ASYMPTOTICS FOR THE SPECTRAL FUNCTION ON ZOLL MANIFOLDS

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Abstract. On a smooth, compact, Riemannian manifold without boundary (M, g), let Δ_g be the Laplace–Beltrami operator. We define the orthogonal projection operator

$$\Pi_{I_\lambda} : L^2(M) \to \bigoplus_{\lambda_j \in I_\lambda} \ker(\Delta_g + \lambda_j^2)$$

for an interval I_λ centered around $\lambda \in \mathbb{R}$ of a small, fixed length. The Schwartz kernel, $\Pi_{I_\lambda}(x, y)$, of this operator plays a key role in the analysis of monochromatic random waves, a model for high energy eigenfunctions. It is expected that $\Pi_{I_\lambda}(x, y)$ has universal asymptotics as $\lambda \to \infty$ in a shrinking neighborhood of the diagonal in $M \times M$ (provided I_λ is chosen appropriately) and hence that certain statistics for monochromatic random waves have universal behavior. These asymptotics are well known for the torus and the round sphere, and were recently proved to hold near points in M with few geodesic loops by Canzani–Hanin. In this article, we prove that the same universal asymptotics hold in the opposite case of Zoll manifolds; that is, manifolds all of whose geodesics are closed with a common period.

1. Introduction

Let (M, g) be a compact, Riemannian manifold without boundary and write Δ_g for the associated (negative definite) Laplace-Beltrami operator. Denote the eigenvalues of $-\Delta_g$ by $0 = \lambda_0^2 < \lambda_1^2 \leqslant \lambda_2^2 \leqslant \cdots$ repeated according to multiplicity. For $I \subset \mathbb{R}$ consider the orthogonal projection operator

$$\Pi_I : L^2(M) \to \bigoplus_{\lambda_j \in I} \ker(\Delta_g + \lambda_j^2).$$

Letting $\{\varphi_j\}_{j=0}^{\infty}$ be an orthonormal basis of $L^2(M)$ such that

$$-\Delta_g \varphi_j = \lambda_j^2 \varphi_j, \quad j = 0, 1, 2, \ldots, \tag{1.1}$$

the Schwarz kernel of Π_I takes the form

$$\Pi_I(x, y) = \sum_{\lambda_j \in I} \varphi_j(x)\overline{\varphi_j(y)}, \qquad x, y \in M. \tag{1.2}$$

Notably, on a general compact smooth manifold with no boundary, the Weyl Law states that

$$\#\{j : \lambda_j \leqslant \lambda\} = \int_M \Pi_{[0,\lambda]}(x, x) d\mathrm{vol}_g(x) = C_n \mathrm{vol}_g(M)\lambda^n + R(\lambda), \tag{1.3}$$

where $R(\lambda) = \mathcal{O}(\lambda^{n-1})$ as $\lambda \to \infty$ [Wey12, Lev53, Ava56, Hö68]. This remainder term is sharp. It is saturated, for example, on the round sphere, $\mathbb{S}^n$. Indeed, it is saturated on any *Zoll manifold* (M, g); i.e. a smooth compact Riemannian manifold without boundary all of whose geodesics are periodic with common minimal period. However, when the set

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of closed geodesics has measure zero in S^*M, the remainder, $R(\lambda)$, can be improved to $o(\lambda^{n-1})$ [DG75].

In this article, we study the asymptotics as $\lambda \to \infty$ of spectral projectors of the form $\Pi_{I_\lambda}(x,y)$, where I_λ is an interval, centered at λ, with length uniformly bounded from above and below (or, possibly, length shrinking slowly with λ). These spectral projectors appear in the field of random waves as the covariance kernels of so-called monochromatic random waves. As with the properties of $\Pi_{[0,\lambda]}(x,y)$, the asymptotics of Π_{I_λ} are intimately connected to the dynamics of the geodesic flow on (M,g).

The most classical random wave studies occur on the round sphere, $\mathbb{S}^n$, and flat torus, $\mathbb{T}^n$ (see, for instance, the survey [Wig22]). In the case of the sphere,

$$\lambda_\ell^2 = \ell(\ell+n-1) \qquad \ell = 0,1,\dots,$$

and it is known that, with $\nu_\ell := \ell + \frac{n-1}{2}$, for $x,y \in \mathbb{S}^n$ with $d_g(x,y) \leqslant r_\ell$ and $\lim_{\ell\to\infty} r_\ell = 0$,

$$\Pi_{\{\lambda_\ell\}}(x,y) = \Pi_{[\nu_\ell-\frac14,\nu_\ell+\frac14]}(x,y) = \frac{\nu_\ell^{n-1}}{(2\pi)^{n/2}}\frac{J_{\frac{n-2}{2}}(|\nu_\ell d_g(x,y)|)}{(\nu_\ell d_g(x,y))^{\frac{n-2}{2}}} + o(\nu_\ell^{n-1}), \quad \ell\to\infty.$$

Here, we write $d_g(x,y)$ for the Riemannian distance between x and y and J_α is the Bessel function of the first kind with index α. We note that for $v \in \mathbb{R}^n$

$$\frac{1}{(2\pi)^{n/2}}\frac{J_{\frac{n-2}{2}}(|v|)}{|v|^{\frac{n-2}{2}}} = \frac{1}{(2\pi)^n}\int_{S^{n-1}} e^{i\langle v,\omega\rangle}\, d\sigma_{\mathbb{S}^{n-1}}(\omega). \tag{1.4}$$

It will be useful below to interpret the Bessel term in (1.7) as an integral over S_y^*M, and we refer the reader to Remark 2.2 for this.

Despite the fact that the dynamics of the geodesic flow on the n-dimensional flat torus, are dramatically different than those on the sphere, there we also have for $x,y \in \mathbb{T}^n$ with $d_g(x,y) \leqslant r_\nu$ and $\lim_{\nu\to\infty} r_\nu = 0$,

$$\Pi_{[\nu-\frac12,\nu+\frac12]}(x,y) = \frac{\nu^{n-1}}{(2\pi)^{n/2}}\frac{J_{\frac{n-2}{2}}(|\nu d_g(x,y)|)}{(\nu d_g(x,y))^{\frac{n-2}{2}}} + o(\nu^{n-1}), \qquad \nu\to\infty.$$

Indeed, perhaps surprisingly, in contrast to the fact that Π_{I_λ} encodes a great deal of dynamical information, one expects that the local behavior of Π_{I_λ} is, in some sense, universal.

Conjecture 1.1. *Let (M,g) be a smooth, compact, Riemannian manifold of dimension n without boundary and $x \in M$. Then, there exist $c>0$, a sequence $\nu_\ell \to \infty$, and a sequence $0<\varepsilon_\ell<c$ such that for $y\in M$ with $d_g(x,y) = r_\ell$ and $\lim_{\ell\to\infty} r_\ell = 0$,*

$$\Pi_{[\nu_\ell-\varepsilon_\ell,\nu_\ell+\varepsilon_\ell]}(x,y) = \Pi_{[\nu_\ell-\varepsilon_\ell,\nu_\ell+\varepsilon_\ell]}(x,x)\frac{(2\pi)^{n/2}}{\mathrm{vol}(\mathbb{S}^{n-1})}\frac{J_{\frac{n-2}{2}}(|\nu_\ell d_g(x,y)|)}{(\nu_\ell d_g(x,y))^{\frac{n-2}{2}}} + o(\nu_\ell^{n-1}), \qquad \ell\to\infty. \tag{1.5}$$

In [CH15, CH18], Canzani–Hanin showed that the asymptotics (1.5) hold whenever x is a non-self focal point. That is, the set of directions $\xi \in S_x^*M$ that generate a geodesic loop that returns to x has Liouville measure zero. As for the flat torus, in the case of non-self focal points, one can take any sequence $\nu_\ell \to \infty$ and $\varepsilon_\ell = 1$. In this article, we study the case of Zoll manifolds which have, in some sense, the opposite dynamical behavior from manifolds without conjugate points. This is a rich class of manifolds that includes compact rank one symmetric spaces. Indeed, while the most well known example of a Zoll manifold is the round sphere, $\mathbb{S}^2$, the moduli space of Zoll metrics on $\mathbb{S}^2$ is infinite dimensional [Gui76].

It is well known that, like it happens for the sphere of radius $\frac{T}{2\pi}$, the eigenvalues of $-\Delta_g$ on a Zoll manifold of period T are strongly clustered near the sequence

$$\nu_\ell := \frac{2\pi}{T}\Big(\ell + \frac{\mathfrak{a}}{4}\Big), \quad \ell = 0, 1, 2, \dots, \tag{1.6}$$

where $\mathfrak{a}$ is the common Maslov index of the closed geodesics [Wei75, Wei77, DG75, CdV79, Cha80].

Because of this sphere-like clustering, it is too much to hope that (1.5) holds for any choice of $\nu_\ell \to \infty$ and we should instead work with spectral projectors for a well chosen sequence ν_ℓ as in the case of the round sphere. In particular, we take ν_ℓ as in (1.6).

Our main theorem shows that Conjecture 1.1 is true on a Zoll manifold.

Theorem 1. *Let (M, g) be a smooth Zoll manifold of dimension $n \geqslant 2$ with uniform period $T > 0$, and let the sequence $\{\nu_\ell\}$ be as in (1.6). Let $R_\varepsilon(\ell; x, y)$ be defined by*

$$\Pi_{[\nu_\ell-\varepsilon,\nu_\ell+\varepsilon]}(x,y) = \frac{2\pi}{T}\frac{\nu_\ell^{n-1}}{(2\pi)^{n/2}}\frac{J_{\frac{n-2}{2}}(|\nu_\ell d_g(x,y)|)}{(\nu_\ell d_g(x,y))^{\frac{n-2}{2}}} + R_\varepsilon(\ell; x, y), \tag{1.7}$$

Then, for any $0 < \varepsilon < \frac{2\pi}{T}$ and every pair of multi-indices $\alpha, \beta \in \mathbb{N}^n$,

$$\lim_{\delta\to 0^+}\limsup_{\ell\to\infty}\sup_{d_g(x,y)\leqslant\delta}\left|\nu_\ell^{1-n-|\alpha|-|\beta|}\partial_x^\alpha\partial_y^\beta R_\varepsilon(\ell; x, y)\right| = 0 \tag{1.8}$$

The on-diagonal version of this result, without derivatives, is proved in [Zel97, Theorem 2] for Zoll manifolds that are simply connected (SC_T manifolds in their language). We also refer the reader to [DG75, Zel92, UZ93, Zel97, Zel09] for related studies on the spectra of Zoll manifolds.

As discussed briefly before, the main motivation for proving Theorem 1 is its application to the theory of random waves on manifolds. A *monochromatic random wave* on (M, g) is a Gaussian random field of the form

$$\psi_{\lambda,\varepsilon}(x) := \frac{1}{\dim(H_{\lambda,\varepsilon})^{1/2}}\sum_{\lambda_j\in[\lambda-\varepsilon,\lambda+\varepsilon]} a_j\varphi_j(x), \tag{1.9}$$

where the a_j are i.i.d. standard Gaussian random variables, the φ_j are the eigenfunctions in (1.1), $\varepsilon > 0$, and

$$H_{\lambda,\varepsilon} := \bigoplus_{\lambda_j\in[\lambda-\varepsilon,\lambda+\varepsilon]} \ker\left(\Delta_g + \lambda_j^2\right).$$

Monochromatic random waves were created to model eigenfunction behavior. Indeed, although $\psi_{\lambda,\varepsilon}$ is not an actual eigenfunction, it is expected to behave like one. (For a careful account of the history, see [Can20, Wig22] and references there.) In particular, much research has been dedicated to understanding the behavior of the zero sets and critical points of random waves. The corresponding features of deterministic eigenfunctions are very difficult to study, and their analysis becomes much more tractable for the monochromatic random counterparts.

The statistics of $\psi_{\lambda,\varepsilon}$ are completely determined by the associated two-point correlation function

$$K_{\lambda,\varepsilon}(x,y) := \mathrm{Cov}\left(\psi_{\lambda,\varepsilon}(x), \psi_{\lambda,\varepsilon}(y)\right) = \frac{1}{\dim(H_{\lambda,\varepsilon})}\Pi_{[\lambda-\varepsilon,\lambda+\varepsilon]}(x,y), \qquad x, y \in M.$$

Most research is typically done on the round sphere or the flat torus since $K_{\lambda,\varepsilon}$ is well understood for these spaces [BMW20, CW17, KKW13, BCW19, Cam19, CMW16, NS09, RW08]. Furthermore, studying features like the zero sets and critical points of $\psi_{\lambda,\varepsilon}$ relies on having asymptotics for $K_{\lambda,\varepsilon}(x,y)$ when $x,y \in B(x_0,\frac{1}{\lambda})$ with x_0 fixed. Although treating $K_{\lambda,\varepsilon}$ on general manifolds is quite challenging, Conjecture 1.1 implies that, when the windows of eigenvalues defining the sum in (1.9) are appropriately chosen,

$$K_{\lambda,\varepsilon}\Big(\exp_{x_0}\big(\tfrac{u}{\lambda}\big), \exp_{x_0}\big(\tfrac{v}{\lambda}\big)\Big)$$

should converge for all $u,v \in T^*_{x_0}M$ to a universal limit that is independent of the topology or geometry of (M,g). Here, $\exp_{x_0}: T^*_{x_0}M \to M$ denotes the exponential map with footpoint at x_0. The following corollary proves this conjecture in the setting of Zoll manifolds.

Corollary 1.2. *Let (M,g) be as in Theorem 1. Let $x_0 \in M$ and $0 < \varepsilon < \frac{2\pi}{T}$. Then, for any function $\ell \mapsto r_\ell$ with $r_\ell = o(\nu_\ell)$ and every pair of multi-indices $\alpha, \beta \in \mathbb{N}^n$,*

$$\lim_{\ell\to\infty} \sup_{|u|,|v|\leqslant r_\ell} \left| \partial_u^\alpha \partial_v^\beta \left(K_{\nu_\ell,\varepsilon}\Big(\exp_{x_0}\big(\tfrac{u}{\nu_\ell}\big), \exp_{x_0}\big(\tfrac{v}{\nu_\ell}\big)\Big) - \frac{(2\pi)^{n/2}}{\mathrm{vol}(S^*M)} \frac{J_{\frac{n-2}{2}}(|u-v|)}{(|u-v|)^{\frac{n-2}{2}}} \right) \right| = 0. \quad (1.10)$$

Results about Conjecture 1.1 yield corresponding asymptotics for the covariance function of monochromatic random wave. Indeed, for a general manifold (M,g), when the interval in (1.9) is $[\lambda-\frac{1}{2},\lambda+\frac{1}{2}]$, the asymptotics from [CH15, CH18] show that (1.10) holds when the point x_0 is non self-focal. In the case where (M,g) has no conjugate points, the asymptotics in (1.10) hold at every point with a logarithmic improvement on the rate of decay to 0 [Kee21].

In the language of Nazarov-Sodin [NS16], if the asymptotics in (1.10) hold at every $x_0 \in M$, then the random waves $\psi_{\lambda,\varepsilon}$ have translation invariant local limits. For ensembles with such translation invariant local limits, Zelditch [Zel09], Nazarov-Sodin [NS16], Sarnak-Wigman [SW19], Gayet-Welschinger [GW16], Canzani-Sarnak [CS19], Canzani-Hanin [CH20] as well as others, prove detailed results on non-integral statistics of the nodal sets of random waves. Such nodal set statistics include the number of connected components, Betti numbers, and topological types. In particular, Corollary 1.2 shows that Zoll manifolds have translation invariant local limits and hence that the corresponding results about statistics of nodal sets and critical points extend to the case of Zoll manifolds.

1.1. **Organization of the paper.** In Section 2 we find asymptotics for the smoothed spectral projector $\rho * \Pi_{[\lambda-\varepsilon,\lambda+\varepsilon]}$, where ρ is an appropriately chosen Schwartz function. In Section 3 we prove some on-diagonal cluster estimates that allow us to concentrate the study of the asymptotics in an $\ell^{-1/2}$ neighborhood of ν_ℓ. Finally, in Section 4 we prove Theorem 1 and Corollary 1.2 by using the on-diagonal estimates obtained in Section 3 to undo the convolution with ρ and hence obtain the desired asymptotics for $\Pi_{[\nu_\ell-\varepsilon,\nu_\ell+\varepsilon]}$.

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2. Analysis of the Smoothed Projector

We begin by analyzing a smoothed version of the spectral function. We observe that $\Pi_{[\lambda-\varepsilon,\lambda+\varepsilon]}(x,y)$ can be rewritten as

$$\Pi_{[\lambda-\varepsilon,\lambda+\varepsilon]}(x,y) = \sum_{j=0}^{\infty} \mathbb{1}_{[-\varepsilon,\varepsilon]}(\lambda-\lambda_j)\varphi_j(x)\overline{\varphi_j(y)},$$

where $\mathbb{1}_{[a,b]}$ denotes the characteristic function of the interval $[a,b]$. Next, we introduce $\rho \in \mathscr{S}(\mathbb{R})$ with the property that $\widehat{\rho}$ is supported in $[-2,2]$ and equal to one on $[-1,1]$. Let ρ_σ denote rescaling by $\sigma > 0$ as before, so that

$$\widehat{\rho}_\sigma(t) = \widehat{\rho}(\sigma t) \tag{2.1}$$

is supported in $[-2/\sigma, 2/\sigma]$ and equal to one on $[-1/\sigma, 1/\sigma]$. The goal of this section is to study the asymptotic behavior of $\rho_\sigma * \Pi_{[\lambda-\varepsilon,\lambda+\varepsilon]}$. This is done in Proposition 2.4 below. In preparation for this result, in Section 2.1 we first rewrite $\rho_\sigma * \Pi_{[\lambda-\varepsilon,\lambda+\varepsilon]}$ in terms of the kernel of the half wave operator and its singularities. Later, in Section 2.2, we find the asymptotic behavior of the kernel when localized to each singularity. We finally state and prove Proposition 2.4 which combines these estimates to obtain asymptotics for the full projector.

2.1. Singularities of the half-wave operator.

To study the smoothed projector, for any $\varepsilon, \sigma > 0$ let

$$\psi_{\varepsilon,\sigma}(\mu) := \rho_\sigma * \mathbb{1}_{[-\varepsilon,\varepsilon]}(\mu),$$

which is Schwartz-class and has Fourier transform

$$\widehat{\psi}_{\varepsilon,\sigma}(t) = \widehat{\rho}_\sigma(t)\frac{2\sin(t\varepsilon)}{t}. \tag{2.2}$$

Then, if $U_t(x,y)$ denotes the kernel of the half-wave operator $U_t = e^{-it\sqrt{-\Delta_g}}$, we have

$$\rho_\sigma * \Pi_{[\lambda-\varepsilon,\lambda+\varepsilon]}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\lambda}\widehat{\psi}_{\varepsilon,\sigma}(t)U_t(x,y)\,dt \tag{2.3}$$

by Fourier inversion. Note that on the left-hand side of (2.3), the convolution is taken with respect to the λ variable. From [DG75], we have that U_t is a Fourier integral operator of class $I^{-\frac{1}{4}}(\mathbb{R}\times M, M; \mathcal{C})$, where the canonical relation $\mathcal{C}$ is given by

$$\begin{aligned}\mathcal{C} = \big\{ \big((t,\tau),(x,\xi),(y,\eta)\big) :\ & (t,\tau) \in T^*\mathbb{R}\backslash\{0\}, \\ & (x,\xi),(y,\eta) \in T^*M\backslash\{0\},\ \tau + |\xi_g| = 0,\ (x,\xi) = \Phi^t(y,\eta)\big\},\end{aligned} \tag{2.4}$$

where $\Phi^t : T^*M \to T^*M$ denotes the geodesic flow.

Since (M,g) is a Zoll manifold with period T, the singularities of $U_t(x,y)$ occur exactly when $t = \pm d_g(x,y) + kT$ for some $k \in \mathbb{Z}$. Thus, for any $0 < \delta < \frac{1}{2}\mathrm{inj}(M,g)$, if $d_g(x,y) \leqslant \delta$, we have that $U_t(x,y)$ is smooth on the support of $1 - \sum_{k\in\mathbb{Z}} \widehat{\rho}_\delta(t-kT)$, and hence

$$\rho_\sigma * \Pi_{[\lambda-\varepsilon,\lambda+\varepsilon]}(x,y) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_{-\infty}^{\infty} e^{it\lambda}\widehat{\psi}_{\varepsilon,\sigma}(t)\widehat{\rho}_\delta(t-kT)U_t(x,y)\,dt + \mathcal{O}(\lambda^{-\infty}), \tag{2.5}$$

where the $\mathcal{O}(\lambda^{-\infty})$ error is uniform for $x, y \in M$ with $d_g(x,y) \leqslant \delta$. We note that the remainder may depend on σ, but this is of no consequence, since we take $\lambda \to \infty$ before sending $\sigma \to 0$. Next, we make the change of variables $t \mapsto t + kT$ to obtain

$$\begin{aligned}\rho_\sigma * \Pi_{[\lambda-\varepsilon,\lambda+\varepsilon]}(x,y) &= \sum_{k\in\mathbb{Z}} \frac{e^{ikT\lambda}}{2\pi} \int_{-\infty}^{\infty} e^{it\lambda}\widehat{\psi}_{\varepsilon,\sigma}(t+kT)\widehat{\rho}_\delta(t)U_{t+kT}(x,y)\,dt + \mathcal{O}(\lambda^{-\infty}) \\ &= \sum_{k\in\mathbb{Z}} e^{ikT\lambda}\mathcal{F}^{-1}_{t\mapsto\lambda}\left(\widehat{f}_k(t)U_{t+kT}(x,y)\right) + \mathcal{O}(\lambda^{-\infty}),\end{aligned}$$

where

$$\widehat{f}_k(t) := \widehat{\psi}_{\varepsilon,\sigma}(t+kT)\widehat{\rho}_\delta(t) \tag{2.6}$$

and $\mathcal{F}^{-1}_{t\mapsto\lambda}$ is the inverse Fourier transform mapping t to λ. Then, we can use that $U_s\varphi_j = e^{-is\lambda_j}\varphi_j$ to obtain

$$\begin{aligned}\mathcal{F}^{-1}_{t\mapsto\lambda}\left(\widehat{f}_k(t)U_{t+kT}(x,y)\right) &= \mathcal{F}^{-1}_{t\mapsto\lambda}\left(\widehat{f}_k(t)\sum_{j=0}^{\infty} e^{-i\lambda_j(t+kT)}\varphi_j(x)\overline{\varphi_j(y)}\right) \\ &= f_k * \left(\sum_{j=0}^{\infty}\delta(\lambda-\lambda_j)e^{-ikT\lambda_j}\varphi_j(x)\overline{\varphi_j(y)}\right) \\ &= f_k * \partial_\lambda\left(\sum_{\lambda_j\leqslant\lambda}\varphi_j(x)\overline{U_{-kT}\varphi_j(y)}\right) \\ &= \partial_\lambda\left(f_k * \Pi_{[0,\lambda]}U_{kT}(x,y)\right).\end{aligned}$$

Therefore, if $d(x,y) \leqslant \delta$,

$$\rho_\sigma * \Pi_{[\lambda-\varepsilon,\lambda+\varepsilon]}(x,y) = \sum_{k\in\mathbb{Z}} e^{ikT\lambda}\partial_\lambda\left(f_k * \Pi_{[0,\lambda]}U_{kT}(x,y)\right) + \mathcal{O}(\lambda^{-\infty}). \tag{2.7}$$

By [DG75, page 53], with $\mathfrak{a}$ as in (1.6) and

$$\mathfrak{b} := \frac{\pi\mathfrak{a}}{2T}, \tag{2.8}$$

we have that $U_t - e^{i\mathfrak{b}T}U_{t+T}$ is a Fourier integral operator of one order lower than U_t, namely $-\frac{1}{4} - 1$. In particular, we have that $U_0 - e^{i\mathfrak{b}T}U_T$ is a pseudodifferential operator of order -1, and

$$U_0 - e^{ik\mathfrak{b}T}U_{kT} \in \Psi^{-1}(M),$$

for any $k \in \mathbb{Z}$. Since U_0 is the identity map, we can write

$$U_{kT} = e^{-ik\mathfrak{b}T}(I + Q_k) \tag{2.9}$$

for $Q_k \in \Psi^{-1}(M)$ with polyhomogeneous symbol. Thus, we obtain

$$\rho_\sigma * \Pi_{[\lambda-\varepsilon,\lambda+\varepsilon]}(x,y) = \sum_{k\in\mathbb{Z}} e^{ikT(\lambda-\mathfrak{b})}\partial_\lambda(f_k * \Pi_{[0,\lambda]}(I+Q_k))(x,y) + \mathcal{O}(\lambda^{-\infty}). \tag{2.10}$$

Therefore, we must determine the asymptotic behavior of the quantity $\partial_\lambda(f_k * \Pi_{[0,\lambda]}(I+Q_k))$, which is handled by the following proposition.

Remark 2.1. Note that for each fixed $\sigma, \delta > 0$, the $\widehat{f}_k$ are identically 0 for sufficiently large k. Therefore, the sum in (2.10) is finite for each $\sigma, \delta > 0$.

2.2. **Pseudodifferential perturbations of the Spectral Projector.** The goal of this section is to find the asymptotic behavior of

$$\partial_\lambda(f_k * \Pi_{[0,\lambda]}(I + Q_k))(x, y)$$

for each k. We are interested in working with points $x, y \in M$ for which $d_g(x, y)$ is small. Therefore, we will assume that we work with coordinates $y = (y_1, \ldots, y_n)$ on M and dual coordinates $(\xi_1, \ldots, \xi_n)$ on T_y^*M. The Riemannian volume form in this coordinates takes the form $\sqrt{|g_y|}dy$, where $|g_y|$ denotes the determinant of the matrix representation of $g(y)$. We also define the function

$$\Theta(x, y) := |\mathrm{det}_g D_{\exp_x^{-1}(y)} \exp_x|,$$

where the subscript g means that we use the metric to choose an orthonormal basis on $T_{\exp_x^{-1}(y)}(T_xM)$ and T_y^*M (c.f. [BGM71, Chapter 2, Proposition C.III.2]). The determinant is then independent of the choice of such a basis. We note that $\Theta(x, y) = \sqrt{|g_x|}$ in normal coordinates centered at y.

If $\xi \in T_y^*M$ is represented as $\xi = r\omega$ with $(r, \omega) \in (0, +\infty) \times S_y^*M$, then we endow S_y^*M with the measure $d\omega$ such that $d\xi = r^{n-1}d\omega dr$.

Remark 2.2. We note that $d\omega$ is not a coordinate invariant measure, but it behaves like a density in y under changes of coordinates. Thus, $d\omega$ should be regarded as a measure taking values in the space of densities on M. Despite this, we note that (1.4) yields

$$\frac{1}{(2\pi)^n} \int\limits_{S_y^*M} e^{i\lambda\langle \exp_y^{-1}(x),\omega\rangle_g} \frac{d\omega}{\sqrt{|g_y|}} = \frac{1}{(2\pi)^{n/2}} \frac{J_{\frac{n-2}{2}}(|\lambda d_g(x, y)|)}{(\lambda d_g(x, y))^{\frac{n-2}{2}}},$$

and the right hand side is clearly coordinate invariant. Here, we used that $d\omega = |g_y|^{1/2} d\sigma_{S^{n-1}}$ and that in local coordinates $\langle \exp_y^{-1}(x), \omega\rangle_g = \langle g_y^{-1/2} \exp_y^{-1}(x), g_y^{-1/2}\omega\rangle_{\mathbb{R}^n}$ with $g_y^{-1/2}\omega \in \mathbb{S}^{n-1}$ and $|g_y^{-1/2} \exp_y^{-1}(x)|_{\mathbb{R}^n} = d_g(x, y)$.

Proposition 2.3. *Let (M, g) be a compact, smooth Riemannian manifold of dimension $n \geqslant 2$ without boundary. Let $Q \in \Psi^{-1}(M)$ with polyhomogeneous symbol $q \sim \sum_{j\geqslant 0} q_{-j-1}$, and $0 < \delta \leqslant \frac{1}{2}\mathrm{inj}(M, g)$. Then, for each pair of multi-indices $\alpha, \beta \in \mathbb{N}^n$, there exist constants $C_1, C_2, \mu_0 > 0$, such that for any function $f \in C^\infty(\mathbb{R})$ with $\widehat{f}$ smooth and compactly supported, and any $x, y \in M$ with $d_g(x, y) \leqslant \delta$ we have*

$$\begin{aligned}&\Theta^{\frac{1}{2}}(x, y)\partial_\mu \left(f * \Pi_{[0,\mu]}(I + Q)\right)(x, y)\\ &\quad= \frac{\mu^{n-1}\widehat{f}(0)}{(2\pi)^n} \int\limits_{S_y^*M} e^{i\mu\langle \exp_y^{-1}(x),\omega\rangle_{g_y}} \left(1 + \frac{1}{\mu}q_{-1}(y, \omega)\right) \frac{d\omega}{\sqrt{|g_y|}} + R(\mu, x, y),\end{aligned}$$

with

$$\sup_{d_g(x,y)\leqslant\delta} |\partial_x^\alpha \partial_y^\beta R(\mu, x, y)| \leqslant C_1\delta\|\partial_t \widehat{f}\|_{L^\infty([-\delta,\delta])}\mu^{n-1+|\alpha|+|\beta|} + C_2\mu^{n-2+|\alpha|+|\beta|} \tag{2.11}$$

for all $\mu \geqslant \mu_0$. Here, C_1 is independent of δ, Q and f.

Proof. We prove the statement first in the case where $\alpha = \beta = 0$. Observe that

$$\partial_\mu \left(f * \Pi_{[0,\mu]}(I+Q) \right)(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\mu} \hat{f}(t) U_t(I+Q)(x,y)\, dt. \tag{2.12}$$

Using the parametrix for U_t constructed in [CH15, Proposition 8], we have that if $d_g(x,y) \leqslant \frac{1}{2}\mathrm{inj}(M,g)$, then

$$U_t(x,y) = \frac{\Theta^{-\frac{1}{2}}(x,y)}{(2\pi)^n} \int_{T_y^*M} e^{i\langle \exp_y^{-1}(x),\xi\rangle_{g_y} - it|\xi|_{g_y}} A(t,y,\xi) \frac{d\xi}{\sqrt{|g_y|}} \tag{2.13}$$

modulo smoothing kernels, for some symbol $A \in S^0$ with a polyhomogeneous expansion $A \sim \sum_{j=0}^{\infty} A_{-j}$. In particular, $A_0(t,y,\xi) \equiv 1$ for all t, and when $t = 0$, $A_{-j}(0,y,\xi) = 0$ for all $j \geqslant 1$. Since Q is pseudodifferential, we can use the same parametrix construction to write

$$U_tQ(x,y) = \frac{\Theta^{-\frac{1}{2}}(x,y)}{(2\pi)^n} \int_{T_y^*M} e^{i\langle \exp_y^{-1}(x),\xi\rangle_{g_y} - it|\xi|_{g_y}} B(t,y,\xi) \frac{d\xi}{\sqrt{|g_y|}} \tag{2.14}$$

for some $B \in S^{-1}$ with $B \sim \sum_{j \geqslant 0} B_{-j-1}$. Note that since Q is pseudodifferential and the principal symbol of U_t is identically 1, the principal symbol of U_tQ is independent of t. At $t = 0$, we have $U_0Q = Q$, and hence the principal symbol of U_tQ is $B_{-1}(t,y,\xi) = q_{-1}(y,\xi)$ for all t. Writing

$$D(t,y,\xi) := A(t,y,\xi) + B(t,y,\xi),$$

from (2.12), (2.13), and (2.14), we obtain

$$\partial_\mu(f*\Pi_\mu(I+Q))(x,y) = \frac{\Theta^{-\frac{1}{2}}(x,y)}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} \int_{T_y^*M} e^{it\mu} e^{i\langle \exp_y^{-1}(x),\xi\rangle_{g_y} - it|\xi|_{g_y}} \hat{f}(t) D(t,y,\xi) \frac{d\xi dt}{\sqrt{|g_y|}} + \mathcal{O}(\mu^{-\infty}). \tag{2.15}$$

To control the first term on the right-hand side above, we change variables via $\xi \mapsto \mu r\omega$ for $(r,\omega) \in \mathbb{R}^+ \times S_y^*M$, which yields that the LHS of (2.15) is

$$\frac{\mu^n}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} \int_0^{\infty} \hat{f}(t) e^{i\mu t(1-r)} r^{n-1} \left(\int_{S_y^*M} e^{i\mu r\langle \exp_y^{-1}(x),\omega\rangle_{g_y}} D(t,y,\mu r\omega) \frac{d\omega}{\sqrt{|g_y|}} \right) dr\, dt. \tag{2.16}$$

Noting that since the phase is nonstationary for $r \neq 1$ we may introduce a cutoff function $\zeta \in C_c^\infty(\mathbb{R})$ which is equal to one on a neighborhood of $r = 1$, and supported in $[\frac{1}{2}, \frac{3}{2}]$. This results in an error which is $\mathcal{O}(\mu^{-\infty})$ as $\mu \to \infty$.

Denote by $S(t,y,\xi)$ the first two terms in the polyhomogeneous expansion of D. Since $A_0(t,y,\xi) \equiv 1$ for all t and $B_0(t,y,\xi) \equiv 0$ for all t,

$$\begin{aligned} S(t,y,\xi) &= 1 + D_{-1}(t,y,\xi) \\ &= 1 + A_{-1}(t,y,\xi) + q_{-1}(y,\xi). \end{aligned}$$

Since $D-S$ is a symbol of order -2, we have $|D(t,y,\mu r\omega)-S(t,y,\mu r\omega)| \leqslant C\mu^{-2}$ uniformly for all t,y. Combining this fact with an application of stationary phase in (t,r), we see that the LHS of (2.15) is equal to

$$\frac{\mu^n}{(2\pi)^{n+1}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\widehat{f}(t)e^{i\mu t(1-r)}r^{n-1}\zeta(r)\left(\int_{S_y^*M} e^{i\mu r\langle \exp_y^{-1}(x),\omega\rangle_{g_y}}S(t,y,\mu r\omega)\,\frac{d\omega}{\sqrt{|g_y|}}\right)dr\,dt+\mathcal{O}(\mu^{n-3}), \tag{2.17}$$

where $\zeta\in C_c^\infty(\mathbb{R})$ is a cut-off function that is equal to 1 near $r=1$ and vanishes for $r\notin[\frac{1}{2},\frac{3}{2}]$. Notice that by homogeneity in the fiber variable, we have that for any $(y,\eta)\in T^*M$,

$$\int_{S_y^*M} e^{i\langle\eta,\omega\rangle_{g_y}}S(t,y,\mu r\omega)\frac{d\omega}{\sqrt{|g_y|}} = \int_{S_y^*M} e^{i\langle\eta,\omega\rangle_{g_y}}\left(1+\frac{1}{\mu r}D_{-1}(t,y,\omega)\right)\frac{d\omega}{\sqrt{|g_y|}}. \tag{2.18}$$

Then, following the proof of [Sog14, Theorem 1.2.1], there exist smooth functions $a_\pm\in C^\infty(T^*M)$ and $b_\pm\in C^\infty(\mathbb{R}\times T^*M)$ such that

$$\int_{S_y^*M} e^{i\langle\eta,\omega\rangle_{g_y}}\,\frac{d\omega}{\sqrt{|g_y|}} = \sum_\pm e^{\pm i|\eta|_{g_y}}a_\pm(y,\eta), \tag{2.19}$$

and

$$\int_{S_y^*M} e^{i\langle\eta,\omega\rangle_{g_y}}D_{-1}(t,y,\omega)\frac{d\omega}{\sqrt{|g_y|}} = \sum_\pm e^{\pm i|\eta|_{g_y}}b_\pm(t,y,\omega), \tag{2.20}$$

satisfying the estimates

$$|\partial_\eta^\gamma a_\pm(y,\eta)| \leqslant C_\gamma(1+|\eta|_{g_y})^{-\frac{n-1}{2}-|\gamma|}, \qquad |\partial_t^k\partial_\eta^\gamma b_\pm(t,y,\eta)| \leqslant C_{\gamma,k}(1+|\eta|_{g_y})^{-\frac{n-1}{2}-|\gamma|} \tag{2.21}$$

for any multi-index γ, any integer $k\geqslant 0$, and some constants C_γ, $C_{\gamma,k}$ which are independent of t, y, and η. Therefore, by (2.12), (2.13), (2.14), (2.19) and (2.20)

$$\partial_\mu(f*\Pi_{[0,\lambda]}(I+Q))(x,y) = \frac{\mu^n}{(2\pi)^{n+1}}\sum_\pm\int_{\mathbb{R}}\int_0^\infty e^{i\mu\psi_\pm(t,r,x,y)}g_\pm(t,r,x,y,\mu)\,dr\,dt, \tag{2.22}$$

where $\psi_\pm(t,r,x,y)=t(1-r)\pm r d_g(x,y)$ and

$$g_\pm(t,r,x,y,\mu) = r^{n-1}\zeta(r)\widehat{f}(t)\left(a_\pm(y,\mu r\exp_y^{-1}(x))+\frac{1}{\mu r}b_\pm(t,y,\mu r\exp_y^{-1}(x))\right). \tag{2.23}$$

Observe that for any fixed $x,y\in M$, the critical points of $\psi_\pm$ occur at $(t_c^\pm,r_c^\pm)=(\pm d_g(x,y),1)$, and that

$$\det\left(\operatorname{Hess}\psi_\pm(t_c^\pm,r_c^\pm,x,y)\right)=1.$$

Therefore, by the method of stationary phase, we see that

$$\begin{aligned}&\partial_\mu(f*\Pi_{[0,\lambda]}(I+Q))(x,y)\\ &\quad= \frac{\mu^{n-1}}{(2\pi)^n}\sum_\pm e^{\pm i\mu d_g(x,y)}\left(g_\pm(t_c^\pm,r_c^\pm,x,y,\mu)-\frac{i}{\mu}\partial_r\partial_t g_\pm(t_c^\pm,r_c^\pm,x,y,\mu)\right)+\mathcal{O}(\mu^{n-3}).\end{aligned}$$

From (2.23) and (2.21) we have that

$$\begin{aligned}\left|\partial_r\partial_t g_\pm(t_c^\pm, r_c^\pm, x, y, \mu)\right| &\leqslant C_1|\partial_t\widehat{f}(\pm d_g(x,y))| + \frac{C_2}{\mu}\left(|\widehat{f}(\pm d_g(x,y))| + |\partial_t\widehat{f}(\pm d_g(x,y))|\right)\\ &\leqslant C_1\|\partial_t\widehat{f}\|_{L^\infty([-\delta,\delta])} + \frac{C_2}{\mu}\|\widehat{f}\|_{C^1([-\delta,\delta])},\end{aligned}$$

and we remark that C_1 is independent of Q due to the definition of $a_\pm$. Therefore,

$$\begin{aligned}&\Theta^{\frac{1}{2}}(x,y)\partial_\mu\left(f * \Pi_{[0,\mu]}(I+Q)\right)(x,y)\\ &\quad= \frac{\mu^{n-1}}{(2\pi)^n}\sum_\pm e^{\pm i\mu d_g(x,y)}\widehat{f}(\pm d_g(x,y))\left(a_\pm(y,\mu\exp_y^{-1}(x)) + \frac{1}{\mu}b_\pm(t_c^\pm, y, \mu\exp_y^{-1}(x))\right)\\ &\quad+ R_1(\mu,x,y),\end{aligned}\tag{2.24}$$

where

$$\sup_{d_g(x,y)\leqslant\delta}|R_1(\mu,x,y)| \leqslant C_1\|\widehat{f}\|_{\dot{C}^1([-\delta,\delta])}\mu^{n-2} + C_2\|\widehat{f}\|_{C^1([-\delta,\delta])}\mu^{n-3} + \mathcal{O}(\mu^{n-3}),$$

with C_1 independent of Q. Next, let us Taylor expand $\widehat{f}$ near 0, which yields

$$\widehat{f}(\pm d_g(x,y)) = \widehat{f}(0) \pm d_g(x,y)\partial_t\widehat{f}(s_\pm)$$

for some $s_\pm$ between 0 and $\pm d_g(x,y)$. Combining this with the fact that

$$\sum_\pm e^{\pm i\mu d_g(x,y)}a_\pm(y,\mu\exp_y^{-1}(x)) = \int_{S_y^*M} e^{i\mu\langle\exp_y^{-1}(x),\omega\rangle}\frac{d\omega}{\sqrt{|g_y|}},\tag{2.25}$$

we obtain

$$\begin{aligned}&\Theta^{\frac{1}{2}}(x,y)\partial_\mu\left(\widehat{f} * \Pi_{[0,\mu]}(I+Q)\right)(x,y)\\ &\quad= \frac{\mu^{n-1}\widehat{f}(0)}{(2\pi)^n}\left(\int_{S_y^*M} e^{i\mu\langle\exp_y^{-1}(x),\omega\rangle_{g_y}}\frac{d\omega}{\sqrt{|g_y|}} + \sum_\pm e^{\pm i\mu d_g(x,y)}b_\pm(t_c^\pm, y, \mu\exp_y^{-1}(x))\right)\\ &\quad+ R_1(\mu,x,y) + R_2(\mu,x,y),\end{aligned}\tag{2.26}$$

where R_1 is as above, and R_2 satisfies

$$\sup_{d_g(x,y)\leqslant\delta}|R_2(\mu,x,y)| \leqslant \delta\|\partial_t\widehat{f}\|_{L^\infty([-\delta,\delta])}(C_0\mu^{n-1} + C_1\mu^{n-2})$$

for some $C_0 > 0$ which is independent of Q and $C_1 > 0$. Next, we Taylor expand

$$b_\pm(t_c^\pm, y, \mu\exp_y^{-1}(x)) = b_\pm(0, y, \mu\exp_y^{-1}(x)) \pm d_g(x,y)\partial_t b_\pm(s'_\pm, y, \mu\exp_y^{-1}(x))$$

for some $s'_\pm$ between 0 and $t_c^\pm = \pm d_g(x,y)$. Recalling (2.21), we have that

$$|\partial_t b_\pm(s_\pm, y, \mu\exp_y^{-1}(x))| \leqslant C(1+\mu d_g(x,y))^{-\frac{n-1}{2}},$$

since $|s_\pm| \leqslant d_g(x,y)$. Therefore, we obtain

$$\frac{\mu^{n-2}\widehat{f}(0)}{(2\pi)^m}\sum_{\pm} e^{\pm i\mu d_g(x,y)} b_\pm(t_c^\pm, y, \mu \exp_y^{-1}(x)) \\ = \frac{\mu^{n-2}\widehat{f}(0)}{(2\pi)^n}\int\limits_{S_y^*M} e^{i\mu\langle \exp_y^{-1}(x),\omega\rangle} D_{-1}(0,y,\omega)\,\frac{d\omega}{\sqrt{|g_y|}} + R_3(\mu,x,y), \tag{2.27}$$

where

$$\sup_{d_g(x,y)\leqslant\delta} |R_3(\mu,x,y)| \leqslant C\delta\widehat{f}(0)\mu^{n-2}.$$

Recalling that $D_{-1}(0,y,\xi) = A_{-1}(0,y,\xi) + q_{-1}(y,\xi) = q_{-1}(y,\xi)$ since all subprincipal terms of A vanish at $t=0$, we have that (2.26) and (2.27) yield

$$\Theta^{\frac{1}{2}}(x,y)\partial_\mu\left(\widehat{f} * \Pi_{[0,\mu]}(I+Q)\right)(x,y) \\ = \frac{\mu^{n-1}\widehat{f}(0)}{(2\pi)^n}\int\limits_{S_y^*M} e^{i\mu\langle \exp_y^{-1}(x),\omega\rangle_{g_y}}\left(1+\frac{1}{\mu}q_{-1}(y,\omega)\right)\frac{d\omega}{\sqrt{|g_y|}} + \widetilde{R}(\mu,x,y),$$

where $\widetilde{R} = R_1 + R_2 + R_3$ satisfies

$$\sup_{d_g(x,y)\leqslant\delta}|\widetilde{R}(\mu,x,y)| \leqslant C_1\delta\|\widehat{f}\|_{\dot{C}^1([-\delta,\delta])}\mu^{n-1} + C_2\|\widehat{f}\|_{\dot{C}^1([-\delta,\delta])}\mu^{n-2} \\ + C_3\delta\widehat{f}(0)\mu^{n-2} + C_4\|\widehat{f}\|_{C^1([-\delta,\delta])}\mu^{n-3} + \mathcal{O}(\mu^{n-3}),$$

for some $C_1, C_2, C_3, C_4 > 0$, with C_1 independent of δ, f, and Q. This completes the proof in the case where $\alpha = \beta = 0$.

To include derivatives in x, y, we observe that

$$\partial_x^\alpha\partial_y^\beta e^{i\langle \exp_y^{-1}(x),\xi\rangle} = \mathcal{O}(|\xi|^{|\alpha|+|\beta|})$$

as $|\xi| \to \infty$. Therefore, we can repeat the preceding argument where the orders of the symbols involved are increased by at most $|\alpha| + |\beta|$ to obtain the desired result.

□

2.3. **Asymptotics for the smoothed spectral projector.** With Proposition 2.3 in hand, we are equipped to prove the main result of this section, namely the asymptotic behavior of the smoothed spectral projector $\rho_\sigma * \Pi_{[\nu_\ell-\varepsilon,\nu_\ell+\varepsilon]}$.

$$R_{\varepsilon,\sigma}(\ell;x,y) := \rho_\sigma * \Pi_{[\nu_\ell-\varepsilon,\nu_\ell+\varepsilon]}(x,y) - \frac{2\pi}{T}\cdot\frac{\nu_\ell^{n-1}}{(2\pi)^n}\int\limits_{S_y^*M} e^{i\nu_\ell\langle \exp_y^{-1}(x),\omega\rangle_g}\frac{d\omega}{\sqrt{|g_y|}}. \tag{2.28}$$

Proposition 2.4. *Let (M,g) be a smooth Zoll manifold with uniform period $T > 0$. Fix $0 < \varepsilon < \frac{2\pi}{T}$. Then, for any multi-indices $\alpha, \beta \in \mathbb{N}^n$ and $R_{\varepsilon,\sigma}$ as in (2.28),*

$$\lim_{\delta\to 0^+}\lim_{\sigma\to 0^+}\limsup_{\ell\to\infty}\sup_{d_g(x,y)\leqslant\delta}\left|\frac{1}{\nu_\ell^{n-1+|\alpha|+|\beta|}}\partial_x^\alpha\partial_y^\beta R_{\varepsilon,\sigma}(\ell;x,y)\right| = 0.$$

Proof. Fix two multi-indices $\alpha, \beta \in \mathbb{N}^n$. First, note that for $\mathfrak{b}$ as in (2.8) we have that for all $k \in \mathbb{Z}$

$$e^{ikT(\mathfrak{b}-\nu_\ell)} = e^{ikT(-2\pi\ell/T)} = e^{-2\pi ik\ell} = 1.$$

Therefore, combining (2.10) with Proposition 2.3 yields

$$\rho_\sigma * \Pi_{[\nu_\ell-\varepsilon,\nu_\ell+\varepsilon]}(x,y) = \nu_\ell^{n-1} L(\ell,x,y) \sum_{k\in\mathbb{Z}} \widehat{f}_k(0) + \nu_\ell^{n-2} \sum_{k\in\mathbb{Z}} \widehat{f}_k(0) W_k(\ell,x,y) + \sum_{k\in\mathbb{Z}} R_k(\ell,x,y), \tag{2.29}$$

where

$$L(\ell,x,y) = \frac{1}{(2\pi)^n \Theta^{\frac12}(x,y)} \int_{S_y^*M} e^{i\nu_\ell\langle \exp_y^{-1}(x),\omega\rangle_g} \frac{d\omega}{\sqrt{|g_y|}}, \tag{2.30}$$

$$W_k(\ell,x,y) = \frac{1}{(2\pi)^n \Theta^{\frac12}(x,y)} \int_{S_y^*M} e^{i\nu_\ell\langle \exp_y^{-1}(x),\omega\rangle_g} \sigma(Q_k)(y,\omega) \frac{d\omega}{\sqrt{|g_y|}}, \tag{2.31}$$

and R_k satisfies

$$\sup_{d_g(x,y)\leqslant\delta} |\partial_x^\alpha \partial_y^\beta R_k(\ell,x,y)| \leqslant C_1 \delta \|\partial_t \widehat{f}_k\|_{L^\infty([-\delta,\delta])} \nu_\ell^{n-1+|\alpha|+|\beta|} + C_2 \nu_\ell^{n-2+|\alpha|+|\beta|}$$

with C_1 independent of δ and k. Recalling that the summation in k is actually finite and that $\sup_{\{\sigma>0,\delta<1,k\in\mathbb{Z}\}} \|\partial_t \widehat{f}_k\|_{L^\infty([-\delta,\delta])} < \infty$ (see Remark 2.1) we have that if we define

$$F_{\delta,\sigma}(x,y,\ell) := \frac{1}{\nu_\ell^{n-1+|\alpha|+|\beta|}} \left(\nu_\ell^{n-2} \sum_{k\in\mathbb{Z}} \widehat{f}_k(0) \partial_x^\alpha \partial_y^\beta W_k(\nu_\ell,x,y) + \sum_{k\in\mathbb{Z}} \partial_x^\alpha \partial_y^\beta R_k(\nu_\ell,x,y) \right),$$

then we have

$$\lim_{\delta\to0^+} \lim_{\sigma\to0^+} \limsup_{\ell\to\infty} \sup_{d_g(x,y)\leqslant\delta} |F_{\delta,\sigma}(x,y,\ell)| = 0. \tag{2.32}$$

To deal with the first term in (2.29), we first claim that

$$\lim_{\sigma\to0^+} \nu_\ell^{n-1} \partial_x^\alpha \partial_y^\beta L(\ell,x,y) \sum_{k\in\mathbb{Z}} \widehat{f}_k(0) = \frac{2\pi}{T} \nu_\ell^{n-1} \partial_x^\alpha \partial_y^\beta L(\ell,x,y), \tag{2.33}$$

uniformly for all $\ell \in \mathbb{N}$ and $x, y \in M$ with $d(x,y) \leqslant \frac12 \mathrm{inj}(M,g)$. To see this, first note that by (2.6) and (2.1) we have

$$\sum_{k\in\mathbb{Z}} \widehat{f}_k(0) = \sum_{k\in\mathbb{Z}} \widehat{\psi}_{\varepsilon,\sigma}(kT) \widehat{\rho}_\delta(0) = \sum_{k\in\mathbb{Z}} \widehat{\psi}_{\varepsilon,\sigma}(kT).$$

Using the Poisson summation formula and (2.2),

$$\sum_{k\in\mathbb{Z}} \widehat{\psi}_{\varepsilon,\sigma}(kT) = \sum_{k\in\mathbb{Z}} \frac{\sin(\varepsilon kT)}{kT} \widehat{\rho}(\sigma kT) = \frac{2\pi}{T} \sum_{k\in\mathbb{Z}} \mathbb{1}_{[-1,1]} * \rho_{\sigma/\varepsilon}(2\pi k/T\varepsilon) = \frac{2\pi}{T} \sum_{k\in\mathbb{Z}} \psi_{1,\sigma/\varepsilon}\left(\frac{2\pi k}{T\varepsilon}\right). \tag{2.34}$$

Motivated by the form of the above expression, we replace σ by $\varepsilon\sigma$, which is permitted since ε is fixed throughout this argument. Thus,

$$\sum_{k\in\mathbb{Z}} \widehat{\psi}_{\varepsilon,\varepsilon\sigma}(kT) = \frac{2\pi}{T} \sum_{k\in\mathbb{Z}} \psi_{1,\sigma}\left(\frac{2\pi k}{T\varepsilon}\right). \tag{2.35}$$

Since $\psi_{1,\sigma} = \mathbb{1}_{[-1,1]} * \rho_\sigma$ and $0 < \varepsilon < \frac{2\pi}{T}$, we have that for $k \neq 0$,

$$\left|\frac{1}{T}\psi_{1,\sigma}\left(\frac{k}{T\varepsilon}\right)\right| \leqslant \frac{C_N}{T}\left(1 + \frac{|k|}{T\varepsilon\sigma}\right)^{-N} \quad \text{for any } N.$$

Thus, if we choose $N \geqslant 2$, we obtain

$$\left|\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} \frac{1}{T}\psi_{1,\sigma}\left(\frac{k}{T\varepsilon}\right)\right| \leqslant C_N \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} (\varepsilon\sigma)^N T^{-1}(\varepsilon\sigma + |k|/T)^{-N} \leqslant C_N(\varepsilon\sigma)^N T^{-1} \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} (|k|/T)^{-N},$$

which converges to 0 as $\sigma \to 0$. Also, when $k = 0$, we have

$$\psi_{1,\sigma}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin t}{t}\widehat{\rho}(\sigma t)\, dt \to \psi(0) = 1$$

as $\sigma \to 0$, and this finishes the proof of the claim in (2.33).

Combining (2.28), (2.29), (2.32), and (2.33) yields that the final step in the proof is to eliminate the factor of $\Theta^{-\frac{1}{2}}(x, y)$ implicit in the definition of L. For this, we observe that $\Theta^{-\frac{1}{2}}(x, x) = 1$ and its differential vanishes on the diagonal in $M \times M$. Hence, for small $d_g(x, y)$, we have

$$\Theta^{-\frac{1}{2}}(x, y) = 1 + d_g(x, y)^2 G(x, y)$$

for some smooth, bounded function G. Thus, it suffices to show that

$$\lim_{\delta\to 0^+} \limsup_{\ell\to\infty} \sup_{d_g(x,y)\leqslant\delta} \left|\frac{1}{\nu_\ell^{|\alpha|+|\beta|}}\partial_x^\alpha \partial_y^\beta \left(d_g(x, y)^2 \int_{S_y^*M} e^{i\nu_\ell\langle \exp_y^{-1}(x),\omega\rangle} \frac{d\omega}{\sqrt{|g_y|}}\right)\right| = 0. \tag{2.36}$$

In the case where at most one derivative falls on the factor of $d_g(x, y)^2$, the above statement holds trivially. If two or more derivatives fall on this factor, then at most $|\alpha| + |\beta| - 2$ factors of ν_ℓ can appear from differentiating the integral over S_y^*M, and so (2.36) also holds in this case.

□

3. On-diagonal analysis of the spectral projector

The goal of this section is to establish a lower bound for the spectral function restricted to the diagonal, which is critical for the purposes of comparing the smoothed projector to the original. In particular, we show that most of the "mass" of the spectral function is concentrated near

$$\bigcup_{\ell\in\mathbb{N}} \left[\nu_\ell - r\ell^{-\frac{1}{2}}, \nu_\ell + r\ell^{-\frac{1}{2}}\right],$$

with ν_ℓ as defined in (1.6). This is similar to the original eigenvalue clustering result of [DG75, Theorem 3.1]. We expect that a stronger cluster estimate with $r\ell^{-\frac{1}{2}}$ replaced with $r\ell^{-1}$ should hold, but we do not prove this here as the refined statement is not needed.

Proposition 3.1. *Let (M, g) be a Zoll manifold with uniform geodesic period $T > 0$ and let $\{\varphi_j\}_j$ be the corresponding Laplace eigenfunctions defined in (1.1). Let $r > 0$ and fix a multi-index $\alpha \in \mathbb{N}^n$. Then, there exist $K, C, \lambda_0 > 0$ so that for all $x \in M$ and $\lambda \geqslant \lambda_0$*

$$\sum_{\lambda_j \in \mathcal{A}(K,r,\lambda)} |\partial_x^\alpha \varphi_j(x)|^2 \geqslant \left(1 - Cr^{-2}\right) \sum_{|\lambda_j - \lambda| \leqslant K} |\partial_x^\alpha \varphi_j(x)|^2,$$

where

$$\mathcal{A}(K, r, \lambda) = \Big\{\lambda_j : \ |\lambda_j - \lambda| \leqslant K, \quad \lambda_j \in \bigcup_{\ell \in \mathbb{N}} \big[\nu_\ell - r\ell^{-\frac{1}{2}}, \nu_\ell + r\ell^{-\frac{1}{2}}\big]\Big\}.$$

Proof. We begin by considering the case where $\alpha = 0$ separately. For this, we proceed in close analogy to the proof of [DG75, Theorem 3.1]. Let $\chi \in \mathscr{S}(\mathbb{R})$ with $\chi \geqslant 0$ and $\widehat{\chi} \in C_c^\infty(\mathbb{R})$ with $\widehat{\chi}(0) > 0$. Repeating previous calculations, we have that for $x \in M$

$$\sum_{j=0}^{\infty} \chi(\lambda - \lambda_j)|\varphi_j(x)|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\lambda} \widehat{\chi}(t) U_t(x, x)\, dt. \tag{3.1}$$

Similarly,

$$\sum_{j=0}^{\infty} e^{i(\mathfrak{b} - \lambda_j)T} \chi(\lambda - \lambda_j)|\varphi_j(x)|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\lambda} \widehat{\chi}(t) e^{i\mathfrak{b}T} U_{t+T}(x, x)\, dt. \tag{3.2}$$

Recalling that $U_t - e^{i\mathfrak{b}T} U_{t+T}$ is an FIO defined by $\mathcal{C}$ of order $-\frac{1}{4} - 1$ (see (2.4)), we know that we can write

$$U_t(x, x) - e^{i\mathfrak{b}T} U_{t+T}(x, x) = \frac{1}{(2\pi)^n} \int_{T_y^* M} e^{i\phi(t,x,x,\xi)} B(t, x, x, \xi)\, d\xi,$$

where B is a symbol of order -1 and ϕ is any admissible phase function which parametrizes $\mathcal{C}$ (c.f. [DG75, p. 45]). As in the proof of Proposition 2.3, we can use the phase function $\phi(t, x, y, \xi) = \langle \exp_y^{-1}(x), \xi\rangle_{g_y} - t|\xi|_{g_y}$. Hence,

$$\begin{aligned}
\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\lambda} \widehat{\chi}(t) \Big(U_t(x, x) - e^{i\mathfrak{b}T} U_{t+T}(x, x)\Big)\, dt &= \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} \int_{T_y^* M} e^{it(\lambda - |\xi|)} \widehat{\chi}(t) B(t, x, x, \xi)\, d\xi\, dt \\
&= \frac{\lambda^n}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} \int_0^{\infty} \int_{S_y^* M} \widehat{\chi}(t) e^{i\lambda t(1-s)} s^{n-1} B(t, x, x, \lambda s\omega)\, ds\, d\omega\, dt \\
&= \mathcal{O}(\lambda^{n-2}).
\end{aligned} \tag{3.3}$$

Here, to obtain the bound in the last line we used the fact that B is a symbol of order -1 and repeated the calculations from the proof of Proposition 2.3 that follow (2.16). From (3.1)and (3.2) it follows that

$$\sum_{j=0}^{\infty} \chi(\lambda - \lambda_j) \left(1 - e^{i(\mathfrak{b} - \lambda_j)T}\right) |\varphi_j(x)|^2 = \mathcal{O}(\lambda^{n-2}). \tag{3.4}$$

Thus, we can take real parts to obtain that

$$\sum_{j=0}^{\infty} (1 - \cos (T(\mathfrak{b} - \lambda_j)))\, \chi(\lambda - \lambda_j)|\varphi_j(x)|^2 = \mathcal{O}(\lambda^{n-2}) \tag{3.5}$$

as $\lambda \to \infty$. For any $r, \ell > 0$ define the set

$$\mathcal{E}(\ell, r) = \{\lambda_j \in \operatorname{Spec}(\sqrt{-\Delta_g}) : \; r\ell^{-1/2} \leqslant T|\lambda_j - \nu_\ell| \leqslant \pi\}$$

Recall that $\nu_\ell = \frac{2\pi\ell}{T} + \mathfrak{b}$ by (2.8). Thus, if $\lambda_j \in \mathcal{E}(\ell, r)$, we have that

$$1 - \cos (T(\mathfrak{b} - \lambda_j)) = 1 - \cos \Big(T(\nu_\ell - \lambda_j) - 2\pi\ell\Big) \geqslant \tfrac{1}{2} r^2 \ell^{-1} - \tfrac{1}{24} r^4 \ell^{-2},$$

since $1 - \cos(\theta - 2\pi\ell) \geqslant \frac{1}{2}\theta^2 - \frac{1}{24}\theta^4$ for $\theta \in [-\pi, \pi]$ and all $\ell \in \mathbb{N}$. Therefore, using that $\nu_\ell \geqslant c\ell$ for ℓ large enough, together with (3.5), we obtain that for every $r > 0$ there exist $C, \ell_0 > 0$ such that for all $\ell \geqslant \ell_0$, we have

$$\begin{aligned}\sum_{\lambda_j \in \mathcal{E}(\ell,r)} \tfrac{1}{2} r^2 \ell^{-1} \min\big(\chi(\mu) : \; |\mu| \leqslant \tfrac{\pi}{T}\big)|\varphi_j(x)|^2 &\leqslant C \sum_{\lambda_j \in \mathcal{E}(\ell,r)} (1 - \cos ((\mathfrak{b} - \lambda_j)))\, \chi(\nu_\ell - \lambda_j)|\varphi_j(x)|^2 \\ &\leqslant C\ell^{n-2}.\end{aligned}$$

If we adjust χ so that $\chi(\mu) > 0$ for all $|\mu| \leqslant \frac{\pi}{T}$, we obtain that

$$\sum_{\lambda_j \in \mathcal{E}(\ell,r)} |\varphi_j(x)|^2 \leqslant C r^{-2} \ell^{n-1} \tag{3.6}$$

for all $r > 0$ and all ℓ large enough.

Next, observe that for any $K, r > 0$,

$$\mathcal{A}(K, r, \lambda) = \{\lambda_j : \; |\lambda_j - \lambda| \leqslant K\} \cap \bigcap_{\ell=1}^{\infty} \mathcal{E}(\ell, r)^c.$$

Therefore,

$$\sum_{\lambda_j \in \mathcal{A}(K,r,\lambda)} |\varphi_j(x)|^2 = \sum_{|\lambda_j - \lambda| \leqslant K} |\varphi_j(x)|^2 - \sum_{\ell=1}^{\infty} \sum_{\lambda_j \in \{|\lambda_j - \lambda| \leqslant K\} \cap \mathcal{E}(\ell,r)} |\varphi_j(x)|^2. \tag{3.7}$$

Note that

$$\{\lambda_j : |\lambda_j - \lambda| \leqslant K\} \cap \mathcal{E}(\ell, r) = \varnothing \qquad \text{if} \quad |\nu_\ell - \lambda| > K + \pi.$$

Thus, if we define

$$\mathcal{V}(\lambda, K) = \{\ell : \; |\nu_\ell - \lambda| \leqslant K + \pi\},$$

by (3.7)

$$\sum_{\lambda_j \in \mathcal{A}(K,r,\lambda)} |\varphi_j(x)|^2 = \sum_{|\lambda_j - \lambda| \leqslant K} |\varphi_j(x)|^2 - \sum_{\ell \in \mathcal{V}(\lambda,K)} \sum_{\lambda_j \in \{|\lambda_j - \lambda| \leqslant K\} \cap \mathcal{E}(\ell,r)} |\varphi_j(x)|^2. \tag{3.8}$$

In addition, for each $\ell \in \mathcal{V}(\lambda, K)$, we have that $\nu_\ell \approx \lambda$, and so by (3.6) that

$$\sum_{\lambda_j \in \{|\lambda_j - \lambda| \leqslant K\} \cap \mathcal{E}(\ell,r)} |\varphi_j(x)|^2 \leqslant C r^{-2} \lambda^{n-1} \tag{3.9}$$

since $\ell \approx \nu_\ell \approx \lambda$. Next, we need the following lemma whose proof we postpone until the end of this section.

Lemma 3.2. *Let (M, g) be any compact smooth manifold of dimension n with Laplace eigenfunctions $\{\varphi_j\}_j$ as in (1.1). Then, for every multi-index $\alpha \in \mathbb{N}$ there exist $K, C, \lambda_0 > 0$ so that*

$$\sum_{|\lambda-\lambda_j|\leqslant K} |\partial_x^\alpha \varphi_j(x)|^2 \geqslant C\lambda^{n-1+2|\alpha|}$$

for all $\lambda \geqslant \lambda_0$.

Returning to the proof of Proposition 3.1, we can combine Lemma 3.2 with (3.9) to obtain that for K sufficiently large,

$$\sum_{\lambda_j\in\{|\lambda_j-\lambda|\leqslant K\}\cap\mathcal{E}(\ell,r)} |\varphi_j(x)|^2 \leqslant Cr^{-2} \sum_{|\lambda_j-\lambda|\leqslant K} |\varphi_j(x)|^2. \tag{3.10}$$

Furthermore, since the cardinality of $\mathcal{V}(\lambda, K)$ is proportional to K, we can combine (3.10) with (3.8) to obtain

$$\sum_{\lambda_j\in\mathcal{A}(K,r,\lambda)} |\varphi_j(x)|^2 \geqslant \left(1 - \frac{C}{r^2}\right) \sum_{|\lambda_j-\lambda|\leqslant K} |\varphi_j(x)|^2,$$

which completes the proof in case where $|\alpha| = 0$.

In order to prove the statement for higher order derivatives ∂_x^α, one need only show the appropriate analogue of (3.4). In particular, this will follow from

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\lambda}\hat{\chi}(t)\partial_x^\alpha\partial_y^\alpha \left(U_t(x,y) - e^{i\mathfrak{b}T}U_{t+T}(x,y)\right)\big|_{y=x}\, dt = \mathcal{O}(\lambda^{n-2+2|\alpha|}). \tag{3.11}$$

This follows directly from the off-diagonal analogue of (3.3), which is given by

$$\begin{aligned} &\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\lambda}\hat{\chi}(t) \left(U_t(x,y) - e^{i\mathfrak{b}T}U_{t+T}(x,y)\right) dt \\ &\qquad = \frac{\lambda^n}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} \int_{T_y^*M} e^{i\lambda\left(\langle \exp_y^{-1}(x),\xi\rangle + t(1-|\xi|)\right)} \hat{\chi}(t)\hat{B}(t,x,y,\lambda\xi)\, d\xi\, dt. \end{aligned}$$

Thus, each derivative in x or y yields at most one additional power of λ, and so by previous arguments we obtain (3.11). The rest of the argument proceeds identically to the $|\alpha| = 0$ case.

□

Proof of Lemma 3.2. The proof of this lower bound relies on the generalized local Weyl law, which states that if A is a classical polyhomogeneous pseudodifferential operator of order zero, then

$$A\Pi_{[0,\lambda]}A^*(x,x) = \sum_{\lambda_j\leqslant\lambda} |A\varphi_j(x)|^2 = L_A(x,\lambda)\lambda^n + R_A(\lambda,x), \tag{3.12}$$

where

$$L_A(x) := C \int_{S_x^*M} |\sigma_0(A)(x,\xi)|^2\, d\xi$$

for some $C > 0$, and $\sup_{x\in M} |R_A(\lambda, x)| \leqslant C_A\lambda^{n-1}$ for some $C_A > 0$ and all $\lambda \geqslant 1$ (c.f. [Sog14, Theorem 5.2.3]). We note that since A is of order zero, $|L_A(x)| \leqslant C'_A$ for some $C'_A > 0$. Given these facts, we define for each multi-index α the operator

$$A = \partial_x^\alpha (1 + \Delta_g)^{-|\alpha|/2} \in \Psi^0_{c\ell}(M)$$

whose principal symbol is a homogeneous function in $C^\infty(T^*M\backslash 0)$ which can be written in local coordinates as

$$\sigma_0(A)(x, \xi) = \frac{i^{|\alpha|}\xi^\alpha}{|\xi|_g^{|\alpha|}}.$$

By the local Weyl law, we have

$$\begin{aligned} A\Pi_{[\lambda-K,\lambda+K]}A^*(x, x) = &\Big(A\Pi_{\lambda+K}A^*(x, x) - L_A(x)(\lambda + K)^n\Big) \\ &- \Big(A\Pi_{\lambda-K}A^*(x, x) - L_A(x)(\lambda - K)^n\Big) \\ &+ L_A(x)\left((\lambda + K)^n - (\lambda - K)^n\right) \\ = &\, R_A(\lambda + K, x) - R_A(\lambda - K, x) + L_A(x)\left(K\lambda^{n-1} + \mathcal{O}_{K,A}(\lambda^{n-2})\right). \end{aligned}$$

Since $|R_A(\lambda, x)| \leqslant C_A\lambda^{n-1}$ and $L_A(x) \geqslant \delta > 0$ for all $x \in M$ and all $\lambda \geqslant 1$, we have that

$$A\Pi_{[\lambda-K,\lambda+K]}A^*(x, x) \geqslant (\delta K - C_A)\,\lambda^{n-1} + \mathcal{O}_{K,A}(\lambda^{n-2}).$$

Thus, if we choose K large enough so that $\delta K - C_A > 0$, there exists a $\lambda_0 > 0$ so that

$$A\Pi_{[\lambda-K,\lambda+K]}A^*(x, x) \geqslant C\lambda^{n-1} \tag{3.13}$$

for some $C > 0$ and all $\lambda \geqslant \lambda_0$. On the other hand, we can use the functional calculus for Δ_g to write

$$A\Pi_{[\lambda-K,\lambda+K]}A^*(x, x) = \sum_{|\lambda-\lambda_j|\leqslant K} (1 + \lambda_j^2)^{-|\alpha|}|\partial_x^\alpha\varphi_j(x)|^2.$$

Observe that

$$\left|\frac{1 + \lambda^2}{1 + \lambda_j^2} - 1\right| = \frac{\left|\lambda^2 - \lambda_j^2\right|}{1 + \lambda_j^2} \leqslant \frac{K(2\lambda + K)}{1 + (\lambda - K)^2},$$

Since $1 + (\lambda - K)^2 \geqslant \frac{1}{2}\lambda^2$ if $\lambda \geqslant \frac{1}{4}K$, we obtain

$$\left|\frac{1 + \lambda^2}{1 + \lambda_j^2} - 1\right| \leqslant CK\lambda^{-1} + \mathcal{O}_K(\lambda^{-2})$$

as $\lambda \to \infty$. Using binomial expansion, we also obtain

$$\left|\frac{(1 + \lambda^2)^{|\alpha|}}{(1 + \lambda_j^2)^{|\alpha|}} - 1\right| \leqslant C_\alpha K\lambda^{-1} + \mathcal{O}_{K,\alpha}(\lambda^{-2})$$

for any α. Therefore,

$$\begin{aligned} &\left|(1 + \lambda^2)^{|\alpha|}A\Pi_{[\lambda-K,\lambda+K]}A^*(x, x) - \sum_{|\lambda_j-\lambda|\leqslant K} |\partial_x^\alpha\varphi_j(x)|^2\right| \\ &\qquad\qquad \leqslant \Big(C_\alpha K\lambda^{-1} + \mathcal{O}_{K,\alpha}(\lambda^{-2})\Big) \sum_{|\lambda_j-\lambda|\leqslant K} |\partial_x^\alpha\varphi_j(x)|^2. \end{aligned} \tag{3.14}$$

Hence, by (3.13),

$$C\lambda^{n-1+2|\alpha|} \leqslant (1+\lambda^2)^{|\alpha|} A\Pi_{[\lambda-K,\lambda+K]}A^*(x,x) \leqslant \Big(1+C_\alpha K\lambda^{-1}+\mathcal{O}_{K,\alpha}(\lambda^{-2})\Big) \sum_{|\lambda_j-\lambda|\leqslant K} |\partial_x^\alpha \varphi_j(x)|^2$$

Since $C_\alpha K\lambda^{-1} + \mathcal{O}_{K,\alpha}(\lambda^{-2})$ tends to zero as $\lambda \to \infty$ for any fixed $K > 0$, this proves the claim.

□

4. Proof of the main results

In this section we prove Theorem 1 and Corollary 1.2.

4.1. **Proof of Theorem 1.** With Proposition 2.4 in place, the proof of Theorem 1 reduces to the claim that for any $\varepsilon < \frac{\pi}{2T}$, and each pair of multi-indices α, β, we have

$$\lim_{\sigma\to 0^+} \limsup_{\ell\to\infty} \nu_\ell^{1-n-|\alpha|-|\beta|} \sup_{x,y\in M} \left|\partial_x^\alpha \partial_y^\beta \left(\Pi_{[\nu_\ell-\varepsilon,\nu_\ell+\varepsilon]}(x,y) - \rho_\sigma * \Pi_{[\nu_\ell-\varepsilon,\nu_\ell+\varepsilon]}(x,y)\right)\right| = 0. \tag{4.1}$$

We proceed to prove (4.1). Noting that

$$\mathcal{F}_{\tau\mapsto t}\left(\mathbb{1}_{[-\varepsilon,\varepsilon]}(\tau)\right) = \int_{-\varepsilon}^{\varepsilon} e^{-it\tau}\, d\tau = \frac{2\sin(t\varepsilon)}{t},$$

we can rewrite

$$\Pi_{[\lambda-\varepsilon,\lambda+\varepsilon]}(x,y) - \rho_\sigma * \Pi_{[\lambda-\varepsilon,\lambda+\varepsilon]}(x,y) = \sum_{j=0}^{\infty} h_{\varepsilon,\sigma}(\lambda-\lambda_j)\varphi_j(x)\overline{\varphi_j(y)}, \tag{4.2}$$

where

$$h_{\varepsilon,\sigma}(\tau) = \mathbb{1}_{[-\varepsilon,\varepsilon]}(\tau) - \frac{1}{\pi}\int_{-\infty}^{\infty} e^{it\tau}\widehat{\rho}_\sigma(t)\frac{\sin(t\varepsilon)}{t}\, dt. \tag{4.3}$$

We claim that $h_{\varepsilon,\sigma}$ satisfies a bound of the form

$$|h_{\varepsilon,\sigma}(\tau)| \leqslant C_N\Big(1 + \frac{||\tau|-\varepsilon|}{\sigma}\Big)^{-N} \qquad \text{for any } N \in \mathbb{N}. \tag{4.4}$$

To see this, recall that ρ is a Schwartz-class function with $\int_{\mathbb{R}} \rho\, dt = \widehat{\rho}(0) = 1$ and $\rho_\sigma(\tau) = \frac{1}{\sigma}\rho(\tau/\sigma)$. Thus,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} e^{it\tau}\widehat{\rho}_\sigma(t)\frac{\sin(t\varepsilon)}{t}\, dt = \int_{-\varepsilon}^{\varepsilon} \frac{1}{\sigma}\rho\left(\frac{\tau-\mu}{\sigma}\right)\, d\mu = \int_{\frac{\tau-\varepsilon}{\sigma}}^{\frac{\tau+\varepsilon}{\sigma}} \rho(\mu)\, d\mu.$$

Suppose $\tau > \varepsilon$. Then,

$$\left|\int_{\frac{\tau-\varepsilon}{\sigma}}^{\frac{\tau+\varepsilon}{\sigma}} \rho(\mu)\, d\mu\right| \leqslant \int_{\frac{\tau-\varepsilon}{\sigma}}^{\infty} |\rho(\mu)|\, d\mu \leqslant C_N\Big(1 + \frac{\tau-\varepsilon}{\sigma}\Big)^{-N}$$

for any N since ρ is Schwartz. The analogous estimate clearly holds in the case where $\tau < -\varepsilon$. If instead $|\tau| < \varepsilon$, then since ρ integrates to 1 and is rapidly decaying, along with the fact that $\mathbb{1}_{[-\varepsilon,\varepsilon]}$ is identically one on $[-\varepsilon, \varepsilon]$, we have that

$$|h_{\varepsilon,\sigma}(\tau)| = \left|\mathbb{1}_{[-\varepsilon,\varepsilon]}(\tau) - \int\limits_{\frac{\tau-\varepsilon}{\sigma}}^{\frac{\tau+\varepsilon}{\sigma}} \rho(\mu)\,d\mu\right| \leqslant \int\limits_{-\infty}^{\frac{\tau-\varepsilon}{\sigma}} |\rho(\mu)|\,d\mu + \int\limits_{\frac{\tau+\varepsilon}{\sigma}}^{\infty} |\rho(\mu)|\,d\mu \leqslant C_N\Big(1 + \frac{||\tau| - \varepsilon|}{\sigma}\Big)^{-N}$$

for any N. Finally, in the case where $|\tau| = \varepsilon$, (4.4) only claims that $h_{\varepsilon,\sigma}(\tau)$ is uniformly bounded in ε, σ, which follows immediately from the fact that

$$|h_{\varepsilon,\sigma}(\varepsilon)| = \left|1 - \int_0^{\frac{2\varepsilon}{\sigma}} \rho(\mu)\,d\mu\right| \leqslant 1$$

along with the analogous statement for $\tau = -\varepsilon$. Therefore, we have proved (4.4).

Observe that by (4.2) and (4.3) we have

$$\begin{aligned}&\left|\partial_x^\alpha \partial_y^\beta \left(\Pi_{[\lambda-\varepsilon,\lambda+\varepsilon]}(x,y) - \rho_\sigma * \Pi_{[\lambda-\varepsilon,\lambda+\varepsilon]}(x,y)\right)\right| \\ &\qquad\leqslant \left(\sum_{j=0}^{\infty} |h_{\varepsilon,\sigma}(\lambda-\lambda_j)||\partial_x^\alpha \varphi_j(x)|^2\right)^{\frac12} \left(\sum_{j=0}^{\infty} |h_{\varepsilon,\sigma}(\lambda-\lambda_j)||\partial_y^\beta \varphi_j(y)|^2\right)^{\frac12}.\end{aligned}$$

Thus, the claim in (4.1) would follow once we prove that given $\alpha \in \mathbb{N}$,

$$\lim_{\sigma\to 0^+} \lim_{\ell\to\infty} \frac{1}{\nu_\ell^{n-1+2|\alpha|}} \sum_{j=0}^{\infty} |h_{\varepsilon,\sigma}(\nu_\ell - \lambda_j)||\partial_x^\alpha \varphi_j(x)|^2 = 0. \tag{4.5}$$

For each ℓ, decompose $\mathbb{N} = J_1(\ell) \cup J_2(\ell) \cup J_3(\ell)$ with

$$J_1(\ell) := \{j : |\lambda_j - \nu_\ell| > \tfrac{\pi}{T}\}, \qquad J_2(\ell) := \{j : |\lambda_j - \nu_\ell| < r\ell^{-1/2}\},$$
$$J_3(\ell) := \{j : r\ell^{-1/2} < |\lambda_j - \nu_\ell| \leqslant \tfrac{\pi}{T}\}.$$

First, note that

$$\sum_{j\in J_1(\ell)} |h_{\varepsilon,\sigma}(\nu_\ell - \lambda_j)||\partial_x^\alpha \varphi_j(x)|^2 = \sum_{m=1}^{\infty} \sum_{|\lambda_j-\nu_\ell|\in[\frac{m\pi}{T},\frac{(m+1)\pi}{T}]} |h_{\varepsilon,\sigma}(\nu_\ell - \lambda_j)||\partial_x^\alpha \varphi_j(x)|^2. \tag{4.6}$$

Whenever $|\lambda_j - \nu_\ell| \in [\frac{m\pi}{T}, \frac{(m+1)\pi}{T}]$ with $m \geqslant 1$ and $\varepsilon < \frac{\pi}{2T}$, we have that

$$|h_{\varepsilon,\sigma}(\nu_\ell - \lambda_j)| \leqslant C_N \left(1 + \frac{1}{\sigma}\left|\frac{m\pi}{T} - \varepsilon\right|\right)^{-N} \leqslant C_N' \left(\frac{m}{\sigma}\right)^{-N}$$

for some $C_N' > 0$ by (4.4). For the same range of λ_j, we also have that

$$\sum_{|\lambda_j-\nu_\ell|\in[m\pi/T,(m+1)\pi/T]} |\partial_x^\alpha \varphi_j(x)|^2 \leqslant C(1 + \nu_\ell + m\pi/T)^{n-1+2|\alpha|}$$

for some $C, C' > 0$ by the local Weyl law (3.12). Therefore, by (4.6)

$$\sum_{j\in J_1(\ell)} |h_{\varepsilon,\sigma}(\nu_\ell - \lambda_j)||\partial_x^\alpha \varphi_j(x)|^2 \leqslant \widetilde{C}_N \sigma^N \sum_{m=1}^{\infty} (1 + \nu_\ell + m\pi/T)^{n-1+2|\alpha|} m^{-N}$$

for some $\widetilde{C}_N > 0$. Taking any $N \geqslant n + 1 + 2\alpha$, we thus obtain

$$\sum_{j \in J_1(\ell)} |h_{\varepsilon,\sigma}(\nu_\ell - \lambda_j)||\partial_x^\alpha \varphi_j(x)|^2 \leqslant C_1 \sigma^N \nu_\ell^{n-1+2|\alpha|} \tag{4.7}$$

for some $C_1 > 0$ and any $\sigma > 0$ small.

Next, to estimate the sum over $J_2(\ell)$ we note that for each fixed $r, \varepsilon > 0$, one can take ℓ sufficiently large so that $|r\ell^{-1/2} - \varepsilon| \geqslant \frac{\varepsilon}{2}$, in which case by (4.4) that

$$|h_{\varepsilon,\sigma}(\nu_\ell - \lambda_j)| \leqslant C_N \left(1 + \frac{\varepsilon}{\sigma}\right)^{-N} \leqslant C_N \left(\frac{\sigma}{\varepsilon}\right)^N$$

for $|\nu_\ell - \lambda_j| \leqslant r\ell^{-1/2}$. By the local Weyl law, we have

$$\sum_{j \in J_2(\ell)} |h_{\varepsilon,\sigma}(\nu_\ell - \lambda_j)||\partial_x^\alpha \varphi_j(x)|^2 \leqslant C_2 \left(\frac{\sigma}{\varepsilon}\right)^N \nu_\ell^{n-1+2|\alpha|} \tag{4.8}$$

for some $C_2 > 0$ and all ℓ sufficiently large.

Finally, to estimate the sum over $J_3(\ell)$ we apply Proposition 3.1, which implies that there exist $K > 0$ and $\ell_0 > 0$ such that for all $\ell \geqslant \ell_0$

$$\sum_{j \in J_3(\ell)} |\partial_x^\alpha \varphi_j(x)|^2 \leqslant Cr^{-2} \sum_{|\lambda_j - \nu_\ell| \leqslant K} |\partial_x^\alpha \varphi_j(x)|^2 \leqslant C' r^{-2} \nu_\ell^{n-1+2|\alpha|},$$

where the final inequality follows from the local Weyl law (3.12). Therefore, since $h_{\varepsilon,\sigma}$ is bounded by a uniform constant for all $\varepsilon, \sigma > 0$, we have

$$\sum_{j \in J_3(\ell)} |h_{\varepsilon,\sigma}(\nu_\ell - \lambda_j)||\partial_x^\alpha \varphi_j(x)|^2 \leqslant C_3 r^{-2} \nu_\ell^{n-1+2|\alpha|} \tag{4.9}$$

for some $C_3 > 0$, all $r > 0$, and all ℓ sufficiently large. Combining (4.7), (4.8), and (4.9),

$$\lim_{\ell \to \infty} \frac{1}{\nu_\ell^{n-1+2|\alpha|}} \sum_{j=0}^{\infty} |h_{\varepsilon,\sigma}(\nu_\ell - \lambda_j)||\partial_x^\alpha \varphi_j(x)|^2 \leqslant C_1 \sigma^N + C_2 (\sigma/\varepsilon)^N + C_3 r^{-2}$$

for all $\varepsilon < \frac{\pi}{2T}$ and all $\sigma, r > 0$. Recalling that $\varepsilon > 0$ was fixed in the statement of the proposition, we may send $\sigma \to 0$ and $r \to \infty$ to obtain (4.5), which completes the proof. $\square$

4.2. **Proof of Corollary 1.2.** The proof of Corollary 1.2 follows from Theorem 1 quite directly. First, we observe that (1.4) yields

$$\frac{1}{(2\pi)^n} \int_{S_y^* M} e^{i\lambda \langle \exp_y^{-1}(x), \omega \rangle_g} \frac{d\omega}{\sqrt{|g_y|}} = \frac{1}{(2\pi)^{n/2}} \frac{J_{\frac{n-2}{2}}(|\lambda d_g(x,y)|)}{(\lambda d_g(x,y))^{\frac{n-2}{2}}}.$$

Next, choose rescaled normal coordinates so that $x = \exp_{x_0}(u/\lambda)$, $y = \exp_{x_0}(v/\lambda)$. If we set $F(\tau) = \frac{J_\nu(\tau)}{\tau^\nu}$ for $\nu = \frac{n-2}{2}$, we then have

$$|F(\lambda d_g(x,y)) - F(|u-v|)| \leqslant |\lambda d_g(x,y) - |u-v||$$

since F' is uniformly bounded on $\mathbb{R}$. Furthermore, by the properties of geodesic normal coordinates, we know that

$$\left| d_g(x,y) - \frac{|u-v|}{\lambda} \right| \leqslant \frac{C|u-v|^2}{\lambda^2}.$$

Hence,

$$\frac{1}{(2\pi)^n}\int_{S_y^*M} e^{i\lambda\langle \exp_y^{-1}(x),\omega\rangle_g}\frac{d\omega}{\sqrt{|g_y|}} = \frac{1}{(2\pi)^{n/2}}\frac{J_{\frac{n-2}{2}}(|u-v|)}{|u-v|^{\frac{n-2}{2}}} + \mathcal{O}\left(\frac{|u-v|^2}{\lambda^2}\right).$$

It only remains to note that by [DG75, Theorem 3.] we have that

$$\dim(H_{\nu_\ell,\varepsilon}) = \frac{2\pi}{T}\cdot\frac{\mathrm{vol}(S^*M)}{(2\pi)^n}\nu_\ell^{n-1} + o(\nu_\ell^{n-1}).$$

□

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