_0)\|_{H_h^{-N}(\partial\Omega)}.\end{aligned}$$

Using the bootstrapping argument as in Section 4.2, we can replace $Ch\|\tilde{X}u(t_0)\|_{H_h^{s-\frac{1}{2}}(\partial\Omega)}$ by $Ch^N\|u(t_0)\|_{H_h^{-N}(\partial\Omega)}$ and estimate (4.27) follows immediately from (4.26). ☐

Finally, we can combine the above estimates to obtain higher regularity in the normal variable.

Lemma 4.14. *Let $\varepsilon > 0$, $B \in \Psi_h^1(\partial\Omega)$, $X, \tilde{X} \in \Psi_{\mathrm{t},h}^0(\Omega)$ such that $\mathrm{WF}_h(X) \subset \mathrm{ell}_h(\tilde{X}) \subset \mathrm{WF}_h(\tilde{X}) \subset \{(x,\xi) : \mathrm{Re}\,\sigma(E)(x,\xi) < -\varepsilon\} \cap \{(x,\xi) : |\sigma(B) - \mathrm{i}\sigma(E)|(x,\xi) > 0\}$. Also, let $k \in \mathbb{N}$, $k \ge 2$, $s \in \mathbb{R}$ and $t_0 < t_1 < t_2$. Then, there exists $h_0, C > 0$ such that*

$$\begin{aligned}&\|Xu\|_{H_h^k\left((t_0,t_1);H_h^s\right)} + \|Xu\|_{H_h^{k-1}\left((t_0,t_1);H_h^{s+1}\right)} + \|Xu\|_{H_h^{k-2}\left((t_0,t_1);H_h^{s+2}\right)} \\ &\quad \le C\sum_{j=0}^{k-2}\|\tilde{X}Pu\|_{H_h^{k-2-j}((t_0,t_2);H_h^{s+j})} + Ch^{\frac{1}{2}}\|\tilde{X}u(t_0)\|_{H_h^{s+k-\frac{1}{2}}(\partial\Omega)} \\ &\quad + Ch^N\|u\|_{H_h^1\left((t_0,t_2);H_h^{-N}(\partial\Omega)\right)} + Ch^N\|Pu\|_{L^2\left((t_0,t_2);H_h^{-N}(\partial\Omega)\right)} + Ch^N\|u(t_0)\|_{H_h^{-N}(\partial\Omega)}\end{aligned} \tag{4.28}$$

and

$$\begin{aligned}&\|Xu\|_{H_h^k((t_0,t_1);H_h^s)} + \|Xu\|_{H_h^{k-1}((t_0,t_1);H_h^{s+1})} + \|Xu\|_{H_h^{k-2}((t_0,t_1);H_h^{s+2})} \\ &\quad \le C\sum_{j=0}^{k-2}\|\tilde{X}Pu\|_{H_h^{k-2-j}((t_0,t_2);H_h^{s+j})} + Ch^{\frac{1}{2}}\|\tilde{X}(hD_{x^1}+B)u(t_0)\|_{H_h^{s+k-\frac{3}{2}}(\partial\Omega)} \\ &\quad + Ch^N\|u\|_{H_h^1((t_0,t_2);H_h^{-N}(\partial\Omega))} + Ch^N\|Pu\|_{L^2((t_0,t_2);H_h^{-N}(\partial\Omega))} + Ch^N\|u(t_0)\|_{H_h^{-N}}\end{aligned} \tag{4.29}$$

for all $0 < h < h_0$.

Proof. First note that for $k \ge 2$

$$\begin{aligned}(hD_{x^1})^kX &= (hD_{x^1})^{k-2}\left(X(hD_{x^1})^2 + 2[hD_{x^1},X] + [hD_{x^1},[hD_{x^1},X]]\right) \\ &= (hD_{x^1})^{k-2}\left(X(P-A_2) + 2[hD_{x^1},X] + [hD_{x^1},[hD_{x^1},X]]\right)\end{aligned}$$

where $A_2 = ha(x)hD_{x^1} - R(x^1, x', hD_{x'})$ and $R \in \Psi^2_{\mathrm{t},h}(\Omega)$ as defined in (2.1). Therefore

$$\begin{aligned}
&\|(hD_{x^1})^k Xu\|_{L^2\left((t_0,t_1);H_h^s\right)} \\
&\quad\le \|(hD_{x^1})^{k-2} XPu\|_{L^2\left((t_0,t_1);H_h^s\right)} + \|(hD_{x^1})^{k-2} Xu\|_{L^2\left((t_0,t_1);H_h^{s+2}\right)} \\
&\quad+ Ch\|\tilde{X}u\|_{H_h^{k-2}\left((t_0,t_1);H_h^{s+2}\right)} + Ch^N\|u\|_{H_h^{k-2}\left((t_0,t_1);H_h^{-N}\right)}
\end{aligned} \tag{4.30}$$

implies

$$\begin{aligned}
&\|(hD_{x^1})^2 Xu\|_{L^2\left((t_0,t_1);H_h^s\right)} \\
&\quad\le \|XPu\|_{L^2\left((t_0,t_1);H_h^s\right)} + \|Xu\|_{L^2\left((t_0,t_1);H_h^{s+2}\right)} \\
&\quad+ Ch\|\tilde{X}u\|_{L^2\left((t_0,t_1);H_h^{s+2}\right)} + Ch^N\|u\|_{L^2\left((t_0,t_1);H_h^{-N}\right)}.
\end{aligned}$$

Then, together with Lemma 4.12, we obtain (4.28) for the case $k = 2$ and all $s \in \mathbb{R}$. Now, suppose (4.28) holds for k and all $s \in \mathbb{R}$. Then (4.30) says

$$\begin{aligned}
&\|(hD_{x^1})^{k+1} Xu\|_{L^2\left((t_0,t_1);H_h^s\right)} \\
&\quad\le \|XPu\|_{H_h^{k-1}\left((t_0,t_1);H_h^s\right)} + \|Xu\|_{H_h^{k-1}\left((t_0,t_1);H_h^{s+2}\right)} \\
&\quad+ Ch\|\tilde{X}u\|_{H_h^{k-1}\left((t_0,t_1);H_h^{s+2}\right)} + Ch^N\|u\|_{H_h^{k-1}\left((t_0,t_1);H_h^{-N}\right)}.
\end{aligned}$$

Moreover, (4.28) with s replacing by $s+1$ says

$$\begin{aligned}
&\|Xu\|_{H_h^k\left((t_0,t_1);H_h^{s+1}\right)} + \|Xu\|_{H_h^{k-1}\left((t_0,t_1);H_h^{s+2}\right)} \\
&\quad\le C\sum_{j=0}^{k-2}\|\tilde{X}Pu\|_{H_h^{k-2-j}((t_0,t_2);H_h^{s+1+j})} + Ch^{\frac12}\|\tilde{X}u(t_0)\|_{H_h^{s+k+\frac12}(\partial\Omega)} \\
&\quad+ Ch^N\|u\|_{H_h^1\left((t_0,t_2);H_h^{-N}(\partial\Omega)\right)} + Ch^N\|Pu\|_{L^2\left((t_0,t_2);H_h^{-N}(\partial\Omega)\right)} + Ch^N\|u(t_0)\|_{H_h^{-N}}.
\end{aligned}$$

This implies

$$\begin{aligned}
&\|Xu\|_{H_h^{k+1}\left((t_0,t_1);H_h^s\right)} + \|Xu\|_{H_h^k\left((t_0,t_1);H_h^{s+1}\right)} + \|Xu\|_{H_h^{k-1}\left((t_0,t_1);H_h^{s+2}\right)} \\
&\quad\le C\sum_{j=0}^{(k+1)-2}\|\tilde{X}Pu\|_{H_h^{(k+1)-2-j}((t_0,t_2);H_h^{s+j})} + Ch^{\frac12}\|\tilde{X}u(t_0)\|_{H_h^{s+(k+1)-\frac12}(\partial\Omega)} \\
&\quad+ Ch^N\|u\|_{H_h^1\left((t_0,t_2);H_h^{-N}(\partial\Omega)\right)} + Ch^N\|Pu\|_{L^2\left((t_0,t_2);H_h^{-N}(\partial\Omega)\right)} + Ch^N\|u(t_0)\|_{H_h^{-N}},
\end{aligned}$$

which completes the proof for (4.28) and (4.29) follows similarly. □

Remark 4.15. *The classical interior elliptic estimate follows immediately from Lemma 4.14. That is, if P is a classical second-order elliptic operator, then for any $V \Subset U$, we have*

$$\|u\|_{H_h^2(V)} \le C(\|Pu\|_{L^2(U)} + \|u\|_{L^2(U)}).$$

4.4. **Exterior problem.** In this section, we apply the results of the previous sections to $P_{\mathcal{O}} - z^2$.

Lemma 4.16. *Let $\varepsilon_0 > 0$, $M > 0$, $X, X_2 \in \Psi^{\mathrm{comp}}_{\mathrm{t},h}(\partial\Omega)$ with $\mathrm{WF}_h(X_2) \subset \{|\xi'|_{g_{\mathcal{O}}} > 1 + \varepsilon_0\}$ and $\mathrm{WF}_h(X) \cap \mathrm{WF}_h(I - X_2) = \emptyset$. Let U be a Fermi normal coordinate neighborhood of $\partial\Omega$ in $\Omega_{\mathcal{O}}$ with coordinates (x^1, x'), E_- as in Proposition 4.2. Then for all $\chi \in C^\infty_c(-1, 1)$ with $\chi \equiv 1$ near 0, $\varepsilon > 0$ small enough, all $k, N \geq 0$, and $\psi \in C^\infty_c(\overline{\Omega}_{\mathcal{O}})$, there is $C > 0$ such that for all $0 < h < 1$, $z \in [1 - \varepsilon_0, 1 + \varepsilon_0] + \mathrm{i}[-Mh, Mh]$ and $u \in L^2(\partial\Omega)$ we have*

$$\|\psi\partial_z^k(\chi(\varepsilon^{-1}x^1)v - G_{\mathcal{O}}Xu)\|_{H^2_h(\Omega_{\mathcal{O}})} \leq Ch^N\|u\|_{L^2(\partial\Omega)},$$

where v satisfies

$$(hD_{x^1} - \Lambda)v = 0, \qquad v|_{x^1} = Xu, \qquad \Lambda := -\mathrm{i}(E_-X_2 - (I - X_2)\,\mathrm{Op}(\langle\xi'\rangle)). \tag{4.31}$$

Moreover, for $X' \in \Psi^{\mathrm{comp}}_{\mathrm{t},h}$ with $\mathrm{WF}_h(I - X') \cap \mathrm{WF}_h(X) = \emptyset$,

$$\|(I - X')\partial_z^k\chi v\|_{H^N_h} \leq C_Nh^N\|u\|_{L^2(\partial\Omega)}.$$

Proof. Let $\varepsilon > 0$ small enough so that $\{d(\partial\Omega, x) < 3\varepsilon\} \subset U$. We will require a few microlocal cutoffs below. Let $X_j \in \Psi^{\mathrm{comp}}_{\mathrm{t},h}(\Omega)$, $j = 0, 1, \ldots, 4$ such that $X_0 = X$,

$$\begin{aligned} &\mathrm{WF}_h(X_j) \subset \{|\xi'|_{g_{\mathcal{O}}} - 1 - \varepsilon_0 > 0\} \quad \text{for} \quad j = 0, 1, \ldots, 4, \\ &\mathrm{WF}_h(X_j) \cap \mathrm{WF}_h(I - X_{j'}) = \emptyset \qquad \text{for} \quad j < j'. \end{aligned}$$

Notice that, with $\Lambda := -\mathrm{i}(E_-X_2 - (I - X_2)\,\mathrm{Op}(\langle\xi'\rangle))$, we have $\mathrm{Im}\,\sigma(\Lambda) > c_0\langle\xi'\rangle > 0$. Define

$$\tilde{v}(x^1, x') := e^{\frac{c_0x^1}{2h}}v(x^1, x'),$$

so that $\tilde{v}$ satisfies

$$(hD_{x^1} - (\Lambda - \mathrm{i}\tfrac{c_0}{2}))\tilde{v} = 0, \qquad \tilde{v}|_{x^1=0} = Xu. \tag{4.32}$$

We claim that for any $X_4 \in \Psi^{\mathrm{comp}}_{\mathrm{t},h}$ with $\mathrm{WF}_h(I - X_4) \cap \mathrm{WF}_h(X_2) = \emptyset$, any $k \geq 0$, and any $0 < \varepsilon' < 3\varepsilon$,

$$\|\partial_z^k\tilde{v}\|_{L^2((0,3\varepsilon);H^N_h(\partial\Omega))} \leq Ch^{\frac{1}{2}}\|Xu\|_{L^2(\partial\Omega)}, \tag{4.33}$$

$$\|(I - X_4)\partial_z^k\tilde{v}\|_{H^N_h((0,\varepsilon')\times\partial\Omega)} \leq C_Nh^N\|u\|_{L^2(\partial\Omega)}. \tag{4.34}$$

For $k = 0$, using that $\mathrm{Im}\,\sigma(\Lambda - \mathrm{i}\frac{c_0}{2}) > \frac{c_0}{2}\langle\xi'\rangle$, we have by Lemma 4.6 with X replaced by I that,

$$\|\tilde{v}\|_{L^2\left((0,3\varepsilon);H^s_h(\partial\Omega)\right)} \leq Ch^{\frac{1}{2}}\|Xu\|_{L^2(\partial\Omega)}. \tag{4.35}$$

Using Lemma 4.6 again, this time with X and $\tilde{X}$ replaced by $(I - X_3)$ and $(I - X_1)$ respectively, we obtain that

$$\begin{aligned} &\|(I - X_3)\tilde{v}\|_{L^2\left((0,\varepsilon');H^s_h(\partial\Omega)\right)} \\ &\qquad \leq C_Nh^N(\|\tilde{v}\|_{L^2\left((0,3\varepsilon);H^{-N}_h(\partial\Omega)\right)} + \|u\|_{L^2(\partial\Omega)}) \leq C_Nh^N\|u\|_{L^2(\partial\Omega)}. \end{aligned} \tag{4.36}$$

Hence, using (4.32) again, many times,

$$\begin{aligned}&\|(I-X_4)(hD_{x^1})^k\tilde{v}\|_{L^2\left((0,\varepsilon');H_h^{N-k}(\partial\Omega)\right)}\\&=\|(I-X_4)(\Lambda-\mathrm{i}\tfrac{c_0}{2})^k\tilde{v}\|_{L^2\left((0,\varepsilon');H_h^{N-k}(\partial\Omega)\right)}\\&\le\|(\Lambda-\mathrm{i}\tfrac{c_0}{2})^k(I-X_4)\tilde{v}\|_{L^2\left((0,\varepsilon');H_h^{N-k}(\partial\Omega)\right)}+Ch\|(I-X_3)\tilde{v}\|_{L^2\left((0,\varepsilon');H_h^{N-1}(\partial\Omega)\right)}.\end{aligned}$$

That is

$$\begin{aligned}&\|(I-X_4)\tilde{v}\|_{H_h^k\left((0,\varepsilon');H_h^{N-k}(\partial\Omega)\right)}\\&\quad\le\|(I-X_4)\tilde{v}\|_{L^2\left((0,\varepsilon');H_h^{N}(\partial\Omega)\right)}+Ch\|(I-X_3)\tilde{v}\|_{L^2\left((0,\varepsilon');H_h^{N-1}(\partial\Omega)\right)}.\end{aligned}$$

Combining with (4.36), one has

$$\|(I-X_4)\tilde{v}\|_{H_h^N((0,\varepsilon')\times\partial\Omega)}\le C_Nh^N\|u\|_{L^2(\partial\Omega)}. \tag{4.37}$$

Now, suppose that (4.33) and (4.37) hold for $0\le k\le K-1$. Then, observe that

$$(hD_{x^1}-(\Lambda+\mathrm{i}\tfrac{c_0}{2}))\partial_z^K\tilde{v}=\sum_{0\le j\le K-1}A_j(z)\partial_z^j\tilde{v},\qquad \partial_z^K\tilde{v}|_{x^1=0}=0,$$

where $A_j(z)\in\Psi_{\mathrm{t},h}^{\mathrm{comp}}$ can be computed from derivatives of Λ in z. Applying Lemma 4.6 as in (4.35) and (4.36), we obtain respectively

$$\|\partial_z^K\tilde{v}\|_{L^2((0,3\varepsilon);H_h^s)}\le C\sum_{j\le K-1}\|\partial_z^j\tilde{v}\|_{L^2((0,3\varepsilon);H_h^{s-\frac12})}\le Ch^{\frac12}\|Xu\|_{L^2(\partial\Omega)},$$

and $\varepsilon'<\varepsilon''<3\varepsilon$,

$$\begin{aligned}\|(I-X_4)\partial_z^K\tilde{v}\|_{L^2\left((0,\varepsilon');H_h^s(\partial\Omega)\right)}&\le C\sum_{0\le j\le K-1}\|(I-X_3)A_j\partial_z^j\tilde{v}\|_{L^2\left((0,\varepsilon'');H_h^{s-1}(\partial\Omega)\right)}\\&\quad+C_Nh^N\left(\|\partial_z^K\tilde{v}\|_{L^2\left((0,\varepsilon'');H_h^{-N}(\partial\Omega)\right)}+\sum_{0\le j\le K-1}\|A_j\partial_z^j\tilde{v}\|_{L^2\left((0,\varepsilon'');H_h^{-N}(\partial\Omega)\right)}\right)\\&\le C_Nh^N\|u\|_{L^2(\partial\Omega)}.\end{aligned}$$

Hence, arguing as we did to obtain (4.37), we have that (4.33) and (4.34) hold for all K.

Now that we have (4.33) and (4.34), we finish the proof of the lemma by understanding $\partial_z^K(P_{\mathcal{O}}-z^2)\chi(x^1)v$. Observe that

$$\begin{aligned}&\partial_z^K(P_{\mathcal{O}}-z^2)\chi(x^1)v\\&=\partial_z^KX_4(hD_{x^1}-h\tilde{a}-\mathrm{i}E_-(x,hD_{x'}))(hD_{x^1}+\mathrm{i}E_-(x,hD_{x'}))\chi(x^1)v\\&\quad+\partial_z^K(I-X_4)(P_{\mathcal{O}}-z^2)\chi(x^1)v+O(h^\infty)_{\Psi_{\mathrm{t},h}^{-\infty}}\chi(x^1)v\\&=\partial_z^KX_4(hD_{x^1}-h\tilde{a}-\mathrm{i}E_-(x,hD_{x'}))\chi(x^1)(hD_{x^1}-\Lambda)v\\&\quad-\mathrm{i}hX_4\partial_z^K(hD_{x^1}-h\tilde{a}-\mathrm{i}E_-(x,hD_{x'}))\chi'(x^1)v\\&\quad+(I-X_4)\partial_z^K(P_{\mathcal{O}}-z^2)\chi(x^1)(I-X_3)v+O(h^\infty)_{\Psi_{\mathrm{t},h}^{-\infty}}\chi(x^1)v.\end{aligned}\tag{4.38}$$

By (4.34), we have

$$\|(I-X_3)\partial_z^K v\|_{H_h^N((0,2\varepsilon)\times\partial\Omega)} = \left\| e^{-\frac{c_0 x^1}{2h}}(I-X_3)\partial_z^K \tilde{v}\right\|_{H_h^N((0,2\varepsilon)\times\partial\Omega)} \le C_N h^N \|u\|_{L^2(\partial\Omega)}, \tag{4.39}$$

by (4.31) we have for any j, and $B_j \in \Psi^{\mathrm{comp}}_{\mathrm{t},h}$,

$$\begin{aligned} &\|X_4(hD_{x^1}-B_j))\chi'\partial_z^j v\|_{L^2(\Omega_{\mathcal{O}})} \\ &\quad\le \|X_4\chi' hD_{x^1}\partial_z^j v\|_{L^2(\Omega_{\mathcal{O}})} + h\|X_4\chi''\partial_z^j v\|_{L^2(\Omega_{\mathcal{O}})} + \|\partial_z^j v\|_{L^2\left((\varepsilon,2\varepsilon);H_h^1(\partial\Omega)\right)} \\ &\quad\le C\sum_{k=0}^{j}\|\partial_z^k v\|_{L^2\left((\varepsilon,2\varepsilon);H_h^1(\partial\Omega)\right)} \le C\left\| e^{-\frac{c_0x^1}{2h}}\partial_z^j\tilde{v}\right\|_{L^2\left((\varepsilon,2\varepsilon);H_h^1(\partial\Omega)\right)} \\ &\quad\le Ce^{-\frac{c}{h}}\|u\|_{L^2(\partial\Omega)}, \end{aligned} \tag{4.40}$$

and by (4.35), we have

$$\|\partial_z^j v\|_{L^2\left((0,3\varepsilon);H_h^s(\partial\Omega)\right)} = \left\| e^{-\frac{c_0x^1}{2h}}\partial_z^j\tilde{v}\right\|_{L^2\left((0,3\varepsilon);H_h^s(\partial\Omega)\right)} \le Ch^{1/2}\|u\|_{L^2(\partial\Omega)}. \tag{4.41}$$

Hence, using (4.39), (4.40), and (4.41) in (4.38), we obtain

$$\|\partial_z^K(P_{\mathcal{O}}-z^2)\chi(x^1)v\|_{L^2(\Omega_{\mathcal{O}})} \le C_N h^N\|u\|_{L^2(\partial\Omega)}. \tag{4.42}$$

Taking $K=0$, using that $G_{\mathcal{O}}(z)Xu$ is outgiong, we first obtain for any $\psi\in C_c^\infty(\overline{\Omega}_{\mathcal{O}})$,

$$\|\psi(G_{\mathcal{O}}(z)Xu-\chi(x^1)v)\|_{H_h^2(\Omega_{\mathcal{O}})} = \|\psi R_{\mathcal{O}}(z)(P_{\mathcal{O}}-z^2)\chi(x^1)v\|_{H_h^2(\Omega_{\mathcal{O}})} \le C_N h^N\|u\|_{L^2(\partial\Omega)}.$$

Now, suppose by induction that for $0\le j\le K-1$, and any $\psi\in C_c^\infty(\overline{\Omega}_{\mathcal{O}})$,

$$\|\psi\partial_z^j(G_{\mathcal{O}}(z)Xu-\chi(x^1)v)\|_{H_h^2(\Omega_{\mathcal{O}})} \le C_N h^N\|u\|_{L^2(\partial\Omega)}.$$

Then, observe that $\partial_z^K(G_{\mathcal{O}}(z)Xu-\chi v)|_{\partial\Omega}=0$, and

$$[(P_{\mathcal{O}}-z^2)\partial_z^K(G_{\mathcal{O}}(z)Xu-\chi v)] = \sum_{j=0}^{K-1}(a_j+b_jz)\partial_z^j(G_{\mathcal{O}}(z)Xu-\chi v) - \partial_z^K[(P_{\mathcal{O}}-z^2)\chi v],$$

where $a_j, b_j\in\mathbb{C}$. By (4.42) and the inductive hypothesis, we have for any $\tilde{\psi}\in C_c^\infty(\overline{\Omega}_{\mathcal{O}})$,

$$\|\tilde{\psi}[(P_{\mathcal{O}}-z^2)\partial_z^K(G_{\mathcal{O}}(z)Xu-\chi v)]\|_{L^2} \le C_N h^N\|u\|_{L^2(\partial\Omega)}.$$

Hence, since $\partial_z^K G(z)Xu$ is outgoing, that for any $\psi\in C_c^\infty(\overline{\Omega}_i)$ with $\operatorname{supp}\psi\cap\operatorname{supp}(1-\tilde{\psi})=\emptyset$,

$$\|\psi\partial_z^K(G_{\mathcal{O}}(z)Xu-\chi v)]\|_{H_h^2(\Omega_{\mathcal{O}})} \le C_N h^N\|u\|_{L^2(\partial\Omega)}.$$

□

Proposition 4.17. *Let $\varepsilon_0\ge 0$, $M>0$, $X\in\Psi_h^0(\partial\Omega)$ with $\mathrm{WF}_h(X)\in\{(x,\xi):|\xi'|_{g_{\mathcal{O}}}>1+\varepsilon_0\}$. Then for all $z\in[1-\varepsilon_0,1+\varepsilon_0]+\mathrm{i}[-Mh,Mh]$, we have $X\Lambda_o(z)\in\Psi_h^1(\partial\Omega)$ and*

$$\sigma(X\Lambda_{\mathcal{O}}(z)) = \sigma(X)\rho_{\mathcal{O}}\left(|\xi'|^2_{g_{\mathcal{O}}}-(\operatorname{Re} z)^2\right)^{\frac{1}{2}}.$$

Moreover, for $X_c \in \Psi_h^{\mathrm{comp}}(\partial\Omega)$ *with* $\mathrm{WF}_h(X_c) \subset \{|\xi'|_{g_{\mathcal{O}}} > 1+\varepsilon_0\}$*, and* $k \geq 0$*,* $X_c \partial_z^k \Lambda_{\mathcal{O}} \in \Psi_h^{\mathrm{comp}}$ *with symbol*

$$\sigma(X_c \partial_z^k \Lambda_{\mathcal{O}}) = \sigma(X_c)\rho_{\mathcal{O}} \partial_z^k \left(|\xi'|^2_{g_{\mathcal{O}}} - z^2\right)^{\frac{1}{2}}. \tag{4.43}$$

Proof. In this case, we recall (4.2). In particular, by (4.19) and Lemma 4.14

$$\begin{aligned}&\|(X(hD_{x^1} + \mathrm{i}E_-)G_{\mathcal{O}}(z)u_0)(0)\|_{H_h^s(\partial\Omega)}\\ &\quad \leq Ch^N \|G_{\mathcal{O}}(z)u_0\|_{H_h^1((t_0,t_2);H_h^{-N}(\partial\Omega))} \leq Ch^N\|u_0\|_{H_h^{-N}(\partial\Omega)},\end{aligned}$$

where we have used the non-trapping estimate of $\|\chi G_{\mathcal{O}}\|_{H_h^{-N}(\partial\Omega)\to H_h^2(\Omega_{\mathcal{O}})} \leq Ch^{-1}$. Hence, since $\Lambda_{\mathcal{O}}(z)u_0 = -\rho_{\mathcal{O}} h\partial_{x^1} u|_{x^1=0} = -\mathrm{i}\rho_{\mathcal{O}} hD_{x^1}u|_{x^1=0}$

$$X\Lambda_o(z)u_0 = (-X\rho_{\mathcal{O}}E_- + O(h^\infty)_{\Psi^{-\infty}})\, u_0,$$

and the first statement follows since $\sigma(E_-) = -\sqrt{|\xi'|^2_{g_{\mathcal{O}}} - (\mathrm{Re}\, z)^2}$. The second statement follows directly from Lemma 4.16.

□

Lemma 4.18. *Let* $\varepsilon_0 > 0$ *and* $M > 0$*. Then for* $X \in \Psi_h^{\mathrm{comp}}(\partial\Omega)$*, with* $\mathrm{WF}_h(X) \subset \{|\xi'|^2_{g_{\mathcal{O}}} > 1+\varepsilon_0\}$ *and* $z \in [1-\varepsilon_0, 1+\varepsilon_0] + \mathrm{i}[-Mh, Mh]$*, we have*

$$-\operatorname{sgn}(\operatorname{Im} z^2)\operatorname{Im}\langle \Lambda_{\mathcal{O}}(z)Xu, Xu\rangle_{L^2(\partial\Omega, \mathrm{dvol}_{g_{\mathcal{O}},\partial\Omega})} \geq C|\operatorname{Im} z^2|\|Xu\|^2_{L^2(\partial\Omega)} + O(h^\infty)\|u\|^2_{L^2(\partial\Omega)}.$$

for some $C > 0$*.*

Proof. First observe that, integration by parts on $B(0,R) \cap \Omega_{\mathcal{O}}$ yields

$$\begin{aligned}&-h\operatorname{Im}\langle \Lambda_{\mathcal{O}}(z)Xu, Xu\rangle_{L^2(\partial\Omega, \mathrm{dvol}_{g_{\mathcal{O}},\partial\Omega})}\\ &\quad = \operatorname{Im} z^2 \|G_{\mathcal{O}}Xu\|^2_{L^2(B(0,R)\cap\Omega_{\mathcal{O}}, \rho_{\mathcal{O}}\mathrm{dvol}_{g_{\mathcal{O}}})} + h\operatorname{Im}\langle \rho_{\mathcal{O}} h\partial_r G_{\mathcal{O}}Xu, G_{\mathcal{O}}Xu\rangle_{L^2(\partial B(0,R), \mathrm{dvol}_{g_{\mathcal{O}},\partial B(0,R)})}.\end{aligned}$$

To simplify the notation, we will omit the dependence on $\rho_{\mathcal{O}}$. In order to complete the proof, we need to understand $G_{\mathcal{O}}Xu$.

We now apply Lemma 4.16

$$-h\operatorname{Im}\langle \Lambda_{\mathcal{O}}(z)G_{\mathcal{O}}Xu, Xu\rangle_{L^2(\partial\Omega)} = \operatorname{Im} z^2\|\chi(x^1)v\|^2_{L^2(\Omega_{\mathcal{O}})} + O(h^\infty)\|u\|^2_{L^2(\partial\Omega)}.$$

Finally, we have, letting $X' \in \Psi^{\mathrm{comp}}_{\mathrm{t},h}$ with $\mathrm{WF}_h(I-X')\cap\mathrm{WF}_h(X)=\emptyset$

$$\begin{aligned}
\|Xu\|^2_{L^2(\partial\Omega)} &= -\int_0^\infty \partial_{x^1}\|\chi(x^1)\tilde{v}(x^1)\|^2_{L^2(\partial\Omega)}\mathrm{d}x^1\\
&= -\int_0^\infty 2\,\mathrm{Re}\left\langle \partial_{x^1}(\chi(x^1)v(x^1)),\chi(x^1)v(x^1)\right\rangle_{L^2(\partial\Omega)}\mathrm{d}x^1\\
&\le Ch^{-1}\|\chi(x^1)hD_{x^1}v\|_{L^2((0,2\varepsilon)\times\partial\Omega)}\|\chi(x^1)v\|_{L^2((0,2\varepsilon)\times\partial\Omega)}+C\|v\|^2_{L^2((0,2\varepsilon)\times\partial\Omega)}\\
&= Ch^{-1}\|\chi(x^1)\Lambda v\|_{L^2((0,2\varepsilon)\times\partial\Omega)}\|\chi(x^1)v\|_{L^2((0,2\varepsilon)\times\partial\Omega)}+C\|v\|^2_{L^2((0,2\varepsilon)\times\partial\Omega)}\\
&\le Ch^{-1}\|\Lambda X'\chi(x^1)v\|_{L^2((0,2\varepsilon)\times\partial\Omega)}\|\chi(x^1)v\|_{L^2((0,2\varepsilon)\times\partial\Omega)}+C_Nh^N\|Xu\|^2_{L^2(\partial\Omega)}\\
&\le Ch^{-1}\|\chi(x^1)v\|^2_{L^2(\Omega_{\mathcal{O}})}+C_Nh^N\|u\|^2_{L^2(\partial\Omega)},
\end{aligned}$$

which completes the proof of the lemma. □

4.5. **Application to the interior problem.** Next, we apply our estimates to $P_{\mathcal{I}}-z^2$.

Lemma 4.19. *Let $\varepsilon_0>0$, $M>0$, U be a Fermi normal coordinate neighborhood of $\partial\Omega$ in $\Omega_{\mathcal{O}}$ with coordinates (x^1,x'), E_- as in Proposition 4.2 with $P_{\mathcal{I}}-z^2=-P(-z^2,g_{\mathcal{I}},L_{\mathcal{I}})$ (as in (4.3)). Then there for all $\chi\in C_c^\infty(-1,1)$ with $\chi\equiv 1$ near 0, $\varepsilon>0$ small enough, and $k,N\ge 0$ there is $C>0$ such that for all $0<h<1$, $z\in[1-\varepsilon_0,1+\varepsilon_0]+\mathrm{i}[-Mh,Mh]$, and $u\in L^2(\partial\Omega)$, we have*

$$\|\partial_z^k(\chi(\varepsilon^{-1}x^1)v-G_{\mathcal{I}}u)\|_{H^2_h(\Omega_{\mathcal{I}})}\le Ch^N\|u\|_{L^2(\partial\Omega)},$$

where v satisfies

$$(hD_{x^1}+\mathrm{i}E_-)v=0,\qquad v|_{x^1}=u.$$

Proof. The proof of this lemma is nearly identical to Lemma 4.16 with the caveat that all cutoffs can be taken to be the identity, which simplifies the proof substantially. □

Proposition 4.20. *Let $\varepsilon_0>0$, $M>0$. Then for all $z\in[1-\varepsilon_0,1+\varepsilon_0]+\mathrm{i}[-Mh,Mh]$, we have $\Lambda_{\mathcal{I}}(z)\in\Psi^1_h(\partial\Omega)$ with principal symbol*

$$\sigma(\Lambda_{\mathcal{I}}(z))=\rho_{\mathcal{I}}\left(|\xi'|^2_{g_{\mathcal{I}}}+(\mathrm{Re}\,z)^2\right)^{\frac{1}{2}}.$$

Moreover

$$\sigma(\partial_z^\alpha\Lambda_{\mathcal{I}}(z))=\rho_{\mathcal{I}}\partial_z^\alpha\left(|\xi'|^2_{g_{\mathcal{I}}}+z^2\right)^{\frac{1}{2}}. \tag{4.44}$$

Proof. The Proposition follows directly from Lemma 4.19 once we calculate the symbol of E_-. Recall that $\sigma(E_-)=-\sqrt{|\xi'|^2_{g_{\mathcal{I}}}+1}$. Recall that $\nu_{g_{\mathcal{I}}}$ is the outward normal with respect to the metric $g_{\mathcal{I}}$ and we have $-\partial_{x^1_{\mathcal{I}}}=\partial_{\nu_{g_{\mathcal{I}}}}$ In particular, the DtN map with respect to $g_{\mathcal{I}}$ is given by $\Lambda_{\mathcal{I}}(z)u_0=\rho_{\mathcal{I}}h\partial_{\nu_{\mathcal{I}}}u_0|_{\partial\Omega_{\mathcal{I}}}$, which can be written as

$$\Lambda_{\mathcal{I}}(z)u_0=(-\rho_{\mathcal{I}}E_-+O(h^\infty)_{\Psi^{-\infty}})\,u_0.$$

□

Lemma 4.21. *Let $\varepsilon_0 \geq 0$ and $M > 0$. Then for $X \in \Psi_h^k(\partial\Omega)$, and $z \in [1-e_0, 1+\varepsilon_0]+\mathrm{i}[-Mh, Mh]$, we have*

$$\operatorname{sgn}(\operatorname{Im} z^2) \operatorname{Im}\langle \Lambda_{\mathcal{I}} Xu, Xu\rangle_{L^2(\partial\Omega, \mathrm{dvol}_{g_{\mathcal{I}},\partial\Omega})} \geq C|\operatorname{Im} z^2| \|Xu\|^2_{L^2(\partial\Omega)},$$

or equivalently,

$$\operatorname{sgn}(\operatorname{Im} z^2) \operatorname{Im}\langle \tau\Lambda_{\mathcal{I}} Xu, Xu\rangle_{L^2(\partial\Omega, \mathrm{dvol}_{g_{\mathcal{O}},\partial\Omega})} \geq C|\operatorname{Im} z^2| \|Xu\|^2_{L^2(\partial\Omega)},$$

for some constant $C > 0$.

Proof. Observe that, from Section 3, one has

$$\begin{aligned} &h \operatorname{Im}\langle \tau\Lambda_{\mathcal{I}} Xu, Xu\rangle_{L^2(\partial\Omega, \mathrm{dvol}_{g_{\mathcal{O}},\partial\Omega})} = h \operatorname{Im}\langle \Lambda_{\mathcal{I}} Xu, Xu\rangle_{L^2(\partial\Omega, \mathrm{dvol}_{g_{\mathcal{I}},\partial\Omega})} \\ &= \operatorname{Im}\langle z^2 G_{\mathcal{I}}(z)Xu, G_{\mathcal{I}}(z)Xu\rangle_{L^2(\Omega_{\mathcal{I}}, \rho_{\mathcal{I}}\mathrm{dvol}_{g_{\mathcal{I}}})} = \operatorname{Im} z^2 \|G_{\mathcal{I}}(z)Xu\|^2_{L^2(\Omega_{\mathcal{I}}, \rho_{\mathcal{I}}\mathrm{dvol}_{g_{\mathcal{I}}})}. \end{aligned} \tag{4.45}$$

We will omit the dependence of $\rho_{\mathcal{I}}$ to ease the notations. Now, $(P_{\mathcal{I}} - z^2)G_{\mathcal{I}}(z)g = 0$ in $\Omega_{\mathcal{I}}$, $G_{\mathcal{I}}(z)g|_{\partial\Omega} = g$ and hence, using the factorization (4.6), we have by Lemma 4.14 with $X = \tilde{X} = I$,

$$\|G_{\mathcal{I}}(z)Xu\|_{H_h^1((0,\varepsilon);L^2(\partial\Omega))} \leq Ch^N \|G_{\mathcal{I}}(z)Xu\|_{H_h^1\left((0,2\varepsilon);H_h^{-N}(\partial\Omega)\right)} + Ch^{\frac{1}{2}}\|Xu\|_{L^2(\partial\Omega)}.$$

Subtracting part of the first term on the left to the right-hand side and using local elliptic regularity for $P_{\mathcal{I}} - z^2$ and applying (3.11) we have

$$\begin{aligned} \|G_{\mathcal{I}}(z)Xu\|_{H_h^1((0,\varepsilon);L^2(\partial\Omega))} &\leq Ch^N \|G_{\mathcal{I}}(z)Xu\|_{H_h^1\left((\varepsilon,2\varepsilon);H_h^{-N}(\partial\Omega)\right)} + Ch^{\frac{1}{2}}\|Xu\|_{L^2(\partial\Omega)} \\ &\leq Ch^N \|G_{\mathcal{I}}(z)Xu\|_{L^2(\Omega_{\mathcal{I}})} + Ch^{\frac{1}{2}}\|Xu\|_{L^2(\partial\Omega)}, \end{aligned} \tag{4.46}$$

where we have used that for $U \Subset \Omega_{\mathcal{I}}$, one has the interior elliptic estimate

$$\|G_{\mathcal{I}}(z)Xu\|_{H_h^N(U)} \leq Ch^N \|G_{\mathcal{I}}(z)Xu\|_{L^2(\Omega_{\mathcal{I}})}. \tag{4.47}$$

Combining (4.46) and (4.47), one obtains

$$\|G_{\mathcal{I}}(z)Xu\|_{H_h^1((0,2\varepsilon);L^2(\partial\Omega))} \leq Ch^N \|G_{\mathcal{I}}(z)Xu\|_{L^2(\Omega_{\mathcal{I}})} + Ch^{\frac{1}{2}}\|Xu\|_{L^2(\partial\Omega)}. \tag{4.48}$$

Hence, letting $\varphi \in C_c^\infty([0, 2\varepsilon))$ with $\varphi \equiv 1$ on $[0, \varepsilon]$,

$$\begin{aligned} \|Xu\|^2_{L^2(\partial\Omega)} &= -\int_0^\infty \partial_{x^1}\left(\varphi(x^1)\|G_{\mathcal{I}}(z)Xu(x^1)\|^2_{L^2(\partial\Omega_{\mathcal{I}})}\right)\mathrm{d}x^1 \\ &\leq C\|G_{\mathcal{I}}(z)Xu\|^2_{L^2(\Omega_{\mathcal{I}},\rho_{\mathcal{I}}\mathrm{dvol}_{g_{\mathcal{I}}})} + Ch^{-1}\int_0^\infty \|hD_{x^1}G_{\mathcal{I}}(z)Xu\|_{L^2(\partial\Omega)}\|G_{\mathcal{I}}(z)Xu\|_{L^2(\partial\Omega)}\mathrm{d}x^1 \\ &\leq Ch^{-1}\left(\delta\|G_{\mathcal{I}}(z)Xu\|^2_{H_h^1((0,2\varepsilon);L^2(\partial\Omega))} + (1+\delta^{-1})\|G_{\mathcal{I}}(z)Xu\|_{L^2(\Omega_{\mathcal{I}},\rho_{\mathcal{I}}\mathrm{dvol}_{g_{\mathcal{I}}})}\right) \\ &\leq Ch^{-1}(\delta h\|Xu\|^2_{L^2(\partial\Omega)} + (2+\delta^{-1})\|G_{\mathcal{I}}(z)Xu\|_{L^2(\Omega_{\mathcal{I}})}, \end{aligned}$$

where we used (4.48) in the last step. Now, choosing δ small enough in the above estimate, we have

$$\|Xu\|^2_{L^2(\partial\Omega)} \leq Ch^{-1}\|G_{\mathcal{I}}(z)Xu\|^2_{L^2(\Omega_{\mathcal{I}})}.$$

The lemma now follows by combining this with (4.45). □

4.6. **Combination of interior and exterior problems.** Thanks to the ellipticity of the interior problem, we have an accurate representation of $\Lambda_{\mathcal{I}}(z)$ and so we work with $\rho_{\mathcal{I}} h\partial_{\nu_{\mathcal{I}}} u_{\mathcal{I}}$ replaced by $\Lambda_{\mathcal{I}}(z)u_{\mathcal{I}}$. However, the exterior Dirichlet-to-Neumann map can only be accurately approximated microlocally in $|\xi'|_{g_{\mathcal{O}}} > 1$ (this is done in Proposition 4.17). Therefore, the difference of the exterior Dirichlet-to-Neumann and the interior Dirichlet-to-Neumann can only be directly analyzed microlocally on microlocally in $|\xi'|_{g_{\mathcal{O}}} > 1$.

Proposition 4.22. *Let $e_0 \geq 0$, $M > 0$, and $X \in \Psi_h^{\mathrm{comp}}(\partial\Omega)$, with $\mathrm{WF}_h(X) \subset \{|\xi'|_{g_{\mathcal{O}}} > 1 + \varepsilon_0\}$. Then, for all $N \in \mathbb{N}$, there exists $C, C_N, h_0 > 0$ such that for $0 < h < h_0$ and $z \in [1-\varepsilon_0, 1+\varepsilon_0] + \mathrm{i}[-Mh, Mh]$,*

$$(|\operatorname{Im} z^2| \|Xu\|^2_{L^2(\partial\Omega)} - C_N h^N \|u\|^2_{L^2(\partial\Omega)}) \leq C \left(|\langle(\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}})Xu, Xu\rangle_{L^2(\partial\Omega, \mathrm{dvol}_{g_{\mathcal{O}}, \partial\Omega})}| \right).$$

Proof. Observe that by Lemmas 4.21 and 4.18

$$|\operatorname{Im}\langle(\tau\Lambda_{\mathcal{I}} - \Lambda_{\mathcal{O}})Xu, Xu\rangle_{L^2(\partial\Omega, \mathrm{dvol}_{g_{\mathcal{O}}, \partial\Omega})}| \geq C(|\operatorname{Im} z^2| \|Xu\|^2_{L^2(\partial\Omega)} - C_N h^N \|u\|^2_{L^2(\partial\Omega)}),$$

which proves the proposition once relabeling the constant C. □

5. Proofs of Theorems 1.8, 1.9, and 1.10

In this section, we prove the first three main theorems on our article.

5.1. **Microlocal estimates for boundary traces.** Before proceeding to the proofs of the theorems, we require some microlocalized apriori estimates on boundary traces.

Lemma 5.1. *Let $\Lambda \in \Psi_h^1(\partial\Omega)$, $X, \tilde{X} \in \Psi_h^0(\partial\Omega)$ with $\mathrm{WF}_h(X) \subset \mathrm{ell}_h(\sigma(R + \Lambda^*\Lambda)) \cap \mathrm{ell}_h(\tilde{X})$, and*

$$P := (hD_{x^1})^2 + hahD_{x^1} - R(x, hD_{x'})$$

be formally self-adjoint. Then for all $s \in \mathbb{R}$, $\varepsilon > 0$, there is $C > 0$ such that

$$\begin{aligned}
&\|Xu|_{x^1=0}\|_{H_h^{s+1}(\partial\Omega)} + \|XhD_{x^1}u|_{x^1=0}\|_{H_h^s(\partial\Omega)} \\
&\leq C\|u\|_{L^2\left((0,\varepsilon);H_h^{s-2}(\partial\Omega)\right)} + Ch^{-1}\|Pu\|_{L^2\left((0,\varepsilon);H_h^s(\partial\Omega)\right)} + C\|\tilde{X}(hD_{x^1} - \Lambda)u|_{x^1=0}\|_{H_h^s(\partial\Omega)} \\
&\quad + Ch^N\|u|_{x^1=0}\|_{H_h^{-N}(\partial\Omega)} + Ch^N\|(hD_{x^1} - \Lambda)u|_{x^1=0}\|_{H_h^{-N}(\partial\Omega)}.
\end{aligned}$$

Proof. We first claim that for any $X, \tilde{X} \in \Psi_h^0(\partial\Omega)$ with $\mathrm{WF}_h(X) \subset \mathrm{ell}_h(\sigma(R + \Lambda^*\Lambda)) \cap \mathrm{ell}_h(\tilde{X})$, we have

$$\begin{aligned}
\|Xu\|^2_{H_h^{s+1}(\partial\Omega)} &\leq C\|u\|^2_{L^2\left((0,\varepsilon);H_h^{s-2}(\partial\Omega)\right)} + Ch^{-2}\|Pu\|^2_{L^2\left((0,\varepsilon);H_h^s(\partial\Omega)\right)} + Ch^j\|\tilde{X}u|_{x^1=0}\|^2_{H_h^{s+1-\frac{j}{2}}(\partial\Omega)} \\
&\quad + \|\tilde{X}(hD_{x^1} - \Lambda)u\|^2_{H_h^s(\partial\Omega)} + C_N h^N\|u\|^2_{H_h^{-N}(\partial\Omega)} + O(h^\infty)\|(hD_{x^1} - \Lambda)u\|^2_{H_h^{-N}(\partial\Omega)}.
\end{aligned} \tag{5.1}$$

Since $\sigma(R + \Lambda^*\Lambda)$ is real valued, we may assume without loss of generality that

$$\mathrm{WF}_h(X) \subset \{\pm\sigma(R + \Lambda^*\Lambda) > 0\}$$

for some choice of $\pm$.

$$\langle (A-B)u,(A-B)u\rangle + \langle Bu,Bu\rangle + 2\operatorname{Re}\langle (A-B)u,Bu\rangle = \langle Au,Au\rangle$$

Let $E_0 \in \Psi_h^s(\partial\Omega)$, with $\mathrm{WF}_h(X) \subset \mathrm{ell}_h(E_0)$ and

$$\mathrm{WF}_h(E_0) \subset \{\pm\sigma(R+\Lambda^*\Lambda) > 0\}.$$

Also, let $\chi \in C_c^\infty(\mathbb{R})$ with $\chi \equiv 1$ near 0 and set $E = \chi(x^1)E_0^*E_0$, and assume that $X_{1,2} \in \Psi_h^0(\partial\Omega)$ with $\mathrm{WF}_h(E_0) \subset \mathrm{ell}_h(X_1) \subset \mathrm{WF}_h(X_1) \subset \mathrm{ell}_h(X_2)$. Then, define

$$\begin{aligned}
Q(u;E) &:= \langle E_0^*E_0 hD_{x^1}u, hD_{x^1}u\rangle_{L^2(\partial\Omega)} + \langle E_0^*E_0Ru,u\rangle_{L^2(\partial\Omega)} + h\langle [a,E_0^*E_0]hD_{x^1}u,u\rangle_{L^2(\partial\Omega)}\\
&= \langle E_0(hD_{x^1}-\Lambda)u, E_0(hD_{x^1}-\Lambda)u\rangle_{L^2(\partial\Omega)} + 2\operatorname{Re}\langle E_0\Lambda u, E_0(hD_{x^1}-\Lambda)u\rangle_{L^2(\partial\Omega)}\\
&\quad \langle (E_0^*E_0R + \Lambda^*E_0^*E_0\Lambda + h[a,E_0^*E_0]\Lambda)u,u\rangle_{L^2(\partial\Omega)} + h\langle [a,E_0^*E_0](hD_{x^1}-\Lambda)u,u\rangle_{L^2(\partial\Omega)}\\
&= \langle E_0(hD_{x^1}-\Lambda)u, E_0(hD_{x^1}-\Lambda)u\rangle_{L^2(\partial\Omega)} + 2\operatorname{Re}\langle (\Lambda E_0 + [E_0,\Lambda])u, E_0(hD_{x^1}-\Lambda)u\rangle_{L^2(\partial\Omega)}\\
&\quad \langle (E_0^*E_0R + \Lambda^*E_0^*E_0\Lambda + h[a,E_0^*E_0]\Lambda)u,u\rangle_{L^2(\partial\Omega)} + h\langle [a,E_0^*E_0](hD_{x^1}-\Lambda)u,u\rangle_{L^2(\partial\Omega)}.
\end{aligned}$$

Next, notice that

$$\begin{aligned}
&\langle (E_0^*E_0R + \Lambda^*E_0^*E_0\Lambda)u,u\rangle_{L^2(\partial\Omega)}\\
&= \langle (E_0^*(RE_0 + [E_0,R]) + (E_0^*\Lambda^* + [\Lambda^*,E_0^*])(\Lambda E_0 + [E_0,\Lambda]))u,u\rangle_{L^2(\partial\Omega)}\\
&= \langle (R+\Lambda^*\Lambda)E_0u, E_0u\rangle + O(h)\|X_1u\|^2_{H_h^{s+\frac12}(\partial\Omega)} + O(h^\infty)\|u\|^2_{H_h^{-N}(\partial\Omega)}.
\end{aligned}$$

Then, using the microlocal Gårding inequality, we obtain

$$\begin{aligned}
\|E_0u\|^2_{H_h^1(\partial\Omega)} &\le |Q(u;E)| + Ch\|X_1u\|^2_{H_h^{s+\frac12}(\partial\Omega)} + C_Nh^N\|u\|^2_{H_h^{-N}(\partial\Omega)}\\
&\qquad + \|X_1(hD_{x^1}-\Lambda)u\|^2_{H_h^s(\partial\Omega)} + C_Nh^N\|(hD_{x^1}-\Lambda)u\|^2_{H_h^{-N}(\partial\Omega)}.
\end{aligned} \tag{5.2}$$

Next, we have

$$\begin{aligned}
|Q(u;E)| &= \Big| -\frac{\mathrm{i}}{h}\langle [P, EhD_{x^1}]u,u\rangle_{L^2(\Omega)} - \frac{2}{h}\operatorname{Im}\big(\langle EhD_{x^1}u, Pu\rangle_{L^2(\Omega)}\big)\\
&\qquad + \frac{\mathrm{i}}{h}\langle Pu, (EhD_{x^1} - (hD_{x^1})^*E)u\rangle_{L^2(\Omega)}\Big|\\
&\le \Big(\|X_1u\|_{H_h^2((0,\varepsilon);H_h^{s-1}(\partial\Omega))} + \|X_1u\|_{H_h^1((0,\varepsilon);H_h^s(\partial\Omega))} + \|X_1u\|_{L^2((0,\varepsilon);H_h^{s+1}(\partial\Omega))}\Big)\\
&\quad \times \|X_1u\|_{L^2((0,\varepsilon);H^s(\partial\Omega))} + Ch^{-1}\|X_1u\|_{H_h^1((0,\varepsilon);H_h^s(\partial\Omega))}\|X_1Pu\|_{L^2((0,\varepsilon);H_h^s(\partial\Omega))}\\
&\quad + Ch^N\Big(\|u\|_{H_h^2((0,\varepsilon);H_h^{-N}(\partial\Omega))} + \|u\|_{H_h^1((0,\varepsilon);H_h^{-N}(\partial\Omega))} + \|u\|_{L^2((0,\varepsilon);H_h^{-N}(\partial\Omega))}\Big).
\end{aligned}$$

Now Lemma 4.14 says

$$\begin{aligned}
&\|X_1u\|_{H_h^2((0,\varepsilon);H_h^{s-1}(\partial\Omega))} + \|X_1u\|_{H_h^1((0,\varepsilon);H_h^s(\partial\Omega))} + \|X_1u\|_{L^2((0,\varepsilon);H_h^{s+1}(\partial\Omega))}\\
&\quad \le \|\tilde{X}Pu\|_{L^2((0,\varepsilon);H_h^{s-1}(\partial\Omega))} + C\|\tilde{X}u\|_{L^2((0,\varepsilon);H_h^{s-1}(\partial\Omega))} + Ch^{\frac12}\|\tilde{X}u|_{x^1=0}\|_{H_h^{s+\frac12}(\partial\Omega)}\\
&\quad + Ch^N\|u\|_{H_h^1((0,\varepsilon);H_h^{-N}(\partial\Omega))} + Ch^N\|Pu\|_{L^2((0,\varepsilon);H_h^{-N}(\partial\Omega))} + Ch^N\|u|_{x^1=0}\|_{H_h^{-N}(\partial\Omega)},
\end{aligned}$$

and

$$\|u\|_{H_h^2\big((0,\varepsilon);H_h^{-N}(\partial\Omega)\big)}+\|u\|_{H_h^1\big((0,\varepsilon);H_h^{-N}(\partial\Omega)\big)}+\|u\|_{L^2\big((0,\varepsilon);H_h^{-N}(\partial\Omega)\big)}$$
$$\leq\|Pu\|^2_{L^2\big((0,\varepsilon);H_h^{-N}(\partial\Omega)\big)}+C\|u\|_{L^2\big((0,\varepsilon);H_h^{-N}(\partial\Omega)\big)}+Ch\|u|_{x^1=0}\|^2_{H_h^{-N+2}(\partial\Omega)}$$
$$+Ch^M\|u|_{x^1=0}\|_{H_h^{-M}(\partial\Omega)}.$$

Hence, the estimate for $|Q(u;E)|$ becomes

$$\begin{aligned}|Q(u;E)|\leq{}&Ch^{-2}\|\tilde{X}Pu\|^2_{L^2\big((0,\varepsilon);H_h^{s}(\partial\Omega)\big)}+C\|X_2u\|^2_{L^2\big((0,\varepsilon);H_h^{s}(\partial\Omega)\big)}+Ch\|\tilde{X}u|_{x^1=0}\|^2_{H_h^{s+\frac{1}{2}}(\partial\Omega)}\\&+Ch^N\|u|_{x^1=0}\|_{H_h^{-N}(\partial\Omega)}.\end{aligned}\tag{5.3}$$

Now apply Lemma 4.12 to $P+\omega'$ for sufficiently large ω', we have

$$\|X_2u\|_{H_h^1\big((0,\varepsilon);H_h^{s-1}(\partial\Omega)\big)}+\|X_2u\|_{L^2\big((0,\varepsilon);H_h^{s}(\partial\Omega)\big)}$$
$$\leq C\|\tilde{X}Pu\|_{L^2((t_0,t_2);H_h^{s-2}(\partial\Omega))}+C_{\omega'}\|\tilde{X}u\|_{L^2((0,\varepsilon);H_h^{s-2}(\partial\Omega))}+Ch^{\frac{1}{2}}\|\tilde{X}u|_{x^1=0}\|_{H_h^{s-\frac{1}{2}}(\partial\Omega)}$$
$$+Ch^N\|u\|_{H_h^1((0,\varepsilon);H_h^{-N}(\partial\Omega))}+Ch^N\|Pu\|_{L^2((0,\varepsilon);H_h^{-N}(\partial\Omega))}+Ch^N\|u|_{x^1=0}\|_{H_h^{-N}(\partial\Omega)}$$

and

$$\|u\|_{H_h^1\big((0,\varepsilon);H_h^{s-1}(\partial\Omega)\big)}+\|u\|_{L^2\big((0,\varepsilon);H_h^{s}(\partial\Omega)\big)}$$
$$\leq C\|Pu\|_{L^2((t_0,t_2);H_h^{s-2}(\partial\Omega))}+C_{\omega'}\|u\|_{L^2((t_0,t_2);H_h^{s-2}(\partial\Omega))}+Ch^{\frac{1}{2}}\|u(t_0)\|_{H_h^{s-\frac{1}{2}}(\partial\Omega)},$$

which implies

$$\begin{aligned}&\|X_2u\|_{L^2\big((0,\varepsilon);H_h^{s}(\partial\Omega)\big)}\\&\quad\leq C\|Pu\|_{L^2((t_0,t_2);H_h^{s-2}(\partial\Omega))}+C_{\omega'}\|u\|_{L^2((0,\varepsilon);H_h^{s-2}(\partial\Omega))}+Ch^{\frac{1}{2}}\|\tilde{X}u|_{x^1=0}\|_{H_h^{s-\frac{1}{2}}(\partial\Omega)}\\&\quad+Ch^N\|u|_{x^1=0}\|_{H_h^{-N}(\partial\Omega)}.\end{aligned}\tag{5.4}$$

Plugging (5.3) and (5.4) into (5.2), we have

$$\|E_0u\|^2_{H_h^1(\partial\Omega)}\leq Ch^{-2}\|Pu\|^2_{L^2\big((0,\varepsilon);H_h^{s}(\partial\Omega)\big)}+C\|u\|^2_{L^2\big((0,\varepsilon);H_h^{s-2}(\partial\Omega)\big)}+Ch\|\tilde{X}u|_{x^1=0}\|^2_{H_h^{s+\frac{1}{2}}(\partial\Omega)}$$
$$+\|\tilde{X}(hD_{x^1}-\Lambda)u\|^2_{H_h^s(\partial\Omega)}+Ch^N\|u|_{x^1=0}\|_{H_h^{-N}(\partial\Omega)}+C_Nh^N\|(hD_{x^1}-\Lambda)u\|^2_{H_h^{-N}(\partial\Omega)},$$

from which (5.1) with $j=1$ follows. Next, suppose that (5.1) holds for some $J\geq 1$. Then, let $X'\in\Psi_h^0(\partial\Omega)$ with

$$\mathrm{WF}_h(X)\subset\mathrm{ell}_h(X'),\qquad\mathrm{WF}_h(X')\subset\mathrm{ell}_h(R+\Lambda^*\Lambda).$$

Then, using (5.1) with $j=J$, and $(X,\tilde{X})$ replaced by (X,X')

$$\begin{aligned}\|Xu\|^2_{H_h^{s+1}(\partial\Omega)}\leq{}&\|u\|^2_{L^2\big((0,\varepsilon);H_h^{s-2}(\partial\Omega)\big)}+Ch^{-2}\|Pu\|^2_{L^2\big((0,\varepsilon);H_h^{s}(\partial\Omega)\big)}+Ch^J\|X'u\|^2_{H_h^{s+1-\frac{J}{2}}(\partial\Omega)}\\&+\|X'(hD_{x^1}-\Lambda)u\|^2_{H_h^s(\partial\Omega)}+C_Nh^N\|u\|^2_{H_h^{-N}}+O(h^\infty)\|(hD_{x^1}-\Lambda)u\|^2_{H_h^{-N}(\partial\Omega)}.\end{aligned}\tag{5.5}$$

Then, applying (5.1) with $j = 1$, $(X, \tilde{X})$ replaced by $(X', \tilde{X})$, and s replaced by $s - \frac{J}{2}$, we obtain

$$\begin{aligned}\|X'u\|^2_{H_h^{s+1-\frac{J}{2}}(\partial\Omega)} \le \|u\|^2_{L^2\left((0,\varepsilon);H_h^{s-2-\frac{J}{2}}(\partial\Omega)\right)} + Ch^{-2}\|Pu\|^2_{L^2\left((0,\varepsilon);H_h^{s-\frac{J}{2}}(\partial\Omega)\right)} + Ch\|\tilde{X}u\|^2_{H_h^{s+\frac{1}{2}-\frac{J}{2}}(\partial\Omega)} \\ + \|\tilde{X}(hD_{x^1} - \Lambda)u\|^2_{H_h^s(\partial\Omega)} + C_N h^N\|u\|^2_{H_h^{-N}(\partial\Omega)} + O(h^\infty)\|(hD_{x^1} - \Lambda)u\|^2_{H_h^{-N}(\partial\Omega)}.\end{aligned} \tag{5.6}$$

Inserting (5.6) in (5.5) then implies (5.1) with $j = J+1$. The proof of the lemma is then completed by the fact that

$$\|XhD_{x^1}u\|_{H_h^s(\partial\Omega)} \le \|\tilde{X}u\|_{H_h^{s+1}(\partial\Omega)} + \|\tilde{X}(hD_{x^1} - \Lambda)u\|_{H_h^s(\partial\Omega)} + C_N h^N\|u|_{x^1=0}\|_{H_h^{-N}(\partial\Omega)}.$$

□

Estimates for the boundary traces of the transmission problem We are now in a position to obtain apriori estimates for the problem (3.19). We start in the simpler situation when (1.6) holds.

Lemma 5.2. *Suppose that* (1.6) *holds. Then for all* $M > 0$, $s \in \mathbb{R}$, *and* $\varepsilon > 0$, *there are* $C, h_0 > 0$ *such that for all* $0 < h < h_0$, $|1 - z| \le Ch$ *and* $u \in L^2\left((0,\varepsilon); H_h^{s-2}(\partial\Omega)\right)$ *solutions to* (3.19), *we have*

$$\|u\|_{H_h^{s+1}(\partial\Omega)} + \|hD_{x^1}u\|_{H_h^s(\partial\Omega)} \le C\big(\|u\|_{L^2\left((0,\varepsilon);H_h^{s-2}(\partial\Omega)\right)} + \|g\|_{H_h^s(\partial\Omega)}\big)$$

for $0 < h < h_0$.

Proof. Recall that in Fermi normal coordinates

$$P_{\mathcal{O}} - z^2 = (hD_{x^1})^2 + ha(x)hD_{x^1} - R(x, hD_{x'})$$

with $\sigma(R) = 1 - |\xi'|^2_{g_{\mathcal{O}}}$.

Let $\Lambda = \mathrm{i}\frac{\tau}{\rho_{\mathcal{O}}}\Lambda_{\mathcal{I}}(z)$ and recall that $\sigma(\Lambda_{\mathcal{I}}(z)) = \rho_{\mathcal{I}}\sqrt{|\xi'|^2_{g_{\mathcal{I}}} + 1}$. Then,

$$(hD_{x^1} - \Lambda)u|_{x^1=0} = (-\mathrm{i}(h\partial_\nu + \tfrac{\tau}{\rho_{\mathcal{O}}}\Lambda_{\mathcal{I}}(z))u = \mathrm{i}\tfrac{\tau}{\rho_{\mathcal{O}}}g,$$

and

$$\sigma(R + \Lambda^*\Lambda) = 1 - |\xi'|^2_{g_{\mathcal{O}}} + \left(\tau\rho_{\mathcal{O}}^{-1}\right)^2\rho_{\mathcal{I}}^2(|\xi'|^2_{g_{\mathcal{I}}} + 1) = \rho_{\mathcal{O}}^{-2}(\rho_{\mathcal{O}}^2 + \tau^2\rho_{\mathcal{I}}^2 + \tau^2\rho_{\mathcal{I}}^2|\xi'|^2_{g_{\mathcal{I}}} - \rho_{\mathcal{O}}^2|\xi'|^2_{g_{\mathcal{O}}}).$$

In particular, $(\tau\rho_{\mathcal{I}})^2|\xi'|^2_{g_{\mathcal{I}}} > \rho_{\mathcal{O}}^2|\xi'|^2_{g_{\mathcal{O}}}$ for all $\xi' \in T^*\partial\Omega$ implies that there exists a positive constant, c_1 such that

$$c_1|\xi'|_{g_{\mathcal{O}}} < (\tau\rho_{\mathcal{I}})^2|\xi'|^2_{g_{\mathcal{I}}} - \rho_{\mathcal{O}}^2|\xi'|^2_{g_{\mathcal{O}}}, \quad \text{for all } \xi' \in T^*\partial\Omega.$$

Now we have, for some constant c_2,

$$\sigma(R + \Lambda^*\Lambda) > c_2\langle|\xi'|_{g_{\mathcal{O}}}\rangle^2 > 0.$$

Hence, Lemma 5.1 yields

$$\begin{aligned}\|u|_{x^1=0}\|_{H_h^{s+1}(\partial\Omega)} + \|hD_{x^1}u|_{x^1=0}\|_{H_h^s(\partial\Omega)} &\le C\|u\|_{L^2\left((0,\varepsilon);H_h^{s-2}(\partial\Omega)\right)} + C\|g\|_{H_h^s(\partial\Omega)} \\ &\quad + C_N h^N\|u|_{x^1=0}\|_{H_h^{-N}(\partial\Omega)} + c_N h^N\|g\|_{H_h^{-N}(\partial\Omega)} \\ &\le C\|u\|_{L^2\left((0,\varepsilon);H_h^{s-2}(\partial\Omega)\right)} + C\|g\|_{H_h^s(\partial\Omega)}.\end{aligned}$$

□

Next, we consider the case of (1.7).

Lemma 5.3. *Suppose that* (1.7) *holds. Then for all* $M > 0$, $s \in \mathbb{R}$, $\varepsilon > 0$, *and*$X, \tilde{X} \in \Psi_h^0(\partial\Omega)$ *satisfying*

$$\mathrm{WF}_h(X) \cap \left\{\rho_{\mathcal{O}}^2|\xi'|^2_{g_{\mathcal{O}}} - \tau^2\rho_{\mathcal{I}}^2|\xi'|^2_{g_{\mathcal{I}}} = \rho_{\mathcal{O}}^2 + \tau^2\rho_{\mathcal{I}}^2\right\} = \emptyset$$

and $\mathrm{WF}_h(X) \subset \mathrm{ell}_h(\tilde{X})$, *there are* $C, h_0 > 0$ *such that for all* $0 < h < h_0$, $|1 - z| \leq Mh$ *and all* $u \in L^2\left((0, \varepsilon); H_h^{s-2}(\partial\Omega)\right)$ *solutions to* (3.19) *we have*

$$\begin{aligned}\|Xu\|_{H_h^{s+1}(\partial\Omega)} &+ \|XhD_{x^1}u\|_{H_h^s(\partial\Omega)}\\ &\leq C\left(\|u\|_{L^2\left((0,\varepsilon);H_h^{s-2}(\partial\Omega)\right)} + \|\tilde{X}g\|_{H_h^s(\partial\Omega)} + C_N h^N\|g\|_{H_h^{-N}(\partial\Omega)} + C_N h^N\|u\|_{H_h^{-N}(\partial\Omega)}\right)\end{aligned}$$

for $0 < h < h_0$.

Proof. As before, we need only consider $R + \Lambda^*\Lambda$ with $\Lambda = \mathrm{i}\tau\rho_{\mathcal{O}}^{-1}\Lambda_{\mathcal{I}}(z)$. Observe that if

$$\sigma(R + \Lambda^*\Lambda) = \rho_{\mathcal{O}}^{-2}(\rho_{\mathcal{O}}^2 + \tau^2\rho_{\mathcal{I}}^2 + \tau^2\rho_{\mathcal{I}}^2|\xi'|^2_{g_{\mathcal{I}}} - \rho_{\mathcal{O}}^2|\xi'|^2_{g_{\mathcal{O}}}) = 0,$$

then

$$\rho_{\mathcal{O}}^2|\xi'|^2_{g_{\mathcal{O}}} - \tau^2\rho_{\mathcal{I}}^2|\xi'|^2_{g_{\mathcal{I}}} = \rho_{\mathcal{O}}^2 + \tau^2\rho_{\mathcal{I}}^2.$$

Hence, Lemma 5.1 yields

$$\begin{aligned}\|Xu|_{x^1=0}\|_{H_h^{s+1}(\partial\Omega)} + \|XhD_{x^1}u|_{x^1=0}\|_{H_h^s(\partial\Omega)} &\leq C\|u\|_{L^2\left((0,\varepsilon);H_h^{s-2}(\partial\Omega)\right)} + C\|\tilde{X}g\|_{H_h^s(\partial\Omega)}\\ &\quad + C_N h^N\|u|_{x^1=0}\|_{H_h^{-N}(\partial\Omega)} + c_N h^N\|g\|_{H_h^{-N}(\partial\Omega)}.\end{aligned}$$

□

Finally, we need an estimate on the high frequencies of a solution to $(P_{\mathcal{O}} - z^2)u = 0$ in terms of the traces of u on the boundary.

Lemma 5.4. *Let* $M > 0$, $N > 0$, $\chi_0, \chi_1 \in C_c^\infty(\mathbb{R}^d)$ *with* $\chi_0 \equiv 1$ *near* $\partial\Omega$, $\mathrm{supp}\,\chi_0 \cap \mathrm{supp}(1-\chi_1) = \emptyset$, $\phi \in C_c^\infty(\mathbb{R})$ *with*

$$\mathrm{supp}(1 - \phi) \cap \{|\xi|_{g_{\mathcal{O}}} : \exists x \in \Omega_{\mathcal{O}} \text{ such that } |\xi|^2_{g_{\mathcal{O}}} \leq 2\} = \emptyset, \tag{5.7}$$

and define $\Phi := \mathrm{Op}(\phi(|\xi|_{g_{\mathcal{O}}}))$. *Then there are* $C, h_0 > 0$ *such that for all* $0 < h < h_0$, $|1-z| < Mh$ *and* $u \in L^2_{\mathrm{loc}}(\Omega)$ *satisfying*

$$(P_{\mathcal{O}} - z^2)u = 0,$$

we have

$$\|(1 - \Phi)\chi_0 u\|_{L^2} \leq C\left(h^{\frac{1}{2}}(\|u\|_{L^2(\partial\Omega)} + \|hD_{\nu_{\mathcal{O}}}u\|_{L^2(\partial\Omega)}) + h^N\|\chi_1 u\|_{L^2(\Omega)}\right).$$

Proof. Let $\tilde{u} \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ $\tilde{u} := 1_\Omega u$. Then gives

$$(P_{\mathcal{O}} - z^2)\tilde{u} = h^2\partial_{\nu_{\mathcal{O}}}^*\delta_{\partial\Omega} \otimes (\rho_{\mathcal{O}}u|_{\partial\Omega}) - h\delta_{\partial\Omega} \otimes \left(\rho_{\mathcal{O}}h\partial_{\nu_{\mathcal{O}}}u|_{\partial\Omega}\right), \tag{5.8}$$

where $\langle \partial_{\nu_{\mathcal{O}}}^* \delta_{\partial\Omega} \otimes (u|_{\partial\Omega}), \varphi\rangle = \int_{\partial\Omega} u \partial_{\nu_{\mathcal{O}}} \varphi \mathrm{d}S$ and $\langle \delta_{\partial\Omega} \otimes \left(h\partial_{\nu_{\mathcal{O}}} u|_{\partial\Omega}\right), \varphi\rangle = \int_{\partial\Omega} \left(h\partial_{\nu_{\mathcal{O}}} u\right) \varphi \mathrm{d}S$ for $\varphi \in C_c^\infty(\mathbb{R}^d)$.

Let $\tilde{\chi} \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp}(1-\tilde{\chi}) \cap \operatorname{supp}\chi_0 = \operatorname{supp}\tilde{\chi} \cap \operatorname{supp}(1-\chi_1) = \emptyset$. Since $\mathrm{WF}_h(I-\Phi) \subset \mathrm{ell}_h(P_{\mathcal{O}} - z^2)$, there is $E \in \Psi_h^{-2}(\mathbb{R}^d)$ with $\mathrm{WF}_h(E) \cap \operatorname{supp}(1-\chi_1) = \emptyset$ such that

$$\begin{aligned}(I-\Phi)\chi_0\tilde{u} &= E(P_{\mathcal{O}} - z^2)\tilde{\chi}\tilde{u} + O(h^\infty)_{\Psi^{-\infty}}\tilde{\chi}\tilde{u} \\ &= E\tilde{\chi}(P_{\mathcal{O}} - z^2)\tilde{u} + E[P, \tilde{\chi}]\chi_1 u + O(h^\infty)_{\Psi^{-\infty}}\chi_1\tilde{u} \\ &= E\tilde{\chi}(P_{\mathcal{O}} - z^2)\tilde{u} + O(h^\infty)_{\Psi^{-\infty}}\chi_1\tilde{u}.\end{aligned}$$

Since $E \in \Psi_h^{-2}(\mathbb{R}^d)$, one has

$$\|(1-\Phi)\chi_0\tilde{u}\|_{L^2(\mathbb{R}^d)} \leq \|(P_{\mathcal{O}} - z^2)\tilde{u}\|_{H_h^{-2}(\mathbb{R}^d)} + C_N h^N \|\chi_1 u\|_{L^2(\Omega)}. \tag{5.9}$$

Using (5.8), we know that

$$\begin{aligned}\|(P_{\mathcal{O}} - z^2)\tilde{u}\|_{H_h^{-2}(\mathbb{R}^d)} &\leq Ch\left(\|h\partial_{\nu_{\mathcal{O}}}^*\delta_{\partial\Omega} \otimes (u|_{\partial\Omega})\|_{H_h^{-2}(\mathbb{R}^d)} + \|\delta_{\partial\Omega} \otimes \left(h\partial_{\nu_{\mathcal{O}}} u|_{\partial\Omega}\right)\|_{H_h^{-2}(\mathbb{R}^d)}\right) \\ &\leq Ch^{\frac{1}{2}}\left(\|u\|_{L^2(\partial\Omega)} + \|h\partial_{\nu_{\mathcal{O}}} u\|_{L^2(\partial\Omega)}\right).\end{aligned} \tag{5.10}$$

Combining with (5.9) and (5.10), one has

$$\|(1-\Phi)\chi_0\tilde{u}\|_{L^2(\mathbb{R}^d)} \leq Ch^{\frac{1}{2}}\left(\|u\|_{L^2(\partial\Omega)} + \|hD_{\nu_{\mathcal{O}}} u\|_{L^2(\partial\Omega)}\right) + C_N h^N \|\chi_1 u\|_{L^2(\Omega)},$$

which completes the proof. □

5.2. **Resolvent estimates - the absence of plasmon resonances.** This section will prove Theorem 1.9. In particular, we obtain the desired resolvent estimates under the condition (1.6).

We start with a lemma that we use repeatedly to prove our estimates. It applies the relevant propagation of defect measures results to obtain estimates.

Lemma 5.5. *Let $X, \tilde{X} \in \Psi_h^0(\partial\Omega)$ with*

$$\{\rho_{\mathcal{O}}^2(|\xi'|_{g_{\mathcal{O}}}^2 - 1) - \tau^2(|\xi'|_{g_{\mathcal{I}}}^2 + 1) = 0\} \cap \mathrm{WF}_h(X) = \emptyset,$$

and $\mathrm{WF}_h(X) \cap (\mathrm{WF}_h(I - \tilde{X})) = \emptyset$. Then, for any $M > 0$, $N > 0$ and $\chi \in C_c^\infty(\overline{\Omega}_{\mathcal{O}})$, there are $h_0 > 0$ and $C > 0$ such that for all $0 < h < h_0$, $|z-1| < Mh$, and $u \in L^2_{\mathrm{loc}}(\Omega)$ satisfying

$$\begin{cases}(P_{\mathcal{O}} - z^2)u = 0 & \text{in} \quad \Omega, \\ \rho_{\mathcal{O}} hD_{\nu_{\mathcal{O}}} u - \tau\Lambda_{\mathcal{I}}(z)u = g & \text{on} \quad \partial\Omega, \\ u \text{ is } z/h \text{ outgoing,}\end{cases} \tag{5.11}$$

we have

$$\|hD_{\nu_{\mathcal{O}}} u\|_{H_h^{\frac{1}{2}}(\partial\Omega)} + \|u\|_{H_h^{\frac{3}{2}}(\partial\Omega)} + \|\chi u\|_{H_h^2(\Omega)} \leq C(\|\tilde{X} g\|_{H_h^{\frac{1}{2}}(\partial\Omega)} + \|(I-X)u\|_{H_h^{3/2}(\partial\Omega)} + h^N \|g\|_{H_h^{-N}(\partial\Omega)}).$$

Remark 5.6. *Notice that for $a > 0$, we have $\Lambda_{\mathcal{I},h}(az) = a^{-1}\Lambda_i(z, ah)$ and hence, by rescaling h, we see that, provided*

$$\{\rho_{\mathcal{O}}^2(|\xi'|_{g_{\mathcal{O}}}^2 - z^2) - \tau^2(|\xi'|_{g_{\mathcal{I}}}^2 + z^2) = 0\} \cap \mathrm{WF}_h(X) = \emptyset,$$

for $z \in [1-\varepsilon_0, 1+\varepsilon_0]$, Lemma 5.5 continues to hold for $z \in [1-\varepsilon_0, 1+\varepsilon_0] + \mathrm{i}[-Mh, Mh]$.

Proof. We first claim it is enough to show that for any $\chi \in C_c^\infty(\overline{\Omega}_{\mathcal{O}})$,

$$\|\chi u\|_{L^2(\Omega)} \le C(\|\tilde{X} g\|_{H_h^{\frac{1}{2}}(\partial\Omega)} + \|(I-X)u\|_{L^2(\partial\Omega)}) + h^N \|g\|_{H_h^{-N}(\partial\Omega)}. \tag{5.12}$$

Indeed, let $\chi_1 \in C_c^\infty(\overline{\Omega})$ with $\operatorname{supp}\chi \cap \operatorname{supp}(1-\chi_1) = \emptyset$. By Lemmas 5.2 and Lemma 5.3, we have

$$\begin{aligned} &\|XhD_{\nu_{\mathcal{O}}} u\|_{H_h^{\frac{1}{2}}(\partial\Omega)} + \|Xu\|_{H_h^{\frac{3}{2}}(\partial\Omega)} \\ &\le C(\|\tilde{X} g\|_{H_h^{\frac{1}{2}}(\partial\Omega)} + \|\chi_1 u\|_{L^2(\Omega)} + C_N h^N \|g\|_{H_h^{-N}(\partial\Omega)} + C_N h^N \|u\|_{H_h^{-N}(\partial\Omega)}), \end{aligned}$$

Hence, using Lemma 2.9 and Proposition 3.1 to control the normal derivative and χu, we have

$$\begin{aligned} &\|hD_{\nu_{\mathcal{O}}} u\|_{H_h^{\frac{1}{2}}(\partial\Omega)} + \|u\|_{H_h^{\frac{3}{2}}(\partial\Omega)} + \|\chi u\|_{H_h^2(\Omega)} \\ &\le C(\|\tilde{X} g\|_{H_h^{\frac{1}{2}}(\partial\Omega)} + C\|(I-X)u\|_{H_h^{3/2}} + \|\chi_1 u\|_{L^2(\Omega)} + C_N h^N \|g\|_{H_h^{-N}(\partial\Omega)}). \end{aligned} \tag{5.13}$$

We will prove (5.12) by contradiction. Suppose that inequality (5.12) is false. That is, there exist sequences of solutions $u_j = u(h_j)$ and z_j such that

$$\begin{aligned} &\|\chi u_j\|_{L^2(\Omega)} = 1, \quad , \text{ and } |1 - z_j| < Mh_j, \\ &\|\tilde{X} g_j\|_{H_h^{\frac{1}{2}}(\partial\Omega)} + \|(I-X)u_j\|_{H_h^{3/2}(\partial\Omega)} + h_j^{-N} \|(I-\tilde{X})g_j\|_{H_h^{-N}(\partial\Omega)} = o(1). \end{aligned} \tag{5.14}$$

Let $\chi_0, \chi_1 \in C_c^\infty(\overline{\Omega})$ with $\chi_0 \equiv 1$ in a neighborhood of $\partial\Omega$ and $\operatorname{supp}\chi_0 \cap \operatorname{supp}(1-\chi) = \operatorname{supp}\chi \cap \operatorname{supp}(1-\chi_1) = \emptyset$. Observe that

$$(P_{\mathcal{O}} - z^2)(1-\chi_0)u_j = [\chi_0, P_{\mathcal{O}}]\chi u_j,$$

and hence

$$(1-\chi_0)u_j = R_{\mathcal{O}}[\chi_0, P_{\mathcal{O}}]\chi u_j.$$

So that, by (3.1)

$$\|(1-\chi_0)u_j\|_{L^2(\Omega)} \le C\|[\chi_0, P_{\mathcal{O}}]\chi u_j\|_{L^2(\Omega)} \le C\|\chi u_j\|_{L^2(\Omega)} \le C,$$

where in the second-to-last inequality, we have used that by elliptic regularity,

$$\|u_j\|_{H_h^1(\operatorname{supp}\partial\chi_0)} \le \|\chi u_j\|_{L^2(\Omega)}.$$

In particular,

$$\|\chi_1 u_j\|_{L^2(\Omega)} \le C. \tag{5.15}$$

From Section 2.3 and the first condition in (5.14), up to extracting a subsequence, we may assume that there is a defect measure μ associated with u_j (See (2.3)). Furthermore, by (5.13), the boundary measures also exist. Since u_j is outgoing, we have

$$\mathrm{WF}_h(u_j) \cap \{(x,\xi) : |x| \ge r_0\} \subset S_+ := \{(x,\xi) : |x| \ge r_0, \langle x, \xi\rangle > 0\}$$

for some $r_0 > 0$.

That is

$$\mu(\chi^2 \Psi_-) = 0, \tag{5.16}$$

where $\mathrm{supp}(\Psi_-) \subset S_- := \{(x,\xi) : |x| \geq r_0, \langle x, \xi\rangle < 0\}$.

By the second condition of (5.14) and Theorem 2.8, we have

$$\pi_*\mu(q_{(x_0,\xi_0)} \circ \varphi_t) = \pi_*\mu(q_{(x_0,\xi_0)}),$$

where $q \in C_c^\infty({}^bT^*\Omega;\mathbb{R})$ and $\mathrm{WF}_h(q_{(x_0,\xi_0)}) \subset B_{(x_0,\xi_0)}(\delta) \cap S_-$ with $B_{(x_0,\xi_0)}(\delta)$ being the ball centered at (x_0,ξ_0) with radius δ. Since $\partial\Omega$ is non-trapping, there exists $t \geq 0$ and $\varphi_t(x_0,\xi_0) = (x_1,\xi_1) \in S_+$. This shows

$$\mu(\chi^2\Psi_+) = 0.$$

Together with (5.16), we have

$$\mu(\chi^2) = 0. \tag{5.17}$$

Let $\phi \in C_c^\infty(\mathbb{R}^d)$ satisfy (5.7) and $\Phi = \phi(hD)$. Then, by (5.15) and (5.13) together with Lemma 5.4, we have

$$\|(1-\Phi)\chi(u1_\Omega)\|_{L^2(\mathbb{R}^d)} \leq Ch^{\frac{1}{2}} + C_N h^N.$$

In particular, using (5.17),

$$\begin{aligned} 1 = \lim_{j\to\infty} \|\chi(u_j1_\Omega)\|_{L^2(\mathbb{R}^d)} &\leq \lim_{j\to\infty} \|\Phi\chi(u_j1_\Omega)\|_{L^2(\mathbb{R}^d)} + \lim_{j\to\infty} \|(I-\Phi)\chi(u_j1_\Omega)\|_{L^2(\mathbb{R}^d)} \\ &= \mu(\chi^2\phi^2) \leq \mu(\chi^2) = 0, \end{aligned}$$

which is a contradiction. □

The next theorem gives the estimates (3.17) and hence proves Theorem 1.8.

Theorem 5.7. *Suppose that* (1.6) *holds. Then for any $M > 0$ and $\chi \in C_c^\infty(\overline{\Omega_{\mathcal{O}}})$, there are $h_0, C > 0$ such that for all $0 < h < h_0$, $|z-1| < Mh$, and $u \in L^2_{\mathrm{loc}}(\Omega)$ satisfying* (5.11) *we have*

$$\|hD_{\nu_{\mathcal{O}}}u\|_{H_h^{\frac{1}{2}}(\partial\Omega)} + \|u\|_{H_h^{\frac{3}{2}}(\partial\Omega)} + \|\chi u\|_{H_h^2(\Omega)} \leq C\|g\|_{H_h^{\frac{1}{2}}(\partial\Omega)}.$$

Proof. The theorem follows from Lemma 5.5 with $X, \tilde{X} = I$.

□

5.3. **Resolvent estimates and plasmonic resonances.** In this subsection, we prove Theorem 1.9. In particular, we obtain the estimates (3.18) under the condition (1.7) and hence prove Theorem 1.9.

Theorem 5.8. *Suppose that* (1.7) *holds. For all $M, N > 0$, $\chi \in C_c^\infty(\overline{\Omega})$, $X \in \Psi_h^{\mathrm{comp}}(\partial\Omega)$ with*

$$\mathrm{WF}_h(I-X) \cap \left\{\rho_{\mathcal{O}}^2|\xi'|^2_{g_{\mathcal{O}}} - \tau^2\rho_{\mathcal{I}}^2|\xi'|^2_{g_{\mathcal{I}}} = \rho_{\mathcal{O}}^2 + \tau^2\rho_{\mathcal{I}}^2\right\} = \emptyset, \qquad \mathrm{WF}_h(X) \subset \{|\xi'|_{g_{\mathcal{O}}} > 1\} \tag{5.18}$$

there are $C, h_0 > 0$ such that for all $0 < h < h_0$, $|1-z| < Mh$, $\operatorname{Im} z < -h^N$, and $u \in L^2_{\mathrm{loc}}(\Omega)$ satisfying (5.11) *we have*

$$\|\chi u\|_{H_h^2(\Omega)} \leq C|\operatorname{Im} z|^{-1}\|Xg\|_{L^2(\partial\Omega)} + C\|(I-X)g\|_{H_h^{\frac{1}{2}}(\partial\Omega)}.$$

Proof. Let $X_i \in \Psi^{\mathrm{comp}}(\partial\Omega)$, $i = 0, 1, 2$ with $\mathrm{WF}_h(X_i) \cap \mathrm{WF}_h(I - X_{i+1}) = \emptyset$, $i = 0, 1$ and

$$\mathrm{WF}_h(I - X_0) \cap \left\{\rho_{\mathcal{O}}^2 |\xi'|^2_{g_{\mathcal{O}}} - \tau^2 \rho_{\mathcal{I}}^2 |\xi'|^2_{g_{\mathcal{I}}} = \rho_{\mathcal{O}}^2 + \tau^2 \rho_{\mathcal{I}}^2\right\} = \emptyset, \qquad \mathrm{WF}_h(X_2) \subset \mathrm{ell}_h(X).$$

Using that $X_2\Lambda_{\mathcal{O}}, \Lambda_{\mathcal{O}}X_2 \in \Psi^{\mathrm{comp}}(\partial\Omega)$, and the wavefront set properties of X_1,

$$\mathrm{WF}_h([\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}}, X_1]) \subset \mathrm{ell}_h(X(\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}}))$$

and hence there is $E \in h\Psi^{\mathrm{comp}}$ such that

$$[\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}}, X_1] = EX_2(\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}})) + O(h^\infty)_{\Psi_h^{-\infty}}.$$

Thus, by Proposition 4.22

$$\begin{aligned}
&|\operatorname{Im} z| \|X_1 u\|^2_{L^2(\partial\Omega)} \\
&\leq |\langle(\Lambda_o - \tau\Lambda_{\mathcal{I}})X_1 u, X_1 u\rangle| + C_N h^N \|u\|^2_{L^2(\partial\Omega)} \\
&\leq |\langle X_1 g, X_1 u\rangle| + |\langle[\Lambda_o - \tau\Lambda_{\mathcal{I}}, X_1]u, X_1 n u\rangle| + C_N h^N \|u\|^2_{L^2(\partial\Omega)} \\
&= |\langle X_1 g, X_1 u\rangle| + |\langle EX_2 g + O(h^\infty)_{\Psi^{-\infty}} u, X' u\rangle| + C_N h^N \|u\|^2_{L^2(\partial\Omega)} \\
&\leq \Big(\|X_2 g\|_{L^2(\partial\Omega)} + O(h^\infty)(\|g\|_{H_h^{-N}(\partial\Omega)} + \|u\|_{H_h^{-N}(\partial\Omega)})\Big) \|X_1 u\|_{L^2(\partial\Omega)} + C_N h^N \|u\|^2_{L^2(\partial\Omega)}.
\end{aligned}$$

Hence, using that $X_1 \in \Psi_h^{\mathrm{comp}}(\partial\Omega)$ for the first inequality

$$\begin{aligned}
|\operatorname{Im} z| \|X_1 u\|_{H_h^{3/2}(\partial\Omega)} &\leq C|\operatorname{Im} z| \|X_1 u\|_{L^2(\partial\Omega)} + C_N h^N \|u\|_{L^2(\partial\Omega)} \\
&\leq C\|X_2 g\|_{L^2(\partial\Omega)} + C_N h^N (\|u\|_{L^2(\partial\Omega)} + \|g\|_{H_h^{-N}(\partial\Omega)}).
\end{aligned} \tag{5.19}$$

Now, by Lemma 5.5 with $X = I - X_1$ and $\tilde{X} = I - X_0$,

$$\begin{aligned}
\|hD_{\nu_{\mathcal{O}}} u\|&_{H_h^{\frac{1}{2}}(\partial\Omega)} + \|u\|_{H_h^{\frac{3}{2}}(\partial\Omega)} + \|\chi u\|_{H_h^2(\Omega)} \\
&\leq C(\|(I - X_0)g\|_{H_h^{\frac{1}{2}}(\partial\Omega)} + \|X_1 u\|_{H_h^{3/2}(\partial\Omega)} + h^N \|g\|_{H_h^{-N}(\partial\Omega)}) \\
&\leq C(\|(I - X_0)g\|_{H_h^{\frac{1}{2}}(\partial\Omega)} + |\operatorname{Im} z|^{-1} \|X_2 g\|_{H_h^{\frac{1}{2}}(\partial\Omega)}) \\
&\leq C(\|(I - X)g\|_{H_h^{\frac{1}{2}}(\partial\Omega)} + |\operatorname{Im} z|^{-1} \|X g\|_{H_h^{\frac{1}{2}}(\partial\Omega)},
\end{aligned}$$

which completes the proof. ∎

5.4. **The plasmonic nature of resonances.** . In this subsection, we show that all resonances close to the real axis are plasmonic. In particular, we prove Theorem 1.10.

Lemma 5.9. *Suppose that* (1.7) *holds. Then for all $M > 0$, $\chi \in C_c^\infty(\overline{\Omega_{\mathcal{O}}})$ with $\chi = 1$ in a neighborhood of $\partial\Omega$ and $\psi \in C_c^\infty(\Omega_{\mathcal{O}})$ (i.e. $\operatorname{supp}\psi \cap \partial\Omega = \emptyset$) the following holds. There is $c > 0$*

such that for all $|1 - z(h)| \leq Mh$ *and* $u = u(h) \in L^2_{\mathrm{loc}}(\Omega)$ *satisfies*

$$\begin{cases} (P_{\mathcal{O}} - z^2)u = 0 & \text{in } \Omega, \\ (\rho_{\mathcal{O}} h\partial_{\nu_{\mathcal{O}}} - \tau\Lambda_{\mathcal{I}}(z))u = 0 & \text{on } \partial\Omega, \\ \|u\|_{L^2(\partial\Omega)} = 1, \\ u \text{ is } z/h \text{ outgoing.} \end{cases}$$

then

$$ch^{\frac{1}{2}} \leq \|\chi u\|_{H_h^2(\Omega)} = O(h^{\frac{1}{2}}), \quad \text{and} \quad \|\psi u\|_{H_h^2(\Omega)} = O(h^\infty).$$

Proof. Let $X_j \in \Psi_h^{\mathrm{comp}}(\partial\Omega)$, $j = 0, 1, 2, 3$ satisfy (5.18) with $\mathrm{WF}_h(I - X_{j+1}) \cap \mathrm{WF}_h(X_j) = \emptyset$. Then from Proposition 4.17, and the elliptic parametrix construction, there is $E \in \Psi_h^{\mathrm{comp}}(\partial\Omega)$ such that

$$(X_2 - X_0) = E(\Lambda_0 - \tau\Lambda_i) + O(h^\infty)_{\Psi_h^{-\infty}}.$$

Hence, using that

$$\mathrm{WF}_h([\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}}, X_1]) \subset \mathrm{WF}_h(X_1) \cap \mathrm{WF}_h(I - X_1) \subset \mathrm{ell}_h(X_2 - X_0),$$

we have

$$\begin{aligned} \|(\rho_{\mathcal{O}} h\partial_{\nu_{\mathcal{O}}} - \tau\Lambda_{\mathcal{I}})G_{\mathcal{O}} X_1 u\|_{H_h^N(\partial\Omega)} &= \|(\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}})X_1 u\|_{H_h^N(\partial\Omega)} \\ &= \|[(\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}}), X_1]u\|_{H_h^N(\partial\Omega)} \leq Ch\|(X_2 - X_0)u\|_{H_h^N(\partial\Omega)} + O(h^\infty) \\ &= Ch\|E(\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}})u\|_{H_h^N(\partial\Omega)} + O(h^\infty) = O(h^\infty). \end{aligned}$$

Define $w := u - G_{\mathcal{O}} X_1 u$. Then, we have

$$\begin{cases} (P_{\mathcal{O}} - z^2)w = 0 & \text{in } \Omega, \\ g := \rho_{\mathcal{O}} h\partial_{\nu_{\mathcal{O}}} w - \tau\Lambda_{\mathcal{I}}(z)w = O(h^\infty) & \text{on } \partial\Omega, \\ w = (I - X_1)u & \text{on } \partial\Omega, \\ w \text{ is } z/h \text{ outgoing.} \end{cases}$$

Now, by Lemma 5.5 with $X = I - X_0$ and $\tilde{X} = I$,

$$\begin{aligned} \|hD_{\nu_{\mathcal{O}}} w\|_{H_h^{\frac{1}{2}}(\partial\Omega)} + \|w\|_{H_h^{\frac{3}{2}}(\partial\Omega)} + \|\chi w\|_{H_h^2(\Omega)} &\leq C(\|g\|_{H_h^{\frac{1}{2}}(\partial\Omega)} + \|X_0 w\|_{H_h^{3/2}(\partial\Omega)}) \\ &= C(\|g\|_{H_h^{\frac{1}{2}}(\partial\Omega)} + \|X_0(I - X_1)u\|_{H_h^{3/2}(\partial\Omega)}) \\ &= O(h^\infty). \end{aligned}$$

Using Lemma 4.16 to bound $\|\chi G_{\mathcal{O}} X_1 u\|_{H_h^2(\Omega)}$, we obtain

$$\|\chi u\|_{L^2(\Omega)} \leq C\|\chi w\|_{H_h^2(\Omega)} + \|\chi G_{\mathcal{O}} X_1 u\|_{H_h^2(\Omega)} \leq Ch^{\frac{1}{2}}.$$

Next, using Lemma 4.16 again, observe that $\psi \in C_c^\infty(\Omega_{\mathcal{O}})$, one has

$$\|\psi G_{\mathcal{O}} X_1 u\|_{H_h^2} = O(h^\infty).$$

Finally, observe that

$$\|\psi u\|_{H_h^2(\Omega)} \leq \|\psi w\|_{H_h^2(\Omega)} + \|\psi G_{\mathcal{O}} X_1 u\|_{H_h^2(\Omega)} = O(h^\infty),$$

which completes the proof. □

We can now complete the proof of Theorem 1.10

Proof of Theorem 1.10. Let $\psi \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\psi \cap \partial\Omega = \emptyset$. Suppose that $\lambda_j \in \mathcal{R}(P)$ with $\operatorname{Re}\lambda_j \to \infty$ and $|\operatorname{Im}\lambda_j| \leq C$ and u_{λ_j} satisfies (1.5) with $f_{\mathcal{I}} = f_{\mathcal{O}} = 0$, and $\|u_{\lambda_j}\|_{L^2(\partial\Omega)} = 1$.

Set $h_j = \operatorname{Re}\lambda_j^{-1}$. Then Lemma 5.9 applies to $u_{\lambda_j,\mathcal{O}}$ and hence

$$\|\psi u_{\lambda_j,\mathcal{O}}\|_{H_h^2(\Omega_{\mathcal{O}})} = O(h^\infty), \qquad \|u_{\lambda_j}\|_{H_h^{3/2}(\partial\Omega)} \leq C.$$

To finish the proof of Theorem 1.10 it suffices apply Lemma 4.19 to see that

$$\|\psi G_{\mathcal{I}} u_{\lambda_j,\mathcal{I}}\|_{H_h^2(\Omega_{\mathcal{I}})} = O(h^\infty).$$

□

6. Counting of Plasmon Resonances

In this section, we prove Theorem 1.11. We start by finding an operator that is uniformly invertible near the real axis and approximates $(\Lambda_{\mathcal{O}}(z) - \tau\Lambda_{\mathcal{I}}(z))^{-1}$ well.

Lemma 6.1. *Suppose that $Q \in \Psi^{\mathrm{comp}}(\partial\Omega)$ satisfy*

$$\left\{\rho_{\mathcal{O}}^2|\xi'|_{g_{\mathcal{O}}}^2 - \tau^2\rho_{\mathcal{I}}^2|\xi'|_{g_{\mathcal{I}}}^2 = \rho_{\mathcal{O}}^2 + \tau^2\rho_{\mathcal{I}}^2\right\} \subset \operatorname{ell}_h(Q), \qquad \operatorname{WF}_h(Q) \subset \{|\xi'|_{g_{\mathcal{O}}} > 1\}. \tag{6.1}$$

Then, there is $\varepsilon > 0$ such that for all $M > 0$ there are $h_0, C > 0$ such that for $0 < h < h_0$, $z \in [1 - 2\varepsilon_0, 1 + 2\varepsilon_0] + \mathrm{i}[-Mh, Mh]$,

$$R_Q(z) := (\Lambda_{\mathcal{O}}(z) - \tau\Lambda_{\mathcal{I}}(z) - \mathrm{i}Q)^{-1}$$

exists and satisfies

$$\|R_Q(z)\|_{H_h^{\frac{1}{2}}(\partial\Omega) \to H_h^{\frac{3}{2}}(\partial\Omega)} \leq C.$$

Proof. Let $X_0, X_1 \in \Psi^{\mathrm{comp}}(\partial\Omega)$ such that $\operatorname{WF}_h(X_0) \subset \operatorname{WF}_h(I - X_1)$, $\operatorname{WF}_h(X_1) \subset \{|\xi'|_{g_{\mathcal{O}}} > 1 + 2\varepsilon\}$, $\operatorname{WF}_h(Q) \cap \operatorname{WF}_h(I - X_0) = \emptyset$, and

$$\operatorname{WF}_h(X_0) \subset \operatorname{ell}_h(X_1(\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}} - \mathrm{i}Q)),$$
$$\operatorname{WF}_h(I - X_0) \cap \left\{\rho_{\mathcal{O}}^2|\xi'|_{g_{\mathcal{O}}}^2 - \tau^2\rho_{\mathcal{I}}^2|\xi'|_{g_{\mathcal{I}}}^2 = \rho_{\mathcal{O}}^2 z^2 + \tau^2\rho_{\mathcal{I}}^2 z^2\right\} = \emptyset, \quad z \in [1 - \varepsilon, 1 + \varepsilon].$$

Then, the elliptic parametrix construction implies

$$\|X_0 u\|_{H_h^s(\partial\Omega)} \leq C\|(\Lambda_{\mathcal{O}}(z) - \tau\Lambda_{\mathcal{I}}(z) - \mathrm{i}Q)u\|_{H_h^{s-1}(\partial\Omega)} + Ch^N\|u\|_{H_h^{-N}(\partial\Omega)}.$$

Hence, by Lemma 5.5 (together with Remark 5.6) with $X = I - X_0$ and $\tilde{X} = I$, we have

$$\begin{aligned}\|hD_{\nu_{\mathcal{O}}}u\|_{H_h^{\frac{1}{2}}(\partial\Omega)} + \|u\|_{H_h^{\frac{3}{2}}(\partial\Omega)} + \|\chi u\|_{H_h^2(\Omega)} &\leq C(\|Qu\|_{H_h^{\frac{1}{2}}(\partial\Omega)} + \|X_0 u\|_{H_h^{3/2}(\partial\Omega)}) \\ &\leq C(\|X_0 u\|_{H_h^{3/2}(\partial\Omega)} + h^N\|u\|_{H_h^{-N}(\partial\Omega)}) \\ &\leq C(\|(\Lambda_{\mathcal{O}}(z) - \tau\Lambda_{\mathcal{I}}(z) - \mathrm{i}Q)u\|_{H_h^{\frac{1}{2}}(\partial\Omega)} + h^N\|u\|_{H_h^{-N}(\partial\Omega)}),\end{aligned}$$

which completes the proof after absorbing the last term on the right-hand side.

□

We now fix Q_0 satisfying (6.1), let $\varepsilon > 0$ as in Lemma 6.1 and are interested in the number of resonances in

$$V_\varepsilon(h) := [1-\varepsilon, 1+\varepsilon] + \mathrm{i}[-h,h].$$

Define

$$\mathcal{Z}_\varepsilon(h) := \{z \in V_\varepsilon(h) \,:\, (\Lambda_{\mathcal{O}}(z) - \tau\Lambda_{\mathcal{I}}(z)) \text{ is not invertible}\}.$$

The next lemma reduces counting the number of resonances in $V_\varepsilon(h)$ to counting the number of zeros of an analytic function and gives a crude upper bound on how many zeros there may be.

Lemma 6.2. *There is $h_0 > 0$ such that for $0 < h < h_0$,*

$$\mathcal{Z}_\varepsilon = \{z \in V_\varepsilon(h) \,:\, F(z) = 0\},$$

where

$$F(z) := \det(I + \mathrm{i}R_Q(z)Q).$$

Moreover,there is $C > 0$ such that

$$|F(z)| \le \exp(Ch^{-d+1}), \qquad z \in V_{2\varepsilon} \tag{6.2}$$

and

$$N(h) := \#\mathcal{Z}_\varepsilon(h) \le Ch^{-d-1}.$$

Proof. Observe that

$$\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}} = (\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}} - \mathrm{i}Q)(I + \mathrm{i}R_Q(z)Q).$$

Therefore, since $(\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}} - \mathrm{i}Q)^{-1}$ exists for all $z \in V_\varepsilon(h)$, $\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}}$ is invertible if and only if $I+\mathrm{i}R_Q(z)Q$ is invertible. Since $Q \in \Psi^{\mathrm{comp}}$, R_QQ is trace class and hence $I+\mathrm{i}R_Q(z)Q$ is invertible if and only if $F(z) \neq 0$.

Now, observe that

$$|F(z)| \le \exp(\|R_Q(z)Q\|_{\mathrm{Tr}}) \le \exp(Ch^{-d+1}), \quad z \in V_{2\varepsilon}(h). \tag{6.3}$$

On the other hand, set $z_s = s + ih$, then

$$(I+\mathrm{i}R_Q(z_s)Q)^{-1} = (\Lambda_{\mathcal{O}}(z_s) - \tau\Lambda_{\mathcal{I}}(z_s))^{-1}(\Lambda_{\mathcal{O}}(z_s) - \tau\Lambda_{\mathcal{I}}(z_s) - \mathrm{i}Q) = I - \mathrm{i}(\Lambda_{\mathcal{O}}(z_s) - \tau\Lambda_{\mathcal{I}}(z_s))^{-1}Q.$$

Therefore, for $s \in [1-\varepsilon, 1+\varepsilon]$, using that $\|(\Lambda_{\mathcal{O}}(z_s) - \tau\Lambda_{\mathcal{I}}(z_s)\| \le Ch^{-1}$, we have

$$|F(z_s)|^{-1} \le \exp(\|(\Lambda_{\mathcal{O}}(z_s) - \tau\Lambda_{\mathcal{I}}(z_s))^{-1}Q\|_{\mathrm{Tr}}) \le \exp(Ch^{-d}). \tag{6.4}$$

Using (6.3) and (6.4) together with [DZ19, (D1.11)]

$$\#\{z \in [s-h, s+h] + \mathrm{i}[-h,h] \,:\, F(z) = 0\} \le Ch^{-d}, \qquad s \in [1-\varepsilon, 1+\varepsilon].$$

Hence,

$$N(h) \le Ch^{-d-1},$$

as claimed. □

With the crude estimate on the number of resonances, in hand, we can now write an effective formula for counting zeros.

Lemma 6.3. *Let* $\chi \in C_c^\infty((1-\varepsilon, 1+\varepsilon))$. *Then,*

$$\sum_{z_j \in \mathcal{Z}_\varepsilon} \chi(\operatorname{Re} z_j) = \frac{1}{2\pi \mathrm{i}} \int_{\partial V_{\varepsilon,N}} \tilde{\chi}(z) \frac{\partial_z F(z)}{F(z)} dz + O(h^\infty), \tag{6.5}$$

where $\tilde{\chi}$ *is an almost analytic extension of* χ.

Proof. First, as in [Dya15, page 375] by [Tit86, Lemma α, Section 3.9] and the estimate (6.2) (splitting the region $V_\varepsilon(h)$ into h by h squares and applying Lemma α to each square, transformed into the unit disk using the Riemann Mapping Theorem), we have

$$\frac{\partial_z F(z)}{F(z)} = \sum_{z_j \in \mathcal{Z}_\varepsilon} \frac{1}{z - z_j} + G(z), \qquad |G(z)| \le C h^{-N}, \qquad z \in V_\varepsilon(h) \cap \operatorname{supp} \tilde{\chi}.$$

Hence, applying Stokes formula in

$$V_\varepsilon(h) \setminus \cup_{z_j \in \mathcal{Z}(h)} B(z_j, r),$$

sending $r \to 0$, and using that $\operatorname{supp} \tilde{\chi} \subset \{\operatorname{Re} z \in (1-\varepsilon, 1+\varepsilon)\}$, we obtain

$$\begin{aligned} \frac{1}{2\pi \mathrm{i}} \int_{\partial V_\varepsilon} \tilde{\chi}(z) \frac{\partial_z F(z)}{F(z)} dz &= \sum_{z_j \in \mathcal{Z}_\varepsilon(h)} \tilde{\chi}(z_j) + \frac{1}{2\pi \mathrm{i}} \int_{V_\varepsilon(h)} \partial_{\bar{z}} \tilde{\chi} \frac{\partial_z F(z)}{F(z)} d\bar{z} \wedge dz \\ &= \sum_{z_j \in \mathcal{Z}_\varepsilon(h)} \tilde{\chi}(z_j) + O(h^\infty), \end{aligned}$$

where the last equality follows from the bound $N(h) \le C h^{-d-1}$, $|G(z)| \le C h^{-N}$, and $\partial_{\bar{z}} \tilde{\chi} = O(|\operatorname{Im} z|^\infty)$.

Finally, since by Theorem 5.8, $|\operatorname{Im} z_j| = O(h^\infty)$, the lemma follows. ☐

In order to obtain an asymptotic formula for the integral in (6.5), we will need to have an accurate description of $(\Lambda_{\mathcal{O}}(z) - \tau \Lambda_{\mathcal{I}}(z))^{-1} X$, for $z \in \Gamma_\varepsilon^\pm(h) := [1-\varepsilon, 1+\varepsilon] \pm \mathrm{i}h$, for any $X \in \Psi^{\mathrm{comp}}(\partial\Omega)$ with $\mathrm{WF}_h(X) \subset \{|\xi'|_{g_{\mathcal{O}}} > (1+\varepsilon)^2\}$.

Lemma 6.4. *Let* $X \in \Psi^{\mathrm{comp}}(\partial\Omega)$ *with* $\mathrm{WF}_h(X) \subset \{|\xi'|_{g_{\mathcal{O}}} > 1+\varepsilon\}$. *Then,*

$$(\Lambda_{\mathcal{O}}(z) - \tau \Lambda_{\mathcal{I}}(z))^{-1} X = -\frac{\mathrm{i}}{h} \int_0^{\pm\infty} W^* U(t,z) W X e^{-\mathrm{i}tz^2/h} dt, \qquad \pm \operatorname{Im} z^2 \le -h^N,$$

where, $W \in \Psi^{\mathrm{comp}}(\partial\Omega)$ *and for some* $\tilde{\chi} \in C^\infty_{\mathrm{comp}}(\{|\xi'|_{g_{\mathcal{O}}} > (1+\varepsilon)^2\}; [0,1])$ *with* $\operatorname{supp}(1-\tilde{\chi}) \cap \mathrm{WF}_h(X) = \emptyset$,

$$\sigma(W) = \sqrt{\frac{\rho_{\mathcal{O}} \sqrt{|\xi'|^2_{g_{\mathcal{O}}} - z^2} + \tau \rho_{\mathcal{I}} \sqrt{|\xi'|^2_{g_{\mathcal{I}}} + z^2}}{\rho_{\mathcal{O}}^2 + \tau^2 \rho_{\mathcal{I}}^2} \tilde{\chi}},$$

there is $c > 0$ *such that for any* α

$$\|D_z^\alpha U(\pm t, z) e^{-\mathrm{i}tz^2/h}\|_{L^2(\partial\Omega) \to L^2(\partial\Omega)} \le C_\alpha e^{-c|\operatorname{Im} z|^2 t/h}, \qquad \pm \operatorname{Im} z^2 \le -h^N, t \ge 0,$$

and

$$(hD_t - B(z)) U(t,z) = 0, \qquad U(0,z) = I$$

for $B \in \Psi^2$ *satisfying*

$$\sigma(B) = \frac{\rho_{\mathcal{O}}^2 |\xi'|_{g_{\mathcal{O}}}^2 - \tau^2 \rho_{\mathcal{I}}^2 |\xi'|_{g_{\mathcal{I}}}^2}{\rho_{\mathcal{O}}^2 + \tau^2 \rho_{\mathcal{I}}^2} \chi^2,$$

where $\chi \in C_{\mathrm{comp}}^\infty(\{|\xi'|_{g_{\mathcal{O}}} > (1+\varepsilon)^2\}; [0,1])$ *with* $\mathrm{supp}(1-\tilde{\chi}) \cap \mathrm{supp}(\chi) = \emptyset$.

Proof. Since the analysis only happens at the boundary, we will omit the argument of Sobolev spaces. Define

$$W := \mathrm{Op}\left(\sqrt{\frac{\rho_{\mathcal{O}} \sqrt{|\xi'|_{g_{\mathcal{O}}}^2 - z^2} + \tau \rho_{\mathcal{I}} \sqrt{|\xi'|_{g_{\mathcal{I}}}^2 + z^2}}{\rho_{\mathcal{O}}^2 + \tau^2 \rho_{\mathcal{I}}^2} \tilde{\chi}} \right).$$

Now, let $X_1 \in \Psi^{\mathrm{comp}}(\partial\Omega)$ with $\mathrm{WF}_h(X_1) \cap \mathrm{supp}(1-\chi) = \emptyset$ and $\mathrm{WF}_h(X) \cap \mathrm{WF}_h(I - X_1) = \emptyset$.

Set

$$B(z) := \mathrm{Op}(\chi)^*(W(\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}})W^* + z^2)\,\mathrm{Op}(\chi).$$

Then,

$$X_1 W(\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}})W^* = X_1(B(z) - z^2) + O(h^\infty)_{\Psi^{-\infty}},$$

and, by (4.44) and (4.43),

$$\sigma(B(z)) = \frac{\rho_{\mathcal{O}}^2 |\xi'|_{g_{\mathcal{O}}}^2 - \tau^2 \rho_{\mathcal{I}}^2 |\xi'|_{g_{\mathcal{I}}}^2}{\rho_{\mathcal{O}}^2 + \tau^2 \rho_{\mathcal{I}}^2} \chi^2, \qquad \partial_z^\alpha B(z) \in h\Psi^{\mathrm{comp}}.$$

Moreover, using Proposition 4.22,

$$\begin{aligned}
&-\mathrm{sgn}(\operatorname{Im} z^2) \operatorname{Im}\langle (B(z) - z^2)u, u\rangle \\
&= -\mathrm{sgn}(\operatorname{Im} z^2) \operatorname{Im}\langle (\mathrm{Op}(\chi)^* W(\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}})W^* \mathrm{Op}(\chi) - z^2(1 - \mathrm{Op}(\chi)^* \mathrm{Op}(\chi)))u, u\rangle \\
&\geq c_0 |\operatorname{Im} z^2| \|W^* \mathrm{Op}(\chi) u\|_{L^2}^2 + |\operatorname{Im} z^2| \langle (1 - \mathrm{Op}(\chi)^* \mathrm{Op}(\chi))u, u\rangle - C_N h^N \|u\|_{L^2}^2 \\
&= |\operatorname{Im} z^2| \langle (1 - \mathrm{Op}(\chi)^* \mathrm{Op}(\chi) + c\,\mathrm{Op}(\chi)^* W W^* \mathrm{Op}(\chi))u, u\rangle - C_N h^N \|u\|_{L^2}^2 \\
&\geq (c|\operatorname{Im} z^2| - C_N h^N)\|u\|_{L^2}^2,
\end{aligned}$$

where the last line follows from Gårding's inequality and they fact that

$$\sigma(1 - \mathrm{Op}(\chi)^* \mathrm{Op}(\chi) + c_0 \mathrm{Op}(\chi)^* W W^* \mathrm{Op}(\chi)) = 1 - \chi^2(1 - c_0 W^2) \geq 2c > 0.$$

Next, observe that, if $(B(z) - z^2)^{-1}$ exists (and is polynomially bounded in h), then for any $A \in \Psi^0$, with $\mathrm{WF}_h(I - A) \cap \mathrm{WF}_h(X) = \emptyset$,

$$(B(z) - z^2)^{-1} X = A(B(z) - z^2)^{-1} X + O(h^\infty)_{\Psi^{-\infty}},$$

and hence, since W is elliptic on $\mathrm{WF}_h(X_1)$,

$$\begin{aligned}
(\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}})W^*(B(z) - z^2)^{-1} W X &= X_1 W(\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}})W^*(B(z) - z^2)^{-1} W X + O(h^\infty)_{\Psi^{-\infty}} \\
&= X + O(h^\infty)_{\Psi^{-\infty}}.
\end{aligned}$$

In particular, since $(\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}})^{-1}$ is tempered and bounded in h,

$$(\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}})^{-1} X = W^*(B(z) - z^2)^{-1} W X + O(h^\infty)_{\Psi^{-\infty}}, \qquad |\operatorname{Im} z| \geq h^N.$$

Therefore, we only need to invert $B(z) - z^2$. For this, define $U(t,z)$ by

$$(hD_t - B(z))(U(t,z)) = 0, \qquad U(0,z) = I.$$

Then, observe that on

$$\begin{aligned}
&-\operatorname{sgn}(\operatorname{Im} z^2)h\partial_t\|U(t,z)e^{-\mathrm{i}tz^2/h}u_0\|_{L^2}^2\\
&= -2\operatorname{sgn}(\operatorname{Im} z^2)\operatorname{Re}\langle h\partial_t(Ue^{-\mathrm{i}tz^2/h}u_0), Ue^{-\mathrm{i}tz^2/h}u_0\rangle\\
&= 2\operatorname{sgn}(\operatorname{Im} z^2)\operatorname{Im}\langle (B(z)-z^2)Ue^{-\mathrm{i}tz^2/h}u_0, Ue^{-\mathrm{i}tz^2/h}u_0\rangle\\
&\le -(c|\operatorname{Im} z^2| - C_N h^N)\|U(t,z)e^{-\mathrm{i}tz^2/h}u_0\|_{L^2}^2.
\end{aligned}$$

In particular,

$$\|U(\pm t,z)e^{-\mathrm{i}tz^2/h}u_0\|_{L^2}^2 \le Ce^{-c|\operatorname{Im} z^2|t/h}\|u_0\|_{L^2}^2, \qquad \pm\operatorname{Im} z^2 \le -h^N,\ t\ge 0.$$

Thus, we have

$$(B(z)-z^2)^{-1} = -\frac{\mathrm{i}}{h}\int_0^{\pm\infty} U(t,z)e^{-\mathrm{i}tz^2/h}\mathrm{d}t, \qquad \pm\operatorname{Im} z^2 \le -h^N.$$

Finally, observe that

$$(hD_t - B)D_zU = (D_zB)U, \qquad D_zU(0) = 0.$$

So that for $\operatorname{Im} z^2 \le -h^N$, and $t \ge 0$,

$$\begin{aligned}
\|D_zU(t,z)e^{-\mathrm{i}tz^2/h}\|_{L^2\to L^2} &\le \frac{1}{h}\int_0^t \|U(t-s,z)e^{-\mathrm{i}(t-s)z^2/h}(D_zB)U(s,z)e^{-\mathrm{i}sz^2/h}\|_{L^2\to L^2}\mathrm{d}s\\
&\le \frac{t}{h}e^{-ct|\operatorname{Im} z^2|/h} \le C_1e^{-ct|\operatorname{Im} z^2|/h}.
\end{aligned}$$

Now, suppose

$$\|D_z^kU(t,z)e^{-\mathrm{i}tz^2/h}\|_{L^2\to L^2} \le C_ke^{-ct|\operatorname{Im} z^2|/h}$$

is true for some C_k and $k \le J$. Now, we have

$$(hD_t - B)D_z^{J+1}U = \sum_{k=1}^{J+1}\binom{J+1}{k}D_z^kBD_z^{J+1-k}U, \qquad D_z^{J+1}U(0) = 0.$$

Then

$$\begin{aligned}
&\|D_z^{J+1}U(t,z)e^{-\mathrm{i}tz^2/h}\|_{L^2\to L^2}\\
&\quad\le \frac{1}{h}\sum_{k=1}^{J+1}\binom{J+1}{k}\int_0^t \|U(t-s,z)e^{-\mathrm{i}(t-s)z^2/h}(D_z^kB)D_z^{J+1-k}U(s,z)e^{-\mathrm{i}sz^2/h}\|_{L^2\to L^2}\mathrm{d}s\\
&\quad\le \frac{C_B}{h}\sum_{k=1}^{J+1}\binom{J+1}{k}C_{J+1-k}e^{-ct|\operatorname{Im} z^2|/h} \le C_{J+1}e^{-ct|\operatorname{Im} z^2|/h},
\end{aligned}$$

which proves the induction step and hence the Lemma.

□

We can now prove Theorem 1.11.

Proof of Theorem 1.11. We start by computing,

$$N_\chi := \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_\pm} \tilde{\chi}(z)\frac{\partial_z F(z)}{F(z)}dz = \sum_{z_j\in\mathcal{Z}_\varepsilon} \chi(\operatorname{Re} z_j) + O(h^\infty),$$

where

$$\Gamma_\pm := [1-\varepsilon, 1+\varepsilon] \pm \mathrm{i}h,$$

oriented to the left and right respectively. Here we have used Lemma 6.3 to obtain the second equality.

Observe first that, using cyclicity of the trace and the fact that $R_Q Q \in \Psi^{\mathrm{comp}}$, we obtain

$$\begin{aligned}\frac{\partial_z F(z)}{F(z)} &= \operatorname{Tr}(I+\mathrm{i}R_Q(z)Q)^{-1}\mathrm{i}\partial_z R_Q(z)Q\\ &= -\mathrm{i}\operatorname{Tr}(\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}})^{-1}\partial_z(\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}})R_Q(z)Q\\ &= -\mathrm{i}\operatorname{Tr}X(\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}})^{-1}X\partial_z(\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}})XR_Q(z)Q + O(h^\infty),\end{aligned}$$

where $X \in \Psi^{\mathrm{comp}}$ with $\mathrm{WF}_h(Q) \cap \mathrm{WF}_h(I-X) = \emptyset$.

We can now use Lemma 6.4 to write

$$N_\chi := \sum_\pm \frac{\mathrm{i}}{2\pi h}\int_{\Gamma_\mp}\tilde{\chi}(z)\int_0^{\pm\infty}\operatorname{Tr}XW^*U(t,z)WXe^{-\mathrm{i}tz^2/h}\partial_z(\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}})XR_Q(z)Q\mathrm{d}t\mathrm{d}z + O(h^\infty).$$

We may now integrate by parts using $-hD_z/(2z)e^{-\mathrm{i}tz^2/h} = e^{-\mathrm{i}tz^2/h}$ to see that for $\rho \in C_c^\infty$ with $1 \notin \operatorname{supp}(1-\rho)$,

$$N_\chi = \sum_\pm \frac{\mathrm{i}}{2\pi h}\int_{\Gamma_\mp}\tilde{\chi}(z)\int_0^{\pm\infty}\rho(t)\operatorname{Tr}XW^*U(t,z)WXe^{-\mathrm{i}tz^2/h}\partial_z(\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}})XR_Q(z)Q\mathrm{d}t\mathrm{d}z + O(h^\infty).$$

Applying Stokes theorem on $[1-\varepsilon, 1+\varepsilon]\times\mathrm{i}[0,h]$for the integral over Γ_+ and on $[1-\varepsilon,1+\varepsilon]\times -\mathrm{i}[0,h]$, we obtain

$$\begin{aligned}N_\chi &= \sum_\pm \mp\frac{\mathrm{i}}{2\pi h}\int_{\mathbb{R}}\chi(z)\int_0^{\pm\infty}\rho(t)\operatorname{Tr}XW^*U(t,z)WXe^{-\mathrm{i}tz^2/h}\partial_z(\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}})XR_Q(z)Q\mathrm{d}t\mathrm{d}z + O(h^\infty)\\ &= \frac{\mathrm{i}}{2\pi h}\int_{\mathbb{R}}\chi(z)\int\rho(t)\operatorname{Tr}XW^*U(t,z)WXe^{-\mathrm{i}tz^2/h}\partial_z(\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}})XR_Q(z)Q\mathrm{d}t\mathrm{d}z + O(h^\infty).\end{aligned} \tag{6.6}$$

We now use [Zwo12, Theorem 10.4] to write in local coordinates

$$U(t,z)e^{-\mathrm{i}tz^2/h}Xu = \frac{1}{(2\pi h)^{d-1}}\int e^{\frac{\mathrm{i}}{h}(\varphi(t,x,\eta)-\langle y,\eta\rangle - tz^2)}a(t,x,\eta,z)\mathrm{d}\eta(Xu)(y)\mathrm{d}y + O\left(h^\infty\|u\|_{H_h^{-N}(\partial\Omega)}\right),$$

where, with $b = \sigma(B)$,

$$\partial_t\varphi(x,\eta) = b(x,\partial_x\varphi), \qquad \varphi(0,x,\eta) = \langle x,\eta\rangle, \qquad a(0,x,\eta,z) = 1.$$

We can then perform stationary phase in (t,z) (6.6) with critical point $t=0$, and $z_c(x,\eta) = \sqrt{b(x,\eta)}$. Or equivalently,

$$z_c^2(x,\eta) = \frac{\rho_{\mathcal{O}}^2|\eta|^2_{g_{\mathcal{O}}} - \tau^2\rho_{\mathcal{I}}^2|\eta|^2_{g_{\mathcal{I}}}}{\rho_{\mathcal{O}}^2 + \tau^2\rho_{\mathcal{I}}^2}.$$

Moreover, $\sigma\left(\partial_z(\Lambda_{\mathcal{O}}-\tau\Lambda_{\mathcal{I}})\right) = -z\left(\frac{\rho_{\mathcal{O}}}{\sqrt{|\xi'|^2_{g_{\mathcal{O}}}-z^2}}+\frac{\tau\rho_{\mathcal{I}}}{\sqrt{|\xi'|^2_{g_{\mathcal{I}}}+z^2}}\right)$ We then compute the trace by restricting to the diagonal and integrating. This calculations yields

$$N_\chi = \frac{1}{(2\pi h)^{d-1}}\int \chi(\sqrt{b(x,\eta)})\mathrm{d}x\mathrm{d}\eta.$$

Taking a sequence of χ approximating $1_{[1-\varepsilon,1+\varepsilon]}$ shows that

$$\#\{z_j\in\mathcal{Z}_\varepsilon\} = (2\pi h)^{1-d}\mathrm{vol}_{T^*\partial\Omega}\Big\{(x,\xi)\,:\,(1-\varepsilon)^2\le \frac{\rho_{\mathcal{O}}^2|\xi'|^2_{g_{\mathcal{O}}}-\tau^2\rho_{\mathcal{I}}^2|\xi'|^2_{g_{\mathcal{I}}}}{\rho_{\mathcal{O}}^2+\tau^2\rho_{\mathcal{I}}^2}\le(1+\varepsilon)^2\Big\}+o(h^{1-d}).$$

Now, set $\alpha=\frac{1+\varepsilon}{1-\varepsilon}$, $h_j:=(1+\varepsilon)\lambda^{-1}\alpha^j$ and observe that

$$\begin{aligned}
&\#\{\lambda_j\in\mathcal{R}(P)\,:\,0<\mathrm{Re}\,\lambda_j\le\lambda\,:\mathrm{Im}\,\lambda_j\ge -M\}\\
&=\sum_{j=0}^{\lfloor\log_\alpha\lambda\rfloor}\#\{\lambda_j\in\mathcal{R}(P)\,:\,\alpha^{-j-1}\lambda<\mathrm{Re}\,\lambda_j\le\alpha^{-j}\lambda,:\mathrm{Im}\,\lambda_j\ge -M\}+O(1)\\
&=\sum_{j=0}^{\lfloor\log_\alpha\lambda\rfloor}(2\pi h_j)^{1-d}\mathrm{vol}_{T^*\partial\Omega}\Big\{(x,\xi)\,:\,(1-\varepsilon)^2\le\frac{\rho_{\mathcal{O}}^2|\xi'|^2_{g_{\mathcal{O}}}-\tau^2\rho_{\mathcal{I}}^2|\xi'|^2_{g_{\mathcal{I}}}}{\rho_{\mathcal{O}}^2+\tau^2\rho_{\mathcal{I}}^2}\le(1+\varepsilon)^2\Big\}+o(h_j^{1-d})\\
&=o(\lambda^{d-1})+\mathrm{vol}_{T^*\partial\Omega}(\mathcal{V})\big((1+\varepsilon)^{d-1}-(1-\varepsilon)^{d-1}\big)\lambda^{d-1}(1+\varepsilon)^{1-d}(2\pi)^{1-d}\sum_{j=0}^{\lfloor\log_\alpha\lambda\rfloor}\alpha^{j(1-d)}\\
&=(2\pi)^{1-d}\mathrm{vol}_{T^*\partial\Omega}(\mathcal{V})\lambda^{d-1}+o(\lambda^{d-1}),
\end{aligned}$$

which completes the proof. □

Appendix A. Properties of the operator P

In this section, we show that P with domain (1.3) is a black-box Hamiltonian.

We begin with a technical lemma

Lemma A.1. *Suppose that M is a smooth, closed manifold (compact without boundary), $m\in\mathbb{R}$, and $A\in\Psi_1^m(M)$ (i.e. a classical pseudodifferential operator of order m) with*

$$|\sigma(A)(x,\xi)|\ge c|\xi|^m,\qquad -[0,\infty)\cap\{\sigma(A)(x,\xi)\,:\,(x,\xi)\in S^*M\}=\emptyset. \tag{A.1}$$

Then A is a Fredholm operator with index 0.

Proof. Let $a=\sigma(A)$. (Recall that a is homogeneous degree m in ξ.)

$$A-\mathrm{Op}_1(a)=:R\in\Psi^{m-1}(M).$$

Let $\chi_i\in C_c^\infty(\mathbb{R};[0,1])$, $i=1,2$ with $\chi_i\equiv 1$ near 0 and $\mathrm{supp}\,\chi_1\cap\mathrm{supp}(1-\chi_2)=\emptyset$. For $h>0$, let

$$A_h:=\mathrm{Op}_1(a(x,h\xi)(1-\chi_1(h|\xi|))+\chi_2(h|\xi|)).$$

Observe that by (A.1)

$$|a(x,h\xi)(1-\chi_1(h|\xi|))+\chi_2(h|\xi|)|\ge c\langle h\xi\rangle^m.$$

Moreover,

$$A_h = \mathrm{Op}_h(a(x,\xi)(1-\chi_1(|\xi|)) + \chi_2(|\xi|)) \in \Psi_h^m(M).$$

Thus, there is $E \in \Psi_h^{-m}$ such that

$$EA_h = I + hR_{-1}, \qquad R_{-1} \in \Psi_h^{-1}(M).$$

In particular, since

$$\|R_{-1}\|_{H_h^s(M)\to H_h^s(M)} \le C_s h,$$

for h small enough A_h is invertible and $A_h^{-1} \in \Psi_h^{-m}(M)$.

Now,

$$\begin{aligned} A_h^{-1}h^m A &= A_h^{-1}(\mathrm{Op}_1(a(x,h\xi)) + h^m R) \\ &= A_h^{-1}((A_h + \mathrm{Op}_1(\chi_1(h|\xi|)a(x,h|\xi|) - \chi_2(h|\xi|)) + h^m R) = I + K, \end{aligned}$$

where $K : H^s(M) \to H^{s+1}(M)$ and hence is compact. In particular,

$$A = h^{-m}A_h + \tilde{K},$$

where $\tilde{K} : H^{s+m}(M) \to H^{s+m+1}(M) \hookrightarrow H^s(M)$ is compact and hence, since $A_h : H^{s+m}(M) \to H^s(M)$ is invertible, A is Fredholm with index 0. □

We now use Lemma A.1 to study the operator P.

Lemma A.2. *The operator P is self-adjoint.*

Proof. We start by showing that P is symmetric..Notice that, integration by parts implies that for $u, v \in H^2(\Omega_{\mathcal{O}})$,

$$\begin{aligned} &\langle \Delta_{g_{\mathcal{O}},\rho_{\mathcal{O}}} u, v\rangle_{L^2(\Omega_{\mathcal{O}},\rho_{\mathcal{O}}\mathrm{dvol}_{g_{\mathcal{O}}})} \\ &\quad = \langle u, \Delta_{g_{\mathcal{O}},\rho_{\mathcal{O}}} v\rangle_{L^2(\Omega_{\mathcal{O}},\rho_{\mathcal{O}}\mathrm{dvol}_{g_{\mathcal{O}}})} + \langle \rho_{\mathcal{O}}\partial_{\nu_{g_{\mathcal{O}}}} u, v\rangle_{L^2(\partial\Omega,\mathrm{dvol}_{g_{\mathcal{O}},\partial\Omega})} - \langle u, \rho_{\mathcal{O}}\partial_{\nu_{g_{\mathcal{O}}}} v\rangle_{L^2(\partial\Omega,\mathrm{dvol}_{g_{\mathcal{O}},\partial\Omega})}. \end{aligned}$$

In addition, for $u, v \in H^2(\Omega_{\mathcal{I}})$,

$$\begin{aligned} &\langle \Delta_{g_{\mathcal{I}},\rho_{\mathcal{I}}} u, v\rangle_{L^2(\Omega_{\mathcal{I}},\rho_{\mathcal{I}}\mathrm{dvol}_{g_{\mathcal{I}}})} \\ &\quad = \langle u, \Delta_{g_{\mathcal{I}},\rho_{\mathcal{I}}} v\rangle_{L^2(\Omega_{\mathcal{I}},\rho_{\mathcal{I}}\mathrm{dvol}_{g_{\mathcal{I}}})} + \langle \tau\rho_{\mathcal{I}}\partial_{\nu_{g_{\mathcal{I}}}} u, v\rangle_{L^2(\partial\Omega,\mathrm{dvol}_{g_{\mathcal{I}},\partial\Omega})} - \langle u, \tau\rho_{\mathcal{I}}\partial_{\nu_{g_{\mathcal{I}}}} v\rangle_{L^2(\partial\Omega,\mathrm{dvol}_{g_{\mathcal{O}},\partial\Omega})}. \end{aligned}$$

In particular, the operator P is symmetric.

We next show that there is z with $\operatorname{Im} z > 0$ such that $(P - z^2) : \mathcal{D}(P) \to L^2$ and $(P - \overline{z^2}) : \mathcal{D}(P) \to L^2$ are surjective. This then implies that $P - \operatorname{Re}(z^2)$ and hence also P is self-adjoint.

To do this, recall the definitions of $R_{\mathcal{O}}(z)$, $G_{\mathcal{O}}(z)$, $R_{\mathcal{I}}(z)$ and $G_{\mathcal{I}}(z)$ from Sections 3 and 3.2, and note that $R_{\mathcal{O}} : L^2(\Omega_{\mathcal{O}}) \to H^2(\Omega_{\mathcal{O}})$, $G_{\mathcal{O}} : H^{3/2}(\partial\Omega) \to H^2(\Omega_{\mathcal{O}})$, $R_{\mathcal{I}} : L^2(\Omega_{\mathcal{I}}) \to H^2(\Omega_{\mathcal{I}})$, and $G_{\mathcal{I}} : H^{3/2}(\partial\Omega) \to H^2(\Omega_{\mathcal{I}})$ are analytic families of operators in $\operatorname{Im} z > 0$.

In particular, $\Lambda_{\mathcal{O}}(z) - \tau\Lambda_{\mathcal{I}}(z) : H^{3/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ is an analytic family of operators in $\operatorname{Im} z > 0$. Moreover, since $\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}} \in \Psi^1(\partial\Omega)$ (i.e. is a non-semiclassical pseudodifferential operator of order 1) has real principal symbol is elliptic in this class Lemma A.1 implies that $\Lambda_{\mathcal{O}} - \tau\Lambda_{\mathcal{I}}$, is an analytic family of Fredholm operators with index 0 in $\operatorname{Im} z > 0$. Thus, by the analytic Fredholm theorem, $(\Lambda_{\mathcal{O}}(z) - \tau\Lambda_{\mathcal{I}}(z))^{-1}$ is a meromorphic family of Fredholm operators with index 0 in $\operatorname{Im} z > 0$.

Using this, we have, in $\operatorname{Im} z > 0$

$$(P - z^2)\left(I + \begin{pmatrix} G_{\mathcal{I}}(z) \\ G_{\mathcal{O}}(z) \end{pmatrix} (\Lambda_{\mathcal{O}}(z) - \tau\Lambda_{\mathcal{I}}(z))^{-1} \begin{pmatrix} \tau\rho_{\mathcal{I}}\partial_{\nu_{\mathcal{I}}} & -\rho_{\mathcal{O}}\partial_{\nu_{\mathcal{O}}} \end{pmatrix}\right) \begin{pmatrix} R_{\mathcal{I}}(z) & 0 \\ 0 & R_{\mathcal{O}}(z) \end{pmatrix} = I_{L^2 \to L^2}.$$

In particular, since the poles of $(\Lambda_{\mathcal{O}}(z) - \tau\Lambda_{\mathcal{I}}(z))^{-1}$ form a discrete set in $\operatorname{Im} z > 0$, one can find z such that $P - z^2 : \mathcal{D}(P) \to L^2$ and $\tilde{P} - (-\bar{z})^2 : \mathcal{D}(P) \to L^2$ are surjective. □

Lemma A.3. *The operator P is a black box Hamiltoniain in the sense of* [DZ19, Definition 4.1].

Proof. The conditions [DZ19, (4.1.4) , (4.1.5), and (4.1.6)] are obviously satisfied. It remains to check that for $1_{B(0,R_0)}(P + \mathrm{i})^{-1}$ is compact, but this follows from the fact that $(P + \mathrm{i})^{-1} : L^2 \to \mathcal{D}(P) \subset (H^2(\Omega_{\mathcal{I}}) \oplus H^2(\Omega_{\mathcal{O}})) \cap H^1(\mathbb{R}^d)$ and the Rellich–Kondrachov embedding theorem. □

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