LOWER BOUNDS FOR CAUCHY DATA ON CURVES IN A NEGATIVELY CURVED SURFACE

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Abstract. We prove a uniform lower bound on Cauchy data on an arbitrary curve on a negatively curved surface using the Dyatlov-Jin(-Nonnenmacher) observability estimate on the global surface. In the process, we prove some further results about defect measures of restrictions of eigenfunctions to a hypersurface.

1. Introduction

The purpose of this article is to prove a positive lower bound for L^2 norms of Cauchy data of Laplace eigenfunctions on curves of a negatively curved surface (M, g) without boundary. Let $\{\varphi_j\}_{j=0}^{\infty}$ be an orthonormal basis of eigenfunctions of the Laplacian,

$$-\Delta_g \varphi_j = \lambda_j^2 \varphi_j, \quad \langle \varphi_j, \varphi_k \rangle = \delta_{jk} \ ,$$

where $\langle f, g \rangle = \int_M f \bar{g} dV$ (dV is the volume form of the metric) Let $H \subset M$ be a smooth hypersurface. The semiclassical Cauchy data along H is defined by

$$CD(\varphi_j) := \{(\varphi_j|_H,\ \lambda_j^{-1} D_\nu \varphi_j|_H)\} \tag{1.1}$$

where $D_\nu := -i\partial_\nu$ and ν is a choice of unit normal to H. For technical reasons, we also introduce the *renormalized Cauchy data*

$$RCD(\varphi_j) = \{(1 + \lambda_j^{-2}\Delta_H)\varphi_j|_H, hD_\nu \varphi_j|_H)\}. \tag{1.2}$$

Here, Δ_H denotes the negative tangential Laplacian for the induced metric on H. Hence, as a semiclassical pseudodifferential operator, the operator $(1 + \lambda_j^{-2}\Delta_H)$ is characteristic precisely on the glancing set S^*H of H and damps out the whispering gallery components of $\varphi_j|_H$.

In what follows, we use semi-classical notation $h_j = \lambda_j^{-1}$, since we will be using semi-classical pseudo-differential calculus. Our first result gives a lower bound on the L^2 norm of (1.2) along H, and in fact a lower bound for more general 'matrix elements' relative to semi-classical pseudo-differential operators $Op_h(a)$ of order 0 on $L^2(H)$ with semi-classical symbol $a \in S^0_{cl}(B^*H)$. We denote by

$$B^*H := \{(q, \xi) \in T^*H : |\xi|_H < 1\}$$

the unit co-ball bundle of H; $|\cdot|_H$ is the restriction of g to T^*H. We also denote by $\gamma_H f := f|_H$ the restriction of $f \in C(M)$ to H.

Theorem 1.1. *Let (M, g) be a compact, negatively curved surface without boundary and $H \subset M$ a smooth curve. Then there exist $h_0 > 0$ and $C_H > 0$ so that,*

$$\|(h_j D_\nu \varphi_j)|_H\|^2_{L^2(H)} + \|\varphi_j|_H\|^2_{L^2(H)} \geq C_H > 0.$$

More generally, this inequality can be mircolocalized: for $a \in S^0_{sc}(H)$*, such that* $Op_h(a)$ *has principal symbol,* $a_0(x', \xi')$*, satisfying* $a_0 \geq 0$ *on* B^*H *and* $\operatorname{supp} a_0 \cap B^*H \neq \emptyset$*, there exist* $h_0 > 0$ *and* $C_{a_0} > 0$ *so that for* $0 < h < h_0$*,*

$$\langle Op_h(a) h_j D_\nu \varphi_j|_H, h_j D_\nu \varphi_j|_H \rangle_{L^2(H)} + \big\langle Op_h(a)(1 + h_j^2 \Delta_H)\varphi_j|_H, \varphi_j|_H \big\rangle_{L^2(H)} \geq C_{a_0}.$$

Note that the second inequality of Theorem 1.1 implies the same result for Cauchy data.

Corollary 1.1. *With the same assumptions and notations as in Theorem 1.1,*

$$\langle Op_h(a) h_j D_\nu \varphi_j|_H, h_j D_\nu \varphi_j|_H \rangle_{L^2(H)} + \langle Op_h(a) \varphi_j|_H, \varphi_j|_H \rangle_{L^2(H)} \geq C_{a_0} > 0.$$

Indeed, this follows from the statement $1 \geq (1 - |\xi'|^2)$ on B^*H relating the symbols of I and of $(I + h^2\Delta_H)$. It is sufficient to consider this region since φ_j concentrates on it in a strong sense (see [CHT15]).

Remark: It is argued in [BHT, p. 3063-3064] that the renormalized Dirichlet data (1.2)

$$\langle (1 + h_j^2 \Delta_H)\varphi_j|_H, \varphi_j|_H \rangle_{L^2(H)}$$

is a closer analogue to the Neumann data $\|h_j D_\nu \varphi_j\|^2_{L^2(H)}$ than is the traditional Dirichlet data. In fact, one can see that $h_j D_\nu$ behaves similarly to $(1+h^2\Delta_H)_+^{\frac{1}{2}} u$ [G16].

We give two proofs of the result, by two rather different approaches whose contrast seems to be of some independent interest. Both are based on the Dyatlov-Jin(-Nonnenmacher) observability estimate [DJ17, DJN19]. The first proof is based on the Rellich identity of [CTZ13] (adapted from [Bu]) relating interior and restricted matrix elements and microlocal lifts of eigenfunctions. The second is based on hyperbolic equations and is closely related to results in [GL17]. It yields the following version of Corollary 1.1.

Theorem 1.2. *Let* (M, g) *be a negatively curved surface and* $H \subset M$ *a smooth curve. Then for all* $0 \neq b_0 \in C^\infty_c(B^*H)$*, there is* $C_{b_0} > 0$ *and* $h_0 > 0$ *such that if*

$$\|(-h_j^2 \Delta_g - 1)\varphi_j\|_{L^2} = o\big(\frac{h_j}{\log h_j^{-1}}\big)\|\varphi_j\|_{L^2},$$

then for $0 < h_j < h_0$

$$0 < C_{b_0}\|\varphi_j\|_{L^2} < \|Op_h(b_0)\varphi_j|_H\|_{L^2(H)} + \|Op_h(b_0) h_j \partial_\nu \varphi_j|_H\|_{L^2(H)}. \tag{1.3}$$

Moreover, for all $U \subset H$ *open, there is* $c > 0$ *such that for all* φ_j *satisfying*

$$(-h_j^2 \Delta_g - 1)\varphi_j = 0,$$

we have

$$0 < c\|\varphi_j\|_{L^2(M)} < \|\varphi_j|_H\|_{L^2(U)} + \|h_j \partial_\nu \varphi_j|_H\|_{L^2(U)}. \tag{1.4}$$

In Section 1.2, we state some further results on microlocal defect measures μ, in particular on the possible case where $\mu(S^*_H M) > 0$. Here (and hereafter) $S^*_H M = \{(x, \xi) \in S^*M : x \in H\}$.

1.1. **Background.** The first proof of Theorem 1.1 develops the Rellich identity approach of [CTZ13] . In that article, it is proved that a sequence of renormalized Cauchy data (1.2) of eigenfunctions is quantum ergodic along any hypersurface $H \subset M$ if the sequence of eigenfunctions is quantum ergodic on the global manifold M. It is not known whether the full orthonormal basis of eigenfunctions of a negatively curved compact surface is quantum ergodic (QE), but a recent result of Dyatlov-Jin(-Nonnenmacher) shows that its microlocal defect measures must have *full support*, that is, they charge (give positive mass to) any open set. The Dyatlov-Jin(-Nonnenmacher) theorem is a microlocal observability estimate for quasimodes u of compact hyperbolic (or more generally negatively curved) surfaces: For all $u \in H^2(M)$, and all $a \in C_0^\infty(T^*M)$ with $a \geq 0$, a not identically zero on S^*M, there exists a constant $C(a) > 0$ so that

$$||u||_{L^2} \leq C(a)||Op_h(a)u||_{L^2} + \frac{C(a)\log(1/h)}{h}||(-h^2\Delta - I)u||_{L^2}. \tag{1.5}$$

In particular, if u is an eigenfunction the second term is zero and one has a lower bound on $\|Op_h(a)u\|_{L^2(M)}$. Theorem 1.1 gives a similar full support property for the restricted microlocal defect measures of the Cauchy data on a curve.

The results may be stated in terms of microlocal defect measures (quantum limits). Several are involved: the global microlocal defect measures on M, and restricted microlocal defect measures on H, both for Neumann data, Dirichlet data and for renormalized Dirichlet data.

We denote by $p(x,\xi) = |\xi|_g^2$ the principal symbol of the Laplacian $-\Delta_g$. We denote by H_p its Hamilton vector field, and by $\varphi_t := \exp(tH_{|\xi|_g^2})$ its Hamiltonian flow, i.e. the geodesic flow $\varphi_t : S^*M \to S^*M$ of (M,g) on its unit cosphere bundle.

We use some notation and background on the semiclassical calculus of pseudo-differential operators as in the references [Bu, DZ, CTZ13, Zw]. On both H and M we fix (Weyl) quantizations $a \to a^w$ of semi-classical symbols to semi-classical pseudo-differential operators. When it is necessary to indicate which manifold is involved, we use Fermi coordinates (x', x_n) with x' coordinates on H (Section 2), and we use capital letters $A^w(x, hD)$ to indicate operators on M and small letters $Op_h(a) = a^w(x', hD_{x'})$ to indicate semi-classical pseudo-differential operators on H (denoted by $\Psi^*_{sc}(H)$).

We recall that microlocal defect measures are the semi-classical limits of the functionals $a \to \langle a^w(x, h_j D_x)u_j, u_j\rangle$, which are often referred to as microlocal lifts or Wigner distributions. Let $\mathcal{Q}^*$ denote the set of all microlocal defect measures for an orthonormal basis $\{\varphi_j\}$ of eigenfunctions. That is, $\mu \in \mathcal{Q}^*$ if there exists a subsequence $\mathcal{S} = \{\varphi_{j_k}\}$ of eigenfunctions for which $\langle A\varphi_{j_k}, \varphi_{j_k}\rangle \to \int_{S^*M} \sigma_A d\mu$. The microlocal defect measures on the global manifold M are invariant probability measures for the geodesic flow $\varphi_t : S^*M \to S^*M$ (see e.g. [Zw, Chapter 5]). Throughout, we denote by $\mathcal{M}(X)$ the space of positive measures on a metric space X, $\mathcal{M}_1(X)$ the space of probability measures and $\mathcal{M}_I(X)$ the space of invariant measures under a flow on X.

The Dyatlov-Jin observability estimate for eigenfunctions implies the following:

Theorem **1.** *Let (M, g) be a compact negatively curved surface without boundary, let $\{\varphi_j\}$ be any choice of orthonormal basis of eigenfunctions, and let $A \in \Psi^0(M)$ be a pseudo-differential operator of order zero with a non-negative principal symbol*

σ_A not vanishing identically on S^*M. Then,

$$\inf_{\mu \in \mathcal{Q}^*} \int_{S^*M} \sigma_A d\mu \geq C_A > 0. \tag{1.6}$$

From (1.6), it follows that all microlocal defect measures of eigenfunctions of compact negatively curved surfaces have *full support*, i.e. charge every open set of S^*M.

Given a quantization $a \to Op_h(a)$ of semi-classical symbols $a \in S^0_{sc}(H)$ of order zero (see [Zw]) to semi-classical pseudo-differential operators on $L^2(H)$, we define the microlocal lifts of the Neumann data as the linear functionals on $a \in S^0_{sc}(H)$ given by

$$\mu_j^N(a) := \int_{B^*H} a\, d\Phi_j^N := \langle Op_h(a) h_j D_\nu \varphi_j|_H, h_j D_\nu \varphi_j|_H \rangle_{L^2(H)}.$$

We define the microlocal lifts of the Dirichlet date by

$$\mu_j^D(a) := \int_{B^*H} a\, d\Phi_j^D := \langle Op_h(a) \varphi_j|_H, \varphi_j|_H \rangle_{L^2(H)}.$$

also define the microlocal lifts of the modified Dirichlet data by

$$\mu_j^{RD}(a) := \int_{B^*H} a\, d\Phi_j^{MD} := \langle Op_h(a)(1 + h_j^2 \Delta_H) \varphi_j|_H, \varphi_j|_H \rangle_{L^2(H)}.$$

Finally, we define the microlocal lift $d\Phi_j^{RCD}$ of the renormalized Cauchy data to be the sum

$$d\Phi_j^{RCD} := d\Phi_j^N + d\Phi_j^{RD}. \tag{1.7}$$

The weak* limits of the above microlocal lifts are termed microlocal defect measures, respectively, of the Neumann, Dirichlet or renormalized Dirichlet type.

The distributions μ_j^N, μ_j^D are asymptotically positive, but are not normalized to have mass one and may tend to infinity. They depend on the choice of quantization, but their possible weak* limits as $h_j \to 0$ do not, and the results of the article are valid for any choice of quantization. We refer to [Zw] for background on semi-classical microlocal analysis.

Theorem 1.3. *Let (M, g) be a compact surface without boundary of negative curvature. Suppose $H \subset M$ is a curve. Then, for any $a \in S^0_{sc}(H)$,*

$$\inf_{\mu \in \mathcal{Q}^*(RCD)} \int_{B^*H} a d\mu \geq C_{a_0} > 0$$

and therefore,

$$\inf_{\mu \in \mathcal{Q}^*(CD)} \int_{B^*H} a d\mu \geq C_{a_0} > 0$$

One should be aware of an obstruction to obtaining lower bounds on (1.2) from lower bounds on (1.1) for hypersurfaces in a general Riemannian manifold. Suppose that the eigenfunctions φ_j are the highest weight spherical harmonics $Y^N_N(\theta, y) \simeq e^{iN\theta} e^{-Ny^2/2}$ on S^2 along the equator γ. The normal y-derivative vanishes since Y^N_N is even under reflection across γ, i.e. $y \to -y$ in Fermi normal coordinates. But the restriction is $e^{iN\theta}$ and it is killed by $(I + N^{-2} \frac{\partial^2}{\partial \theta^2})$. Hence, the renormalized Cauchy data vanishes. As this example shows, Rellich-type identities for renormalized Cauchy data are not necessarily equivalent to bounds on Cauchy data (1.1) because of the effects of concentration in tangential directions to H. Theorem 1.1 nevertheless gives a positive lower bound for curves on a negatively curved surface, because the Dyatlov-Jin lower bound (1.5) implies that restrictions of eigenfunctions on such surfaces cannot concentrate entirely in the tangential directions.

Remark: The proof of Theorem 1.1 using the Rellich identity is a continuation of the proof in [CTZ13] in the case where the microlocal defect measure of a sequence is Liouville measure μ_L. Since the calculations in the case of a general microlocal defect measure of independent interest, we present most of the details in all dimensions and in more detail than is strictly necessary for the proof of Theorem 1.1. We only specialize to curves in a negatively curved surface at the end of the proof.

1.2. **Results on microlocal defect measures.** In the process of proving Theorem 1.1 via the Rellich formula we will obtain some facts about the collection of defect measures for eigenfunctions that are of independent interest. First, we study the restrictions of $\mu \in \mathcal{M}_I \cap \mathcal{M}_1$ to a hypersurface H

Theorem 1.4. *Let $H \subset M$ be a hypersurface and $\mu \in \mathcal{M}_1(S^*M) \cap \mathcal{M}_I(S^*M)$ where the relevant flow is φ_t. Then,*

- $\mu|_{S^*_H M} = \mu|_{S^*H}$*;*
- *the support of $\mu|_{S^*H}$ is is contained in the null-space of the second fundamental form $Q(0, x', \xi')$, i.e. $Q(0, x', \xi') = 0$ for $(x', \xi') \in \mathrm{Supp}(\mu|_{\mathrm{S^*H}})$.*
- *Hence, $\mu|_{S^*H}$ is supported on the subset of S^*H where the Hamilton vector field H_p coincides with the Hamilton vector field $H_{p|_{S^*H}}$ of the submanifold metric norm, $g_H(x', \xi') = |\xi'|^2_{g_H}$.*

In fact, we will see in Section 4 that such measures have $\mu|_{S^*H}$ supported where H_p is tangent to H to infinite order.

The following immediate corollary was mentioned in [TZ17].

Corollary 1.2. *If the second fundamental form of H is non-degenerate, $\mu|_{S^*_H M} = 0$.*

Finally, we obtain an expression for $\mu \in \mathcal{Q}^*(RCD)$.

Theorem 1.5. *Suppose that φ_h has defect measure μ and that its renormalized Cauchy data has defect measure $\mu^{RCD} \in \mathcal{Q}^*(RCD)$. Then, for $a \in C(T^*H)$,*

$$\mu^{RCD}(a) = \int_{S^*_H M \setminus S^*H} a(\pi(\zeta)) |\xi_n(\zeta)| d\nu^{\perp}(\zeta)$$

where for $A \subset S^*_H M \setminus S^*H$,

$$\nu^{\perp}(A) := \lim_{T \to 0^+} \frac{1}{2T} \mu\Big(\bigcup_{|t| \leq T} \varphi_t(A) \Big),$$

and $\pi : S^*_H M \to B^*H$ denotes the orthogonal projection.

1.3. **Outline of the article.** Section 2 contains some preliminary facts about the Laplacian. This is followed by Section 3 which reviews the basic calculations for Rellich's formula. Section 4 then contains the study of invariant measures and in particular, the proof of Theorem 1.4. Section 5 contains the proof of theorem 1.5 and the proof of Theorem 1.1 to from Theorem 1.5. Finally, Section 6 contains the proof of Theorem 1.2 via a factorization method.

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2. Laplacian in Fermi normal coordinates along a hypersurface

Let (M, g) be any Riemannian manifold. We recall that in any coordinate system,

$$\Delta_g = \frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x_i}(g^{ij}\sqrt{g}\frac{\partial}{\partial x_j}),$$

where $g = \det(g_{ij})$. Here $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ and g^{ij} is the inverse matrix.

Let $H \subset M$ be a hypersurface, and let $x = (x_1, ..., x_{n-1}, x_n) = (x', x_n)$ be Fermi normal coordinates in a small tubular neighbourhood $H(\epsilon)$ of H defined near a point $x_0 \in H$. Thus, $H = \{x_1 = 0\}$ with coordinates x' on H. Fermi coordinates use the charts $\exp_{x'} x_n\nu$ where ν is a choice of unit normal field. In these coordinates we can locally write

$$H(\epsilon) := \{(x', x_n) \in U \times \mathbb{R},\, |x_n| < \epsilon\}. \tag{2.1}$$

Here $U \subset \mathbb{R}^{n-1}$ is a coordinate chart containing $x_0 \in H$ and $\epsilon > 0$ is small but for the moment, fixed. We also denote by ξ_n, ξ' the symplectically dual coordinates on $T^*H(\epsilon)$.

In Fermi normal coordinates, the metric is given by

$$g = g_H(x, dx') + dx_n^2.$$

where $g_H(x, dx')$ is a metric in x' depending on x_n. In particular, the metric induced on H is $g_H(0, x', dx')$. Therefore,

$$\begin{aligned} -h^2\Delta_g &= \frac{1}{\sqrt{g(x)}}(hD_{x_n}\sqrt{g(x)}hD_{x_n} + hD_{x_i}g_H^{ij}(x)\sqrt{g(x)}hD_{x_j}) \\ &= \frac{1}{\sqrt{g(x)}}hD_{x_n}\sqrt{g(x)}hD_{x_n} + R(h, x', x_n, hD_{x'}) \\ &= (hD_{x_n})^2 + R(h, x', x_n, hD_{x'}) + hr_{1n}(x, hD_{x_n}) \end{aligned} \tag{2.2}$$

where R is a second-order h-differential operator along H with coefficients depending on x_n, r_{1n} is a first order normal operator, and

$$R = R_2(x', x_n, hD_{x'}) + hr_1(x', x_n, hD_{x'})$$

where $r_1(x_n, x', hD_{x'})$ is a first order operator along H with coefficients depending on x_n, and

$$R_2(x', x_n, hD_{x'}) = R_2(x', 0, hD_{x'}) + 2x_nQ(x', x_n, hD_{x'}), \tag{2.3}$$

Here, $Q(x', 0, \xi')$ is the second fundamental form of H and $R_2(x', 0, hD_{x'}) = -h^2\Delta_H$ (the induced tangential semiclassical Laplacian on H). The semi-classical principal symbol $\sigma(-h^2\Delta_g)$ of $-h^2\Delta_g$ is given by

$$p(x, \xi) = \xi_n^2 + |\xi'|^2_{g_H}. \tag{2.4}$$

We recall that the second fundamental form $II(X, Y)$ of a hypersurface H is the symmetric tensor on TH defined by $II(X, Y) = \nabla^M_X Y - \nabla^H_X Y$ where ∇^M is the covariant derivative for (M, g) and ∇^H is the covariant derivative for $(H, g|_H)$. The second fundamental form defines a quadratic form on $T_{x'}H$ for every $x' \in H$. Hence it is given by a quadratic polynomial $Q(0, x', \xi')$ in ξ' at each x'.

The first order terms r_1, r_{1n} play no role in the calculations of this paper since they only contribute to the $O(h)$ remainder.

2.1. **Calculations on a surface.** Although x_n denotes the nth Fermi coordinate in dimension n, we often use the same notation on a surface when we want to have uniform notation in all dimensions, since x_n also indicates 'normal coordinate'. In the case of a curve H in a surface M, we use the notation $(s, y) = (x_1, x_n)$ for the Fermi normal coordinates along H, with $x_n = y, x' = s$ (where s is arc-length); the symplectically dual coordinates are denoted by (σ, η). The metric components are given by $g^{00} = g(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}), g^{11} = g(\frac{\partial}{\partial y}, \frac{\partial}{\partial y})$ and $g^{01} = g^{10} = g(\frac{\partial}{\partial y}, \frac{\partial}{\partial s}) \equiv 0$. Since the vector fields $\frac{\partial}{\partial y}$ are tangent to unit speed geodesics, $g^{11} \equiv 1$ and in particular, $\partial_{x_n} g^{11} = 0$. The Taylor expansion of $g^{00}(s, y)$ around $y = 0$ has the form,

$$g^{00}(s, y) = 1 + 2y\kappa_\nu(s) + C_1\tau(s)y^2 + O(y^3),$$

where $\kappa_\nu(s)$ is the geodesic curvature of H, and where τ is the scalar curvature of (M, g).

3. Rellich identity

The result of Theorem 1.1 is local on H, and with no loss of generality we may assume that H is the boundary of a smooth open domain $M_+ \subset M$, $H = \partial M_+$, and $x_n > 0$ in M_+. We then use a Rellich identity to write the integral of a commutator over M_+ as a sum of integrals over the boundary (of course the same argument would apply on $M_- = M \setminus M_+$). We follow the exposition of [CTZ13] in the following and continue to use the notation φ_h for a sequence of eigenfunctions and allow $H \subset M$ to be a hypersurface in a manifold of any dimension.

Let $A(x, hD_x) \in \Psi^0_{sc}(M)$ be an order zero semiclassical pseudodifferential operator on M (see [Zw]). Also denote by γ_H the restriction operator $\gamma_H f = f|_H$. If φ_h is a Laplace eigenfunction of eigenvalue $-h^{-2}$, then by Green's formula,

$$\begin{aligned} -\frac{i}{h}\int_{M_+} &\big([-h^2\Delta_g,\, A(x, hD_x)]\, \varphi_h(x)\big)\, \overline{\varphi_h(x)}\, dx \\ &= \int_H \big(\gamma_H (hD_\nu\, A(x', x_n, hD_x)\varphi_h\big)\, \overline{\varphi_h}_H\, d\sigma_H \\ &\quad + \int_H \big(\, A(x', x_n, hD_x)\, \gamma_h\varphi_h\big)\, \big(\gamma_H \overline{hD_\nu\varphi_h}\big)\, d\sigma_H. \end{aligned} \tag{3.1}$$

Here, $D_{x_j} = \frac{1}{i}\frac{\partial}{\partial x_j}$, $D_{x'} = (D_{x_1}, ..., D_{x_{n-1}})$, $D_\nu = \frac{1}{i}\partial_\nu$ where ∂_ν is the interior unit normal to M_+. Henceforth we often abbreviate $D_{x'}$ by D'. Also, $d\sigma_H$ is the surface measure on H.

Let $Op_h(a) = a^w(x', hD_{x'})$ be a semi-classical pseudo-differential operator on $L^2(H)$. We wish to choose $A(x, hD_x)$ to so that $A(x', 0, hD_x)$ is close to $a^w(x', hD_{x'})$ and so that $[-h^2\Delta_g, A(x, hD_x)]$ has good positivity properties when $a^w(x', hD_{x'}) \geq 0$.

We let $\chi \in C_0^\infty(\mathbb{R})$ be a cutoff with $\chi(x) = 0$ for $|x| \geq 1$ and $\chi(x) = 1$ for $|x| \leq 1/2$. Given $a \in S^{0,0}(T^*H \times (0, h_0])$, we define the pseudo-differential operator

A on M by,

$$A(x', x_n, hD_x) = \chi(\frac{x_n}{\epsilon})\, hD_{x_n} a^w(x', hD'). \tag{3.2}$$

We now calculate the two sides of (3.1) following [CTZ13], in particular showing that matrix elements of the commutator $[-h^2\Delta_g, A(x, hD_x)]$ of this 'extension' of $a^w(x', D_{x'})$ with $-h^2\Delta_g$ have good positivity properties. Of course, A is not truly an extension because it is not totally characteristic, i.e. it also contains normal derivatives hD_{x_n}.

3.0.1. *The right hand side.* Since $\chi(0) = 1$, the second term

$$\int_H \gamma_H\, (\, A(x', x_n, hD_x)\, \varphi_h)\, (\gamma_H \overline{hD_\nu \varphi_h})\, d\sigma_H = \int_H \gamma_H \chi(\frac{x_n}{\epsilon})\, hD_{x_n} a^w(x', hD')\varphi_h) \overline{hD_\nu \varphi_h} d\sigma_H$$

on the right side of (3.1) is the Neumann data matrix element,

$$\langle a^w(x', hD') hD_{x_n} \varphi_h|_H, hD_{x_n}\varphi_h|_H \rangle\, . \tag{3.3}$$

We now show that the first term on the right hand side of (3.1) is the renormalized Dirichlet data. Using that $\chi'(0) = 0$ and $-h^2\Delta_g \varphi_h = \varphi_h$, the first term equals

$$\begin{aligned}
&\int_H \gamma_H\, (hD_n(\chi(x_n/\epsilon) hD_n a^w(x', hD')\varphi_h)\, (\gamma_H \overline{\varphi_h})\, d\sigma_H \\
&= \int_H \gamma_H \chi(x_n/\epsilon) a^w(x', hD')(hD_n)^2 \varphi_h \gamma_H \overline{\varphi_h} d\sigma_H \\
&\qquad + \int_H \frac{h}{i\epsilon} \chi'(x_n/\epsilon) hD_n a^w(x', hD')\varphi_h\Big) \gamma_H \overline{\varphi_h} d\sigma_H \\
&= \int_H \gamma_H (\chi(x_n/\epsilon) a^w(x', hD')(1 - R(x_n, x', hD'))\varphi_h) \gamma_H \overline{\varphi_h} d\sigma_H \\
&\qquad + O_\epsilon(h)(\|\gamma_H \varphi_h\|^2 + \|\gamma_H hD_{x_n}\varphi_h\|^2) \\
&= \int_H a^w(x', hD')(1 - h^2\Delta_H)\gamma_H \varphi_h \cdot \gamma_H \overline{\varphi_h} d\sigma_H \\
&\qquad + O(h)(\|\gamma_H \varphi_h\|^2 + \|\gamma_H hD_{x_n}\varphi_h\|^2).
\end{aligned} \tag{3.4}$$

In the last line we use that $\chi(0) = 1$ and the expansions (2.2)-(2.3) together with the fact that, since φ_h is a Laplace eigenfunction

$$\|\gamma_H r_1(x', 0, hD_{x;})\varphi_h\|_{L^2(H)} \leq C\|\gamma_H \varphi_h\|_{L^2}.$$

3.0.2. *Left hand side of* (3.1). Since the semi-classical principal symbol of $\frac{i}{h}[-h^2\Delta_g,\, A(x, hD_x)]$ equals the Poisson bracket $\{\xi_n^2 + R_2(x_n, x', \xi'), \chi(\frac{x_n}{\epsilon})\xi_n a(x', \xi')\}$, we have

(3.5)
$$\begin{aligned}
-\frac{i}{h}\int_{M_+} &\big([-h^2\Delta_g,\, A(x, hD_x)]\, \varphi_h(x)\big)\, \overline{\varphi_h(x)}\, dx \\
&= -\Big\langle \Big(\Big\{\xi_n^2 + R_2(x', x_n, \xi'),\, \chi(\frac{x_n}{\epsilon})\xi_n a(x', \xi')\Big\}\Big)^w \varphi_h,\ \varphi_h \Big\rangle_{L^2(M_+)} + \mathcal{O}_\epsilon(h).
\end{aligned}$$

3.1. **Some Poisson bracket calculations.** Since ξ_n^2 is only non-trivially paired with x_n,

$$-\left\{\xi_n^2 + R_2(x', x_n, \xi'),\ \chi(\tfrac{x_n}{\epsilon})\xi_n a(x', \xi')\right\} \\ = -\tfrac{2}{\epsilon}\chi'(\tfrac{x_n}{\epsilon})\xi_n^2 a(x', \xi') + \chi(\tfrac{x_n}{\epsilon})P_2(x', x_n, \xi', \xi_n), \tag{3.6}$$

where $P_2 = -\{R_2(x', x_n, \xi'), \xi_n a(x', \xi')\}$. In general dimensions,

$$P_2 = \tfrac{\partial R_2(x', x_n, \xi')}{\partial_{x_n}} a(x', \xi') - \xi_n\{R_2(x_n, x', \xi), a(x', \xi')\}. \tag{3.7}$$

When restricted to S^*H the second term is zero and one gets

$$\frac{\partial R_2(x_n, x', \xi')}{\partial_{x_n}}|_{x_n=0}\ \ a(x', \xi') = 2Q(0, x', \xi')a(x', \xi').$$

In the case of a curve H and in Fermi normal coordinates,

$$P_2(s, 0, \sigma, \eta) = 2\kappa_\nu(s)a(s, \sigma) - 2\sigma\eta\frac{\partial(a(s, \sigma))}{\partial s}. \tag{3.8}$$

The first term vanishes when $y = 0$, since $g^{00}(s, 0) = 1$, while $\frac{\partial g^{00}(s,y)\sigma^2}{\partial y}a(s, \sigma) = 2\kappa_\nu(s)a(s, \sigma)$. Hence, The second term vanishes when $\eta = 0$, i.e. on S^*H.

3.2. **Semi-classical limit of the Rellich formula.** We consider any sequence $\{\varphi_h\}$ with a single microlocal defect measure $\mu \in \mathcal{Q}^*$. It will be convenient to extend some integrals from M_+ to M. For this, we introduce a cutoff $\tilde{\chi} \in \mathcal{C}^\infty(M)$ such that

$$\chi'(x_n/\epsilon)|_{M_+} = \tilde{\chi}'(x_n/\epsilon), \qquad \text{supp}\,(1 - \tilde{\chi}) \cap x_n \geq 0 = \emptyset \tag{3.9}$$

Also recall that P_2 is defined in (3.7).

Proposition 3.1. *Let (M, g) be a compact Riemannian manifold and let $H \subset M$ be a smooth, embedded, orientable hypersurface. Then,*

$$\left|\lim_{h\to 0}\begin{pmatrix} \langle Op_h(a)hD_\nu\varphi_h|_H, hD_\nu\varphi_h|_H\rangle_{L^2(H)} \\ + \left\langle Op_h(a)(1 + h^2\Delta_H)\varphi_h|_H, \varphi_h|_H\right\rangle_{L^2(H)}\end{pmatrix} - I_0(a, \epsilon, \mu)\right| \\ \leq II_0(a, \epsilon, \mu). \tag{3.10}$$

where

$$\begin{cases} I_0(a, \epsilon, \mu) := -2\int_{S^*M} \frac{1}{\epsilon}\tilde{\chi}'(\frac{x_n}{\epsilon})\, \xi_n^2 a(x', \xi')d\mu, \\ \\ II_0(a, \epsilon, \mu) := \sqrt{\int \chi(\frac{x_n}{\epsilon})^2 P_2^2(x', x_n, \xi')d\mu}.. \end{cases} \tag{3.11}$$

Proof. The Rellich identity and the calculations (3.1)-(3.4)-(3.6) in Section 3.1 show that, for any hypersurface $H \subset M$,

$$\begin{matrix}\langle Op_h(a)hD_\nu\varphi_h|_H, hD_\nu\varphi_h|_H\rangle_{L^2(H)} \\ + \langle Op_h(a)(1 + h^2\Delta_H)\varphi_h|_H, \varphi_h|_H\rangle_{L^2(H)}\end{matrix} = I_h(a, \epsilon) + II_h(a, \epsilon) + \mathcal{O}_\epsilon(h) \tag{3.12}$$

where

$$I_h = -\left\langle \left(\tfrac{2}{\epsilon}\chi'(\tfrac{x_n}{\epsilon})\xi_n^2 a(x', \xi')\right)^w \varphi_h,\ \varphi_h\right\rangle_{L^2(M_+)} \\ II_h = \left\langle\left(\chi(\tfrac{x_n}{\epsilon})P_2(x', x_n, \xi', \xi_n)\right)^w \varphi_h, \varphi_h\right\rangle_{L^2(M_+)} \tag{3.13}$$

and P_2 is given by (3.7) - (3.8).

Now, $\chi'(x_n/\epsilon)|_{M_+} = \tilde{\chi}'(x_n/\epsilon)$ where $\tilde{\chi}$ is as in (3.9). Therefore, Since $\tilde{\chi}'$ and χ' are supported inside M_+,

$$\Big\langle \big(\tfrac{1}{\epsilon}\chi'(\tfrac{x_n}{\epsilon})\,\xi_n^2 a(x',\xi')\big)^w \varphi_h,\ \varphi_h\Big\rangle_{L^2(M_+)} = \Big\langle \big(\tfrac{1}{\epsilon}\tilde{\chi}'(\tfrac{x_n}{\epsilon})\,\xi_n^2 a(x',\xi')\big)^w \varphi_h,\ \varphi_h\Big\rangle_{L^2(M)}.$$

Sending $h \to 0$ in the right hand side yields $I_0(a,\epsilon,\mu)$.

Next, observe that by Cauchy-Schwarz,

$$\begin{aligned} II_h(a,\epsilon) &= \big\langle \big(\chi(\tfrac{x_n}{\epsilon})P_2(x',x_n,\xi')\big)^w \varphi_h,\ \varphi_h\big\rangle_{L^2(M_+)} \\ &\le \|\big(\chi(\tfrac{x_n}{\epsilon})P_2(x',x_n,\xi')\big)^w\varphi_h\|_{L^2(M)}. \end{aligned}$$

Then,

$$\lim_{h\to 0}\|\big(\chi(\tfrac{x_n}{\epsilon})P_2(x',x_n,\xi')\big)^w\varphi_h\|^2_{L^2(M)} = \int \chi(\tfrac{x_n}{\epsilon})^2 P_2^2(x',x_n,\xi')d\mu.$$

This completes the proof.

□

4. Decompositions of microlocal defect measures

The next two sections are devoted to the calculation of the limits $I_0(a,\epsilon,\mu)$ resp. $II_0(a,\epsilon,\mu)$ ((3.11)) as $\epsilon\to 0$. We first make the decomposition

$$\mu = \mu|_{S^*_H M} + \mu^{\perp}, \quad \text{where } \mu^{\perp}(S^*_H M) = 0, \tag{4.1}$$

and where $\mu|_{S^*_H M} = \mathbf{1}_{S^*_H M}\mu$ is the restriction of μ to $S^*_H M$. Here, $S^*_H M$ is the set of unit co-vectors to M with footpoint on H and $S^*H \subset S^*_H M$ are those (co-)tangent to H. In this section, we first study the measure $\mu|_{S^*_H M}$, showing that it is supported in in S^*H at points which are 'nearly' totally geodesic (See Lemma 4.3 and Corollary 4.4). We then calculate the limits of (3.11) in Proposition 5.1.

4.1. Disintegration of μ with respect to the geodesic flow.

We next briefly recall the theory of disintegration of measures along a fibration [Du19, Theorems 2.1.22, 4.1.17].

Proposition 4.1 (Disintegration Theorem). *Suppose that $(Y,\mathcal{Y},\mu)$ is a probability space, X is a Borel subset of a complete separable metric space, endowed with the Borel sigma algebra, and $\pi : Y\to X$ is measurable. Define $\nu := \pi_*\mu$. Then there is a ν a.e. unique family of probability measure $\{\mu_x\}_{x\in X}$ on Y such that*

(i) *for all Borel $A\subset Y$, $x\mapsto \mu_x(A)$ is measurable.*
(ii) $\mu_x(Y\setminus \pi^{-1}(x)) = 0$
(iii) *for any Borel measurable function $f : Y \to \mathbb{R}_+$,*

$$\int_Y f(y)d\mu(y) = \int_X \left(\int_{\pi^{-1}(x)} f(y)d\mu_x(y)\right)d\nu(x). \tag{4.2}$$

In the case of interest, we fix $\delta>0$ small and define

$$Y = FL_\delta(S^*_H M) := \bigcup_{|t|\le\delta} \exp(tH_p)(S^*_H M),$$

and the map

$$\pi_\delta : FL_\delta(S^*_H M) \to \mathfrak{G}_\delta := Y/\sim$$

where $\sim$ denotes the relation of belonging to the same orbit. Then, let μ^δ_x and ν^δ be the measures guaranteed by Proposition 4.1.

Note that the quotient space $\mathfrak{G}_\delta$ is not equal to S^*_HM; e.g. $H=\gamma$ is a closed geodesic, then $S^*\gamma$ is a single orbit and a single point in the quotient. There is, however, a large subset of $\mathfrak{G}_\delta$ which can be easily identified with S^*_HM. In particular, if an orbit in $\mathfrak{G}_\delta$ intersects S^*_HM only once, we may identify this orbit with its intersection with S^*_HM.

Lemma 4.2. *Suppose that μ is invariant under $\exp(tH_p)$ and $\rho_0\in S^*_HM$ such that there is a neighborhood, U of ρ_0 such that for $\rho\in U$*

$$\bigcup_{|t|\leq\delta}\exp(tH_p)(\rho)\cap S^*_HM=\rho.$$

Then, identifying $\rho\in S^*_HM$ with its orbit in $\mathfrak{G}_\delta$, for all $\rho\in U$, using $[-\delta,\delta]\times U\ni t\mapsto\exp(tH_p)(\rho)\in FL_\delta(S^*_HM)$ as coordinates on their image, $\mu^\delta_\rho(t,\zeta)=\frac{1}{2\delta}1_{[-\delta,\delta]}dt\delta_\rho(\zeta)$ and in particular, for $A\subset U$ Borel,

$$\nu^\delta(A)=\mu\Big(\bigcup_{|t|\leq\delta}\exp(tH_p)(A)\Big).$$

Proof. First observe that the given coordinates are valid. Next, μ^δ_ρ is clearly supported on $\pi^{-1}(\rho)$ and is Borel measurable. Therefore, we need only check that (4.2) holds with f supported in

$$FL_\delta(U):=\bigcup_{|t|\leq\delta}\exp(tH_p)(U).$$

For that, observe that on $FL_\delta(U)$, $\mu=\nu^\perp(\zeta)dt$ for some $\nu^\perp$. Therefore, for all $0<T\leq\delta$,

$$\nu^\perp(A)=\frac{1}{2T}\mu\Big(\bigcup_{|t|\leq T}\exp(tH_p)(A)\Big)=\frac{1}{2\delta}\nu^\delta(A),$$

and the lemma follows. ☐

For future use, we define

$$\nu^\perp(A):=\frac{1}{2\delta}\nu^\delta(A)=\lim_{T\to0^+}\frac{1}{2T}\mu\Big(\bigcup_{|t|\leq T}\exp(tH_p)(A)\Big). \tag{4.3}$$

4.2. **Disintegration of μ with respect to the normal fibration.** It is also possible to disintegrate μ with respect to the Fermi normal fibration over H. Let

$$S^*H(\epsilon):=\{(x',x_n,\xi)\in S^*M\mid|x_n|<\epsilon\}.$$

Let $H_{\xi_n}=\frac{\partial}{\partial x_n}$ be the Hamilton vector field of ξ_n on $|x_n|<\epsilon$. Its Hamilton flow is given by $\psi_t(x',x_n,\xi',\xi_n)=(x',x_n+t,\xi',\xi_n)$. In these coordinates S^*_HM is defined by $x_n=0$ and the integral curves of ψ_t define a fibration over S^*_HM. Given $(x,\xi)\in S^*_xM$ with $x\in\mathcal{T}_\delta(H)$, parallel translate ξ along the normal geodesic from x to H. Denote the result by $P^{x'}_x\xi$. Define

$$\begin{cases}\pi_\delta:S^*\mathcal{T}_\delta(H)\to S^*_HM,\ \ \pi_\delta(x',x_n,\xi):=(x',P^{x'}_x\xi),\\ \\ \mu^\delta_H=\pi_{\delta*}d\mu|_{S^*H(\epsilon)}.\end{cases} \tag{4.4}$$

For δ very small, this map is well-approximated by the map,

$$\widetilde{\pi}_\delta:S^*H(\epsilon)\to T^*_HM,\ \ \widetilde{\pi}_\delta(x',x_n,\xi',\xi_n):=(x',0,\xi',\xi_n),\ \ . \tag{4.5}$$

which however is not normalized so that the image lies in S^*M.

Applying Proposition 4.1 there exist finite fiber measures $d\mu_\rho^\epsilon$ on the fiber of (4.4) over $\rho \in S_H^*M$ such that

$$\int_{S^*H(\epsilon)} f d\mu = \int_{S_H^*M} \left(\int_{\pi_\epsilon^{-1}(\rho)} f d\mu_\rho^\epsilon \right) d\mu_H^\epsilon. \tag{4.6}$$

The principal defect of this disintegration is that the Fermi normal fibration is not invariant under φ_t, and thus the disintegrated measures are more difficult to compute. For instance, if $\mu = \delta_\gamma$ is a periodic orbit measure, the fiber measure $d\mu_\zeta$ is singular with respect to Lebesgue measure along the Fermi normal fibers, and does not possess a derivative at $x_n = 0$.

This type of fibration could be used in Section 5 for the proof of Lemma 5.3, but we find it simpler to use the geodesic fibration.

4.3. **The behavior of $\mu|_{S_H^*M}$ and $\mu|_{S_H^*M}$.** The purpose of this section is to prove Theorem 1.4

Proof. The proof consists of several Lemmas which yield stronger versions of the conclusions.

Lemma 4.3. *The measure $\mu|_{S_H^*M}$ satisfies*

$$\mu|_{S_H^*M}(\mathcal{T}_+) = 0$$

where

$$\mathcal{T}_+ := \{\rho \in S_H^*M \mid T_{S^*H}(\rho) > 0\}, \qquad T_{S^*H}(\rho) := \inf\{t > 0 \mid \varphi_t(\rho) \in S^*H\}.$$

Moreover,

$$\mu|_{S_H^*M}(A) \le \liminf_{T\to 0} \frac{|\{t \in [\min(0,T), \max(0,T)] \mid \varphi_{-t}(A \cap S^*H) \cap A \cap S^*H\}|}{|T|}.$$

Proof. Let $A \subset S_H^*M$ Borel measurable. Then, for any $T > 0$

$$\begin{aligned} \mu|_{S_H^*M}(A) = \mu(A) = \frac{1}{T}\int_0^T \mu(1_{\varphi_{-t}(A)})dt &= \frac{1}{T}\mu\Big(\int_0^T 1_{\varphi_{-t}(A)}(\rho)dt\Big)d\mu(\rho) \\ &\le \frac{|\{t \in [0,T] \mid \varphi_t(A) \cap A \ne \emptyset\}|}{T} \\ &\le \frac{|\{t \in [0,T] \mid \varphi_t(A) \cap S_H^*M \ne \emptyset\}|}{|T|} \end{aligned} \tag{4.7}$$

Now, define

$$(S_H^*M)_\delta = \{\zeta \in S_H^*M : T_{S_H^*M}(\zeta) > 2\delta\},$$

where

$$T_{S_H^*M}(\zeta) := \inf_{t>0}\{\varphi_t(\zeta) \in S_H^*M\}.$$

Then, since $T_{S_H^*M}$ is lower semincontinuous, $(S_H^M)_\delta$ is open and hence measurable. Therefore by (4.7)

$$\mu|_{S_H^*M}((S_H^*M)_\delta) = 0$$

and hence

$$\mu|_{S_H^*M}(\{\zeta \mid T_{S_H^*M}(\zeta) > 0\}) = 0.$$

Note that $S_H^*M \setminus S^*H \subset \{\zeta \mid T_{S_H^*M}(\zeta) > 0\}$ and hence,

$$\mu|_{S_H^*M} = \mu|_{S^*H}.$$

Therefore, arguing as in (4.7),

(4.8)
$$\mu|_{S^*_HM}(A) \le \liminf_{T\to 0} \frac{|\{t\in[\min(0,T),\max(0,T)] \mid \varphi_{-t}(A\cap S^*H)\cap A\cap S^*H\}|}{|T|}$$

Now, with T_{S^*H} as above, define

$$(S^*H)_\delta := \{\zeta \in S^*_HM \mid T_{S^*H}(\zeta) > \delta\}.$$

Then (4.8) implies $\mu|_{S^*_HM}(T_{S^*H} > 0) = 0$.

□

Corollary 4.4. *Define*

$$\mathcal{G}^k := \{\rho \in S^*_HM \mid [H_p^k x_n](\rho) \neq 0,\ [H_p^j x_n](\rho) = 0, j<k\}.$$

Then,

$$\mu|_{S^*_HM}\Big(\bigcup_{k=0}^{\infty} \mathcal{G}^k\Big) = 0.$$

In particular,

$$\mu|_{S^*_HM} = \mu|_{S^*H}$$

and

$$\mu|_{S^*_HM}\big(\{(0,x',\xi') \mid Q(0,x',\xi')\neq 0\}\big) = 0.$$

Proof. Observe that if $\rho \in \mathcal{G}^k$, then,

$$|x_n(\varphi_t(\rho))| \ge ct^k + O(t^{k+1})$$

and in particular, $T_{S^*H}(\rho) > 0$. Therefore, $\mathcal{G}^k \subset \mathcal{T}_+$ and the claim follows. □

This concludes the proof of Theorem 1.4.

□

4.4. **A conjecture.** Theorem 1.4 and accompanying Lemmas leave open some purely dynamical questions concerning the the restriction $\mu|_{S^*_HM} = \mu|_{S^*H}$ of an invariant measure. We state a conjecture which we hope to explore in the future.

We denote by γ a geodesic of (M,g) and also (by abuse of notation) the corresponding orbit of the geodesic flow in S^*M. When (and only when) γ is a periodic geodesic, we denote by δ_γ the normalized periodic orbit measure $\delta_\gamma(f) = \frac{1}{L_\gamma}\int_\gamma f\,ds$ where L_γ is the length of γ.

Conjecture 4.1. *Suppose that μ is an invariant probability measure for the geodesic flow of a compact Riemannian manifold. Suppose that $H \subset M$ is a smooth hypersurface and that $\mu(S^*H) > 0$. Then $\mu|_{S^*H}$ is supported on a union of periodic geodesics γ such that $\gamma \cap S^*H$ has positive arc-length measure and $\mu|_{S^*H\cap\gamma} \ll \delta_\gamma$ (i.e. is absolutely continuous).*

The conjecture is simplest in dimension two, when $\dim H = \dim\gamma$. In that case one consequence of the conjecture is that if $\mu(S^*H) > 0$, then H has positive measure intersection with a periodic geodesic. In the case where (M,g) is of negative curvature, each invariant measure is an orbital averaging measure over the orbit through a quasi-regular point. This orbit may touch S^*H repeatedly, in a quasi-periodic fashion, or it may spiral in to S^*H over a part of the orbit. If the orbital average charges S^*H, we conjecture that it must contain a periodic orbit measure as an ergodic component.

5. Rellich proof of Theorem 1.1

We can now state the main ingredient in the proof of Theorem 1.5. Once that theorem is proved, we will finish the section by proving Theorem 1.1.

Proposition 5.1. *Let $H \subset M$ be a hypersurface in a Riemannian manifold and $I_0(a,\delta,\mu)$ and $II_0(a,\delta,\mu)$ be as in Proposition 3.1. Then*

$$\begin{cases} (i) \liminf_{\delta \downarrow 0} I_0(a,\delta,\mu) = \int_{S^*_H M} a(\zeta)|\xi_n(\zeta)| d\nu^\perp(\zeta); \\ (ii) \lim_{\delta\downarrow 0} II_0(a,\delta,\mu) = 0. \end{cases}$$

where $\nu^\perp$ is defined in (4.3).

5.1. **Proof of Proposition 5.1(ii).** By (3.11), Proposition 5.1(ii) asserts the following:

Lemma 5.2. *Let $H \subset M$ be a hypersurface. For any fixed $\epsilon > 0$,*

$$\lim_{\delta\to 0} \int_{S^*M} \chi^2(\tfrac{x_n}{\delta}) P_2^2(x',\xi',x_n) d\mu = 0.$$

Proof. Note that by the dominated convergence theorem,

$$\lim_{\delta\to 0} \int_{S^*H} \chi^2(\frac{x_n}{\delta}) P_2^2(x',\xi',x_n) d\mu = \int Q^2(0,x',\xi') a^2(x',\xi') d\mu|_{S^*H} = 0$$

where the last equality follows from Corollary 4.4. □

The following lemma completes the proof of Proposition 5.1.

Lemma 5.3. *We have*

$$\lim_{\epsilon\to 0} I_0(a,\epsilon,\mu) = \int_{S^*_H M \setminus S^*H} |\xi_n| a(\pi(q)) d\nu^\perp(q).$$

where $\nu^\perp$ is defined in (4.3).

Proof. Let $\chi_1 \in C_c^\infty(-2,2)$ with $\chi_1 \equiv 1$ on $[-1,1]$. Then, observe that by the dominated convergence theorem, for any $\delta > 0$,

$$\lim_{\epsilon\to 0} \int_{S^*M} \frac{1}{\epsilon} \tilde{\chi}'\Big(\frac{x_n}{\epsilon}\Big) \xi_n^2 \chi_1(\delta \epsilon^{-\frac{1}{2}} \xi_n) a(x',\xi') d\mu = 0.$$

since the integrand is bounded by δ^{-2} on S^*M. Next, observe that for $(x(t),\xi(t)) = \exp(tH_p)(x_0', 0, \xi_0', \xi_n)$,

$$\dot{x}_n(t) = 2\xi_n(t), \qquad |\xi_n(t) - \xi_n(0)| \le C_1|t|, \qquad C_1 := \sup_{S^*M} |H_p \xi_n|. \tag{5.1}$$

Therefore, the map

$$\Psi(t,\zeta) \in \{(t,\zeta) \in \mathbb{R} \times S^*_H M \mid |t| < C_1^{-1}|\xi_n(\zeta)|\} \mapsto \exp(tH_p)(\zeta) \in S^*M$$

is one to one. Suppose that $\zeta_0 \in S^*M$ with $0 < |x_n(\zeta_0)| \le \frac{1}{3C_1^{-1}} |\xi_n(\zeta_0)|^2$. We will show that ζ_0 lies in the image of Ψ. Since the arguments in other cases are the same, we assume $x_n(\zeta_0)$, $\xi_n(\zeta_0) > 0$. Then with $(x(t),\xi(t)) = \exp(tH_p(\zeta_0))$,

$$x_n(t) = x_n(\zeta_0) + 2\int_0^t \xi_n(s) ds, \qquad |\xi_n(s) - \xi_n(\zeta_0)| \le C_1|s|.$$

Therefore, for $-\frac{1}{2C_1}|\xi_n(\zeta_0)| \le t \le 0$,

$$x_n(t) \le x_n(\zeta_0) + \xi_n(\zeta_0) t$$

In particular,

$$x_n(-\frac{1}{3C_1}|\xi_n(\zeta_0)||) \leq x_n(\zeta_0) - \frac{1}{3C_1}|\xi_n(\zeta_0)|^2 \leq 0,$$

and there is $t \in [-\frac{1}{3C_1}|\xi_n(\zeta_0)|, 0]$ such that $x_n(t) = 0$ and $\xi_n(t) \geq \xi_n(\zeta_0)/2$. Therefore, ζ_0 lies in the image of Ψ. In particular, $(t,\zeta) \mapsto \Psi(t,\zeta)$ can be used as coordinates on

$$\{|x_n| \leq \frac{1}{3C_1}|\xi_n|^2\}.$$

Choosing $\delta > 0$ small enough these coordinates are valid on

$$\operatorname{supp} \frac{1}{\epsilon}\tilde{\chi}'\Big(\frac{x_n}{\epsilon}\Big)\xi_n^2(1 - \chi_1(\delta\epsilon^{-\frac{1}{2}}\xi_n)a(x',\xi').$$

Next, recall that by Lemma 4.2 in these coordinates $\mu = dtd\nu^{\perp}(\zeta)$.

$$\begin{aligned}&\int_{S^*M} \frac{1}{\epsilon}\tilde{\chi}'\Big(\frac{x_n}{\epsilon}\Big)\xi_n^2[1 - \chi_1(\delta\epsilon^{-\frac{1}{2}}\xi_n)]a(x',\xi')d\mu\\ &= \int_{S^*_HM}\int_{\mathbb{R}} \frac{1}{\epsilon}\tilde{\chi}'\Big(\frac{x_n(t,\zeta)}{\epsilon}\Big)[\xi_n(t,\zeta)]^2[1 - \chi_1(\delta\epsilon^{-\frac{1}{2}}\xi_n(t,\zeta)]a((x',\xi')(t,\zeta))dtd\nu^{\perp}(\zeta)\end{aligned}$$

Now, since on the support of the integrant $c|\xi_n(0)| \leq |\xi_n(t)| \leq C|\xi_n(0)|$, and $\dot{x_n}(t) = 2\xi_n(t)$, we can change variables $w = \epsilon^{-1}x_n(t,q)$ to obtain

$$\begin{aligned}&\int_{S^*M} \tilde{\chi}'\Big(\frac{x_n}{\epsilon}\Big)\xi_n^2[1 - \chi_1(\delta\epsilon^{-\frac{1}{2}}\xi_n)]a(x',\xi')d\mu\\ &= \frac{1}{2}\int_{S^*_HM}\int_{\mathbb{R}} \tilde{\chi}'(w)|\xi_n(\epsilon w,\zeta)|[1 - \chi_1(\delta\epsilon^{-\frac{1}{2}}\xi_n(\epsilon w,\zeta)]a((x',\xi')(\epsilon w,\zeta))dwd\nu^{\perp}(\zeta).\end{aligned}$$

Then, sending $\epsilon \to 0$ and applying the dominated convergence theorem, we obtain

$$\begin{aligned}\lim_{\epsilon\to 0^+}\int_{S^*M} &\tilde{\chi}'\Big(\frac{x_n}{\epsilon}\Big)\xi_n^2[1 - \chi_1(\delta\epsilon^{-\frac{1}{2}}\xi_n)]a(x',\xi')d\mu\\ &= \frac{1}{2}\int_{S^*_HM}\int_{\mathbb{R}} \tilde{\chi}'(w)|\xi_n(\zeta)|1_{|\xi_n|>0}a(\pi(\zeta))dwd\nu^{\perp}(\zeta)\\ &= -\frac{1}{2}\int_{S^*_HM}\int_{\mathbb{R}} |\xi_n(\zeta)|1_{|\xi_n|>0}a(\pi(\zeta))d\nu^{\perp}(\zeta)\end{aligned}$$

where $\pi : S^*_HM \to B^*H$ denotes the orthogonal projection map. □

Completion of the proof of Theorems 1.5. Observe that by Proposition 3.1,

$$\left|\lim_{h\to 0}\begin{pmatrix}\langle Op_h(a)hD_\nu\varphi_h|_H, hD_\nu\varphi_h|_H\rangle_{L^2(H)}\\ \quad + \big\langle Op_h(a)(1+h^2\Delta_H)\varphi_h|_H, \varphi_h|_H\big\rangle_{L^2(H)}\end{pmatrix} - I_0(a,\epsilon,\mu)\right|$$
$$\leq II_0(a,\epsilon,\mu).$$

Therefore, since by Lemma 5.2, $II_0(q,\epsilon,\mu) \underset{\epsilon\to 0}{\longrightarrow} 0$,

$$\lim_{\epsilon\to 0}\lim_{h\to 0}\begin{pmatrix}\langle Op_h(a)hD_\nu\varphi_h|_H, hD_\nu\varphi_h|_H\rangle_{L^2(H)}\\ \quad + \big\langle Op_h(a)(1+h^2\Delta_H)\varphi_h|_H, \varphi_h|_H\big\rangle_{L^2(H)}\end{pmatrix} - I_0(a,\epsilon,\mu) = 0$$

and, since the term in parentheses is independent of ϵ,

$$\begin{aligned}
&\lim_{h\to 0}\left(\begin{array}{l}\langle Op_h(a)hD_\nu\varphi_h|_H, hD_\nu\varphi_h|_H\rangle_{L^2(H)} \\ \qquad + \left\langle Op_h(a)(1+h^2\Delta_H)\varphi_h|_H, \varphi_h|_H\right\rangle_{L^2(H)}\end{array}\right) \\
&= \lim_{\epsilon\to 0} I_0(a,\epsilon,\mu) \\
&= \int_{S^*_H M\setminus S^*H} |\xi_n| a(\pi(\zeta))d\nu^\perp(\zeta).
\end{aligned}$$

where the last equality follows from Lemma 5.3. This completes the proof of Theorem 1.5. □

5.2. **Completion of the proof of Theorem 1.1.** To complete the proof of Theorem 1.1 we need to combine Theorem 1.5 the Dyatlov-Jin(-Nonnenmacher) theorem. Suppose Theorem 1.1 is false. Then, there exists $a \in C_c^\infty(T^*H)$ with $a \geq 0$ and $\operatorname{supp} a \cap B^*H \neq \emptyset$, $h_j \to 0$ such that φ_{h_j} is a Laplace eigenfunction with eigenvalue h_j^{-2} and

$$\lim_{j\to\infty} \langle Op_{h_j}(a)(1+h_j^2\Delta_H)\varphi_{h_j}, \varphi_{h_j}\rangle_{L^2(H)} + \langle Op_{h_j} h_j D_\nu\varphi_{h_j}, h_j D_\nu \varphi_{h_j}\rangle_{L^2(H)} = 0.$$

Now, we can extract a subsequence such that u has defect measure μ and its renormalized Cauchy data has defect measure μ^{RCD}. Then, by Theorem 1.5

$$\begin{aligned}
&\lim_{j\to\infty} \langle Op_{h_j}(a)(1+h_j^2\Delta_H)\varphi_{h_j}, \varphi_{h_j}\rangle_{L^2(H)} + \langle Op_{h_j} h_j D_\nu\varphi_{h_j}, h_j D_\nu \varphi_{h_j}\rangle_{L^2(H)} \\
&\qquad = \int a d\mu^{RCD} = \int_{S^*_H M\setminus S^*H} a(\pi(\zeta))|\xi_n(\zeta)|d\nu^\perp(\zeta).
\end{aligned}$$

The proof of Theoerm 1.1 will be completed by the following lemma.

Lemma **2.** *Let (M,g) be a negatively curved surface, and let $H \subset M$ be a smooth curve. Then for any $a \in C(B^*H)$ satisfying $a \geq 0$ and $a \neq 0$, there exists $C_{a,H} > 0$ so that*

$$\int_{S^*_H M\setminus S^*H} |\xi_n| a(\pi(\zeta))d\nu^\perp(\zeta) > C_{a,H} > 0.$$

Proof. By the Dyatlov-Jin(-Nonnenmacher)theorem, the support of μ is S^*M. Hence, $\mu \neq \mu|_{S^*H}$, and the support of $\mu^\perp$ is also S^*M. It then follows from the invariance of μ under the geodesic flow that the support of $d\nu^\perp$ is equal to $S^*_H M$. Since a is continuous, there is $q \in \operatorname{supp} a$ with $|\xi_n(\zeta)| > 0$. In particular, there is an open neighborhood U of ζ such that $a(\pi(\zeta))|\xi_n(\zeta)| > c > 0$ In particular, since $a \geq 0$,

$$\int_{S^*_H M\setminus S^*H} |\xi_n| a(\pi(\zeta))d\nu^\perp(\zeta) \geq c\nu^\perp(U) C_{a,H} > 0$$

.

□

6. Proof of Theorem 1.2 via hyperbolic equations

In this section, we use hyperbolic equations to prove Theorem 1.2. The idea is that H is a Cauchy surface for a hyperbolic problem. In particular, when geodesics intersect H transversally, we can think of H as a Cauchy surface for the problem

$$((hD_{x_n})^2 - R(x, hD_{x'}))u = 0,$$

where $H = \{x_n = 0\}$. Therefore, microlocally in this region, $(u|_H, hD_\nu u|_H) \mapsto u$ is a continuous map.

We start by factoring the operator $-h^2\Delta_g - 1$ in the hyperbolic region. This lemma is a semiclassical version of [HoIII, Lemma 23.2.8].

Lemma **3.** *For all $\epsilon > 0$ and $\delta > 0$ small, there are*

$$\Lambda_\pm = \Lambda_\pm \in C^\infty((-\delta, \delta); \Psi^{\mathrm{comp}}(\mathbb{R}^{n-1})),$$

$\tilde{\Lambda}_\pm = \tilde{\Lambda}_\pm(x, hD_{x'})$ *with*

$$\sigma(\Lambda) = \sigma(\tilde{\Lambda}) = \sqrt{1 - r(x, \xi')}, \qquad r(x, \xi') < 1 - \epsilon^2$$

such that for all $b \in C^\infty((-\delta, \delta); S^{\mathrm{comp}}(T^*\mathbb{R}^{n-1}))$ with $\operatorname{supp} b \subset \{r(x, \xi') < 1 - \epsilon\}$,

$$\begin{aligned} b(x, hD_{x'})(-h^2\Delta_g - 1) &= b(x, hD_{x'})(hD_{x_n} - \Lambda_-)(hD_{x_n} + \Lambda_+) + O(h^\infty)_{L^2 \to L^2} \\ &= b(x, hD_{x'})(hD_{x_n} + \tilde{\Lambda}_+)(hD_{x_n} - \tilde{\Lambda}_-) + O(h^\infty)_{L^2 \to L^2} \end{aligned}$$

Proof. Fix $\chi = \chi(x, \xi') \in C_c^\infty(\mathbb{R}^{2n-1})$ with

$$\chi \equiv 1 \text{ on } \{r(x, \xi') < 1 - \epsilon^2\}, \qquad \operatorname{supp} \chi \subset \{r(x, \xi') < 1 - \frac{\epsilon^2}{2}\}$$

and set

$$\lambda_0 = \chi\sqrt{1 - r(x, \xi')}, \qquad \Lambda_0 = \lambda_0(x, hD_{x'}).$$

Recall that

$$-h^2\Delta_g - 1 = Op_h(|\xi_n|^2 + r(x, \xi') - 1) + h(a(x)hD_{x_n} + e(x, hD_{x'})).$$

Then,

$$\begin{aligned} &b(x, hD_{x'})(hD_{x_n} - \Lambda_0)(hD_{x_n} + \Lambda_0) \\ &= b(x, hD_{x'})(hD_{x_n}^2 - \Lambda_0^2 + [hD_{x_n}, \Lambda_0]) \\ &= b(x, hD_{x'})(-h^2\Delta_g - 1 - ha(x)hD_{x_n} - hr_0(x, hD_{x'})) + O(h^\infty)_{L^2 \to L^2}. \end{aligned}$$

To obtain a finer factorization for $(-h^2\Delta_g - 1)$, we put

$$\lambda_1^- = a(x) + \frac{r_0\chi^2}{2\lambda_0}, \qquad \lambda_1^+ = \frac{r_0\chi^2}{2\lambda_0}$$

and write

$$\Lambda_1^- = \Lambda_0 + h\lambda_1^-(x, hD_{x'}), \qquad \Lambda_1^+ = \lambda_0 + h\lambda_1^+(x, hD_{x'}).$$

Then, we have

$$b(x, hD_{x'})(hD_{x_n} - \Lambda_1^-)(hD_{x_n} + \Lambda_1^+) = b(x, hD_{x'})(-h^2\Delta_g - 1 - h^2 r_1(x, hD_{x'})) + O(h^\infty)_{L^2 \to L^2}.$$

Define $r_j(x, \xi')$, $j \geq 1$ iteratively by

$$b(x, hD_{x'})(hD_{x_n} - \Lambda_j^-)(hD_{x_n} + \Lambda_j^+) = b(x, hD_{x'})(-h^2\Delta_g - 1 - h^{j+1} r_j(x, hD_{x'})) + O(h^\infty)_{L^2 \to L^2}.$$

Then, for $j \geq 2$, define $\Lambda_j^\pm$ by

$$\Lambda_j^\pm = \Lambda_{j-1}^\pm + h^j\lambda_j(x, hD_{x'}), \qquad \lambda_j(x, hD_{x'}) = \frac{r_{j-1}\chi^2}{2\lambda_0}.$$

Letting $\Lambda_\pm \sim \sum_{j=0}^\infty h^j\lambda_j^\pm(x, hD_{x'})$, the claim is proved for the first factorization.

The proof is nearly identical for the other factorization.

$\square$

Next, we use the factorization from Lemma 3 to produce propagation estimates for the Cauchy problem posed on $\{x_n = 0\}$.

LEMMA **4.** *Let $b_0 \in C_c^\infty(T^*H)$ with $\operatorname{supp} b_0 \subset B^*H := \{(x',\xi') \mid r(0,x',\xi') < 1\}$. Then there is $\delta > 0$ such that if $a \in C_c^\infty(T^*M)$ with*

$$\operatorname{supp} a \cap S^*M \subset \bigcup_{|t|<\delta} \varphi_t(\pi^{-1}(\{|b_0| > 0\}))$$

where $\pi : S_H^*M \to T^*H$ denotes the projection and $\varphi_t := \exp(tH_{|\xi|_g^2})$, we have the estimate

$$\begin{aligned}\|Op_h(a)u\|_{L^2(M)} \leq C\|Op_h(b_0)u|_H\|_{L^2(H)} + C\|Op_h(b_0)h\partial_\nu u|_H\|_{L^2(H)} \\ + Ch^{-1}\|(-h^2\Delta_g - 1)u\|_{L^2}) + O(h^\infty)\|u\|_{L^2}.\end{aligned}$$

Proof. Fix b_0 as above and let $\epsilon > 0$ such that $\operatorname{supp} b_0 \subset \{r(x,\xi') < 1-\epsilon\}$. Next, let $\Lambda_\pm$, $\tilde{\Lambda}_\pm$ as in Lemma 3 and $\lambda = \sqrt{1 - r(x,\xi')}$. Finally, let $\tilde{b}_0 \in C_c^\infty(T^*H)$ with $\operatorname{supp} \tilde{b} \subset \operatorname{supp} b_0$, and $c > 0$ such that

$$\operatorname{supp} a \cap S^*M \subset \bigcup_{|t|<\delta} \varphi_t(\pi^{-1}(\{|\tilde{b}| > c/2\})). \tag{6.1}$$

We start by defining $b^- \in C^\infty((-3\delta_0, 3\delta_0); S^{\mathrm{comp}}(T^*\mathbb{R}^{n-1})$ such that

$$\mathrm{WF_h}([Op_h(b^-), D_{x_n} - \Lambda_-] \cap \{|x_n| < 2\delta_0\} = \emptyset. \tag{6.2}$$

and $b^-(0,x',\xi') = \tilde{b}$. To do this, define $\tilde{b}_0 = \tilde{b}_0(x,\xi')$ by

$$\tilde{b}_0(0,x',\xi') = \tilde{b}, \qquad (\partial_{x_n} - H_\lambda)\tilde{b}_0 = 0, \tag{6.3}$$

Next, define iteratively for $j \geq 1$,

$$h^j Op_h(e_{j-1}) = ih^{-1}\Big[hD_{x_n} - \Lambda_-, Op_h\Big(\sum_{k=0}^{j-1} h^k \tilde{b}_k\Big)\Big], \qquad \begin{aligned}(\partial_{x_n} - \lambda)\tilde{b}_j = e_j, \\ \tilde{b}_j(0,x',\xi') = 0.\end{aligned}$$

Then, putting $b^- \sim \sum_j h^j \tilde{b}_j$, we have (6.2).

Note that there is $\delta_0 > 0$ depending only on $\epsilon > 0$ such that a solution to (6.3) exists for $|x_n| < \delta_0$ and,

$$\operatorname{supp} b^- \cap \{|x_n| < 3\delta_0\} \subset \{r(x,\xi') < 1 - \epsilon^2\}.$$

By standard energy estimates (see e.g [HoIII, Lemma 23.1.1])

$$\begin{aligned}&\|b^-(x,hD_{x'})(hD_{x_n} + \Lambda_+)u\|_{L^2} \\ &\leq C(\|Op_h(\tilde{b})(hD_{x_n} + \Lambda_+)\|_{L^2(H)} + h^{-1}\|(hD_{x_n} - \Lambda_-)b^-(x,hD_{x'})(hD_{x_n} + \Lambda_+)u\|_{L^2(|x_n|<\delta_0)}.\end{aligned}$$

Next, observe that by (6.2), for $\chi \in C_c^\infty((-2\delta_0, 2\delta_0))$

$$\chi(x_n)[hD_{x_n} - \Lambda, b^-(x,hD_{x'})] = O(h^\infty)_{L^2 \to L^2}$$

and hence

$$\begin{aligned}
&\|b^-(x,hD_{x'})(hD_{x_n}+\Lambda_+)u\|_{L^2}\\
&\leq C(\|Op_h(\tilde{b})(hD_{x_n}+\Lambda_+)\|_{L^2(H)}+h^{-1}\|b^-(x,hD_{x'})(hD_{x_n}-\Lambda_-)(hD_{x_n}+\Lambda_-)u\|_{L^2}\\
&\qquad+O(h^\infty)\|(hD_{x_n}+\Lambda_+)u\|_{L^2}\\
&=C(\|Op_h(\tilde{b})(hD_{x_n}+\Lambda_+)\|_{L^2(H)}+h^{-1}\|(-h^2\Delta_g-1)u\|_{L^2}\\
&\qquad+O(h^\infty)\|(hD_{x_n}+\Lambda_+)u\|_{L^2}+O(h^\infty)\|u\|_{L^2}
\end{aligned}$$

Next, by the elliptic parametrix construction

$$\|(hD_{x_n}-\Lambda_-)u\|_{L^2}\leq C\|(-h^2\Delta_g+1)u\|_{L^2}\leq C(\|(-h^2\Delta_g-1)u\|_{L^2}+\|u\|_{L^2})$$

Therefore,

$$\begin{aligned}
(6.4)\quad&\|b^-(x,hD_{x'})(hD_{x_n}+\Lambda_+)u\|_{L^2}\\
&\leq C(\|Op_h(\tilde{b})(hD_{x_n}+\Lambda_+)u\|_{L^2(H)}+h^{-1}\|(-h^2\Delta_g-1)u\|_{L^2}+O(h^\infty)\|u\|_{L^2}).
\end{aligned}$$

Next, we construct $b^+\in C^\infty((-3\delta_0,\delta_0);S^{\mathrm{comp}}T^*\mathbb{R}^{n-1})$, such that defining b^+ such that

$$\mathrm{WF_h}([Op_h(b^+),D_{x_n}+\tilde{\Lambda}_+]\cap\{|x_n|<2\delta_0\}=\emptyset. \tag{6.5}$$

In particular, we start with $\tilde{b}_0=\tilde{b}_0(x,\xi')$ such that

$$\tilde{b}_0(0,x',\xi')=\tilde{b},\qquad(\partial_{x_n}+H_\lambda)\tilde{b}_0=0 \tag{6.6}$$

and proceed as in the construction of b^-.

We then obtain the estimate

$$\begin{aligned}
(6.7)\quad&\|b^+(x,hD_{x'})(hD_{x_n}-\tilde{\Lambda}_-)u\|_{L^2}\\
&\leq C(\|Op_h(\tilde{b})(hD_{x_n}-\Lambda_-)u\|_{L^2(H)}+h^{-1}\|(-h^2\Delta_g-1)u\|_{L^2}+O(h^\infty)\|u\|_{L^2}).
\end{aligned}$$

Next, observe that on $\{\mp\xi_n>0\}\cap S^*M$,

$$(\partial_{x_n}\pm H_\lambda)=(\xi_n\mp\lambda)^{-1}H_{|\xi|_g^2-1}.$$

Therefore, by (6.3) and (6.6), $\sigma(b^\pm)$ is locally invariant under the geodesic flow on $S^*M\cap\{\mp\xi_n>0\}$. In particular, there is $\delta_1>0$ such that

$$\{|b^\pm|>c>0\}\cap S^*M\supset\bigcup_{|t|<\delta_1}\varphi_t(\{(x,\xi)\in S^*_HM\mid\mp\xi_n>0,\,|\tilde{b}(x,\xi')|>c/2>0\}.$$

Therefore, there is $c>0$ such that

$$[b^+(\xi_n-\lambda)]^2+[b^-(\xi_n+\lambda)]^2>c>0\qquad\text{on }\bigcup_{|t|<\delta_1}\varphi_t(\{(x,\xi)\in S^*_HM\mid|\tilde{b}(x,\xi')|>c/2\}$$

In particular, by the elliptic parametrix construction and (6.1) there are $e_i\in C^\infty_c(T^*M)$, $i=1,2,3$ such that

$$\begin{aligned}
(6.8)\quad&Op_h(a)=Op_h(e_1)b^+(x,hD_{x'})(hD_{x_n}-\tilde{\Lambda}_-)\\
&+Op_h(e_1)b^-(x,hD_{x'})(hD_{x_n}+\Lambda_+)+Op_h(e_2)(-h^2\Delta_g-1)+O(h^\infty)_{L^2\to L^2}.
\end{aligned}$$

Therefore, by (6.4), (6.7), and (6.8)

$$\begin{aligned}\|Op_h(a)u\|_{L^2} \leq C\|Op_h(\tilde{b})(hD_{x_n}+\Lambda_+)u\|_{L^2(H)} + C\|Op_h(\tilde{b})(hD_{x_n}-\tilde{\Lambda}_-)u\|_{L^2(H)}\\ + Ch^{-1}\|(-h^2\Delta_g-1)u\|_{L^2} + O(h^\infty)\|u\|_{L^2}\end{aligned}$$

Finally, note that

$$\begin{aligned}\|Op_h(\tilde{b})(hD_{x_n}+\Lambda_+)u\|_{L^2(H)} &\leq \|Op_h(\tilde{b})hD_\nu u\|_{L^2(H)} + \|Op_h(\tilde{b})\Lambda_+ u\|_{L^2(H)}\\ &\leq \|Op_h(b_0)hD_\nu u\|_{L^2(H)} + \|Op_h(b_0)u\|_{L^2(H)} + O(h^\infty)\|u\|_{L^2(H)}\\ &\leq \|Op_h(b_0)hD_\nu u\|_{L^2(H)} + \|Op_h(b_0)u\|_{L^2(H)}\\ &\qquad + O(h^\infty)(\|(-h^2\Delta_g-1)u\|_{L^2}+\|u\|_{L^2})\end{aligned}$$

where in the next to last line we use that $\operatorname{supp}\tilde{b} \subset \operatorname{supp} b_0$ and in the last line, we use that Sobolev embedding. Similarly,

$$\begin{aligned}\|Op_h(\tilde{b})(hD_{x_n}-\tilde{\Lambda}_-)u\|_{L^2(H)} \leq \|Op_h(b_0)hD_\nu u\|_{L^2(H)} + \|Op_h(b_0)u\|_{L^2(H)}\\ + O(h^\infty)(\|(-h^2\Delta_g-1)u\|_{L^2}+\|u\|_{L^2}),\end{aligned}$$

which completes the proof. □

Proof of Theorem 1.2. Suppose (1.3) does not hold. Then,

$$\lim_{h\to 0}\|Op_h(b_0)u|_H\|_{L^2(H)} + \|Op_h(b_0)h\partial_\nu u|_H\|_{L^2(H)} = 0.$$

In particular, by Lemma 4, there is $a \in C_c^\infty(T^*M)$ with $a \cap S^*M \neq 0$ such that

$$\lim_{h\to 0}\|Op_h(a)u\|_{L^2(M)} = 0.$$

But, this contradictions the results of [DJ17, DJN19].

To prove the second claim, observe that by unique continuation, see e.g. [GL17, Theorem 1.7], there is $c > 0$ such that for all $h > 0$,

$$\|u\|_{L^2(M)} < Ce^{C/h}(\|u|_H\|_{L^2(U)} + \|h\partial_\nu u|_H\|_{L^2(U)}). \tag{6.9}$$

Combining (6.9) with (1.3) proves (1.4). □

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