IMPROVEMENTS FOR EIGENFUNCTION AVERAGES: AN APPLICATION OF GEODESIC BEAMS

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ABSTRACT. Let (M,g) be a smooth, compact Riemannian manifold and $\{\phi_\lambda\}$ an L^2-normalized sequence of Laplace eigenfunctions, $-\Delta_g \phi_\lambda = \lambda^2\phi_\lambda$. Given a smooth submanifold $H \subset M$ of codimension $k \geq 1$, we find conditions on the pair (M,H), even when $H = \{x\}$, for which

$$\Big|\int_H \phi_\lambda d\sigma_H\Big| = O\Big(\frac{\lambda^{\frac{k-1}{2}}}{\sqrt{\log \lambda}}\Big) \qquad \text{or} \qquad |\phi_\lambda(x)| = O\Big(\frac{\lambda^{\frac{n-1}{2}}}{\sqrt{\log \lambda}}\Big),$$

as $\lambda \to \infty$. These conditions require no global assumption on the manifold M and instead relate to the structure of the set of recurrent directions in the unit normal bundle to H. Our results extend all previously known conditions guaranteeing improvements on averages, including those on sup-norms. For example, we show that if (M,g) is a surface with Anosov geodesic flow, then there are logarithmically improved averages for any $H \subset M$. We also find weaker conditions than having no conjugate points which guarantee $\sqrt{\log \lambda}$ improvements for the L^∞ norm of eigenfunctions. Our results are obtained using geodesic beam techniques, which yield a mechanism for obtaining general quantitative improvements for averages and sup-norms.

1. Introduction

On a smooth compact Riemannian manifold without boundary of dimension n, (M,g), we consider sequences of Laplace eigenfunctions $\{\phi_\lambda\}$ solving

$$(-\Delta_g - \lambda^2)\phi_\lambda = 0, \qquad \|\phi_\lambda\|_{L^2(M)} = 1.$$

We study the average oscillatory behavior of ϕ_λ when restricted to a submanifold $H \subset M$ without boundary. In particular, we examine the behavior of the integral average $\int_H \phi_\lambda d\sigma_H$ as $\lambda \to \infty$, where σ_H is the volume measure on H induced by the Riemannian metric. Since we allow H to consist of a single point, our results include the study of sup-norms $\|\phi_\lambda\|_{L^\infty(M)}$.

The study of these quantities has a long history. In general

$$\int_H \phi_\lambda d\sigma_H = O(\lambda^{\frac{k-1}{2}}) \qquad \text{and} \qquad \|\phi_\lambda\|_{L^\infty(M)} = O(\lambda^{\frac{n-1}{2}}), \tag{1.1}$$

where k is the codimension of H, and H is any smooth embedded submanifold. The sup-norm bound in (1.1) is a consequence of the well known works [Ava56, Lev52, Hör68]. The bound on averages was first obtained in [Goo83] and [Hej82], for the case in which H is a periodic geodesic in a compact hyperbolic surface. The general bound in (1.1) for integral averages was proved by Zelditch in [Zel92, Corollary 3.3].

Since it is easy to find examples on the round sphere which saturate the estimate (1.1), it is natural to ask whether the bound is typically saturated, and to understand conditions under which the estimate may be improved.

In [CG19, Gal19, CGT18, GT17], the authors (together with Toth in the latter two cases) gave bounds on integral averages based on understanding microlocal concentration as measured by defect measures (see [Zwo12, Chapter 5] or [Gér91] for a description of defect measures). In particular, [CG19] gave a new proof of (1.1) and studied conditions on $(\{\phi_\lambda\}, H)$ guaranteeing

$$\int_H \phi_\lambda d\sigma_H = o\big(\lambda^{\frac{k-1}{2}}\big). \tag{1.2}$$

These conditions generalized and weakened the assumptions in [SZ02, STZ11, CS15, SXZ17, Wym17, Wym20a, Wym19, GT17, Gal19, CGT18, Bér77, SZ16a, SZ16b] which guarantee at least the improvement (1.2). However, the results in [CG19] neither recovered the bound

$$\int_H \phi_\lambda d\sigma_H = O\left(\frac{\lambda^{\frac{k-1}{2}}}{\sqrt{\log \lambda}}\right), \tag{1.3}$$

obtained in [SXZ17, Wym20a, Wym20b] under various conditions on H when M has non-positive curvature, nor recovered the improvement on sup-norms given in [Bér77, Bon17, Ran78] when $k = n$ and M has no conjugate points. In the present article, we address such quantitative improvements.

To the authors' knowledge, this article improves and extends *all* existing bounds on averages over submanifolds for eigenfunctions of the Laplacian, including those on L^∞ norms (without additional assumptions on the eigenfunctions; see Remark 1 for more detail on other types of assumptions). The estimates from [CG20a] imply those of [CG19] and therefore can be used to obtain all previously known improvements of the form (1.2). In this article, we make the geometric arguments necessary to apply geodesic beam techniques and improve upon the results of [Wym20b, Wym20a, SXZ17, Bér77, Bon17, Ran78].

These improvements are possible because the geodesic beam techniques developed in [CG20a] give an explicit bound on averages over submanifolds, H, which depends only on microlocal information about ϕ_λ near the unit conormal bundle to H, SN^*H. In particular, microlocally near the conormal bundle to H, the quasimodes are decomposed into what we call geodesic beams: $\phi_\lambda = \sum_{j\in\mathcal{J}} \chi_{\mathcal{T}_j}\phi_\lambda$ near H. Each geodesic beam, $\chi_{\mathcal{T}_j}\phi_\lambda$, is obtained by localizing ϕ_λ to a length ~ 1 geodesic tube $\mathcal{T}_j$ of radius $R(\lambda) \sim \lambda^{-1/2+\delta}$ around a geodesic through SN^*H. The contributions of these tubes are then estimated using an energy estimate due to Koch–Tataru–Zworski [KTZ07]. After recombining, the estimate reads (for the case $H = \{x\}$)

$$|\phi_\lambda(x)| \leq CR(\lambda)^{(n-1)/2}\lambda^{(n-1)/2}\sum_{j\in\mathcal{J}} \|\chi_{\mathcal{T}_j}\phi_\lambda\|_{L^2(M)}.$$

This estimate requires no assumptions on the geometry of H or M and is purely local. It is only with this bound in place that [CG20a] applies Egorov's theorem to $\log\lambda$ time in order to obtain a purely dynamical estimate (see also Theorem 5) of the form

$$|\phi_\lambda(x)| \leq CR(\lambda)^{(n-1)/2}\lambda^{(n-1)/2}\Big(|\mathcal{B}|^{1/2} + \frac{|\mathcal{G}|^{1/2}}{|\log\lambda|^{1/2}}\Big)\|\phi_\lambda\|_{L^2(M)}, \tag{1.4}$$

where $\cup_{j\in\mathcal{G}}\mathcal{T}_j$ is non-self looping for $\log\lambda$ time (see (1.16)) and $\mathcal{J} = \mathcal{G}\cup\mathcal{B}$. See Section 1.1 for a more detailed explanation of the techniques which includes estimates similar to (1.4) which allow for multiple non-looping sets, and [CG20a] for the proofs of these analytic statements.

In this article, we apply dynamical arguments to draw conclusions about the pairs $((M,g),H)$ supporting eigenfunctions with maximal averages. While previous works on eigenfunction averages rely on explicit parametrices for the kernel of the half wave-group for large times, the authors' techniques [GT17, Gal19, CGT18, CG19, CG20a], show that improvements can be effectively obtained by understanding the microlocalization properties of eigenfunctions.

Remark 1. Note that in this paper we study averages of relatively weak quasimodes for the Laplacian with no additional assumptions on the functions. This is in contrast with results which impose additional conditions on the functions such as: that they be Laplace eigenfunctions that simultaneously satisfy additional equations [IS95, GT20, Tac19]; that they be eigenfunctions in the very rigid case of the flat torus [Bou93, Gro85]; or that they form a density one subsequence of Laplace eigenfunctions [JZ16].

We now state the main results of this article. In order to match the language of [CG20a], we will semiclassically rescale, setting $h = \lambda^{-1}$ and sending $h \to 0^+$. Relabeling, ϕ_λ as ϕ_h, the eigenfunction equation becomes

$$(-h^2\Delta_g - 1)\phi_h = 0, \qquad \|\phi_h\|_{L^2} = 1.$$

We also recall the notation for the semiclassical Sobolev norms:

$$\|u\|^2_{H^s_{\mathrm{scl}}(M)} := \big\langle(-h^2\Delta_g+1)^s u, u\big\rangle_{L^2(M)}. \tag{1.5}$$

Let Ξ denote the collection of maximal unit speed geodesics for (M,g). For m a positive integer, $r>0$, $t\in\mathbb{R}$, and $x\in M$ define

$$\Xi_x^{m,r,t} := \big\{\gamma\in\Xi : \gamma(0) = x,\ \exists \text{ at least } m \text{ conjugate points to } x \text{ in } \gamma(t-r,t+r)\big\},$$

where we count conjugate points with multiplicity. Next, for a set $V\subset M$ write

$$\mathcal{C}_V^{m,r,t} := \bigcup_{x\in V}\{\gamma(t) : \gamma\in\Xi_x^{m,r,t}\}.$$

Note that if $r_t\to 0^+$ as $|t|\to\infty$, then saying that $x\in\mathcal{C}_x^{n-1,r_t,t}$ for t large indicates that x behaves like a point that is maximally self-conjugate. This is the case for every point on the sphere. The following result applies under the assumption that this does not happen and obtains quantitative improvements in that setting.

Theorem 1. *Let $V\subset M$ and assume that there exist $t_0>0$ and $a>0$ so that*

$$\inf_{x\in V} d\big(x,\mathcal{C}_x^{n-1,r_t,t}\big) \geq r_t, \qquad \textit{for } t\geq t_0$$

with $r_t = \frac{1}{a}e^{-at}$. Then, there exist $C > 0$ and $h_0 > 0$ so that for $0 < h < h_0$ and $u \in \mathcal{D}'(M)$

$$\|u\|_{L^\infty(V)} \le Ch^{\frac{1-n}{2}}\left(\frac{\|u\|_{L^2(M)}}{\sqrt{\log h^{-1}}} + \frac{\sqrt{\log h^{-1}}}{h}\big\|(-h^2\Delta_g - 1)u\big\|_{H_{\rm scl}^{\frac{n-3}{2}}(M)}\right).$$

In fact a generalization of Theorem 1 holds not just for $H = \{x\}$, but for any $H \subset M$ of large enough codimension.

Theorem 2. *Let $H \subset M$ be a closed embedded submanifold of codimension $k > \frac{n+1}{2}$ and assume that there exist $t_0 > 0$ and $a > 0$ such that*

$$d\big(H, \mathcal{C}_H^{2k-n-1, r_t, t}\big) \ge r_t, \qquad \text{for } t \ge t_0 \tag{1.6}$$

with $r_t := \frac{1}{a}e^{-at}$. Then, there exists $C > 0$, so that for all $w \in C_c^\infty(H)$ the following holds. There exists $h_0 > 0$ such that for all $0 < h < h_0$ and $u \in \mathcal{D}'(M)$,

$$\Big|\int_H w u d\sigma_H\Big| \le Ch^{\frac{1-k}{2}}\|w\|_\infty\left(\frac{\|u\|_{L^2(M)}}{\sqrt{\log h^{-1}}} + \frac{\sqrt{\log h^{-1}}}{h}\big\|(-h^2\Delta_g - 1)u\big\|_{H_{\rm scl}^{\frac{k-3}{2}}(M)}\right). \tag{1.7}$$

Remark 2. One should think of the assumption in Theorem 1 as ruling out maximal self-conjugacy of a point with itself uniformly up to time ∞. In fact, in order to obtain an L^∞ bound of $o(h^{\frac{1-n}{2}})$ on $u(x)$, it is enough to assume that there is not a positive measure set of directions $A \subset S_x^*M$ so that for each element $\xi \in A$ there is a sequence of geodesics starting at x in the direction of ξ with length tending to infinity along which x is maximally conjugate to itself.

Before stating our next theorem, we recall that if (M, g) has strictly negative sectional curvature, then it also has Anosov geodesic flow [Ano67]. Also, both Anosov geodesic flow and non-positive sectional curvature imply that (M, g) has no conjugate points [Kli74].

When (M, g) is non-positively curved (indeed when it has no focal points), if every geodesic encounters a point of negative curvature, then (M, g) has Anosov geodesic flow [Ebe73a, Corollary 3.4]. In particular, there are manifolds for which the curvature is positive in some places while the geodesic flow is Anosov. However, even in non-positive curvature some geodesics may fail to encounter negative curvature and thus the geodesic flow may not be Anosov. To study this situation, we introduce an integrated curvature condition inspired by that in [SXZ17]: There are $T > 0$, and $c_K > 0$ so that for every geodesic γ of length $t \ge T$ in the universal cover $(\tilde{M}, \tilde{g})$ of (M, g), and for all $0 \le s \le 1$,

$$\int_{\Omega_\gamma(s)} K dv_{\tilde{g}} \le -c_K e^{-\frac{1}{c_K\sqrt{s}}} \tag{1.8}$$

where $\Omega_\gamma(s) := \{x \in \tilde{M} : \ d(x, \gamma) \le s\}$, and K is the scalar curvature for $(\tilde{M}, \tilde{g})$. Note that, unlike the curvature conditions in [SXZ17], the assumption in (1.8) allows the curvature to vanish in open sets so long as no geodesic lies entirely in such an open set. Moreover, it allows the curvature to vanish to infinite order at the geodesic.

Theorem 3. *Let (M,g) be a smooth, compact Riemannian surface. Let $H \subset M$ be a closed embedded curve or a point. Suppose one of the following assumptions holds:*

- ***A.*** *(M,g) has Anosov geodesic flow.*
- ***B.*** *(M,g) has non-positive curvature and satisfies the integrated curvature condition (1.8), and H is a geodesic.*

Then, there exists $C>0$ so that for all $w \in C_c^\infty(H)$ the following holds. There is $h_0>0$ so that for $0<h<h_0$ and $u \in \mathcal{D}'(M)$

$$\Big|\int_H wud\sigma_H\Big| \le Ch^{\frac{1-k}{2}}\|w\|_\infty\Big(\frac{\|u\|_{L^2(M)}}{\sqrt{\log h^{-1}}} + \frac{\sqrt{\log h^{-1}}}{h}\|(-h^2\Delta_g - 1)u\|_{H_{\rm scl}^{\frac{k-3}{2}}(M)}\Big). \quad (1.9)$$

Remark 3. In fact, the proof Theorem 3.B shows that it is enough to have (1.8) for every geodesic γ normal to H.

For manifolds of arbitrary dimensions, we also obtain quantitative improvements for averages in a variety of situations.

Theorem 4. *Let (M,g) be a smooth, compact Riemannian manifold of dimension n and $H \subset M$ be a closed embedded submanifold of codimension k. Suppose one of the following assumptions holds:*

- ***A.*** *(M,g) has no conjugate points and H has codimension $k > \frac{n+1}{2}$.*
- ***B.*** *(M,g) has no conjugate points and H is a geodesic sphere.*
- ***C.*** *(M,g) is non-positively curved and has Anosov geodesic flow, and H has codimension $k>1$.*
- ***D.*** *(M,g) is non-positively curved and has Anosov geodesic flow, and H is totally geodesic.*
- ***E.*** *(M,g) has Anosov geodesic flow and H is a subset of M that lifts to a horosphere in the universal cover.*

Then, there exists $C>0$ so that for all $w \in C_c^\infty(H)$ the following holds. There is $h_0>0$ so that for $0<h<h_0$ and $u \in \mathcal{D}'(M)$

$$\Big|\int_H wud\sigma_H\Big| \le Ch^{\frac{1-k}{2}}\|w\|_\infty\Big(\frac{\|u\|_{L^2(M)}}{\sqrt{\log h^{-1}}} + \frac{\sqrt{\log h^{-1}}}{h}\|(-h^2\Delta_g - 1)u\|_{H_{\rm scl}^{\frac{k-3}{2}}(M)}\Big). \quad (1.10)$$

We note here that Theorem 3.B includes the bounds of [SXZ17] as a special case (see Remark 12 for an explanation). The bounds in [Wym20a, Wym20b] are special cases of Theorem 3.A, Theorem 4.C, and the results of Theorem 6 below (see the discussion that follows Theorem 6). We also note that for any smooth compact embedded submanifold, $H_0 \subset M$, satisfying one of the conditions in Theorem 4, there is a neighborhood U of H_0, in the C^∞ topology, so that the constants C and h_0 in Theorem 4 are uniform over $H \in U$ and w taken in a bounded subset of $C_c^\infty(H)$. In particular, the sup-norm bounds from [Bér77, Bon17, Ran78] are a special case of Theorem 4.A. Similar to the $o(h^{\frac{1-k}{2}})$ bounds in [CG19], we conjecture that (1.10) holds whenever (M,g) is a manifold with Anosov geodesic flow, regardless of the geometry of H.

Geodesic beam techniques can also be used to study L^p norms of eigenfunctions [CG20b] and to give quantitatively improved remainder estimates for the kernel of the spectral projector and for Kuznecov sum type formulae [CG20c]. The authors are currently studying how to give polynomial improvements for L^∞ norms on certain manifolds with integrable geodesic flow. To our knowledge the only other case where polynomial improvements are available is in [IS95] for Hecke–Maase forms on arithmetic surfaces or when (M, g) is the flat torus [Bou93, Gro85].

1.1. **Results on geodesic beams.** The main estimate from [CG20a] gives control on eigenfunction averages in terms of microlocal data. We now review the necessary notation to state that result.

Let $p(x,\xi) = |\xi|_{g(x)}$ defined on T^*M and consider the geodesic flow on T^*M,

$$\varphi_t := \exp(tH_p). \tag{1.11}$$

Next, fix a hypersurface

$$\mathcal{H}_\Sigma \subset T^*M \text{ transverse to } H_p \text{ with } SN^*H \subset \mathcal{H}_\Sigma, \tag{1.12}$$

define $\Psi : \mathbb{R} \times \mathcal{H}_\Sigma \to T^*M$ by $\Psi(t, q) = \varphi_t(q)$, and let

$$\tau_{{}_{\mathrm{inj}H}} := \sup\{\tau \le 1 : \ \Psi|_{(-\tau,\tau)\times\mathcal{H}_\Sigma} \text{ is injective}\}. \tag{1.13}$$

Given $A \subset T^*M$ define

$$\Lambda^\tau_{{}_A} := \bigcup_{|t|\le\tau} \varphi_t(A).$$

For $r > 0$ and $A \subset SN^*H$ we define

$$\Lambda^\tau_{{}_A}(r) := \Lambda^{\tau+r}_{{}_{A_r}}, \qquad A_r := \{\rho \in \mathcal{H}_\Sigma : d(\rho, A) < r\}. \tag{1.14}$$

where d denotes the distance induced by the Sasaki metric on TM (see e.g. Appendix 6 or [Bla10, Chapter 9] for an explanation of the Sasaki metric).

Throughout the paper we adopt the notation

$$K_{{}_H} > 0 \tag{1.15}$$

for a constant so that all sectional curvatures of H are bounded by $K_{{}_H}$ and the second fundamental form of H is bounded by $K_{{}_H}$. Note that when H is a point, we may take $K_{{}_H}$ to be arbitrarily close to 0.

We next recall [CG20a, Theorem 11] which controls eigenfunction averages by covers of $\Lambda^\tau_{SN^*H}(h^\delta)$ by "good" tubes that are non self-looping and "bad" tubes whose number is controlled. In fact, Theorems 1, 2, and 4 are reduced to a purely dynamical argument together with an application of Theorem 5.

For $0 < t_0 < T_0$, we say that $A \subset T^*M$ is $[t_0, T_0]$ *non-self looping* if

$$\bigcup_{t=t_0}^{T_0} \varphi_t(A) \cap A = \emptyset \qquad \text{or} \qquad \bigcup_{t=-T_0}^{-t_0} \varphi_t(A) \cap A = \emptyset. \tag{1.16}$$

We define the *maximal expansion rate*

$$\Lambda_{\max} := \limsup_{|t|\to\infty} \frac{1}{|t|} \log \sup_{S^*M} \|d\varphi_t(x,\xi)\|. \tag{1.17}$$

Then, the Ehrenfest time at frequency h^{-1} is

$$T_e(h) := \frac{\log h^{-1}}{2\Lambda_{\max}}. \tag{1.18}$$

Note that $\Lambda_{\max} \in [0,\infty)$ and if $\Lambda_{\max} = 0$, we may replace it by an arbitrarily small positive constant.

Definition 1. Let $A \subset SN^*H$, $r > 0$, $\tau > 0$, and $\{\rho_j\}_{j=1}^{N_r} \subset A$. We say that the collection of tubes $\{\Lambda^\tau_{\rho_j}(r)\}_{j=1}^{N_r}$ is a (τ, r)-*cover* of a set $A \subset SN^*H$ provided

$$\Lambda^\tau_A(\tfrac{1}{2}r) \subset \bigcup_{j=1}^{N_r} \Lambda^\tau_{\rho_j}(r).$$

It will often be useful to have a notion of (τ, r) cover of SN^*H without too many overlapping tubes. To that end, we make the following definition.

Definition 2. Let $A \subset SN^*H$, $r > 0$, $\mathfrak{D} > 0$, and $\{\rho_j\}_{j=1}^{N_r} \subset A$. We say that the collection of tubes $\{\Lambda^\tau_{\rho_j}(r)\}_{j=1}^{N_r}$ is a $(\mathfrak{D}, \tau, r)$-*good cover* of a set $A \subset SN^*H$ provided that it is a (τ, r)-cover for A and there exists a partition $\{\mathcal{J}_\ell\}_{\ell=1}^{\mathfrak{D}}$ of $\{1, \dots, N_r\}$ so that for every $\ell \in \{1, \dots, \mathfrak{D}\}$

$$\Lambda^\tau_{\rho_j}(3r) \cap \Lambda^\tau_{\rho_i}(3r) = \emptyset \qquad i, j \in \mathcal{J}_\ell, \quad i \neq j.$$

We recall that [CG20a, Proposition 3.3] shows the existence of $\mathfrak{D}_n > 0$, depending only on n, so that for all sufficiently small (τ, r) there are of $(\mathfrak{D}_n, \tau, r)$ good covers of SN^*H. We will use this fact freely throughout this article.

For convenience we state [CG20a, Theorem 11]. The theorem involves many parameters. These provide flexibility when applying the theorem, but make the statement involved. We refer the reader to the comments after the statement of the theorem for a heuristic explanation of its contents.

Theorem 5 ([CG20a, Theorem 11])**.** *Let $H \subset M$ be a submanifold of codimension k. Let $0 < \delta < \frac{1}{2}$, $N > 0$ and $\{w_h\}_h$ with $w_h \in S_\delta \cap C_c^\infty(H)$. There exist positive constants $\tau_0 = \tau_0(M, g, \tau_{\mathit{injH}}, H)$, $R_0 = R_0(M, g, K_H, k, \tau_{\mathit{injH}})$, $C_{n,k}$ depending only on n and k, and $h_0 = h_0(M, g, \delta, H)$, and for each $0 < \tau \leq \tau_0$ there exist $C = C(M, g, \tau, \delta, , H) > 0$ and $C_N = C_N(M, g, N, \tau, \delta, \{w_h\}_h, H) > 0$, so that the following holds.*

Let $8h^\delta \leq R(h) \leq R_0$, $0 \leq \alpha < 1 - 2\limsup_{h\to 0} \frac{\log R(h)}{\log h}$, and suppose $\{\Lambda^\tau_{\rho_j}(R(h))\}_{j=1}^{N_h}$ is a $(\mathfrak{D}, \tau, R(h))$ cover of SN^*H for some $\mathfrak{D} > 0$.

In addition, suppose there exist $\mathcal{B} \subset \{1, \dots, N_h\}$ and a finite collection $\{\mathcal{G}_\ell\}_{\ell \in \mathcal{L}} \subset \{1, \dots, N_h\}$ with

$$\mathcal{J}_h(w_h) \ \subset\ \mathcal{B} \cup \bigcup_{\ell \in \mathcal{L}} \mathcal{G}_\ell,$$

where

$$\mathcal{J}_h(w_h) := \{j :\ \Lambda^\tau_{\rho_j}(2R(h)) \cap \pi^{-1}(\operatorname{supp} w_h) \neq \emptyset\}, \tag{1.19}$$

and so that for every $\ell \in \mathcal{L}$ there exist $t_\ell = t_\ell(h) > 0$ and $T_\ell = T_\ell(h) \le 2\alpha T_e(h)$ so that

$$\bigcup_{j\in\mathcal{G}_\ell} \Lambda^\tau_{\rho_j}(R(h)) \quad \textit{is} \quad [t_\ell, T_\ell] \textit{ non-self looping for } \varphi_t := \exp(tH_{|\xi|_g}).$$

Then, for $u \in \mathcal{D}'(M)$ and $0 < h < h_0$,

$$\begin{aligned} h^{\frac{k-1}{2}}\Big|\int_H w_h u\, d\sigma_H\Big| \le{}& \frac{C_{n,k}\mathfrak{D}\|w_h\|_\infty R(h)^{\frac{n-1}{2}}}{\tau^{\frac12}}\Bigg(|\mathcal{B}|^{\frac12} + \sum_{\ell\in\mathcal{L}} \frac{(|\mathcal{G}_\ell| t_\ell)^{\frac12}}{T_\ell^{\frac12}}\Bigg)\|u\|_{L^2(M)} \\ &+ \frac{C_{n,k}\mathfrak{D}\|w_h\|_\infty R(h)^{\frac{n-1}{2}}}{\tau^{\frac12}} \sum_{\ell\in\mathcal{L}} \frac{(|\mathcal{G}_\ell| t_\ell T_\ell)^{\frac12}}{h}\ \|(-h^2\Delta_g - 1)u\|_{L^2(M)} \\ &+ Ch^{-1}\|w_h\|_\infty \|(-h^2\Delta_g-1)u\|_{H^{\frac{k-3}{2}}_{\mathrm{scl}}(M)} \\ &+ C_N h^N\big(\|u\|_{L^2(M)} + \|(-h^2\Delta_g-1)u\|_{H^{\frac{k-3}{2}}_{\mathrm{scl}}(M)}\big). \end{aligned}$$

Here, the constant C_N depends on $\{w_h\}_h$ only through finitely many S_δ seminorms of w_h. The constants τ_0, C, C_N, h_0 depend on H only through finitely many derivatives of its curvature and second fundamental form.

Remark 4. The estimates in Theorem 5 are uniform in H. For a precise description see [CG20a, Theorem 11]. In particular, when $H = \{x\}$ and $w = 1$, then $k = 0$ and $|\int_H w_h u\, d\sigma_H|$ is replaced with $\|u\|_{L^\infty(B(x,h^\delta))}$.

Theorem 5 reduces estimates on averages to construction of covers of $\Lambda^\tau_{SN^*H}(h^\delta)$ by sets with appropriate structure. To understand the statement, we first ignore the extra structure requirement and assume $(-h^2\Delta_g - 1)u = 0$. With these simplifications, and ignoring an $h^\infty\|u\|_{L^2(M)}$ term, if there is a cover of $\Lambda^\tau_{SN^*H}(h^\delta)$ by "good" sets $\{G_\ell(h)\}_{\ell\in L}$ and a "bad" set $B(h)$ with G_ℓ, $[t_\ell(h), T_\ell(h)]$ non-self looping, the estimate reads

$$h^{\frac{k-1}{2}}\Big|\int_H wud\sigma_H\Big| \le \frac{C_{n,k}\|w\|_\infty}{\tau^{\frac12}}\Bigg([\sigma_{SN^*H}(B)]^{\frac12} + \sum_{\ell\in\mathcal{L}} \frac{[\sigma_{SN^*H}(G_\ell)]^{\frac12} t_\ell^{\frac12}}{T_\ell^{\frac12}(h)}\Bigg)\|u\|_{L^2(M)},$$

where σ_{SN^*H} denotes the volume induced on SN^*H by the Sasaki metric on T^*M and for $A \subset T^*M$, we write $\sigma_{SN^*H}(A) = \sigma_{SN^*H}(A \cap SN^*H)$. The additional structure required on the sets G_ℓ and B is that they consist of a union tubes $\Lambda^\tau_{\rho_i}(h^\delta)$ for some $0 \le \delta < \frac12$ and that $T_\ell(h) < 2(1-2\delta)T_e(h)$. With this in mind, Theorem 5 should be thought of as giving non-recurrent condition on SN^*H which guarantees quantitative improvements over (1.1). This type of non-recurrence was exploited in [GT20] to understand L^∞ norms for eigenfunctions at the umbillic points of the tri-axial ellipsoid, a *quantum-completely integrable* situation. Taking t_ℓ, T_ℓ, G_ℓ and B to be h-independent can be used to recover the dynamical consequences in [CG19, Gal19] (see [Gal18]).

Remark 5. Note that it is possible to use Theorem 5 to obtain quantitative estimates which are strictly between $O(h^{\frac{1-k}{2}})$ and $O(h^{\frac{1-k}{2}}/\sqrt{\log h^{-1}})$. For example, this happens if r_t is replaced by e.g. $a^{-1}e^{-at^2}$ in (1.6). We expect that the construction in [BP96] can be used to generate examples where this type of behavior is optimal.

1.2. **Manifolds with no focal points or Anosov geodesic flow.** In parts 3.A, 4.C, 4.D and 4.E of Theorem 4 we assume either that (M, g) has no focal points or that it has Anosov geodesic flow. We show that these structures allow us to construct non-self looping covers away from the points $\mathcal{S}_H \subset SN^*H$ at which the tangent space to SN^*H splits into a sum of stable and unstable directions. To make this sentence precise we introduce some notation.

If (M, g) has no conjugate points, then for any $\rho \in S^*M$ there exist a weak stable subspace $E^w_+(\rho) \subset T_\rho S^*M$ and a weak unstable subspace $E^w_-(\rho) \subset T_\rho S^*M$ so that

$$d\varphi_t : E^w_\pm(\rho) \to E^w_\pm(\varphi_t(\rho)),$$

and

$$|d\varphi_t(\mathbf{v})| \leq C|\mathbf{v}| \ \text{ for } \mathbf{v} \in E^w_\pm \ \text{ and } t \to \pm\infty.$$

(see e.g. [Ebe73a, Proposition 2.13] which is based on [Gre58]) We also define the stable $(+)$ and unstable $(-)$ subspaces as $E_\pm(\rho) = E^w_\pm(\rho) \cap (\mathbb{R}H_p)^\perp$ where the orthogonal complement is taken with respect to the Sasaki metric. These subspaces then have the property that

$$T_\rho S^*M = (E_+(\rho) + E_-(\rho)) \oplus \mathbb{R}H_p(\rho).$$

While this particular decomposition happens to be an orthogonal sum, throughout the article we will use $A = A_1 \oplus A_2$ to mean direct sum i.e. that $A = A_1 + A_2$ and $A_1 \cap A_2 = \{0\}$.

We recall that a manifold has no focal points if for every geodesic γ, and every Jacobi field $Y(t)$ along γ with $Y(0) = 0$ and $Y'(0) \neq 0$, $Y(t)$ satisfies $\frac{d}{dt}\|Y(t)\|^2 > 0$ for $t > 0$, where $\|\cdot\|$ denotes the norm with respect to the Riemannian metric. In particular, if (M, g) has non-positive curvature, then it has no focal points (see e.g. [Ebe73a, page 440]). It is also known that if (M, g) has no focal points then (M, g) has no conjugate points and that $E_\pm(\rho)$ vary continuously with ρ. (See for example [Ebe73a, Proposition 2.13 and remarks thereafter].) See e.g. [Rug07, Ebe73b, Pes77] for further discussions of manifolds without focal points.

The geodesic flow is said to be Anosov [Ano67] if there exist $E_\pm(\rho) \subset T_\rho S^*M$ and $\mathbf{B} > 0$ so that for all $\rho \in S^*M$,

$$|d\varphi_t(\mathbf{v})| \leq \mathbf{B}e^{\mp \frac{t}{\mathbf{B}}}|\mathbf{v}|, \qquad \mathbf{v} \in E_\pm(\rho), \quad t \to \pm\infty, \tag{1.20}$$

and

$$T_\rho S^*M = E_+(\rho) \oplus E_-(\rho) \oplus \mathbb{R}H_p. \tag{1.21}$$

Recall that a manifold with Anosov geodesic flow does not have conjugate points [Kli74] and hence we use the same notation $E_\pm(\rho)$ as in that case. In fact, a manifold has Anosov geodesic flow if and only if it has no conjugate points and (1.21) holds [Ebe73a, Theorem 3.2]. One consequence of having Anosov geodesic flow is that the spaces $E_\pm(\rho)$ are Hölder continuous in ρ [KH95, Theorem 19.1.6].

In order to find examples of manfiolds with Anosov geodesic flow, we recall that any manifold with no focal points in which every geodesic encounters a point of negative curvature has Anosov geodesic flow [Ebe73a, Corollary 3.4]. In particular, the class of manifolds with Anosov geodesic flows includes those with negative curvature [Ano67].

Below we write

$$N_\pm(\rho) := T_\rho(SN^*H) \cap E_\pm(\rho), \tag{1.22}$$

and define the *mixed* and *split* subsets of SN^*H respectively by

$$\mathcal{M}_H := \Big\{\rho \in SN^*H : \;\; N_-(\rho) \neq \{0\} \text{ and } N_+(\rho) \neq \{0\}\Big\}, \tag{1.23}$$

$$\mathcal{S}_H := \Big\{\rho \in SN^*H : \;\; T_\rho(SN^*H) = N_-(\rho) + N_+(\rho)\Big\}. \tag{1.24}$$

Then we write

$$\mathcal{A}_H := \mathcal{M}_H \cap \mathcal{S}_H \tag{1.25}$$

where we will use $\mathcal{A}_H$ when considering manifolds with Anosov geodesic flow and $\mathcal{S}_H$ when considering those with no focal points.

In what follows, π continues to be the canonical projection $\pi : SN^*H \to H$.

Theorem 6. *Let $H \subset M$ be a closed embedded submanifold of codimension k. Suppose that $A \subset H$ and one of the following two conditions holds:*

- *(M, g) has no focal points and $\pi^{-1}(A) \cap \mathcal{S}_H = \emptyset$.*
- *(M, g) has Anosov geodesic flow and $\pi^{-1}(A) \cap \mathcal{A}_H = \emptyset$.*

Then, there exists $C > 0$ so that for all $w \in C_c^\infty(H)$ with $\operatorname{supp} w \subset A$ the following holds. There exists $h_0 > 0$ so that for $0 < h < h_0$ and $u \in \mathcal{D}'(M)$

$$\Big|\int_H wud\sigma_H\Big| \leq Ch^{\frac{1-k}{2}}\|w\|_\infty \left(\frac{\|u\|_{L^2(M)}}{\sqrt{\log h^{-1}}} + \frac{\sqrt{\log h^{-1}}}{h}\|(-h^2\Delta_g - 1)u\|_{H_{\rm scl}^{\frac{k-3}{2}}(M)}\right).$$

Theorem 6 also comes with some uniformity over the constants (C, h_0). In particular, for (A_0, H_0) satisfying one of the conditions in Theorem 6, there is a neighborhood U of (A_0, H_0) in the C^∞ topology so that the constants (C, h_0) are uniform for $(A, H) \in U$ and w in a bounded subset of C_c^∞. Here and below when we refer to the C^∞ topology on (A, H) we mean the following. Fix coordinate charts $\{U_j\}_j$ near H_0 such that $H_0 \subset \cup_j U_j$ and in each U_j, H_0 is given by $\{(x', x'') \mid x' = 0\}$. We define a neighborhood basis near (A_0, H_0) by saying for ε, k that (A, H) is ε close to H_0 if H is given by $\{(x', x'') \mid x' = f(x'')\}$ for some $f \in C^k$ with $\|f\|_{C^k} \leq \varepsilon$ and

$$\sup_{x\in A}\inf_{y\in A_0} d(x,y) + \sup_{x\in A_0}\inf_{y\in A} d(x,y) < \varepsilon.$$

Note in particular that since $E_\pm(\rho)$ are continuous in ρ, if (A_0, H_0) satisfies the assumptions of Theorem 6, then for $\varepsilon > 0$ small enough, k large enough, and (A, H), ε, k close to (A_0, H_0), the pair (A, H) satisfies the assumptions of Theorem 6.

We note that the conclusion of Theorem 6 holds when (M, g) is a surface with Anosov geodesic flow, since in this case $\mathcal{A}_H = \emptyset$ regardless of H. To see this note that if $\dim M = 2$, then $\mathcal{S}_H = \mathcal{A}_H$ since $\dim T_\rho(SN^*H) = 1$. Indeed, it is not possible to have both $N_+(\rho) \neq \{0\}$ and $N_-(\rho) \neq \{0\}$ unless $N_+(\rho) = N_-(\rho) = T_\rho(SN^*H)$ and hence $\mathcal{S}_H \subset \mathcal{A}_H$. Moreover, in the Anosov case, since $E_+(\rho) \cap E_-(\rho) = \{0\}$, $\mathcal{A}_H = \emptyset$.

In [Wym17, Wym20a] Wyman works with (M, g) non-positively curved (and hence having no focal points), $\dim M = 2$ and $H = \{\gamma(s)\}$ a curve. He then imposes the condition that for all s the curvature of γ, $\kappa_\gamma(s)$, avoids two special values $\mathbf{k}_\pm(\gamma'(s))$

determined by the tangent vector to $\gamma(s)$. He shows that under this condition, when ϕ_h is an eigenfunction of the Laplacian,

$$\int_\gamma \phi_h d\sigma_\gamma = O\Big(\frac{1}{\sqrt{\log h^{-1}}}\Big).$$

We note that if $\kappa_\gamma(s) = \mathbf{k}_\pm(\gamma'(s))$, then the lift of γ to the universal cover of M is tangent to a stable or unstable horosphere at $\gamma(s)$, and $\kappa_\gamma(s)$ is equal to the curvature of that horosphere. Since this implies that $T_{(\gamma(s),\gamma'(s))}SN^*\gamma$ is stable or unstable, the condition there is that $\mathcal{S}_\gamma = \emptyset$. Thus, the condition $\mathcal{S}_H = \emptyset$ is the generalization to higher codimensions and more general geometries of that in [Wym17, Wym20a].

We also point out that through a small improvement in a dynamical argument, we have replaced the set

$$\mathcal{N}_H := \mathcal{S}_H \cup \mathcal{M}_H$$

in [CG19, Theorem 8] with $\mathcal{S}_H$ when considering manifolds without focal points.

1.3. **Outline of the paper.** Sections 2 and 3 build technical tools for constructing non-self looping covers. Then, Sections 4, and 5 apply these tools to build non-self looping covers under certain geometric assumptions. In particular, Theorems 1 and 2 are proved in Section 4. In Section 5, we prove Theorem 6 and the remaining cases in Theorem 4. The reader will find below that there are *many* parameters explicitly named in the propositions. We understand that keeping track of these may be painful (and encourage the reader to treat them as some positive constant in most cases). However, it is important to keep of track of the dependence of our estimates on many of these constants e.g. in the proof of Theorem 1.

1.4. **Index of Notation.** In general we denote points in T^*M by ρ, and vectors in $T_\rho(T^*M)$ in boldface (e.g. $\mathbf{v} \in T_\rho(T^*M)$). Sets of indices are denoted in calligraphic font (e.g $\mathcal{I}$). When position and momentum need to be distinguished we write $\rho = (x, \xi)$ for $x \in M$ and $\xi \in T^*_xM$. Next, we list symbols that are used repeatedly in the text along with the location where they are first defined.

φ_t	(1.11)	$\Lambda_{\max}$	(1.17)	F, δ_F	(2.2)
$\mathcal{H}_\Sigma$	(1.12)	$T_e(h)$	(1.18)	ψ	(2.3)
$\tau_{\mathrm{inj}H}$	(1.13)	$N_\pm(\rho)$	(1.22)	J_t	(3.1)
$\Lambda_A^\tau(r)$	(1.14)	$\mathcal{M}_H$	(1.23)	$\mathbf{D}$	(3.4)
K_H	(1.15)	$\mathcal{S}_H$	(1.24)	C_φ	(3.3)
$\mathbf{B}$	(1.20)	$\mathcal{A}_H$	(1.25)	$\Theta_\pm$	(5.7)
$H^m_{\mathrm{scl}}(M)$	(1.5)				

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2. Partial invertibility of $d\varphi_t|_{TSN^*H}$ and looping sets

The aim of this section is to study the set of geodesic loops in SN^*H under conditions on the structure of the set of conjugate points of (M,g). However, we work in the general setting in which the Hamiltonian flow is not necessarily the geodesic one. We do this since we plan to use the results for general Hamiltonian flows in future work. In particular, let $p \in S^m$ be real valued with

$$|p| \geq |\xi|^m/C, \qquad |\xi| \geq C$$

and define $\varphi_t := \exp(tH_p)$ and $\Sigma_{H,p} := \{p=0\} \cap N^*H$ so that in the case $p = |\xi|_g - 1$, $\Sigma_{H,p} = SN^*H$. We assume that H is *conormally transverse* for p in the sense that for any defining functions $f_1, \dots f_k$ for H, i.e. $f_i \in C^\infty(M;\mathbb{R})$ with $H = \{x \in M \mid f_i(x) = 0,\, i = 1, \dots, k\}$ and $df_i|_H$ are linearly independent, we have

$$N^*H \subset \{p \neq 0\} \cup \bigcup_{i=1}^{k} \{H_p f_i \neq 0\}. \tag{2.1}$$

Note that with this definition the $\tau_{\mathrm{inj}H}$ as in (1.13) continues to make sense for general p and conormally transvers H. For such H, we define $r_H : T^*M \to \mathbb{R}$ by $r_H(\rho) = d(\pi(\rho), H)$, and let

$$\mathfrak{I}_H := \inf_{\rho \in \Sigma_{H,p}} \lim_{t \to 0^+} |H_p r_H(\varphi_t(\rho))|$$

We now fix once and for all a defining function $F : T^*M \to \mathbb{R}^{n+1}$ for $\Sigma_{H,p}$ and $\delta_F > 0$ so that:

For $q \in T^*M$ with $d(q, \Sigma_{H,p}) < \delta_F$,

$$\begin{aligned}
&\bullet\ \Sigma_{H,p} = F^{-1}(0)\\
&\bullet\ \tfrac{1}{2} d(q, \Sigma_{H,p}) \leq |F(q)| \leq 2 d(q, \Sigma_{H,p}),\\
&\bullet\ dF(q) \text{ has a right inverse } R_F(q) \text{ with } \|R_F(q)\| \leq 2,\\
&\bullet\ \max_{|\alpha| \leq 2}(|\partial^\alpha F(q)|) \leq 2.
\end{aligned} \tag{2.2}$$

Define also $\psi : \mathbb{R} \times T^*M \to \mathbb{R}^{n+1}$

$$\psi(t,\rho) = F \circ \varphi_t(\rho). \tag{2.3}$$

Working under the assumption that the set of conjugate points can be controlled and that the dimension of $\dim H < \frac{n-1}{2}$ will allow us to say that if $\varphi_{t_0}(\rho_0)$ is exponentially close to $\Sigma_{H,p} = SN^*H$ for some time t_0 and some $\rho_0 \in SN^*H$, then there exists a tangent vector $\mathbf{w} \in T_{\rho_0} SN^*H$ for which the restriction

$$d\psi_{(t_0,\rho_0)} : \mathbb{R}\partial_t \times \mathbb{R}\mathbf{w} \to T_{\psi(t_0,\rho_0)}\mathbb{R}^{n+1} \tag{2.4}$$

has a left inverse whose norm we control. This is proved in Lemma 4.1 and is the cornerstone in the proof of Theorems 2 and 1. Note, however, that asking (2.4) to hold is a very general condition that may not need the control of the structure of the set of conjugate points. We will use this in Section 5.

The goal of this section is to prove Proposition 2.2 below, whose purpose is to control the number of tubes that emanate from a subset of $\Sigma_{H,p}$ and loop back to $\Sigma_{H,p}$. This is done under the assumption that the restriction of $d\psi_{(t_0,\rho_0)}$ in (2.4) has a left inverse. To state this proposition we first need a lemma that describes a convenient system of coordinates near $\Sigma_{H,p}$. The statement of this lemma is illustrated in Figure 1.

Observe that by [DG14, (C.3)] for any $\Lambda > \Lambda_{\max}$ and α multiindex, there exists $C_{M,p,\alpha} > 0$ depending only on M, p, α so that

$$|\partial^\alpha \varphi_t| \leq C_{M,p,\alpha} e^{|\alpha|\Lambda t}. \tag{2.5}$$

Lemma 2.1 (Coordinates near $\Sigma_{H,p}$)**.** *There exists $\tau_1 = \tau_1(M,p,\mathfrak{I}_H) > 0$ and $\mathfrak{c}_0 = \mathfrak{c}_0(M,p,\mathfrak{I}_H)$ so that for $\Lambda > \Lambda_{max}$ the following holds. Let $\rho_0 \in \Sigma_{H,p}$, $t_0 \in \mathbb{R}$ be so that*

- *there exists $\mathbf{w} = \mathbf{w}(t_0,\rho_0) \in T_{\rho_0}\Sigma_{H,p}$ so that the restriction*
$$d\psi_{(t_0,\rho_0)} : \mathbb{R}\partial_t \times \mathbb{R}\mathbf{w} \to T_{\psi(t_0,\rho_0)}\mathbb{R}^{n+1}$$
has left inverse $L_{(t_0,\rho_0)}$ with $\|L_{(t_0,\rho_0)}\| \leq A$ for some $A \geq 1$,
- *$d(\varphi_{t_0}(\rho_0), \Sigma_{H,p}) \leq \min\left\{\frac{e^{-2\Lambda|t_0|}}{16\mathfrak{c}_0^2 A^2.}, \delta_F\right\}$*

Then, points ρ in a neighborhood of ρ_0 can be written in coordinates $\rho = \rho(y_1,\dots,y_{2n})$, with $\rho_0 = \rho(0,\dots,0)$ and $\Sigma_{H,p} = \{y_n = \cdots = y_{2n} = 0\}$, so that

$$\frac{1}{2} d(\rho(y),\rho(y')) \leq |y - y'| \leq 2d(\rho(y),\rho(y')).$$

In addition, there exists a smooth real valued function f defined in a neighborhood of $0 \in \mathbb{R}^{2n-1}$ so that letting $r_{t_0} := \frac{8e^{-3\Lambda|t_0|}}{\mathfrak{c}_0^2 A^2}$ and $0 < r < \frac{1}{128}e^{\Lambda|t_0|} r_{t_0}$, if

$$|y| < r_{t_0} \qquad \text{and} \qquad d(\varphi_t(\rho(y)), \Sigma_{H,p}) < r \quad \text{for some } t \in [t_0 - \tau_1, t_0 + \tau_1],$$

then

$$|y_1 - f(y_2,\dots y_{2n})| < 2(1+\mathfrak{c}_0)Ar \qquad \text{and} \qquad |\partial_{y_j} f| < \mathfrak{c}_0 A e^{\Lambda|t_0|}.$$

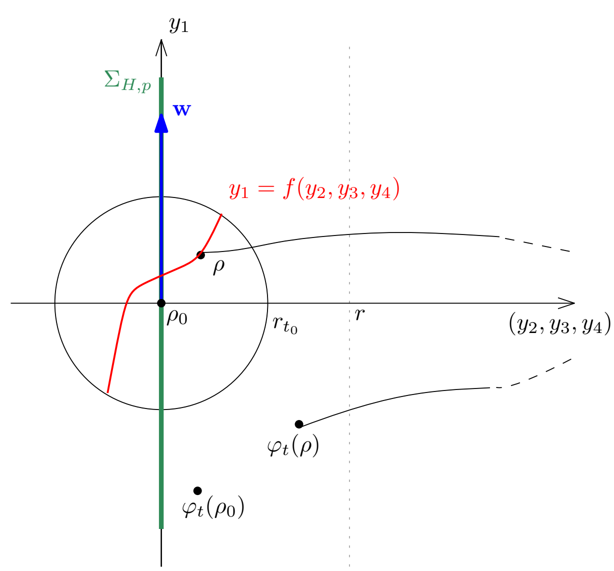

Figure 1. Illustration of the statement in Lemma 2.1 when H is a curve and M is a surface.

Proof. Since $d\psi_{(t_0,\rho_0)}:\mathbb{R}\partial_t\times\mathbb{R}\mathbf{w}\to\mathbb{R}^{n+1}$ has a left inverse, we may find an orthogonal matrix O such that $O\circ F=(f_1,\dots,f_{n+1})$ and with $\tilde{F}=(f_1,f_2)$,

$$\Psi:\mathbb{R}\times T^*M\to\mathbb{R}^2,\qquad \Psi(t,\rho):=\tilde{F}\circ\varphi_t(\rho),$$

the restriction $d\Psi:\mathbb{R}\partial_t\times\mathbb{R}\mathbf{w}\to\mathbb{R}^2$ is invertible with inverse L having $\|L\|\leq A$. Note that since O is orthogonal, $O\circ F$ is a defining function satisfying (2.2) with the same δ_F. Moreover, since

$$d\psi_{(t_0,\rho_0)}:\mathbb{R}\partial_t\to T_{\psi(t_0,\rho_0)}\mathbb{R}^{n+1}$$

has a left inverse, $L_1\in\mathbb{R}$ with $|L_1|<2\mathfrak{I}_H^{-1}:=A_0$ we may choose O so that with $\Psi(t,\rho)=(\Psi_1(t,\rho),\Psi_2(t,\rho))$, we have $|\partial_t\Psi_1(t_0,\rho_0)|\geq A_0^{-1}$ and $\partial_t\Psi_2(t_0,\rho_0)=0$.

Let $(t,y)=(t,y_1,y_2,\dots,y_{n-1},y_n,\dots y_{2n})$ be coordinates on $\mathbb{R}\times T^*M$ near (t_0,ρ_0) so that $(t_0,0)\mapsto(t_0,\rho_0)$, $\partial_{y_1}\mapsto\mathbf{w}/\|\mathbf{w}\|$ at $(t_0,0)$, and $(y_n,y_{n+1},\dots,y_{2n})$ define $\Sigma_{H,p}$. Finally, let $\tilde{y}=e^{\Lambda|t_0|}y$. We will work with these coordinates on $\mathbb{R}\times T^*M$ for the remainder of the proof.

Applying the implicit function theorem (see Lemma A.1) with $x_0=t$, $x_1=\tilde{y}$ and $\tilde{f}:\mathbb{R}\times\mathbb{R}^{2n}\times\mathbb{R}\to\mathbb{R}$ with $\tilde{f}(x_0,x_1,x_2)=\Psi_1(x_0,x_1)-x_2$ gives that there exists a neighborhood $U\subset\mathbb{R}^{2n}\times\mathbb{R}$ of $(0,x_2^0)$, where $x_2^0:=\Psi_1(t_0,0)$, and a function $x_0=\mathfrak{t}:U\to\mathbb{R}$, so that for $(\tilde{y},x_2)\in U$,

$$x_2=\Psi_1\big(\mathfrak{t}(\tilde{y},x_2),\tilde{y}\big)$$

with

$$|\partial_{x_2}\mathfrak{t}|\leq A_0,\qquad\qquad \max_{1\leq j\leq 2n}|\partial_{\tilde{y}_j}\mathfrak{t}|\leq\tfrac{c_{M,p}}{64n}A_0,$$

where $c_{M,p}$ is a positive constant depending only on (M,p). Here, the t_0 independent bounds follow from the chain rule. Moreover, we have $|\partial^2_{t,\tilde{y}}\tilde{f}|\leq\frac{c_{M,p}}{64n}$, $|\partial_t^2\tilde{f}|\leq\frac{c_{M,p}}{64n}$, and $|\partial_{\tilde{y}_j}\tilde{f}|\leq\frac{c_{M,p}}{64n}$ for all $j=1,\dots,2n$. Then, working with

$$r_0=\tfrac{8}{c_{M,p}A_0},\qquad r_1=\min\Big\{\tfrac{32}{c_{M,p}^2A_0^2},\ \tfrac{8}{c_{M,p}A_0}\Big\},\qquad r_2=\tfrac{2}{c_{M,p}A_0^2},$$

$$B_0=\tfrac{c_{M,p}}{32},\qquad B_1=\tfrac{c_{M,p}}{64n},\qquad B_2=0,\qquad \tilde{B}_1=\tfrac{c_{M,p}}{64n},\qquad \tilde{B}_2=1,$$

for r_0,r_1,r_2 and $B_0,B_1,B_2,\tilde{B}_1,\tilde{B}_2$ as in Lemma A.1, we obtain that U can be chosen so that $B(0,r_1)\times B(x_2^0,r_2)\subset U$. In particular, it follows that if

$$|\mathfrak{t}-t_0|<\tfrac{8}{c_{M,p}A_0},\qquad |\tilde{y}|\leq\min\Big\{\tfrac{32}{c_{M,p}^2A_0^2},\tfrac{8}{c_{M,p}A_0}\Big\},\qquad |x_2-x_2^0|<\tfrac{2}{c_{M,p}A_0^2},\tag{2.6}$$

then

$$|\mathfrak{t}(\tilde{y},x_2)-\mathfrak{t}(\tilde{y},0)|\leq A_0|x_2|.$$

Next, since $d\Psi:\mathbb{R}\partial_t\times\mathbb{R}\mathbf{w}\to\mathbb{R}^2$ is invertible with inverse L satisfying $\|L\|\leq A$, we have $|\partial_{\tilde{y}_1}\tilde{f}|^{-1}\leq Ae^{\Lambda|t_0|}$ where now we write $\tilde{f}$ for

$$\tilde{f}(\tilde{y},x_2,x_3)=\Psi_2(\mathfrak{t}(\tilde{y},x_2),\tilde{y})-x_3.$$

Next, we write $\tilde{y}=(\tilde{y}_1,\tilde{y}')$ and once again apply the implicit function theorem (Lemma A.1) with $x_0=\tilde{y}_1$, $x_1=(x_2,\tilde{y}')$, $x_3\in\mathbb{R}$, to see that there exists $U\subset\mathbb{R}^{2n}\times\mathbb{R}$

of $(0,x_3^0)$, with $x_3^0=\Psi_2(t_0,0)$, and a function $x_0=\tilde{\mathbf{y}}_1:U\to\mathbb{R}$, so that for $(\tilde y',x_3)\in U$,

$$x_3=\Psi_2\Big(\mathfrak{t}\big(\tilde{\mathbf{y}}_1(\tilde y',x_2,x_3),\tilde y',x_2\big),\tilde{\mathbf{y}}_1(\tilde y',x_2,x_3),\tilde y'\Big)$$

with

$$|\partial_{x_3}\tilde{\mathbf{y}}_1|\le Ae^{\Lambda|t_0|},\qquad |\partial_{x_2}\tilde{\mathbf{y}}_1|<\mathfrak{c}_0Ae^{\Lambda|t_0|},\qquad \max_{2\le j\le 2n}|\partial_{\tilde y_j}\tilde{\mathbf{y}}_1|\le\mathfrak{c}_0\,Ae^{\Lambda|t_0|}$$

where $\mathfrak{c}_0$ is a positive constant depending only on (M,p,A_0), so that $|\partial^2_{(x_2,\tilde y)}\tilde f|\le\frac{\mathfrak{c}_0}{64n}$ and $|\partial_{x_2}\tilde f|$, $|\partial_{\tilde y_j}\tilde f|\le\frac{\mathfrak{c}_0}{64n}$ for all $j=2,\dots,2n$. Without loss of generality we assume that $\mathfrak{c}_0\ge c_{M,p}A_0$ and that $\mathfrak{c}_0>1$. Then, working with

$$r_0=\tfrac{8e^{-\Lambda|t_0|}}{\mathfrak{c}_0A},\qquad r_1=\min\Big\{\tfrac{32e^{-2\Lambda|t_0|}}{\mathfrak{c}_0^2A^2},\tfrac{8e^{-\Lambda|t_0|}}{\mathfrak{c}_0A}\Big\},\qquad r_2=\tfrac{2e^{-2\Lambda|t_0|}}{\mathfrak{c}_0A^2},$$
$$B_0=\tfrac{\mathfrak{c}_0}{32},\qquad B_1=\tfrac{\mathfrak{c}_0}{64n},\qquad B_2=0,\qquad \tilde B_1=\tfrac{\mathfrak{c}_0}{64n},\qquad \tilde B_2=1,$$

for r_0,r_1,r_2 and $B_0,B_1,B_2,\tilde B_1,\tilde B_2$ as in Lemma A.1, we obtain that U can be chosen so that $B((x_2^0,0),r_1)\times B(x_3^0,r_2)\subset U$. In particular, it follows that if

$$|\tilde{\mathbf{y}}_1|<\tfrac{8e^{-\Lambda|t_0|}}{\mathfrak{c}_0A},\quad |(\tilde y',x_2-x_2^0)|\le\min\Big\{\tfrac{32e^{-2\Lambda|t_0|}}{\mathfrak{c}_0^2A^2},\tfrac{8e^{-\Lambda|t_0|}}{\mathfrak{c}_0A}\Big\},\quad |x_3-x_3^0|<\tfrac{2e^{-\Lambda|t_0|}}{\mathfrak{c}_0A^2},\tag{2.7}$$

then

$$|\tilde{\mathbf{y}}_1(\tilde y',x_2,x_3)-\tilde{\mathbf{y}}_1(\tilde y',x_2,0)|\le Ae^{\Lambda|t_0|}|x_3|.$$

Note that this can be done since by assumption $\mathfrak{c}_0>1$ and

$$|0-x_3^0|=|\Psi_2(t_0,\rho_0)|\le 2d(\varphi_{t_0}(\rho_0),\Sigma_{H,p})<\tfrac{2e^{-2\Lambda|t_0|}}{\mathfrak{c}_0A^2}.\tag{2.8}$$

It follows, after undoing the change $\tilde y=e^{\Lambda|t_0|}y$, that if

- $\max\{|x_2-x_2^0|,|x_3-x_3^0|\}<\min\Big\{\frac{2}{c_{M,p}A_0^2},\ \frac{32e^{-2\Lambda|t_0|}}{\mathfrak{c}_0^2A^2},\ \frac{8e^{-\Lambda|t_0|}}{\mathfrak{c}_0A},\ \frac{2e^{-\Lambda|t_0|}}{\mathfrak{c}_0A^2}\Big\}$,
- $|y|<\min\Big\{\frac{8e^{-2\Lambda|t_0|}}{\mathfrak{c}_0A},\ \frac{32e^{-3\Lambda|t_0|}}{\mathfrak{c}_0^2A^2},\ \frac{8e^{-2\Lambda|t_0|}}{\mathfrak{c}_0A},\ \frac{32e^{-\Lambda|t_0|}}{c_{M,p}^2A_0^2},\ \frac{8e^{-\Lambda|t_0|}}{c_{M,p}A_0}\Big\}$,
- $|t-t_0|<\frac{8}{c_{M,p}A_0}$,

then

$$|\mathbf{y}_1(y',x_2,x_3)-\mathbf{y}_1(y',0,0)|\le(1+\mathfrak{c}_0)A\,|(x_2,x_3)|.$$

Next, note that since $d(\varphi_t(\rho(y)),\Sigma_{H,p})\le r$ and $r<\frac{e^{-2\Lambda|t_0|}}{16\mathfrak{c}_0^2A^2}$, then

$$|x_2-x_2^0|\le|x_2|+|x_2^0|\le 2d(\varphi_t(\rho(y)),\Sigma_{H,p})+2d(\varphi_{t_0}(\rho_0),\Sigma_{H,p})\le\tfrac{2e^{-2\Lambda|t_0|}}{\mathfrak{c}_0A^2},$$

and similarly, $|x_3-x_3^0|\le\frac{2e^{-2\Lambda|t_0|}}{\mathfrak{c}_0A^2}$. In addition, we can assume $c_{M,p}>1$. Since $\mathfrak{c}_0\ge c_{M,p}A_0$, with the above definition of r_{t_0}, we obtain that if $r<\frac{1}{128}e^{\Lambda|t_0|}r_{t_0}$ and $|y|<r_{t_0}$, then

$$|\mathbf{y}_1(y',x_2,x_3)-\mathbf{y}_1(y',0,0)|\le 2(1+\mathfrak{c}_0)Ar.$$

To finish the argument, we note that we may define $f(y'):=\mathbf{y}_1(y',0,0)$ satisfying $|\partial_{y'}f|\le\mathfrak{c}_0Ae^{\Lambda|t_0|}$ as claimed. Where, as argued in (2.8), this can be done since $|0-x_2^0|<\frac{2e^{-2\Lambda|t_0|}}{\mathfrak{c}_0A^2}$ and using that $A\ge 1$, $\mathfrak{c}_0\ge c_{M,p}A_0$.

□

Remark 6. We proceed to study the number of looping directions and prove the main result of this section. In what follows $\mathfrak{c}_0$ denotes the constant from Lemma 2.1.

Proposition 2.2. *Let $0 \le t_0 < T_0$, $0 < \tilde{c} < \delta_F$, $a > 0$, $\Lambda > \Lambda_{max}$, $c > 0$, $\beta \in \mathbb{R}$, $A \subset \Sigma_{H,p}$, and $B \subset A$ a ball of radius $R > 0$ satisfy the following assumption: for all $(t,\rho) \in [t_0, T_0] \times B$ such that $d(\varphi_t(\rho), A) \le \tilde{c}\, e^{-a|t|}$, there exists $\mathbf{w} \in T_\rho \Sigma_{H,p}$ for which the restriction*

$$d\psi_{(t,\rho)} : \mathbb{R}\partial_t \times \mathbb{R}\mathbf{w} \to T_{\psi(t,\rho)}\mathbb{R}^{n+1}$$

has left inverse $L_{(t,\rho)}$ with $\|L_{(t,\rho)}\| \le ce^{\beta|t|}$.

There exist $\alpha_1 = \alpha_1(M,p) > 0$ and $\alpha_2 = \alpha_2(M,p,c,\tilde{c},\delta_F,\mathfrak{I}_H)$ so that the following holds.

Let $r_0, r_1, r_2 > 0$ satisfy

$$r_0 < r_1, \qquad r_1 < \alpha_1 r_2, \qquad r_2 \le \min\{R, 1, \alpha_2\, e^{-\gamma T_0}\}, \qquad r_0 < \tfrac{1}{3}\, e^{-\Lambda T_0} r_2,$$

where $\gamma = \max\{a, 3\Lambda + 2\beta\}$. Let $0 < \tau_0 < \frac{\tau_{injH}}{2}$, $0 < \tau \le \tau_0$, and $\{\rho_j\}_{j=1}^N \subset \Sigma_{H,p}$ be a family of points so that

$$\Lambda^\tau_{\rho_j}(r_1) \cap \Lambda^\tau_B(r_0) \neq \emptyset, \qquad \Lambda^\tau_B(r_0) \subset \bigcup_{j=1}^N \Lambda^\tau_{\rho_j}(r_1),$$

and $\big\{\Lambda^\tau_{\rho_j}(r_1)\big\}_{j=1}^N$ can be divided into $\mathfrak{D}$ sets of disjoint tubes.

Then, there exist a partition of the indices $\mathcal{G} \cup \mathcal{B} = \{1, \dots, N\}$ and a constant $\mathbf{C_0} = \mathbf{C_0}(M,p,k,c,\beta,\mathfrak{I}_H) > 0$ so that

- *$\bigcup_{j \in \mathcal{G}} \Lambda^\tau_{\rho_j}(r_1)$ is non-self looping for times in $[t_0, T_0]$. Moreover,*

$$d\Big(\Lambda^\tau_A(r_0)\,,\ \bigcup_{t \in [t_0,T_0]} \bigcup_{j \in \mathcal{G}} \varphi_t(\Lambda^\tau_{\rho_j}(r_1))\Big) > 2r_1.$$

- *$|\mathcal{B}| \le \mathbf{C_0}\mathfrak{D}\; r_2\, \frac{R^{n-1}}{r_1^{n-1}}\, T_0\, e^{4(\Lambda+\beta)T_0}$.*

Remark 7. Note that we will typically apply Proposition 2.2 with $\{\Lambda^\tau_{\rho_j}(r_1)\}_j$ a subset of a $(\mathfrak{D}_n, \tau, r)$ good cover for $\Sigma_{H,p}$. In this case the constant $\mathfrak{D}$ can be absorbed into $\mathbf{C_0}$ since it depends only on n.

Proof. Let $\tau_1 = \tau_1(M,p,\mathfrak{I}_H)$ be the minimum of 1 and the constant from Lemma 2.1, and let L be the largest integer with $L \le \frac{1}{\tau_1}(T_0 - t_0) + 1$. Cover $[t_0, T_0]$ by

$$[t_0, T_0] \subset \bigcup_{\ell=0}^{L} \big[s_\ell - \tfrac{\tau_1}{2}, s_\ell + \tfrac{\tau_1}{2}\big],$$

where $s_\ell := t_0 + (\ell + \frac{1}{2})\tau_1$. We claim that for each $\ell = 0, \dots, L$ there exists a partition of indices $\mathcal{G}_\ell \cup \mathcal{B}_\ell = \{1, \dots, N\}$ so that

$$|\mathcal{B}_\ell| \le \mathbf{C_0}\mathfrak{D}\frac{r_2 R^{n-1}}{r_1^{n-1}} e^{4(\Lambda+\beta)|s_\ell|} \tag{2.9}$$

and

$$d\left(\Lambda_A^\tau(r_0)\,,\bigcup_{s=s_\ell-\frac{\tau_1}{2}}^{s_\ell+\frac{\tau_1}{2}}\varphi_t\big(\Lambda_{\rho_k}^\tau(r_1)\big)\right)\geq\frac{1}{C_S}r_2-C_Sr_0\qquad\forall k\in\mathcal{G}_\ell.\tag{2.10}$$

Here,

$$C_S:=\sup\big\{\|d\varphi_t(q)\|:\ q\in\Lambda^1_{\{p=0\}}(\varepsilon_0),\,|t|\leq\tfrac{4}{3}\big\},$$

where $\varepsilon_0>R$ is a constant independent of r_0,r_1,r_2,R. The result then follows from setting

$$\mathcal{B}:=\bigcup_{\ell=0}^{L}\mathcal{B}_\ell\qquad\text{and}\qquad\mathcal{G}:=\{1,\dots,N\}\backslash\mathcal{B},$$

together with asking for $\alpha_1<\frac{1}{2C_S+C_S^2}$ so that $\frac{1}{C_S}r_2-C_Sr_0>2r_1$. Note that the adjustment depends only on (M,p).

We have reduced the proof of the lemma to establishing the claims in (2.9) and (2.10). We next explain that it suffices to prove (2.10) with $\Lambda_A^\tau(r_0)$ replaced by A. To see this, let $\{t_j\}$ be so that

$$[-(3\tau+\tau_1{+}r_0),3\tau+\tau_1+r_0]=\bigcup_{j=1}^{J}[t_j-\tfrac{\tau_1}{2},t_j+\tfrac{\tau_1}{2}],$$

where J is the largest integer with $J\leq(6\tau+2r_0)/\tau_1+2$. Note that since $\tau<\tau_0<1$, $r_0<\frac{1}{3}$ and τ_1 depends only on $(M,p,\mathfrak{I}_H)$, the same is true for J. Fix $\ell\in\{1,\dots,L\}$. We claim that for each $j\in\{1,\dots,J\}$ there exists a partition $\mathfrak{g}_j^\ell\cup\mathfrak{b}_j^\ell=\{1,\dots,N\}$ with

$$|\mathfrak{b}_j^\ell|\leq\mathbf{C_0}\mathfrak{D}\frac{r_2R^{n-1}}{r_1^{n-1}}e^{4(\Lambda+\beta)|s_\ell|},\tag{2.11}$$

and

$$d\Big(A,\bigcup_{t=s_\ell+t_j-\frac{\tau_1}{2}}^{s_\ell+t_j+\frac{\tau_1}{2}}\varphi_t\big(\rho\big)\Big)\geq r_2\qquad\text{for all }\ \rho\in\bigcup_{k\in\mathfrak{g}_j^\ell}\Lambda_{\rho_k}^\tau(r_1).\tag{2.12}$$

Suppose the claims in (2.11) and (2.12) hold and let

$$\mathcal{B}_\ell:=\bigcup_{j=1}^{J}\mathfrak{b}_j^\ell\qquad\text{and}\qquad\mathcal{G}_\ell=\{1,\dots,N\}\backslash\mathcal{B}_\ell.$$

Then, by construction, after possibly adjusting $\mathbf{C_0}$ to take into account the bound on J (which only depends on $(M,p,\mathfrak{I}_H)$), we obtain that (2.9) also holds. To derive (2.10) suppose $\rho\in\Lambda_{\rho_k}^\tau(r_1)$ for some $k\in\mathcal{G}_\ell$. In particular, since $k\in\mathfrak{g}_j^\ell$ for all $j=1,\dots,J$, relations (2.12) yield that

$$d\Big(A,\bigcup_{t=s_\ell-3\tau-\tau_1-r_0}^{s_\ell+3\tau+\tau_1+r_0}\varphi_t(\rho)\Big)\geq r_2.$$

In particular, using the definition of C_{S}, that $\tau < \tau_{\mathrm{inj}H} \leq 1$, and $r_0 < \frac{1}{3}$

$$d\Big(\Lambda_A^{\tau+r_0}, \bigcup_{t=s_\ell-2\tau-\tau_1}^{s_\ell+2\tau+\tau_1} \varphi_t(\rho)\Big) \geq \frac{r_2}{C_{S}},$$

and this proves (2.10) after using the definition of C_S once again.

We have then reduced the proof of the proposition to establishing the claims in (2.11) and (2.12). Fix $\ell \in \{1,\dots,L\}$, $j \in \{1,\dots,J\}$, and set

$$s := s_\ell + t_j.$$

To prove these claims we start by covering B by balls $B^s_\alpha \subset T^*M$ of radius $\mathbf{R_s} > 0$ (to be determined later) and centers in B,

$$B \subset \bigcup_{\alpha=1}^{I_s} B^s_\alpha,$$

so that $I_s \leq C_n R^{n-1}\mathbf{R}_s^{-(n-1)}$ for some $C_n > 0$. Fix B^s_α and suppose there exists $\rho_0 \in B^s_\alpha$ such that

$$d(\Sigma_{H,p}, \rho_0) < r_0 \qquad \text{and} \qquad d\Big(A, \bigcup_{t=s-\frac{\tau_1}{2}}^{s+\frac{\tau_1}{2}} \varphi_t(\rho_0)\Big) < r_2. \tag{2.13}$$

Then there exists $\tilde{s} \in [s-\frac{\tau_1}{2}, s+\frac{\tau_1}{2}]$ with $d(\varphi_{\tilde{s}}(\rho_0), A) < r_2$. Next, since $d(\rho_0, \Sigma_{H,p}) < r_0$, there exists $\rho_\alpha \in \Sigma_{H,p}$ with

$$\varphi_{\tilde{s}}(\rho_\alpha) \in B(\varphi_{\tilde{s}}(\rho_0), c_{M,p}e^{\Lambda|\tilde{s}|}r_0), \qquad d(\rho_0,\rho_\alpha) < r_0,$$

for some $c_{M,p} > 0$. In addition, letting $\bar{\mathbf{r}}_s = c_{M,p}e^{\Lambda|\tilde{s}|}r_0$,

$$d(\Sigma_{H,p}, \varphi_{\tilde{s}}(\rho_\alpha)) \leq d(A, \varphi_{\tilde{s}}(\rho_\alpha)) \leq d(A, \varphi_{\tilde{s}}(\rho_0)) + d(\varphi_{\tilde{s}}(\rho_0), \varphi_{\tilde{s}}(\rho)) < r_2 + \bar{\mathbf{r}}_{\mathbf{s}}.$$

We then assume that $\alpha_2 < \frac{3}{3+c_{M,p}}\min\{\frac{\tilde{c}}{2}, \frac{\delta_F}{2}, \frac{1}{32\mathfrak{c}_0^2c^2}\}$ so that

$$r_2 + \bar{\mathbf{r}}_{\mathbf{s}} < \min\left\{\tilde{c}e^{-a|\tilde{s}|}, \frac{e^{-2(\Lambda+\beta)|\tilde{s}|}}{16\mathfrak{c}_0^2c^2}, \delta_F\right\}$$

where $\mathfrak{c}_0$ is from Lemma 2.1. Then, by assumption there exists $\mathbf{w} = \mathbf{w}(\tilde{s},\rho_\alpha) \in T_{\rho_\alpha}\Sigma_{H,p}$ so that the restriction $d\psi_{(\tilde{s},\rho_\alpha)} : \mathbb{R}\partial_t \times \mathbb{R}\mathbf{w} \to T_{\psi(\tilde{s},\rho_\alpha)}\mathbb{R}^{n+1}$ has left inverse $L_{(\tilde{s},\rho_\alpha)}$ with $\|L_{(\tilde{s},\rho_\alpha)}\| \leq ce^{\beta|\tilde{s}|}$. By Lemma 2.1 the points ρ in a neighborhood of ρ_α can be written in coordinates $\rho = \rho(y_1,\dots,y_{2n})$ with $\rho_\alpha = \rho(0,\dots,0)$ and $\Sigma_{H,p} = \{y_n = \dots = y_{2n} = 0\}$ so that $\frac{1}{2}d(\rho(y),\rho(y')) < |y-y'| < 2d(\rho(y),\rho(y'))$. Let

$$r_{\tilde{s}} := \frac{8e^{-(3\Lambda+2\beta)|\tilde{s}|}}{c^2\mathfrak{c}_0^2}.$$

These coordinates are built with the property that there exists a smooth real valued function f defined in a neighborhood of $0 \in \mathbb{R}^{2n-1}$ so that if $0 < r < \frac{1}{128}e^{\Lambda|\tilde{s}|}r_{\tilde{s}}$,

$$|y| < r_{\tilde{s}} \qquad \text{and} \qquad d(\varphi_t(\rho(y)), \Sigma_{H,p}) < r \quad \text{for some } t \in \big[\tilde{s}-\tau_1, \tilde{s}+\tau_1\big],$$

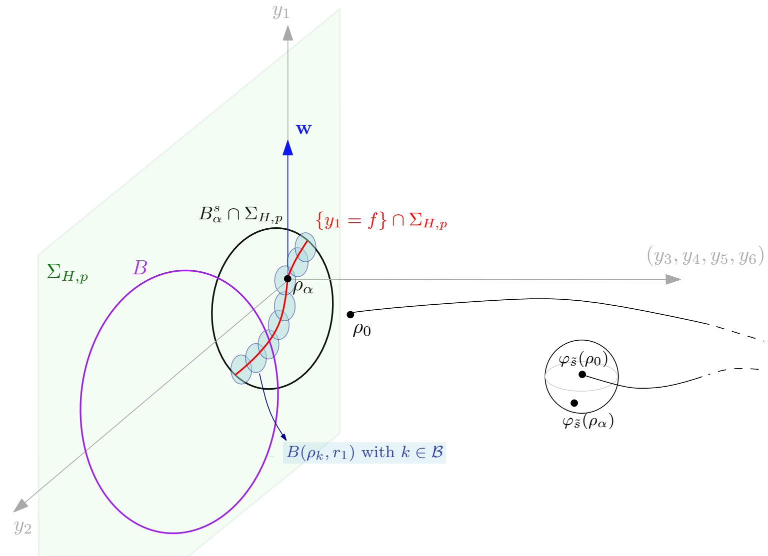

FIGURE 2. Illustration, when $n = 3$, of the covering balls that intersect B^s_α and loop back for times $\tilde{s}$ near s.

then

$$|y_1 - f(y_2, \dots y_{2n})| < 2(1 + \mathfrak{c}_0)ce^{\beta|\tilde{s}|}r \qquad \text{and} \qquad |\partial_{y_j} f| < \mathfrak{c}_0\, ce^{\beta|\tilde{s}|}e^{\Lambda|\tilde{s}|}$$

Assume $\alpha_2 < \frac{1}{128}$ so that $r_2 < \frac{1}{128}e^{\Lambda|\tilde{s}|}r_{\tilde{s}}$. Since $\tilde{s} \in [s - \frac{\tau_1}{2}, s + \frac{\tau_1}{2}]$, we may choose $r := r_2$ to get that, if $\rho = \rho(y) \in B(\Sigma_{H,p}, r_0)$ satisfies $d(\rho, \rho_\alpha) < \frac{r_{\tilde{s}}}{2}$ and

$$d\Big(\Sigma_{H,p}, \bigcup_{t=s-\frac{\tau_1}{2}}^{s+\frac{\tau_1}{2}} \varphi_t(\rho)\Big) < r_2, \tag{2.14}$$

then with $\bar{y} = (y_n, \dots y_{2n})$

$$\begin{aligned} |y_1 - f(y_2, \dots, y_{n-1}, 0)| &\le |y_1 - f(y_2, \dots, y_{n-1}, \bar{y})| + |\partial_{y_j} f(y_2, \dots, y_{n-1}, 0)||\bar{y}| \\ &< 2(1 + \mathfrak{c}_0)ce^{\beta|\tilde{s}|}r_2 + \mathfrak{c}_0 ce^{\beta|\tilde{s}|}e^{\Lambda|\tilde{s}|}2r_0 \\ &< C_0 e^{\beta|\tilde{s}|}r_2. \end{aligned}$$

Here, we have used that the assumption $r_0 < \frac{1}{3}\,e^{-\Lambda T_0}r_2$ implies $e^{\Lambda|\tilde{s}|}2r_0 < r_2$, and we have written $C_0 = (2 + 3\mathfrak{c}_0)c$. Also, we used that $|\bar{y}| \le 2d(\rho(y), \rho(y_2, \dots, y_{n-1}, 0)) = 2d(\rho(y), \Sigma_{H,p}) \le 2r_0$.

Next, we let $\mathbf{R_s} = \frac{r_{\tilde{s}}}{8}$ and use that $\alpha_2 < \frac{1}{16c^2\mathfrak{c}_0^2}$ to obtain that since $\rho_0 \in B^s_\alpha$, for $\rho \in B^s_\alpha$,

$$d(\rho, \rho_\alpha) \le d(\rho_0, \rho_\alpha) + d(\rho, \rho_0) < r_0 + 2\mathbf{R_s} < \frac{r_{\tilde{s}}}{2}. \tag{2.15}$$

In particular, (2.15) implies

$$B^s_\alpha \subset \{\rho \in T^*M :\ d(\rho, \rho_\alpha) < \frac{r_{\tilde{s}}}{2}\}.$$

Therefore, we have showed that if $\rho \in B^s_\alpha \cap B(\Sigma_{H,p}, r_0)$ satisfies (2.14), then $\rho \in \mathcal{U}^s_{\rho_\alpha} \cap B(\Sigma_{H,p}, r_0)$ where

$$\mathcal{U}^s_{\rho_\alpha} = \Big\{\rho :\ |y_1 - f(y_2, \dots, y_{n-1}, 0)| < C_0 e^{\beta|\tilde{s}|} r_2, \quad d(\rho, \rho_\alpha) < \tfrac{r_{\tilde{s}}}{2}\Big\}.$$

This is illustrated in Figure 2. Next, note that, the number of disjoint tubes in $\{\Lambda^\tau_{\rho_j}(r_1)\}_{j=1}^N$ that intersect $\mathcal{U}^s_{\rho_\alpha} \cap B(\Sigma_{H,p}, r_0)$ is controlled by the number of disjoint balls in the collection $\{B(\rho_j, r_1)\}_{j=1}^N$ that intersect $\mathcal{U}^s_{\rho_\alpha} \cap \Sigma_{H,p}$. In addition, for each $j \in \{1, \dots, N\}$ the intersection $B(\rho_j, r_1) \cap \Sigma_{H,p}$ is entirely contained in $\tilde{\mathcal{U}}^s_{\rho_\alpha} \cap \Sigma_{H,p}$ where

$$\tilde{\mathcal{U}}^s_{\rho_\alpha} = \Big\{\rho : |y_1 - f(y_2, \dots, y_{n-1}, 0)| < C_0 e^{\beta|\tilde{s}|} r_2 + 4r_1, \quad d(\rho, \rho_\alpha) < \frac{r_{\tilde{s}}}{2} + 4r_1\Big\}.$$

In particular,

$$\mathrm{vol}(\tilde{\mathcal{U}}^s_{\rho_\alpha} \cap \Sigma_{H,p}) \le (C_0 e^{\beta|\tilde{s}|} r_2 + 4r_1) \int_{B(0, \frac{r_{\tilde{s}}}{2} + 4r_1)} \sqrt{1 + |\nabla f|^2}\, dy_2 \dots dy_{n-1}.$$

Hence, the number of disjoint balls in the collection $\{B(\rho_j, r_1)\}_{j=1}^N$ that intersect $\mathcal{U}^s_{\rho_\alpha} \cap \Sigma_{H,p}$ is controlled by

$$2\sqrt{n-1}\, \mathfrak{c}_0 c (C_0 e^{\beta(|s|+\tau_1)} r_2 + 4r_1)\, e^{(\beta+\Lambda)(|s|+\tau_1)} \big(\frac{r_{\tilde{s}}}{2} + 4r_1\big)^{n-2} r_1^{-(n-1)}.$$

Here, we used the bound $|\partial_{y_j} f| < \mathfrak{c}_0\, c e^{(\beta+\Lambda)|\tilde{s}|}$ and that $e^{\beta|\tilde{s}|} \le e^{\beta(|s|+\tau_1)}$.

Finally, note that since $\alpha_2 < \frac{1}{c^2 \mathfrak{c}_0^2}$ and $\gamma \ge 3\Lambda + 2\beta$, by choosing $\alpha_1 < 1$, we have $r_1 < \min\{r_2, r_{\tilde{s}}\}$. Hence, the number of disjoint balls in the collection $\{B(\rho_j, r_1)\}_{j=1}^N$ that intersect $\mathcal{U}^s_{\rho_\alpha} \cap \Sigma_{H,p}$ is controlled by $e^{2\beta\tau_1} e^{(2\beta+\Lambda)|s|} r_2 \tilde{r}_s^{n-2} r_1^{-(n-1)}$ up to a constant that depends only on $(M, p, k, c, \mathfrak{I}_H)$. In addition, note that in the collection $\{\Lambda^\tau_{\rho_j}(r_1)\}_{j=1}^N$ there are $\mathfrak{D}$ sets of disjoint tubes of radius r_1. Therefore, since there are $I_s \le C_n R^{n-1} \mathbf{R_s}^{-(n-1)}$ balls B^s_α, for $s = s_\ell + t_j$ we can build $\mathfrak{b}^\ell_j$ so that

$$\rho \notin \bigcup_{k \in \mathfrak{b}^\ell_j} \Lambda^\tau_{\rho_k}(r_1) \quad \Longrightarrow \quad d\Big(A, \bigcup_{t = s_\ell + t_j - \frac{\tau_1}{2}}^{s_\ell + t_j + \frac{\tau_1}{2}} \varphi_t(\rho)\Big) \ge r_2,$$

and so that for some $\mathbf{C_0} = \mathbf{C_0}(M, p, k, c, \beta, \mathfrak{I}_H) > 0$

$$|\mathfrak{b}^\ell_j| \le \mathbf{C_0} \mathfrak{D} \frac{e^{(2\beta+\Lambda)|s|} r_2 r_{\tilde{s}}^{n-2} R^{n-1}}{r_1^{n-1} \mathbf{R}_s^{n-1}}.$$

Here, we have used that $e^{2\beta\tau_1} \le e^{2\beta}$ since $\tau_1 \le 1$. Using that $\frac{r_{\tilde{s}}^{n-2}}{\mathbf{R}_s^{n-1}} = \frac{8^{n-1}}{r_{\tilde{s}}}$ and adjusting $\mathbf{C_0}$, we obtain (2.11). This concludes the proofs of the claims in (2.11) and (2.12).

□

3. Contraction of φ_t and non-self looping sets

The proofs of Theorems 4 and 6 hinge on controlling how the geodesic flow changes the volume of sets contained in SN^*H. As in the previous section, we work with a general Hamiltonian p such that H is conormally transverse for p. Let

$$J_t := d\varphi_t|_{T_\rho \Sigma_{H,p}} : T_\rho \Sigma_{H,p} \to d\varphi_t(T_\rho \Sigma_{H,p}). \tag{3.1}$$

When the Hamiltonian flow is assumed to be Anosov, we have that for $A_0 \subset \mathcal{S}_H \setminus \mathcal{M}_H$, we can split A_0 into pieces $A_{\pm,0}$ such that there is $C_0 \geq 1$ satisfying

$$\sup_{\rho \in A_{\pm,0}} |\det J_t| \leq C_0 e^{-|t|/C_0}, \qquad \pm t \geq 0. \tag{3.2}$$

The analysis in this section will be used in Section 5 to prove Theorem 6 and in particular, to handle $\mathcal{S}_H \setminus \mathcal{M}_H$. This, for instance, is the step which allows us to show that averages over subsets of horospheres have improvements.

Note, however, that the condition in (3.2) is very general and that it may hold in situations where the Hamiltonian flow is not Anosov. For example, such an estimate holds for the geodesic flow at the umbillic points of the triaxial ellipsoid (see e.g. [GT20]). This section is dedicated to study the structure of the set of looping tubes under the assumption that (3.2) holds.

By (2.5), there exists $C_\varphi > 0$ depending only on (M, p), so that for all $\Lambda > \Lambda_{\max}$

$$\|d\varphi_t\| \leq C_\varphi e^{\Lambda |t|}, \qquad t \in \mathbb{R}. \tag{3.3}$$

Let $\mathbf{D} > 1$ be so that

$$e^{-\Lambda \mathbf{D}} < \min\Big\{ \frac{e^{-\Lambda(1+\tau_{\mathrm{inj}H})}}{C_\varphi}, \frac{\alpha_1}{4}, \frac{1}{4} \Big\}, \tag{3.4}$$

where $\alpha_1 = \alpha_1(M, p)$ is the constant introduced in Proposition 2.2.

Definition 3. Let $A_0 \subset \Sigma_{H,p}$, $\varepsilon_0 > 0$, $\digamma > 0$, $\mathfrak{t}_0 : [\varepsilon_0, \infty) \to [1, \infty)$, and $T_0 > 1$. If the following conditions are satisfied, we say that

$$A_0 \text{ can be } (\varepsilon_0, \mathfrak{t}_0, \digamma)\text{-controlled up to time } T_0.$$

Let $\varepsilon \geq \varepsilon_0$, $\Lambda > \Lambda_{\max}$,

$$0 < R_0 \leq \tfrac{1}{\digamma} e^{-\digamma \Lambda |T_0|}, \qquad 0 < r_0 < R_0,$$

and balls $\{B_{0,i}\}_{i=1}^N \subset \Sigma_{H,p}$ centered in A_0 with radii $\{R_{0,i}\}_{i=1}^N \subset [r_0, R_0]$. Then, for $0 < \tau < \frac{1}{2}\tau_{\mathrm{inj}H}$ and all

$$A_1 \subset \bigcup_{i=1}^N B_{0,i} \subset A_0 \qquad \text{and} \qquad 0 < r < \tfrac{1}{\digamma} e^{-\digamma \Lambda T_0} r_0,$$

there are balls $\{\tilde{B}_{1,k}\}_k \subset \Sigma_{H,p}$ with radii $\{R_{1,k}\}_k \subset [0, \frac{1}{4} R_0]$ so that

(1) $\Lambda^\tau_{A_1 \setminus \cup_k \tilde{B}_{1,k}}(r)$ is non self-looping for times in $[\mathfrak{t}_0(\varepsilon), T_0]$,

(2) $\sum_k R_{1,k}^{n-1} \leq \varepsilon \sum_i R_{0,i}^{n-1}$,

(3) $\inf_k R_{1,k} \geq e^{-\mathbf{D}\Lambda T_0} \inf_i R_{0,i}$.

We observe that when we write $A_1 \setminus \cup_k \tilde{B}_{1,k}$ we mean $A_1 \cap (\Sigma_{H,p} \setminus \cup_k \tilde{B}_{1,k})$.

Note that Definition 3 is vacuous if $T_0 \le \mathfrak{t}_0(\varepsilon_0)$.

Lemma 3.1. *There exists $\digamma > 0$ depending only on (M, p, K_H) so that for every monotone decreasing function $f : [0,\infty) \to [0,\infty)$ with $f \in L^1([0,\infty))$ and $\Lambda > \Lambda_{\max}$, there exists a function $\mathfrak{t}_0 : (0,\infty) \to [1,\infty)$ with the following properties.*
If $A_0 \subset \Sigma_{H,p}$ is so that

$$\sup_{\rho \in A_0} |\det J_t| \le f(|t|) \tag{3.5}$$

for all $t \in (0, T_0)$ or for all $t \in (-T_0, 0)$, then, for all $\varepsilon_0 > 0$,

$$A_0 \text{ can be } (\varepsilon_0, \mathfrak{t}_0, \digamma)\text{-controlled up to time } T_0$$

in the sense of Definition 3. Furthermore, in addition to conditions (1), (2) and (3) in Definition 3 being satisfied, either

$$\bigcup_{t=\mathfrak{t}_0(\varepsilon)}^{T_0} \varphi_t(\Lambda^\tau_{A_1 \setminus \cup_k \tilde{B}_{1,k}}(r)) \cap \Lambda^\tau_{\Sigma_{H,p} \setminus \cup_k \tilde{B}_{1,k}}(r) = \emptyset,$$

or

$$\bigcup_{t=-T_0}^{-\mathfrak{t}_0(\varepsilon)} \varphi_t(\Lambda^\tau_{A_1 \setminus \cup_k \tilde{B}_{1,k}}(r)) \cap \Lambda^\tau_{\Sigma_{H,p} \setminus \cup_k \tilde{B}_{1,k}}(r) = \emptyset.$$

Note that the last conclusion of Lemma 3.1 differs from condition (1) in Definition 3 since we insist that, after flowing, not only does $\Lambda^\tau_{A_1 \setminus \cup_k \tilde{B}_{1,k}}(r)$ not self-intersect (as in (1) of Definition 3, but it does not even intersect $\Sigma_{H,p} \setminus \cup_k \tilde{B}_{1,k}$.

Proof. We prove the case in which (3.5) holds for all $t \in (0, T_0)$ (the case in which it holds for all $t \in (-T_0, 0)$ is identical after sending $t \to -t$). Let $\Lambda > \Lambda_{\max}$ and t_0 be large enough so that $t_0 > \tau_{\text{inj}H} + 2$ and

$$C_\varphi e^{\Lambda} e^{-\mathbf{D}\Lambda(t_0 - \tau_{\text{inj}H} - 1)} \le 1, \tag{3.6}$$

where C_φ is as in (3.4). We will assume, without loss of generality, that $f(|t|) \ge \frac{1}{C_\varphi} e^{-\Lambda t}$. Define

$$\mathfrak{t}_0 : (0,\infty) \to [1,\infty) \qquad \mathfrak{t}_0(\varepsilon) = \inf\left\{ s \ge t_0 : \int_s^\infty f(s)ds \le \frac{\varepsilon \tau_{\text{inj}H}}{4\alpha} \right\},$$

where

$$\alpha := 2^{3n-1}\gamma^{n-1} \qquad \text{and} \qquad \gamma := \tfrac{1}{4} C_\varphi e^{\Lambda}.$$

Here, $\mathfrak{t}_0(\varepsilon) \ge 2$ since $t_0 > \tau_{\text{inj}H} + 2 > 2$.

Fix $\varepsilon_0 > 0$ and let $\varepsilon \ge \varepsilon_0$. Let $0 < \tau < \frac{1}{2}\tau_{\text{inj}H}$, $R_0 > 0$, $0 < r_0 < R_0$ and let $\{B_{0,i}\}_{i=1}^N \subset \Sigma_{H,p}$ be a collection of balls centered in A_0 with radii $\{R_{0,i}\}_{i=1}^N \subset [r_0, R_0]$. Let $A_1 \subset \bigcup_{i=1}^N B_{0,i}$ and $0 < r < 1$. For each $i \in \{1,\dots,N\}$ let $\{I_{0,i,j}\}_{j=1}^{N_i}$ be a collection of disjoint intervals $I_{0,i,j} \subset [\mathsf{t}_0(\varepsilon) - 2\tau - r, T_0 + 2\tau + r]$ so that $\frac{\tau_{\operatorname{inj}H}}{4} \leq |I_{0,i,j}| < \frac{\tau_{\operatorname{inj}H}}{2}$ and

$$\{t \in [\mathsf{t}_0(\varepsilon) - 2\tau - r, T_0 + 2\tau + r] : \ \varphi_t(\Lambda^0_{B_{0,i}}(r)) \cap \Lambda^0_{\Sigma_{H,p}}(r) \neq \emptyset\} \subset \bigcup_{j=1}^{N_i} I_{0,i,j},$$
$$\text{and} \tag{3.7}$$
$$\bigcup_{t \in I_{0,i,j}} \varphi_t(\Lambda^0_{B_{0,i}}(r)) \cap \Lambda^0_{\Sigma_{H,p}}(r) \neq \emptyset.$$

For $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, N_i\}$ define

$$D_{0,i,j} := \bigcup_{t \in I_{0,i,j}} \varphi_t(\Lambda^0_{B_{0,i}}(r)) \cap \Lambda^0_{\Sigma_{H,p}}(r). \tag{3.8}$$

We claim that for each pair (i, j)

$$D_{0,i,j} \subset \bigcup_{\ell=1}^{L_{i,j}} \Lambda^0_{B_{0,i,j,\ell}}(r) \tag{3.9}$$

where $\{B_{0,i,j,\ell}\}_{\ell=1}^{L_{i,j}}$ are balls centered in $\Sigma_{H,p}$ with radii $R_{0,i,j,\ell} := \gamma e^{-\mathbf{D}\Lambda t_{0,i,j}} R_{0,i}$ satisfying

$$L_{i,j} R_{0,i,j,\ell}^{n-1} \leq \alpha f(t_{0,i,j}) R_{0,i}^{n-1} \tag{3.10}$$

(see Figure 3 for an illustration of this covering), where $t_{0,i,j} := \min\{t : \ t \in I_{0,i,j}\}$. Note that $t_{0,i,j} > 1$ for all (i, j) since $r < 1$ and $\mathsf{t}_0(\varepsilon) \geq t_0 > \tau_{\operatorname{inj}H} + 2$, and so $\mathsf{t}_0(\varepsilon) - 2\tau - r > \mathsf{t}_0(\varepsilon) - \tau_{\operatorname{inj}H} - 1 > 1$.

Note that, since we take $0 < r < R_0 < \digamma^{-1} e^{-\digamma \Lambda T_0}$, if we let $\digamma_0 = \digamma_0(M, p, K_H)$ large enough and assume $\digamma \geq \digamma_0$, then $\Sigma_{H,p}$ is almost flat as a submanifold of T^*M at scale R_0. In particular, we have

$$\mathcal{B}(\rho, \tfrac{1}{2}R) \cap \Lambda^0_{\Sigma_{H,p}}(r) \ \subset \ \Lambda^0_{B(\rho,R)}(r),$$

for all $\rho \in \Sigma_{H,p}$ and $0 \leq R \leq R_0$. Here we are using $\mathcal{B}$ to denote a ball in T^*M and B to denote a ball in $\Sigma_{H,p}$. Therefore, it suffices to show that

$$D_{0,i,j} \subset \bigcup_{\ell=1}^{L_{i,j}} \mathcal{B}_{0,i,j,\ell}. \tag{3.11}$$

where $\{\mathcal{B}_{0,i,j,\ell}\}_{\ell=1}^{L_{i,j}} \subset T^*M$ are balls with radii $\mathcal{R}_{0,i,j,\ell} = \frac{1}{2} R_{0,i,j,\ell}$ with $R_{0,i,j,\ell}$ as in (3.10).

Let $\rho_{0,i} \in A_0$ be the center of $B_{0,i}$ and fix $j \in \{1, \dots, N_i\}$. To prove the claim in (3.11) fix $t_{\rho_{0,i}} \in I_{0,i,j}$ so that $\varphi_{t_{\rho_{0,i}}}(\rho_{0,i}) \in \Lambda^0_{\Sigma_{H,p}}(r)$. Observe that choosing coordinates near $\rho_{0,i}$ and $\varphi_{t_{\rho_{0,i}}}(\rho_{0,i})$, we have for t near $t_{\rho_{0,i}}$ and ρ near $\rho_{0,i}$,

$$\varphi_t(\rho) = \varphi_t(\rho_{0,i}) + d\varphi_t(\rho - \rho_{0,i}) + O(|\rho - \rho_{0,i}|^2 e^{2\Lambda|t|}).$$

If $|\rho - \rho_{0,i}| \leq R_{0,i}$ and $\rho \in \Sigma_{H,p}$, this gives

$$\varphi_t(\rho) = \varphi_t(\rho_{0,i}) + J_t(\rho - \rho_{0,i}) + O(R_{0,i}^2 e^{2\Lambda|t|}).$$

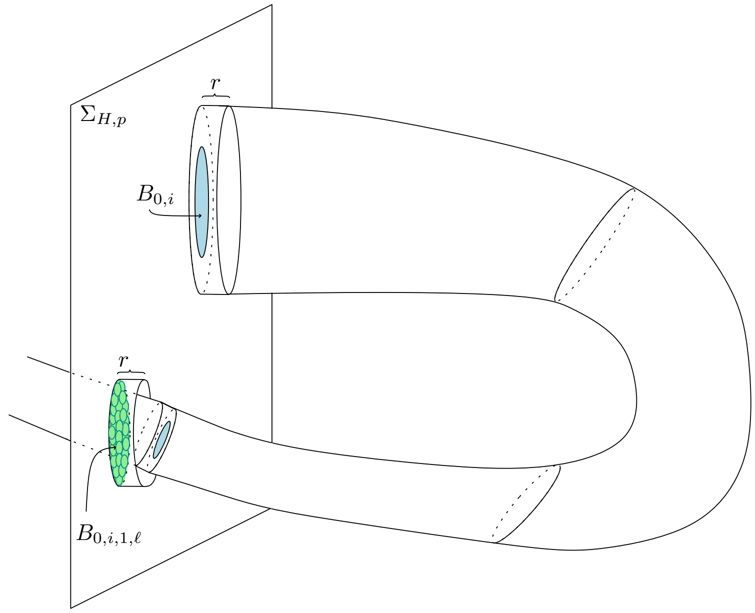

FIGURE 3. Illustration of a contracting ball and the cover by much smaller balls for the proof of Lemma 3.1.

Now, let $\{\lambda_i(t)\}_{i=1}^{n-1}$ be the singular values of J_t ordered so that $\lambda_i(t) \leq \lambda_{i+1}(t)$. Then, modulo perturbations controlled by $R_0^2 e^{2\Lambda|t|}$, the set $\varphi_t(B_{0,i})$ is an $n-1$ dimensional ellipsoid with axes of length $\lambda_i(t)R_{0,i}$. Also, observe that

$$\frac{e^{-\Lambda t}}{C_\varphi} \leq \lambda_1(t) \leq \lambda_{n-1}(t) \leq C_\varphi e^{\Lambda t},$$

where C_φ is as in (3.3). Since $\mathfrak{t}_0(\varepsilon) \geq 1$, we note that $e^{-\Lambda \mathfrak{t}_0(\varepsilon)(\mathbf{D}-1)} < \frac{1}{C_\varphi}$. This ensures that $e^{-\mathbf{D}\Lambda t} < \frac{e^{-\Lambda t}}{C_\varphi}$ for all $t \geq \mathfrak{t}_0(\varepsilon)$.

Also, note that there exists a constant $\alpha_{M,p} > 0$ so that for all $i \in \{1, \dots, N\}$ and $\rho \in \varphi_{t_{\rho_{0,i}}}(\Lambda^0_{B_{0,i}}(r))$ we have $d(\rho, \varphi_{t_{\rho_{0,i}}}(B_{0,i})) \leq \alpha_{M,p} e^{\Lambda t_{\rho_{0,i}}} r$. Define $\digamma$ by

$$\digamma := \max\{8\alpha_{M,p}\,,\, \mathbf{D}+1\,,\, \digamma_0\},$$

and from now on work with $R_0 \leq \frac{1}{\digamma} e^{-\digamma \Lambda |T_0|}$. Then, if $0 < r < \frac{1}{\digamma} e^{-\digamma \Lambda T_0} r_0$, we have that r is small enough so that $\alpha_{M,p} e^{\Lambda T_0} r \leq \frac{1}{8} e^{-\mathbf{D}\Lambda T_0} r_0$. In particular, $\alpha_{M,p} e^{\Lambda t_{\rho_{0,i}}} r < \frac{1}{8} e^{-\mathbf{D}\Lambda t_{0,i,j}} R_{0,i}$ for all $i \in \{1, \dots, N\}$ and there are points $\{q_\ell\}_{\ell=1}^{L_{i,j}} \subset \varphi_{t_{\rho_{0,i}}}(B_{0,i})$ so that

$$\varphi_{t_{\rho_{0,i}}}(\Lambda^0_{B_{0,i}}(r)) \subset \bigcup_{\ell=1}^{L_{i,j}} \mathcal{B}(q_\ell, \tfrac{1}{8} e^{-\mathbf{D}\Lambda t_{0,i,j}} R_{0,i}), \tag{3.12}$$

where the balls in the right hand side are balls in T^*M. Furthermore,

$$\begin{aligned}\operatorname{vol}(\varphi_{t_{\rho_{0,i}}}(B_{0,i})) &\leq \operatorname{vol}(B_{0,i})(|\det(J_{t_{\rho_{0,i}}})| + C_{M,p} R_0^2 e^{2\Lambda t_{\rho_{0,i}}}) \\ &\leq C_n R_{0,i}^{n-1}(f(t_{\rho_{0,i}}) + C_{M,p} R_0^2 e^{2\Lambda t_{\rho_{0,i}}})\end{aligned}$$

for some $C_n > 0$ and $C_{M,p} > 0$. Next, adjust $\digamma$ so that $\digamma^2 > C_\varphi C_{M,p}$. Then, since $f(|t|) \geq \frac{1}{C_\varphi} e^{-\Lambda t}$,

$$\operatorname{vol}(\varphi_{t_{\rho_{0,i}}}(B_{0,i})) \leq 2C_n R_{0,i}^{n-1} f(t_{\rho_{0,i}}).$$

Observe that by (3.4) and $t_{\rho_{0,i}} - \frac{\tau_{\mathrm{inj}H}}{2} \leq t_{0,i,j} \leq t_{\rho_{0,i}}$, we have $e^{-\mathbf{D}\Lambda t_{0,i,j}} < \lambda_1(t_{\rho_{0,i}})$. Therefore, using that $t_{0,i,j} \leq t_{\rho_{0,i}}$ again, the points $\{q_\ell\}_{\ell=1}^{L_{i,j}}$ can be chosen so that

$$\begin{aligned}L_{i,j} C_n (\tfrac{1}{8} e^{-\mathbf{D}\Lambda t_{0,i,j}} R_{0,i})^{n-1} &\leq 2 \operatorname{vol}\Big(\varphi_{t_{\rho_{0,i}}}(B_{0,i}) \bigcap \cup_{\ell=1}^{L_{i,j}} \mathcal{B}(q_\ell, \tfrac{1}{8} e^{-\mathbf{D}\Lambda t_{0,i,j}} R_{0,i})\Big) \\ &\leq 4 C_n R_{0,i}^{n-1} f(t_{0,i,j}). \qquad (3.13)\end{aligned}$$

Note that this yields $L_{i,j}(\frac{1}{8} e^{-\mathbf{D}\Lambda t_{0,i,j}})^{n-1} \leq 4 f(t_{0,i,j})$.

Since $|I_{0,i,j}| < 1$, it follows that for every choice of indices ℓ, (i,j) we have

$$\operatorname{diam}\Big(\bigcup_{t \in I_{0,i,j}} \varphi_{t - t_{\rho_{0,i}}}(\mathcal{B}(q_\ell, \tfrac{1}{8} e^{-\mathbf{D}\Lambda t_{0,i,j}} R_{0,i})) \cap \Lambda^0_{\Sigma_{H,p}}(r)\Big) \leq \frac{1}{8} C_\varphi e^{\Lambda} e^{-\mathbf{D}\Lambda t_{0,i,j}} R_{0,i} \leq \frac{1}{8} R_{0,i} \qquad (3.14)$$

where in the last inequality, we use the definition of $\mathbf{D}$. Without loss of generality, we may assume that $C_\varphi \geq 4$ (redefining $\mathbf{D}$ in the process) and hence that $\gamma = \frac{1}{4} C_\varphi e^{\Lambda} \geq 1$ (see (3.10)). This implies that we can find a point $\rho_{0,i,j,\ell} \in \Sigma_{H,p}$ so that the ball $\mathcal{B}_{0,i,j,\ell} \subset T^*M$ of center $\rho_{0,i,j,\ell}$ and radius $\mathcal{R}_{0,i,j,\ell} = \frac{1}{2} \gamma e^{-\mathbf{D}\Lambda t_{0,i,j}} R_{0,i} = \frac{1}{2} R_{0,i,j,\ell}$ contains the set in (3.14) whose diameter is being bounded. Thus, by the definition (3.8) of $D_{0,i,j}$ together with (3.12), we conclude that (3.11) and (3.9) hold. Also, by the definition of $R_{0,i,j,\ell}$, the definition of α, and (3.13), for each choice of (i,j)

$$\sum_{\ell=1}^{L_{i,j}} R_{0,i,j,\ell}^{n-1} = L_{i,j} \gamma^{n-1} (e^{-\mathbf{D}\Lambda t_{0,i,j}} R_{0,i})^{n-1} \leq \alpha f(t_{0,i,j}) R_{0,i}^{n-1},$$

and hence (3.10) holds. Therefore, from the definition of $\mathfrak{t}_0(\varepsilon)$ it follows that

$$\sum_{i,j,\ell} R_{0,i,j,\ell}^{n-1} \leq \alpha \sum_{i,j} f(t_{0,i,j}) R_{0,i}^{n-1} \leq \frac{4\alpha}{\tau_{\mathrm{inj}H}} \int_{\mathfrak{t}_0(\varepsilon)}^{\infty} f(s)\,ds \sum_i R_{0,i}^{n-1} \leq \varepsilon \sum_i R_{0,i}^{n-1}, \qquad (3.15)$$

where to get the second inequality we used that $t_{0,i,j+1} - t_{0,i,j} \geq \tau_{\mathrm{inj}H}/4$ implies

$$\sum_j \frac{\tau_{\mathrm{inj}H}}{4} f(t_{0,i,j}) \leq \int_{\mathfrak{t}_0(\varepsilon)}^{\infty} f(s)\,ds.$$

Let $k = k(i,j,\ell)$ be an index reassignment and write $\tilde{B}_{1,k} = B_{0,i,j,\ell}$ and $R_{1,k} = R_{0,i,j,\ell}$. Note that by the definition of $R_{0,i,j,\ell}$ in (3.10) and the first inequality in (3.6) we know $R_{1,k} \leq \frac{1}{4} R_0$. In addition, $\cup_{i,j} D_{0,i,j} \subset \cup_k \tilde{B}_{1,k}$. According to (3.7) and (3.8) we proved that

$$\bigcup_{t=\mathfrak{t}_0(\varepsilon)-2\tau-r}^{T_0+2\tau+r} \varphi_t(\Lambda^0_{A_1 \setminus \cup_k \tilde{B}_{1,k}}(r)) \cap \Lambda^0_{\Sigma_{H,p} \setminus \cup_k \tilde{B}_{1,k}}(r) = \emptyset. \tag{3.16}$$

We claim that this implies

$$\bigcup_{t=\mathfrak{t}_0(\varepsilon)}^{T_0} \varphi_t(\Lambda^\tau_{A_1 \setminus \cup_k \tilde{B}_{1,k}}(r)) \cap \Lambda^\tau_{\Sigma_{H,p} \setminus \cup_k \tilde{B}_{1,k}}(r) = \emptyset. \tag{3.17}$$

Indeed, if ρ belongs to the set in (3.17), then there exist times $t \in [\mathfrak{t}_0(\varepsilon)-\tau-r, T_0+\tau+r]$, $s \in [-\tau - r, \tau + r]$, and points $q_0, q_1 \in \mathcal{H}_\Sigma$ (see (1.12)) with

$$d(q_0, A_1 \setminus \cup_k \tilde{B}_{1,k}) < r, \qquad d(q_1, \Sigma_{H,p} \setminus \cup_k \tilde{B}_{1,k}) < r$$

so that $\rho = \varphi_t(q_0) = \varphi_s(q_1)$. Let $\tau' \in [-\tau, \tau]$ be so that $|s - \tau'| < r$. Then, $\varphi_{-\tau'}(\rho) = \varphi_{s-\tau'}(q_1) = \varphi_{t-\tau'}(q_0)$ belongs to the set in (3.16) since $|s - \tau'| < r$ and $t - \tau' \in [\mathfrak{t}_0(\varepsilon) - 2\tau - r, T_0 + 2\tau + r]$. This means that if the set in (3.16) is empty, then so is the set in (3.17). Finally, (3.17) implies that

$$\Lambda^\tau_{A_1}(r) \setminus \bigcup_k \Lambda^\tau_{\tilde{B}_{1,k}}(r)$$

is non self looping for times in $[\mathfrak{t}_0(\varepsilon), T_0]$. Furthermore, (3.15) now reads

$$\sum_k R^{n-1}_{1,k} \leq \varepsilon \sum_i R^{n-1}_{0,i}.$$

$\square$

Lemma 3.2. *Let $E \subset \Sigma_{H,p}$ be a ball of radius $\delta > 0$. Let $\varepsilon_0 > 0$, $\mathfrak{t}_0 : [\varepsilon_0, +\infty) \to [1, +\infty)$, $T_0 > 0$, and $\digamma > 0$, have the property that E can be $(\varepsilon_0, \mathfrak{t}_0, \digamma)$-controlled up to time T_0 in the sense of Definition 3. Let $0 < m < \frac{\log T_0 - \log \mathfrak{t}_0(\varepsilon)}{\log 2}$ be a positive integer,*

$$0 \leq R_0 \leq \min\Big\{\tfrac{1}{\digamma} e^{-\digamma \Lambda T_0}, \tfrac{\delta}{10}\Big\}, \qquad 0 < r_1 < \tfrac{1}{5\digamma} e^{-(\digamma + 2\mathbf{D})\Lambda T_0} R_0,$$

and $E_0 \subset E$ with $d(E_0, E^c) > R_0$. Let $0 < \tau < \frac{1}{2}\tau_{injH}$ and suppose that $\Lambda^\tau_{\rho_j}(r_1)$ is a $(\mathfrak{D}, \tau, r_1)$ good cover of $\Sigma_{H,p}$ and set

$$\mathcal{E} := \{j \in \{1, \dots, N_{r_1}\} : \Lambda^\tau_{\rho_j}(r_1) \cap \Lambda^\tau_{E_0}(\tfrac{r_1}{5}) \neq \emptyset\}.$$

Then, there exist $C_{M,p} > 0$ depending only on (M, p) and sets $\{\mathcal{G}_\ell\}_{\ell=0}^m \subset \{1, \dots N_{r_1}\}$, $\mathcal{B} \subset \{1, \dots N_{r_1}\}$ so that

$$\mathcal{E} \ \subset \ \mathcal{B} \cup \bigcup_{\ell=0}^m \mathcal{G}_\ell,$$

- $\bigcup_{i\in\mathcal{G}_\ell} \Lambda^\tau_{\rho_i}(r_1)$ *is* $[\mathfrak{t}_0, 2^{-\ell}T_0]$ *non-self looping for every* $\ell \in \{0,\dots,m\}$, (3.18)
- $|\mathcal{G}_\ell| \le C_{M,p}\mathfrak{D}\varepsilon_0^\ell \delta^{n-1} r_1^{1-n}$ *for every* $\ell \in \{0,\dots,m\}$, (3.19)
- $|\mathcal{B}| \le C_{M,p}\mathfrak{D}\varepsilon_0^{m+1}\delta^{n-1}r_1^{1-n}$. (3.20)

Proof. Choose balls $\{B_{0,i}\}_{i=1}^N$ centered in E_0 so that $E_0 \subset \bigcup_{i=1}^N B_{0,i}$ where $B_{0,i}$ has radius $R_{0,i} = R_0$ built so that $NR_0^{n-1} \le C_n\delta^{n-1}$. This can be done since $R_0 < \frac{\delta}{10}$. Let $r_0 := e^{-2\mathbf{D}\Lambda T_0}R_0$. Since E can be $(\varepsilon_0, \mathfrak{t}_0, \digamma)$-controlled up to time T_0, for

$$0 < r < \tfrac{1}{\digamma}e^{-\digamma\Lambda T_0}r_0 = \tfrac{1}{\digamma}e^{-(\digamma+2\mathbf{D})\Lambda T_0}R_0$$

there are balls $\{\tilde{B}_{1,k}\}_k \subset \Sigma_{H,p}$ of radii $\{R_{1,k}\}_k \subset [0, \frac14 R_0]$, so that

$$\inf_k R_{1,k} \ge e^{-\mathbf{D}\Lambda T_0}R_0 \ge r_0, \qquad \sum_k R_{1,k}^{n-1} \le \varepsilon_0 N R_0^{n-1},$$

and with $G_0 := \Lambda^\tau_{E_0\setminus\tilde{E}_1}(r)$ non-self-looping for times in $[\mathfrak{t}_0(\varepsilon), T_0]$, where we have set $\tilde{E}_1 = \cup_k \tilde{B}_{1,k}$. Note that we may assume that $E_0 \cap \tilde{B}_{1,k} \neq \emptyset$ for all k. Now, since $R_{1,k} \le \frac14 R_0$, the ball $\tilde{B}_{1,k}$ is centered at a distance no more than $\frac14 R_0$ from E_0. So, letting $E_1 := \cup_k B_{1,k}$ with $B_{1,k}$ the ball of radius $2R_{1,k}$ with the same center as $\tilde{B}_{1,k}$, we have

$$d(E_1, E^c) \ge d(E_0, E^c) - \tfrac34 R_0 > (1 - \tfrac34)R_0.$$

Next, we set $T_1 := 2^{-1}T_0$ and use that E_0 can be $(\varepsilon_0, \mathfrak{t}_0, \digamma)$-controlled up to time T_1 (indeed up to time $2T_1$). By definition $E_1 \subset \bigcup_k B_{1,k}$ and $R_0 \le \digamma^{-1}e^{-\digamma\Lambda T_0} \le \digamma^{-1}e^{-\digamma\Lambda T_1}$. Therefore, since $0 < r < \digamma^{-1}e^{-\digamma\Lambda T_0}r_0 < \digamma^{-1}e^{-\digamma\Lambda T_1}r_0$, there are balls $\{\tilde{B}_{2,k}\}_k \subset \Sigma_{H,p}$ of radii $0 < R_{2,k} \le \frac{1}{4^2}R_0$ with

$$\inf_k R_{2,k} \ge e^{-\mathbf{D}\Lambda T_1}\inf_i R_{1,i} \quad \text{and} \quad \sum_k R_{2,k}^{n-1} \le \varepsilon_0 \sum_k R_{1,k}^{n-1} \le \varepsilon_0^2 N R_0^{n-1}, \tag{3.21}$$

so that $G_1 := \Lambda^\tau_{E_1\setminus\tilde{E}_2}(r)$ is non-self-looping for times in $[\mathfrak{t}_0(\varepsilon), T_1]$, where we have set $\tilde{E}_2 = \cup_k \tilde{B}_{2,k}$. Since we may assume that $E_1 \cap \tilde{B}_{2,k} \neq \emptyset$ for all k, the balls $\tilde{B}_{2,k}$ are centered at a distance smaller than $\frac{1}{4^2}R_0$ from E_1. In particular, letting $E_2 = \cup_k B_{2,k}$ where $B_{2,k}$ is the ball of radius $2R_{2,k}$ centered at the same point as $\tilde{R}_{2,k}$, we have

$$d(E_2, E^c) \ge d(E_1, E^c) - \tfrac{3}{4^2}R_0 > R_0\big(1 - \tfrac34 - \tfrac{3}{4^2}\big).$$

Continuing this way we claim that one can construct a collection of sets $\{G_\ell\}_{\ell=1}^m \subset \Lambda^\tau_E(r)$ so that

A) G_ℓ is non-self-looping for times in $[\mathfrak{t}_0(\varepsilon), T_\ell]$ with $T_\ell = 2^{-\ell}T_0$.

B) There are balls $B_{\ell,k}, \tilde{B}_{\ell,k} \subset \Sigma_{H,p}$ centered at $\rho_{\ell,k} \in E$ of radii $2R_{\ell,k}$, $R_{\ell,k}$ respectively so that

$$G_\ell = \Lambda^\tau_{E_\ell\setminus\tilde{E}_{\ell+1}}(r),$$

where $E_\ell = \bigcup_k B_{\ell,k}$ and $\tilde{E}_\ell = \bigcup_k \tilde{B}_{\ell,k}$.

C) For all $\ell \geq 1$, the radii satisfy $\sup_\ell R_{\ell,k} \leq \frac{1}{4^\ell} R_0$,

$$\inf_k R_{\ell,k} \geq e^{-2\mathbf{D}\Lambda T_0} R_0 = r_0 \qquad \text{and} \qquad \sum_k R_{\ell,k}^{n-1} \leq \varepsilon_0^\ell N R_0^{n-1}. \tag{3.22}$$

The claim in (A) follows by construction of G_ℓ. For the claim in (B), we only need to check that the balls $B_{\ell,k}$ are centered in E. For this, note that since $R_{\ell,k} \leq \frac{1}{4^\ell} R_0$, by induction

$$d(E_\ell, E^c) > d(E_{\ell-1}, E^c) - \tfrac{3}{4^\ell} R_0 > R_0 \Big(1 - \sum_{j=1}^{\ell} \tfrac{3}{4^j}\Big) \geq \frac{1}{4^\ell} R_0.$$

Remark 8. Note that this actually gives $E_\ell \subset E$ and so all of $B_{\ell,k}$ is inside E (not just its center).

We proceed to justify the first inequality in (3.22). Note that the construction yields that $\inf_k R_{\ell,k} \geq e^{-\mathbf{D}\Lambda T_\ell} \inf_i R_{\ell-1,i}$ for every ℓ. Therefore, since $T_\ell = 2^{-\ell} T_0$ and $\inf_k R_{\ell,k} \geq e^{-\mathbf{D}\Lambda T_\ell} \inf_i R_{\ell-1,i}$ (see (3.21)), we obtain

$$\inf_k R_{\ell,k} \geq \prod_{j=0}^{\ell} e^{-\mathbf{D}\Lambda \frac{T_0}{2^j}} R_0 = e^{-\mathbf{D}\Lambda T_0 (2 - \frac{1}{2^\ell})} R_0 \geq e^{-2\mathbf{D}\Lambda T_0} R_0.$$

The construction also yields that $\sum_k R_{\ell,k}^{n-1} \leq \varepsilon_0 \sum_k R_{\ell-1,k}^{n-1}$ for all ℓ. Therefore, the upper bound (3.22) on the sum of the radii follows by induction. Indeed,

$$\sum_k R_{\ell,k}^{n-1} \leq \varepsilon_0^\ell \sum_k R_{0,k}^{n-1} = \varepsilon_0^\ell N R_0^{n-1}.$$

Set $r := 5r_1$ in the above argument, and define

$$\mathcal{G}_\ell := \{i \in \mathcal{E} : \Lambda^\tau_{\rho_i}(r_1) \subset G_\ell\}, \qquad \mathcal{B} := \mathcal{E} \setminus \bigcup_{\ell=0}^{m} \mathcal{G}_\ell.$$

Then, since G_ℓ is $[\mathfrak{t}_0(\varepsilon_0), 2^{-\ell} T_0]$ non-self looping, (3.18) holds. Furthermore, $\mathcal{E} \subset \mathcal{B} \cup \bigcup_{\ell=0}^m \mathcal{G}_\ell$ by construction.

We proceed to prove (3.19). Since the cover by tubes can be decomposed into $\mathfrak{D}$ sets of disjoint tubes,

$$|\mathcal{G}_\ell| \leq \mathfrak{D} \frac{\operatorname{vol}(G_\ell \cap \Lambda^\tau_{E_0}(r_1))}{\min_i \operatorname{vol}(\Lambda^\tau_{\rho_i}(r_1))} \leq C_{M,p} \mathfrak{D} r_1^{1-n} \sum_k R_{\ell,k}^{n-1} \leq C_{M,p} \mathfrak{D} r_1^{1-n} \varepsilon_0^\ell N R_0^{n-1},$$

for some $C_{M,p} > 0$ that depends only on (M,p). Then, (3.19) follows since $N R_0^{n-1} \leq C_n \delta^{n-1}$.

The rest of the proof is dedicated to obtaining (3.20). For each ℓ note that $E_\ell \subset (G_\ell \cup \tilde{E}_{\ell+1})$ and $\Lambda^\tau_{E_\ell}(\frac{r_1}{5}) \subset \Lambda^\tau_{\Sigma_{H,p}}(\frac{r_1}{5}) \subset \cup_i \Lambda^\tau_{\rho_i}(r_1)$. We claim that for every pair of indices (ℓ, i) with $\Lambda^\tau_{E_\ell}(\frac{r_1}{5}) \cap \Lambda^\tau_{\rho_i}(r_1) \neq \emptyset$, either

$$\Lambda^\tau_{\rho_i}(r_1) \subset \Lambda^\tau_{E_\ell \setminus \tilde{E}_{\ell+1}}(5r_1) \qquad \text{or} \qquad \Lambda^\tau_{\rho_i}(r_1) \cap \Lambda^\tau_{\tilde{E}_{\ell+1}}(\tfrac{r_1}{5}) \neq \emptyset.$$

Indeed, suppose that $\Lambda^\tau_{\rho_i}(r_1) \cap \Lambda^\tau_{\tilde{E}_{\ell+1}}(\frac{r_1}{5}) = \emptyset$. Then, there exists $q \in \mathcal{H}_\Sigma \cap \Lambda^\tau_{\rho_i}(r_1)$ so that $d(q, \rho_i) < r_1$, $d(q, E_\ell) < \frac{r_1}{5}$, $d(q, \tilde{E}_{\ell+1}) \geq \frac{r_1}{5}$. In particular, $d(q, E_\ell \setminus \tilde{E}_{\ell+1}) < \frac{r_1}{5}$. Now, suppose that $q_1 \in \mathcal{H}_\Sigma \cap \Lambda^\tau_{\rho_i}(r_1)$. Then,

$$d(q_1, E_\ell \setminus \tilde{E}_{\ell+1}) \leq d(q_1, \rho_i) + d(\rho_i, q) + d(q, E_\ell \setminus \tilde{E}_{\ell+1}) < \tfrac{11}{5} r_1 < 5r_1.$$

In particular, $\Lambda^\tau_{\rho_i}(r_1) \subset \Lambda^\tau_{E_\ell \setminus \tilde{E}_{\ell+1}}(5r_1)$ as claimed.

Now, suppose that $\Lambda^\tau_{\rho_i}(r_1) \cap \Lambda^\tau_{\tilde{E}_{\ell+1}}(\frac{r_1}{5}) \neq \emptyset$. Then, since $r_1 < \frac{r_0}{5}$ and $R_{\ell,k} \geq r_0$, we have

$$\Lambda^\tau_{\rho_i}(r_1) \cap \mathcal{H}_\Sigma \ \subset E'_{\ell+1}$$

where $E'_{\ell+1} = \cup_j \frac{3}{2} \tilde{B}_{\ell+1,j}$. Observe then that for all ℓ

$$\Lambda^\tau_{E_\ell}(\tfrac{r_1}{5}) \cap \Big(\bigcup_{i \in \mathcal{G}_\ell} \Lambda^\tau_{\rho_i}(r_1) \Big)^c \ \subset \ \Lambda^\tau_{E'_{\ell+1}}(\tfrac{r_1}{5}). \tag{3.23}$$

By induction in $k \geq 2$ we assume that $\Lambda^\tau_{E_0}(\frac{r_1}{5}) \cap \Big(\bigcup_{\ell=0}^{k-1} \bigcup_{i \in \mathcal{G}_\ell} \Lambda^\tau_{\rho_i}(r_1) \Big)^c \subset \Lambda^\tau_{E'_k}(\frac{r_1}{5})$. Note that the base case $k = 1$ is covered by setting $\ell = 0$ in (3.23). Then, using (3.23) with $\ell = k$ together with the inclusion $\tilde{E}_k \subset E'_k \subset E_k$ (in fact the balls defining each set have the same center and radii given respectively by $R_{\ell,k}$, $\frac{3}{2} R_{l,k}$ and $2R_{l,k}$) we obtain

$$\Lambda^\tau_{E_0}(\tfrac{r_1}{5}) \cap \Big(\bigcup_{\ell=0}^{k} \bigcup_{i \in \mathcal{G}_\ell} \Lambda^\tau_{\rho_i}(r_1) \Big)^c \ \subset \ \Lambda^\tau_{E'_{k+1}}(\tfrac{r_1}{5}).$$

In particular, if $i \in \mathcal{B}$, then $\Lambda^\tau_{E_0}(\frac{r_1}{5}) \cap \Lambda^\tau_{\rho_i}(r_1) \subset \Lambda^\tau_{E_{m+1}}(\frac{r_1}{5})$.

Therefore,

$$|\mathcal{B}| \leq C_{M,p} \mathfrak{D} r_1^{1-n} \sum_i R^{n-1}_{m+1,i} \leq C_{M,p} r_1^{1-n} \varepsilon_0^{m+1} N R_0^{n-1},$$

for some $C_{M,p}$ that depends only on (M, p). This proves (3.20) since $N R_0^{n-1} \leq C_n \delta^{n-1}$. $\square$

4. No Conjugate points: Proof of Theorems 1 and 2

We dedicate this section to the proofs of Theorems 1 and 2. We work with the Hamiltonian $p : T^*M \to \mathbb{R}$ given by $p(x, \xi) = |\xi|_{g(x,\xi)} - 1$. The Hamiltonian flow φ_t associated to it is the geodesic flow, and for any $H \subset M$ we have $\Sigma_{H,p} = SN^*H$.

Let $\Lambda > \Lambda_{\max}$, $t_0 \in \mathbb{R}$, $\varepsilon > 0$, and $x \in M$. The study of the behavior of the geodesic flow near SN^*H under the no conjugate points assumption hinges on the fact that if there are no more than m conjugate points (counted with multiplicity) along φ_t for $t \in (t_0 - 2\varepsilon, t_0 + 2\varepsilon)$, then for every $\rho \in S^*_x M$ there is a subspace $\mathbf{V}_\rho \subset T_\rho S^*_x M$ of dimension $n - 1 - m$ so that for all $\mathbf{v} \in \mathbf{V}_\rho$,

$$\|\mathbf{v}\| \leq C \varepsilon^{-1} e^{\Lambda |t_0|} \|(d\pi \circ d\varphi_t)_\rho \mathbf{v}\|, \qquad t \in (t_0 - \varepsilon, t_0 + \varepsilon).$$

In particular, this yields that the restriction $(d\pi \circ d\varphi_t)_\rho : \mathbf{V}_\rho \to T_{\pi\varphi_t(\rho)} M$ is invertible onto its image with

$$\|(d\pi \circ d\varphi_t)^{-1}_\rho\| \leq C \varepsilon^{-1} e^{\Lambda |t_0|}. \tag{4.1}$$

The proof of this result is included in Section 6 as Proposition 6.1 and it holds as long as

$$0<\varepsilon<e^{-C\Lambda|t_0|}/C \tag{4.2}$$

for $C>0$, depending only on (M,g) as defined in as in Proposition 6.1.

In what follows we continue to write $F:T^*M\to\mathbb{R}^{n+1}$ for the defining function of SN^*H satisfying (2.2) and we continue to work with

$$\psi:\mathbb{R}\times T^*M\to\mathbb{R}^{n+1},\qquad\qquad \psi(t,\rho)=F\circ\varphi_t(\rho).$$

The following lemma is dedicated to finding a suitable left inverse for $d\psi$.

Lemma 4.1. *Suppose $k>\frac{n+1}{2}$, $\Lambda>\Lambda_{\max}$. There exists $c_H>0$ depending only on K_H (as defined in (1.15)) such that the following holds. Let $t_0\in\mathbb{R}$ and $a>0$ satisfy*

$$d(H,\mathcal{C}_H^{2k-n-1,r_{t_0},t_0})>r_{t_0},$$

where $r_t=\frac{1}{a}e^{-a|t|}$. Then, if $\rho_0\in SN^*H$ and

$$d(SN^*H,\varphi_{t_0}(\rho_0))<\min(r_{t_0},c_H),$$

there exists $\mathbf{w}_0\in T_{\rho_0}SN^*H$ so that the restriction

$$d\psi_{(t_0,\rho_0)}:\mathbb{R}\partial_t\times\mathbb{R}\mathbf{w}_0\to T_{\psi(t_0,\rho_0)}\mathbb{R}^{n+1}$$

has left inverse $L_{(t_0,\rho_0)}$ with

$$\|L_{(t_0,\rho_0)}\|\le C_{M,g}(1+a)e^{C_{M,g}(a+\Lambda)|t_0|}$$

where $C_{M,g}>0$ is a constant depending only on (M,g).

Note that the assumption $k>\frac{n+1}{2}$ is needed for $\mathcal{C}_H^{2k-n-1,r_{t_0},t_0}$ to be defined. The reason why $2k-n-1$ appears in the exponent of $\mathcal{C}_H$ is explained in Remark 9.

Proof. Let $\tilde{F}:=(f_1,\dots,f_k)\in C^\infty(M;\mathbb{R}^k)$ be a defining function for $H\subset M$ such that $d\tilde{F}_y$ has right inverse $R_{\tilde{F},y}$ with $\|R_{\tilde{F},y}\|\le 2$ for all y such that $d(y,H)<c_H$. Note that c_H can be chosen uniformly depending only on K_H as in (1.15). Next, define

$$\tilde{\psi}:\mathbb{R}\times T^*M\to\mathbb{R}^k,\qquad\qquad \tilde{\psi}(t,\rho):=\tilde{F}\circ\pi\circ\varphi_t(\rho).$$

We claim that there exists $\mathbf{w}_0\in T_{\rho_0}SN^*H$ so that

$$d\tilde{\psi}_{(t_0,\rho_0)}:\mathbb{R}\partial_t\times\mathbb{R}\mathbf{w}_0\to\mathbb{R}^k$$

is injective and has a left inverse bounded by $C_{M,g}(1+a)e^{C_{M,g}(a+\Lambda)|t_0|}$. Note that this is sufficient as this produces a left inverse for ψ itself.

Observe that for $s\in\mathbb{R}$, $\rho\in SN^*H$, and $\mathbf{w}\in T_\rho SN^*H$,

$$d\tilde{\psi}_{(t,\rho)}(s\partial_t,\mathbf{w})=d(\tilde{F}\circ\pi)_{\varphi_t(\rho)}\big(s\,H_p+(d\varphi_t)_\rho\,\mathbf{w}\big). \tag{4.3}$$

Note also that since H is conormally transverse for p, there exists a neighborhood $W\subset T^*M$ of SN^*H and $c>0$ so that for $\tilde{\rho}\in W$,

$$\|d(\tilde{F}\circ\pi)_{\varphi_t(\tilde{\rho})}H_p\|\ge\frac{1}{2}. \tag{4.4}$$

In particular, the restriction

$$d\tilde{\psi}_{(t_0,\rho_0)}:\mathbb{R}\partial_t\to\mathbb{R}^k$$

has a left inverse bounded by 2.

We proceed to find $\mathbf{w}_0 \in T_{\rho_0} SN^*H$ as claimed.

Suppose $d(H, \mathcal{C}_H^{2k-n-1, r_{t_0}, t_0}) > r_{t_0}$. Then, by definition, for all $x \in H$, and every unit speed geodesic γ with $\gamma(0) = x$, there the number of conjugate points to x (counted with multiplicity) along $\gamma(t_0 - r_{t_0}, t_0 + r_{t_0})$ is smaller than or equal to $m := 2k - n - 2$ whenever $d(\gamma(t_0), H) < r_{t_0}$. In particular, since $d(\varphi_{t_0}(\rho_0), SN^*H) < r_{t_0}$, we have $d(\pi(\varphi_{t_0}(\rho_0)), H) < r_{t_0}$. Therefore, by setting $\varepsilon = \min(r_{t_0}/2, e^{-C\Lambda|t_0|}/C)$ in (4.1) with C as in (4.2), we have that there is a $n - 1 - m$ dimensional subspace $\mathbf{V}_{\rho_0} \subset T_{\rho_0} S^*_{x_0} M$ so that $d\pi \circ d\varphi_{t_0}|_{\mathbf{V}_{\rho_0}}$ is invertible onto its image with

$$\|(d\pi \circ d\varphi_{t_0}|_{\mathbf{V}_{\rho_0}})^{-1}\| \leq C\varepsilon^{-1} e^{\Lambda|t_0|} \leq C_{M,g}(1 + a) e^{C_{M,g}(a+\Lambda)|t_0|}, \tag{4.5}$$

for some $C_{M,g} > 0$ depending only on (M, g), and where $x_0 := \pi(\rho_0)$.

Let

$$V = d(\pi \circ \varphi)_{(t_0, \rho_0)}\big(\mathbb{R}\partial_t \times (T_{\rho_0}(SN^*_{x_0}H) \cap \mathbf{V}_{\rho_0})\big).$$

Note that since $\dim \mathbf{V}_{\rho_0} = n - 1 - m$, $\dim T_{\rho_0} SN^*_{x_0} H = k - 1$, $\dim S^*_{x_0} M = n - 1$, we know that $\dim(T_{\rho_0} SN^*_{x_0} H \cap \mathbf{V}_{\rho_0}) \geq k - 1 - m$, and so $\dim V \geq k - m$. Also, the restriction

$$d(\pi \circ \varphi)_{(t_0, \rho_0)} : \mathbb{R}\partial_t \times (T_{\rho_0}(SN^*_x H) \cap \mathbf{V}_{\rho_0}) \to V$$

is invertible with inverse $\tilde{L}_{(t_0, \rho_0)}$ satisfying

$$\|\tilde{L}_{(t_0, \rho_0)}\| \leq C_{M,g}(1 + a) e^{C_{M,g}(a+\Lambda)|t_0|}.$$

Next, there exists a neighborhood $U \subset M$ of H so that for $y \in U$, $d\tilde{F}_y : T_y M \to \mathbb{R}^k$ is surjective with right inverse R_y. By assumption, R_y is bounded by 2. Furthermore, we may assume without loss of generality that for $\rho \in T^*U \cap W$, $d\pi_\rho H_p$ lies in the range of $R_{\pi(\rho)}$. Since $\dim(\operatorname{ran} R_{\pi(\varphi_{t_0}(\rho_0))}) = k$, $\dim V \geq k - m$, and both V and $\operatorname{ran} R_{\pi(\varphi_{t_0}(\rho_0))}$ are contained in $T_{\pi(\varphi_{t_0}(\rho_0))} M$, we know that

$$\dim(\operatorname{ran} R_{\pi(\varphi_{t_0}(\rho_0))} \cap V) \geq 2k - m - n = 2.$$

Then, this guarantees the existence of $\mathbf{w}_0 \in T_{\rho_0}(SN^*_{x_0} H) \cap \mathbf{V}_{\rho_0} \backslash \{0\}$, so that

$$(d\pi \circ d\varphi_{t_0})_{\rho_0} \mathbf{w}_0 \in \operatorname{ran} R_{\pi(\varphi_{t_0}(\rho_0))}.$$

Remark 9. Note that having $\dim(\operatorname{ran} R_{\pi(\varphi_{t_0}(\rho_0))} \cap V) \geq 1$ would not have been sufficient as ∂_t is a component we cannot ignore. It is here where we need that $2k - m - n = 2$. In particular, this step explains why the assumption in the lemma is written for the space $\mathcal{C}_H^{m+1, r_{t_0}, t_0}$ with $m = 2k - n - 2$.

Then, there exists $\mathbf{x} \in \mathbb{R}^k$ so that

$$(d\pi \circ d\varphi_{t_0})_{\rho_0} \mathbf{w}_0 = R_{\pi(\varphi_{t_0}(\rho_0))} \mathbf{x}.$$

Since $\sup_{y \in U} \|R_y\| \leq 2$,

$$\|(d\pi \circ d\varphi_{t_0})_{\rho_0} \mathbf{w}_0\| \leq 2\|\mathbf{x}\|$$

and by (4.5) we have

$$\|\mathbf{w}_0\| \leq C_{M,g}\, a e^{(a+\Lambda)|t_0|} \|\mathbf{x}\|.$$

which implies the desired claim since $(d\tilde{F}\circ d\pi\circ d\varphi_{t_0})_{\rho_0}\mathbf{w}_0=\mathbf{x}$ and so

$$\|d(\tilde{F}\circ\pi)_{\varphi_{t_0}(\rho_0)}((d\varphi_{t_0})_{\rho_0}\mathbf{w}_0)\|\geq (C_{M,g}a)^{-1}e^{-(a+\Lambda)|t_0|}\|\mathbf{w}_0\|. \tag{4.6}$$

Combining (4.4) and (4.6) with (4.3) gives the desired bound on the left inverse for $d\tilde{\psi}$ restricted to $\mathbb{R}\partial_t\times\mathbb{R}\mathbf{w}_0$ provided we impose $C_{M,g}\geq 2$. □

Proof of Theorem 2. Let $t_0>0$, $a>\delta_F^{-1}$ so that for $t\geq t_0$,

$$d\Big(H,\mathcal{C}_H^{2k-n-1,r_t,t}\Big)>r_t, \tag{4.7}$$

where $r_t=\frac{1}{a}e^{-at}$. By Lemma 4.1, for $t\geq t_0$, if $\rho\in SN^*H$ and $d(\varphi_t(\rho),SN^*H)<\min(\frac{1}{a}e^{-at},c_H)$, then there exists a $\mathbf{w}=\mathbf{w}(t,\rho)\in T_\rho SN^*H$ so that $d\psi$ restricted to $\mathbb{R}\partial_t\times\mathbb{R}\mathbf{w}$ has left inverse $L_{(t,\rho)}$ with

$$\|L_{(t,\rho)}\|\leq C_{M,g}(1+a)e^{C_{M,g}(a+\Lambda)|t|},$$

for some $C_{M,g}>0$ and any $\Lambda>\Lambda_{\max}$. For the purposes of the proof of Theorem 2 fix $\Lambda=2\Lambda_{\max}+1$. Let $c:=(1+a)C_{M,g}$, $\beta:=C_{M,g}(a+\Lambda)$, and let $t_1=t_1(a,t_0)\geq t_0$ be so that

$$\|L_{(t,\rho)}\|\leq ce^{\beta|t|}\qquad t\geq t_1.$$

In particular, we may cover SN^*H by finitely many balls $\{B_i\}_{i=1}^N$ of radius $R>0$ (independent of h) so that $NR^{n-1}<C_n\operatorname{vol}(SN^*H)$, and the hypotheses of Proposition 2.2 hold for each B_i choosing $\tilde{c}=a^{-1}$.

Let $\alpha_1=\alpha_1(M,g)$ and $\alpha_2=\alpha_2(M,g,a,\delta_F)$ be as in Proposition 2.2. Fix $0<\varepsilon<\frac{1}{4}$ and set

$$r_0:=h^{2\varepsilon},\qquad r_1:=h^{\varepsilon},\qquad r_2:=\tfrac{2}{\alpha_1}h^{\varepsilon}.$$

Let

$$T_0(h)=b\log h^{-1}$$

with $b>0$ to be chosen later. Then, the assumptions in Proposition 2.2 hold provided

$$h^{\varepsilon}<\min\Big\{\tfrac{2}{3\alpha_1}e^{-\Lambda T_0}\ ,\ \tfrac{\alpha_1\alpha_2}{2}e^{-\gamma T_0},\tfrac{\alpha_1 R}{2}\Big\}$$

where $\gamma=\max\{a,3\Lambda+2\beta\}=5\Lambda+2a$. In particular, if we set $\alpha_3:=\min\{\frac{2}{3\alpha_1},\frac{\alpha_1\alpha_2}{2}\}$, the assumptions in Proposition 2.2 hold provided $h<\big(\frac{\alpha_1R}{2}\big)^{\frac{1}{\varepsilon}}$ and

$$T_0(h)<\frac{\varepsilon}{\gamma}\log h^{-1}+\frac{\log\alpha_3}{\gamma}. \tag{4.8}$$

We will choose T_0 satisfying (4.8) later.

Let $0<\tau_0<\tau_{\mathrm{inj}H}$, $R_0=R_0(n,k,g,K_H)>0$ be as in Theorem 5. Note that $\tau_0=\tau_0(M,g,\tau_{\mathrm{inj}H})$. Also let $h_0=h_0(M,g)>0$ be the constant given by Theorem 5 and possibly shrink it so that $h_0<\big(\frac{\alpha_1R}{2}\big)^{\frac{1}{\varepsilon}}$. Let $\{\rho_j\}_j\subset SN^*H$ be so that $\{\Lambda^{\tau}_{\rho_j}(h^{\varepsilon})\}_j$ is a $(\mathfrak{D}_n,\tau_0,h^{\varepsilon})$ good cover of SN^*H (existence of such a cover follows from [CG20a, Proposition 3.3] - see Remark 7). Then, for each $i \in \{1, \dots, K\}$ we apply Proposition 2.2 to obtain a cover of $\Lambda^{\tau_0}_{B_i}(h^{2\varepsilon})$ by tubes $\{\Lambda^{\tau_0}_{\rho_{i_j}}(h^{\varepsilon})\}_{j=1}^{N_i}$ with $\rho_{i_j} \in B_i$ and so that $\{1, \dots, N_i\} = \mathcal{G}_i \cup \mathcal{B}_i$,

$$\bigcup_{j \in \mathcal{G}_i} \Lambda^{\tau_0}_{\rho_j}(h^{\varepsilon}) \quad \text{is} \quad [t_0, T_0(h)] \quad \text{non-self looping,}$$

$$h^{\varepsilon(n-1)}|\mathcal{B}_i| \leq \mathbf{C_0} \tfrac{2}{\alpha_1}\, h^{\varepsilon}\, R^{n-1}\, T_0 e^{4(\Lambda+\beta)T_0},$$

where $\mathbf{C}_0 = \mathbf{C}_0(M, g, k, a) > 0$. We choose $b > 0$ so that $b < \frac{\varepsilon}{12(\Lambda+\beta)}$ and (4.8) is satisfied for all $h < h_0$. Note that this implies that $b = b(M, g, a, \delta_F)$. In particular, there exists $h_0 = h_0(\tau_0, \mathbf{C}_0)$, so that for all $0 < h < h_0$,

$$h^{\varepsilon(n-1)}|\mathcal{B}_i| < h^{\frac{\varepsilon}{3}} R^{n-1}. \tag{4.9}$$

We next apply Theorem 5 $\delta := 2\varepsilon$, and $R(h) := h^{\varepsilon}$ (not to be confused with R). If needed, we shrink h_0 so that $5h^{2\varepsilon} \leq R(h) < R_0$ for all $0 < h < h_0$. We let $\alpha < 1 - 2\varepsilon$ and let b be small enough so that $T_0(h) \leq 2\alpha T_e(h)$ for all $0 < h < h_0$. We also let $\mathcal{B} = \cup_{i=1}^{K} \mathcal{B}_i$, and work with only one set of good indices $\mathcal{G} := \mathcal{I}_h(w) \backslash \mathcal{B}$. We choose $t_\ell(h) = t_1$ and $T_\ell(h) = T_0(h)$. Note that (4.9) gives

$$R(h)^{\frac{n-1}{2}} |\mathcal{B}|^{\frac{1}{2}} \leq h^{\frac{\varepsilon}{6}} (K R^{n-1})^{\frac{1}{2}} \leq h^{\frac{\varepsilon}{6}} C_n^{\frac{1}{2}} \operatorname{vol}(SN^*H)^{\frac{1}{2}}.$$

Since in addition

$$|\mathcal{G}| \leq |\mathcal{I}_h(w)| \leq K(\max_{1 \leq i \leq K} N_i) \leq \operatorname{vol}(SN^*H) C_n h^{-\varepsilon(n-1)},$$

Let $N > 0$. Theorem 5 yields the existence of constants $C_{n,k} > 0$, $\tilde{C} = \tilde{C}(M, g, \tau_0, \varepsilon) > 0$ and $C_N > 0$ so that for all $0 < h < h_0$

$$\begin{aligned}
&h^{\frac{k-1}{2}} \Big| \int_H w u \, d\sigma_H \Big| \\
&\leq \frac{C_{n,k} \operatorname{vol}(SN^*H)^{\frac{1}{2}} \|w\|_\infty C_n^{\frac{1}{2}}}{\tau_0^{\frac{1}{2}}} \left(\left[h^{\frac{\varepsilon}{6}} + \frac{t_1^{\frac{1}{2}}}{T_0^{\frac{1}{2}}(h)} \right] \|u\|_{L^2(M)} + \frac{T_0^{\frac{1}{2}}(h) t_1^{\frac{1}{2}}}{h} \|(-h^2\Delta_g - I)u\|_{H^{-2}_{\mathrm{scl}}(M)} \right) \\
&+ \frac{\tilde{C}}{h} \|w\|_\infty \|(-h^2\Delta_g - I)u\|_{H^{\frac{k-3}{2}}_{\mathrm{scl}}(M)} + C_N h^N \big(\|u\|_{L^2(M)} + \|(-h^2\Delta_g - I)u\|_{H^{\frac{k-3}{2}}_{\mathrm{scl}}(M)} \big)
\end{aligned} \tag{4.10}$$

$$\leq C \|w\|_\infty \left(\frac{\|u\|_{L^2(M)}}{\sqrt{\log h^{-1}}} + \frac{\sqrt{\log h^{-1}}}{h} \|(-h^2\Delta_g - I)u\|_{H^{\frac{k-3}{2}}_{\mathrm{scl}}(M)} \right) \tag{4.11}$$

where $C = C(M, g, k, t_0, a, \delta_F, \operatorname{vol}(SN^*H), \tau_{\mathrm{inj}H}) > 0$ is some positive constant and $h_0 = h_0(\delta, M, g, \tau_0, k, a, w, R_0)$ is chosen small enough so that the last term on the right of (4.10) can be absorbed. Note that the ε dependence of C and h_0 is resolved by fixing any $\varepsilon < \frac{1}{4}$. □

Proof of Theorem 1. Note that if $H = \{x\}$ then $SN^*H = S^*_x M$ and $\operatorname{vol}(S^*_x M) = c_n$ for some $c_n > 0$ that depends only on n. Next, note that $\tau_{\mathrm{inj}H}(\{x\})$ and δ_F can be chosen uniform on M and that $H_p r_H = 2$. Moreover, in this case, $w = 1$ and K_H can be taken arbitrarily small so $R_0 = R_0(n, k, p, K_H)$ can be taken to be uniform on M.

Therefore, since the constant in (4.11) and h_0 depends only on

$$M,\ g,\ k,\ t_0,\ a,\ \delta_F,\ \mathrm{vol}(SN^*H),\ \tau_{\mathrm{inj}H},$$

all of the terms on the right hand side of (4.11) are uniform for $x \in M$ completing the proof of Theorem 1. □

5. No focal points or Anosov geodesic flow: Proof of Theorems 4 and 6

Next we analyze the cases in which (M, g) has no focal points or Anosov geodesic flow. For $\rho \in SN^*H$ we continue to write $N_\pm(\rho) = T_\rho(SN^*H) \cap E_\pm(\rho)$ and define the functions $m, m_\pm : SN^*H \to \{0, \dots, n-1\}$

$$m(\rho) := \dim(N_+(\rho) + N_-(\rho)), \qquad m_\pm(\rho) := \dim N_\pm(\rho), \tag{5.1}$$

and note that the continuity of $E_\pm(\rho)$ implies that m, $m_\pm$ are upper semicontinuous (see e.g. [CG19, Lemma 20]). We will need extensions of $N_\pm(\rho)$, $m_\pm(\rho)$ to neighborhoods of SN^*H for our next lemma. To have this, for each ρ in a neighborhood of SN^*H define the set

$$\mathcal{F}_\rho := \{q \in T^*M :\ F(q) = F(\rho)\},$$

where F is the defining function for SN^*H introduced in (2.2). Since for $\rho \in SN^*H$, $\mathcal{F}_\rho = SN^*H$, $\mathcal{F}_\rho$ can be thought of as a family of 'translates' of SN^*H. We then define

$$\tilde{N}_\pm(\rho) := T_\rho \mathcal{F}_\rho \cap E_\pm(\rho) \qquad \text{and} \qquad \tilde{m}_\pm(\rho) := \dim \tilde{N}_\pm(\rho).$$

Note that since $T_\rho \mathcal{F}_\rho$ is smooth in ρ and agrees with $T_\rho(SN^*H)$ for $\rho \in SN^*H$, $\tilde{m}_\pm(\rho)$ is upper semicontinuous with $\tilde{m}_\pm|_{SN^*H} = m_\pm$. In what follows we continue to write $\mathcal{S}_H = \{\rho \in SN^*H :\ T_\rho(SN^*H) = N_-(\rho) + N_+(\rho)\}$.

The following lemma shows that if $\rho \in SN^*H$ does not belong to $\mathcal{S}_H$ and $\varphi_t(\rho)$ is close enough to ρ for t sufficiently large, then $(d\varphi_t)_\rho \mathbf{w}$ leaves $T_{\varphi_t(\rho)}\mathcal{F}_{\varphi_t(\rho)}$ for some $\mathbf{w} \in T_\rho SN^*H$.

Lemma 5.1. *Suppose (M, g) has Anosov geodesic flow or no focal points and let $K \subset (SN^*H \backslash \mathcal{S}_H)$ be a compact set. Then there exist positive constants $c_K, t_K, \delta_K > 0$ so that if $d(\rho, K) \le \delta_K$, $|t| \ge t_K$, and*

$$\varphi_t(\rho) \in \overline{B(\rho, \delta_K)},$$

then there is $\mathbf{w} = \mathbf{w}(t, \rho) \in T_\rho(SN^*H) \backslash \{0\}$ with

$$\inf\{\|d\varphi_t(\mathbf{w}) + \mathbf{v}\| :\ \mathbf{v} \in T_{\varphi_t(\rho)}\mathcal{F}_{\varphi_t(\rho)} + \mathbb{R}H_p\} \ge c_K \|\mathbf{w}\|. \tag{5.2}$$

Proof. First note that since $\tilde{m}_\pm$ are upper semi-continuous, K is compact, and $K \cap \mathcal{S}_H$ is empty, there exists $\delta_{\tilde{K}} > 0$ so that $d(K, \mathcal{S}_H) > \delta_{\tilde{K}}$. Therefore, to prove the lemma we work with the compact set $\tilde{K} := \{\rho \in SN^*H :\ d(\rho, K) \le \frac{\delta_{\tilde{K}}}{2}\}$ and insist that $\delta_K < \frac{\delta_{\tilde{K}}}{2}$.

Let $\rho \in \tilde{K}$. Since $T_\rho(SN^*H) \neq N_+(\rho) + N_-(\rho)$, we may choose $\mathbf{u} = \mathbf{u}(\rho)$ such that

$$\mathbf{u} \in T_\rho(SN^*H) \setminus (N_+(\rho) + N_-(\rho)), \qquad \|\mathbf{u}\| = 1.$$

Now, let $\mathbf{u}_+ \in E_+(\rho)$ and $\mathbf{u}_- \in E_-(\rho)$ be so that

$$\mathbf{u} = \mathbf{u}_+ + \mathbf{u}_-.$$

In particular, $\mathbf{u}_\pm \notin N_\pm(\rho)$.

When studying the case $t > t_K$, we will use that $\mathbf{u}_-$ grows along the positive time flow, while for $t < -t_K$ we will use that $\mathbf{u}_+$ grows along the negative time flow. Since the arguments are identical, except with time reversed and the roles of $\mathbf{u}_+$ and $\mathbf{u}_-$ switched, we only explicitly write that for $t > t_K$.

We claim that there is $C_{_K} > 0$ such that for all $\rho \in \tilde{K}$, we may in addition choose $\mathbf{u} = \mathbf{u}(\rho)$ such that

$$\begin{gathered}\mathbf{u}_- \in E_-(\rho) \cap (N_-(\rho))^\perp \cap (E_+(\rho) \cap E_-(\rho))^\perp, \\ C_{_K}^{-1}\|\mathbf{u}_+\| \leq \|\mathbf{u}_-\| \leq C_{_K}\|\mathbf{u}_+\|.\end{gathered} \tag{5.3}$$

For this, we set

$$\begin{gathered}\bar{N}_\pm(\rho) := N_\pm(\rho) \cap \Big(E_+(\rho) \cap E_-(\rho)\Big)^\perp, \\ U_\pm(\rho) := E_\pm(\rho) \cap (N_\pm(\rho))^\perp \cap \Big(E_+(\rho) \cap E_-(\rho)\Big)^\perp.\end{gathered}$$

We then observe that

$$(\mathbb{R}H_p(\rho))^\perp = U_+(\rho) \oplus \bar{N}_+(\rho) \oplus \Big(E_+(\rho) \cap E_-(\rho)\Big) \oplus \bar{N}_-(\rho) \oplus U_-(\rho)$$

and decompose a vector $\mathbf{v} \in (\mathbb{R}H_p)^\perp$ correspondingly as

$$\mathbf{v} = \mathbf{v}_{U_+} + \mathbf{v}_{\bar{N}_+} + \mathbf{v}_0 + \mathbf{v}_{\bar{N}_-} + \mathbf{v}_{U_-}.$$

Suppose the claim in (5.3) fails. Then, for all $n \in \mathbb{N}$, there are $\rho_n \in \tilde{K}$ such that for all $\mathbf{v} \in T_{\rho_n}SN^*H$,

$$n^{-1}\|\mathbf{v}_{U_+} + \mathbf{v}_{\bar{N}_+} + \mathbf{v}_0\| > \|\mathbf{v}_{U_-} + \mathbf{v}_{\bar{N}_-}\|, \qquad \text{or} \qquad n\|\mathbf{v}_{U_+} + \mathbf{v}_{\bar{N}_+} + \mathbf{v}_{\mathbf{0}}\| < \|\mathbf{v}_{U_-} + \mathbf{v}_{\bar{N}_-}\|.$$

In particular, since $\mathbf{v}_{\bar{N}_-} \in T_{\rho_n}SN^*H$, we have $\mathbf{v} - \mathbf{v}_{\bar{N}_-} \in T_{\rho_n}SN^*H$, and hence, for all $\mathbf{v} \in T_{\rho_n}SN^*H$,

$$n^{-1}\|\mathbf{v}_{U_+} + \mathbf{v}_{\bar{N}_+} + \mathbf{v}_0\| > \|\mathbf{v}_{U_-}\|, \qquad \text{or} \qquad n\|\mathbf{v}_{U_+} + \mathbf{v}_{\bar{N}_+} + \mathbf{v}_{\mathbf{0}}\| < \|\mathbf{v}_{U_-}\|.$$

Since $\tilde{K}$ is compact, we may assume $\rho_n \to \rho \in \tilde{K}$. Then, for all $\mathbf{v} \in T_\rho SN^*H$, there are $\mathbf{v}_n \in T_{\rho_n}SN^*H$ such that $\mathbf{v}_n \to \mathbf{v}$. Let $\mathbf{v} \in T_\rho SN^*H \setminus (N_+(\rho) + N_-(\rho))$ and $\mathbf{v}_n \to \mathbf{v}$ as above.

Then,

$$n^{-1}\|\mathbf{v}_{n,U_+} + \mathbf{v}_{n,\bar{N}_+} + \mathbf{v}_{n,0}\| > \|\mathbf{v}_{n,U_-}\|, \qquad \text{or} \qquad n\|\mathbf{v}_{U_+} + \mathbf{v}_{n,\bar{N}_+} + \mathbf{v}_{n,0}\| < \|\mathbf{v}_{n,U_-}\|.$$

Extracting a subsequence again, we may assume that one of these inequalities holds for all n. We consider first the case

$$n^{-1}\|\mathbf{v}_{n,U_+} + \mathbf{v}_{n,\bar{N}_+} + \mathbf{v}_{n,0}\| > \|\mathbf{v}_{n,U_-}\|.$$

Now, since $\mathbf{v}_n \to \mathbf{v}$, and $E_+(\rho)$ is continuous,

$$\mathbf{v}_{n,U_+} + \mathbf{v}_{n,\bar{N}_+} + \mathbf{v}_{n,0} \to \tilde{\mathbf{v}}_+ \in E_+(\rho)$$

In particular, this implies that $\mathbf{v}_{n,U_-} \to 0$ and hence $\mathbf{v}_{n,\bar{N}_-} \to \mathbf{v} - \tilde{\mathbf{v}}_+$. Using that $\rho \mapsto T_\rho SN^*H$ and $\rho \mapsto E_-(\rho)$ are continuous maps, and that $\mathbf{v}_{n,\bar{N}_-} \in E_-(\rho_n) \cap T_{\rho_n}SN^*H$, we have $\mathbf{v} - \tilde{\mathbf{v}}_+ \in N_-(\rho)$ and hence also $\mathbf{v}_+ \in N_+(\rho)$. Therefore, $\mathbf{v} \in N_+(\rho) + N_-(\rho)$, a contradiction.

Next, we consider the other case:

$$n\|\mathbf{v}_{U_+} + \mathbf{v}_{n,\bar{N}_+} + \mathbf{v}_{n,0}\| < \|\mathbf{v}_{n,U_-}\|.$$

Then, since $\mathbf{v}_n \to \mathbf{v}$, $\mathbf{v}_{n,U_-}$ is bounded and hence $\mathbf{v}_{U_+} + \mathbf{v}_{n,\bar{N}_+} + \mathbf{v}_{n,0} \to 0$. In particular, $\mathbf{v}_{n,U_-} + \mathbf{v}_{n,\bar{N}_-} \to \mathbf{v}$, so $\mathbf{v} \in E_-(\rho)$ and hence $\mathbf{v} \in N_-(\rho)$, a contradiction. Since both cases lead to a contradiction, we have proved the claim (5.3).

Since $d\varphi_t : E_-(\rho) \to E_-(\varphi_t(\rho))$ and $d\varphi_t : E_+(\rho) \cap E_-(\rho) \to E_+(\varphi_t(\rho)) \cap E_-(\varphi_t(\rho))$ are isomorphisms, we have

$$\dim \operatorname{span}\big(d\varphi_t(\mathbf{u}_-), \quad d\varphi_t(N_-(\rho))\big) = 1 + \dim N_-(\rho).$$

Also, note that since $\tilde{m}_-$ is upper semicontinuous and integer valued, we may choose $\delta > 0$ uniform in $\rho \in SN^*H$ so that $\dim \tilde{N}_-(q) \le \dim N_-(\rho)$ for all $q \in B(\rho, \delta)$. For any t and $q \in B(\rho, \delta)$ we then have

$$\dim \operatorname{span}\big(d\varphi_t(\mathbf{u}_-), \quad d\varphi_t(N_-(\rho))\big) \ge 1 + \dim \tilde{N}_-(q). \tag{5.4}$$

Next, note that $\operatorname{span}\big(d\varphi_t(\mathbf{u}_-), \quad d\varphi_t(N_-(\rho))\big) \subset E_-(\varphi_t(\rho))$. Suppose now that $\varphi_t(\rho) \in B(\rho, \delta)$ for some t and note that if $d\varphi_t(\mathbf{w}) \in E_-(\varphi_t(\rho)) \backslash \tilde{N}_-(\varphi_t(\rho))$, then $d\varphi_t(\mathbf{w}) \notin T_{\varphi_t(\rho)}\mathcal{F}_{\varphi_t(\rho)}$. In particular, relation (5.4) gives that there exists a linear combination

$$\mathbf{w_t} = a_t\, \mathbf{u}_- + \mathbf{e}_-(t) \in E_-(\rho),$$

with $\mathbf{e}_-(t) \in N_-(\rho)$, so that $d\varphi_t \mathbf{w_t} \in (\tilde{N}_-(\varphi_t(\rho)))^\perp$ with $\|d\varphi_t \mathbf{w_t}\| = 1$. If we had that $\mathbf{w_t}$ was a tangent vector in $T_\rho(SN^*H)$ and we had control on $\|\mathbf{w}_t\|$ we would be done proving (5.2). Note that to say this we are using that $d\varphi_t \mathbf{w}_t \in E_-(\varphi_t(\rho))$ and that $E_-(\varphi_t(\rho)) \perp \mathbb{R}H_p$. However, since $\mathbf{u}_-$ is not in $T_\rho SN^*H$ we have to modify $\mathbf{w_t}$. Consider the vector

$$\tilde{\mathbf{w}}_\mathbf{t} = a_t\, \mathbf{u} + \mathbf{e}_-(t),$$

and note that $\tilde{\mathbf{w}}_\mathbf{t} \in T_\rho(SN^*H)$ and

$$d\varphi_t(\tilde{\mathbf{w}}_\mathbf{t}) = d\varphi_t(\mathbf{w_t}) + a_t\, d\varphi_t(\mathbf{u}_+).$$

Let $\delta_1 > 0$ be so that $1 - \delta_1 \tilde{\mathbf{B}} C_{_K} > \frac{1}{2}$. We claim that there is $t_{_K} > 0$, depending only on (M, p, K), so that for $t > t_{_K}$,

$$\|\mathbf{w_t}\| \le \delta_1 \qquad \text{and} \qquad |a_t| < \delta_1 \|\mathbf{u}_-\|^{-1}. \tag{5.5}$$

Note that this yields that for t large enough, $d\varphi_t(\tilde{\mathbf{w}}_\mathbf{t})$ approaches $d\varphi_t(\mathbf{w_t}) \notin T_{\varphi_t(\rho)}\mathcal{F}_{\varphi_t(\rho)}$. In particular, the t-flowout of the $\tilde{\mathbf{w}}_\mathbf{t}$ direction in $T_\rho(SN^*H)$ approaches $E_-(\varphi_t(\rho))$ (see Figure 4). We postpone the proof of (5.5) until the end, and show how to finish the proof assuming it holds.

We next observe that there exists $\tilde{\mathbf{B}} > 0$ so that if $\mathbf{w} \in E_\pm(\rho)$ then $\|d\varphi_t \mathbf{w}\| \le \tilde{\mathbf{B}}\|\mathbf{w}\|$ as $t \to \pm\infty$. Indeed, in the Anosov case $\tilde{\mathbf{B}} = \mathbf{B}$, where $\mathbf{B}$ is defined in (1.20), and in the no focal point case the existence of $\tilde{\mathbf{B}}$ is guaranteed by [Ebe73a, Proposition 2.13, Corollary 2.14]. We can therefore conclude from (5.3) and (5.5) that

$$\|\pi_{t,\rho}(d\varphi_t \tilde{\mathbf{w}}_\mathbf{t})\| \ge \|\pi_{t,\rho}(d\varphi_t \mathbf{w_t})\| - \|a_t\, \pi_{t,\rho}(d\varphi_t \mathbf{u}_+)\| > 1 - \delta_1 \tilde{\mathbf{B}} C_{_K},$$

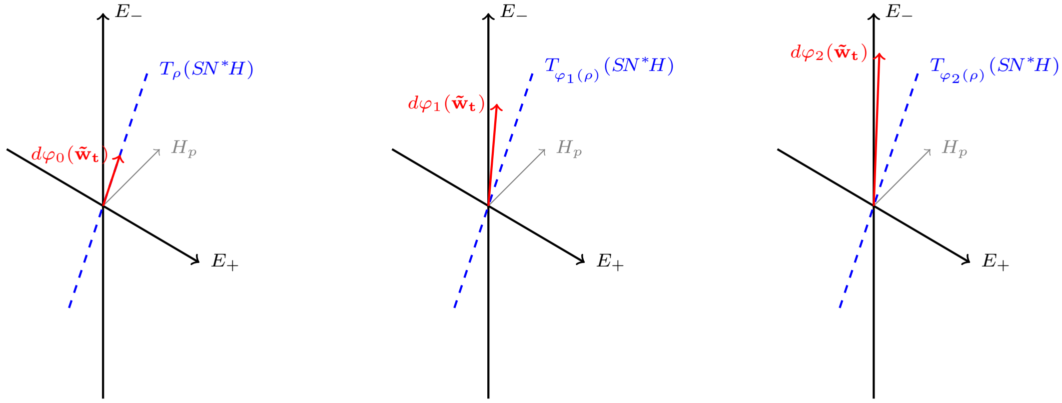

FIGURE 4. Schematic of the rotation of $\tilde{\mathbf{w}}_\mathbf{t}$ under the geodesic flow.

and

$$\|\tilde{\mathbf{w}}_\mathbf{t}\| = \|\mathbf{w}_t + a_t\mathbf{u}_+\| \leq \|\mathbf{w}_t\| + |a_t|\|\mathbf{u}_+\| \leq \delta_1(1 + C_{{}_K}),$$

where $\pi_{t,\rho}$ denotes orthogonal projection onto $E_-(\varphi_t(\rho)) \cap (\tilde{N}_-(\varphi_t(\rho)))^\perp$. In particular,

$$\|\pi_{t,\rho}(d\varphi_t\tilde{\mathbf{w}}_\mathbf{t})\| \geq \frac{1 - \delta_1\tilde{\mathbf{B}}C_{{}_K}}{\delta_1(1 + C_{{}_K})}\|\tilde{\mathbf{w}}_\mathbf{t}\|.$$

Therefore, there exist positive constants $c_{{}_K}$, $\delta_{{}_K}$ and $t_{{}_K}$ (uniform for $\rho \in K$) so that if $\varphi_t(\rho) \in B(\rho, \delta_{{}_K})$ for some t with $|t| > t_{{}_K}$, then there is $\mathbf{w} = \tilde{\mathbf{w}}_t \in T_\rho(SN^*H)$ so that

$$\|d\varphi_t(\mathbf{w}) + \mathbb{R}H_p + T_{\varphi_t(\rho)}\mathcal{F}_{\varphi_t(\rho)}\| \geq c_{{}_K}\|\mathbf{w}\|. \tag{5.6}$$

This would finish the proof assuming that the claim in (5.5) holds.

We proceed to prove (5.5). We start with the Anosov case. By the definition of Anosov geodesic flow,

$$\|(d\varphi_t|_{E_-})^{-1}\| \leq \mathbf{B}e^{-t/\mathbf{B}}, \quad t \geq 0.$$

Thus, since $\mathbf{w}_\mathbf{t} \in E_-(\rho)$ and $\|d\varphi_t\mathbf{w}_\mathbf{t}\| = 1$, we find $\|\mathbf{w}_\mathbf{t}\| \leq \mathbf{B}e^{-t/\mathbf{B}}$. In particular, since $\mathbf{u}_-$ and $\mathbf{e}_-(t)$ are orthogonal, we have

$$|a_t| \leq \mathbf{B}e^{-t/\mathbf{B}}\|\mathbf{u}_-\|^{-1}, \qquad t \geq 0.$$

This proves the claim (5.5) in the Anosov flow case after choosing $t_{{}_K} > 0$ large enough so that $\mathbf{B}e^{-t/\mathbf{B}} \leq \delta_1$.

We next consider the non-focal points case. Define $\mathcal{C}_+^\alpha(\rho) \subset T_\rho(S^*M)$ to be the conic set of vectors forming an angle larger than or equal to $\alpha > 0$ with $E_+(\rho)$. Let $\alpha_{{}_K} > 0$ be so that $\mathbf{w}_t \in E_-(\rho) \cap \mathcal{C}_+^{\alpha_K}(\rho)$ for all $\rho \in \tilde{K}$. By [Ebe73a, Proposition 2.6] $(d\pi)_\rho : E_\pm(\rho) \oplus H_p(\rho) \to T_{\pi(\rho)}M$ is an isomorphism for each ρ. In particular, letting $V(\rho) \subset T_\rho(S^*M)$ denote the vertical vectors, we have that $E_\pm(\rho) \cap V(\rho) = \emptyset$ and $V(\rho) \oplus E_+(\rho) \oplus H_p(\rho) = T_{\pi(\rho)}S^*M$. In addition, since (M, g) has no focal points, $\cup_{\rho \in S^*M}E_\pm(\rho)$ is closed [Ebe73a, see right before Proposition 2.7] and hence there exists $c_{\alpha_K} > 0$ depending only on $\alpha_{{}_K}$ so that

$$\mathbf{w}_t = \mathbf{e}_+ + \mathbf{v}$$

with

$$c_{\alpha_K}\|\mathbf{e}_+\| \leq \|\mathbf{w}_t\| \leq \frac{1}{c_{\alpha_K}}\|\mathbf{v}\|.$$

and $\mathbf{e}_+ \in E_+(\rho)$, $\mathbf{v} \in V(\rho)$. By [Ebe73a, Remark 2.10], for all $R > 0$ there exists $T(R) > 0$ so that $\|Y(t)\| \geq R\|Y'(0)\|$ for all $t > T(R)$, where $Y(t)$ is any Jacobi field with $Y(0) = 0$ and perpendicular to a unit speed geodesic γ with $\gamma(0) \in \tilde{K}$. Since $\mathbf{v}$ is a vertical vector, we may consider $Y(t) = d\pi \circ d\varphi_t(\mathbf{v})$, and this implies that $Y'(0) = \mathbf{K}\mathbf{v}^\sharp$ (see Appendix 6 for an explanation of the connection map $\mathbf{K}$, and the $\sharp$ operator). We therefore have that $\|d\varphi_t \mathbf{v}\| \geq R\|\mathbf{v}\|$ for all $t > T(R)$. In particular, then

$$\|d\varphi_t\mathbf{w}_t\| = \|d\varphi_t\mathbf{v} + d\varphi_t\mathbf{e}_+\| \geq R\|\mathbf{v}\| - \tilde{\mathbf{B}}\|\mathbf{e}_+\| \geq (Rc_{\alpha_K} - c_{\alpha_K}^{-1}\tilde{\mathbf{B}})\|\mathbf{w}_t\|.$$

So, choosing $R(\alpha_K) = c_{\alpha_K}^{-1}(\delta_1^{-1} + c_{\alpha_K}^{-1}\tilde{\mathbf{B}})$, we have that for $t \geq t_K := T(R(\alpha_K))$,

$$1 = \|d\varphi_t\mathbf{w}_t\| \geq \delta_1^{-1}\|\mathbf{w}_t\|.$$

In particular, for $t \geq t_K$, since $\mathbf{u}_-$ is orthogonal to $\mathbf{e}_-(t)$, we obtain $1 = \|d\varphi_t\mathbf{w}_t\| \geq \delta_1^{-1}\|\mathbf{w}_t\| \geq \delta_1^{-1}|a_t|\|\mathbf{u}_-\|$, completing the proof of the lemma in the case of manifolds without focal points.

□

When (M, g) has Anosov geodesic flow, we need to define a notion of angle between a vector and $E_\pm(\rho)$. Let $\pi_\pm : T_\rho S^*M \to E_\pm(\rho)$ be the projection onto $E_\pm(\rho)$ along $E_\mp(\rho) \oplus H_p(\rho)$ i.e. if $\mathbf{u} = \mathbf{v}_+ + \mathbf{v}_- + rH_p$ with $r \in \mathbb{R}$, $\mathbf{v}_\pm \in E_\pm(\rho)$, then $\pi_\pm(\mathbf{u}) = \mathbf{v}_\pm$. For $\rho \in S^*M$, define $\Theta_\rho^\pm : (\mathbb{R}H_p(\rho))^\perp \setminus \{0\} \to [0, \infty]$ by

$$\Theta_\rho^\pm(\mathbf{u}) := \frac{\|\pi_\mp\mathbf{u}\|}{\|\pi_\pm\mathbf{u}\|}. \tag{5.7}$$

Note that $\Theta_\rho^\pm$ should be thought of as measuring the tangent of the angle from $E_\pm(\rho)$, and that given a compact subset K of $T^*M\setminus\{0\}$ there exists $C_K > 0$ so that for all $\rho \in K$, $t \in \mathbb{R}$, and $\mathbf{u} \in T_\rho S^*M$, we have

$$\frac{e^{\pm t/C_K}}{C_K}\,\Theta_\rho^\pm(\mathbf{u}) \leq \Theta_\rho^\pm(d\varphi_t\mathbf{u}) \leq C_K e^{\pm C_K t}\,\Theta_\rho^\pm(\mathbf{u}). \tag{5.8}$$

In what follows we will use the fact that by [CG20a, Proposition 3.3] there are $\mathfrak{D}_n > 0$ depending only on n, $\tau_{SN^*H} > 0$ depending only on $\tau_{\text{inj}\,H}$, and $R_0 > 0$ depending only on (n, k, K_H) and finitely many derivatives of the curvature and second fundamental form of H, so that for $0 < \tau < \tau_{SN^*H}$ and $0 < r < R_0$, there is a $(\mathfrak{D}_n, \tau, r)$ good cover of SN^*H.

Lemma 5.2. *Let (M, g) have Anosov geodesic flow and $H \subset M$ satisfy $\mathcal{A}_H = \emptyset$. Then, there exist $c = c(M, g, H) > 0$, $C = C(M, g, H) > 2$, $I > 0$, $t_0 > 1$, so that for all $\Lambda > \Lambda_{max}$ the following holds.*

Let $T_0 \geq t_0$, $m = \lfloor \frac{\log T_0 - \log t_0}{\log 2} \rfloor$, $0 < \tau_0 < \tau_{SN^*H}$, $0 < \tau \leq \tau_0$,

$$0 \leq r_1 \leq \min\{e^{-CT_0}, R_0\},$$

and $\{\Lambda^\tau_{\rho_j}(r_1)\}_{j=1}^{N_{r_1}}$ *be a* $(\mathfrak{D}_n,\tau,r_1)$ *good cover of* SN^*H*. Then, for each* $i\in\{1,\dots,I\}$ *there are sets of indices* $\{\mathcal{G}_{i,\ell}\}_{\ell=0}^m\subset\{1,\dots,N_{r_1}\}$ *and* $\mathcal{B}\subset\{1,\dots,N_{r_1}\}$ *so that*

$$\bigcup_{i=1}^I\bigcup_{\ell=0}^m\mathcal{G}_{i,\ell}\cup\mathcal{B}=\{1,\dots,N_{r_1}\},$$

and for every $i\in\{1,\dots,I\}$ *and every* $\ell\in\{0,\dots,m\}$

- $\bigcup_{j\in\mathcal{G}_{i,\ell}}\Lambda^\tau_{\rho_j}(r_1)$ *is* $[t_0,2^{-\ell}T_0]$ *non-self looping,*
- $|\mathcal{G}_{i,\ell}|\le c\,5^{-\ell}\,r_1^{1-n}$,
- $|\mathcal{B}|\le c\left(\frac{t_0}{T_0}\right)^{\frac{\log 5}{\log 2}}r_1^{1-n}$.

We note that if $H_0\subset M$ is an embedded submanifold, there exists a neighborhood U of H_0 (in the C^∞ topology) so that the constants $c=c(M,p,H)$ and $C=C(M,p,H)$ in Lemma 5.2 are uniform for $H\in U$.

Proof. Let $0\le r_0\le\frac{1}{C}e^{-\Lambda T_0}r_1$. Then $\{\Lambda^\tau_{\rho_j}(r_1)\}_{j=1}^{N_{r_1}}$ covers $\Lambda^\tau_{SN^*H}(r_0)$ since $r_0\le\frac12 r_1$. Throughout this proof we will repeatedly use that if $F:T^*M\to\mathbb{R}^{n+1}$ is the defining function for SN^*H, then there exist $\delta_0,c_0>0$ so that for $q\in T^*M$

$$d(q,SN^*H)\le\delta_0\quad\Longrightarrow\quad\|dF\mathbf{v}\|\ge c_0\inf\left\{\|\mathbf{v}+\mathbf{u}\|:\ \mathbf{u}\in T_q\mathcal{F}_q\right\}\quad\forall\mathbf{v}\in T_q(T^*M).\tag{5.9}$$

In addition, let $\nu>0$ be so that $\rho\mapsto E_\pm(\rho)\in C^\nu$ and define $c_{_H}>0$ so that

$$\sup_{q_1,q_2\in SN^*H}\Big(\|\tan^{-1}\circ\Theta^\pm_{q_1}\|_{L^\infty(T_{q_1}SN^*H)}-\|\tan^{-1}\circ\Theta^\pm_{q_2}\|_{L^\infty(T_{q_2}SN^*H)}\Big)\le\frac{1}{c_{_H}}d(q_1,q_2)^\nu.\tag{5.10}$$

This implies that that for all $\varepsilon>0$, there exists $\delta_\varepsilon>0$ so that for every ball $\tilde B\subset SN^*H$ of radius δ_ε we have that

$$\sup_{\rho_1,\rho_2\in\tilde B}\Big|\|\tan^{-1}\Theta^\pm_{\rho_1}\|_{L^\infty(T_{\rho_1}SN^*H)}-\|\tan^{-1}\Theta^\pm_{\rho_2}\|_{L^\infty(T_{\rho_2}SN^*H)}\Big|<\varepsilon.\tag{5.11}$$

Also, since $\mathcal{A}_H=\emptyset$, we know that for every $\rho\in\mathcal{S}_H$ we must have that either $m_+(\rho)=0$ or $m_-(\rho)=0$, where we continue to write $m_\pm(\rho)=\dim N_\pm(\rho)$. Therefore, choosing

$$\varepsilon=\varepsilon(M,p,H)<1\tag{5.12}$$

small enough, depending only on (M,g,H), and shrinking δ_ε if necessary, we may also assume that if $\tilde B\cap\mathcal{S}_H\neq\emptyset$ then either

$$\begin{gathered}m_-(\rho)=0\ \text{ and }\ \Theta^+_\rho\le\varepsilon\quad\text{for all }\ \rho\in\tilde B,\\ \text{or}\\ m_+(\rho)=0\ \text{ and }\ \Theta^-_\rho\le\varepsilon\quad\text{for all }\ \rho\in\tilde B.\end{gathered}\tag{5.13}$$

Furthermore, we assume that $\delta_\varepsilon\le\frac29\left[\varepsilon c_{_H}\right]^{\frac1\nu}$.

Next, let $\{B_i\}_{i=1}^{N_\varepsilon} \subset SN^*H$ be a cover of SN^*H with

$$SN^*H \subset \bigcup_{i=1}^{N_\varepsilon} B_i, \qquad B_i \text{ ball of radius } \tfrac{1}{2}\delta_\varepsilon.$$

Let $\mathcal{I}_{\mathcal{S}_H} := \{i \in \{1, \dots, N_\varepsilon\} : \ B_i \cap \mathcal{S}_H \neq \emptyset\}$, and define $K = K_\varepsilon$ by

$$K := \bigcup_{i \in \mathcal{I}_{\mathcal{S}_H}} (SN^*H \backslash B_i).$$

Since $K \subset (SN^*H \backslash \mathcal{S}_H)$ is compact and the geodesic flow is Anosov, by Lemma 5.1 there exist positive constants $c_{_K}, t_{_K}, \delta_{_K}$ so that $d(K, \mathcal{S}_H) > \delta_{_K}$ and, if $d(\rho, K) \leq \delta_{_K}$ and $\varphi_t(\rho) \in \overline{B(\rho, \delta_{_K})}$ for some $|t| > t_{_K}$, then there exists $\mathbf{w} = \mathbf{w}(t, \rho) \in T_\rho(SN^*H)$ so that

$$\inf\{\|d\varphi_t(\mathbf{w}) + \mathbf{v}\| : \ \mathbf{v} \in T_{\varphi_t(\rho)} \mathcal{F}_{\varphi_t(\rho)} + \mathbb{R}H_p\} \geq c_{_K} \|\mathbf{w}\|. \tag{5.14}$$

We then introduce a cover $\{D_i\}_{i \in I_K} \subset SN^*H$ of K by balls with

$$K \subset \bigcup_{i \in I_K} D_i, \qquad D_i \text{ ball of radius } \tfrac{1}{4}R,$$

where

$$R := \min\{\delta_{_K}, \delta_0, \tfrac{1}{2}\delta_\varepsilon, \delta_F\}$$

and δ_F is as in (2.2). Note that R depends only on (M, p, H, K). It follows that,

$$SN^*H \subset \left(\bigcup_{i \in \mathcal{I}_{\mathcal{S}_H}} B_i \ \cup \ \bigcup_{i \in \mathcal{I}_K} D_i \right) \tag{5.15}$$

where each ball B_i satisfies (5.11) and (5.13), and each ball D_i satisfies (5.14). Also,

$$\mathcal{S}_H \cap D_i = \emptyset \quad \forall i \in \mathcal{I}_K \qquad \text{and} \qquad \mathcal{S}_H \cap B_i \neq \emptyset \quad \forall i \in \mathcal{I}_{\mathcal{S}_H}.$$

Since SN^*H can be split as in (5.15), we present how to treat D_i with $i \in \mathcal{S}_H$ and B_i with $i \in \mathcal{I}_K$ separately.

Treatment of $D \in \{D_i\}_{i \in \mathcal{I}_K}$.

Let $D \in \{D_i\}_{i \in \mathcal{I}_K}$. Note that since $R \leq \min\{\delta_{_K}, \delta_0\}$, by (5.14) we know that if $\rho \in D$ and $|t| \geq t_{_K}$ are so that $d(\varphi_t(\rho), \rho) < R$, then there exists $\mathbf{w} = \mathbf{w}(t, \rho) \in T_\rho(SN^*H)$ so that for all $s \in \mathbb{R}$

$$\begin{aligned} \|dF(d\varphi_t \mathbf{w} + sH_p)\| &\geq c_0 \inf\left\{\|d\varphi_t \mathbf{w} + sH_p + \mathbf{u}\| : \ \mathbf{u} \in T_{\varphi_t(\rho)} \mathcal{F}_{\varphi_t(\rho)}\right\} \\ &\geq c_0 \inf\left\{\|d\varphi_t \mathbf{w} + \mathbf{v}\| : \ \mathbf{v} \in T_{\varphi_t(\rho)} \mathcal{F}_{\varphi_t(\rho)} + \mathbb{R}H_p\right\} \\ &\geq c_0 c_{_K} \|\mathbf{w}\|, \end{aligned}$$

where we used (5.9) to get the first inequality and (5.14) for the third one. This implies that if $|t| \geq t_{_K}$ and $\rho \in D$ are so that $d(\varphi_t(\rho), \rho) < R$, then $d\psi(t, \rho) := d(F \circ \varphi_t)(t, \rho)$ has a left inverse $L_{(t,\rho)}$ when restricted to $\mathbb{R}\partial_t \oplus \mathbb{R}\mathbf{w}$ with $\|L_{(t,\rho)}\| \leq (c_0 c_{_K})^{-1}$.

Let α_1, α_2 be as in Proposition 2.2, and note that they only depend on (M, g, H, K). We aim to apply this proposition with $A = D$, $B = D$, $\beta = 0$, $c = (c_0 c_{_K})^{-1}$, $a = 0$, $\tilde{c} = \frac{R}{4}$. Let t_1 satisfy

$$t_1 \geq \max\{1, t_{_K}\}. \tag{5.16}$$

Note that t_1 depends only on (M, p, H, K).

Next, let $T_0 \geq t_1$. By construction, if $(t, \rho) \in [t_1, T_0] \times D$ are so that $d(\varphi_t(\rho), D) \leq \tilde{c}$, by (5.16) we have

$$d(\varphi_t(\rho), \rho) \leq d(\varphi_t(\rho), D) + \operatorname{diam}(D) \leq \tilde{c} + 2(\tfrac{1}{4}R) < R.$$

In this case there exists $\mathbf{w} = \mathbf{w}(t, \rho) \in T_\rho(SN^*H)$ so that $d\psi(t, \rho)$ has a left inverse $L_{(t,\rho)}$ when restricted to $\mathbb{R}\partial_t \oplus \mathbb{R}\mathbf{w}$ with $\|L_{(t,\rho)}\| \leq c_0 c_{\scriptscriptstyle K} \leq c$.

Let $C > 0$ be so that

$$\frac{1}{C} < \min\{\tfrac{1}{2}, \tfrac{1}{3\alpha_1}\} \qquad \text{and} \qquad e^{-CT_0} \leq \min\{\tfrac{1}{8}\alpha_1 R,\ \tfrac{1}{2}\alpha_1\alpha_2 e^{-3\Lambda T_0}\}. \tag{5.17}$$

Set $r_2 := \frac{2}{\alpha_1} r_1$ and note that by construction, and the assumptions on the pair (r_0, r_1), we have

$$r_1 < \alpha_1\, r_2, \qquad r_2 \leq \min\{\tfrac{1}{4}R, \alpha_2\, e^{-3\Lambda T_0}\}, \qquad r_0 < \tfrac{1}{3}\, e^{-\Lambda T_0} r_2.$$

Also, note that we work with $0 < \tau < \tau_0 < \tau_{\scriptscriptstyle SN^*H}$, and that by definition $\tau_{\scriptscriptstyle SN^*H} < \frac{1}{2}\tau_{\operatorname{inj}H}$ as requested by Proposition 2.2. We apply Proposition 2.2 to the cover $\{\Lambda^\tau_{\rho_j}(r_1)\}_{j \in \mathcal{E}_D}$ of $\Lambda^\tau_{\scriptscriptstyle D}(r_0)$ where

$$\mathcal{E}_{\scriptscriptstyle D} := \{j : \Lambda^\tau_{\rho_j}(r_1) \cap \Lambda^\tau_{\scriptscriptstyle D}(r_0) \neq \emptyset\}. \tag{5.18}$$

Then, there is a partition $\mathcal{E}_{\scriptscriptstyle D} = \mathcal{G}_{\scriptscriptstyle D} \cup \mathcal{B}_{\scriptscriptstyle D}$ with

$$|\mathcal{B}_{\scriptscriptstyle D}| \leq \mathbf{C_0}\, \frac{R^{n-1}}{r_1^{n-2}}\, T_0 e^{4\Lambda T_0}, \tag{5.19}$$

where $\mathbf{C_0} = \mathbf{C_0}(M, g, k, c_0, c_{\scriptscriptstyle K}) > 0$, and so that

$$\bigcup_{j \in \mathcal{G}_D} \Lambda^\tau_{\rho_j}(r_1) \qquad \text{is} \quad [t_1, T_0] \text{ non-self looping.} \tag{5.20}$$

Treatment of $B \in \{B_i\}_{i \in \mathcal{I}_{\mathcal{S}_H}}$

Let $B \in \{B_i\}_{i \in \mathcal{I}_{\mathcal{S}_H}}$. Since (5.13) is satisfied for all $\rho \in B$, we shall focus on the case where $m_-(\rho) = 0$ for all $\rho \in B$; the other being similar after sending $t \mapsto -t$ in the arguments below.

Suppose B is the ball $B(\rho_{\scriptscriptstyle B}, \frac{1}{2}\delta_\varepsilon)$ for some $\rho_{\scriptscriptstyle B} \in SN^*H$ and let

$$E := B(\rho_{\scriptscriptstyle B}, \tfrac{3}{4}\delta_\varepsilon) \subset SN^*H, \qquad \tilde{B} := B(\rho_{\scriptscriptstyle B}, \delta_\varepsilon) \subset SN^*H.$$

Note that $B \subset E \subset \tilde{B}$, and that $\Theta^+_\rho \leq \varepsilon$ for all $\rho \in \tilde{B}$ by (5.13).

We claim that there exist a function $\mathfrak{t}_2 : [\frac{1}{5}, +\infty) \to [1, +\infty)$ that depends only on (M, p), and a constant $\digamma > 0$ depending on $(M, p, K_{\scriptscriptstyle H})$, so that

$$E \ \text{ can be } \ (\tfrac{1}{5}, \mathfrak{t}_2, \digamma)\text{-controlled up to time } T_0. \tag{5.21}$$

If the claim in (5.21) holds, setting $R_0 := \min\{\frac{1}{\digamma} e^{-\digamma \Lambda T_0}, \frac{1}{8}\delta_\varepsilon\}$ and noting that $d(B, E^c) = \frac{1}{4}\delta_\varepsilon > R_0$, we may apply Lemma 3.2 to the ball E with $E_0 = B$ and $\varepsilon_0 = \frac{1}{5}$. Indeed, by possibly enlarging $C > 0$ in (5.17) so that

$$e^{-CT_0} < \tfrac{1}{5\digamma} e^{-(\digamma + 2\mathbf{D})\Lambda T_0} R_0, \tag{5.22}$$

by the assumption that $r_1 \le e^{-CT_0}$ we conclude $0 < r_1 < \frac{1}{5\digamma} e^{-(\digamma+2\mathbf{D})\Lambda T_0} R_0$. Therefore, letting

$$\mathcal{E}_{B} := \{j : \Lambda^\tau_{\rho_j}(r_1) \cap \Lambda^\tau_{B}(r_0) \neq \emptyset\}, \tag{5.23}$$

there exists $C_{M,g} > 0$ depending only on (M, g), so that for every integer $0 < m < \frac{\log T_0 - \log t_0(\frac{1}{5})}{\log 2}$ there are sets $\{\mathcal{G}_{B,\ell}\}_{\ell=0}^m \subset \{1, \dots N_{r_1}\}$, $\mathcal{B}_{B} \subset \{1, \dots N_{r_1}\}$ satisfying

$$\mathcal{E}_{B} \subset \mathcal{B}_{B} \cup \bigcup_{\ell=0}^{m} \mathcal{G}_{B,\ell}, \qquad \bigcup_{i \in \mathcal{G}_{B,\ell}} \Lambda^\tau_{\rho_i}(r_1) \text{ is } [t_2(\tfrac{1}{5}), 2^{-\ell} T_0] \text{ non-self looping}$$

$$|\mathcal{G}_{B,\ell}| \le C_{M,p} \frac{\delta_\varepsilon^{n-1}}{5^\ell} \frac{1}{r_1^{n-1}}, \qquad \text{and} \qquad |\mathcal{B}_{B}| \le C_{M,p} \frac{\delta_\varepsilon^{n-1}}{5^{m+1}} \frac{1}{r_1^{n-1}}, \tag{5.24}$$

for all $\ell \in \{0, \dots, m\}$. We shall use this construction later in the proof, namely below the "Constructing the complete cover" title, to build the complete cover.

We dedicate the rest of the argument to proving the claim in (5.21). Let $\digamma > 0$ satisfy

$$\frac{1}{\digamma} < \min\Big\{\frac{\alpha}{4}, \frac{\alpha^2}{4}, \frac{\alpha}{60\mathbf{C_0}}, \frac{[\varepsilon \mathfrak{c}_H]^{\frac{1}{\nu}}}{3}, \frac{\varepsilon^{\frac{1}{\nu}}}{C_{\Theta}^{\frac{1}{\nu}}}, \frac{1}{11}, \frac{\nu}{2}\Big\}, \tag{5.25}$$

where $\alpha := \min\{\frac{1}{3}, \alpha_1, \alpha_2\}$, $\mathfrak{c}_H$ is defined in (5.10), $\mathbf{C_0}$ is the positive constant introduced in Proposition 2.2 (that depends only on (M, g, H, ε) when the left inverse is bounded by $\frac{2C_\varphi}{c_0\,\varepsilon}$), and C_{Θ} is so that for all $\rho_1, \rho_2 \in SN^*H$

$$\sup_{\substack{\mathbf{w}_1 \in T_{\rho_1} SN^*H \\ \Theta^+(\mathbf{w}_1) \le \varepsilon}} \inf_{\substack{\mathbf{w}_2 \in T_{\rho_2} SN^*H \\ \Theta^+(\mathbf{w}_2) \le \varepsilon}} |\Theta^+_{\varphi_t(\rho_1)}(d\varphi_t)_{\rho_1}\mathbf{w}_1 - \Theta^+_{\varphi_t(\rho_2)}(d\varphi_t)_{\rho_2}\mathbf{w}_2\| \le C_{\Theta} d(\rho_1, \rho_2)^\nu e^{2\Lambda|t|} \tag{5.26}$$

for all $t \in \mathbb{R}$. Next, Let $0 < \tau < \tau_0$, $\varepsilon_1 \ge \frac{1}{5}$,

$$0 < \tilde{R}_0 \le \tfrac{1}{\digamma} e^{-\digamma \Lambda T_0} \qquad \text{and} \qquad 0 < \tilde{r}_0 < \tilde{R}_0.$$

Also, let $\{B_{0,i}\}_{i=1}^N \subset SN^*H$ be a collection of balls with centers in E and radii $R_{0,i} = \tilde{R}_0 \ge 0$ so that

$$E \subset \bigcup_{i=1}^N B_{0,i} \subset \tilde{B}.$$

Using (5.8) we let $L \ge 1$ be so that for all $q \in SN^*H$ and all $\mathbf{u} \in T_\rho S^*M \backslash \{0\}$ we have $\Theta^+_{\varphi_s(q)}(d\varphi_s \mathbf{u}) \ge \frac{1}{L} \Theta^+_q(\mathbf{u})$ provided $s \ge 0$. Next, for each $i \in \{1, \dots, N\}$ let

$$T_{B_{0,i}} := \inf_{\rho \in B_{0,i}} T(\rho) \qquad \text{for} \qquad T(\rho) := \inf\big\{t \ge 0 : \sup_{\mathbf{w} \in T_\rho SN^*H} \Theta^+_\rho(d\varphi_t \mathbf{w}) > 5L\varepsilon\big\},$$

where $\varepsilon = \varepsilon(M, g, H)$ as defined in (5.12). Note that since $\Theta^+_\rho \le \varepsilon$ for $\rho \in \tilde{B}$, then $T_{B_{0,i}} > 0$ for all $i \in \{1, \dots, N\}$.

Control of $B_{0,i}$ before time $T_{B_{0,i}}$. We claim that for all $\rho \in B_{0,i}$ and $\mathbf{w} \in T_\rho S^*M$

$$\|d\varphi_t \mathbf{w}\| \le \mathbf{B}(1 + 5L\varepsilon) e^{-t/\mathbf{B}} \|\mathbf{w}\| \qquad 0 \le t < T_{B_{0,i}}. \tag{5.27}$$

Indeed, suppose that $0 \leq t < T(\rho)$ for some $\rho \in B_{0,i}$. Then, $\Theta^+_{\varphi_t(\rho)}(d\varphi_t \mathbf{w}) \leq 5L\varepsilon$ for all $\mathbf{w} \in T_\rho SN^*H$ and so, using that $\pi_\pm d\varphi_t = d\varphi_t \pi_\pm$, we have

$$\|d\varphi_t \mathbf{w}\| \leq \|d\varphi_t \pi_+ \mathbf{w}\| + \|d\varphi_t \pi_- \mathbf{w}\| \leq (1+5L\varepsilon)\|d\varphi_t \pi_+ \mathbf{w}\| \leq (1+5L\varepsilon)\mathbf{B} e^{-t/\mathbf{B}}\|\mathbf{w}\|.$$

From (5.27) it follows that there exists $C_0 > 0$, depending only on (M, g, H), so that

$$\sup_{\rho \in B_{0,i}} |\det J_t| \leq C_0\, e^{-|t|/C_0} \qquad \text{for all } \; t \in (0, T_{B_{0,i}}).$$

Suppose that $T_{B_{0,i}} > 1$. By Lemma 3.1, for all $\varepsilon_0 > 0$ there exists $\digamma_{M,g,K_H} > 0$ and a function $\mathfrak{t}_0 : [\varepsilon_0, +\infty) \to [1, +\infty)$ depending only on $(M, g, H, \varepsilon_0, C_0)$ so that the set $B_{0,i}$ can be $(\varepsilon_0, \mathfrak{t}_0, \digamma_{M,p})$-controlled up to time $T_{B_{0,i}}$ in the sense of Definition 3. In addition, by Lemma 3.1, given $\varepsilon_1 > 0$ and any $0 < r \leq \frac{1}{\digamma} e^{-\digamma \Lambda T_0} \tilde{r}_0$, there exist balls $\{\tilde{B}_{1,k}\}_k \subset SN^*H$ with radii $R_{1,k} \in [0, \frac{1}{4}\tilde{R}_0]$ so that

$$\bigcup_{t=\mathfrak{t}_0(\frac{1}{5})}^{T_{B_{0,i}}} \varphi_t(\Lambda^\tau_{B_{0,i} \setminus \cup_k \tilde{B}_{1,k}}(r)) \bigcap \Lambda^\tau_{SN^*H \setminus \cup_k \tilde{B}_{1,k}}(r) = \emptyset, \tag{5.28}$$

$$\sum_k \tilde{R}^{n-1}_{1,k} \leq \frac{\varepsilon_1}{2} \tilde{R}^{n-1}_0 \qquad \text{and} \qquad \inf_k \tilde{R}_{1,k} \geq e^{-\mathbf{D}\Lambda T_0} \tilde{R}_0. \tag{5.29}$$

In the case in which $T_{B_{0,i}} \leq 1$ we will not attempt to control $B_{0,i}$ for times smaller than $T_{B_{0,i}}$. Indeed, we will set $t_0 = 1$, interpret (5.28) and (5.29) as empty statements, and define every ball $\tilde{B}_{1,k}$ as the empty set.

We now set $\varepsilon_0 = \frac{1}{10}$ so that $\varepsilon_1 \geq \frac{1}{5}$.

Control of $B_{0,i}$ after time $T_{B_{0,i}}$. Set $A := \bigcup_{i=1}^N B_{0,i}$. Next, suppose that $\rho \in B_{0,i}$ and $t \geq T_{B_{0,i}}$ are so that $d(\varphi_t(\rho), A) \leq \tilde{c}\, e^{-2\Lambda|t|}$ where

$$\tilde{c} := \min\Big\{\tfrac{1}{3}\big[\varepsilon c_{_H}\big]^{\frac{1}{\nu}}, \delta_0, \delta_F\Big\},$$

with δ_F defined in (2.2), δ_0 defined in (5.9), and $c_{_H}$ defined in (5.10).

Since by (5.25) the parameter $\digamma$ is chosen so that $\frac{1}{\digamma} \leq \min\{\frac{\varepsilon^{\frac{1}{\nu}}}{C_\Theta^{\frac{1}{\nu}}}, \frac{1}{11}\}$ and $\tilde{R}_0 < \frac{1}{\digamma} e^{-\digamma \Lambda T_0}$, we have $\tilde{R}_0 \leq \frac{\varepsilon^{\frac{1}{\nu}}}{C_\Theta^{\frac{1}{\nu}}} e^{-\frac{2}{\nu}\Lambda T_0}$. Thus, using (5.26), $L \geq 1$, and that $\rho \in B_{0,i}$, there exists $\mathbf{w} \in T_\rho SN^*H$ for which

$$\Theta^+_{\varphi_{T_{B_{0,i}}}(\rho)}(d\varphi_{T_{B_{0,i}}} \mathbf{w}) \geq 4L\varepsilon.$$

It then follows by the definition of L that, if $t = T_{B_{0,i}} + s$ for some $s > 0$, then $\Theta^+_{\varphi_t(\rho)}(d\varphi_t \mathbf{w}) = \Theta^+_{\varphi_s(\varphi_{T_{B_{0,i}}}(\rho))}(d\varphi_s(d\varphi_{T_{B_{0,i}}} \mathbf{w})) \geq \frac{1}{L}\Theta^+_{\varphi_{T_{B_{0,i}}}(\rho)}(d\varphi_{T_{B_{0,i}}} \mathbf{w}) \geq 4\varepsilon$. In particular,

$$\Theta^+_{\varphi_t(\rho)}(d\varphi_t \mathbf{w} + rH_p) \geq 4\varepsilon \qquad \text{for all } \; r \in \mathbb{R}. \tag{5.30}$$

In addition, we note that

$$\Theta^+_{\varphi_t(\rho)}(\mathbf{v}) \le 2\varepsilon \qquad \text{for all } \mathbf{v} \in T_{\varphi_t(\rho)}\mathcal{F}_{\varphi_t(\rho)}. \tag{5.31}$$

Indeed, this follows from the estimate in (5.10) together with the facts that $\Theta^+_\rho \le \varepsilon$, $B_{0,i}$ is a ball with radius $\tilde{R}_0$ and center in E, and

$$d(\varphi_t(\rho), \rho) \le d(\varphi_t(\rho), A) + \operatorname{diam}(E) + \tilde{R}_0 \le \tilde{c}\, e^{-2\Lambda|t|} + 2(\tfrac{3}{4})\delta_\varepsilon + \tfrac{1}{F} \le [\varepsilon c_{_H}]^{\frac{1}{\nu}}.$$

We have also used that $\tilde{c} \le \frac{1}{3}[\varepsilon c_{_H}]^{\frac{1}{\nu}}$, $\delta_\varepsilon \le \frac{2}{9}[\varepsilon c_{_H}]^{\frac{1}{\nu}}$, and $\frac{1}{F} \le \frac{1}{3}[\varepsilon c_{_H}]^{\frac{1}{\nu}}$ by (5.25).

From (5.30) and (5.31) it follows that for all $r \in \mathbb{R}$ and $(\rho, t) \in B_{0,i} \times [T_{B_{0,i}}, \infty)$ with $d(\varphi_t(\rho), A) \le \tilde{c}\, e^{-2\Lambda|t|}$ we have

$$\inf\{|\Theta^+_{\varphi_t(\rho)}(d\varphi_t \mathbf{w} + rH_p) - \Theta^+_{\varphi_t(\rho)}(\mathbf{v})| : \mathbf{v} \in T_{\varphi_t(\rho)}\mathcal{F}_{\varphi_t(\rho)}\} \ge 2\varepsilon\|\mathbf{w}\|.$$

Moreover, we claim that there is $c_{_{M,g}} > 0$ depending only on (M, g) so that

$$\|d\varphi_t \mathbf{w} + \mathbf{v}\| \ge \frac{\varepsilon c_{_{M,g}}}{2C_\varphi}\, e^{-\Lambda t}\|\mathbf{w}\|, \tag{5.32}$$

for all $\mathbf{v} \in T_{\varphi_t(\rho)}\mathcal{F}_{\varphi_t(\rho)} \oplus \mathbb{R}H_p$.

To see this, first observe that by continuity of $E_\pm$ and the fact that $E_+ \cap E_- = \{0\}$, there exists $c_{_{M,g}} > 0$ depending only on (M, g) so that for all $\mathbf{v} \in TT^*M$

$$c_{_{M,g}}(\|\pi_+\mathbf{v}\| + \|\pi_-\mathbf{v}\|) \le \|\mathbf{v}\| \le \|\pi_+\mathbf{v}\| + \|\pi_-\mathbf{v}\|. \tag{5.33}$$

Next, suppose that $\|\pi_+\mathbf{v}\| < \frac{3}{2}\|\pi_+ d\varphi_t\mathbf{w}\|$. Then, by (5.30), (5.31), and (5.33)

$$\begin{aligned}\|d\varphi_t\mathbf{w} + \mathbf{v}\| &\ge c_{_{M,g}}(\|\pi_- d\varphi_t\mathbf{w}\| - \|\pi_-\mathbf{v}\|)\\ &\ge c_{_{M,g}}(4\varepsilon\|\pi_+ d\varphi_t\mathbf{w}\| - 2\varepsilon\|\pi_+\mathbf{v}\|) \ge c_{_{M,g}}\varepsilon\|\pi_+ d\varphi_t\mathbf{w}\|.\end{aligned}$$

On the other hand, assuming that $\varepsilon \le \frac{1}{2}$ we have $\|\pi_+\mathbf{v}\| \ge \frac{3}{2}\|\pi_+ d\varphi_t\mathbf{w}\|$, then

$$\|d\varphi_t\mathbf{w} + \mathbf{v}\| \ge c_{_{M,g}}(\|\pi_+\mathbf{v}\| - \|\pi_+ d\varphi_t\mathbf{w}\|) \ge c_{_{M,g}}\tfrac{1}{2}\|\pi_+ d\varphi_t\mathbf{w}\| \ge c_{_{M,g}}\varepsilon\|\pi_+ d\varphi_t\mathbf{w}\|.$$

Also, note that

$$\|\pi_+ d\varphi_t\mathbf{w}\| = \|d\varphi_t \pi_+\mathbf{w}\| \ge \tfrac{1}{C_\varphi}e^{-\Lambda|t|}\|\pi_+\mathbf{w}\|,$$

and

$$\|\mathbf{w}\| \le \|\pi_+\mathbf{w}\| + \|\pi_-\mathbf{w}\| \le (1 + \Theta^+_\rho(\mathbf{w}))\|\pi_+\mathbf{w}\| \le (1+\varepsilon)\|\pi_+\mathbf{w}\|.$$

The proof of (5.32) follows from noticing that $\frac{\varepsilon}{1+\varepsilon} \ge \frac{\varepsilon}{2}$ since $\varepsilon < 1$.

Since $d(\varphi_t(\rho), A) \le \tilde{c}\, e^{-2\Lambda|t|} \le \delta_0$, we conclude by (5.9) and (5.32) that for all $s \in \mathbb{R}$

$$\begin{aligned}\|dF(d\varphi_t\mathbf{w} + sH_p)\| &\ge c_0 \inf\{\|d\varphi_t\mathbf{w} + \mathbf{v}\| : \mathbf{v} \in T_{\varphi_t(\rho)}\mathcal{F}_{\varphi_t(\rho)} \oplus \mathbb{R}H_p\}\\ &\ge \frac{c_0\, \varepsilon\, c_{_{M,g}}}{2C_\varphi} e^{-\Lambda t}\|\mathbf{w}\|.\end{aligned}$$

This means that if $\psi = F \circ \varphi_t$, then $d\psi(t, \rho)$ has a left inverse $L_{(t,\rho)}$ when restricted to $\mathbb{R}\partial_t \oplus \mathbb{R}\mathbf{w}$ with $\|L_{(t,\rho)}\| \le \frac{2C_\varphi}{c_0\,\varepsilon\, c_{_{M,g}}}e^{t\Lambda}$.

In particular, for any $t \geq T_{B_{0,i}}$ so that $d(\varphi_t(\rho), A) \leq \tilde{c}\, e^{-2\Lambda|t|}$, the hypotheses of Proposition 2.2 apply to the set A with $t_0 = T_{B_{0,i}}$, $B = B_{0,i}$, $R = \tilde{R}_0$, $\beta = \Lambda$, and $c = c_0\, C_{M,g}\, \varepsilon^{-1}$, $a = 2\Lambda$. Fix $0 < \tilde{r}_0 < \tilde{R}_0$ and $0 < r \leq \frac{1}{F} e^{-F \Lambda T_0} \tilde{r}_0$. Let

$$\tilde{r}_2 := \max\Big\{6e^{\Lambda T_0} r,\ \tfrac{4}{\alpha_1} r,\ \tfrac{4}{\alpha_1} e^{-\mathbf{D}\Lambda T_0} \tilde{R}_0\Big\},$$

and note by the definition (5.25) of F we have

$$\tilde{r}_2 < \min\Big\{\tilde{R}_0,\ \alpha_2 e^{-5\Lambda T_0},\ \tfrac{1}{10\mathbf{C_0}} e^{-10\Lambda T_0}\Big\}.$$

This can be done since $T_0 > 1$ and $e^{-\mathbf{D}\Lambda} < \frac{\alpha_1}{4}$ by the definition (3.4) of $\mathbf{D}$.

Setting $\tilde{r}_1 := \max\{2r, e^{-\mathbf{D}\Lambda T_0}\}$ we have

$$r < \tilde{r}_1, \qquad \tilde{r}_1 < \alpha_1\, \tilde{r}_2, \qquad \tilde{r}_2 \leq \min\{\tilde{R}_0,\, \alpha_2\, e^{-5\Lambda T_0}\}, \qquad r < \tfrac{1}{3} e^{-\Lambda T_0} \tilde{r}_2.$$

Therefore, we may apply Proposition 2.2 to the cover $\{\Lambda^\tau_{\rho_j}(\tilde{r}_1)\}_{j \in \mathcal{E}_{B_{0,i}}}$ of $\Lambda^\tau_{B_{0,i}}(r)$ where

$$\mathcal{E}_{B_{0,i}} := \{j : \Lambda^\tau_{\rho_j}(\tilde{r}_1) \cap \Lambda^\tau_{B_{0,i}}(r) \neq \emptyset\}. \tag{5.34}$$

Then, there is a partition $\mathcal{E}_{B_{0,i}} = \mathcal{G}_{B_{0,i}} \cup \mathcal{B}_{B_{0,i}}$ with

$$|\mathcal{B}_{B_{0,i}}| \leq \mathbf{C_0}\, \tilde{r}_2 \frac{R_0^{n-1}}{\tilde{r}_1^{n-1}}\, T_0 e^{8\Lambda T_0}, \tag{5.35}$$

and so that

$$\bigcup_{t=T_{B_{0,i}}}^{T_0} \varphi_t\Big(\Lambda^\tau_{B_{0,i}}(r) \backslash \bigcup_{j \in \mathcal{B}_{B_{0,i}}} \Lambda^\tau_{\rho_j}(\tilde{r}_1)\Big) \bigcap \Lambda^\tau_A(r) = \emptyset. \tag{5.36}$$

Here $\mathbf{C_0}$ coincides with the positive constant used in the definition (5.25) of F. Combining (5.28) with (5.36), and using that $E \subset A$ and $0 < r < \frac{1}{F} e^{-F\Lambda T_0} \tilde{r}_0$, we obtain

$$\bigcup_{t=t_0}^{T_0} \varphi_t\Big(\Lambda^\tau_{B_{0,i} \backslash \cup_k \tilde{B}_{1,k}}(r) \backslash \bigcup_{j \in \mathcal{B}_{B_{0,i}}} \Lambda^\tau_{\rho_j}(\tilde{r}_1)\Big) \bigcap \Lambda^\tau_{E \backslash \cup_k \tilde{B}_{1,k}}(r) = \emptyset, \tag{5.37}$$

In particular, there are balls $\{\tilde{B}_{2,j}\}_j$ with radii $R_{2,j} = \tilde{r}_1$ so that

$$\bigcup_{t=t_0}^{T_0} \varphi_t(\Lambda^\tau_{B_{0,i} \backslash [\cup_{k,j} \tilde{B}_{1,k} \cup \tilde{B}_{2,j}]}(r)) \cap \Lambda^\tau_{E \backslash \cup_k \tilde{B}_{1,k}}(r) = \emptyset.$$

In addition,

$$\sum_j R_{2,j}^{n-1} \leq \mathbf{C_0} \tilde{r}_2 R_0^{n-1}\, T_0 e^{8\Lambda T_0} \leq \frac{\varepsilon_1}{2} R_0^{n-1}, \tag{5.38}$$

where the first inequality is due to (5.35) and the second one is a consequence of the fact that $\tilde{r}_2 < \frac{1}{10\mathbf{C_0}} e^{-9\Lambda T_0}$ and $\frac{\varepsilon_1}{2} \geq \frac{1}{10}$.

Repeating this argument with $B_{0,i}$ for every $i \in \{1, \dots, N\}$ we conclude that there exist balls $\tilde{B}_\ell$ of radius R_ℓ centered in E so that

$$\Lambda^\tau_{E \backslash \cup_\ell \tilde{B}_\ell}(r) \quad \text{is} \quad [t_0(\tfrac{1}{5}), T_0] \text{ non-self looping.} \tag{5.39}$$

Note that $R_\ell = \tilde{r}_1 \in [0, \frac{1}{4}\tilde{R}_0]$ since $\tilde{r}_1 = \max\{2r, e^{-\mathbf{D}\Lambda T_0}\tilde{R}_0\}$ while $2r \leq \frac{2}{\digamma}\tilde{r}_0 \leq \frac{2}{11}\tilde{r}_0 \leq \frac{1}{4}\tilde{R}_0$ and $e^{-\mathbf{D}\Lambda} < \frac{1}{4}$ by the definition (3.4) of $\mathbf{D}$. Also, by (5.29) and (5.38),

$$\sum_\ell R_\ell^{n-1} \leq \sum_{i=1}^N \Big(\sum_k R_{1,k}^{n-1} + \sum_j R_{2,j}^{n-1} \Big) \leq \varepsilon_1 \sum_{i=1}^N R_0^{n-1}. \tag{5.40}$$

Finally, since $R_{1,k} \geq e^{-\mathbf{D}\Lambda T_0} R_0$ for all k and $R_{2,j} = \tilde{r}_1 \geq e^{-\mathbf{D}\Lambda T_0}\tilde{R}_0$ for all j,

$$R_\ell \geq e^{-\mathbf{D}\Lambda T_0} R_0. \tag{5.41}$$

Relations (5.39), (5.40) and (5.41) show that E can be $(\frac{1}{5}, \digamma)$-controlled up to time T_0 as claimed in (5.21).

Constructing the complete cover

We now partition $\{\rho_j\}_{j=1}^{N_{r_1}}$. Let $t_0 = \max\{t_1, \mathfrak{t}_2(\frac{1}{5})\}$ where t_1 is defined in (5.16) and t_2 is defined in (5.21). By (5.19) and (5.20), for each $i \in \mathcal{I}_K$ we have constructed a partition $\mathcal{E}_{D_i} = \mathcal{G}_{D_i} \cup \mathcal{B}_{D_i}$ of $\mathcal{E}_{D_i} = \{j : \Lambda^\tau_{\rho_j}(r_1) \cap \Lambda^\tau_{D_i}(r_0) \neq \emptyset\}$ where

$$|\mathcal{B}_{D_i}| \leq \mathbf{C_0}\, \frac{R^{n-1}}{r_1^{n-2}}\, T_0 e^{4\Lambda T_0} \quad \text{and} \quad \bigcup_{j \in \mathcal{G}_{D_i}} \Lambda^\tau_{\rho_j}(r_1) \text{ is } [t_0, T_0] \text{ non-self looping.} \tag{5.42}$$

Moreover, by (5.24), for each $i \in \mathcal{I}_{\mathcal{S}_H}$ and $m > 0$ integer we have constructed a partition of $\mathcal{E}_{B_i} = \{j : \Lambda^\tau_{\rho_j}(r_1) \cap \Lambda^\tau_{B_i}(r_0) \neq \emptyset\}$ by sets $\{\mathcal{G}_{B_i,\ell}\}_{\ell=0}^m \subset \{1, \dots N_{r_1}\}$, $\mathcal{B}_{B_i} \subset \{1, \dots N_{r_1}\}$ satisfying

$$\mathcal{E}_{B_i} \subset \mathcal{B}_{B_i} \cup \bigcup_{\ell=0}^m \mathcal{G}_{B_i,\ell}, \qquad \bigcup_{j \in \mathcal{G}_{B_i,\ell}} \Lambda^\tau_{\rho_j}(r_1) \text{ is } [t_0, 2^{-\ell}T_0] \text{ non-self looping,}$$

$$|\mathcal{G}_{B_i,\ell}| \leq C_{M,p} \frac{\delta_\varepsilon^{n-1}}{5^\ell} \frac{1}{r_1^{n-1}} \qquad \text{and} \qquad |\mathcal{B}_{B_i}| \leq C_{M,p} \frac{\delta_\varepsilon^{n-1}}{5^{m+1}} \frac{1}{r_1^{n-1}}. \tag{5.43}$$

Next, define

$$m := \Big\lfloor \frac{\log T_0 - \log t_0}{\log 2} \Big\rfloor \qquad \text{and} \qquad \mathcal{B} := \bigcup_{i \in I_K} \mathcal{B}_{D_i} \cup \bigcup_{i \in \mathcal{I}_{\mathcal{S}_H}} \mathcal{B}_{B_i}.$$

For each $i \in \mathcal{I}_K$ set $\mathcal{G}_{i,0} := \mathcal{G}_{D_i}$ and $\mathcal{G}_{i,\ell} := \mathcal{G}_{B_{i,\ell-1}}$ for $\ell \geq 1$. Then, there exists $I < \infty$, depending only on (M, H, p), so that after relabelling the indices $i \in I_K \cup I_{\mathcal{S}_H}$ there are sets $\{\mathcal{G}_{i,\ell} : 1 \leq \ell \leq m,\ 1 \leq i \leq I\}$ so that

$$\bigcup_{i=1}^I \bigcup_{\ell=1}^m \mathcal{G}_{i,\ell} \cup \mathcal{B} = \{1, \dots N_{r_1}\}, \qquad \bigcup_{j \in \mathcal{G}_{i,\ell}} \Lambda^\tau_{\rho_j}(r_1) \text{ is } [t_0, 2^{-\ell}T_0] \text{ non-self looping.}$$

In addition, there exists $c > 0$, which may change from line to line, so that

$$|\mathcal{B}| \le c\, r_1^{1-n}\Big(|I_K|\; r_1 R^{n-1}\; T_0 e^{4\Lambda T_0} + |I_{\mathcal{S}_H}| \frac{\delta_\varepsilon^{n-1}}{5^{m+1}}\Big)$$
$$\le c\, r_1^{1-n}\Big(r_1 T_0 e^{4\Lambda T_0} + \Big(\frac{t_0}{T_0}\Big)^{\frac{\log 5}{\log 2}}\Big).$$

Here, we have used that $|\mathcal{I}_K| \le c\, R^{-(n-1)}$ and $|\mathcal{I}_{\mathcal{S}_H}| \le c\, \delta_\varepsilon^{-(n-1)}$. Since $r_1 \le e^{-CT_0}$ and we may enlarge C so that $C > 4\Lambda + 1 + \log 5$, we conclude that

$$|\mathcal{B}| \le c\Big(\frac{t_0}{T_0}\Big)^{\frac{\log 5}{\log 2}} r_1^{1-n},$$

as claimed. In addition, note that $|\mathcal{G}_{D_i}| \le |\mathcal{E}_{D_i}| \le c\, R^{n-1} r_1^{-(n-1)}$ for each $i \in \mathcal{I}_K$. Therefore, since $R \le 1$ and $\delta_\varepsilon \le 1$, for all $\ell \in \{1, \dots, m\}$ and all $i \in \{1, \dots, L\}$

$$|\mathcal{G}_{i,\ell}| \le c\, \frac{1}{5^\ell}\, r_1^{1-n}.$$

Finally, we note that by construction the constants $c = c(M, g, H)$ and $C = C(M, g, H)$ are uniform for for H varying in a small neighborhood of a fixed submanifold $H_0 \subset M$. □

Lemma 5.3. *Suppose that (M, g) has no focal points and $\mathcal{S}_H = \emptyset$. Then, the conclusions of Lemma 5.2 hold.*

Proof. Since SN^*H is compact by Lemma 5.1 there exist positive constants c_K, t_K, δ_K so that if $\rho \in K$ and $\varphi_t(\rho) \in \overline{B(\rho, \delta_K)}$ for some $|t| > t_K$, then there exists $\mathbf{w} = \mathbf{w}(t, \rho) \in T_\rho(SN^*H)$ so that

$$\inf\{\|d\varphi_t(\mathbf{w}) + \mathbf{v}\| :\ \mathbf{v} \in T_{\varphi_t(\rho)}\mathcal{F}_{\varphi_t(\rho)} \oplus \mathbb{R}H_p\} \ge c_K \|\mathbf{w}\|. \tag{5.44}$$

Cover SN^*H with finitely many balls $\{D_i\}_{i\in I} \subset SN^*H$ of radius equal to δ_K. The remainder of the proof of this lemma is identical to that in the Anosov case since $\mathcal{S}_H = \emptyset$ implies that $D_i \cap \mathcal{S}_H = \emptyset$ for all i. □

5.1. **Proof of Theorem 6.** We first apply Lemma 5.2 when (M, g) has Anosov geodesic flow, or Lemma 5.3 when (M, g) has no focal points. Let $c > 0$, $C > 2$, $I > 0$, $t_0 > 1$ be the constants whose existence is given by the lemmas. Then, let $\Lambda > \Lambda_{\max}$, $0 < \tau_0 < \tau_{SN^*H}$, $0 < \tau < \tau_0$,

$$0 < \varepsilon < \tfrac{1}{2}, \qquad 0 < a < \tfrac{1-2\varepsilon}{\varepsilon}, \qquad \tilde{c} \ge \max\{C, \tfrac{\Lambda_{\max}}{a}\}, \qquad \varepsilon\big(1 + \tfrac{\Lambda}{\tilde{c}}\big) < \delta < \tfrac{1}{2},$$

$$T_0(h) = \tfrac{\varepsilon}{\tilde{c}} \log h^{-1}, \qquad r_1(h) = h^\varepsilon, \qquad r_0(h) = h^\delta,$$

and let $\{\Lambda^\tau_{\rho_j}(h^\varepsilon)\}_{j=1}^{N_{h^\varepsilon}}$ be a $(\mathfrak{D}_n, \tau, h^\varepsilon)$-good cover of SN^*H. Then, since $\tilde{c} \ge C$, Lemmas 5.2 and 5.3 give that for each $i \in \{1, \dots, I\}$, and

$$m := \Big\lfloor \frac{\log T_0(h) - \log t_0}{\log 2} \Big\rfloor,$$

there are sets of indices $\{\mathcal{G}_{i,\ell}\}_{\ell=0}^m \subset \{1,\dots,N_{h^\varepsilon}\}$ and $\mathcal{B}\subset\{1,\dots,N_{h^\varepsilon}\}$ so that

$$\bigcup_{i=1}^{I}\bigcup_{\ell=0}^{m}\mathcal{G}_{i,\ell}\cup\mathcal{B}=\{1,\dots,N_{h^\varepsilon}\},$$

and for every $i\in\{1,\dots,I\}$ and every $\ell\in\{0,\dots,m\}$

$$\bigcup_{j\in\mathcal{G}_{i,\ell}}\Lambda^\tau_{\rho_j}(h^\varepsilon)\ \text{ is }\ [t_0,2^{-\ell}T_0(h)]\quad\text{non-self looping,}$$

$$|\mathcal{G}_{i,\ell}|\le c\,5^{-\ell}\,h^{\varepsilon(1-n)},\qquad\qquad |\mathcal{B}|\le c\Big(\frac{\tilde c}{\varepsilon\log h^{-1}}\Big)^{\frac{\log 5}{\log 2}}h^{\varepsilon(1-n)}.$$

Next, we apply Theorem 5 with $R(h)=h^\varepsilon$, $\alpha=a\varepsilon$, $t_\ell(h)=t_0$ for all ℓ, $T_\ell(h)=2^{-\ell}T_0(h)$ for all ℓ. Note that $R_0>R(h)\ge 5h^\delta$ for h small enough since $\delta>\varepsilon$, and that $\alpha<1-2\varepsilon$ as needed. In addition, $T_\ell(h)\le 2\alpha T_e(h)$ since $\tilde c\ge\frac{\Lambda_{\max}}{a}$. It follows that there exists $C>0$, and for all $N>0$ there exists C_N so that

$$\begin{aligned}
&h^{\frac{k-1}{2}}\Big|\int_H wud\sigma_H\Big|\\
&\quad\le C\|w\|_\infty\Big(\Big[\Big(\frac{\tilde c}{\varepsilon\log h^{-1}}\Big)^{\frac{\log 5}{2\log 2}}+\frac{1}{\sqrt{\log h^{-1}}}\sum_\ell(\tfrac{2}{5})^{\frac{\ell}{2}}\Big]\|u\|_{L^2(M)}+\frac{\sqrt{\log h^{-1}}}{h}\sum_\ell(\tfrac{1}{10})^{\frac{\ell}{2}}\|Pu\|_{L^2(M)}\Big)\\
&\qquad+Ch^{-1}\|w\|_\infty\|Pu\|_{H_{\rm scl}^{\frac{k+1}{2}}(M)}+C_Nh^N\big(\|u\|_{L^2(M)}+\|Pu\|_{H_{\rm scl}^{\frac{k+1}{2}}(M)}\big),
\end{aligned}$$

which gives the desired result after choosing h_0 to be small enough. We note that if $H_0\subset M$, there is a neighborhood U of H_0 (in the C^∞ topology) so that the constants C, C_N and h_0 are uniform over $H\in U$, w taken in a bounded subset of C_c^∞, and N bounded above. □

5.2. **Proof of Theorem 4.** We have already proved Theorem 4.A in Theorem 2. For Theorem 3.A, Theorem 4.D, Theorem 4.E we refer the reader to [CG19, Section 5.4] where it is shown that either $\mathcal{A}_H=\emptyset$ in Theorem 3.A, $\mathcal{S}_H=\emptyset$ in Theorem 4.D, and $\mathcal{A}_H=\emptyset$ in Theorem 4.E. Therefore, Theorem 6 can be applied to all these setups yielding the desired conclusions.

Proof of Theorem 4.B. Let H be a geodesic sphere. Then, $H=\pi(\varphi_s(S^*_xM))$ for some $x\in M$ and $s>0$. Next, we observe, using that (M,g) has no conjugate points, the proof of Theorem 2 (when the submanifold is the point $\{x\}$) yields the existence of a cover for S^*_xM, with some choices of $(R(h),t_\ell(h),T_\ell(h))$, so that Theorem 5 implies the outcome in Theorem 2 (which coincides with that of Theorem 4). Then, since $\varphi_s(S^*_xM)=SN^*H$, the result follows from flowing out the cover for time s to obtain a cover for SN^*H. This cover will have the same desired properties as the original one, but possibly with $R(h)$ replaced by $m_sR(h)$ for some $m_s>0$ independent of h. The result follows from applying Theorem 5 to the new cover. □

Remark 10. This proof in fact shows that there is a certain invariance of estimates under fixed time geodesic flow. That is, if one uses Theorem 5 to conclude an estimate on H, then essentially the same estimate will hold on $\pi\varphi_s(SN^*H)$ for any $s\in\mathbb{R}$ independent of h provided that $\pi\varphi_s(SN^*H)$ is a finite union of submanifolds of codimension k for some k.

Proof of Theorem 4.C. For this part we assume that (M,g) has Anosov geodesic flow, non-positive curvature, and H is a submanifold with codimension $k>1$. We will prove that $\mathcal{A}_{\mathcal{H}}=\emptyset$, and by Theorem 6 this will imply the desired conclusion. In what follows we write π for both $\pi: TM\to M$ and $\pi: T^*M\to M$ since it should be clear from context which map is being used.

We proceed by contradiction. Suppose there exists $\rho\in\mathcal{A}_H\subset SN^*H$. We write $\rho^\sharp\in SNH$ and note

$$T_{\rho^\sharp}NH=\{\mathbf{w}:\ \ \exists\, N\!:\!(-\varepsilon,\varepsilon)\to NH \text{ smooth field, } N(0)=\rho^\sharp, N'(0)=\mathbf{w}\}.$$

Moreover, for $v\in T_{\pi(\rho^\sharp)}H$ and $\mathbf{w}\in T_{\rho^\sharp}NH$ with $d\pi\mathbf{w}\in T_{\rho^\sharp}H\backslash\{0\}$ and $\mathbf{w}=N'(0)$ with N as before,

$$\langle\tilde\nabla_{d\pi\mathbf{w}}N\,,\,v\rangle_{g(\pi(\rho^\sharp))}=-\langle\rho^\sharp\,,\,\Pi_H(d\pi\mathbf{w},v)\rangle_{g(\pi(\rho^\sharp))}.$$

Here, $\tilde\nabla$ denotes the Levi–Civita connection on M and $\Pi_H: TH\times TH\to NH$ is the second fundamental form of H. The equality follows from the definition of the second fundamental form, together with the fact that N is a normal vector field.

We will derive a contradiction fromthe assumption that $T_\rho SN^*H=N_+(\rho)\oplus N_-(\rho)$, by showing that the stable and unstable manifolds at $\rho^\sharp$ have signed second fundamental forms. In particular, note that $E^\sharp_\pm(\rho^\sharp)$ are given by $T\mathcal{W}_\pm(\rho^\sharp)$ where $\mathcal{W}_\pm(\rho^\sharp)$ are respectively the stable and unstable manifolds through $\rho^\sharp$. Furthermore, these manifolds are $\mathcal{W}_\pm(\rho^\sharp)=N\mathcal{H}_\pm$ where $\mathcal{H}_\pm\subset M$ are smooth submanifolds given by the stable/unstable horospheres in M so that $\rho^\sharp\in N\mathcal{H}_\pm$ [Rug07, Section 4.1]. The signed curvature of $\mathcal{H}_\pm$ implies that there is $c>0$ so that

$$\pm\,\Pi_{\mathcal{H}_\pm}\geq c>0. \tag{5.45}$$

We postpone the proof of this fact until the end of the lemma and first derive our contradiction.

Since $T_\rho SN^*H=N_+(\rho)\oplus N_-(\rho)$, then $T_{\rho^\sharp}SNH=N^\sharp_+(\rho)\oplus N^\sharp_-(\rho)$. In addition, since $k>1$, for any $u\in TH$, there exist $\mathbf{w}_1,\mathbf{w}_2\in T_{\rho^\sharp}SNH$ linearly independent with $d\pi\mathbf{w}_i=u$ for $i=1,2$. In particular, since $T_{\rho^\sharp}(SNH)=N^\sharp_+(\rho)\oplus N^\sharp_-(\rho)$, we have $\mathbf{w}_i=\mathbf{w}_{+,i}+\mathbf{w}_{-,i}$, with $\mathbf{w}_{\pm,i}\in N^\sharp_\pm(\rho)$. Thus, $d\pi\mathbf{w}_+=d\pi\mathbf{w}_-$ where $\mathbf{w}_+=\mathbf{w}_{+,1}-\mathbf{w}_{+,2}\in N^\sharp_+(\rho)$ and $\mathbf{w}_-=\mathbf{w}_{-,2}-\mathbf{w}_{-,1}\in N^\sharp_-(\rho)$. Since $d\pi: E^\sharp_\pm(\rho)\to T_{\pi(\rho)}M$ is injective where $\pi: TM\to M$ is the standard projection, $v:=d\pi\mathbf{w}_\pm\neq 0$.

Now, since $\mathbf{w}_\pm\in T_{\rho^\sharp}(SN\mathcal{H}_\pm)$, using (5.45),

$$-\langle\tilde\nabla_v N\,,\,v\rangle_{g(\pi(\rho^\sharp))}=-\langle\tilde\nabla_{d\pi\mathbf{w}_-}N\,,\,v\rangle_{g(\pi(\rho^\sharp))}=\langle\rho^\sharp,\Pi_{\mathcal{H}_+}(v,v)\rangle\geq c\|v\|^2,$$

and

$$\langle\tilde\nabla_v N\,,\,v\rangle_{g(\pi(\rho^\sharp))}=-\langle\tilde\nabla_{d\pi\mathbf{w}_+}N\,,\,v\rangle_{g(\pi(\rho^\sharp))}=\langle\rho^\sharp,\Pi_{\mathcal{H}_-}(v,v)\rangle\leq -c\|v\|^2.$$

This is a contradiction since $\|v\|>0$.

We now prove (5.45). We have by [Ebe73b, Theorem 1, part (6)] that since (M, g) has Anosov flow and non-positive curvature, there are $c, t_0 > 0$ so that for any perpendicular Jacobi field $Y(t)$ with $Y(0) = 0$, and $t \geq t_0$,

$$\langle Y'(t), Y(t)\rangle \geq c\|Y(t)\|^2. \tag{5.46}$$

By [Rug07, Proof of Lemma 4.2] the second fundamental form to $\mathcal{H}_\pm$ at $\pi(\rho^\sharp) \in \mathcal{H}_\pm$ is given by

$$\pm\Pi_{\mathcal{H}_\pm} = \mp \lim_{r\to\mp\infty} U_r(0)$$

where $U_r(t) = Y_r'(t)Y_r^{-1}(t)$ and $Y_r(t)$ is a matrix of perpendicular Jacobi fields along $t \mapsto \pi\varphi_t(\rho)$ satisfying $Y_r(r) = 0$ and $Y_r(0) = \mathrm{Id}$. In particular, by (5.46), applied to the Jacobi field $\tilde{Y}(t) = Y_r(r - t)$, at $t = r$ gives for $r \geq t_0$,

$$\langle U_r(0)x, x\rangle = \langle Y_r'(0)x, Y_r(0)x\rangle = -\langle \tilde{Y}'(r)x, \tilde{Y}(r)x\rangle \leq -c\|Y_r(0)x\|^2 = -c\|x\|^2.$$

Similarly, for $r \leq -t_0$, we apply (5.46) to $\tilde{Y}(t) = Y_r(r + t)$ at $t = |r|$ to obtain

$$\langle U_r(0)x, x\rangle = \langle \tilde{Y}'(|r|)x, \tilde{Y}(|r|)x\rangle \geq c\|x\|^2$$

This yields that $\pm\Pi_{\mathcal{H}_\pm} = \mp\lim_{r\to\pm\infty} U_r(0) \geq c > 0$ as claimed. □

5.3. **Proof of Theorem 3.** For Theorem 3.A we refer the reader to [CG19, Section 5.4] where it is shown that $\mathcal{A}_H = \emptyset$. Therefore, Theorem 6 can be applied to this setup yielding the desired conclusions.

We proceed to prove Theorem 3.B. Fix a geodesic $H \subset M$.We prove that Theorem 3.B holds under the following curvature assumption. Suppose there exist $T > 0$, and $c_1, c_2, c_3 > 0$ so that for all $\rho_0, \rho_1 \in SN^*H$ with $d(\rho_0, \rho_1) = s \leq c_3$, and all $t_0, t_1 \geq T$ with $\varphi_{t_0}(\rho_0), \varphi_{t_1}(\rho_1) \in SN^*H$, we have

$$-\int_{Q_s} K dv_{\tilde{g}} \geq c_1 e^{-c_2/\sqrt{s}}, \tag{5.47}$$

where Q_s is the quadrilateral domain in the universal cover, $(\tilde{M}, \tilde{g})$, whose sides are the geodesics that join the points, $\pi(\rho_0), \pi(\rho_1), \pi(\varphi_{t_0}(\rho_0)), \pi(\varphi_{t_1}(\rho_1))$. At the end of the proof we shall show that the integrated curvature assumption (1.8) implies the assumption in (5.47).

The first step in the proof is to show that there exist $r_0 > 0$ and $c_4 > 0$ so that the following holds. If $0 < r \leq r_0$ and $\rho_0, \rho_1 \in SN^*H$ are such that there are $t_0, t_1 \geq T$ with $|t_0 - t_1| < \frac{\tau_{\mathrm{inj}H}}{2}$ and

$$d(\varphi_{t_0}(\rho_0), SN^*H) < r, \qquad d(\varphi_{t_1}(\rho_1), SN^*H) < r,$$

then either

$$d(\rho_0, \rho_1) < c_2^2 \ln\left(\tfrac{c_4}{r}\right)^{-2} \qquad \text{or} \qquad d(\rho_0, \rho_1) > c_3. \tag{5.48}$$

To prove the claim in (5.48) suppose that there is $\rho_0 \in SN^*H$ with $d(\varphi_{t_0}(\rho_0), SN^*H) < r$ for some $r > 0$. Then, there exists $C = C(M, g, H) \geq 1$ so that by changing t_0 to $\tilde{t_0}$ with $|t_0 - \tilde{t_0}| \leq Cr$ and $r > 0$ small enough, we may assume that $\pi(\varphi_{\tilde{t}_0}(\rho_0)) \in H$ and $d(\varphi_{\tilde{t}_0}(\rho_0), SN^*H) < 2Cr$. Now, let $\rho_s \in SN^*H$, with $d(\rho_0, \rho_s) = s$ and suppose there is t_s with $|t_0 - t_s| < \frac{\tau_{\mathrm{inj}H}}{2}$ and $d(\varphi_{t_s}(\rho_s), SN^*H) < r$. As before, we can adjust t_s to $\tilde{t}_s$, with $|t_s - \tilde{t}_s| \leq Cr$, in order to have $\pi(\varphi_{\tilde{t}_s}(\rho_s)) \in H$ and $d(\varphi_{\tilde{t}_s}(\rho_s), SN^*H) < 2Cr$. Let

$$\gamma_0(t) := \pi(\varphi_t(\rho_0)), \qquad \gamma_s(t) := \pi(\varphi_t(\rho_s)).$$

Note that, in the universal cover of M, $\tilde{M}$, γ_s does not intersect γ_0 unless $\rho_0 = \rho_s$. Indeed, suppose they did intersect at an angle β. Then, by the Gauss–Bonnet theorem, we would have

$$0 \geq \int_{\Delta_s} K \, dv_{\tilde{g}} = \beta \geq 0,$$

where Δ_s is the triangular region enclosed by γ_0, γ_s and H. In particular, this would give $\beta = 0$ and hence $\gamma_s = \gamma_0$ and $s = 0$.

Next, suppose that γ_0 and γ_s do not cross in the universal cover. Let α_s denote the angle between $\dot{\gamma}_s(\tilde{t}_s)$ and H, and let α_0 denote the angle between $\dot{\gamma}_0(\tilde{t}_0)$ and H. This can be done since $\pi(\varphi_{\tilde{t}_0}(\rho_0)) \in H$ and $\pi(\varphi_{\tilde{t}_s}(\rho_s)) \in H$. Then, by the Gauss–Bonnet theorem,

$$\pi - \alpha_0 - \alpha_s = -\int_{Q_s} K \, dv_{\tilde{g}}$$

where Q_s is the quadrilateral formed by γ_0, γ_s, the copy of H in $\tilde{M}$ that contains $\pi(\rho_0), \pi(\rho_s)$, and the copy of H that contains $\pi(\varphi_{\tilde{t}_0}(\rho_0)), \pi(\varphi_{\tilde{t}_s}(\rho_s))$. Since $d(\varphi_{\tilde{t}_0}(\rho_0), SN^*H) \leq 2Cr$, we have $0 < \frac{\pi}{2} - \alpha_0 \leq 2Cr$. Hence,

$$\frac{\pi}{2} - \alpha_s \geq -\int_{Q_s} K \, dv_{\tilde{g}} - 2Cr.$$

In particular, by the curvature assumption (5.47) we have that if $s \leq c_3$,

$$\frac{\pi}{2} - \alpha_s \geq c_1 e^{-c_2/\sqrt{s}} - 2Cr.$$

Let $\tilde{C} = \tilde{C}(H, M, g) > 0$ be so that if $\frac{\pi}{2} - \alpha_s \geq 2\tilde{C}r$, then $d(\varphi_{\tilde{t}_s}(\rho_s), SN^*H) > 2Cr$. Then, for $c_2^2 \ln(c_4 r^{-1})^{-2} < s \leq c_3$, with $c_4 = c_1/2(C + \tilde{C})$, we have

$$\frac{\pi}{2} - \alpha_s > 2\tilde{C}r.$$

This implies that $d(\varphi_{\tilde{t}_s}(\rho_s), SN^*H) > 2Cr$, and hence proves (5.48).

Let τ_0 be the positive constant given in Theorem 5 and $0 < r \leq r_0$. Next, we prove that there exists $C > 0$ so that if $0 < r_1 < r$, then for every $0 < \tau \leq \tau_0$, $T_0 > T$, and every $(\mathfrak{D}_n, \tau, r_1)$-good cover of SN^*H by tubes $\{\Lambda^\tau_{\rho_j}(r_1)\}_{j=1}^{N_{r_1}}$, there is a partition $\{1, \ldots, N_{r_1}\} = \mathcal{B} \cup \mathcal{G}$ so that

$$\bigcup_{j \in \mathcal{G}} \Lambda^\tau_{\rho_j}(r_1) \text{ is } (T, T_0) \text{ non-self looping} \quad \text{and } |\mathcal{B}| \leq C \frac{T_0}{T} \ln\left(\tfrac{c_4}{r}\right)^{-2} r_1^{-1}. \tag{5.49}$$

Note that by splitting $[T, T_0]$ into intervals of length τ the claim in (5.49) is implied by showing that for each $\tilde{t} \in [T, T_0]$

$$\#\left\{\rho_j : \bigcup_{|t - \tilde{t}| < \frac{\tau}{2}} \varphi_t(\Lambda^\tau_{\rho_j}(r_1)) \cap \Lambda^\tau_{SN^*H}(r_1) \neq \emptyset\right\} \leq C \ln\left(\tfrac{c_4}{r}\right)^{-2} r_1^{-1}. \tag{5.50}$$

To prove (5.50) we start by covering SN^*H by balls $\{B_\ell\}_{\ell=1}^L$ of radius $\frac{c_3}{2}$. Fix $\tilde{t}\geq T+\frac{\tau}{2}$. It follows from (5.48) that for each $\ell\in\{1,\dots,L\}$, if

$$N_\ell := B_\ell \cap \{\rho:\ \exists\, t\in(\tilde{t}-\tfrac{\tau}{2},\tilde{t}+\tfrac{\tau}{2}),\quad d(SN^*H,\varphi_t(\rho))<r\},$$

then there is $\rho_\ell\in N_\ell$ such that

$$N_\ell\subset\{\rho\in SN^*H:\ d(\rho,\rho_\ell)<c_2^2(\ln(c_4r^{-1}))^{-2}\}.$$

In particular, since $\{\Lambda^\tau_{\rho_j}(r_1)\}_{j=1}^{N_{r_1}}$ is a $(\mathfrak{D}_n,\tau,r_1)$ good cover for SN^*H and $r_1<r$ there exists $C_n>0$ so that for each $\ell\in\{1,\dots,L\}$,

$$\#\Big\{\rho_j:\ \Lambda^\tau_{\rho_j}(r_1)\cap B_\ell\neq\emptyset,\ \bigcup_{|t-\tilde{t}|<\frac{\tau}{2}}\varphi_t(\Lambda^\tau_{\rho_j}(r_1))\cap\Lambda^\tau_{SN^*H}(r_1)\neq\emptyset\Big\}\leq C_nc_2^2\ln(\tfrac{c_4}{r})^{-2}r_1^{-1}.$$

The claim in (5.50) follows from taking the union in ℓ over all the balls B_ℓ.

Finally, let $\varepsilon>0$ and $\delta>0$ with $\varepsilon<\delta$. Also, set $r=h^\varepsilon$, $r_1=8h^\delta$ and

$$T_0=\gamma\log h^{-1}-\beta,\qquad 0<\gamma<\tfrac{\delta-\varepsilon}{\Lambda_{\max}},\qquad \beta<-\tfrac{\log C}{\Lambda_{\max}}.$$

We have obtained a splitting of $\{1,\dots,N_h\}$ into $\mathcal{B}\cup\mathcal{G}$ with the tubes in $\mathcal{G}$ being $[T,T_0]$ non-self looping and such that

$$|\mathcal{B}|\leq C\frac{T_0}{T}(\varepsilon\ln c_4h^{-1})^{-2}h^{-\delta}.$$

Using this cover in Theorem 5 completes the proof of Theorem 4 part 4.C since $\frac{T_0}{T}\leq\log h^{-1}$ and hence $h^\delta|\mathcal{B}|\leq\frac{C}{\log h^{-1}}$ for some $C>0$ and h small enough.

To see that (5.47) holds, let $s\mapsto\rho_s=(x(s),\xi(s))\in SN^*H$ be a smooth map, where $x(s)$ parametrizes H with $|\dot{x}(s)|_g=1$ and $\langle\dot{\xi}(s),\xi(s)\rangle=0$ for all s. Next, let $\Gamma(s,t)=\pi(\varphi_t(\rho_s))$ so that $t\mapsto\Gamma(s,t)$ is a geodesic with $\langle\partial_t\Gamma(s,t),\dot{x}(s)\rangle_g=0$ and $\Gamma(s,0)=x(s)$.

In particular, if we let

$$Y(t)=\partial_s\Gamma(s,t)|_{s=0},$$

then $Y(t)$ is a Jacobi field along γ_0 with $Y(0)=\dot{x}(0)$ and

$$\tfrac{D}{dt}Y(0)=\tfrac{D}{ds}\partial_t\Gamma(s,t)\Big|_{(0,0)}=0.$$

Indeed, observe that the angle between $\partial_t\Gamma(s,t)|_{t=0}$ and $\dot{x}(s)$ is constant and $|\partial_t\Gamma(s,t)|_g=1$. Therefore, since $x(s)$ is a unit speed geodesic, $\frac{D}{ds}\partial_t\Gamma(s,t)|_{t=0}=0$ and hence $\frac{D}{dt}Y(0)=0$.

Now, let $\gamma_0^\perp(t)$ be a parallel vector field along $\gamma_0(t)$ with $\langle\dot{\gamma_0}(t),\gamma_0^\perp(t)\rangle_g=0$ and $|\gamma_0^\perp(t)|_g=1$, we then have $Y(t)=J(t)\gamma_0^\perp(t)$ with $J(0)=1$, $J'(0)=0$, and

$$J''(t)+R(t)J(t)=0.$$

Since, $R(t)\leq 0$ and $J''(t)\geq 0$,

$$J(t)\geq 1.$$

In particular,

$$\partial_s(\pi\circ\varphi_t(\rho_s))|_{s=0}=d(\pi\circ\varphi_t)|_{\rho_0}\partial_s\rho_s|_{s=0}=Y(t),$$

and hence

$$d(\pi\circ\varphi_t(\rho_s),\exp_{\pi\circ\varphi_t(\rho_0)}(sY(t))\leq C_1e^{2\Lambda t}s^2.$$

Therefore, for $t \in [0, 4T]$,

$$d(\gamma_s(t), \exp_{\gamma_0(t)}(sY(t))) \leq C_1 e^{8\Lambda T} s^2.$$

Since $J(t) \geq 1$, it follows that Q_s contains $\Omega_{\tilde{\gamma}}(\frac{s}{4})$ for $s < \frac{1}{8C_1} e^{-8\Lambda T}$ where $\tilde{\gamma} := \{\gamma_{\frac{s}{2}}(t) : t \in [T, 2T]\}$. Therefore,

$$-\int_{Q_s} K dv_{\tilde{g}} \geq -\int_{\Omega_{\tilde{\gamma}}(\frac{s}{4})} K dv_{\tilde{g}} \geq c_1 e^{-c_2/\sqrt{s}},$$

as claimed. $\square$

Remark 11. We note that the proof of Theorem 3.B essentially shows that, while horospheres on M may not be positively curved everywhere, their curvature can only vanish at a fixed exponential rate.

Remark 12. This remark explains how Theorem 3.B implies the results of [SXZ17]. Note that the condition in [SXZ17] is that there are $c_1 > 0$, and $N > 0$ such that for every ball B_s in M of radius $s < 1$ one has $\int_{B_s} K \leq -c_1 s^N$. This remains true if we replace M by its universal cover, $\tilde{M}$, and implies that $\tilde{M}$ has non-positive curvature. To see that this condition implies those in Theorem 3.B, one needs to check that there is $c > 0$ such that $\int_{\Omega_\gamma(s)} K \leq -ce^{-\frac{1}{c\sqrt{s}}}$ where $\Omega_\gamma(s) := \{x \in \tilde{M} \mid d(x, \gamma) \leq s\}$. Now, observe that $\Omega_{\gamma(s)}$ contains at least one ball, B_s of radius s and hence, since $\tilde{M}$ has non-positive curvature,

$$\int_{\Omega_\gamma(s)} K \leq \int_{B_s} K \leq -c_1 s^N \ll -ce^{-\frac{1}{c\sqrt{s}}},$$

for some $c > 0$.

6. On vanishing of Jacobi of fields

This section is dedicated to the proof of Proposition 6.1 below. The proof of this proposition hinges on showing that given a geodesic $\gamma(t)$, if there is an r-dimensional vector space of perpendicular Jacobi fields along the geodesic that vanish at $\gamma(0)$ and that nearly vanish at $\gamma(t_0)$, then there must be r conjugate points to $\gamma(0)$ (counted with multiplicity) near $\gamma(t_0)$. See Lemma 6.4 for a precise statement of the required degree of vanishing. There, each $A(t)u_j$ denotes a Jacobi field.

In what follows $\pi : T^*M \to M$ is the natural projection and φ_t denotes the geodesic flow on S^*M.

Proposition 6.1. *Let $\Lambda > \Lambda_{\max}$. There exists $C > 0$ so that for any $t_0 \in \mathbb{R}$, $\rho \in S^*M$, and $0 < \varepsilon < \frac{1}{C} e^{-C\Lambda|t_0|}$, the following holds. If there are no more than m conjugate points to $\pi(\rho)$ (counted with multiplicity) along the geodesic $t \mapsto \pi(\varphi_t(\rho))$ for $t \in (t_0 - 2\varepsilon, t_0 + 2\varepsilon)$, then there is a subspace $\mathbf{V}_\rho \subset T_\rho S^*_x M$ of dimension $n - 1 - m$ so that for all $\mathbf{v} \in \mathbf{V}_\rho$,*

$$\|\mathbf{v}\| \leq C\varepsilon^{-1} e^{\Lambda|t_0|} \|d\pi \circ d\varphi_t \mathbf{v}\|, \qquad t \in (t_0 - \varepsilon, t_0 + \varepsilon).$$

In particular, $d\pi \circ d\varphi_t : \mathbf{V}_\rho \to T_{\pi\varphi_t(\rho)}M$ *is invertible onto its image with*

$$\|(d\pi \circ d\varphi_t)^{-1}\| \leq C\varepsilon^{-1}e^{\Lambda|t_0|},$$

for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$*.*

The proof of Proposition 6.1 can be found at the end of this section.

6.1. **Preliminaries on the Jacobi equation.** The argument relies on the fact that given $v \in T_\rho S_x M$ the vector field $Y(t) = d\pi \circ dT_t(v)$ is a Jacobi vector field along the geodesic $\gamma(t)$ in M whose initial conditions are given by ρ. Here, T_t denotes the geodesic flow on TM. Note that [Ebe73a, Proposition 1.7] gives $\|dT_t v\|^2 = \|Y(t)\|^2 + \|Y'(t)\|^2$ where $'$ denotes the covariant derivative of Y along γ.

Let $\{E_1(t), \dots, E_{n-1}(t)\}$ be a parallel orthonormal frame along a geodesic γ spanning the orthogonal complement of $E_n(t) := \gamma'(t)$. Then for $Y(t) = \sum_{i=1}^{n-1} y_i(t)E_i(t)$ a perpendicular vector field along γ, we identify Y with $t \mapsto (y_1(t), \dots, y_{n-1}(t))$. The covariant derivative of Y is then given by $t \mapsto (y_1'(t), \dots y_{n-1}'(t))$. Conversely, for each such curve in $\mathbb{R}^{n-1}$, there is a perpendicular vector field along γ. Now, for $t \in \mathbb{R}$, we define a symmetric $(n-1) \times (n-1)$ matrix $R(t) = (R_{ij}(t))$ where

$$R_{ij}(t) = \langle R(E_n(t), E_i(t))E_n(t), E_j(t)\rangle_{g(\gamma(t))} \tag{6.1}$$

and $R(X, Y)$ denotes the curvature tensor. Then we consider the Jacobi equation

$$Y''(t) + R(t)Y(t) = 0. \tag{6.2}$$

Let $A(t) \in \mathbb{M}_{n-1\times n-1}$ solve (6.2) with

$$A(0) = 0, \qquad A'(0) = \mathrm{Id}\,. \tag{6.3}$$

Then, the perpendicular Jacobi fields on γ with $Y(0) = 0$ and $\|Y'(0)\| = 1$, are given by

$$Y(t) = A(t)v,$$

with $\|v\| = 1$. In particular, $A(t)$ is nonsingular if and only if $\gamma(0)$ is not conjugate to $\gamma(t)$ along γ (at time t).

Before proceeding further, we relate $d\varphi_t$ to $A(t)$. To do this, we introduce the horizontal and vertical decomposition of TM. Let $\pi : TM \to M$ be projection to the base. Then $d\pi : TTM \to TM$ has kernel equal to the *vertical subspace* of TTM. We define the *connection map*

$$\mathbf{K} : TTM \to TM$$

by the following procedure. Let $V \in TM$ and $v \in T_V(TM)$, let $Z : (-\varepsilon, \varepsilon) \to TM$ be a smooth curve with initial velocity v and position V. Let $\alpha = \pi \circ Z : (-\varepsilon, \varepsilon) \to M$ and define $\mathbf{K}(v) = Z'(0)$ where $Z'(0)$ denotes the covariant derivative of $Z(t)$ along α evaluated at $t = 0$. The kernel of $\mathbf{K}$ is called the *horizontal subspace*. The *Sasaki metric*, g_s, on TM is defined for $v, w \in T_V TM$ by

$$\langle v, w\rangle_{g_s(V)} := \langle d\pi v, d\pi w\rangle_{g(\pi(V))} + \langle \mathbf{K}v, \mathbf{K}w\rangle_{g(\pi(V))}.$$

Under the Sasaki metric, TTM decomposes into the orthogonal sum of the horizontal and vertical subspaces.

Define the map $^\sharp : T^*M \to TM$ and its inverse $^\flat : TM \to T^*M$ by

$$g(\rho^\sharp, W) = \rho(W), \qquad V^\flat(W) = g(V, W).$$

Next, we define a map ${}^\sharp : TT^*M \to TTM$ and its inverse ${}^\flat : TTM \to TT^*M$ as follows. Let $\rho(t) : (-\varepsilon, \varepsilon) \to T^*M$ be a smooth curve with initial velocity $\mathbf{v} \in T_\rho T^*M$. Then,

$$\mathbf{v}^\sharp = \frac{d}{dt}\Big|_{t=0} \rho^\sharp(t).$$

Similarly, let $V(t) : (-\varepsilon, \varepsilon) \to TM$ be a smooth curve with initial velocity $v \in T_q TM$. Then,

$$v^\flat = \frac{d}{dt}\Big|_{t=0} V^\flat(t).$$

Using these identifications, we define the *Sasaki metric on* T^*M, g_s^*, by

$$\langle \mathbf{v}, \mathbf{w} \rangle_{g_s^*} = \langle \mathbf{v}^\sharp, \mathbf{w}^\sharp \rangle_{g_s}.$$

Note also that

$$d\pi V^\flat = d\pi V.$$

The geodesic flow on TM, $T_t : TM \to TM$, is given by

$$T_t V := (\varphi_t V^\flat)^\sharp.$$

Now, if $v \in T_V TM$, then by [Ebe73a, Proposition 1.7]

$$Y_v(t) = d\pi \circ dT_t(v), \qquad Y_v'(t) = \mathbf{K} \circ dT_t(v)$$

where $Y_v(t)$ is the unique solution to (6.2) with $Y_v(0) = d\pi v$ and $Y_v'(0) = \mathbf{K}v$. In particular,

$$|dT_t v|_{g_s}^2 = |Y_v(t)|^2 + |Y_v'(t)|^2.$$

Finally, this implies that for $\mathbf{v} \in TT^*M$,

$$|d\varphi_t \mathbf{v}|_{g_s^*}^2 = |Y_{\mathbf{v}^\sharp}(t)|^2 + |Y_{\mathbf{v}^\sharp}'(t)|^2. \tag{6.4}$$

Lemma 6.2. *For all $x \in M$ and $\rho \in S_x^*M$ the map ${}^\sharp$ is an isomorphism from $T_\rho S_x^*M$ to the subspace of $T_{\rho^\sharp}SM$ consisting of vertical vectors v such that $\mathbf{K}v$ is perpendicular to $\gamma'(0)$ where $\gamma(t) = \pi \circ \varphi_t(\rho)$.*

Proof. Let $\mathbf{v} \in T_\rho S_x^*M$. Then $d\pi\mathbf{v} = 0$ and in particular $\mathbf{v}^\sharp$ is vertical. Let $\rho(s) : (-\varepsilon, \varepsilon) \to S_x^*M$ with velocity equal to $\mathbf{v}$ at 0 and $\rho(0) = \rho$. Then, using geodesic normal coordinates with $x = 0$, and $\rho = dx^1$, we have

$$\rho(t) = \sum_{i=1}^n \rho_i(t) dx^i$$

with $\rho_1(0) = 1$, and $\sum_{i=1}^n |\rho_i(t)|^2 = 1$. Therefore, $\sum_{i=1}^n 2\rho_i(0)\rho_i'(0) = 0$, and hence, since $\rho_i(0) = 0$ for $i = 2, \dots n$ and $\rho_1(0) = 1$, we have $\rho_1'(0) = 0$. Next, since $\pi \circ \rho(s) = x$, we have in geodesic normal coordinates at x that $\rho(t)^\sharp = \sum_{i=1}^n \rho_i(t)\partial_{x_i}$. In particular, since $\gamma(t) = (t, 0, \dots, 0)$,

$$\langle \mathbf{K}\mathbf{v}^\sharp, \gamma'(0) \rangle_{g(x)} = \partial_t \langle \rho^\sharp(t), \gamma'(0) \rangle_{g(x)}\big|_{t=0} = \partial_t \rho_1(s)|_{t=0} = \rho_1'(0) = 0.$$

Therefore, $\mathbf{K}\mathbf{v}^\sharp$ is perpendicular to $\gamma'(0)$.

Since $\dim T_\rho S_x^*M = n - 1$, the set of vectors in T_xM orthogonal to $\gamma'(0)$ has dimension $n - 1$, and ${}^\sharp$ is an isomorphism, this completes the proof of the lemma. $\square$

Now, fix $\rho \in S^*M$, and let $\gamma(t) := \pi(\varphi_t(\rho))$. Observe that by Lemma 6.2 for $\mathbf{v} \in T_\rho S^*_x M$, $d\pi \mathbf{v}^\sharp = 0$ and $\mathbf{K}\mathbf{v}^\sharp$ is perpendicular to $\gamma'(0)$. Therefore,

$$d\pi(d\varphi_t \mathbf{v})^\sharp = A(t)\mathbf{K}\mathbf{v}^\sharp, \qquad \mathbf{K}(d\varphi_t \mathbf{v})^\sharp = A'(t)\mathbf{K}\mathbf{v}^\sharp. \tag{6.5}$$

The next lemma shows that if $A(t)v$ is small, then $A'(t)v$ cannot be very small.

Lemma 6.3. *Let $\Lambda > \Lambda_{\max}$. Then there is $c > 0$ such that for all γ geodesic, $A(t)$ solving* (6.3)*, $t_0 \in \mathbb{R}$ and $v \in (\gamma'(0))^\perp$ such that $\|A(t_0)v\| \leq \frac{c}{2} e^{-\Lambda|t_0|}\|v\|$, we have*

$$\|A'(t_0)v\| \geq \tfrac{c}{2} e^{-\Lambda|t_0|}\|v\|.$$

Proof. Let $x = \gamma(0)$, $\rho = (x, \gamma'(0))^\flat$, and $v \in (\gamma'(0))^\perp$. Then, by Lemma 6.2, there exists $\mathbf{v} \in T_\rho S^*_x M$ such that $d\pi\mathbf{v}^\sharp = 0$, and $\mathbf{K}\mathbf{v}^\sharp = v$. In particular, by (6.5)

$$d\pi(d\varphi_t\mathbf{v})^\sharp = A(t)v, \qquad \mathbf{K}(d\varphi_t\mathbf{v})^\sharp = A'(t)v.$$

Since there exists $C > 0$ such that $\|(d\varphi_t)^{-1}\| \leq Ce^{\Lambda|t|}$ for all t, the maps $\flat$, $\sharp$ are isomorphisms, and (6.4) holds, there exists $c > 0$ such that

$$\|A(t)v\| + \|A'(t)v\| \geq ce^{-\Lambda|t|}\|v\|. \tag{6.6}$$

In particular, if $\|A(t)v\| \leq \frac{c}{2}e^{-\Lambda|t_0|}\|v\|$, the conclusion holds. $\square$

6.2. **Finding conjugate points.** The goal of this section is to prove that if there is a vector space $\mathcal{V}$ of dimension r such that $\|A(t_0)|_{\mathcal{V}}\|$ is small, then there are at least r conjugate points to $\gamma(0)$ (counted with multiplicity) near the point $\gamma(t_0)$. That is, we show that if there is an r-dimensional vector space consisting of perpendicular Jacobi fields along $\gamma(t)$ that vanish at $\gamma(0)$ and nearly vanish at $\gamma(t_0)$, then there are r conjugate points to $\gamma(0)$ (counted with multiplicity) near the point $\gamma(t_0)$.

Lemma 6.4. *There are $c, C > 0$ such that the following holds. Let γ be a geodesic and $A(t)$ solve* (6.3) *and suppose there are $t_0 \in \mathbb{R}$, $\{u_j\}_{j=1}^r \subset (\gamma'(0))^\perp$ orthonormal and $\beta_0 > 0$ such that*

$$\|A(t_0)u_j\| \leq \beta_0, \qquad \beta_0 \leq ce^{-(r+2)\Lambda|t_0|}.$$

Then, there exist $t_1, \dots, t_r \in \mathbb{R}\backslash\{0\}$ such that

$$\sum_{j=1}^r \dim\ker A(t_j) \geq r \qquad \textit{and} \qquad \max_j |t_j - t_0| < C\beta_0 e^{\Lambda|t_0|}.$$

To ease notation, for any t such that $A^{-1}(t)$ exists, we introduce the matrix

$$U(t) := A'(t)A^{-1}(t), \tag{6.7}$$

and note that $U(t)$ is symmetric for all such t [Ebe73a]. This matrix was also used by Green [Gre58] and Eberlein [Ebe73a, Ebe73b] in the case of no conjugate points, for which it exists for all $t \neq 0$ and solves a certain Ricatti equation.

Recall that in the Newton iteration algorithm for finding zeros of a function, f, one starts with x_0 where $f(x_0)$ is small, and searches for the zero by defining the sequence $x_{n+1} = x_n - \frac{f'(x_n)}{f(x_n)}$. Under appropriate conditions $x_n \to x_*$ and $f(x_*) = 0$.

In this section, we implement a Newton-type algorithm for finding non-zero solutions, (t_*, v_*), of the equation $A(t_*)v_* = 0$. The sequence $\{x_n\}_n$ is defined so that the linearization of f at x_n will be zero at x_{n+1}. In the same spirit, we start at some time t_0 where $\|A(t_0)|_{\mathcal{V}}\| \ll 1$ for some vector space $\mathcal{V}$ and look for solutions to

$$A(t_0)v - \lambda_0 A'(t_0)v = 0 \tag{6.8}$$

such that $|\lambda_0| \ll 1$ and $v \in \mathcal{V}$. Since we can rephrase the problem as solving $(\mathrm{Id} - \lambda U(t))A(t)v = 0$, the matrix $U(t)$ will be used to do this. In particular, finding solutions to (6.8) will amount to finding eigenvalues and eigenvectors for U. It is here that the self-adjointness of U plays a crucial role. After this step, we put $t_1 = t_0 - \lambda_0$ and repeat the process as in Newton iteration.

In the next lemma, we show that if $\|A(t_0)|_{\mathcal{V}}\| \ll 1$, for some r-dimensional vector space $\mathcal{V}$, then we can find r large eigenvalues of the matrix $U(t_0)$.

Lemma 6.5. *There is $C > 0$ such that the following holds. Let $t_0 \in \mathbb{R}$ and $\beta > 0$ such that $A(t_0)^{-1}$ exists, $C\beta e^{\Lambda|t_0|} < 1$, and there are $\{u_j\}_{j=1}^r \subset (\gamma'(0))^\perp \backslash \{0\}$ orthogonal with*

$$\max_j \frac{\|A(t_0)u_j\|}{\|u_j\|} \leq \beta.$$

Then, there exist eigenvalues $\{\lambda_j^{-1}\}_{j=1}^r$ of $U(t_0)$ with $\max_j |\lambda_j| \leq C\beta e^{\Lambda|\tilde{t}|}$ for all $|\tilde{t} - t_0| \leq 1$.

Proof. First, we check that $A(t_0)u$ is small for all $u \in \operatorname{span}\{u_1, \dots, u_r\}$. This follows since there exists $C_n > 0$ depending only on n such that

$$\Big\|A(t_0)\sum_{j=1}^r b_j u_j\Big\| \leq \beta \sum_{j=1}^r |b_j| \leq \beta C_n \Big\|\sum_{j=1}^r b_j u_j\Big\|.$$

In particular, provided $\beta C_n \leq \frac{c}{2} e^{-\Lambda|t_0|}$, by Lemma 6.3, we have

$$\frac{\|U(t_0)A(t_0)u\|}{\|A(t_0)u\|} = \frac{\|A'(t_0)u\|}{\|A(t_0)u\|} \geq \beta^{-1}C_n^{-1} c e^{-\Lambda|t_0|}.$$

We now apply the max-min principle to $U(t_0)$ using the fact that $A(t_0)$ applied to $\operatorname{span}\{u_1, \dots, u_r\}$ is an r dimensional vector space. That is, observe that if we order the eigenvalues of $U(t_0)$ as $|\lambda_1|^{-1} \geq |\lambda_2|^{-1} \geq \cdots \geq |\lambda_{n-1}|^{-1}$, then,

$$|\lambda_k|^{-2} = \max_{\mathcal{V}} \Big\{ \min\big\{ \tfrac{\|Av\|^2}{\|v\|^2} \,:\, v \in \mathcal{V}\big\} \,:\, \dim \mathcal{V} = k \Big\},$$

where the maximum is taken over all subspaces $\mathcal{V}$ of dimension k. Taking $\mathcal{V}_r = \operatorname{span}\{A(t_0)u_1, \dots, A(t_0)u_r\}$, $\dim \mathcal{V}_r = r$, and

$$\min\big\{ \tfrac{\|U(t_0)v\|^2}{\|v\|^2} \,:\, v \in \mathcal{V}_r\big\} \geq \beta^{-2}C_n^{-2}c^2 e^{-2\Lambda|t_0|}.$$

In particular,

$$|\lambda_j|^{-1} \geq \beta^{-1}C_n^{-1} c e^{-\Lambda|t_0|}, \qquad j = 1, \dots, r.$$

The bound can be rewritten as a bound in terms of $\tilde{t}$ by modifying the constant C. $\square$

The next lemma will be used to make steps in the Newton iteration. In particular, starting from time t_0, where $U(t_0)$ has large eigenvalues, we find a new time, $t_0 - s$, where $U(t_0 - s)$ has substantially larger eigenvalues.

Lemma 6.6. *There are $c, C > 0$ such that the following holds. Suppose that $A(t_0)^{-1}$ exists, $U(t_0)$ has eigenvalues $\{1/\lambda_j\}_{j=1}^r$ with $|\lambda_1| = \max_j |\lambda_j|$ and orthonormal eigenvectors $\{e_j\}_{j=1}^r$. Let $B \geq 0$ and $|s| \leq 2|\lambda_1|$ such that*

$$\max_j |s - \lambda_j| \leq B|\lambda_1|^3 \quad \text{and} \quad C(1+B)|\lambda_1|^3 \leq \tfrac{c}{2} e^{-2\Lambda|t_0|}.$$

Then, for $v \in \operatorname{span}\{A(t_0)^{-1}e_j\}_{j=1}^r$,

$$\|A(t_0 - s)v\| \leq C(1+B)|\lambda_1|^3 e^{\Lambda|t_0|}\|v\|.$$

Moreover, if $A(t_0 - s)^{-1}$ exists, $U(t_0 - s)$ has eigenvalues $\{1/\lambda_j(s)\}_{j=1}^r$ satisfying

$$|\lambda_j(s)| \leq C(1+B)|\lambda_1|^3 e^{2\Lambda|\tilde{t}|}, \qquad \text{for} \quad |\tilde{t} - t| \leq 1.$$

Proof. We claim that for all $w \in \operatorname{span}\{e_1, \dots e_r\}$ we have

$$\frac{\|U(t_0 - s)A(t_0 - s)A^{-1}(t_0)w\|}{\|A(t_0 - s)A^{-1}(t_0)w\|} \geq C^{-1}(1+B)^{-1}|\lambda_1|^{-3}e^{-2\Lambda|t_0|}. \tag{6.9}$$

This would complete the proof after an application of the max-min principle since $A(t_0 - s)A^{-1}(t_0)$ applied to $\operatorname{span}\{e_1, \dots, e_r\}$ is an r dimensional vector space. Note that (6.9) yields a bound on $|\lambda_j(s)|$ in terms of t_0. This can be rewritten as a bound in terms of $\tilde{t}$ by modifying the constant C.

Note that $U(t_0 - s)A(t_0 - s)A^{-1}(t_0)w = A'(t_0 - s)A^{-1}(t_0)w$ for $w \in \operatorname{span}\{e_1, \dots, e_r\}$. Therefore, by Lemma 6.3, proving (6.9) amounts to finding an upper bound on its denominator.

Given any $t \in \mathbb{R}$, a Taylor expansion near $s = 0$ combined with (6.2) yield that for all $v \in (\gamma'(0))^\perp$

$$A(t - s)v = A(t)v - sA'(t)v - \tfrac{s^2}{2}R(t)A(t)v + Q(t, s, v), \tag{6.10}$$

with $\|Q(t, s, v)\| \leq Cs^3 e^{\Lambda|t|}\|v\|$ for some $C > 0$ depending only on R (c.f. (6.1)).

Let $w = \sum_{j=1}^r b_j e_j$ for some $\{b_j\}_{j=1}^r \subset \mathbb{R}$ and set $v = A^{-1}(t_0)w$. Then, by (6.10)

$$A(t_0 - s)v = \sum_{j=1}^r \frac{\lambda_j - s}{\lambda_j} b_j e_j - \tfrac{1}{2}s^2 R(t_0)w + Q(t_0, s, v).$$

Next, using that $|\lambda_1| = \max_{1 \leq j \leq r} |\lambda_j|$ and orthogonality, there is $C > 0$ such that

$$\frac{1}{|\lambda_1|}\|w\| \leq \Big\| \sum_{j=1}^r \frac{b_j}{\lambda_j} e_j \Big\| = \big\|U(t_0)w\big\| \leq \|A'(t_0)\| \big\|A^{-1}(t_0)w\big\| \leq Ce^{\Lambda|t_0|}\|v\|.$$

Then,

$$\Big\| \sum_{j=1}^r \frac{\lambda_j - s}{\lambda_j} b_j e_j \Big\|^2 = \sum_{j=1}^r \frac{|\lambda_j - s|^2}{\lambda_j^2} b_j^2 \leq B^2|\lambda_1|^6 \sum_{j=1}^r \frac{b_j^2}{\lambda_j^2} \leq C^2 e^{2\Lambda|t_0|}B^2|\lambda_1|^6\|v\|^2.$$

In particular, together these imply $\|A(t_0 - s)v\| \leq (C + B)|\lambda_1|^3 e^{\Lambda|t_0|}\|v\|$. Thus, using Lemma 6.3, provided that $C(1+B)|\lambda_1|^3 e^{\Lambda|t_0|} \leq \frac{c}{2}e^{-\Lambda|t_0|}$, the claim in (6.9) holds. $\square$

The first step in proving Lemma 6.4 is to show that given u_0 such that $\|A(t_0)u_0\| \ll \|u_0\|$, we can find t near t_0 such that $\ker A(t) \neq \{0\}$. This lemma uses the simplest version of our Newton iteration scheme where we do not keep track of multiplicities.

Lemma 6.7. *There are $c, C > 0$ such that the following holds. Suppose that there are $t_0 \in \mathbb{R}$ and $u_0 \in (\gamma'(0))^{\perp} \setminus \{0\}$ such that*

$$\|A(t_0)u_0\| \leq \beta\|u_0\|, \qquad 0 \leq \beta \leq ce^{-3\Lambda|t_0|}. \tag{6.11}$$

Then, there exist $t \in \mathbb{R}$ such that

$$|t - t_0| \leq C\beta e^{\Lambda|t_0|} \qquad \text{and} \qquad \dim\ker A(t) \geq 1.$$

Proof. We assume by contradiction that $A(s)^{-1}$ exists if $|s - t_0| \leq C_1\beta e^{\Lambda|t_0|}$. Then, by Lemma 6.5, there is an eigenvector v_0 of $U(t_0)$ with eigenvalue λ_0^{-1} satisfying

$$|\lambda_0| \leq C\beta e^{\Lambda|t_0|}. \tag{6.12}$$

Let $t_1 := t_0 - \lambda_0$, $\lambda_{-1} := \beta^{1/3}e^{-\Lambda|t_0|/3}$, and assume we have found $(t_{k+1}, \lambda_k, v_k)$ for $k = 0, \dots, m$ such that $\|v_k\| = 1$,

$$t_{k+1} = t_k - \lambda_k, \qquad U(t_k)v_k = \lambda_k^{-1}v_k, \qquad |\lambda_k| \leq Ce^{2\Lambda|t_0|}|\lambda_{k-1}|^3. \tag{6.13}$$

By induction, one checks that

$$|\lambda_k| \leq \left(Ce^{2\Lambda|t_0|}\right)^{\sum_{\ell=0}^{k-1}3^\ell}\left(C\beta e^{\Lambda|t_0|}\right)^{3^k}, \qquad k = 1, \dots, m. \tag{6.14}$$

In particular,

$$|t_{m+1} - t_0| \leq \sum_{k=0}^{m}|t_{k+1} - t_k| \leq 2C\beta e^{\Lambda|t_0|} \leq 1.$$

Next, by Lemma 6.6 with $t_0 = t_m$, $s = \lambda_m$ and $B = 0$, there are (v_{m+1}, λ_{m+1}) such that $\|v_{m+1}\| = 1$, $U(t_{m+1})v_{m+1} = \lambda_{m+1}^{-1}v_{m+1}$, and

$$|\lambda_{m+1}| \leq Ce^{2\Lambda|t_0|}|\lambda_m|^3.$$

Finally, letting $t_{m+2} = t_{m+1} - \lambda_{m+1}$ completes the inductive step.

Therefore, for all $k \geq 0$ there are (t_k, λ_k, v_k) satisfying (6.13). In particular,

$$|t_k - t_0| \leq C\beta e^{2\Lambda|t_0|}.$$

Hence, there exists $t \in \mathbb{R}$ such that $t_k \to t$ and $|t - t_0| \leq C\beta e^{2\Lambda|t_0|}$. Next, note that

$$|\lambda_k|^{-1} = \|U(t_k)v_k\| = \|A'(t_k)A^{-1}(t_k)v_k\| \leq Ce^{\Lambda|t_k|}\|A^{-1}(t_k)v_k\| \leq Ce^{\Lambda|t_0|}\|A^{-1}(t_k)v_k\|.$$

In particular, since $|\lambda_k| \to 0$, we conclude $\|A(t_k)^{-1}v_k\| \to \infty$. On the other hand, by assumption $A(t)$ is invertible and hence, there exists $C > 0$ and an open interval I around t such that

$$\|A(s)^{-1}\| \leq C < \infty, \qquad s \in I,$$

which gives a contradiction if we choose C_1 large enough.

$\square$

In the proof of Lemma 6.4, we will induct on the number of times at which $A(t)$ is not invertible in a small neighborhood of the time t_0 where $\|A(t_0)|_{\mathcal{V}}\| \ll 1$. To begin, we implement Newton iteration to handle the case when we apriori have at most one such time and control the multiplicity of the conjugate point at that time in terms of $\dim \mathcal{V}$.

Lemma 6.8. *There are $c, C > 0$ such that the following holds. Let $\beta_0 > 0$ and $t_0 \in \mathbb{R}$ with $\beta_0 \leq ce^{-3\Lambda|t_0|}$. Suppose there exists $t_* \neq t_0$ so that*

$$A(t) \text{ is invertible for } \ t \neq t_* \ \text{ with } \ |t - t_0| \leq 2C\beta_0 e^{\Lambda|t_0|},$$

and that there are $\{u_j\}_{j=1}^r$ orthogonal such that $\|A(t_0)u_j\| \leq \beta_0\|u_j\|$ for $j = 1, \dots, r$. Then,

$$\dim\ker(A(t_*)) \geq r.$$

Proof. By Lemma 6.5, since $C\beta_0 e^{\Lambda|t_0|} < 1$, $U(t_0)$ has eigenvalues $\{\lambda_{0,j}^{-1}\}_{j=1}^r$ such that

$$|\lambda_{0,j}| \leq C\beta_0 e^{\Lambda|t_0|}.$$

Let $\{e_{0,j}\}_{j=1}^r$ be the eigenvectors of $U(t_0)$ with eigenvalues $\{1/\lambda_{0,j}\}_{j=1}^r$. Here, we set $\lambda_{0,j} = \lambda_j$ for all $j = 1, \dots, r$. Note that, by Lemma 6.6, for all $j = 1, \dots, r$

$$\|A(t_0 - \lambda_{0,j})A^{-1}(t_0)e_{0,j}\| \leq C^4\beta_0^3 e^{4\Lambda|t_0|}\|A^{-1}(t_0)e_{0,j}\|.$$

Then, by Lemma 6.7 there are $t \in \mathbb{R}$ and $w \in (\gamma'(0))^\perp \setminus \{0\}$ such that $A(t)w = 0$ and $\max_j |t - t_0 + \lambda_{0,j}| \leq C^5\beta_0^3 e^{5\Lambda|t_0|}$. In particular, since $|t - t_0| \leq 2\beta_0$, we have must have $t = t_*$ and so

$$|t_0 - t_*| \leq C\beta_0 e^{\Lambda|t_0|} + C^5\beta_0^3 e^{5\Lambda|t_0|}.$$

Set $\beta_{-1} := (\beta_0(C^3(1 + 2C^2e^{2\Lambda|t_0|}))^{-1}e^{-4\Lambda|t_0|})^{1/3}$. Let $m \geq 0$ and for $0 \leq k \leq m$ suppose that we have found $(t_k, \{\lambda_{k,j}\}_{j=1}^r, \beta_k)$ such that

(1) $U(t_k)$ has eigenvalues $\{\lambda_{k,j}^{-1}\}_{j=1}^r$ with $\max_j |\lambda_{k,j}| \leq C\beta_k e^{\Lambda|t_0|}$,
(2) $A(t)$ is invertible on $I(t_k, \beta_k) \setminus \{t_*\}$,
(3) $0 < |t_k - t_*| \leq C\beta_k e^{\Lambda|t_0|} + C^5\beta_k^3 e^{5\Lambda|t_0|}$,
(4) $\beta_k \leq C^3(1 + 2C^2e^{2\Lambda|t_0|})\beta_{k-1}^3 e^{4\Lambda|t_0|}$,

where

$$I(t_k, \beta_k) := (t_k - 2C\beta_k e^{\Lambda|t_0|}, t_k + 2C\beta_k e^{\Lambda|t_0|}).$$

Then, for each $0 \leq k \leq m$ let $\{e_{k,j}\}_{j=1}^r$ be the eigenvectors of $U(t_k)$ with eigenvalues $\{1/\lambda_{k,j}\}_{j=1}^r$. Note that, by Lemma 6.6 with $B = 0$,

$$\|A(t_k - \lambda_{k,j})A^{-1}(t_k)e_{k,j}\| \leq C|\lambda_{k,j}|^3 e^{\Lambda|t_0|}\|A^{-1}(t_k)e_{k,j}\|.$$

Thus, by Lemma 6.7 there are $t \in \mathbb{R}$ and $w \in (\gamma'(0))^\perp \setminus \{0\}$ such that $A(t)w = 0$ and $|t - t_k + \lambda_{k,j}| \leq C^2|\lambda_{k,j}|^3 e^{2\Lambda|t_0|}$ for $j = 1, \dots, r$. In particular, since $t \in I(t_k.\beta_k)$, we must have $t = t_*$ and so

$$\max_j |t_* - t_k + \lambda_{k,j}| \leq C^2 e^{2\Lambda|t_0|}|\lambda_{k,j}|^3 \qquad \text{and} \qquad \max_{j,\ell} |\lambda_{k,j} - \lambda_{k,\ell}| \leq 2C^5\beta_k^3 e^{5\Lambda|t_0|}. \tag{6.15}$$

Next, we define $t_{k+1} \in \mathbb{R}$ such that

$$0 < |t_* - t_{k+1}| \leq C^2|\lambda_{k,1}|^3 e^{2\Lambda|t_0|}, \tag{6.16}$$

where $\lambda_{k,1}$ is chosen so that $\max_j |\lambda_{k,j}| = |\lambda_{k,1}|$. Then, with $s_k = t_k - t_{k+1}$,

$$\max_j |s_k - \lambda_{k,j}| = \max_j |t_* - t_{k+1} + t_k - t_* - \lambda_{k,j}| \leq 2C^2 e^{2\Lambda|t_0|}|\lambda_{k,1}|^3.$$

Thus, we may apply Lemma 6.6 with $B = 2C^2e^{2\Lambda|t_0|}$, $s = s_k$, and $t_0 = t_k$ to obtain that $U(t_{k+1})$ has eigenvalues $\{1/\lambda_{k+1,j}\}_{j=1}^r$ satisfying

$$|\lambda_{k+1,j}| \leq C(1 + 2C^2e^{2\Lambda|t_0|})|\lambda_{k,1}|^3 e^{2\Lambda|t_0|} \leq C\beta_{k+1}e^{\Lambda|t_0|},$$

where we set $\beta_{k+1} := C^3(1 + 2C^2e^{2\Lambda|t_0|})\beta_k^3 e^{4\Lambda|t_0|}$.

Next, we claim that A is invertible on $I(t_{k+1}, \beta_{k+1})\backslash\{t_*\}$. Indeed, for $t \in I(t_{k+1}, \beta_{k+1})$, assumptions (3) and (4) in the induction hypotheses and (6.16) yield, since $|t - t_k| \leq |t - t_{k+1}| + |t_* - t_k| + |t_* - t_{k+1}|$,

$$|t - t_k| < 2C\beta_{k+1}e^{\Lambda|t_0|} + C\beta_k e^{\Lambda|t_0|} + 2C^5\beta_k^3 e^{5\Lambda|t_0|} \leq 2C\beta_k e^{\Lambda|t_0|}.$$

Therefore, $I(t_{k+1}.\beta_{k+1}) \subset I(t_k.\beta_k)$ and hence A is invertible on $I(t_{k+1}.\beta_{k+1}) \setminus \{t_*\}$.

Thus, by induction, there are $(t_k, \{\lambda_{k,j}\}_{j=1}^r, \beta_k)$ such that (1)-(4) above hold. In particular, $\beta_k \to 0$, $t_k \to t_*$, and, by (6.15), we may choose $\tilde{t}_k \in I(t_k.\beta_k)$ such that $A(\tilde{t}_k)$ is invertible and

$$\max_j |\tilde{t}_k - t_k + \lambda_{k,j}| \leq 2C^2e^{2\Lambda|t_0|}|\lambda_{k,1}|^3.$$

Note that $\tilde{t}_k \to t_*$ and by Lemma 6.6 (with $t_0 = t_k$, $s = t_k - \tilde{t}_k$, and $B = 2C^2e^{2\Lambda|t_0|}$), for $v \in \mathcal{V}_k := \operatorname{span}\{A(t_k)^{-1}e_{k,j}\}_{j=1}^r$,

$$\begin{aligned} \|A(\tilde{t}_k)v\| &\leq C(1 + 2C^2e^{2\Lambda|t_0|})|\lambda_{k,1}|^3 e^{\Lambda|t_0|}\|v\| \\ &\leq C^4(1 + 2C^2e^{2\Lambda|t_0|})\beta_k^3 e^{4\Lambda|t_0|}\|v\|. \end{aligned} \tag{6.17}$$

Choosing any orthonormal basis $\{v_{k,1}, \ldots, v_{k,r}\}$ for $\mathcal{V}_k$ we may extract a convergent subsequence $\{v_{k_\ell,j}\}_\ell$ such that

$$\lim_{\ell\to\infty} v_{k_\ell,j} = v_j$$

for all $j = 1, \ldots, r$, and where $\{v_j\}_{j=1}^r \subset (\gamma'(0))^\perp$ are orthonormal vectors. Since the map $t \mapsto A(t)$ is continuous, and by (6.17) $\lim_{\ell\to\infty} \|A(\tilde{t}_{k_\ell})v_{k_\ell,j}\| = 0$ for all $j = 1, \ldots, r$, we conclude

$$A(t_*)v_j = 0, \qquad j = 1, \ldots, r,$$

and hence $\dim\ker A(t_*) \geq r$.

$\square$

We now prove Lemma 6.4. The problem that needs to be addressed is that Lemma 6.8 only applies when there is a single time, t_*, in an interval proportional to the smallness of $\beta := \|A(t_0)|_{\mathcal{V}}\|$ such that $A(t_*)$ is not invertible. To explain how to handle this, first observe that there are at most $n-1$ times t_i in a small interval around t_0 such that $A(t_i)$ is not invertible. If these times are well separated relative to β, then we can apply Lemma 6.8. If they are not, however, we note that there *is* some $\tilde{\beta} > 0$ such that the times t_i are well-separated relative to $\tilde{\beta}$. Therefore, our main goal in the proof below is to effectively decrease β by finding new times, $s_0 - \lambda_{0,j}$, in the original interval at which $\|A(s_0-\lambda_{0,j})|_{\mathcal{V}_j}\| \lesssim \beta^3 \ll \beta$ with $\sum_j \dim \mathcal{V}_j \geq \dim \mathcal{V}$. We then regroup the times t_i at scale $\sim \beta^3$ to finish the proof.

Proof of Lemma 6.4. Let C be the maximum of the constants C found in Lemmas 6.4, 6.5, 6.6, 6.7, and 6.8. Similarly, we let c be the minimum of all the constants c given by the same lemmas. To ease the presentation, for $t,\beta>0$ we again write

$$I(t,\beta) := (t-2C\beta e^{\Lambda|t_0|}, t+2C\beta e^{\Lambda|t_0|}).$$

If there are $\{t_i\}_{i=1}^r \subset I(t_0,\frac{1}{2}\beta_0)$ with $t_i\neq t_j$, $i\neq j$ such $A(t_i)$ is not invertible, then the proof is complete since $\dim\ker A(t_i)\geq 1$. Therefore, we may assume there are at most $r-1$ such times.

For $t\in\mathbb{R}$, $\beta>0$, $l\in\mathbb{N}$, $\{t_i\}_{i=1}^l\subset\mathbb{R}$ we introduce the following statements:

- $\mathcal{P}(t,\beta,l,\{t_i\}_{i=1}^l)$ is the statement: If A is invertible on $I(t,\beta)\setminus\{t_i\}_{i=1}^l$ and there are $\{u_j\}_{j=1}^r$ orthonormal with $\max_{1\leq j\leq r}\|A(t)u_j\|\leq\beta$, then $\sum_{i=1}^l \dim\ker A(t_i)\geq r$.
- $\mathcal{P}(t,\beta,l)$ is the statement: $\mathcal{P}(t,\beta,l,\{t_i\}_{i=1}^l)$ holds for all collections $\{t_i\}_{i=1}^l\subset I(t,\frac{1}{2}\beta)$ with t_i distinct.

The goal is to prove that there is $c_r>0$ such that $\mathcal{P}(t,c_re^{-(r+1)\Lambda|t|},r-1)$ holds for all t since this would yield the lemma. To do this, first note $\mathcal{P}(t,ce^{-3\Lambda|t|},1)$ holds for all t by Lemma 6.8.

Step 1. Fix (t,β) and a collection of distinct times $\{t_i\}_{i=1}^k\subset I(t,\frac{1}{2}\beta)$. In this step we prove that $\mathcal{P}(t,\beta,k,\{t_i\}_{i=1}^k)$ holds if the times $\{t_i\}_{i=1}^k$ are well-separated relative to β^3. That is, if

$$\min_{i_1\neq i_2}|t_{i_1}-t_{i_2}|\geq 27C^5(1+2C^2e^{2\Lambda|t_0|})\beta^3e^{5\Lambda|t_0|}. \tag{6.18}$$

Let $s\in\mathbb{R}$ such that

$$|s-t|<c_0\beta e^{-\Lambda|t_0|}, \qquad A(s)^{-1}\ \text{ exists.} \tag{6.19}$$

Then, since $\max_{1\leq j\leq r}\|A(t)u_j\|\leq\beta$ and $\|A'(t)\|\leq Ce^{\Lambda|t_0|}$, for c_0 small eneough,

$$\max_{1\leq j\leq r}\|A(s)u_j\|\leq\beta+C|s-t|e^{\Lambda|t_0|}\leq\tfrac{3}{2}\beta.$$

In particular, by Lemma 6.5, $U(s)$ has eigenvalues $\{\lambda_j^{-1}\}_{j=1}^r$ with orthonormal eigenvectors $\{e_j\}_{j=1}^r$ such that

$$|\lambda_1|=\max_{1\leq j\leq r}|\lambda_j|\leq\tfrac{3}{2}C\beta e^{\Lambda|t_0|}. \tag{6.20}$$

If β is small enough so that (6.18) holds, we have that (6.20) implies

$$\min_{i_1\neq i_2}|t_{i_1}-t_{i_2}|\geq 4C\beta_1e^{\Lambda|t_0|}, \qquad \beta_1:=C(1+2C^2e^{2\Lambda|t_0|})|\lambda_1|^3e^{\Lambda|t_0|}. \tag{6.21}$$

Next, observe that for all $j=1,\dots,r$, by Lemma 6.6 (with $t_0=s$, $B=0$ and $s=\lambda_j$),

$$\|A(s-\lambda_j)A^{-1}(s)e_j\|\leq C|\lambda_1|^3e^{\Lambda|t_0|}\|A^{-1}(s)e_j\|,$$

where $\{e_j\}_{j=1}^r$ are orthonormal eigenvectors for $U(s)$. Then, we apply Lemma 6.7 to obtain $\{\tilde{t}_j\}_{j=1}^r$ such that $A(\tilde{t}_j)$ is not invertible and

$$\max_j |\tilde{t}_j - s + \lambda_j| \le C^2|\lambda_1|^3 e^{2\Lambda|t_0|}. \tag{6.22}$$

In particular, this yields $\max_{1\le j\le r}|s-\tilde{t}_j| \le C^2|\lambda_1|^3e^{2\Lambda|t_0|} + \max_{1\le j\le r}|\lambda_j|$, and so by (6.19) and (6.20),

$$\max_{1\le j\le r}|t-\tilde{t}_j| \le c\beta e^{-\Lambda|t_0|} + \tfrac{27}{8}C^5\beta^3e^{5\Lambda|t_0|} + \tfrac{3}{2}C\beta e^{\Lambda|t_0|} < 2C\beta e^{\Lambda|t_0|},$$

where for last inequality we may have shrunk c in terms of how it compares to C. Since this shows that $\{\tilde{t}_j\}_{j=1}^r \in I(t,\beta)$ and we know that $A(\tilde{t}_j)$ is not invertible, we must have $\{\tilde{t}_j\}_{j=1}^r \subset \{t_i\}_{i=1}^k$. Note that then (6.22) translates to

$$\max_{1\le j\le r}\min_{1\le i\le k}|s-\lambda_j - t_i| \le C^2|\lambda_1|^3e^{2\Lambda|t_0|}. \tag{6.23}$$

Next, observe that A is invertible on $I(t,\beta)\setminus\{t_i\}_{i=1}^k$ by assumption, and that then (6.18) implies that A is invertible on $I(t_\ell, 2\beta_1)\setminus\{t_i\}_{i=1}^k$ for all $\ell = 1,\dots,k$. Next, for $i = 1,\dots,k$ let s_i such that

$$0 < |s_i - t_i| < C^2|\lambda_1^3|e^{2\Lambda|t_0|}.$$

Since $I(s_i,\beta_1)\subset I(t_i, 2\beta_1)$ we have

$$A \;\text{ is invertible on }\; \bigcup_{i=1}^k I(s_i,\beta_1)\setminus\{t_i\}. \tag{6.24}$$

Then, for $i = 1,\dots,k$ there are $\{r_i\}_{i=1}^k$ with $\sum_{i=1}^k r_i = r$ and distinct $\{j_{i,\ell}\}_{\ell=1}^{m_i}$ with $\bigcup_{i=1}^k\{j_{i,\ell}\}_{\ell=1}^{r_i} = \{1,\dots,r\}$, such that the definition of s_i together with (6.23) yield that for all $1\le i\le k$ and $1\le \ell\le r_i$

$$|s_i - s + \lambda_{j_{i,\ell}}| \le 2C^2|\lambda_1|^3e^{2\Lambda|t_0|}.$$

Therefore, for each $1\le i\le k$, by Lemma 6.6 (with $t_0 = s$, $s = s_0 - s_i$ and $B = 2C^2e^{2\Lambda|t_0|}$),

$$\|A(s_i)v\| \le \beta_1\|v\|,$$

for $v \in \operatorname{span}\{A^{-1}(s)e_{j_{i,\ell}}\}_{\ell=1}^{r_i}$. Next, using (6.24) we apply Lemma 6.8 (with $t = t_i$, $t_0 = s_i$, $\beta_0 = \beta_1$) to obtain

$$\dim\ker(A(t_i)) \ge r_i.$$

Then, since $\sum_{i=1}^k r_i = r$, the statement $\mathcal{P}(t,\beta,k,\{t_i\}_{i=1}^k)$ holds provided we had that β is small enough that (6.18) is true.

Step 2. In this step we show how to deal with times $\{t_i\}_{i=1}^k$ that may not be well-separated.

Suppose that $0<\beta<ce^{-(k+2)\Lambda|t_0|}$ and $(t,\beta,\{t_i\}_{i=1}^k)$ are such that the hypothesis of $\mathcal{P}(t,\beta,k,\{t_i\}_{i=1}^k)$ hold and $\{t_i\}_{i=1}^k\subset I(t,\frac{1}{2}\beta)$. We claim that there exist a collection of indices $\{i_\alpha\}_{\alpha=1}^m\subset\{1,\dots k\}$, indices $\{\ell_{i_\alpha}^\infty\}_{\alpha=1}^m\subset\{1,\dots k\}$, times $\{t_{i_\alpha}^\infty\}_\alpha\subset$

$I(t, \frac{1}{2}\beta)$, numbers $\{\beta_{\ell^\infty_{i_\alpha}}\}_\alpha \subset (0, \frac{1}{2}\beta)$, non-empty disjoint sets $\mathcal{I}_{i_\alpha} \subset \{1, \dots k\}$, sets $\mathcal{J}_{i_\alpha} \subset \{1, \dots, r\}$, and intervals

$$U^\infty_{i_\alpha} = I(t^\infty_{i_\alpha}, \beta_{\ell^\infty_{i_\alpha}}), \qquad U^\infty_{i_{\alpha_1}} \cap U^\infty_{i_{\alpha_2}} = \emptyset \quad \alpha_1 \neq \alpha_2, \tag{6.25}$$

satisfying

$$\{1, \dots, k\} = \bigcup_{\alpha=1}^m \mathcal{I}^\infty_{i_\alpha}, \qquad \{1, \dots, r\} = \bigcup_{\alpha=1}^m \mathcal{J}^\infty_{i_\alpha}, \tag{6.26}$$

and such that $\{t_i\}_{i \in \mathcal{I}^\infty_\alpha} \subset I(t^\infty_{i_\alpha}, \frac{1}{2}\beta_{\ell^\infty_{i_\alpha}})$, A is invertible on $U^\infty_{i_\alpha} \setminus \{t_i\}_{i \in \mathcal{I}^\infty_\alpha}$, and for all $v \in \operatorname{span}\{A^{-1}(s)e_j\}_{j \in \mathcal{J}^\infty_{i_\alpha}}$

$$\|A(t^\infty_{i_\alpha})v\| \leq \beta_{\ell^\infty_{i_\alpha}} \|v\|.$$

Let s and $\{\lambda_j\}_j$ be built as in the discussion leading to (6.20). Then, with β_1 as in (6.21) and for $i_0 \in \{1, \dots, k\}$ define $U^1_{i_0} := I(t_{i_0}, 3\beta_1)$,

$$\mathcal{I}^1_{i_0} := \{i \mid t_i \in U^1_{i_0}\}, \qquad \mathcal{J}^1_{i_0} := \{j \mid \min_{i \in \mathcal{I}^1_{i_0}} |s - \lambda_j - t_i| \leq C^2 |\lambda_1|^3 e^{2\Lambda|t_0|}\}. \tag{6.27}$$

If $\mathcal{I}^1_{i_0} = \{i_0\}$, then A is invertible on $I(t_{i_0}, \beta_1) \setminus \{t_{i_0}\}$ and for all $v \in \operatorname{span}\{A^{-1}(s)e_j\}_{j \in \mathcal{J}^1_{i_0}}$,

$$\|A(t_{i_0})v\| \leq \beta_1 \|v\|.$$

We then define $\ell^\infty_{i_0} = 1$, $t^\infty_{i_0} = t_{i_0}$,

$$U^\infty_{i_0} := I(t_{i_0}, \beta_{\ell^\infty_{i_0}}), \qquad \mathcal{J}^\infty_{i_0} := \mathcal{J}^1_{i_0}, \qquad \mathcal{I}^\infty_{i_0} := \mathcal{I}^1_{i_0}.$$

If $\{i_0\} \subsetneq \mathcal{I}^1_{i_0}$, let $\bar{t}^1_{i_0} := \frac{1}{|\mathcal{I}^1_{i_0}|} \sum_{i \in \mathcal{I}^1_{i_0}} t_i \in I(t, \frac{1}{2}\beta)$. Then, by Lemma 6.6 (with $t_0 = s$, $B = C^2 e^{2\Lambda|t_0|} + 12Ce^{\Lambda|t_0|} \frac{\beta_1}{|\lambda_1|^3}$), for all $v \in \operatorname{span}\{A^{-1}(s)e_j\}_{j \in \mathcal{J}^1_{i_0}}$,

$$\|A(\bar{t}^1_{i_0})v\| \leq \beta_2 \|v\|, \qquad \beta_2 := C\Big(1 + C^2 e^{2\Lambda|t_0|} + 12Ce^{\Lambda|t_0|} \frac{\beta_1}{|\lambda_1|^3}\Big) |\lambda_1|^3 e^{\Lambda|t_0|}.$$

Next, let $U^2_{i_0} := I(\bar{t}^1_{i_0}, 3\beta_2)$ and define $\mathcal{J}^2_{i_0}, \mathcal{I}^2_{i_0}$ as in (6.27). Note that $\mathcal{I}^1_{i_0} \subset \mathcal{I}^2_{i_0}$.

If $\mathcal{I}^2_{i_0} = \mathcal{I}^1_{i_0}$, then, since $\beta_2 \geq 6\beta_1$, A is invertible on $I(\bar{t}^1_{i_0}, \beta_2) \setminus \{t_i\}_{i \in \mathcal{I}^2_{i_0}}$, and $\{t_i\}_{i \in \mathcal{I}^2_{i_0}} \subset I(\bar{t}^1_{i_0}, \frac{\beta_2}{2})$. We let $\ell^\infty_{i_0} = 2$, $t^\infty_{i_0} = \bar{t}^1_{i_0}$,

$$U^\infty_{i_0} := I(\bar{t}^1_{i_0}, \beta_2), \qquad \mathcal{I}^\infty_{i_0} := \mathcal{I}^2_{i_0}, \qquad \mathcal{J}^\infty_{i_0} := \mathcal{J}^2_{i_0}.$$

Otherwise, we continue the process until we find $\mathcal{I}^\ell_{i_0} = \mathcal{I}^{\ell-1}_{i_0}$ for some ℓ and set $\ell^\infty_{i_0} = \ell$, $t^\infty_{i_0} = \bar{t}^{\ell-1}_{i_0} \in I(t, \frac{1}{2}\beta)$,

$$U^\infty_{i_0} := I(\bar{t}^{\ell-1}_{i_0}, \beta_\ell), \qquad \mathcal{I}^\infty_{i_0} := \mathcal{I}^\ell_{i_0}, \qquad \mathcal{J}^\infty_{i_0} := \mathcal{J}^\ell_{i_0}.$$

Note that for all i_0, $\ell^\infty_{i_0} \leq k$.

Next, we claim that if i_1, i_2 are such that $U^\infty_{i_1} \cap U^\infty_{i_2} \neq \emptyset$, then

$$\mathcal{I}^\infty_{i_1} \subset \mathcal{I}^\infty_{i_2} \qquad \text{or} \qquad \mathcal{I}^\infty_{i_2} \subset \mathcal{I}^\infty_{i_1}. \tag{6.28}$$

Indeed, suppose $U^\infty_{i_1} \cap U^\infty_{i_2} \neq \emptyset$. Without loss, assume $\ell^\infty_{i_2} \geq \ell^\infty_{i_1}$. Then, $\beta_{\ell^\infty_{i_2}} \geq \beta_{\ell^\infty_{i_1}}$, and so $U^\infty_{i_1} \subset I(t^\infty_{i_2}, 3\beta_{\ell^\infty_{i_2}})$. In particular, since $\mathcal{I}^{\ell^\infty_{i_1}}_{i_1} = \{i \mid t_i \in U^\infty_{i_1}\}$ and $\mathcal{I}^{\ell^\infty_{i_2}}_{i_2} = \{i \mid t_i \in U^\infty_{i_2}\}$, we have $\mathcal{I}^\infty_{i_2} \supset \mathcal{I}^\infty_{i_1}$, proving the claim in (6.28).

From the claim in (6.28) it follows that there exist $1 \le m \le k$ and $\{i_\alpha\}_{\alpha=1}^m \subset \{1,\dots,k\}$ such that (6.25) and (6.26) hold.

To prove that $\beta_\ell < \frac{1}{2}\beta$, we actually show that for all ℓ,

$$\beta_\ell \le C^{2(\ell-1)} 13^{\ell-1} e^{(2(\ell-1)+1)\Lambda|t_0|}(1 + C^2 e^{2\Lambda|t_0|})|\lambda_1|^3. \tag{6.29}$$

This implies $\beta_{\ell^\infty_{i_\alpha}} < \frac{1}{2}\beta$ since $\ell^\infty_{i_\alpha} \le k$, $|\lambda_1| \le 2C\beta e^{\Lambda|t_0|}$, and we are assuming $\beta < ce^{-(k+2)\Lambda|t|}$. To see the claim in (6.29) first note that, with $\tilde\beta_\ell = \beta_\ell\big((1 + C^2e^{2\Lambda|t_0|})|\lambda_1|^3\big)^{-1}$,

$$\tilde\beta_{\ell+1} = C(1 + 12Ce^{\Lambda|t_0|}\tilde\beta_\ell)e^{\Lambda|t_0|}, \qquad \tilde\beta_1 = e^{\Lambda|t_0|}.$$

Therefore, since $\tilde\beta_1 \ge 1$, and we may assume $C \ge 1$, $\tilde\beta_\ell \ge \tilde\beta_{\ell-1}$, and $\tilde\beta_\ell \le 13C^2e^{2\Lambda|t_0|}\tilde\beta_{\ell-1}$. Hence, the claim in (6.29) follows.

Step 3. The goal is to prove that $\mathcal{P}(t, ce^{-(r+1)\Lambda|t|}, r-1)$ holds for all t since this would yield the lemma. To do this, first note $\mathcal{P}(t, ce^{-3\Lambda|t|}, 1)$ holds for all t by Lemma 6.8. We will prove the lemma by induction. Let $2 \le k \le r-1$. We assume

$$\mathcal{P}(t, ce^{-(l+2)\Lambda|t|}, l) \quad \text{holds for all } t \text{ and all } 1 \le l \le k-1. \tag{6.30}$$

Let $\beta < ce^{-(k+2)\Lambda|t|}$, and suppose that the assumptions of $\mathcal{P}(t,\beta,k,\{t_i\}_{i=1}^k)$ hold and $\{t_i\}_{i=1}^k \subset I(t, \frac{1}{2}\beta)$. To finish the proof we must show that the conclusions of $\mathcal{P}(t,\beta,k,\{t_i\}_{i=1}^k)$ hold. We do this by contradiction.

Suppose that the conclusions of $\mathcal{P}(t,\beta,k,\{t_i\}_{i=1}^k)$ do not hold. We will arrive at a contradiction after one further induction. Let $(s_0,\beta_0) = (t,\beta)$, and $N \ge 1$. Assume we have found $\{(s_n,\beta_n)\}_{n=0}^N$ such that for all $1 \le n < N$

$$\begin{gathered} \text{the assumptions of } \mathcal{P}(s_n,\beta_n,k,\{t_i\}_{i=1}^k) \text{ hold} \\ \text{and} \\ s_n \in I(s_{n-1}, \tfrac{1}{2}\beta), \qquad \beta_n < \tfrac{1}{2}\beta_{n-1}. \end{gathered} \tag{6.31}$$

We claim that there are (s_N,β_N) such that (6.31) holds with $n = N$. To see this, we apply Step 2 above with $(t,\beta) = (s_{N-1},\beta_{N-1})$.

Observe that in this case we must have $m = 1$ for m as in (6.26). Indeed, if $m > 1$, then $|\mathcal{I}^\infty_{i_1}| < k$. In particular, since $\mathcal{P}(t, ce^{-(l+2)\Lambda|t|}, l)$ holds for $1 \le l \le k-1$, the conclusions of $\mathcal{P}(t,\beta,k,\{t_i\}_{i=1}^k)$ hold (and this is a contradiction).

Next, note that having $m = 1$ implies $|\mathcal{I}^\infty_{i_1}| = k$. In particular, the assumptions of $\mathcal{P}(t^\infty_{i_1}, \beta_{\ell^\infty_{i_1}}, k, \{t_i\}_{i=1}^k)$ hold with $\beta_{\ell^\infty_{i_1}} < \frac{1}{2}\beta_{N-1}$. Defining, $(s_N,\beta_N) = (t^\infty_{i_1}, \beta_{\ell^\infty_{i_1}})$ we have found (s_N,β_N) such that (6.31) holds with $n = N$.

Finally, by induction, if the conclusions of $\mathcal{P}(t,\beta,k,\{t_i\}_{i=1}^k)$ do not hold, then for all $n = 1,2,\dots,$ there are (s_n,β_n) such that (6.31) holds. In particular, $\beta_n \le 2^{-n}\beta$ and, for n large enough, (6.18) is satisfied with $\beta = \beta_n$. Therefore, by Step 1, the conclusions of $\mathcal{P}(t,\beta,k,\{t_i\}_{i=1}^k)$ hold and we have finished the proof. □

Proof of Proposition 6.1. Let c, C be as in Lemma 6.4. Let γ be a geodesic and $A(t)$ solve (6.3). Let $t_* \in (t_0-\varepsilon, t_0+\varepsilon)$ and $\beta := \frac{1}{C}\varepsilon e^{-\Lambda|t_*|}$. Without loss of generality assume that C is large enough that $\beta < ce^{-n\Lambda|t_*|}$.

By assumption, there are no more than m conjugate points to $\gamma(0)$ in $(t_*-\varepsilon,t_*+\varepsilon)$. In particular, for all $r>m$ there is no collection of times $t_1,\dots,t_r$ with $\max_j|t_j-t_*|<C\beta e^{\Lambda|t_*|}$ such that $\sum_{j=1}^r\dim\ker A(t_j)\geq r$. By Lemma 6.4 this implies that

$$\|A(t_*)|_{\mathcal{V}}\|>\tfrac{1}{C}\varepsilon e^{-\Lambda|t_*|} \tag{6.32}$$

for all subspaces $\mathcal{V}\subset(\gamma(0)')^\perp$ with $\dim\mathcal{V}>m$.

We claim there is a subspace $\mathcal{V}$ of dimension $n-1-m$ and $C>0$ such that

$$\|A(t_*)w\|\geq\tfrac{1}{C}\varepsilon e^{-\Lambda|t_*|}\|w\|,\qquad w\in\mathcal{V}. \tag{6.33}$$

To prove this, suppose there is no such subspace $\mathcal{V}$ or $C>0$. Then, for all $\delta>0$ there is $v_\delta\neq0$ such that

$$\|A(t_*)v_\delta\|<\delta\varepsilon e^{-\Lambda|t_*|}\|v_\delta\|,$$

Let $\mathcal{V}_0=\{0\}$, $\mathcal{V}_1=\mathbb{R}v_\delta$, $1\leq k\leq m$, and suppose that we have found $C_j>0$ and $\{\mathcal{V}_j\}_{j=1}^k$ such that $\mathcal{V}_{j-1}\subset\{\mathcal{V}_j\}$, $\dim\mathcal{V}_j=j$, and

$$\|A(t_*)|_{\mathcal{V}_j}\|\leq\delta\varepsilon C_je^{-\Lambda|t_*|}.$$

Note that $\dim\mathcal{V}_j^\perp=n-1-j$, and hence, since $n-1-j\geq n-1-m$, by assumption there is $w_k\in\mathcal{V}_k^\perp$ such that

$$\|A(t_*)w_k\|<\delta\varepsilon e^{-\Lambda|t_*|}\|w_k\|.$$

Now, put $\mathcal{V}_{k+1}=\mathcal{V}_k\oplus\mathbb{R}w_k$ and let $v=(v_k,\lambda w_k)\in\mathcal{V}_{k+1}$ with $\lambda\in\mathbb{R}$. Then,

$$\|Av\|\leq\|Av_k\|+|\lambda|\|Aw_k\|\leq\delta\varepsilon e^{-\Lambda|t_*|}(C_k\|v_k\|+|\lambda|\|w_k\|)\leq\delta\varepsilon C_{k+1}e^{-\Lambda|t_*|}\|v\|,$$

where in the last inequality we use that v_k and w_k are orthogonal. In particular,

$$\|A(t_*)|_{\mathcal{V}_{k+1}}\|\leq\delta\varepsilon C_{j+1}e^{-\Lambda|t_*|}.$$

Finally, $\dim\mathcal{V}_{m+1}=m+1$, and

$$\|A(t_*)|_{\mathcal{V}_{m+1}}\|\leq\delta\varepsilon C_me^{-\Lambda|t_*|},$$

which contradicts (6.32), provided δ is small enough. This proves the claim in (6.33).

Now, let $\mathcal{V}$ as in (6.33). Then, by Lemma 6.2 there is $\mathbf{V}_\rho\subset T_\rho S^*_xM$ of dimension $n-1-m$ such that

$$d\pi\mathbf{V}_\rho^\sharp=0,\qquad\mathbf{K}\mathbf{V}_\rho^\sharp=\mathcal{V}.$$

For $\mathbf{v}\in\mathbf{V}_\rho$,

$$d\pi(d\varphi_{t_*}\mathbf{v})^\sharp=A(t)\mathbf{K}\mathbf{v}^\sharp,$$

and, since $\mathbf{K}\mathbf{v}^\sharp\in\mathcal{V}$, (6.33) implies that for $\mathbf{v}\in\mathbf{V}_\rho$

$$\|d\pi d\varphi_{t_*}\mathbf{v}\|=\|d\pi(d\varphi_{t_*}\mathbf{v})^\sharp\|\geq\varepsilon e^{-\Lambda|t_*|}\|\mathbf{K}\mathbf{v}^\sharp\|/C=\varepsilon e^{-\Lambda|t_*|}\|\mathbf{v}^\sharp\|/C\geq\varepsilon e^{-\Lambda|t_*|}\|\mathbf{v}\|/C.$$

Modifying the constant C, we can replace $|t_*|$ by $|t_0|$ in the previous estimate.

□

Appendix A.

A.1. **Implicit function theorem with estimates on the size.**

Lemma A.1. *Suppose that* $f(x_0, x_1, x_2) : \mathbb{R}^{m_0} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}^{m_0}$ *so that* $f(0,0,0) = 0$*,*

$$L := (D_{x_0} f(0,0))^{-1} \text{ exists}, \qquad \sup_{|\alpha|=1} |\partial^\alpha_{x_i} f| \le \tilde{B}_i, \qquad \sup_{|\alpha|=1, |\beta|=1} |\partial^\alpha_{x_i} \partial^\beta_{x_0} f| \le B_i.$$

Suppose further that $r_0, r_1, r_2 > 0$ *satisfy*

$$S := \|L\| \sum_{i=0}^{2} m_i B_i r_i < 1, \qquad \text{and} \qquad S r_0 + \|L\| \sum_{i=1}^{2} m_i \tilde{B}_i r_i \le r_0. \tag{A.1}$$

Then there exists a neighborhood $U \subset \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ *a function* $x_0 : U \to \mathbb{R}^n$ *so that*

$$f(x_0(x_1, x_2), x_1, x_2) = 0$$

and $B(0, r_1) \times B(0, r_2) \subset U$*.*

Proof. We employ the usual proof of the implicit function theorem. Let $G : \mathbb{R}^n \to \mathbb{R}^n$ have

$$G(x_0; x_1, x_2) = x_0 - L f(x_0, x_1, x_2).$$

Our aim is to choose $r_0, r_1 > 0$ so small that G is a contraction for $x_1 \in B(0, r_1)$, $x_0 \in B(0, r_0)$ and $x_2 \in B(0, r_2)$. Note that Note that

$$|G(x_0; x_1, x_2) - G(w; x_1, x_2)| \le \sup \|D_{x_0} G\| |x_0 - w|$$

and

$$|G(x_0; x_1, x_2)| \le \sup \|D_{x_0} G\| |x_0| + |G(0; x_1, x_2)|.$$

Therefore, we need to choose r_i small enough that

$$S_G := \sup\{\|D_{x_0} G\| : \ (x_0, x_1, x_2) \in B(0, r_0) \times B(0, r_1) \times B(0, r_2)\} < 1 \tag{A.2}$$

and

$$|G(x_0; x_1, x_2)| \le S_G r_0 + \|L\| |f(0, x_1, x_2)| \le S_G r_0 + \|L\| (m_1 \tilde{B}_1 r_1 + m_2 \tilde{B}_2 r_2) < r_0. \tag{A.3}$$

Now,

$$D_{x_0} G = \mathrm{Id} - L D_{x_0} f(x_0, x_1, x_2)$$

and $L D_{x_0} f(0,0,0) = \mathrm{Id}$. Therefore,

$$\|D_{x_0} G\| \le \|L\| (m_0 B_0 r_0 + m_1 B_1 r_1 + m_2 B_2 r_2) = S < 1.$$

In particular, $S_G < S$ and for r_i as in (A.1), we have that (A.2), (A.3) hold. In particular, G is a contraction and the proof is complete. ∎

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