Non-uniform finite-element meshes defined by ray dynamics for Helmholtz problems

Martin Averseng*, Jeffrey Galkowski†, Euan A. Spence‡

June 18, 2025

Abstract

The h-version of the finite-element method (h-FEM) applied to the high-frequency Helmholtz equation has been a classic topic in numerical analysis since the 1990s. It is now rigorously understood that (using piecewise polynomials of degree p on a mesh of a maximal width h) the conditions "$(hk)^p\rho$ sufficiently small" and "$(hk)^{2p}\rho$ sufficiently small" guarantee, respectively, k-uniform quasioptimality (QO) and bounded relative error (BRE), where ρ is the norm of the solution operator with $\rho \sim k$ for non-trapping problems. Empirically, these conditions are observed to be optimal in the context of h-FEM with a uniform mesh. This paper demonstrates that QO and BRE can be achieved using certain non-uniform meshes that violate the conditions above on h and involve coarser meshes away from trapping and in the perfectly matched layer (PML). The main theorem details how varying the meshwidth in one region affects errors both in that region and elsewhere. One notable consequence is that, for any scattering problem (trapping or nontrapping), in the PML one only needs hk to be sufficiently small; i.e. there is no pollution in the PML.

The motivating idea for the analysis is that the Helmholtz data-to-solution map behaves differently depending on the locations of both the measurement and data, in particular, on the properties of billiards trajectories (i.e. rays) through these sets. Because of this, it is natural that the approximation requirements for finite-element spaces in a subset should depend on the properties of billiard rays through that set. Inserting this behaviour into the latest duality arguments for the FEM applied to the high-frequency Helmholtz equation allows us to retain detailed information about the influence of *both* the mesh structure *and* the behaviour of the true solution on local errors in FEM.

Contents

*Laboratoire Angevin de Recherche Mathématique, Université d'Angers, 49045 Angers, France, martin.averseng@univ-angers.fr

†Department of Mathematics, University College London, London, WC1H 0AY, UK, J.Galkowski@ucl.ac.uk

‡Department of Mathematical Sciences, University of Bath, Bath, BA2 7AY, UK, E.A.Spence@bath.ac.uk

1 Introduction

1.1 The main result in its simplest form

The scattering problem and its finite-element approximation using a PML. We study computing approximations to the solution of sound-soft or sound-hard scattering problems using the finite-element method with non-uniform meshes. We consider scattering by an open obstacle $\Omega_- \Subset \mathbb{R}^d$ with smooth boundary and connected complement, $\Omega_+ := \mathbb{R}^d \setminus \overline{\Omega_-}$: given $f \in L^2_{\mathrm{comp}}(\overline{\Omega_+})$, find $u \in H^1_{\mathrm{loc}}(\overline{\Omega_+})$ such that

$$-k^{-2}\operatorname{div}(A\nabla u) - nu = f \text{ in } \Omega_+, \qquad (Bu)|_{\partial\Omega_+} = 0, \qquad (k^{-1}\partial_r - i)u = o_{r\to\infty}(r^{\frac{1-d}{2}}), \tag{1.1}$$

where A is a smooth, symmetric, positive-definite matrix with real coefficients, $n \in C^\infty(\overline{\Omega_+};\mathbb{R}_+)$, $\operatorname{supp}(A-I)\cup\operatorname{supp}(n-1) \Subset \overline{\Omega}_+$, and $Bu = u$ in the sound-soft case and $Bu = \partial_\nu u$, with ν the normal to $\partial\Omega_+$ in the sound-hard case.

We approximate the Sommerfeld radiation condition using a radial perfectly matched layer (PML): let $\Omega_{\mathrm{tr}} \Subset \mathbb{R}^d$ be open and contain the closed convex hull of $\Omega_- \cup \operatorname{supp}(A-I) \cup \operatorname{supp}(n-1)$. We truncate the problem (1.1) to the computational domain $\Omega := \Omega_+ \cap \Omega_{\mathrm{tr}}$ and apply the finite-element method to the problem: given $f \in L^2(\Omega)$, find $u \in H^1(\Omega)$ such that

$$P_k u := -k^{-2}\operatorname{div}(A_\theta \nabla u) + k^{-2} b_\theta \cdot \nabla u - n_\theta u = f \text{ in } \Omega, \qquad (Bu)|_{\partial\Omega_+} = 0, \qquad u|_{\partial\Omega_{\mathrm{tr}}} = 0, \tag{1.2}$$

where A_θ, b_θ, and n_θ are defined in §A (and A_θ, b_θ, and n_θ are respectively A, 0, and n in the non-PML region). Let $a_k(\cdot,\cdot)$ be the sesquilinear form associated with (1.2).

Definition 1.1 *Given a subspace $V \subset H^1(\Omega) \cap H^1_0(\Omega_+)$ (or $H^1_0(\Omega)$ in the sound-soft case), a* finite-element/Galerkin solution *of (1.2) is an element $u_h \in V$ such that*

$$a_k(u_h, w_h) = \langle f, w_h \rangle \quad \text{for all } w_h \in V. \tag{1.3}$$

Let

$$\rho = \rho(k) := \sup\Big\{ \|u\|_{L^2(\Omega)} \,:\, u \text{ solves (1.2) with } \|f\|_{L^2(\Omega)} = 1 \Big\}.$$

Recall that, with the normalisation used in (1.1), for all $k_0 > 0$ there exists $c > 0$ such that $\rho(k) \geq ck$ for $k > k_0$. By [GLS23, Theorem 1.6], for a radial PML (defined in §A), there exist $C, k_1 > 0$ such that for $k > k_1$ and $\chi \equiv 1$ on the convex hull of Ω,

$$\rho \leq C \sup\Big\{ \|\chi u\|_{L^2(\Omega^+)} \,:\, u \text{ solves (1.1) with } \|\chi f\|_{L^2(\Omega^+)} = 1 \Big\};$$

i.e. the PML solution operator is controlled by the scattering solution operator.

Assumption 1.2 *The set $\mathcal{J} \subset \mathbb{R}_+$, Ω_-, A_θ, b_θ, and n_θ are such that there are $C > 0$, $N > 0$ such that $\rho(k) \leq Ck^N$ for $k \in \mathbb{R}_+ \setminus \mathcal{J}$.*

By [LSW21] and [GLS23, Theorem 1.6], for any $\delta > 0$, Assumption 1.2 holds for a radial PML, any (Ω_-, A, n), and some $\mathcal{J}_\delta$ with $|\mathcal{J}_\delta| < \delta$.

State-of-the-art analysis of the h-FEM. The h-version of the finite-element method (FEM) considers the Galerkin solution to (1.2) with V given by the space of piecewise polynomials of a fixed degree, p, on a mesh with maximum width h. The accuracy of the solution is then increased by decreasing h.

Many authors have studied k-explicit conditions on the meshwidth guaranteeing that the finite-element solution exists and has controlled error. The best existing result is the following: if $(hk)^{2p}\rho$ is sufficiently small, then for $m \in \{0, \ldots, p-1\}$,

$$\|u - u_h\|_{H^{1-m}_k(\Omega)} \leq C\Big((hk)^m + \rho(hk)^p\Big) \inf_{w_h \in V^p_{\mathcal{T}_k}} \|u - w_h\|_{H^1_k(\Omega)}. \tag{1.4}$$

This estimate was proved for general Helmholtz problems and general $p \in \mathbb{Z}^+$ in [GS25] (with earlier work in [FW09, MS10, FW11, MS11, Wu14, DW15, BCFG17, LW19, CFN20, Pem20, CFGT22, LSW22a]) and is empirically sharp when the mesh considered has uniform width h. The bound (1.4) implies that if $\rho(hk)^p$ is bounded then the FE solution is quasi-optimal (QO) in the sense that $\|u - u_h\|_{H^1_k(\Omega)}$ is, up to a constant, the best-approximation error. Since $\rho \gtrsim k$, the requirement $\rho(hk)^p \lesssim 1$ implies that $hk \lesssim \rho^{1/p} \ll k^{-1/p}$– this fact that hk must decrease with k is the *pollution effect* [BS00].

Using standard piecewise-polynomial approximation results in the right-hand side of (1.4), one obtains that

$$\|u - u_h\|_{H^{1-m}_k(\Omega)} \leq C\Big((hk)^m + \rho(hk)^p\Big)(hk)^p\|u\|_{H^{p+1}_k(\Omega)}. \tag{1.5}$$

If the data is k-oscillatory, then so is the solution (by elliptic regularity; see [GS25, Page 9]), with $\|u\|_{H^{p+1}_k(\Omega)} \leq C\|u\|_{H^1_k(\Omega)}$. In this case, (1.5) implies that the Galerkin solution has bounded relative error (BRE) if $(hk)^{2p}\rho$ is sufficiently small. We highlight that this threshold for BRE was famously identified for 1-d problems in the work of Ihlenburg and Babuška [IB95, IB97] (see [IB97, Page 350, penultimate displayed equation], [Ihl98, Equation 4.7.41]).

To date, all k-explicit a priori analyses of the h-FEM consider uniform meshes. The goal of this paper is to study non-uniform meshes, designed by considering the ray dynamics in Ω_+, and give local – as opposed to global – criteria on the meshwidths. In particular, we show that there exist meshes that obtain QO/BRE while severely violating the mesh thresholds above, and thus involve many fewer degrees of freedom (see Table 1.1 below).

Subsets of Ω defined by ray dynamics. We define billiard trajectories to be geodesics for the metric $g^{-1} = A/n$ in Ω_+ continued by reflection with respect to g at the boundary of Ω_+[1]– when $A = I$ and $n = 1$, these are straight line paths continued using the Snell–Descartes law at the boundary. Next, we define the *cavity* $\mathcal{K} \subset \overline{\Omega}_+$ as the set of points $x \in \overline{\Omega}_+$ such that there is a billiard trajectory passing over x that remains in a compact set for all positive and negative times. We also define the *visible set* $\mathcal{V} \subset \overline{\Omega}_+$ as those points $x \in \overline{\Omega}_+$ such that there is a billiard trajectory passing over x that remains in a compact set for all positive times or all negative times. Finally, we define the *invisible set* $\mathcal{I} := \overline{\Omega}_+ \setminus (\mathcal{V} \cup \mathcal{K})$ (the adjectives visible and invisible are relative to the cavity). Let $\Omega_\mathcal{P} \subset \Omega$ be an open neighbourhood of $\partial\Omega_{\rm tr}$ that is strictly contained in the PML. Next, let $\Omega_\mathcal{K}$, $\Omega_\mathcal{V}$ and $\Omega_\mathcal{I}$ be open neighbourhoods of the intersections with $\overline{\Omega}$ of, respectively, $\mathcal{K}$, $\mathcal{V} \setminus (\mathcal{K} \cup \Omega_\mathcal{P})$, and $\mathcal{I} \setminus \Omega_\mathcal{P}$ in the subspace topology of $\overline{\Omega}$ such that $\Omega_\mathcal{K} \cap \partial\Omega_{\rm tr} = \Omega_\mathcal{V} \cap \partial\Omega_{\rm tr} = \Omega_\mathcal{I} \cap \partial\Omega_{\rm tr} = \emptyset$.

The finite-element space. Given a mesh, $\mathcal{T}$ of Ω, we define $h_\mathcal{K}, h_\mathcal{V}, h_\mathcal{I}, h_\mathcal{P} > 0$, to be upper bounds for the diameter of any mesh element that intersects $\Omega_\mathcal{K}$, $\Omega_\mathcal{V}$, $\Omega_\mathcal{I}$, and $\Omega_\mathcal{P}$ respectively, and let $h := \max\{h_\mathcal{K}, h_\mathcal{V}, h_\mathcal{I}, h_\mathcal{P}\}$. Since Ω is C^∞, some elements of the mesh need to be curved; however, our results can, in principle, be combined with those of [CFS25] to prove results about simplicial meshes. Let $\gamma(\mathcal{T})$ denote the shape-regularity constant of the mesh $\mathcal{T}$ (see, e.g., [BS08, Equation (4.4.16)]). We define the following measure of local uniformity of the mesh at scale $\epsilon > 0$:

$$U(\mathcal{T}, \epsilon) := \sup_{x \in \Omega} \sup_{\substack{T_1, T_2 \in \mathcal{T} \\ T_1 \cap B(x,\epsilon) \neq \emptyset \\ T_2 \cap B(x,\epsilon) = \emptyset}} \frac{\operatorname{diam}(T_1)}{\operatorname{diam}(T_2)}.$$

[1] In fact, we use a somewhat more complicated notion, the *generalised broken bicharacteristic* [Hör85, Section 24.3].

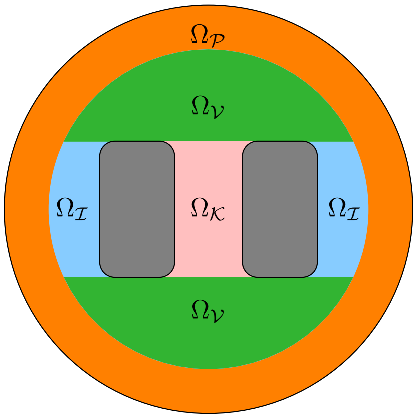


Figure 1.1: The domains $\Omega_{\mathcal{K}}, \Omega_{\mathcal{V}}, \Omega_{\mathcal{I}}$, and $\Omega_{\mathcal{P}}$, when Ω_- consists of two (rounded) aligned rectangles

We say that a family of meshes $(\mathcal{T}_k)_{k>0}$ is *wavelength-scale quasiuniform with constant* $\gamma_0 > 0$ if the mesh is shape regular with $\gamma(\mathcal{T}_k) \geq \gamma_0$ and $U(\mathcal{T}, (1+k)^{-1}) \leq \gamma_0^{-1}$ for all $k > 0$.

For a mesh $\mathcal{T}$ and $p \in \{1, 2, \dots\}$, we denote by $V_{\mathcal{T}}^p \subset H_0^1(\Omega)$ (or $H_0^1(\Omega_{\rm tr}) \cap H^1(\Omega)$ in the sound-hard case) the space of Lagrange piecewise polynomials of degree p on the mesh $\mathcal{T}$.

The main result in its simplest form. Define

$$\mathcal{C} := \begin{pmatrix} \rho & \sqrt{k\rho} & 0 & 0 \\ \sqrt{k\rho} & k & k & 0 \\ 0 & k & k & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{H} := \begin{pmatrix} h_{\mathcal{K}} & 0 & 0 & 0 \\ 0 & h_{\mathcal{V}} & 0 & 0 \\ 0 & 0 & h_{\mathcal{I}} & 0 \\ 0 & 0 & 0 & h_{\mathcal{P}} \end{pmatrix}, \quad \mathscr{F} := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$
$$\mathscr{T} := \begin{pmatrix} 1 & (h_{\mathcal{V}}k)^{2p}\sqrt{k\rho} & (h_{\mathcal{V}}k)^{2p}\sqrt{k\rho}(h_{\mathcal{I}}k)^{2p}k & 0 \\ (h_{\mathcal{K}}k)^{2p}\sqrt{k\rho} & 1 & (h_{\mathcal{I}}k)^{2p}k & 0 \\ (h_{\mathcal{K}}k)^{2p}\sqrt{k\rho}(h_{\mathcal{V}}k)^{2p}k & (h_{\mathcal{V}}k)^{2p}k & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.6}$$

Conceptually, $\mathcal{C}$ is the norm of the localised data-to-solution map (with $\mathcal{C}$ standing for "communication") and $\mathscr{T}$ controls the propagation of Galerkin errors between subdomains according to the graph in Figure 9.1 (with a simplified version – Figure 1.2 – given in the sketch of the proof in §1.3.2).

We work in k-weighted Sobolev spaces defined for $U \subset \mathbb{R}^d$ by

$$\|u\|_{H_k^n(U)}^2 := \sum_{|\alpha| \leq n} k^{-2|\alpha|} \|\partial^\alpha u\|_{L^2(U)}^2, \quad n \in \mathbb{N}, \tag{1.7}$$

and let $H_k^{-n}(U)$ be the normed dual of $H_k^n(U)$.

The following is a particular case of our main result (Theorem 3.11 below).

Theorem 1.3 *Let $k_0, N, \gamma_0 > 0$, $p \in \mathbb{N} \setminus \{0\}$, $\mathcal{J} \subset \mathbb{R}_+$ such that Assumption 1.2 holds, and let $\Omega'_\star$ be compactly contained in $\Omega_\star$ with respect to the subspace topology of $\overline{\Omega}$, $\star \in \{\mathcal{K}, \mathcal{V}, \mathcal{I}, \mathcal{P}\}$.*

There exist $c, C > 0$ such that for all families of meshes $(\mathcal{T}_k)_{k>0}$ that are wavelength-scale quasiuniform with constant γ_0 and satisfy

$$(h_{\mathcal{K}}k)^{2p}\rho(k) + (h_{\mathcal{V}}k)^{2p}k + (h_{\mathcal{I}}k)^{2p}k + (h_{\mathcal{P}}k)^{2p} \leq c, \tag{1.8}$$

all $k \in (k_0, \infty) \setminus \mathcal{J}$, and all $w_{h,\star} \in V_{\mathcal{T}_k}^p$, with $\star \in \{\mathcal{K}, \mathcal{V}, \mathcal{I}, \mathcal{P}\}$, the Galerkin solution, $u_h \in V_{\mathcal{T}_k}^p$, to (1.2) exists, is unique, and satisfies, for $0 \leq m \leq p$,

$$\begin{pmatrix} \|u-u_h\|_{H_k^{1-m}(\Omega'_{\mathcal{K}})} \\ \|u-u_h\|_{H_k^{1-m}(\Omega'_{\mathcal{V}})} \\ \|u-u_h\|_{H_k^{1-m}(\Omega'_{\mathcal{I}})} \\ \|u-u_h\|_{H_k^{1-m}(\Omega'_{\mathcal{P}})} \end{pmatrix} \leq C\Big[(\mathcal{H}k)^m + \mathscr{T}\mathcal{C}(\mathcal{H}k)^p + k^{-N}(hk)^m\mathscr{F}\Big] \begin{pmatrix} \|u-w_{h,\mathcal{K}}\|_{H_k^1(\Omega_{\mathcal{K}})} \\ \|u-w_{h,\mathcal{V}}\|_{H_k^1(\Omega_{\mathcal{V}})} \\ \|u-w_{h,\mathcal{I}}\|_{H_k^1(\Omega_{\mathcal{I}})} \\ \|u-w_{h,\mathcal{P}}\|_{H_k^1(\Omega_{\mathcal{P}})} \end{pmatrix}, \tag{1.9}$$

where the inequality in (1.9) *is understood component-wise.*

Remark 1.4 *From the estimate* (1.9) *and the interpretation of $\mathscr{T}$ as the propagation of Galerkin errors, the matrix $\mathcal{C}(\mathcal{H}k)^p$ should be viewed as mapping best approximation errors to Galerkin errors. This appears more concretely in the proof of Theorem 1.3 and we discuss this interpretation in §1.3.3.*

To the best of the authors' knowledge, Theorem 1.3 and its more sophisticated analogue Theorem 3.11 are the first results concerning k-dependent, non-uniform finite-element meshes in the context of the Helmholtz equation. For a uniform mesh ($h_{\mathcal{K}} = h_{\mathcal{V}} = h_{\mathcal{I}} = h_{\mathcal{P}} = h$) (1.9) implies the strongest previously-known bound (1.4). Indeed, for a uniform mesh with $(hk)^{2p}\rho$ sufficiently small, all the elements of the matrix $\mathscr{T}$ are bounded by a constant, and all the elements of $\mathscr{C}(\mathcal{H}k)^p$ are bounded by $\rho(hk)^p$. However, Theorem 1.3 provides much more information than (1.4): it describes how the best approximation errors and local meshwidth in each region affect the Galerkin error in all other regions. Section 1.2 highlights some notable consequences of this description, with Section 2 illustrating these numerically.

Theorem 1.3 is most interesting when $\rho(k) \gg k$, which is equivalent to the problem being trapping, i.e., $\mathcal{K} \neq \emptyset$ (see [BBR10], [DZ19, Theorem 7.1]). In particular, Theorem 1.3 shows that in the trapping case there exist meshes with $(hk)^p\rho \gg 1$ whose finite-element solutions have guaranteed k-uniform quasioptimality (see Corollary 1.11). Even when $\mathcal{K} = \emptyset$, Theorem 1.3 gives new information including that one needs only a fixed number of points per wavelength in the PML.

To compare with the estimate (1.5) on relative error, we state the following corollary of Theorem 1.3 which follows from standard piecewise-polynomial approximation estimates.

Corollary 1.5 *Let $k_0, N, \gamma_0 > 0$, $p \in \mathbb{N} \setminus \{0\}$, $\mathcal{J} \subset \mathbb{R}_+$ such that Assumption 1.2 holds, and let $\Omega'_\star$ be compactly contained in $\Omega_\star$ with respect to the subspace topology of $\overline{\Omega}$, $\star \in \{\mathcal{K}, \mathcal{V}, \mathcal{I}, \mathcal{P}\}$.*

There exist $c, C > 0$ such that for all families of meshes $(\mathcal{T}_k)_{k>0}$ that are wavelength-scale quasiuniform with constant γ_0 and satisfy (1.8) *and all $k \in (k_0, \infty) \setminus \mathcal{J}$, the Galerkin solution, $u_h \in V^p_{\mathcal{T}_k}$, to* (1.2) *exists, is unique, and satisfies, for $0 \leq m \leq p$,*

$$\begin{pmatrix} \|u - u_h\|_{H_k^{1-m}(\Omega'_{\mathcal{K}})} \\ \|u - u_h\|_{H_k^{1-m}(\Omega'_{\mathcal{V}})} \\ \|u - u_h\|_{H_k^{1-m}(\Omega'_{\mathcal{I}})} \\ \|u - u_h\|_{H_k^{1-m}(\Omega'_{\mathcal{P}})} \end{pmatrix} \leq C\Big[(\mathcal{H}k)^m + \mathscr{T}\mathcal{C}(\mathcal{H}k)^p + k^{-N}(hk)^m \mathscr{F}\Big](\mathcal{H}k)^p \begin{pmatrix} \|u\|_{H_k^{p+1}(\Omega_{\mathcal{K}})} \\ \|u\|_{H_k^{p+1}(\Omega_{\mathcal{V}})} \\ \|u\|_{H_k^{p+1}(\Omega_{\mathcal{I}})} \\ \|u\|_{H_k^{p+1}(\Omega_{\mathcal{P}})} \end{pmatrix}. \tag{1.10}$$

As with (1.5), when the data, f, is k-oscillatory, so is the solution u, and in this case, $\|u\|_{H_k^{p+1}(U')} \leq C\|u\|_{H_k^1(U)}$ for $U' \Subset U$. Hence, one can use (1.10) to find meshes with $(hk)^{2p}\rho \gg 1$ that nevertheless have guaranteed control on the relative error (see Corollary 1.12).

Remark 1.6 (Improvements in Theorem 3.11) *Theorem 3.11 below is stronger than Theorem 1.3 in that it considers arbitrary covers of Ω, and bounds the high ($\gg k$) and low ($\lesssim k$) frequencies of the Galerkin error separately. Two situations in which a more complicated cover is advantageous are the following. 1) There are two or more cavities that are dynamically separated, i.e., for which there is no billiard trajectory whose closure insects both cavities. 2) One has a priori information about the data and/or solution and hence can obtain good control on the right-hand side of* (1.9)*. Even when $\mathcal{K} = \emptyset$, such information combined with Theorem 1.3 allows one to define meshes with a priori improved accuracy in some regions, without the need to choose a small meshwidth everywhere.*

1.2 Special cases of Theorem 1.3

We now apply Theorem 1.3 in several special cases, and derive consequences regarding quasi-optimality and bounded relative errors. For simplicity, we state these results for $m = 0$, i.e., we bound the H_k^1 norm of the Galerkin error. The results of this section are summarised in Table 1.1.

Define

$$\mathscr{M} := I + \mathscr{T}\mathcal{C}(\mathcal{H}k)^p, \qquad \mathscr{M}_{\mathrm{RE}} := \mathscr{M}(\mathcal{H}k)^p,$$

and set $\mathscr{M}_\Omega := \mathscr{M}\begin{pmatrix}1 & \dots & 1\end{pmatrix}^T$, $\mathscr{M}_{\mathrm{RE},\Omega} := \mathscr{M}_{\mathrm{RE}}\begin{pmatrix}1 & \dots & 1\end{pmatrix}^T$. With these definitions, the terms in square brackets on the right-hand sides of (1.9) and (1.10) become, respectively, $[\mathscr{M} + k^{-N}\mathscr{F}]$ and $[\mathscr{M}_{\mathrm{RE}} + k^{-N}\mathscr{F}(\mathcal{H}k)^p]$ and, in particular, imply that

$$\begin{pmatrix}\|u-u_h\|_{H^1_k(\Omega'_{\mathcal{K}})}\\ \|u-u_h\|_{H^1_k(\Omega'_{\mathcal{V}})}\\ \|u-u_h\|_{H^1_k(\Omega'_{\mathcal{I}})}\\ \|u-u_h\|_{H^1_k(\Omega'_{\mathcal{P}})}\end{pmatrix} C \leq \begin{cases}\left[\mathscr{M}_\Omega + k^{-N}\begin{pmatrix}1 & \dots & 1\end{pmatrix}^T\right] \inf_{w_h\in V^p_{\mathcal{T}_k}} \|u-w_h\|_{H^1_k(\Omega)} \\ \\ \left[\mathscr{M}_{\mathrm{RE},\Omega} + k^{-N}(hk)^p\begin{pmatrix}1 & \dots & 1\end{pmatrix}^T\right] \|u\|_{H^{p+1}_k(\Omega)}\,.\end{cases}$$

Bounds for the coarsest meshes allowed by Theorem 1.3.

Corollary 1.7 (Bound on the quasi-optimality constant) *Under the same assumptions as Thoerem 1.3,*

$$\mathscr{M} \leq C\begin{pmatrix}\sqrt{\rho} & \sqrt{\rho} & \sqrt{\rho} & 0\\ \sqrt{k} & \sqrt{k} & \sqrt{k} & 0\\ \sqrt{k} & \sqrt{k} & \sqrt{k} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \qquad \mathscr{M}_\Omega \leq C\begin{pmatrix}\sqrt{\rho}\\ \sqrt{k}\\ \sqrt{k}\\ 1\end{pmatrix}.$$

Corollary 1.8 (Bound on the relative error) *Under the same assumptions as in Theorem 1.3, for all $\varepsilon \leq c$, if $(\mathcal{T}_k)_{k>0}$ satisfies*

$$(h_{\mathcal{K}}k)^{2p}\rho(k) + (h_{\mathcal{V}}k)^{2p}k + (h_{\mathcal{I}}k)^{2p}k + (h_{\mathcal{P}}k)^{2p} \leq \varepsilon,$$

then

$$\mathscr{M}_{\mathrm{RE}} \leq C\sqrt{\varepsilon}\begin{pmatrix}1 & \sqrt{\frac{\rho}{k}} & \sqrt{\frac{\rho}{k}} & 0\\ \sqrt{\frac{k}{\rho}} & 1 & 1 & 0\\ \sqrt{\frac{k}{\rho}} & 1 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \qquad \mathscr{M}_{\mathrm{RE},\Omega} \leq C\sqrt{\varepsilon}\begin{pmatrix}\sqrt{\rho/k}\\ 1\\ 1\\ 1\end{pmatrix}.$$

Estimates for uniform meshes. For quasi-uniform meshes $h_{\mathcal{K}} = h_{\mathcal{V}} = h_{\mathcal{I}} = h_{\mathcal{P}} =: h$, recall from §1.1 that the known mesh conditions for ensuring k-uniform quasioptimality or a controllably-small relative error of the Galerkin solution are, respectively

$$(hk)^p\rho(k) < c, \qquad (hk)^{2p}\rho(k) < c,$$

for $c > 0$ sufficiently small, with the former regime known as the *asymptotic regime*. Here we show that these thresholds also ensure better error estimates for the Galerkin error away from trapping.

Corollary 1.9 (Asymptotic estimates) *Under the same assumptions as Theorem 1.3, if $\mathcal{T}_k$ satisfies $h_{\mathcal{K}} = h_{\mathcal{V}} = h_{\mathcal{I}} = h_{\mathcal{P}} = h$, with $(hk)^p\rho < c$, then*

$$\mathscr{M} \leq C\begin{pmatrix}1 & \sqrt{\frac{k}{\rho}} & (\frac{k}{\rho})^{\frac{3}{2}}\frac{1}{\rho} & 0\\ \sqrt{\frac{k}{\rho}} & 1 & \frac{k}{\rho} & 0\\ (\frac{k}{\rho})^{\frac{3}{2}}\frac{1}{\rho} & \frac{k}{\rho} & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Corollary 1.10 (Preasymptotic estimates) *Under the same assumptions as Theorem 1.3, there exists $C > 0$ such that for all $0 < \varepsilon < c$ if $(\mathcal{T}_k)_{k>0}$ satisfies $h_{\mathcal{K}} = h_{\mathcal{V}} = h_{\mathcal{I}} = h_{\mathcal{P}} = h$, with $(hk)^{2p}\rho < \varepsilon$, then*

$$\mathscr{M}_{\mathrm{RE}} \leq C\sqrt{\varepsilon}\begin{pmatrix}1 & \sqrt{\frac{k}{\rho}} & \left(\frac{k}{\rho}\right)^{3/2} & 0\\ \sqrt{\frac{k}{\rho}} & \frac{k}{\rho} + \frac{1}{\sqrt{\rho}} & \frac{k}{\rho} & 0\\ \left(\frac{k}{\rho}\right)^{3/2} & \left(\frac{k}{\rho}\right)^2 & \frac{k}{\rho} + \frac{1}{\sqrt{\rho}} & 0\\ 0 & 0 & 0 & \frac{1}{\sqrt{\rho}}\end{pmatrix}. \tag{1.11}$$

Weakest conditions guaranteeing k-uniform quasi-optimality and controllably-small relative error.

We proceed by identifying the minimal thresholds under which Theorem 1.3 guarantees that (i) the Galerkin solution is quasi-optimal, uniformly in k, and (ii) the relative error is controllably small.

Corollary 1.11 (Threshold for k-uniform quasi-optimality) *Under the same assumptions as Theorem 1.3, if $(\mathcal{T}_k)_{k>0}$ satisfies*

$$(h_{\mathcal{K}}k)^p\rho + (h_{\mathcal{V}}k)^p\sqrt{k\rho} + (h_{\mathcal{I}}k)^p k + (h_{\mathcal{P}}k)^p < c, \tag{1.12}$$

then

$$\mathscr{M} \leq C\begin{pmatrix} 1 & 1 & \frac{1}{\sqrt{k\rho}} & 0 \\ \sqrt{\frac{k}{\rho}} & 1 & 1 & 0 \\ \frac{1}{\rho}\sqrt{\frac{k}{\rho}} & \sqrt{\frac{k}{\rho}} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.13}$$

Corollary 1.12 (Threshold for bounded relative error) *Under the same assumptions as Theorem 1.3, there exists $C>0$ such that for all $0<\varepsilon<c$ if $(\mathcal{T}_k)_{k>0}$ satisfies*

$$(h_{\mathcal{K}}k)^{2p}\rho + (h_{\mathcal{V}}k)^{2p}\sqrt{\rho k} + (h_{\mathcal{I}}k)^{2p} k + (h_{\mathcal{P}}k)^{2p} < \varepsilon, \tag{1.14}$$

then

$$\mathscr{M}_{\mathrm{RE}} \leq C\sqrt{\varepsilon}\begin{pmatrix} 1 & 1 & 1 & 0 \\ \sqrt{\frac{k}{\rho}} & \sqrt{\frac{k}{\rho}} + \frac{1}{(\rho k)^{1/4}} & 1 & 0 \\ \frac{k}{\rho} & \frac{k}{\rho} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.15}$$

In particular, $\|u-u_h\|_{H^1_k(\Omega)}/\|u\|_{H^{p+1}_k(\Omega)}$ is bounded.

Remark 1.13 *In Corollaries 1.8 and 1.10 one can track how the matrix entries depend on ε, but we do not do this here for simplicity.*

Condition guaranteeing quasi-optimality away from trapping. We finally give the weakest condition under which Theorem 1.3 ensures that the quantities

$$\frac{\|u-u_h\|_{H^1_k(\Omega'_{\mathcal{V}})}}{\min\limits_{w_h\in V^p_{\mathcal{T}}}\|u-w_h\|_{H^1_k(\Omega)}}, \quad \frac{\|u-u_h\|_{H^1_k(\Omega'_{\mathcal{I}})}}{\min\limits_{w_h\in V^p_{\mathcal{T}}}\|u-w_h\|_{H^1_k(\Omega)}} \tag{1.16}$$

remain k-uniformly bounded. We refer to these quantities as the quasi-optimality constants "away from trapping". These quantities should not be confused with "local quasi-optimality" constants (which would be defined with $\Omega_{\mathcal{V}}$ and $\Omega_{\mathcal{I}}$ instead of Ω in the denominators of (1.16)).

Corollary 1.14 (Threshold for k-uniform "quasi-optimality away from trapping") *Under the same assumptions as Theorem 1.3, if $(\mathcal{T}_k)_{k>0}$ satisfies*

$$(h_{\mathcal{K}}k)^p\sqrt{k\rho} + (h_{\mathcal{V}}k)^p k + (h_{\mathcal{I}}k)^p k + (h_{\mathcal{P}}k)^p < c, \tag{1.17}$$

then

$$\mathscr{M} \leq C\begin{pmatrix} \sqrt{\frac{\rho}{k}} & \sqrt{\frac{\rho}{k}} & \frac{1}{k^2}\sqrt{\frac{\rho}{k}} & 0 \\ 1 & 1 & 1 & 0 \\ \frac{1}{k} & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathscr{M}_\Omega \leq C\begin{pmatrix} \sqrt{\frac{\rho}{k}} \\ 1 \\ 1 \\ 1 \end{pmatrix}. \tag{1.18}$$

Remark 1.15 *It is natural to look for the analogous weakest condition guaranteeing a controllably small k-uniform bound on the "relative error away from trapping", defined by*

$$\frac{\|u-u_h\|_{H^1_k(\Omega'_{\mathcal{V}})}}{\|u\|_{H^{p+1}_k(\Omega)}}, \quad \frac{\|u-u_h\|_{H^1_k(\Omega'_{\mathcal{I}})}}{\|u\|_{H^{p+1}_k(\Omega)}} \tag{1.19}$$

(again not to be confused with "local relative errors" which would involve $\|u\|_{H^1_k(\Omega_{\mathcal{V}}\cap\Omega)}$ and $\|u\|_{H^1_k(\Omega_{\mathcal{I}}\cap\Omega)}$ in the denominators). However, these quantities are already bounded under the weakest possible condition in Theorem 1.3, see Corollary 1.8.

	Mesh threshold	Asymptotic DoFs	Theoretical guarantee	Name
↑ more DoFs	$(h_{\mathcal{K}}k)^p\rho + (h_{\mathcal{V}}k)^p\rho + (h_{\mathcal{I}}k)^p\rho = c$	$\mathrm{vol}(\Omega)k^d\rho^{\frac{d}{p}}$	k-QO	U1
	$(h_{\mathcal{K}}k)^p\rho + (h_{\mathcal{V}}k)^p\sqrt{k\rho} + (h_{\mathcal{I}}k)^pk = c$	$\mathrm{vol}(\Omega_{\mathcal{K}})k^d\rho^{\frac{d}{p}}$	k-QO	QO
	$(h_{\mathcal{K}}k)^p\sqrt{k\rho} + (h_{\mathcal{V}}k)^pk + (h_{\mathcal{I}}k)^pk = c$	$\mathrm{vol}(\Omega_{\mathcal{K}})k^{d+\frac{1}{2p}}\rho^{\frac{d}{2p}}$	k-QO away from trapping	QO away
	$(h_{\mathcal{K}}k)^{2p}\rho + (h_{\mathcal{V}}k)^{2p}\rho + (h_{\mathcal{I}}k)^{2p}\rho = c$	$\mathrm{vol}(\Omega)k^d\rho^{\frac{d}{2p}}$	CRE	U2
	$(h_{\mathcal{K}}k)^{2p}\rho + (h_{\mathcal{V}}k)^{2p}\sqrt{k\rho} + (h_{\mathcal{I}}k)^{2p}k = c$	$\mathrm{vol}(\Omega_{\mathcal{K}})k^d\rho^{\frac{d}{2p}}$	CRE	RE
fewer DoFs ↓	$(h_{\mathcal{K}}k)^{2p}\rho + (h_{\mathcal{V}}k)^pk + (h_{\mathcal{I}}k)^pk = c$	$\mathrm{vol}(\Omega_{\mathcal{K}})k^d\rho^{\frac{d}{2p}}$	CRE away from trapping	RE away

Table 1.1: Summary of the special cases of Theorem 1.3 discussed in this section, with $\mathcal{K}\neq\emptyset$. Note that in all cases we require $h_Pk = c$ which does not contribute to the asymptotic number of degrees of freedom (DoFs). Here, k-QO stands for k-uniform quasioptimality, and CRE stands for controllably-small relative error.

1.3 Discussion of the ideas behind Theorem 1.3 and a sketch of the proof

1.3.1 The ideas behind Theorem 1.3

The following two important phenomena motivate Theorem 1.3.

1. *The solution operator P_k^{-1} reflects the billiard dynamics in Ω.* In particular, for $\chi_1, \chi_2 \in C^\infty(\Omega)$ the operator $\chi_1 P_k^{-1}\chi_2$ behaves differently depending on the locations of $\operatorname{supp}\chi_j$; e.g., $\|P_k^{-1}\| \gg k$ when $\Omega_{\mathcal{K}}\neq\emptyset$, but if both χ_1 and χ_2 are away from $\Omega_{\mathcal{K}}$ then $\|\chi_1 P_k^{-1}\chi_2\|_{L^2\to L^2} \lesssim k$.
2. *The Galerkin error propagates.* The best possible situation would be local quasioptimality i.e., there exists $C>0$ such that the Galerkin solution u_h satisfies, for every $U\subset\Omega$,

$$\|u-u_h\|_{H^1_k(U)} \le C \inf_{w_h\in V^p_{\mathcal{T}}} \|u-w_h\|_{H^1_k(U)}. \tag{1.20}$$

In this case, since approximation of oscillatory functions by piecewise polynomials is well understood (see [Gal25] and the references therein), the properties of the data and behaviour of P_k^{-1} would dictate the meshwidth in each region. Unfortunately (1.20) cannot hold for general meshes. Indeed, suppose that (1.20) holds and let $\phi, \phi_1, \phi_2 \in C^\infty(\Omega)$ be such that $\operatorname{supp}\phi \subset \{\phi_1 \equiv 1\}$, $\phi \neq 0$, and $\phi_1+\phi_2\equiv 1$ on Ω. Then,

$$\phi(u-u_h) = \sum_{j=1}^{2} \phi P_k^{-1}\phi_j P_k(u-u_h). \tag{1.21}$$

We now consider a situation where $\mathcal{T}$ has arbitrarily small elements on $\operatorname{supp}\phi_1 =: \Omega_1$ so that, by (1.20),

$$\|u-u_h\|_{H^1_k(\Omega_1)} \le C \inf_{w_h\in V^p_{\mathcal{T}}} \|u-w_h\|_{H^1_k(\Omega_1)} \ll 1.$$

In particular,

$$\|\phi(u-u_h)\|_{H^1_k} + \|P_k^{-1}\phi_1 P_k(u-u_h)\|_{H^1_k} \ll 1$$

(by continuity of P_k and P_k^{-1} and locality of P_k). Then, (1.21) implies that

$$\|\phi P_k^{-1}\phi_2 P_k(u-u_h)\|_{H^1_k} \ll 1,$$

which cannot be true unless the meshwidth is also sufficiently small on Ω_2 or $\phi P_k^{-1}\phi_2 \approx 0$. By Item 1, the latter is not the case whenever $\operatorname{supp}\phi$ and $\operatorname{supp}\phi_2$ are connected by a billiard trajectory. (For a striking illustration of this propagation of error, see [AGS24, Figure 3].)

This argument indicates, not only that the Galerkin error propagates, but that the norm of the operator $\phi P_k^{-1}\phi_2$ determines the strength of propagation from $\operatorname{supp}\phi_2$ to $\operatorname{supp}\phi$.

Item 1 motivates varying the meshwidth from one location to another, but Item 2 shows that, to be effective, this strategy must take into account the global behaviour of billiard trajectories. In particular, by Item 2, the error in the cavity is *not* just dictated by the meshwidth in the cavity – the meshwidth also needs to be sufficiently small away from the cavity to control the propagating error.

1.3.2 Sketch of the proof of Theorem 1.3

For simplicity, we consider here the bound (1.9) with $m = p$ and ignore improvements that are possible in the overlaps between subdomains, in the PML region, and by splitting the frequencies of the Galerkin error into those $\gg k$ and $\lesssim k$.

The proofs of Theorem 1.3 and Theorem 3.11 are, at heart, localised versions of the elliptic projection-type argument introduced in [GS25]. We first recap this argument and prove (1.4) for $m = p$. The key insight in [GS25] is the existence of a self-adjoint smoothing operator S_k so that $P_k^\sharp := P_k + S_k$ is coercive (uniformly in k) and for all N there exists $C > 0$ such that for $k \geq k_0$

$$\|S_k\|_{H_k^{-N}\to H_k^N} \leq C$$

(see (5.12) for the definition of the operator S_k). Since $P_k^\sharp$ is coercive there is an *elliptic projection* $\Pi_k^\sharp : H_k^1 \to V^p_{\mathcal{T}_k}$ such that

$$\left\langle P_k^\sharp w_h, (I-\Pi_k^\sharp)u\right\rangle = 0 \quad \text{for all } w_h \in V^p_{\mathcal{T}_k} \tag{1.22}$$

and there exists $C > 0$ such that for all $k > k_0$

$$\|(\mathrm{I}-\Pi_k^\sharp)v\|_{H^1_k} \leq C \inf_{w_h\in V^p_{\mathcal{T}_k}} \|v-w_h\|_{H^1_k} \tag{1.23}$$

(i.e. $\Pi_k^\sharp$ is the adjoint Galerkin projection associated to $P_k^\sharp$). Moreover, by an Aubin–Nitsche-type duality argument

$$\|(\mathrm{I}-\Pi_k^\sharp)v\|_{H_k^{-p+1}} \leq C(hk)^p \inf_{w_h\in V^p_{\mathcal{T}_k}} \|v-w_h\|_{H^1_k}. \tag{1.24}$$

It follows from (1.22) and Galerkin orthogonality (1.3) that for all $w_h \in V^p_{\mathcal{T}_k}$, $v \in H_k^{p-1}$,

$$\begin{aligned}
\langle u-u_h, v\rangle &= \left\langle P_k(u-u_h), R_k^* v\right\rangle \\
&= \left\langle P_k(u-u_h), (I-\Pi_k^\sharp)R_k^* v\right\rangle \\
&= \left\langle P_k^\sharp(u-u_h), (I-\Pi_k^\sharp)R_k^* v\right\rangle - \left\langle S(u-u_h), (I-\Pi_k^\sharp)R_k^* v\right\rangle \\
&= \left\langle P_k^\sharp(u-w_h), (I-\Pi_k^\sharp)R_k^* v\right\rangle - \left\langle S(u-u_h), (I-\Pi_k^\sharp)R_k^* v\right\rangle.
\end{aligned} \tag{1.25}$$

By (1.23), (1.24), and the mapping properties $S_k : H_k^{-p+1} \to H_k^{p-1}$ and $P_k^\sharp : H_k^1 \to H_k^{-1}$,

$$\begin{gathered}
|\langle u-u_h, v\rangle| \leq C\eta_p\Big(\inf_{w_h\in V^p_{\mathcal{T}_k}} \|u-w_h\|_{H^1_k} + (hk)^p\|u-u_h\|_{H_k^{-p+1}}\Big)\|v\|_{H_k^{p-1}}, \\
\text{where} \quad \eta_p := \sup_{0\neq v\in H_k^{p-1}} \inf_{w_h\in V^p_{\mathcal{T}_k}} \frac{\|R_k^* v - w_h\|_{H^1_k}}{\|v\|_{H_k^{p-1}}}.
\end{gathered} \tag{1.26}$$

By duality, (1.26) implies

$$\|u-u_h\|_{H_k^{-p+1}} \le C\Big(b \inf_{w_h\in V^p_{\mathcal{T}_k}} \|u-w_h\|_{H^1_k} + \omega\|u-u_h\|_{H_k^{-p+1}}\Big),\qquad b:=\eta_p,\qquad \omega:=(hk)^p\eta_p. \tag{1.27}$$

By a frequency splitting argument similar to that in Lemma 8.5 below and the fact that $\rho \ge ck$,

$$\eta_p \le C(hk)^p\big(1+\|R_k\|_{L^2\to L^2}\big) = C(hk)^p(1+\rho) \le C(hk)^p\rho.$$

Thus, from (1.27), when $(hk)^{2p}\rho$ is sufficiently small, $(1-C\omega)^{-1}$ exists and is positive, and then

$$\|u-u_h\|_{H_k^{-p+1}} \le C(1-C\omega)^{-1} b \inf_{w_h\in V^p_{\mathcal{T}_k}} \|u-w_h\|_{H^1_k} \le C(hk)^p\rho \inf_{w_h\in V^p_{\mathcal{T}_k}} \|u-w_h\|_{H^1_k}, \tag{1.28}$$

which is the preasymptotic estimate (1.4) for $m=p$.

We now sketch the localised version of the above argument, which is used to prove Theorems 1.3 and 3.11. In this sketch, we treat $\Pi^\sharp_k$ and S_k as though they are local; i.e., for $\chi,\psi\in C^\infty(\overline{\Omega})$ with $\operatorname{supp}\chi\cap\operatorname{supp}\psi=\emptyset$, we neglect the terms

$$\chi\Pi^\sharp_k\psi \text{ and } \chi S_k\psi.$$

In general these terms are nonzero, but Sections 5 to 7, which contain the bulk of the technical work of this paper, show that they are $O(k^{-\infty})$ and smoothing. Using these properties, Section 8 shows that these terms only contribute to the remainder term in Theorem 1.3.

To localise the elliptic-projection argument, we introduce an open cover of Ω, $\{\Omega_j\}_{j=1}^M$ and $\{\phi_j\}_{j=1}^M \subset C^\infty(\overline{\Omega})$ a partition of unity subordinate to this cover. (In Theorem 1.3, $M=4$ and $(\Omega_1,\Omega_2,\Omega_3,\Omega_4) := (\Omega_{\mathcal{K}},\Omega_{\mathcal{V}},\Omega_{\mathcal{I}},\Omega_{\mathcal{P}})$.) Next, let $\chi_j\in C^\infty(\overline{\Omega})$, $j=1,\ldots,M$ such that

$$\operatorname{supp}\chi_j\subset\Omega_j\cup\partial\Omega,\qquad \chi_j\equiv 1 \text{ in a neighbourhood of } \operatorname{supp}\phi_j.$$

Arguing as in (1.25), for all $w_{h,j}\in V_k$, $j=1,\ldots,M$, and $v\in H_k^{p-1}$, we obtain

$$\begin{aligned}
&\langle\chi_i(u-u_h),v\rangle\\
&=\big\langle P_k(u-u_h), R_k^*\chi_i v\big\rangle\\
&=\big\langle P^\sharp_k(u-u_h),(I-\Pi^\sharp_k)R_k^*\chi_i v\big\rangle - \big\langle S_k(u-u_h),(I-\Pi^\sharp_k)R_k^*\chi_i v\big\rangle\\
&=\sum_{j=1}^M\Big(\big\langle P^\sharp_k(u-w_{h,j}),(I-\Pi^\sharp_k)\phi_jR_k^*\chi_i v\big\rangle - \big\langle S_k(u-u_h),(I-\Pi^\sharp_k)\phi_jR_k^*\chi_i v\big\rangle\Big)\\
&=\sum_{j=1}^M\Big(\big\langle P^\sharp_k\chi_j(u-w_{h,j}),\chi_j(I-\Pi^\sharp_k)\phi_jR_k^*\chi_i v\big\rangle - \big\langle S_k\chi_j(u-u_h),\chi_j(I-\Pi^\sharp_k)\phi_jR_k^*\chi_i v\big\rangle\Big),
\end{aligned} \tag{1.29}$$

where we have neglected the nonlocal parts of $\Pi^\sharp_k$, $P^\sharp_k$, and S_k in the last line. The local Aubin–Nitsche–type argument in Lemma 8.2 shows that (modulo remainder terms)

$$\|\chi_j(\mathrm{I}-\Pi^\sharp_k)v\|_{H_k^{-p+1}} \le C(h_jk)^p \inf_{w_h\in V^p_{\mathcal{T}_k}}\|v-w_h\|_{H^1_k},\qquad \text{where } h_j := \max_{\substack{K\in\mathcal{T}\\ K\cap\Omega_j\neq\emptyset}} h_K \tag{1.30}$$

(compare to (1.24)). By (1.23), (1.30) and the mapping properties $S_k: H_k^{-p+1}\to H_k^{p-1}$ and $P^\sharp_k: H^1_k\to H^{-1}_k$,

$$\begin{aligned}
&|\langle\chi_i(u-u_h),v\rangle|\\
&\quad\le C\sum_j \eta_p(j\to i)\Big(\inf_{w_{h,j}\in V^p_{\mathcal{T}_k}}\|\chi_j(u-w_{h,j})\|_{H^1_k} + (h_jk)^p\|\chi_j(u-u_h)\|_{H_k^{-p+1}}\Big)\|v\|_{H_k^{p-1}},\\
&\qquad\text{where}\quad \eta_p(j\to i) := \sup_{0\neq v\in H_k^{p-1}}\inf_{w_h\in V^p_{\mathcal{T}_k}}\frac{\|\chi_jR_k^*\chi_i v-w_h\|_{H^1_k}}{\|v\|_{H_k^{p-1}}}
\end{aligned} \tag{1.31}$$

(compare to (1.26)). By duality, (1.31) implies

$$\|\chi_i(u-u_h)\|_{H_k^{-p+1}} \le \sum_j C\eta_p(j\to i)\Big(\inf_{w_{h,j}\in V^p_{\mathcal{T}_k}} \|\chi_j(u-w_{h,j})\|_{H^1_k} + (h_jk)^p\|\chi_j(u-u_h)\|_{H_k^{-p+1}}\Big).$$

We then use a frequency splitting argument (see Lemma 8.5) to obtain (neglecting remainder terms)

$$\eta_p(j\to i)\le C(h_jk)^p\|\chi_jR_k^*\chi_i\|_{L^2\to L^2} + C1_{\{\Omega_i\cap\Omega_j\neq\emptyset\}}(h_{ij}k)^p, \quad \text{where } h_{ij} := \min(h_i,h_j).$$

In particular, this yields the system of inequalities

$$\big(\|\chi_i(u-u_h)\|_{H_k^{-p+1}}\big)_{i=1}^M \le CB\Big(\inf_{w_{h,j}\in V^p_{\mathcal{T}_k}} \|\chi_j(u-w_{h,j})\|_{H^1_k}\Big)_{j=1}^M + CW\big(\|\chi_j(u-u_h)\|_{H_k^{-p+1}}\big)_{j=1}^M, \tag{1.32}$$

where

$$\begin{aligned} B_{ij} &:= \eta_p(j\to i) \le C(h_jk)^p\|\chi_jR_k^*\chi_i\|_{L^2\to L^2} + C1_{\{\Omega_i\cap\Omega_j\neq\emptyset\}}(h_{ij}k)^p\\ W_{ij} &:= (h_jk)^p\eta_p(j\to i) \le C(h_jk)^{2p}\|\chi_jR_k^*\chi_i\|_{L^2\to L^2} + C(h_jk)^p1_{\{\Omega_i\cap\Omega_j\neq\emptyset\}}(h_{ij}k)^p \end{aligned}$$

(compare to (1.27)). Under the condition that

$$\sum_{n=0}^{\infty}(CW)^n < \infty, \tag{1.33}$$

$(I-CW)^{-1}$ exists and has non-negative entries. Hence (1.32) implies that

$$\|\underline{u-u_h}\|_{H_k^{-p+1}} \le C(I-CW)^{-1}B\|\underline{u-w_h}\|_{H^1_k}.$$

(compare to (1.28)).

To understand when $\sum_n (CW)^n$ converges, consider W as the weighted adjacency matrix of a directed graph with M nodes representing $\{\Omega\}_{j=1}^M$. Observe that the entry in the i^{th} row and j^{th} column of $(CW)^\ell$ is given by C^ℓ times the sum of the weights over all paths of length ℓ from j to i in this graph. Hence, the sum converges if for any i and j the sum of the weights of all paths from j to i multiplied by $C^{\text{path length}}$ is finite. Using elementary graph analysis this condition can be reduced to the requirement that all the sum of such weights for all non-self intersecting loops is less than 1 (see Appendix B).

In the setting of Theorem 1.3, $M=4$ and $(\Omega_1,\Omega_2,\Omega_3,\Omega_4) := (\Omega_{\mathcal{K}},\Omega_{\mathcal{V}},\Omega_{\mathcal{I}},\Omega_{\mathcal{P}})$. For $k\notin\mathcal{J}$, Section 4 obtains the bounds of Table 1.2 on $\psi R_k^*\chi$ according to the support of ψ and χ.

supp ψ \ supp χ	$\Omega_{\mathcal{K}}$	$\Omega_{\mathcal{V}}$	$\Omega_{\mathcal{I}}$	$\Omega_{\mathcal{P}}$
$\Omega_{\mathcal{K}}$	ρ	$\sqrt{k\rho}$	$O(k^{-\infty})$	$O(k^{-\infty})$
$\Omega_{\mathcal{V}}$	$\sqrt{k\rho}$	k	k	1
$\Omega_{\mathcal{I}}$	$O(k^{-\infty})$	k	k	1
$\Omega_{\mathcal{P}}$	$O(k^{-\infty})$	1	1	1

Table 1.2: Bounds on $\|\psi R_k^*\chi\|_{L^2\to L^2}$ (up to k-independent constants) proved in Section 4 for $k\notin\mathcal{J}$.

As a result, the graph corresponding to W is the one in Figure 1.2, and the requirement that the sum of weights on all non-self intersecting loops be less than 1 reduces to (1.8).

1.3.3 Interpretation as error propagation

To properly interpret the matrices appearing in (1.9), we return to (1.29), which is equivalent to

$$\chi_i(u-u_h) = \sum_{j=1}^M \chi_iR_k\phi_j\big((I-\Pi_k^\sharp)^*\chi_jP_k^\sharp\chi_j(u-w_{h,j}) - (I-\Pi_k^\sharp)^*\chi_jS_k\chi_j(u-u_h)\big). \tag{1.34}$$

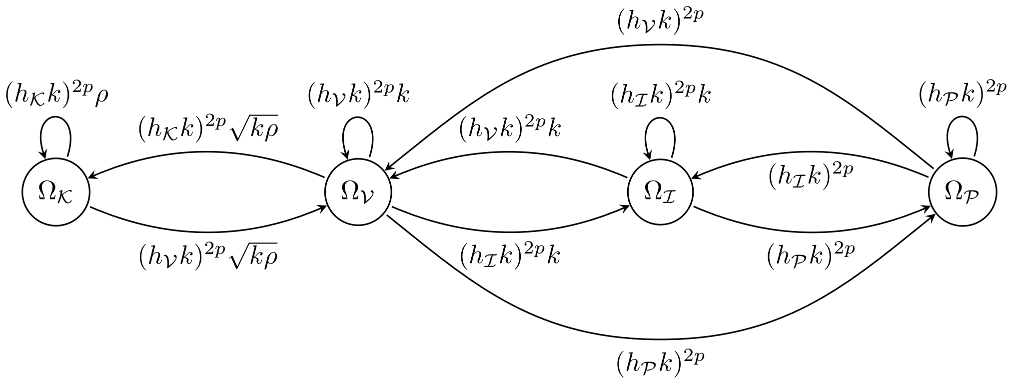


Figure 1.2: The graph showing propagation of errors for the decomposition into $\Omega_\mathcal{K}$, $\Omega_\mathcal{V}$, $\Omega_\mathcal{I}$, and $\Omega_\mathcal{P}$ in the simplified setup of Section 1.3.2. Note that this can be improved using the analysis in Section 8. The graph corresponding to Theorem 1.3 is shown in Figure 9.1 and that for Theorem 3.11 is shown in Figure 3.1.

We are interested in $\|\chi_i(u-u_h)\|_{H_k^{-p+1}}$, which we think of as the low frequencies of $\chi_i(u-u_h)$; these low frequencies are captured by $S_k\chi_i(u-u_h)$. For purposes of this discussion, we assume S_k commutes with $\chi_i R_k \phi_j$. This is not quite true, but (away from the PML), since

$$\|S_k\chi_i R_k\phi_j\|_{H_k^{-p+1}\to L^2} \le C\|\chi_i R_k \phi_j\|_{L^2\to L^2},$$

$S_k\chi_i R_k\phi_j$ acts like $\chi_i R_k\phi_j\mathcal{L}$ where $\mathcal{L}$ is a lowpass filter. We show in Theorem 4.2 that near the PML there is no propagation and so we ignore the PML here.

With these caveats, (1.34) implies

$$S_k\chi_i(u-u_h) = \sum_{j=1}^{M}\chi_i R_k\phi_j\Big(S_k\chi_j(I-\Pi_k^\sharp)^*\chi_j P_k^\sharp\chi_j(u-w_{h,j}) - S_k\chi_j(I-\Pi_k^\sharp)^*\chi_j\tilde{S}_k S_k\chi_j(u-u_h)\Big).$$

The operator $\chi_i R_k\phi_j$ has the effect of propagating between domains. The operator $(I-\Pi_k^\sharp)^*$ essentially takes the best approximation in H_k^1 norm and S_k then returns only the frequency $\lesssim k$ components. This process is represented in the graph in Figure 1.3.

To find $S_k\chi_i(u-u_h)$ in terms of the local best approximations to u, one inserts $\chi_j P^\sharp\chi_j(u-w_{h,j})$ at node 1 in Figure 1.3 and follows the cycle to node 4, producing the first approximation to $S_k\chi_i(u-u_h)$. One then continues around the cycle arbitrarily many times, adding $\chi_j P^\sharp\chi_j(u-w_{h,j})$ in each cycle. This process converges under the condition (1.33) and the final result at node 4 is $(S_k\chi_i(u-u_h))_i$. The W and B matrices in (1.32) are respectively one full cycle from node 4 to node 4 and a path from node 1 to node 4 in Figure 1.3.

Acknowledgements: MA was supported by EPSRC grant EP/R005591/1, JG was supported by EPSRC grants EP/V001760/1 and EP/V051636/1, Leverhulme Research Project Grant RPG-2023-325, and ERC Synergy Grant PSINumScat - 101167139, and EAS was supported by EPSRC grant EP/R005591/1 and ERC Synergy Grant PSINumScat - 101167139.

2 Numerical experiments illustrating the main result

We illustrate Theorem 1.3 with numerical results in a selection of asymptotic regimes and in two different geometric settings, in which we solve the PDE (1.1) with constant coefficients $A, n \equiv 1$.

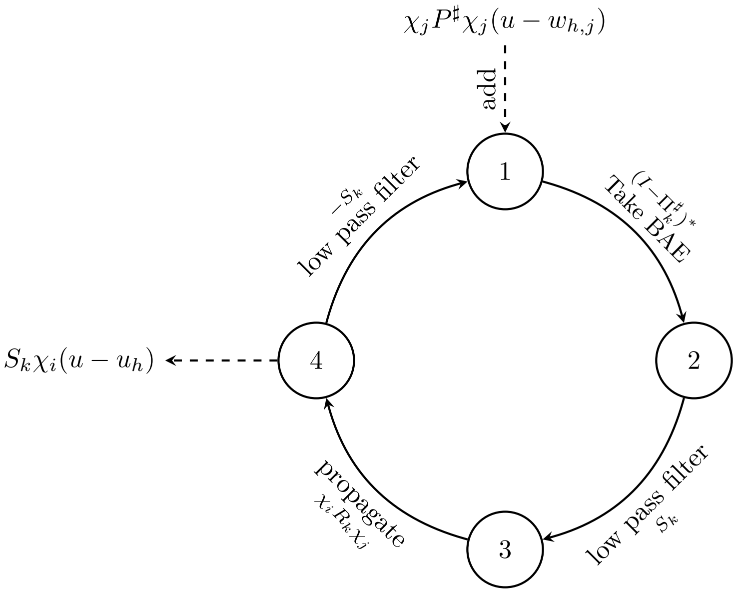


Figure 1.3: The graph showing the process of error propagation when determining the low frequencies of $u - u_h$ from the local best approximation errors. The arrows are labelled first with the type of operation (low pass filter etc.) and then with the operator whose action gives this effect. These operators are applied multiplicatively.

2.1 Experimental setup

2.1.1 Geometric setup

The first geometric setting involves a scatterer with two parallel "walls" obtained by placing two rectangular obstacles next to each other. The second geometric setting is similar, but has one of the two rectangles shifted slightly upwards to "make way" for the wave to come inside the cavity.

In the first setting (without shifting one of the two rectangles) the obstacle Ω_- is the union of two congruent rectangles of sides $L_1 = 0.7\sqrt{2}$ and $L_2 = 1.3\sqrt{2}$ with rounded corners so that they have a C^∞ boundary (this is done using the technique from [EO16]). The four vertices of the first (respectively second) rectangle are located at the coordinates $[-X_1/2 \pm \frac{L_1}{2}, \pm\frac{L_2}{2}]$ (respectively, $[X_1/2 \pm \frac{L_1}{2}, \pm\frac{L_2}{2}]$) where $X_1 = 3\frac{\sqrt{2}}{2}$. Therefore, the two rectangles have parallel sides and are separated by a gap in the x-axis equal to $L_{\text{gap}} = X_1 - L_1 = 0.8\sqrt{2}$.

For any $\delta > 0$ small enough (e.g. smaller than, $\min(L_1, L_2/2)$) the cavity $\mathcal{K}$ is contained in the rectangle $\Omega_{\mathcal{K}} := \overline{\Omega}_+ \cap \Big((-\frac{L_{\text{gap}}}{2} - \delta, \frac{L_{\text{gap}}}{2} + \delta) \times (-\frac{L_2}{2}, \frac{L_2}{2})\Big)$. A neighbourhood of $\mathcal{V} \setminus \mathcal{K}$ is given by $\Big\{(x,y) \in \overline{\Omega}_+ \,:\, |y| > \frac{L_2}{2} - \delta\Big\}$. Finally, $\Omega_{\mathcal{I}} := \Big\{(x,y) \in \overline{\Omega_+} \,:\, |x| > \frac{L_{\text{gap}}}{2} + \delta\Big\}$ is a neighbourhood of $\mathcal{I}$. For these geometries, and since the wave speed is constant, we can identify the regions $\mathcal{K}, \mathcal{V}$, and $\mathcal{I}$ "by eye". For more complicated geometries and wave speeds, one would need to identify these regions using ray tracing.

We use the radial PML with coefficients defined in (A.3), with the PML scaling function given by $f_\theta(r) = (r - R_{\mathcal{P}})^3/(3(R_{\text{tr}} - R_{\mathcal{P}})^2)$ for $r > R_{\mathcal{P}}$, with $R_{\mathcal{P}} = 2$ and $R_{\text{tr}} = 2.5$.

2.1.2 Discussion of $\rho(k)$ in the experimental setup

For the wavenumbers

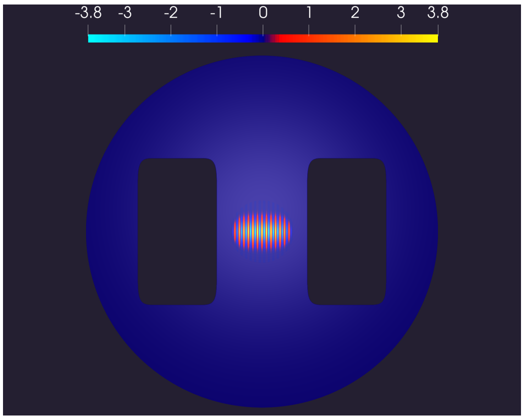
$$k_n := \frac{n\pi}{L_{\text{gap}}},$$

one can show that there exists $c > 0$ such that $\rho(k_n) \geq ck_n^2$ (e.g. by considering (1.1) with the right hand side f obtained by applying the operator $-k^{-2}\Delta - 1$ to $u(x,y) := \chi(x,y)\sin\Big[k_n\Big(x - \frac{L_{\text{gap}}}{2}\Big)\Big]$ where χ is a smooth compactly supported function which is identically 1 in the set $[-\frac{L_{\text{gap}}}{2}, \frac{L_{\text{gap}}}{2}] \times$

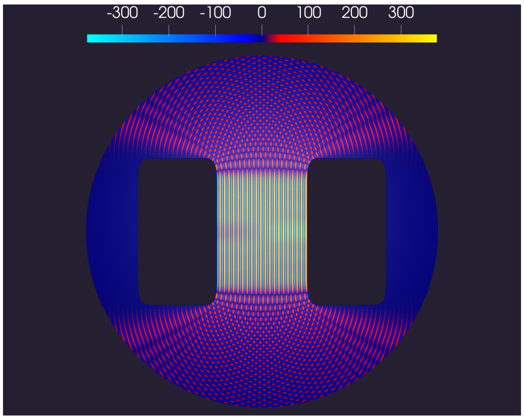
$[-\varepsilon, \varepsilon]$ for some sufficiently small $\varepsilon > 0$. The best known upper bound for all $k \in \mathbb{R}_+$ is $\rho(k) \leq Ck^3$ for all $k \geq k_0$ [CWSGS20], but it is conjectured that $\rho(k) \leq Ck^2$, and we assume this from now on.

2.1.3 Description of the sources

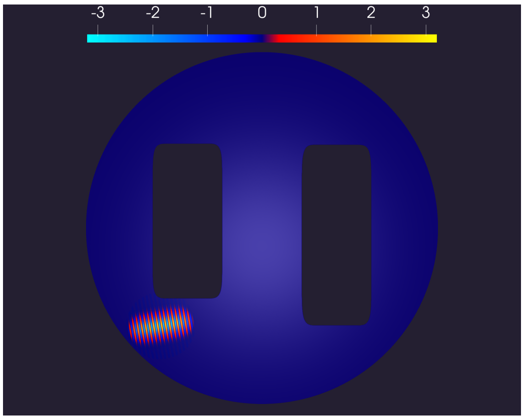
For these two geometries, we consider k-dependent right-hand sides (source terms) f_{in} and f_{out} that are "Gaussian beams" (or "wave-packets") of width $k^{-1/2}$ both in the physical and frequency space, and centered in physical space either at the origin $(0,0)$, i.e. *inside* the cavity $\mathcal{K}$, propagating in the x-direction, or *outside* the cavity, propagating in the direction of angle $\theta(k) = O(k^{-1/2})$ with respect to the x-axis. For the outside beam, the physical position (x_0, y_0) is chosen so that the central ray of the beam hits the bottom of the right-hand "wall" of the cavity (this ensures that the beam can coherently stay in the cavity as long as possible, with $O(\sqrt{k})$ reflections on the cavity's boundaries), see Figure 2.1 (c). The beams are normalized so that $\|f_{\text{in}}\|_{L^2(\mathbb{R}^2)}, \|f_{\text{out}}\|_{L^2(\mathbb{R}^2)} = 1$. The obstacle, the right-hand sides f_{in} and f_{out}, and the corresponding solutions u_{in} and u_{out} are represented in Figure 2.1.



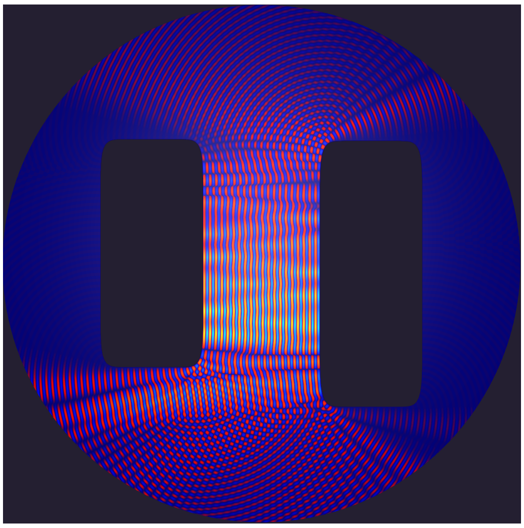
(a) Data f_{in}



(b) Solution $\text{Re}(u_{\text{in}})$



(c) Data f_{out}



(d) Solution $\text{Re}(u_{\text{out}})$

Figure 2.1: Top left: right-hand side f_{in}. Top right: numerical approximation of the solution u_{in} with data f_{in}. Bottom left: right-hand side f_{out}. Bottom right: numerical approximation of the solution u_{out} with data f_{out}. In these figures, $k = 40\frac{\pi}{L_{\text{gap}}} \approx 111$ and the functions f and u are truncated to a domain $B_R \cap \Omega_+$ where $R = 2.2$.

2.1.4 Reference solutions and their k-dependence

Numerical approximations of the exact solutions u_{in} and u_{out} are computed using the FEM with piecewise polynomials of degree $p_{\text{ref}} = 4$. These numerical solutions are used as reference solutions to analyze the error in the FEM with $p = 2$ throughout numerical experiments.

By Theorem 4.1, there exists a constant $C > 0$ and, for each $N > 0$, a constant $C_N > 0$ such that $\|u_{\text{in}}\|_{H^1_k(\Omega_{\mathcal{V}})} \le C\sqrt{k\rho}$, $\|u_{\text{in}}\|_{H^1_k(\Omega_{\mathcal{I}})} \le C_N k^{-N}$, $\|u_{\text{out}}\|_{H^1_k(\Omega_{\mathcal{K}})} \le C\sqrt{k\rho}$ and $\|u_{\text{out}}\|_{H^1_k(\Omega_{\mathcal{V}}\cup\Omega_{\mathcal{I}})} \le Ck$ for all $k \ge k_0$. This is illustrated in Figure 2.2, where we observe the empirical rates $\|u_{\text{in}}\|_{H^1_k(\Omega_{\mathcal{K}})} \approx Ck^{1.7} \le Ck^2$, $\|u_{\text{in}}\|_{H^1_k(\Omega_{\mathcal{V}})} \approx Ck^{1.2} \le Ck^{3/2}$, and $\|u_{\text{out}}\|_{H^1_k(\Omega_{\mathcal{K}})} \approx Ck^{1.4} \le Ck^{3/2}$, $\|u_{\text{out}}\|_{H^1_k(\Omega_{\mathcal{V}})} \approx Ck^{0.75} \le Ck$. The regimes that we consider are those of Table 1.1.

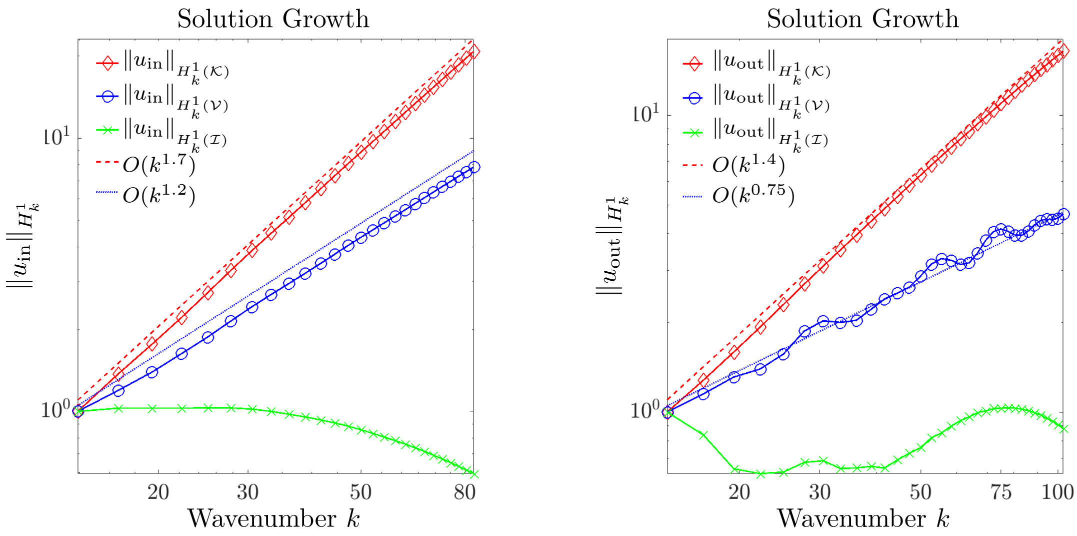


Figure 2.2: Left: growth of the solution u_{in}. Right: growth of the solution u_{out}. Solid red line (resp. blue, yellow): growth in the cavity (resp. the visible set, the invisible set).

Region	$\|u_{\text{in}}\|_{H^1_k}$	$\|u_{\text{out}}\|_{H^1_k}$
$\Omega_{\mathcal{K}}$	$\approx k^{1.7}$	$\approx k^{1.4}$
$\Omega_{\mathcal{V}}$	$\approx k^{1.2}$	$\approx k^{0.75}$

Table 2.1: Bounds on the H^1_k norms of u_{in} and u_{out} inferred from Figure 2.2.

2.1.5 Non-uniform meshing

The non-uniform meshes used in the experiments are created using a feature of FreeFem++ allowing one to "adapt" a mesh according to a custom metric. For our purposes, we only require an isotropic metric, which is described by a scalar function $h : \Omega \to \mathbb{R}_+$ describing the local required mesh size. This function h can be passed – along with an initial, uniform mesh – as an optional argument to the FreeFem++ "adaptMesh" routine, which uses the BAMG algorithm [Hec98]. We define $h \in C^\infty(\overline{\Omega})$ so that

$$\max_{x\in\Omega_\star} h(x) \le h_\star,$$

where $\star \in \{\mathcal{K}, \mathcal{V}, \mathcal{I}, \mathcal{P}\}$, with $h_\star$ the corresponding mesh threshold. In some parts of $\Omega_\star$, the function h can be significantly smaller than $h_\star$, for instance in intersections between two subdomains. However, we enforce that $h(x) \equiv h_\star$ for all x in a k-independent subset $\Omega'_\star \subset \Omega_\star$. Therefore, up to smooth transitions across regions, the metric is sharply described by $h_\star$. In all the experiments, we take $h_{\mathcal{P}}k$ to be constant. Since the solution in the PML region is not physically relevant, we do not display the errors in this region.

2.2 Numerical results

In the numerical results, we compute a few important quantities under a variety of mesh conditions. The *local quasioptimality (QO) constants* for $u_{\text{in/out}}$ are given by

$$\left\|u_{\text{in/out}} - u_h\right\|_{H^1_k(\Omega_\star)} \Big/ \left\|u_{\text{in/out}} - w_h\right\|_{H^1_k(\Omega_\star)}, \qquad \star \in \{\mathcal{K}, \mathcal{V}, \mathcal{I}\},$$

where u_h is the Galerkin solution and w_h is the best approximation of $u_{\text{in/out}}$ in the finite-element space. The *local-global relative error* is the Galerkin error in the H^1_k norm in these regions, normalized by the global H^1_k norm of the solution is given by

$$\left\|u_{\text{in/out}} - u_h\right\|_{H^1_k(\Omega_\star)} \Big/ \left\|u_{\text{in/out}}\right\|_{H^1_k(\Omega)}, \qquad \star \in \{\mathcal{K}, \mathcal{V}\},$$

2.2.1 Regime Uniform 1 (U1)

The first numerical experiment uses the uniform mesh guaranteeing k-uniform quasioptimality. We choose

$$(h_{\mathcal{K}}k)^p k^2 = (h_{\mathcal{V}}k)^p k^2 = (h_{\mathcal{I}}k)^p k^2 =: (hk)^p k^2 = C$$

where C is independent of k. Figure 2.3 plots the local QO constants and Figure 2.4 plots the local-global relative errors.

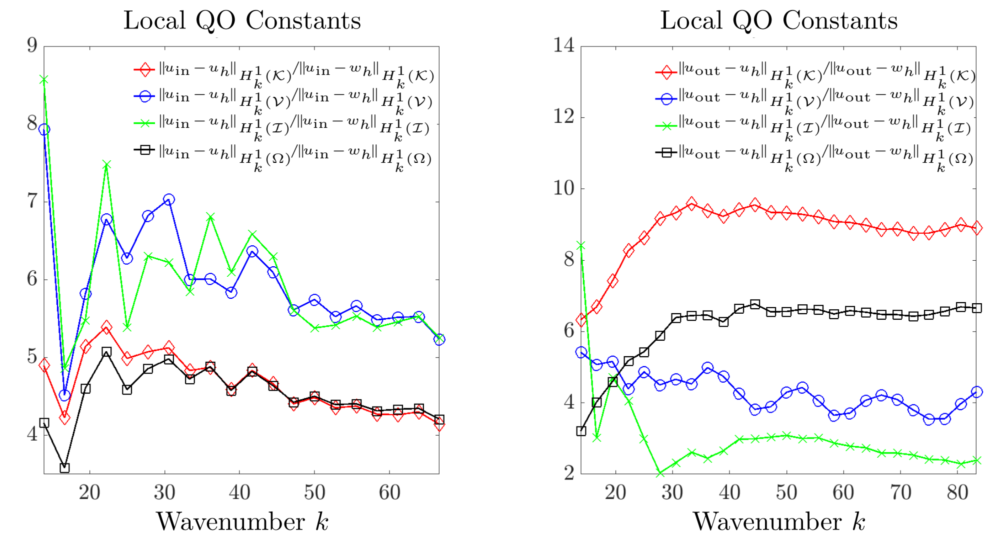


Figure 2.3: QO constants for u_{in} (left) and u_{out} (right) in regime U1. Black squares: global QO constant. Red diamonds: local QO constant in the cavity. Blue circles: local QO constant in the visible set. Green crosses: local QO constant in the invisible set.

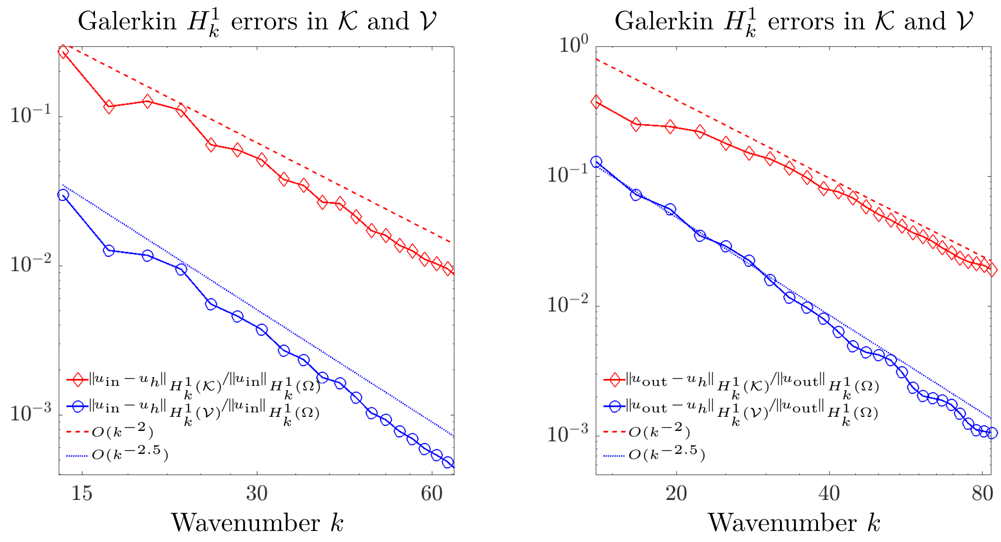


Figure 2.4: Local Galerkin errors in the H^1_k norm in $\mathcal{K}$ and $\mathcal{V}$ for the approximation of $u_{\rm in}$ (left) and $u_{\rm out}$ (right) in regime U1. Red diamonds: Galerkin error in K. Blue circles: Galerkin error in $\mathcal{V}$. A priori bounds represented as red dashed lines (for the cavity) and blue dotted lines (away from cavity).

It is well-known that in U1, the Galerkin solution is globally k-uniformly quasi-optimal (see Table 1.1), and this also follows from Theorem 1.3 (see Corollary 1.9, using that all matrix entries are $\lesssim 1$). This fact is illustrated by the solid black curves in Figure 2.5. By Corollary 1.9, the inferred rates in Table 2.1, and the fact that $\rho(k) \geq Ck^2$ at the wavenumbers chosen in the experiments, the following a priori bounds for $u_{\rm in}$ and $u_{\rm out}$ can be obtained:

$$\begin{aligned}
\frac{\|u_{\rm in} - u_h\|_{H^1_k(\Omega_\mathcal{K})}}{\|u_{\rm in}\|_{H^1_k(\Omega)}} &\lesssim (hk)^p \frac{\|u_{\rm in}\|_{H^1_k(\Omega_\mathcal{K})}}{\|u_{\rm in}\|_{H^1_k(\Omega)}} + \sqrt{\frac{k}{\rho}}(hk)^p \frac{\|u_{\rm in}\|_{H^1_k(\Omega_\mathcal{V})}}{\|u_{\rm in}\|_{H^1_k(\Omega)}} + \left(\frac{k}{\rho}\right)^{\frac{3}{2}} \frac{1}{\rho}(hk)^p \frac{\|u_{\rm in}\|_{H^1_k(\Omega_\mathcal{I})}}{\|u_{\rm in}\|_{H^1_k(\Omega)}} \\
&\lesssim k^{-2}, \\
\frac{\|u_{\rm in} - u_h\|_{H^1_k(\Omega_\mathcal{V})}}{\|u_{\rm in}\|_{H^1_k(\Omega)}} &\lesssim (hk)^p \underbrace{\sqrt{\frac{k}{\rho}}}_{k^{-0.5}} \frac{\|u_{\rm in}\|_{H^1_k(\Omega_\mathcal{K})}}{\|u_{\rm in}\|_{H^1_k(\Omega)}} + (hk)^p \underbrace{\frac{\|u_{\rm in}\|_{H^1_k(\Omega_\mathcal{V})}}{\|u_{\rm in}\|_{H^1_k(\Omega)}}}_{k^{-0.5}} + \left(\frac{k}{\rho}\right)^{\frac{3}{2}} \frac{1}{\rho}(hk)^p \frac{\|u_{\rm in}\|_{H^1_k(\Omega_\mathcal{I})}}{\|u_{\rm in}\|_{H^1_k(\Omega)}} \\
&\lesssim k^{-\frac{5}{2}}, \\
\frac{\|u_{\rm out} - u_h\|_{H^1_k(\Omega_\mathcal{K})}}{\|u_{\rm out}\|_{H^1_k(\Omega)}} &\lesssim (hk)^p \frac{\|u_{\rm out}\|_{H^1_k(\Omega_\mathcal{K})}}{\|u_{\rm out}\|_{H^1_k(\Omega)}} + \sqrt{\frac{k}{\rho}}(hk)^p \frac{\|u_{\rm out}\|_{H^1_k(\Omega_\mathcal{V})}}{\|u_{\rm out}\|_{H^1_k(\Omega)}} + \left(\frac{k}{\rho}\right)^{\frac{3}{2}} \frac{1}{\rho}(hk)^p \frac{\|u_{\rm out}\|_{H^1_k(\Omega_\mathcal{I})}}{\|u_{\rm out}\|_{H^1_k(\Omega)}} \\
&\lesssim k^{-2}, \\
\frac{\|u_{\rm out} - u_h\|_{H^1_k(\Omega_\mathcal{V})}}{\|u_{\rm out}\|_{H^1_k(\Omega)}} &\lesssim (hk)^p \underbrace{\sqrt{\frac{k}{\rho}}}_{k^{-0.5}} \frac{\|u_{\rm out}\|_{H^1_k(\Omega_\mathcal{K})}}{\|u_{\rm out}\|_{H^1_k(\Omega)}} + (hk)^p \underbrace{\frac{\|u_{\rm out}\|_{H^1_k(\Omega_\mathcal{V})}}{\|u_{\rm out}\|_{H^1_k(\Omega)}}}_{k^{-0.65}} + \left(\frac{k}{\rho}\right)^{\frac{3}{2}} \frac{1}{\rho}(hk)^p \frac{\|u_{\rm out}\|_{H^1_k(\Omega_\mathcal{I})}}{\|u_{\rm out}\|_{H^1_k(\Omega)}} \\
&\lesssim k^{-\frac{5}{2}}.
\end{aligned}$$

Figure 2.4 shows that, at least experimentally, these rates are sharp. Furthermore, since u is k-oscillatory, the results of [Gal25] imply that the standard polynomial approximation bounds are locally sharp, i.e. the local best approximation errors satisfy

$$\|u - w_{h,\mathcal{K}}\|_{H^1_k(\Omega_\mathcal{K})} \geq C(hk)^p \|u\|_{H^1_k(\Omega_\mathcal{K})}, \qquad \|u - w_{h,\mathcal{V}}\|_{H^1_k(\Omega_\mathcal{V})} \geq C(hk)^p \|u\|_{H^1_k(\Omega_\mathcal{V})}.$$

Corollary 1.9 then implies that the local quasi-optimality constants in each region are k-uniformly bounded as well, i.e.,

$$\frac{\|u-u_h\|_{H^1_k(\Omega_\mathcal{K})}}{\|u-w_{h,\mathcal{K}}\|_{H^1_k(\Omega_\mathcal{K})}} \lesssim 1, \qquad \frac{\|u-u_h\|_{H^1_k(\Omega_\mathcal{V})}}{\|u-w_{h,\mathcal{V}}\|_{H^1_k(\Omega_\mathcal{V})}} \lesssim 1.$$

This is consistent with the behavior observed in Figure 2.3.

2.2.2 Regime Quasioptimality (QO)

In QO, we choose

$$(h_\mathcal{K} k)^p k^2 + (h_\mathcal{V} k)^p k^{\frac{3}{2}} + (h_\mathcal{I} k)^p k = C,$$

where C is independent of k. By Corollary 1.11, the Galerkin solution is again k-uniformly globally quasi-optimal, see Table 1.1. Figure 2.5 shows the local quasi-optimality constants in each regions for the problems involving u_{in} and u_{out}. Figure 2.6 plots local–global relative errors.

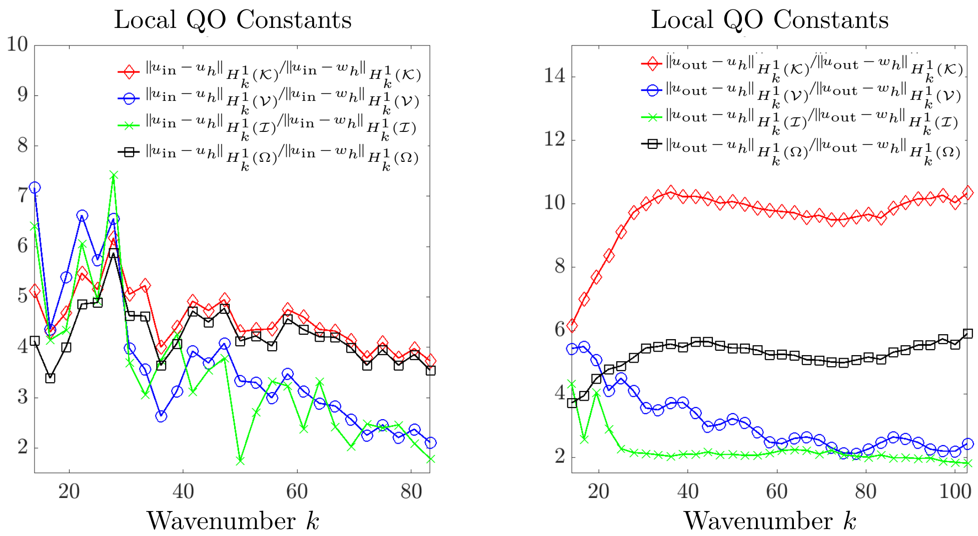


Figure 2.5: Local QO constants for u_{in} (left) and u_{out} (right) in the regime QO. Black squares: global QO constant. Red diamonds: local QO constant in the cavity. Blue circles: local QO constant in the visible set. Green crosses: local QO constant in the invisible set.

By Corollary 1.11, the inferred rates in Table 2.1, and the fact that $\rho(k) \geq Ck^2$, the a priori bounds for u_{in} and u_{out} in each region are given by

$$\frac{\|u_{\text{in}}-u_h\|_{H^1_k(\Omega_\mathcal{K})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}} \lesssim \underbrace{(h_\mathcal{K} k)^p}_{k^{-2}} \frac{\|u_{\text{in}}\|_{H^1_k(\Omega_\mathcal{K})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}} + \underbrace{(h_\mathcal{V} k)^p}_{k^{-3/2}} \underbrace{\frac{\|u_{\text{in}}\|_{H^1_k(\Omega_\mathcal{V})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}}}_{k^{-\frac{1}{2}}} + (k\rho)^{-\frac{1}{2}} (h_\mathcal{I} k)^p \frac{\|u_{\text{in}}\|_{H^1_k(\Omega_\mathcal{I})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}}$$

$$\lesssim k^{-2},$$

$$\frac{\|u_{\text{in}}-u_h\|_{H^1_k(\Omega_\mathcal{V})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}} \lesssim \underbrace{(h_\mathcal{K} k)^p}_{k^{-2}} \frac{\|u_{\text{in}}\|_{H^1_k(\Omega_\mathcal{K})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}} + \underbrace{(h_\mathcal{V} k)^p}_{k^{-1}} \underbrace{\frac{\|u_{\text{in}}\|_{H^1_k(\Omega_\mathcal{V})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}}}_{k^{-0.5}} + (h_\mathcal{I} k)^p \frac{\|u_{\text{in}}\|_{H^1_k(\Omega_\mathcal{I})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}} \lesssim k^{-1.5},$$

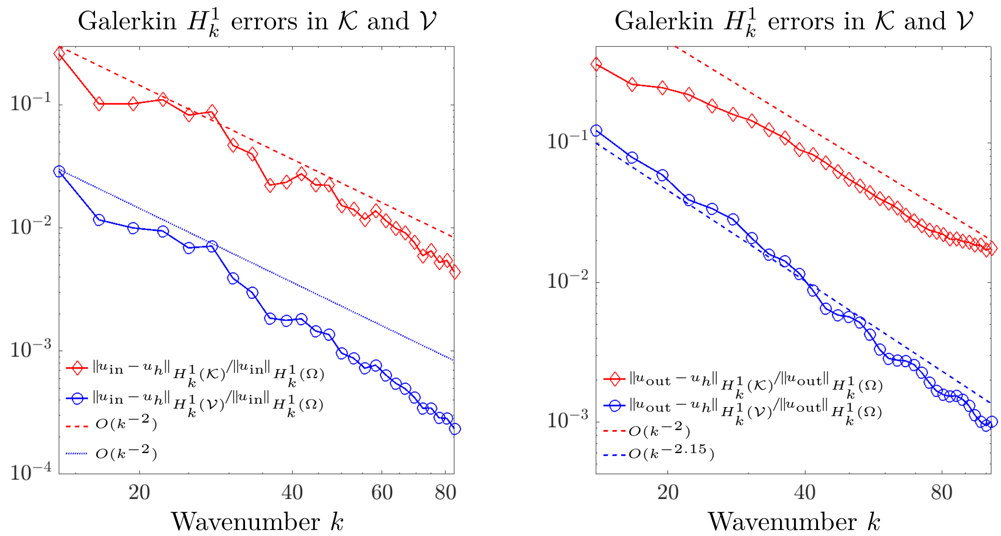


Figure 2.6: Local Galerkin errors in the H^1_k norm in $\mathcal{K}$ and $\mathcal{V}$ for the approximation of u_{in} (left) and u_{out} (right) in the regime QO. Red diamonds: error in the cavity. Blue circles: error away from the cavity. A priori bounds represented as red dashed lines (for the cavity) and blue dotted lines (away from cavity).

$$\frac{\|u_{\text{out}} - u_h\|_{H^1_k(\Omega_\mathcal{K})}}{\|u_{\text{out}}\|_{H^1_k(\Omega)}} \lesssim \underbrace{(h_\mathcal{K} k)^p}_{k^{-2}} \frac{\|u_{\text{out}}\|_{H^1_k(\Omega_\mathcal{K})}}{\|u_{\text{out}}\|_{H^1_k(\Omega)}} + \underbrace{(h_\mathcal{V} k)^p}_{k^{-\frac{3}{2}}} \underbrace{\frac{\|u_{\text{out}}\|_{H^1_k(\Omega_\mathcal{V})}}{\|u_{\text{out}}\|_{H^1_k(\Omega)}}}_{k^{-0.65}} + (k\rho)^{-\frac{1}{2}} (h_\mathcal{I} k)^p \frac{\|u_{\text{out}}\|_{H^1_k(\Omega_\mathcal{I})}}{\|u_{\text{out}}\|_{H^1_k(\Omega)}}$$
$$\lesssim k^{-2},$$
$$\frac{\|u_{\text{out}} - u_h\|_{H^1_k(\Omega_\mathcal{V})}}{\|u_{\text{out}}\|_{H^1_k(\Omega)}} \lesssim \underbrace{(h_\mathcal{K} k)^p}_{k^{-2}} \underbrace{\sqrt{\frac{k}{\rho}}}_{k^{-0.5}} \frac{\|u_{\text{out}}\|_{H^1_k(\Omega_\mathcal{K})}}{\|u_{\text{out}}\|_{H^1_k(\Omega)}} + \underbrace{(h_\mathcal{V} k)^p}_{k^{-\frac{3}{2}}} \underbrace{\frac{\|u_{\text{out}}\|_{H^1_k(\Omega_\mathcal{V})}}{\|u_{\text{out}}\|_{H^1_k(\Omega)}}}_{k^{-0.65}} + (h_\mathcal{I} k)^p \frac{\|u_{\text{out}}\|_{H^1_k(\Omega_\mathcal{I})}}{\|u_{\text{out}}\|_{H^1_k(\Omega)}}$$
$$\lesssim k^{-2.15}.$$

Again, these rates are experimentally verified in Figure 2.6.

2.2.3 Regime Quasioptimality away (QO away)

In QO away, we choose

$$(h_\mathcal{K} k)^p k^{3/2} = (h_\mathcal{V} k)^p k = (h_\mathcal{I} k)^p k =: (hk)^p k^2 = C,$$

where C is independent of k. Theorem 1.3 no longer guarantees k-uniform quasi-optimality, but Corollary 1.14 and the conjecture that $\rho(k) \leq Ck^2$ imply the following bounds for the "QO constants" (not to be confused with "local QO constants" – notice the global norm of the best approximation error in the denominator instead of the local norm in the local QO constants)

$$\frac{\|u - u_h\|_{H^1_k(\Omega_\mathcal{K})}}{\|u - w_h\|_{H^1_k(\Omega)}} \lesssim \sqrt{k}, \quad \frac{\|u - u_h\|_{H^1_k(\Omega_\mathcal{V})}}{\|u - w_h\|_{H^1_k(\Omega)}} \lesssim 1, \quad \frac{\|u - u_h\|_{H^1_k(\Omega_\mathcal{I})}}{\|u - w_h\|_{H^1_k(\Omega)}} \lesssim 1, \tag{2.1}$$

hence, the Galerkin solution remains k-uniform quasioptimal away from the cavity in the regime QO away (see also Table 1.1).

Figure 2.7 shows the QO constants in each regions for the problems involving u_{in} and u_{out}. The QO constant in the invisible set is orders of magnitude smaller than the other quantities, so it is not displayed. Figure 2.8 plots the local-global relative errors.

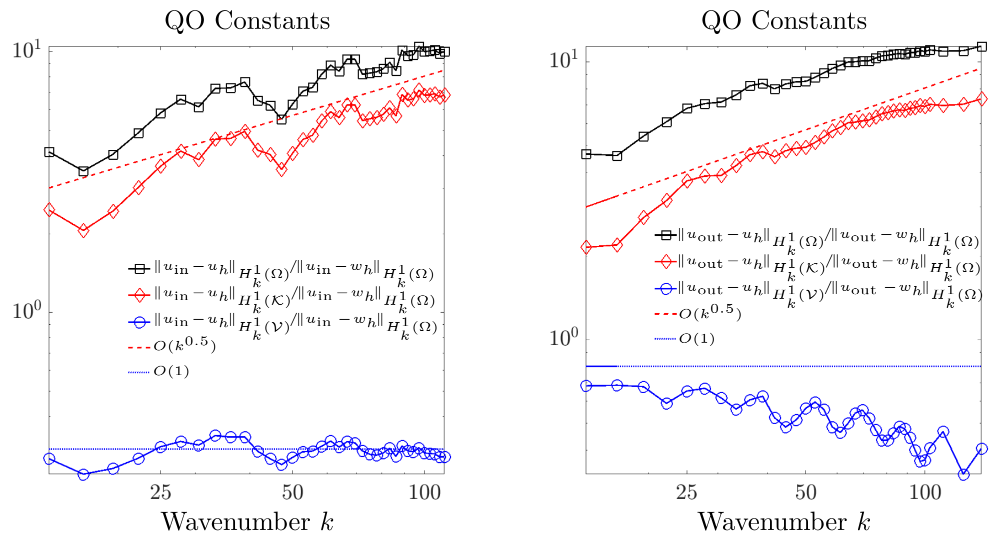


Figure 2.7: QO constants for u_{in} (left) and u_{out} (right) in the regime QO away. Black squares: global QO constant. Red diamonds: QO constant in $\mathcal{K}$. Blue circles: QO constant in Γ. A priori bounds represented as red dashed lines (for the cavity) and blue dotted lines (away from cavity).

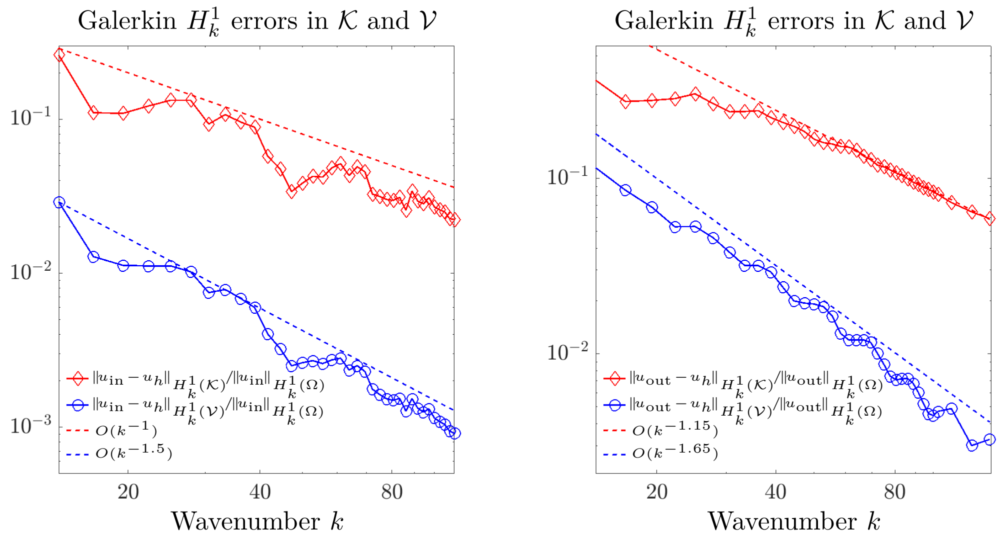


Figure 2.8: Local Galerkin errors in the H^1_k norm in $\mathcal{K}$ and $\mathcal{V}$ for the approximation of u_{in} (left) and u_{out} (right) in the regime QO away. Red diamonds: error in the cavity. Blue circles: error away from the cavity. The a priori bounds derived from Corollary 1.11, the lower bound $\rho(k) \geq Ck^2$, and the inferred rates in Table 2.1, are represented as red dashed lines (for the cavity) and blue dotted lines (away from cavity).

The numerical results in Figure 2.7 illustrate that the bounds in (2.1) are, at least experimentally, sharp. Furthermore, by Corollary 1.14, the inferred rates in Table 2.1, and the fact that $\rho(k) \geq Ck^2$, the a priori bounds for u_{in} and u_{out} in $\mathcal{K}$ and $\mathcal{V}$ are given by

$$\begin{aligned}\frac{\|u_{\text{in}}-u_h\|_{H^1_k(\Omega_\mathcal{K})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}} &\lesssim \underbrace{(h_\mathcal{K}k)^p}_{k^{-2}} \underbrace{\sqrt{\frac{\rho}{k}}}_{k^{\frac{1}{2}}} \frac{\|u_{\text{in}}\|_{H^1_k(\Omega_\mathcal{K})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}} + \underbrace{\sqrt{\frac{\rho}{k}}}_{k^{\frac{1}{2}}} \underbrace{(h_\mathcal{V}k)^p}_{k^{-1}} \underbrace{\frac{\|u_{\text{in}}\|_{H^1_k(\Omega_\mathcal{V})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}}}_{k^{-\frac{1}{2}}} + (k\rho)^{-\frac{1}{2}}(h_\mathcal{I}k)^p \frac{\|u_{\text{in}}\|_{H^1_k(\Omega_\mathcal{I})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}}\\
&\lesssim k^{-1},\\
\frac{\|u_{\text{in}}-u_h\|_{H^1_k(\Omega_\mathcal{V})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}} &\lesssim \underbrace{(h_\mathcal{K}k)^p}_{k^{-2}} \underbrace{\sqrt{\frac{k}{\rho}}}_{k^{-0.5}} \frac{\|u_{\text{in}}\|_{H^1_k(\Omega_\mathcal{K})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}} + \underbrace{(h_\mathcal{V}k)^p}_{k^{-\frac{3}{2}}} \underbrace{\frac{\|u_{\text{in}}\|_{H^1_k(\Omega_\mathcal{V})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}}}_{k^{-0.5}} + (h_\mathcal{I}k)^p \frac{\|u_{\text{in}}\|_{H^1_k(\Omega_\mathcal{I})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}} \lesssim k^{-2},\end{aligned}$$

and similarly,

$$\frac{\|u_{\text{out}}-u_h\|_{H^1_k(\Omega_\mathcal{K})}}{\|u_{\text{out}}\|_{H^1_k(\Omega)}} \lesssim k^{-1.15}, \qquad \frac{\|u_{\text{out}}-u_h\|_{H^1_k(\Omega_\mathcal{V})}}{\|u_{\text{out}}\|_{H^1_k(\Omega)}} \lesssim k^{-1.65}.$$

These rates are experimentally verified in Figure 2.8.

2.2.4 Regime Uniform 2 (U2)

In U2, we choose

$$(h_\mathcal{K}k)^{2p}k^2 = (h_\mathcal{V}k)^{2p}k^2 = (h_\mathcal{I}k)^{2p}k^2 =: (hk)^p k^2 = C,$$

where C is independent of k. Figure 2.9 shows the QO constants in each regions for the problems involving u_{in} and u_{out}. Figure 2.10 plots local-global relative errors.

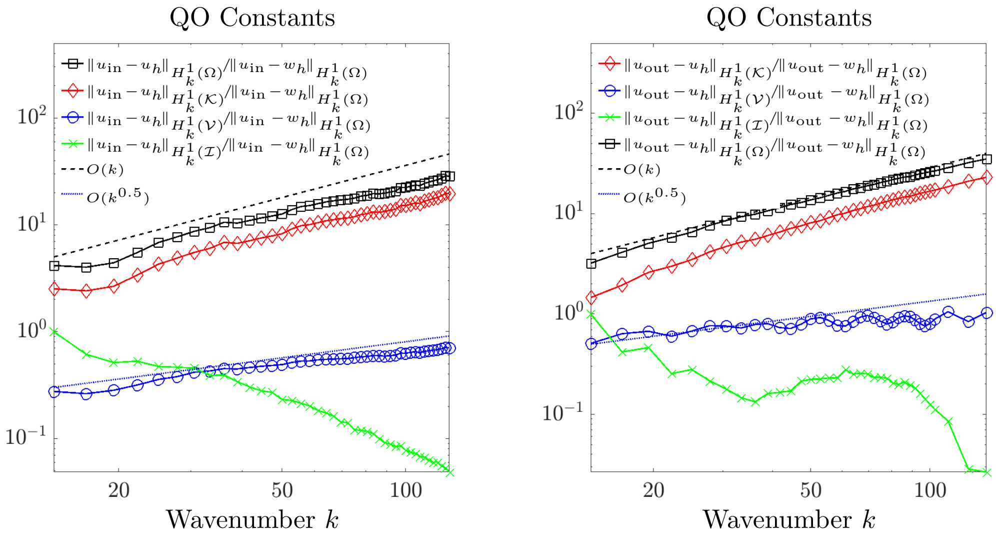


Figure 2.9: QO constants for u_{in} (left) and u_{out} (right) in the regime U2. Black squares: global QO constant. Red diamonds: QO constant in $\mathcal{K}$. Blue circles: QO constant in the visible set. Green crosses: QO constant in Γ. A priori bounds represented as red dashed lines (for the cavity) and blue dotted lines (away from cavity).

The Galerkin solution is no longer k-uniformly quasi-optimal, but the relative error is bounded in terms of C, see Table 1.1. The latter fact is illustrated by the black solid lines in Figure 2.10. Furthermore, Corollary 1.10 and the conjecture that $\rho(k) = O(k^2)$ imply the a priori bound

$$\|u-u_h\|_{H^1_k(\Omega)} \lesssim k\|u-w_h\|_{H^1_k(\Omega)},$$

for $u = u_{\text{in}}$ or $u = u_{\text{out}}$, as well as the following bounds on the QO constants

$$\|u-u_h\|_{H^1_k(\Omega_\mathcal{K})} \lesssim k\|u-w_h\|_{H^1_k(\Omega)}, \qquad \|u-u_h\|_{H^1_k(\Omega_\mathcal{V})} \lesssim \sqrt{k}\|u-w_h\|_{H^1_k(\Omega)},$$

$$\|u - u_h\|_{H^1_k(\Omega_{\mathcal{I}})} \lesssim \sqrt{k}\|u - w_h\|_{H^1_k(\Omega)}.$$

These bounds are in line with the results shown in Figure 2.9. The inferred rates in Table 2.1 additionally give the following a priori bounds

$$\frac{\|u_{\text{in}} - u_h\|_{H^1_k(\Omega_{\mathcal{K}})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}} \lesssim 1 + \sqrt{\frac{k}{\rho}} \frac{\|u_{\text{in}}\|_{H^1_k(\Omega_{\mathcal{V}})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}} + \left(\frac{k}{\rho}\right)^{3/2} \frac{\|u_{\text{in}}\|_{H^1_k(\Omega_{\mathcal{I}})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}} \lesssim 1,$$

$$\frac{\|u_{\text{in}} - u_h\|_{H^1_k(\Omega_{\mathcal{V}})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}} \lesssim \underbrace{\sqrt{\frac{k}{\rho}}}_{k^{-\frac{1}{2}}} \frac{\|u_{\text{in}}\|_{H^1_k(\Omega_{\mathcal{K}})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}} + \underbrace{\frac{\|u_{\text{in}}\|_{H^1_k(\Omega_{\mathcal{V}})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}}}_{k^{-0.5}} + \underbrace{\frac{k}{\rho}}_{k^{-1}} \frac{\|u_{\text{in}}\|_{H^1_k(\Omega_{\mathcal{I}})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}} \lesssim k^{-1/2}.$$

Similarly,

$$\frac{\|u_{\text{out}} - u_h\|_{H^1_k(\Omega_{\mathcal{K}})}}{\|u_{\text{out}}\|_{H^1_k(\Omega)}} \lesssim 1, \quad \frac{\|u_{\text{out}} - u_h\|_{H^1_k(\Omega_{\mathcal{V}})}}{\|u_{\text{out}}\|_{H^1_k(\Omega)}} \lesssim k^{-0.5}.$$

These rates are experimentally verified in Figure 2.10.

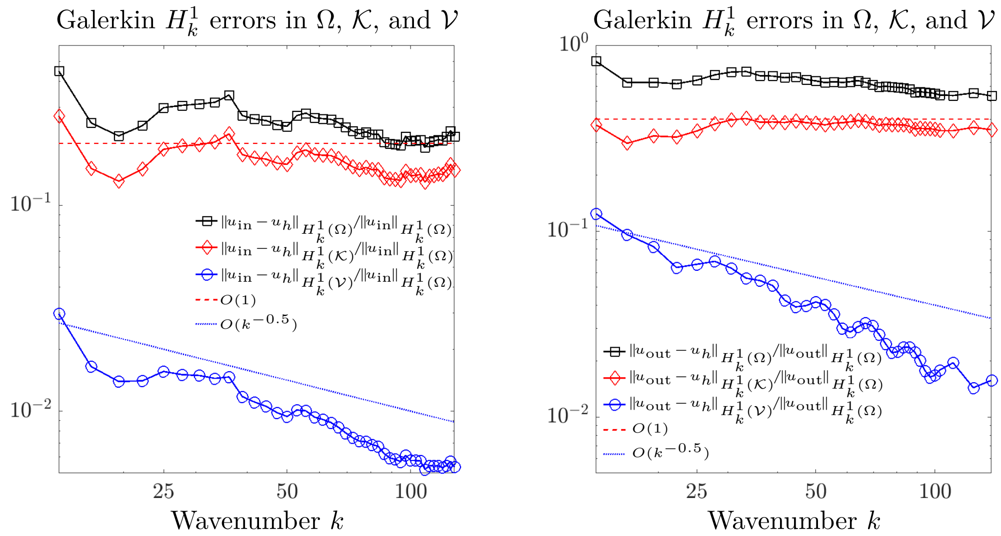


Figure 2.10: Local Galerkin errors in the H^1_k norm in $\mathcal{K}$ and $\mathcal{V}$ for the approximation of u_{in} (left) and u_{out} (right) in the regime U2. Black squares: global relative error. Red diamonds: error in the cavity. Blue circles: error away from the cavity. A priori bounds represented as red dashed lines (for the cavity) and blue dotted lines (away from cavity).

2.2.5 Regime Relative error (RE)

In RE, we choose

$$(h_{\mathcal{K}}k)^{2p}k^2 = (h_{\mathcal{V}}k)^{2p}k^{3/2} = (h_{\mathcal{I}}k)^{2p}k = C,$$

where C is independent of k. Figure 2.11 shows the QO constants in each regions for the problems involving u_{in} and u_{out}. Figure 2.12 plots local-global relative error.

By Corollary 1.12, the relative error is k-uniformly bounded, and this is illustrated by the black solid lines in Figure 2.12. Furthermore,

$$\frac{\|u_{\text{in}} - u_h\|_{H^1_k(\Omega_{\mathcal{K}})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}} \lesssim 1 + \frac{\|u_{\text{in}}\|_{H^1_k(\Omega_{\mathcal{V}})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}} + \frac{\|u_{\text{in}}\|_{H^1_k(\Omega_{\mathcal{K}})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}} \lesssim 1,$$

$$\frac{\|u_{\text{in}} - u_h\|_{H^1_k(\Omega_{\mathcal{V}})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}} \lesssim \underbrace{\sqrt{\frac{k}{\rho}}}_{k^{-\frac{1}{2}}} \frac{\|u_{\text{in}}\|_{H^1_k(\Omega_{\mathcal{K}})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}} + \underbrace{\frac{\|u_{\text{in}}\|_{H^1_k(\Omega_{\mathcal{V}})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}}}_{k^{-0.5}} + \frac{\|u_{\text{in}}\|_{H^1_k(\Omega_{\mathcal{I}})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}} \lesssim k^{-1/2}.$$

Similarly,

$$\frac{\|u_{\text{out}} - u_h\|_{H^1_k(\Omega_\mathcal{K})}}{\|u_{\text{out}}\|_{H^1_k(\Omega)}} \lesssim 1, \qquad \frac{\|u_{\text{out}} - u_h\|_{H^1_k(\Omega_\mathcal{V})}}{\|u_{\text{out}}\|_{H^1_k(\Omega)}} \lesssim k^{-\frac{1}{2}}.$$

These rates are verified in Figure 2.12.

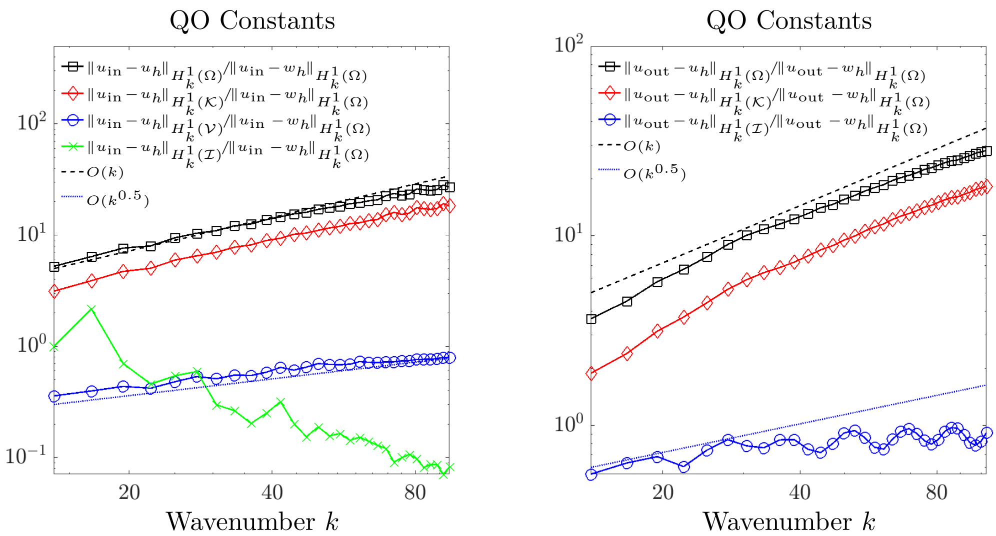


Figure 2.11: QO constants for u_{in} (left) and u_{out} (right) in the regime RE. Black squares: global QO constant. Red diamonds: QO constant in $\mathcal{K}$. Blue circles: QO constant in the visible set. Green crosses: QO constant in Γ. A priori bounds represented as red dashed lines (for the cavity) and blue dotted lines (away from cavity).

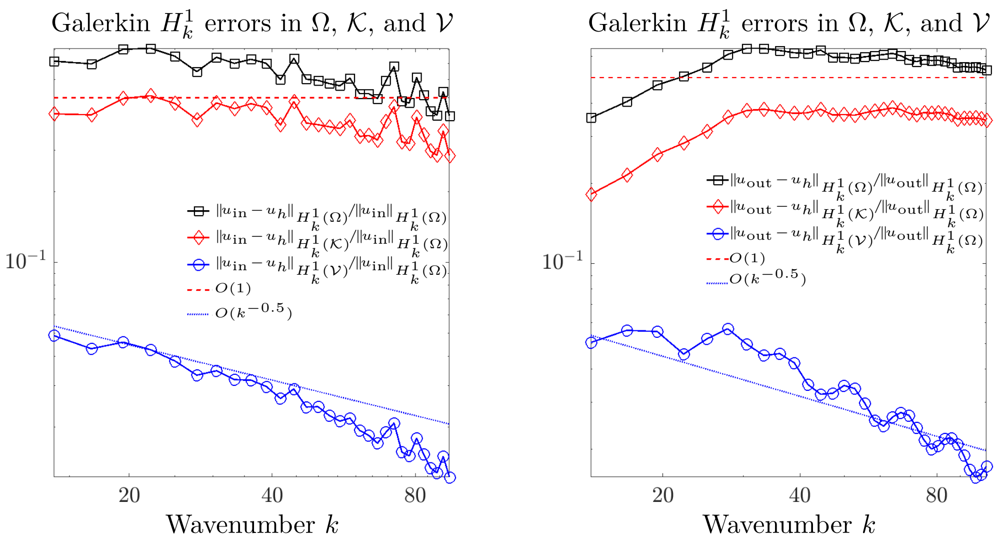


Figure 2.12: Local Galerkin errors in the H^1_k norm in $\mathcal{K}$ and $\mathcal{V}$ for the approximation of u_{in} (left) and u_{out} (right) in the regime RE. Black squares: global relative error. Red diamonds: error in the cavity. Blue circles: error away from the cavity. A priori bounds represented as red dashed lines (for the cavity) and blue dotted lines (away from cavity).

2.2.6 Regime Relative error away (RE away)

In RE away, we choose

$$(h_{\mathcal{K}}k)^{2p}k^2 = (h_{\mathcal{V}}k)^{2p}k = (h_{\mathcal{I}}k)^{2p}k = C,$$

where C is independent of k. Figure 2.13 shows the QO constants in each regions for the problems involving u_{in} and u_{out}. Figure 2.14 plots the local-global relative error.

This is the coarsest regime for which Theorem 1.3 applies. By Corollary 1.7 and the conjecture that $\rho(k) \le Ck^2$, one has the following a priori bounds on the local QO factors:

$$\|u - u_h\|_{H^1_k(\Omega_{\mathcal{K}})} \lesssim k\|u - w_h\|_{H^1_k(\Omega)}, \qquad \|u - u_h\|_{H^1_k(\Omega_{\mathcal{V}})} \lesssim \sqrt{k}\|u - w_h\|_{H^1_k(\Omega)},$$
$$\|u - u_h\|_{H^1_k(\Omega_{\mathcal{I}})} \lesssim \sqrt{k}\|u - w_h\|_{H^1_k(\Omega)}.$$

These bounds are experimentally verified in Figure 2.13. By Corollary 1.8, we also have the following a priori bounds on the local relative errors:

$$\frac{\|u_{\text{in}} - u_h\|_{H^1_k(\Omega_{\mathcal{K}})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}} \lesssim 1 + \underbrace{\sqrt{\frac{\rho}{k}}}_{k^{1/2}} \underbrace{\frac{\|u_{\text{in}}\|_{H^1_k(\Omega_{\mathcal{K}})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}}}_{k^{-\frac{1}{2}}} + \frac{\|u_{\text{in}}\|_{H^1_k(\Omega_{\mathcal{I}})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}} \lesssim 1,$$

$$\frac{\|u_{\text{in}} - u_h\|_{H^1_k(\Omega_{\mathcal{V}})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}} \lesssim \underbrace{\sqrt{\frac{k}{\rho}}}_{k^{-\frac{1}{2}}} \frac{\|u_{\text{in}}\|_{H^1_k(\Omega_{\mathcal{K}})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}} + \underbrace{\frac{\|u_{\text{in}}\|_{H^1_k(\Omega_{\mathcal{V}})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}}}_{k^{-\frac{1}{2}}} + \frac{\|u_{\text{in}}\|_{H^1_k(\Omega_{\mathcal{I}})}}{\|u_{\text{in}}\|_{H^1_k(\Omega)}}, \lesssim k^{-1/2}.$$

Similarly,

$$\frac{\|u_{\text{out}} - u_h\|_{H^1_k(\Omega_{\mathcal{K}})}}{\|u_{\text{out}}\|_{H^1_k(\Omega)}} \lesssim 1 \quad \text{and} \quad \frac{\|u_{\text{out}} - u_h\|_{H^1_k(\Omega_{\mathcal{V}})}}{\|u_{\text{out}}\|_{H^1_k(\Omega)}} \lesssim k^{-1/2}.$$

These bounds are also verified in our numerical experiments, see Figure 2.14.

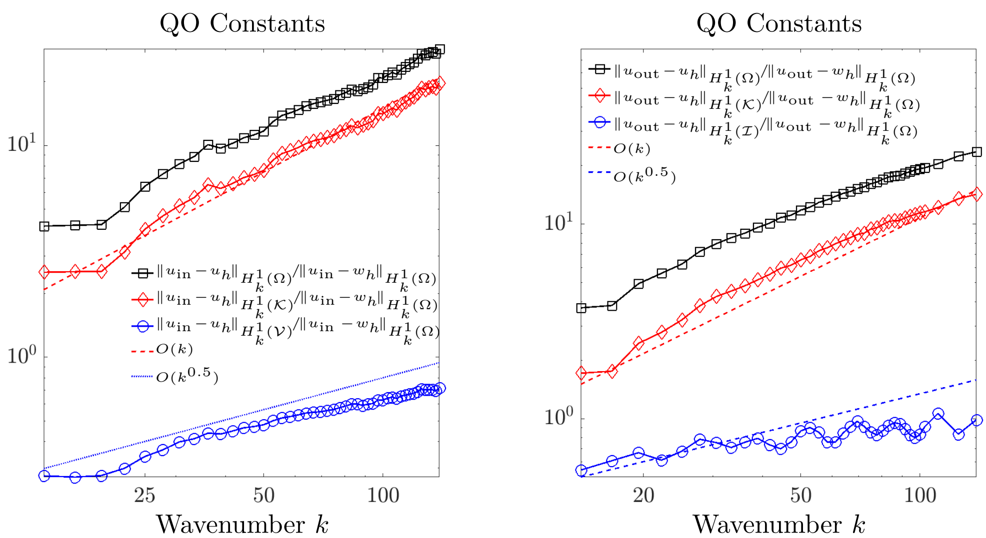


Figure 2.13: QO constants for u_{in} (left) and u_{out} (right) in the regime RE away. Black squares: global QO constant. Red diamonds: QO constant in $\mathcal{K}$. Blue circles: QO constant in the visible set. A priori bounds represented as red dashed lines (for the cavity) and blue dotted lines (away from cavity).

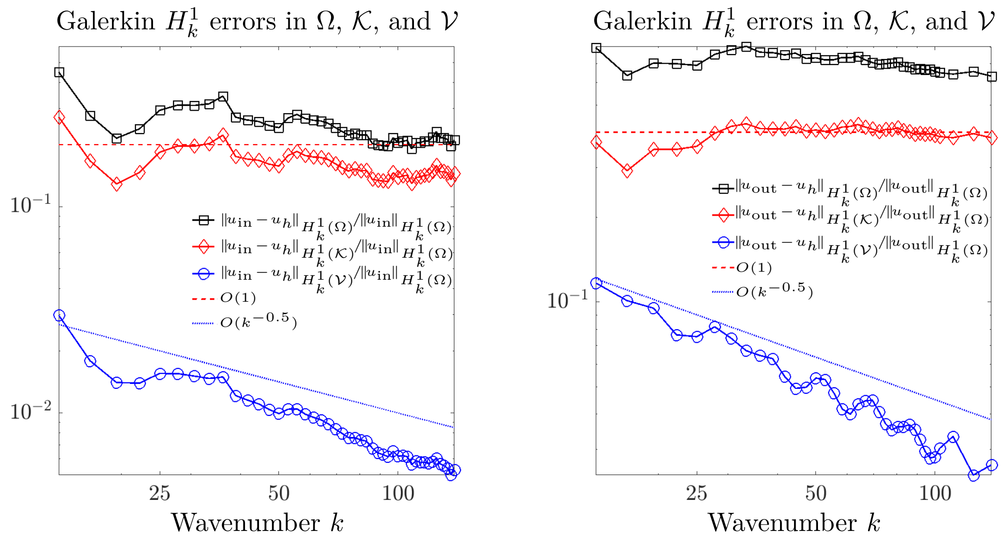


Figure 2.14: Local Galerkin errors in the H^1_k norm in $\mathcal{K}$ and $\mathcal{V}$ for the approximation of u_{in} (left) and u_{out} (right) in the regime RE away. Black squares: global relative error. Red diamonds: error in the cavity. Blue circles: error away from the cavity. A priori bounds represented as red dashed lines (for the cavity) and blue dotted lines (away from cavity).

2.3 An adaptive mesh-refinement algorithm

The numerical experiments show that Theorem 1.3 accurately captures the effect of local best approximation errors on the local Galerkin errors. It is therefore natural to use Theorem 1.3 to inform an adaptive refinement algorithm. This will be investigated elsewhere, but we sketch the main steps here. Given a set $U \subset \Omega$ on which one wants an accurate solution, implement the following.

1. Use ray tracing to (a) identify the regions $\mathcal{K}, \mathcal{V}$, and $\mathcal{I}$ and (b) estimate ρ.
2. Compute a Galerkin solution, u_h^0 on a rough mesh and let $j = 0$.
3. Compute $\|u_h^j\|_{\Omega_{\mathcal{K}}}$, $\|u_h^j\|_{\Omega_{\mathcal{V}}}$, $\|u_h^j\|_{\Omega_{\mathcal{I}}}$, and $\|u_h^j\|_{\Omega_{\mathcal{P}}}$.
4. Assuming that $\|u\|_{\Omega_j} \propto \|u_h\|_{\Omega_j}$, use the standard approximation property $\|u - w_h\|_{H^1_k} \leq C(hk)^p \|u\|_{H^{p+1}_k}$ and associated lower bounds [Gal25] to obtain bounds on the vector of best approximation errors on the hand side of (1.9).
5. Put the bounds from Step 3 into Theorem 1.3 to give an estimator for the map
$$(h_{\mathcal{K}}, h_{\mathcal{V}}, h_{\mathcal{I}}, h_{\mathcal{P}}) \mapsto \Big(\|u - u_h\|_{H^1_k(\Omega'_{\mathcal{K}})}, \|u - u_h\|_{H^1_k(\Omega'_{\mathcal{V}})}, \|u - u_h\|_{H^1_k(\Omega'_{\mathcal{I}})}, \|u - u_h\|_{H^1_k(\Omega'_{\mathcal{P}})} \Big).$$
Use this map (e.g., via a penalised optimisation process) to determine what mesh refinement will be effective for reducing the error in U.
6. Solve the problem on the new mesh to obtain u_h^{j+1}.
7. Set $j = j + 1$ and repeat Steps 3-6 until the desired accuracy is achieved.

The constants in Theorem 1.3 are not given explicitly. However, we believe that replacing all the constants by one (or possibly adaptively tuning the constants) will produce an effective adaptive refinement for a fixed k (large enough).

Remark 2.1 *To accomplish Step 1 (a), one can use a set of sample points and directions* $x_i \in \Omega$, $i = 1, \dots, N$ *and* $\xi_j \in S^{d-1}$, $j = 1, \dots, M$ *at a fine scale,* δ *and fix a maximal time* $T_{\max} > 0$ *and* *then run a ray tracing algorithm from each sample point $t_{ij} : \min(T_{\max}, T_{ij,\mathcal{E}})$, where $T_{ij,\mathcal{E}}$ is the time at which the ray from (x_i, ξ_j) enters the PML region. Define $I_{\mathcal{V}} := \{i : \max_j t_{ij} = T_{\max}\}$. We set $\Omega_{\mathcal{K}}$ to be a neighbourhood of $\{x_i : i \in I_{\mathcal{V}}\}$, and $\Omega_{\mathcal{V}}$ a neighbourhood of the complement of $\Omega_{\mathcal{K}}$ intersected with the rays from $\{x_i\}_{i\in I_{\mathcal{K}}}$.*

For Step 1(b) one can use the heuristic that

$$\rho \lesssim kV^{-1}(k^{-1})\,, \tag{2.2}$$

where V^{-1} is the inverse of $t \mapsto V(t)$, and $V(t)$ is the volume of the set points in $\Omega \times S^{d-1}$ that do not enter the PML in time t. To estimate $V(t)$, we use the approximation

$$V(t) \sim \widetilde{V}(t) := \delta^{2d-1} \#\{(x_i, \xi_j) \,:\, t_{ij} \geq t\}.$$

The heuristic (2.2) is valid at least for some special cases, including some cases of the weakest form of trapping where trajectories escape exponentially fast, and certain geometries that are warped products [CW13]. Furthermore, one place where $V(t)$ rigorously appears is in fractal upper bounds on the number of resonances near the real axis [DG17]. However, a precise characterisation of ρ via billiard dynamics is a challenging open problem.

3 Assumptions and statement of the main result

We now gather some definitions and assumptions and state our main result, Theorem 3.11.

3.1 The Helmholtz PML operators

Throughout this paper, $\Omega_- \subset \mathbb{R}^d$ (the obstacle) denotes a bounded open set with C^∞ boundary and connected complement. Let $\Omega_{\rm tr} \Subset \mathbb{R}^d$ (the truncation domain) be a bounded open set with $\Omega_- \Subset \Omega_{\rm tr}$ and define $\Omega := \Omega_{\rm tr} \setminus \overline{\Omega_-}$ (the computational domain). Let $\Gamma_{\rm tr} = \partial\Omega_{\rm tr}$ so that $\partial\Omega = \partial\Omega_- \sqcup \Gamma_{\rm tr}$. For all $k > 0$ and $n \geq 0$, and given $U \subset \mathbb{R}^d$, let $H^n_k(U)$ (abbreviated H^n_k when $U = \Omega$) be the completion of $C^\infty(U)$ with respect to the norm (1.7), and let $H^{-n}_k(U)$ be the normed dual of $H^n_k(U)$, with, as usual, $L^2(\Omega)$ identified through the L^2 pairing with a subspace of H^{-n}_k for all $n \geq 0$. [2]

Let $a_k : H^1_k \times H^1_k \to \mathbb{C}$ be the sesquilinear form defined by

$$a_k(u, v) := \int_\Omega \Big(k^{-2} A_\theta(x) \nabla u(x) \cdot \overline{\nabla v(x)} + k^{-2} \langle b_\theta(x), \nabla u\rangle \overline{v(x)} - n_\theta(x) u(x) \overline{v(x)} \Big)\, dx, \tag{3.1}$$

where A_θ, b_θ, and n_θ are defined in §A.

To cover both Dirichlet (so-called "sound-soft") and Neumann ("sound-hard") obstacles, we consider a subspace $\mathcal{Z}_k \subset H^1_k$, which can be either given by $\mathcal{Z}_k = \mathcal{Z}_{k,d}$ or $\mathcal{Z}_k = \mathcal{Z}_{k,n}$, where

$$\mathcal{Z}_{k,d} := H^1_{0,k}(\Omega), \qquad \text{and} \qquad \mathcal{Z}_{k,n} := \overline{\{u \in C^\infty(\overline{\Omega}) \,:\, \operatorname{supp} u \cap \Gamma_{\rm tr} = \emptyset\}}^{H^1_k}, \tag{3.2}$$

and let $\mathcal{Z}^*_k$ be the normed dual of $\mathcal{Z}_k$. The Helmholtz operator $P_k : \mathcal{Z}_k \to (\mathcal{Z}_k)^*$ is then defined as the linear operator associated to a_k, i.e.

$$P_k : \mathcal{Z}_k \to \mathcal{Z}^*_k\,, \quad \langle P_k u, v\rangle := a_k(u, v) \quad \text{for all } u, v \in \mathcal{Z}_k.$$

If P_k is invertible, we denote by $R_k := (P_k)^{-1} : \mathcal{Z}^*_k \to \mathcal{Z}_k$ its inverse (also known as the resolvent), and let

$$\rho(k) := \|R_k\|_{L^2 \to L^2} = \sup_{f \in L^2(\Omega)\setminus\{0\}} \frac{\|P_k^{-1} f\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}}$$

Our main result holds for k ranging in a subset $\mathbb{R}_+ \setminus \mathcal{J}$ of the positive real numbers on which $\rho(k)$ is polynomially bounded; i.e., we make the following assumption on $\mathcal{J}$.

[2]We highlight that this is not the standard notation, as H^{-n} usually denotes the dual of $H^n_0 = \overline{C^\infty_c(\Omega)}^{H^n_k}$.

Assumption 3.1 (Polynomial bound on the resolvent) *There exist $C > 0$, $N > 0$ such that, for all $k \in \mathbb{R}_+ \setminus \mathcal{J}$,*

$$\rho(k) \leq Ck^N.$$

In this paper, we are then interested in the error in the finite-element approximation solution (see next paragraph) of the variational problem, for $k \in \mathbb{R}_+ \setminus \mathcal{J}$:

$$\text{find } u \in \mathcal{Z}_k \text{ such that, for all } v \in \mathcal{Z}_k,\ a_k(u, v) = F(v), \tag{3.3}$$

where $F : \mathcal{Z}_k \to \mathcal{Z}_k^*$ is a continuous anti-linear form.

3.2 Finite-element approximation

We consider a Galerkin approximation of the variational problem (3.3). Following the practice in the local FEM error analysis literature (see in particular [NS74, Assumptions A.1–A.3], and also [Wah91, DGS11, Bre20]), and following closely [AGS24], we describe V_k through a set of standard assumptions as follows. Throughout, a fixed positive integer p, modelling the polynomial degree of the finite-element subspace, is chosen independently of k; hence this setting models a "h-version" of the FEM.

If $U \subset \Omega$ is an open set, define

$$C^\infty_<(U) := \big\{\chi \in C^\infty(\overline{\Omega}) \text{ such that } \operatorname{supp}\chi \subset \overline{U} \text{ and } \partial_<(\operatorname{supp}\chi, U) > 0\big\}$$

$$\mathcal{Z}_k^{1,<}(U) := \overline{\big\{v \in \mathcal{Z}_k \text{ s.t. } \operatorname{supp} v \subset \overline{U} \text{ and } \partial_<(\operatorname{supp} v, U) > 0\big\}},$$

where the closure is taken with respect to the $\mathcal{Z}_k^j$ norm, and for any subsets $\Omega_0 \subset \Omega_1 \subset \Omega$,

$$\partial_<(\Omega_0, \Omega_1) := \operatorname{dist}(\,\partial\Omega_0 \setminus \partial\Omega\,,\ \partial\Omega_1 \setminus \partial\Omega\,). \tag{3.4}$$

A *triangulation* $\mathcal{T}$ of Ω is a set of pairwise disjoint open subsets $K \subset \Omega$ such that

$$\bigcup_{K \in \mathcal{T}} \overline{K} = \overline{\Omega}.$$

We denote by h_K the diameter of $K \in \mathcal{T}$. For $k > 0$, a *finite-element space* V_k over a triangulation $\mathcal{T}$ of Ω is a finite-dimensional subspace $V_k \subset \mathcal{Z}_k$ such that for every $u \in V_k$ and $K \in \mathcal{T}$, $u_{|K} \in C^\infty(\overline{K})$. If V_k is a finite-element space and $U \subset \Omega$, define

$$V_k^<(U) := \mathcal{Z}_k^{1,<}(U) \cap V_k.$$

In what follows, for each $k > 0$, $\mathcal{T}_k$ is a given triangulation and V_k is a finite-element space over $\mathcal{T}_k$. We denote

$$h = h(k) := \max_{K \in \mathcal{T}_k} h_K$$

the global meshwidth, and make the following standard assumptions.

Assumption 3.2 (Sub-wavelength grid) *For all $k_0 > 0$, there exists a positive constant $C > 0$ such that for all $k \geq k_0$*

$$h \leq Ck^{-1}.$$

Assumption 3.3 (Wavelength-scale quasi-uniformity) *For all $R > 0$ and all $k_0 > 0$, there exists $C > 0$ such that for all $k \geq k_0$ and any elements K, K' of $\mathcal{T}_k$ such that $\operatorname{dist}(K, K') \leq Rk^{-1}$,*

$$\frac{1}{C} \leq \frac{h_K}{h_{K'}} \leq C.$$

Assumption 3.4 (Approximation property) *There exists $\kappa > 0$ such that for every $k_0 > 0$, there exists $C > 0$ such that for all $j \in \{1, \ldots, p+1\}$, all $m \in \{0, \ldots, j\}$, all $k \geq k_0$ and all $u \in \mathcal{Z}_k^1 \cap \mathcal{H}_k^j$, there exists $u_h \in V_k$ such that*

$$\sum_{K \in \mathcal{T}_k} (h_K k)^{2(m-j)} \|u - u_h\|^2_{H_k^m(K)} \leq C\|u\|^2_{H_k^j}.$$

Furthermore, given subsets $U_0 \subset U_1 \subset \Omega$ such that

$$\partial_<(U_0, U_1) \geq \kappa \max\{h_K \mid K \in \mathcal{T}_k \text{ s.t. } K \cap U_1 \neq \emptyset\},$$

if $\operatorname{supp} u \subset U_0 \cup \partial\Omega$, *then u_h can be chosen such that* $\operatorname{supp} u_h \subset U_1 \cup \partial\Omega$.

Assumption 3.5 (Super-approximation property) *There exists $\kappa > 0$ such that for all $k_0 > 0$ and $C_\dagger > 0$, there exists $C > 0$ such that for all $k \geq k_0$ and any subsets $U_0 \subset U_1 \subset \Omega$ such that*

$$d := \partial_<(U_0, U_1) \geq \kappa \max\{h_K \mid K \in \mathcal{T}_k \text{ s.t. } K \cap U_1 \neq \emptyset\},$$

if $\chi \in C^\infty_<(U_0)$ is such that,

$$\max_{|\alpha|=n} \|\partial^\alpha \chi\|_\infty \leq \frac{C_\dagger}{d^n}, \text{ for } n = 0, \ldots, p,$$

then for any $u_h \in V_k$, there exists $v_h \in V_k^<(U_1)$ such that

$$\|\chi^2 u_h - v_h\|_{H^1_k(K)} \leq C \frac{h_K}{d} \left[\left(1 + \frac{1}{kd}\right) \|u_h\|_{L^2(K)} + \|\chi u_h\|_{H^1_k(K)} \right] \qquad \text{for all } K \in \mathcal{T}_k.$$

Assumption 3.6 (Inverse inequality on elements) *There exists C such that for all $k > 0$, all $K \in \mathcal{T}_k$, all $u_h \in V_k$ and all $j \in \{0, 1, \ldots, p\}$,*

$$\|u_h\|_{H^1_k(K)} \leq \frac{C}{h_K k} \|u_h\|_{L^2(K)} \quad \text{and} \quad \|u_h\|_{L^2(K)} \leq \frac{C}{(h_K k)^j} \|u_h\|_{H_k^{-j}(K)},$$

where $\|u_h\|_{H_k^{-j}(K)} := \sup_{v \in C_c^\infty(K)} \left(\left| \int_K u_h v \, dx \right| / \|v\|_{H^j_k(K)} \right)$.

Definition 3.7 (Well-behaved finite-element of order p) *We say that $(V_k)_{k>0}$ is a* well-behaved finite-element of order p *if it satisfies Assumptions 3.2-3.6 above.*

Remark 3.8 *Since Ω has a C^∞ boundary, under the assumptions of the present section, the elements K must be curved, ruling out from our settings the standard simplicial Lagrange finite-element discretizations. However,*

- *this type of assumptions is common in the high-frequency error analysis for the finite-element method for the Helmholtz equation, see e.g. [MS10, Appendix B], and*
- *the "geometric error" incurred by using simplicial elements instead of curved elements is studied in [CFS25], and shown to be smaller than the pollution error.*

For each $k \in \mathbb{R}_+ \setminus \mathcal{J}$, the Galerkin solution $u_h = u_h(k) \in V_k$ (where the subscript h emphasizes the dependence of u_h with respect to the meshwidth of the triangulation) is defined by

$$a_k(u - u_h, v_h) = 0 \quad \text{for all } v_h \in V_k,$$

and our main result, Theorem 3.11, describes the (micro-)local behaviour of the error $u - u_h$.

3.3 Frequency splitting of the error

We consider a splitting of the Galerkin error $u - u_h$ into "low-frequencies" and "high-frequencies". To define these notions, we introduce frequency cutoffs as follows.

The following Gårding inequality holds (see §6.1 and §A): there exists $\omega \in \mathbb{R}$ (with $\omega = 0$ for the most commonly-used PML) such that for all $k_0 > 0$ there are $c_{\mathrm{Ga}}, C_{\mathrm{Ga}} > 0$ such that for all $k \geq k_0$,

$$\Re(\mathrm{e}^{\mathrm{i}\omega} a_k(u, u)) \geq c_{\mathrm{Ga}} \|u\|^2_{\mathcal{Z}_k} - C_{\mathrm{Ga}} \|u\|^2_{L^2}. \tag{3.5}$$

We deduce from this (see §5) that $\sigma(\mathcal{P}_k) \subset [-C_{\mathrm{Ga}}, +\infty)$, and thus, for each k_0, there exists a function $\psi^\sharp$ such that

$$\psi^\sharp(x) \geq \frac{-x + C_{\mathrm{Ga}}}{2} \qquad \text{for all } x \in \sigma(\mathcal{P}_k). \tag{3.6}$$

Let $\mathcal{P}_k : \mathcal{Z}_k \to (\mathcal{Z}_k)^*$ be defined by

$$\mathcal{P}_k = \frac{1}{2}\Big(\mathrm{e}^{\mathrm{i}\omega} P_k + \mathrm{e}^{-\mathrm{i}\omega} P_k^*\Big).$$

We show in Section 6 that $\mathcal{P}_k$ is self-adjoint on $L^2(\Omega)$ with domain $\mathcal{Z}_k^2 = \mathcal{Z}_{k,d}^2$ in the Dirichlet case, and $\mathcal{Z}_k^2 = \mathcal{Z}_{k,n}^2$ in the Neumann case, where

$$\mathcal{Z}_{k,d}^2 = H_0^1(\Omega) \cap H^2(\Omega), \qquad \mathcal{Z}_{k,n}^2 = \{u \in H^2(\Omega) \,:\, \partial_{\nu,A_\theta} u|_{\partial\Omega_-} = 0,\ u|_{\Gamma_{\mathrm{tr}}} = 0\}. \tag{3.7}$$

Thus, if $f : \mathbb{R} \to \mathbb{R}$ is a bounded, continuous function, we may consider $f(\mathcal{P}_k) : L^2(\Omega) \to L^2(\Omega)$ defined by the functional calculus.

Low-frequency cutoffs will then be defined as $\Psi = \psi(\mathcal{P}_k)$ where $\psi \in C_c^\infty(\mathbb{R})$ is such that $\psi \equiv 1$ on the support of $\psi^\sharp$, and $1 - \Psi$ will correspond to high-frequency cutoffs.

3.4 Spatial splitting of the error

In addition to considering the Galerkin error locally in frequency space, we also localize it spatially. We fix a neighbourhood U_{P} of Γ_{tr} in which Theorem 4.2 holds (that is, sufficiently "deep" in the PML region so that the resolvent R_k^* in this region behaves like a pseudolocal, uniformly bounded operator with respect to k). Let

$$\Omega = \bigcup_{j=1}^{M} \Omega_j$$

be an open cover of Ω by $M = M_{\mathrm{I}} + M_{\mathrm{P}}$ subdomains. We assume that the "interior" domains $\Omega_1, \ldots, \Omega_{M_{\mathrm{I}}}$ do not intersect the truncation boundary, while the "PML" domains $\Omega_{M_{\mathrm{I}}+1}, \ldots, \Omega_{M_{\mathrm{I}}+M_{\mathrm{P}}}$ all lie inside the deep PML region, i.e.

$$\bigcup_{j=1}^{M_{\mathrm{I}}} \Omega_j \cap \Gamma_{\mathrm{tr}} = \emptyset\,, \qquad \bigcup_{j=M_{\mathrm{I}}+1}^{M_{\mathrm{I}}+M_{\mathrm{P}}} \Omega_j \subset U_{\mathrm{P}}. \tag{3.8}$$

For $i, j \in \{1, \ldots, M\}$, define

$$h_i := \max_{K \in \mathcal{T}_k} \{\mathrm{diam}(K) \mid K \cap \Omega_i \neq \emptyset\} \quad \text{and} \quad h_{ij} := \min(h_i, h_j), \tag{3.9}$$

the local mesh sizes on Ω_j and $\Omega_i \cap \Omega_j$.

3.5 Matrix quantities

In Theorem 3.11, the description of the local error in subdomains is given in terms of matrices $\mathcal{H}$, $\mathcal{H}^{\min}$, $\mathcal{C}$, T and B that we define now.

For every natural number ℓ, define the following $M \times M$ matrices

$$\mathcal{H} := \mathrm{diag}(h_1, \ldots, h_M), \qquad \mathcal{H}^{\min}(\ell) := 1_{\{\Omega_i \cap \Omega_j \neq \emptyset\}} (h_{ij}^\ell)_{1 \le i,j \le M}. \tag{3.10}$$

Furthermore, let $\mathcal{C}$ be the $M \times M$ matrix defined by

$$\mathcal{C}_{ij} := \|1_{\Omega_j} R_k^* 1_{\Omega_i}\|_{L^2 \to L^2} = \|1_{\Omega_i} R_k 1_{\Omega_j}\|_{L^2 \to L^2}, \quad i, j = 1, \ldots, M \tag{3.11}$$

For an $M \times M$ matrix A (either $\mathcal{H}$ or $\mathcal{H}^{\min}$), we write

$$A =: \begin{pmatrix} A_{\mathrm{I,I}} & A_{\mathrm{I,P}} \\ A_{\mathrm{P,I}} & A_{\mathrm{P,P}} \end{pmatrix}, \quad A_{i,j} \in \mathbb{M}(M_i \times M_j).$$

where $M_1 := M_{\mathrm{I}}$ and $M_2 := M_{\mathrm{P}}$. Let $B \in \mathbb{M}((2M_{\mathrm{I}} + M_{\mathrm{P}}) \times M)$ be defined by

$$B := \begin{pmatrix} \mathcal{C}_{\mathrm{I,I}}(\mathcal{H}_{\mathrm{I,I}} k)^p & 0 \\ (\mathcal{H}_{\mathrm{I,I}} k)^p & 0 \\ 0 & (\mathcal{H}_{\mathrm{P,P}} k)^p \end{pmatrix}, \tag{3.12}$$

and let $W \in \mathbb{M}((2M_{\mathrm{I}} + M_{\mathrm{P}}) \times (2M_{\mathrm{I}} + M_{\mathrm{P}}))$ be defined by

$$W := \begin{pmatrix} \mathcal{C}_{\mathrm{I,I}}(\mathcal{H}_{\mathrm{I,I}} k)^{2p} & \mathcal{C}_{\mathrm{I,I}}(\mathcal{H}_{\mathrm{I,I}} k)^{2p} & \mathcal{H}^{\min}_{\mathrm{I,P}}(N) k^N \\ \mathcal{H}^{\min}_{\mathrm{I,I}}(2p) k^{2p} & \mathcal{H}^{\min}_{\mathrm{I,I}}(N) k^N & \mathcal{H}^{\min}_{\mathrm{I,P}}(N) k^N \\ \mathcal{H}^{\min}_{\mathrm{P,I}}(N) k^N & \mathcal{H}^{\min}_{\mathrm{P,I}}(N) k^N & \mathcal{H}^{\min}_{\mathrm{P,P}}(N) k^N \end{pmatrix}, \tag{3.13}$$

3.6 Simple-path matrix

To any square matrix $W \in \mathbb{M}(N \times N)$, one can associate a matrix $V = V(W)$ defined from the coefficients W in terms of simple paths on a graph. To define this, let $\mathcal{G} = \mathcal{G}(W)$ be the (complete) directed graph, with node set $\mathcal{N} := \{1, \ldots, N\}$, and with edge set $\mathcal{E}$ the set of ordered pairs $(i, j) \in \{1, \ldots, N\}^2$. A *path* p in $\mathcal{G}$ is a finite (and possibly empty) sequence of edges

$$p = (i_1, j_1)(i_2, j_2) \ldots (i_{L-1}, j_{L-1})(i_L, j_L)$$

satisfying the conditions $j_\ell = i_{\ell+1}$ for $1 \le \ell \le L - 1$. Let $\mathbf{0}$ stand for the empty path. We write $|p| := L$ and denote by $p(\ell)$ the ℓ-th node visited by p, i.e., $p(\ell) := i_\ell$ if $1 \le \ell \le |p|$ and $p(|p| + 1) := j_L$. Let $\mathbb{P}_{ij}$ be the set of paths from i to j, i.e., such that $p(1) = i$ and $p(|p| + 1) = j$.

A path p is *non-intersecting* if the map $\ell \mapsto p(\ell)$ is injective. For $i, j \in \{1, \ldots, M\}$, let $\mathbb{V}_{ij}$ be the set of non-intersecting paths from i to j. Observe that $\mathbb{V}_{ii} := \{\mathbf{0}\}$.

A non-empty path p is a *loop* if it starts and ends at the same node, i.e., if $p(1) = p(|p| + 1)$. It is a *simple loop* if it is a loop but otherwise does not intersect itself, i.e.,

$$p(\ell) = p(m) \implies \big(\ell = m \ \text{ or } \ \{\ell, m\} = \{1, |p| + 1\}\big).$$

We denote by $\mathbb{SL}$ the set of simple loops.

To each edge $e = (i, j)$ of $\mathcal{G}$, we associated the weight $W_e := W_{ij}$ (the (i, j)-th coefficient of the matrix W). We also define the weight of the path p as the product of the weights of its edges, i.e., $W_{\mathbf{0}} := 1$ and

$$W_{e_1 e_2 \ldots e_L} := W_{e_1} W_{e_2} \ldots W_{e_L}.$$

Definition 3.9 (Simple-path matrix) *The* simple-path matrix $T^\star = T^\star(W) \in \mathbb{M}(N \times N)$ *of a matrix* $W \in \mathbb{M}(N \times N)$ *is defined by*

$$T^\star_{ij} := \sum_{p \in \mathbb{V}_{ij}} W_p, \quad 1 \le i, j \le N.$$

Observe that the diagonal entries of $T^\star$ *are* 1 *since* $\mathbb{V}_{ii} = \{\mathbf{0}\}$.

Remark 3.10 *We show in Theorem 8.13 that, provided the simple loops of of* $\mathcal{G}$ *carry weights bounded by* $c < 1$*, then the* $I - W$ *is invertible and* $(I - W)^{-1} \le T^\star$ *coefficientwise.*

3.7 Statement of the main result

Theorem 3.11 (The main result) *Let* a_k *be defined by* (3.1) *and let* $\mathcal{J} \subset \mathbb{R}_+$ *be such that Assumption 3.1 holds. Let* p *be a positive integer and let* $(V_k)_{k>0}$ *be a well-behaved Finite-Element of order* p *in the sense of Definition 3.7. Let* $\{\Omega_i\}_{i=1}^M$ *be an open cover of* Ω *such that the conditions* (3.8) *hold. For every* $i \in \{1, \ldots, M\}$*, let* $\chi_i \in C^\infty(\overline{\Omega})$ *be such that*

$$\operatorname{supp}(\chi_i) \subset \Omega_i \cup \partial\Omega \quad \text{and} \quad \Omega = \bigcup_{i=1}^M \operatorname{int}\big(\{\chi_i \equiv 1\}\big), \tag{3.14}$$

where the interior is taken in the subspace topology of Ω*. Let* $k_0, N > 0$*, let* $\psi^\sharp$ *satisfy* (3.6) *and let* $\psi \in C^\infty_c(\mathbb{R})$ *be such that* $\operatorname{supp} \psi^\sharp \cap \operatorname{supp}(1 - \psi) = \emptyset$.

Then, there exist constants $h_0, C_\dagger > 0$ *and, for any* $0 < c < 1$*, a constant* $C > 0$ *such that the following holds. For any* $k \in (k_0, \infty) \setminus \mathcal{J}$*, if* $\mathcal{T}_k$ *satisfies* $h(k) \le h_0$ *and*

$$\sum_{L \in \mathbb{SL}} C_\dagger^{|L|} W_L \le c, \tag{3.15}$$

where W *is defined by* (3.13)*, then for all* $u \in H^1_k$*, there exists a unique solution* $u_h \in V_k$ *to the Galerkin problem*

$$a_k(u - u_h, v_h) = 0 \quad \text{for all } v_h \in V_k. \tag{3.16}$$

Moreover, for any $w_{h,1},\ldots,w_{h,M}\in V_k$, $m\in\{0,\ldots,p\}$, and $i\in\{1,\ldots,M_{\rm I}\}$,

$$
\begin{pmatrix}
\big(\|\chi_i\Psi(u-u_h)\|_{H_k^{1-m}}\big)_{i=1}^{M_{\rm I}}\\
\big(\|\chi_i(1-\Psi)(u-u_h)\|_{H_k^{1-m}}\big)_{i=1}^{M_{\rm I}}\\
\big(\|\chi_i(u-u_h)\|_{H_k^{1-m}}\big)_{i=M_{\rm I}+1}^{M}
\end{pmatrix}
\leq \left[\begin{pmatrix} 0 & 0\\ (\mathcal{H}_{\rm I,I}k)^m & 0\\ 0 & (\mathcal{H}_{\rm P,P}k)^m\end{pmatrix} + \begin{pmatrix} {\rm I} & 0 & 0\\ (\mathcal{H}_{\rm I,I}k)^{p+m} & (\mathcal{H}_{\rm I,I}k)^N & 0\\ 0 & 0 & (\mathcal{H}_{\rm P,P}k)^N\end{pmatrix} T^\star B\right]\Big(\|u-w_{h,j}\|_{H_k^1(\Omega_j)}\Big)_{j=1}^{M} + CR. \tag{3.17}
$$

where $\mathcal{H}$ is defined by (3.10), B is defined by (3.12), $T^\star$ is the simple-path matrix of $C_\dagger W$ in the sense of Definition 3.9, and

$$
R := k^{-N}(hk)^m\sum_{j=1}^{M}\|u-w_{h,j}\|_{H_k^1}.
$$

In particular, the local Galerkin errors satisfy

$$
\big(\|\chi_i(u-u_h)\|_{H_k^{1-m}}\big)_{i=1}^{M}
\leq \left[\begin{pmatrix} (\mathcal{H}_{\rm I,I}k)^m & 0\\ 0 & (\mathcal{H}_{\rm P,P}k)^m\end{pmatrix} + \begin{pmatrix} {\rm I} & (\mathcal{H}_{\rm I,I}k)^N & 0\\ 0 & 0 & (\mathcal{H}_{\rm P,P}k)^N\end{pmatrix} T^\star B\right]\Big(\|u-w_{h,j}\|_{H_k^1(\Omega_j)}\Big)_{j=1}^{M} + CR. \tag{3.18}
$$

The proof of Theorem 3.11 is given in §8. Figure 3.1 shows the weighted graph associated to the matrix W in the setting of §1; i.e., $M_{\rm I}=3$ (with domains $\Omega_{\mathcal{K}},\Omega_{\mathcal{V}}$, and $\Omega_{\mathcal{I}}$) and $M_{\rm P}=1$.

4 Local bounds on the Helmholtz solution operator

This section describes two results showing how the Helmholtz solution operator has improved k-dependence based on the data and measurement locations. The first result (Theorem 4.1) considers locations relative to the cavity or the ray dynamics, and the second (Theorem 4.2) considers locations relative to the PML. Both results are proved in Appendix C with the first result a special case of a more general result phrased in terms of semiclassical pseudodifferential operators.

Theorem 4.1 (Improved behaviour away from trapping) *Let $k_0>0$ and let $\mathcal{J}$ be such that Assumption 1.2 holds.*

(i) For all $\chi\in C^\infty(\overline{\Omega})$ with $\operatorname{supp}\chi\cap\mathcal{K}=\emptyset$, there exists $C>0$ such that for all $k\in(k_0,\infty)\setminus\mathcal{J}$

$$
\|\chi R_k\|_{L^2\to L^2}+\|R_k\chi\|_{L^2\to L^2}\leq C\sqrt{k\rho},\qquad \|\chi R_k\chi\|_{L^2\to L^2}\leq Ck. \tag{4.1}
$$

(ii) For all $\chi,\psi\in C^\infty(\overline{\Omega})$ with $\operatorname{supp}\chi\subset\mathcal{I}$ and $\operatorname{supp}\psi\subset\mathcal{K}$, and all $N>0$, there exists $C>0$ such that for all $k\in(k_0,\infty)\setminus\mathcal{J}$

$$
\|\chi R_k\psi\|_{L^2\to L^2}+\|\psi R_k\chi\|_{L^2\to L^2}\leq Ck^{-N}. \tag{4.2}
$$

In the case of scattering without boundaries, the result analogous to (4.1) was proved in [DV12a, DV12b].

Theorem 4.2 (Improved behaviour in the PML) *Let $k_0>0$ and let $\mathcal{J}$ be such that Assumption 1.2 holds. Then there is $U\subset\Omega$ a neighbourhood of $\Gamma_{\rm tr}$ such that for all $\chi\in C^\infty(\overline{\Omega})$ with $\operatorname{supp}\chi\subset U$, there exists $C>0$ such that, for all $k\in(k_0,\infty)\setminus\mathcal{J}$,*

$$
\|\chi R_k\|_{L^2\to L^2}+\|R_k\chi\|_{L^2\to L^2}\leq C. \tag{4.3}
$$

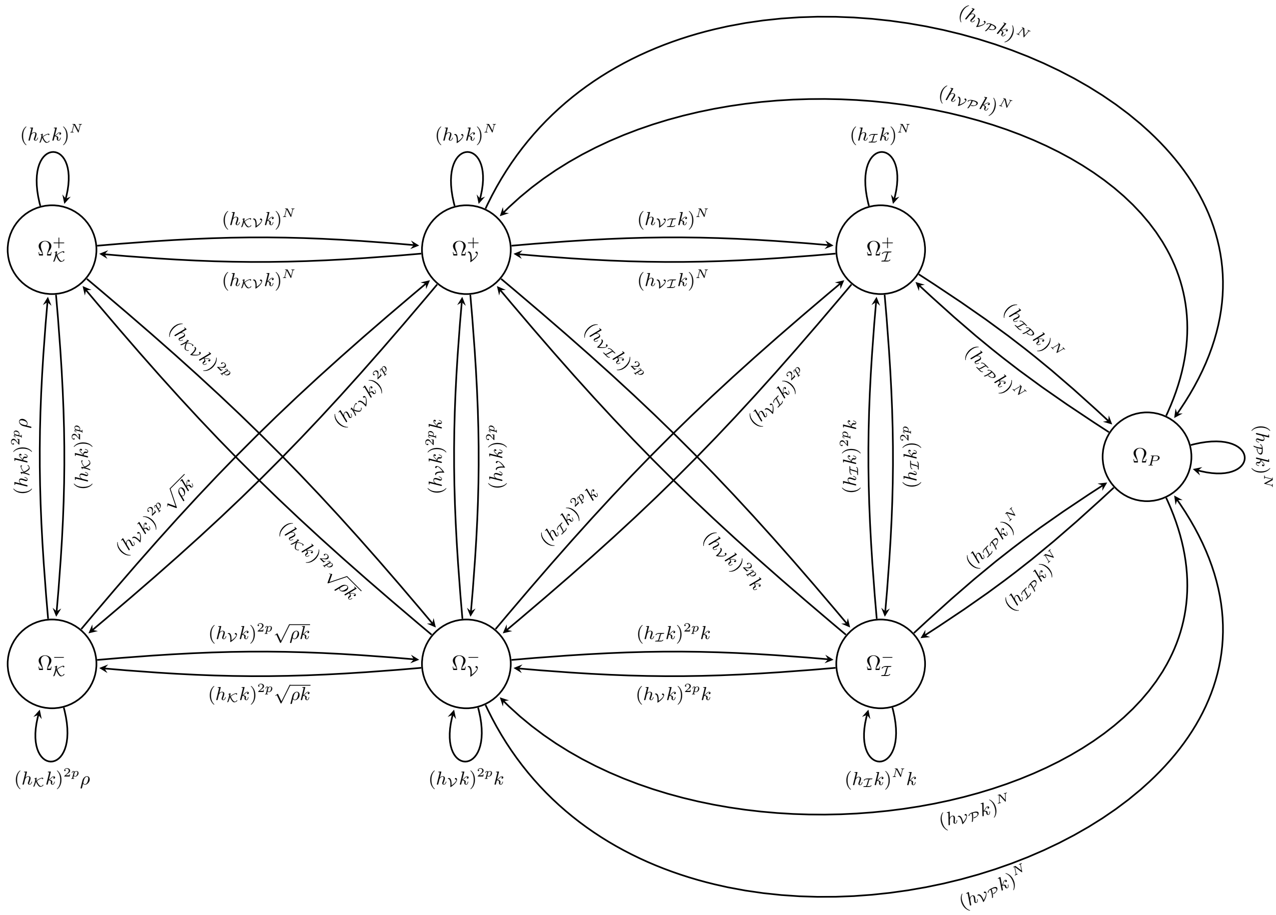


Figure 3.1: The weighted graph associated to the matrix W in the case when $M_{\rm I} = 3$ (with domains $\Omega_{\mathcal{K}}, \Omega_{\mathcal{V}}$, and $\Omega_{\mathcal{I}}$), $M_{\rm P} = 1$, and with the k-dependence of $\mathcal{C}$ (3.11) given by the results of §4. Edges with zero weight (between $\Omega_{\mathcal{K}}^{\pm}$ and $\Omega_{\mathcal{P}}$) or $O(k^{-\infty})$ weight (between $\Omega_{\mathcal{K}}^{\pm}$ and $\Omega_{\mathcal{I}}^{\pm}$) are not displayed. Finally $h_{\mathcal{VP}} := \min\{h_{\mathcal{V}}, h_{\mathcal{P}}\}$ etc.

Moreover, if $\operatorname{supp}\chi \subset U$, *and* $\psi \in C^\infty(\overline{\Omega})$ *with* $\operatorname{supp}\chi \cap \operatorname{supp}\psi = \emptyset$, *then for any* N *there exists* $C > 0$ *such that for all* $k > k_0$,

$$\|\chi R_k \psi\|_{L^2 \to H_k^N} + \|\psi R_k \chi\|_{L^2 \to H_k^N} \leq C k^{-N}. \tag{4.4}$$

Theorem 4.2 is based on ellipticity in the PML region.

5 Abstract pseudolocality results

As described in §1.3.2, the proof of the main result requires pseudolocaity of the operators S_k and $\Pi_k^\sharp$ (see (1.29)); furthermore, although not stated in §1.3.2, the proof also requires pseudolocality of $(P_k^\sharp)^{-1}$, where $P_k^\sharp := P_k + S_k$. This section proves pseudolocality of S_k and $(P_k^\sharp)^{-1}$, and §7 proves pseudolocality of $\Pi_k^\sharp$.

The operator S_k is defined as a function of the self-adjoint operator $\mathcal{P}_k := \Re P_k$ via the functional calculus. This section therefore studies general Helmholtz operators (satisfying continuity, a Gårding inequality, and elliptic regularity), proves that $\mathcal{P}_k$ is self-adjoint, and then proves that both functions of $\mathcal{P}_k$ and $(P_k^\sharp)^{-1}$ are pseudolocal; i.e., when sandwiched by disjoint "spatial" or "frequency" cutoffs, the result is $O(k^{-\infty})$ and infinitely smoothing.

Abstract setting	Model Dirichlet setting
$\mathcal{H}^n_k$	$H^n(\Omega)$ with k-weighted norm
$a_k(u,v)$	$\int_\Omega k^{-2}\nabla u\cdot\overline{\nabla v} - u\overline{v}$
$\mathcal{Z}_k$	$H^1_0(\Omega)$
$\mathcal{Z}^2_k$	$H^2(\Omega)\cap H^1_0(\Omega)$
$\mathcal{Z}^n_k$	$H^n(\Omega)\cap H^1_0(\Omega)$
$\mathcal{D}^{2n}_k$	$\{u\in H^{2n}(\Omega) : \gamma u, ..., \gamma\Delta^{n-1}u = 0 \text{ on } \partial\Omega\}$
Abstract setting	**Model Neumann setting**
$\mathcal{H}^n_k$	$H^n(\Omega)$ with k-weighted norm
$a_k(u,v)$	$\int_\Omega k^{-2}\nabla u\cdot\overline{\nabla v} - u\overline{v}$
$\mathcal{Z}_k$	$H^1(\Omega)$
$\mathcal{Z}^2_k$	$\{u\in H^2(\Omega)\ :\ \partial_\nu u = 0 \text{ on } \partial\Omega\}$
$\mathcal{Z}^n_k$	$\{u\in H^n(\Omega)\ :\ \partial_\nu u = 0 \text{ on } \partial\Omega\}$
$\mathcal{D}^{2n}_k$	$\{u\in H^{2n}(\Omega) : \partial_\nu u, ..., \partial_\nu\Delta^{n-1}u = 0 \text{ on } \partial\Omega\}$

Table 5.1: Model examples for the spaces in Section 5

For the "spatial cutoffs", we require some control over their repeated commutators with $\mathcal{P}_k$ in a scale of Hilbert spaces $(\mathcal{H}^n_k)_{n\geq 0}$ (which will be taken as $H^n_k(\Omega)$). Checking these assumptions in the concrete setting will require the construction of suitable cutoff functions with a special behavior near the boundary

For the "frequency cutoffs", we require that, in addition, the repeated commutators act in domains $\mathcal{D}^n$ of powers of the self-adjoint operators $\mathcal{P}_k$. This essentially asks that repeated commutators preserve an arbitrary number of boundary conditions, which in practice, will be achieved by requiring the frequency cutoffs to be constant near the boundary and 0 near the PML truncation boundary.

5.1 Abstract formulation of Helmholtz operators with smooth coefficients on smooth domains

In what follows, $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is a Hilbert space and for every $k\in\mathbb{R}_+$, $(\mathcal{H}^n_k, \|\cdot\|_{\mathcal{H}^n_k})_{n\in\mathbb{N}}$ is a decreasing sequence of Hilbert spaces with continuous and dense inclusions $\mathcal{H}^n_k\subset\mathcal{H}^m_k$ for all $m\leq n$, with

$$\|u\|_{\mathcal{H}^m_k}\leq\|u\|_{\mathcal{H}^n_k}\qquad\text{for all } u\in\mathcal{H}^n_k$$

and such that $\mathcal{H}^0_k=\mathcal{H}$ with equal norms.[3] For all $n\in\mathbb{N}$, $\mathcal{H}^{-n}_k$ denotes the *anti*-dual of $\mathcal{H}^n_k$, i.e., the set of continuous complex-valued *anti*-linear forms on $\mathcal{H}^n_k$. For any $u\in\mathcal{H}$, one may define an element $L^n_u\in\mathcal{H}^{-n}_k$ by

$$L^n_u(v) := \langle u, v\rangle := (u,v)_{\mathcal{H}}\qquad\text{for all } v\in\mathcal{H}^n_k.$$

By density of the embeddings $\mathcal{H}^n_k\subset\mathcal{H}^m_k\subset\mathcal{H}$ for $n\geq m$, the mapping $u\mapsto L^n_u$ is injective and L^n_u coincides with L^m_u on $\mathcal{H}^m_k$, so we may identify u with L_u. Under this identification, the continuous embeddings $\mathcal{H}^n_k\subset\mathcal{H}^m_k$ for $n\geq m$ extend to all $m,n\in\mathbb{Z}$ and $\langle\cdot,\cdot\rangle$ extends to a continuous sesquilinear (linear on the left, anti-linear on the right) pairing $\mathcal{H}^{-n}_k\times\mathcal{H}^n_k$ for all n.

Let $(a_k)_{k\in\mathbb{R}_+}$ be a family of sesquilinear forms

$$a_k : \mathcal{H}^1_k\times\mathcal{H}^1_k\to\mathbb{C}\quad\text{ for all } k\geq 0,$$

[3]Notice that the abstract setup is in many parts similar to [GS25], but here it is not assumed that the inclusions $\mathcal{H}^n_k\subset\mathcal{H}^m_k$ are compact for $n>m$.

Assumption 5.1 (k-uniform Continuity) *For every $k_0 \in \mathbb{R}$, there exists a positive constant $C_0(k_0) > 0$ such that, for all $k \geq k_0$ and all $u, v \in \mathcal{H}_k^1$,*

$$|a_k(u,v)| \leq C_0(k_0)\|u\|_{\mathcal{H}_k^1}\|v\|_{\mathcal{H}_k^1}.$$

Assumption 5.2 (Gårding inequality) *For every $k_0 \in \mathbb{R}$, there exist positive constants $c_{\mathrm{Ga}}(k_0)$ and $C_{\mathrm{Ga}}(k_0) > 0$ such that, for all $k \geq k_0$,*

$$\Re(a_k(u,u)) \geq c_{\mathrm{Ga}}(k_0)\|u\|^2_{\mathcal{H}_k^1} - C_{\mathrm{Ga}}(k_0)\|u\|^2_{\mathcal{H}} \qquad \text{for all } u \in \mathcal{H}_k^1.$$

Let $\Re a_k$ denote the Hermitian part of a_k, i.e.

$$(\Re a_k)(u,v) := \frac{1}{2}\left(a_k(u,v) + \overline{a_k(v,u)}\right).$$

We fix a closed subspace $\mathcal{Z}_k \subset \mathcal{H}_k^1$ (possibly $\mathcal{H}_k^1$ itself) which is dense in $\mathcal{H}$ with respect to the $\mathcal{H}$ norm, and make the following assumption:

Assumption 5.3 (Domain symmetry) *The spaces*

$$\left\{u \in \mathcal{Z}_k \ : \sup_{v \in \mathcal{Z}_k, \|v\|_{\mathcal{H}}=1} |a_k(u,v)| < +\infty\right\}, \qquad \left\{u \in \mathcal{Z}_k \ : \sup_{v \in \mathcal{Z}_k, \|v\|_{\mathcal{H}}=1} |a_k(v,u)| < +\infty\right\},$$

$$\text{and} \quad \left\{u \in \mathcal{Z}_k \ : \sup_{v \in \mathcal{Z}_k, \|v\|_{\mathcal{H}}=1} \left|(\Re a_k)(u,v)\right| < +\infty\right\}$$

are equal and contained in $\mathcal{H}_k^2$. We denote their common value by $\mathcal{Z}_k^2$.

Remark 5.4 (Boundary conditions) *In practice, the space $\mathcal{Z}_k^2$ will be a subset of $H^2(\Omega)$ with Dirichlet/Neumann conditions on (parts of) $\partial\Omega$. Dirichlet conditions will be enforced "essentially" by the choice of $\mathcal{Z}_k$, and Neumann conditions will appear "naturally" in $\mathcal{Z}_k^2$ as a result of a lack of Dirichlet condition.*

Due to the density of $\mathcal{Z}_k$ in $\mathcal{H}$ and the Riesz representation theorem, this allows to state the following definition

Definition 5.5 (The operators P_k, P_k^* and $\mathcal{P}_k$) *For all $u \in \mathcal{Z}_k^2$, define $P_k u$ and $P_k^* u$ as the unique elements of $\mathcal{H}$ such that for all $v \in \mathcal{Z}_k$*

$$\langle P_k u, v\rangle = a_k(u,v) \text{ and } \langle P_k^* u, v\rangle = \overline{a_k(v,u)}. \tag{5.1}$$

Furthermore, let

$$\mathcal{P}_k u := \frac{1}{2}\left(P_k u + P_k^* u\right).$$

Proposition 5.6 *The space $\mathcal{Z}_k^2$ is dense in $\mathcal{H}$ and $\mathcal{Z}_k$ for their respective norms, i.e.*

$$\overline{\mathcal{Z}_k^2}^{\|\cdot\|_{\mathcal{H}}} = \mathcal{H} \quad \text{and} \quad \overline{\mathcal{Z}_k^2}^{\|\cdot\|_{\mathcal{Z}_k}} = \mathcal{Z}_k$$

Moreover, $\mathcal{P}_k : \mathcal{Z}_k^2 \to \mathcal{H}$ is an unbounded self-adjoint operator. Its spectrum satisfies

$$\sigma(\mathcal{P}_k) \subset [-C_{\mathrm{Ga}}(k_0), +\infty).$$

Proof. The continuity and Gårding inequality, and the fact that $\mathcal{Z}_k$ is a closed subspace of $\mathcal{H}_k^1$ that is dense in $\mathcal{H}$ (for the $\|\cdot\|_{\mathcal{H}}$ norm) imply that the restriction of $\Re a_k$ to $\mathcal{Z}_k$ is a lower semi-bounded closed Hermitian form in the sense of [Sch12, Chap. 10], and $\mathcal{P}_k$ is the operator associated to $\Re a_k$ in the sense of [Sch12, Definition 10.4]. The density of $\mathcal{Z}_k^2$ in $\mathcal{H}$ and the self-adjointness of $\mathcal{P}_k$ then follows from [Sch12, Theorem 10.7]. The density of $\mathcal{Z}_k^2$ in $\mathcal{Z}_k$ is Proposition 10.5(iv) in the same reference. The lower bound on the spectrum is by Proposition 10.4 in the same reference. ■

Proposition 5.7 *$\mathcal{Z}_k^2$ is a Hilbert space under the norm*

$$\|u\|_{\mathcal{Z}_k^2} := \left\|\left(\mathcal{P}_k + (C_{\mathrm{Ga}}(k_0)+1)\,\mathrm{I}\right)u\right\|_{\mathcal{H}}. \tag{5.2}$$

Proof. The operator $A := \big(\mathcal{P}_k + (C_{\rm Ga}(k_0)+1)\,\mathrm{I}\big)u$ is self-adjoint, thus closed (since the adjoint operator is closed by, e.g., [Sch12, Prop. 1.6]), so its graph norm makes $\mathcal{Z}_k^2$ a Hilbert space. Furthermore, its spectrum is contained in $[1,+\infty)$, so that

$$(Au,u)_{\mathcal{H}} \geq \|u\|_{\mathcal{H}}^2 \implies \|Au\|_{\mathcal{H}} \geq \|u\|_{\mathcal{H}}.$$

Thus,

$$\|Au\|_{\mathcal{H}} \leq \|Au\|_{\mathcal{H}} + \|u\|_{\mathcal{H}} \leq 2\|Au\|_{\mathcal{H}},$$

concluding the proof. ∎

We denote the dual of $\mathcal{Z}_k^2$ by $\mathcal{Z}_k^{-2}$, and identify $\mathcal{H}$ and $\mathcal{Z}_k$ as subspaces of $\mathcal{Z}_k^{-2}$ – this identification is possible by the density of $\mathcal{Z}_k^2$ in $\mathcal{H}$. There are then unique linear continuous extensions of the operators P_k, P_k^* and $\mathcal{P}_k$ from $\mathcal{H}$ to $\mathcal{Z}_k^{-2}$ by

$$\langle P_k u, v\rangle := \langle u, P_k^* v\rangle, \quad \langle P_k^* u, v\rangle := \langle u, P_k v\rangle\,, \quad \langle \mathcal{P}_k u, v\rangle := \langle u, \mathcal{P}_k v\rangle,$$

for all $u \in \mathcal{H}$ and $v \in \mathcal{Z}_k^2$. With these definitions, the operator P_k^* is indeed the conjugate adjoint of P_k, as the notation (5.1) suggests.

Remark 5.8 (P_k is not a differential operator) *In the model settings of Table 5.1, P_k is not a differential operator. For instance, in the case of the Neumann Laplacian, although P_k agrees with the differential operator $-k^{-2}\Delta - \mathrm{I}$ on $\mathcal{Z}_k^2$, its extension to $L^2(\Omega)$ differs from it (even when Δ is interpreted in the sense of distributions). Indeed, for $u \in C^\infty(\overline{\Omega}) \subset L^2(\Omega)$, integration by parts reveals that*

$$P_k u = -k^{-2}\Delta u - u + k^{-2}\gamma' \cdot \partial_\nu u$$

where $\gamma : H^\ell(\Omega) \to H^{\ell-1/2}(\partial\Omega)$, $\ell > \frac{1}{2}$, is the trace operator and γ' is its adjoint. In particular, even if $u \in H^n(\Omega)$ for a large n, $P_k u$ is only in $(H^{1/2+\varepsilon}(\Omega))^*$ for all $\varepsilon > 0$, instead of $H^{n-1}(\Omega)$, unless $\partial_\nu u = 0$.

Proposition 5.9 *The operators P_k and P_k^* map $\mathcal{Z}_k$ to $\mathcal{Z}_k^*$ continuously, and they satisfy*

$$\langle P_k u, v\rangle = a_k(u,v), \quad \langle P_k^* u, v\rangle = \overline{a_k(v,u)} \qquad \text{for all } u,v \in \mathcal{Z}_k,$$

$$\max(\|P_k u\|_{\mathcal{Z}_k^*}, \|P_k^* u\|_{\mathcal{Z}_k^*}) \leq C_0(k_0)\|u\|_{\mathcal{Z}_k} \qquad \text{for all } u \in \mathcal{Z}_k.$$

Proof. We observe that for $u \in \mathcal{Z}_k$ and $v \in \mathcal{Z}_k^2$,

$$\langle P_k u, v\rangle = \langle u, P_k^* v\rangle = (u, P_k^* v)_{\mathcal{H}} = \overline{(P_k^* v, u)_{\mathcal{H}}} = \overline{\langle P_k^* v, u\rangle} = \overline{\overline{a_k(u,v)}} = a_k(u,v)$$

and thus

$$|\langle P_k u, v\rangle| \leq C_0(k_0)\|u\|_{\mathcal{Z}_k}\|v\|_{\mathcal{Z}_k}.$$

The conclusion follows by density of $\mathcal{Z}_k^2$ in $\mathcal{Z}_k$ for the $\mathcal{Z}_k$ norm. The reasoning for P_k^* is similar. ∎

Definition 5.10 (The resolvent norm $\rho(k)$) *Given $k \geq 0$, if $P_k : \mathcal{Z}_k \to \mathcal{Z}_k^*$ is invertible, we define*

$$\rho(k) := \sup_{f \in \mathcal{H}\setminus\{0\}} \frac{\|P_k^{-1} f\|_{\mathcal{H}}}{\|f\|_{\mathcal{H}}}. \tag{5.3}$$

Proposition 5.11 *Suppose that $P_k : \mathcal{Z}_k \to \mathcal{Z}_k^*$ is invertible. Then $P_k^* : \mathcal{Z}_k \to \mathcal{Z}_k^*$ is also invertible and for all $k_0 > 0$, there exists $C > 0$ such that for all $k \geq k_0$,*

$$\|P_k^{-1}u\|_{\mathcal{Z}_k} + \|(P_k^*)^{-1}u\|_{\mathcal{Z}_k} \leq C\big(1+\rho(k)\big)\|u\|_{\mathcal{Z}_k^*}.$$

Moreover, for all $z \in \mathbb{C}\setminus\mathbb{R}$, $(\mathcal{P}_k - z) : \mathcal{Z}_k \to \mathcal{Z}_k^*$ is invertible and

$$\|(\mathcal{P}_k - z)^{-1}u\|_{\mathcal{Z}_k} \leq C\frac{\langle z\rangle}{|\Im(z)|}\|u\|_{\mathcal{Z}_k^*}$$

where $\langle z\rangle = (1+|z|^2)^{1/2}$.

Proof. Let $k_0 > 0$ be fixed and denote by C a generic constant depending only on k_0. For $u \in \mathcal{Z}_k^*$, the Gårding inequality gives

$$\begin{aligned}\|P_k^{-1}u\|_{\mathcal{Z}_k}^2 &\leq C\left(\mathrm{Re}(a_k(P_k^{-1}u, P_k^{-1}u)) + \|P_k^{-1}u\|_{\mathcal{H}}^2\right) \\ &\leq C\left(\mathrm{Re}(\langle u, P_k^{-1}u\rangle) + \|P_k^{-1}u\|_{\mathcal{H}}^2\right) \end{aligned} \tag{5.4}$$

Now if, moreover, $u \in \mathcal{H}$, then $\|P_k^{-1}u\|_{\mathcal{Z}_k}^2 \leq C\left(\|u\|_{\mathcal{H}}\|P_k^{-1}u\|_{\mathcal{H}} + \|P_k^{-1}u\|_{\mathcal{H}}^2\right)$ Thus,

$$\|P_k^{-1}u\|_{\mathcal{Z}_k} \leq C(\|u\|_{\mathcal{H}}^2 + \|(P_k^{-1}u)\|_{\mathcal{H}}^2),$$

which implies

$$\|P_k^{-1}\|_{\mathcal{H}\to\mathcal{Z}_k} \leq C\big(1+\rho(k)\big).$$

By the same argument, using that $\rho(k) = \|P_k^{-1}\|_{\mathcal{H}\to\mathcal{H}} = \|(P_k^*)^{-1}\|_{\mathcal{H}\to\mathcal{H}}$ (since $P_k^{-1} : \mathcal{H} \subset \mathcal{Z}_k^* \to \mathcal{Z}_k \subset \mathcal{H}$),

$$\|(P_k^*)^{-1}\|_{\mathcal{H}\to\mathcal{Z}_k} \leq C\big(1+\rho(k)\big).$$

Thus by duality,

$$\|P_k^{-1}\|_{\mathcal{Z}_k^*\to\mathcal{H}} \leq C(1+\rho(k)).$$

Using this in the right-hand side of (5.4) as well as the inequality

$$2ab \leq \epsilon a^2 + \epsilon^{-1}b^2 \quad \text{ for all } a, b, \epsilon > 0, \tag{5.5}$$

we obtain for all $\varepsilon \in (0,1)$ sufficiently small,

$$\|P_k^{-1}u\|_{\mathcal{Z}_k} \leq C\left(\varepsilon\|P_k^{-1}u\|_{\mathcal{Z}_k}^2 + (\varepsilon^{-1}+\rho(k))\|u\|_{\mathcal{Z}_k^*}^2\right)$$

and thus

$$\|P_k^{-1}\|_{\mathcal{Z}_k^*\to\mathcal{Z}_k} \leq C(1+\rho(k)).$$

We obtain the analogous bound for $(P_k^*)^{-1}$ by duality. The proof of the bound $\|(\mathcal{P}_k - z)^{-1}\|_{\mathcal{Z}_k^*\to\mathcal{Z}_k}$ is similar, first estimating $\|(\mathcal{P}_k - z)^{-1}\|_{\mathcal{H}\to\mathcal{Z}_k}$, using that

$$\|(\mathcal{P}_k - z)^{-1}u\|_{\mathcal{H}} \leq \frac{1}{|\Im(z)|}\|u\|_{\mathcal{H}},$$

since $\mathcal{P}_k$ is self-adjoint on $\mathcal{H}$. ∎

Definition 5.12 (The spaces $\mathcal{Z}_k^n$) *Let*

$$\mathcal{Z}_k^n := \begin{cases}\mathcal{H} & \text{if } n = 0\\ \mathcal{Z}_k & \text{if } n = 1\\ \mathcal{Z}_k^2 \cap \mathcal{H}_k^n & \text{if } n \geq 2\end{cases}$$

(recalling for $n = 2$ that $\mathcal{Z}_k^2 \subset \mathcal{H}_k^2$ by Assumption 5.3). For $n \geq 2$, the norm

$$\|u\|_{\mathcal{Z}_k^n}^2 := \|u\|_{\mathcal{Z}_k^2}^2 + \|u\|_{\mathcal{H}_k^n}^2,$$

makes $\mathcal{Z}_k^n$ a Hilbert space. We denote the dual of $\mathcal{Z}_k^n$ by $\mathcal{Z}_k^{-n}$ for all $n \geq 0$.

Assumption 5.13 (Continuity of P_k and P_k^*) *For all $n \in \mathbb{N}$, the operators P_k and P_k^* define continuous maps*

$$P_k, P_k^* : \mathcal{Z}_k^{n+2} \to \mathcal{H}_k^n.$$

For all $k_0 > 0$ and $n \in \mathbb{N}$, there exists $C(k_0) > 0$ such that for all $k \geq k_0$,

$$\|P_k u\|_{\mathcal{H}_k^n} + \|P_k^* u\|_{\mathcal{H}_k^n} \leq C(k_0)\|u\|_{\mathcal{Z}_k^{n+2}}^2.$$

Remark 5.14 *The idea behind Assumption 5.13 is that* P_k *and* P_k^* *can only act on functions satisfying the chosen boundary condition, and they remove this boundary condition as well as decreasing the regularity index by* 2*. Thus, given* $a_k(\cdot,\cdot)$*,* $\mathcal{H}_k, \mathcal{H}_k^1, \mathcal{H}_k^2$ *and* $\mathcal{Z}_k$*, Assumption 5.13 can be considered as constraining the spaces* $\mathcal{H}_k^n$ *for* $n \geq 3$*.*

Because of this, we can extend P_k and P_k^* uniquely into continuous linear maps from $\mathcal{H}_k^{-n}$ to $\mathcal{Z}_k^{-n-2}$ for $n \geq 0$ by setting

$$\begin{cases} \langle P_k u, v\rangle := \langle u, P_k^* v\rangle, \\ \langle P_k^* u, v\rangle := \langle u, P_k v\rangle, \end{cases} \qquad \text{for all } v \in \mathcal{Z}_k^{n+2}.$$

To state more conveniently the mapping properties of P_k and P_k^*, we define the Hilbert spaces

$$\mathcal{W}_k^n := \mathcal{Z}_k^n \cap \mathcal{H}_k^n, \qquad \mathcal{Y}_k^n := \mathcal{Z}_k^n + \mathcal{H}_k^n. \tag{5.6}$$

By the inclusion $\mathcal{Z}_k^n \subset \mathcal{H}_k^n$ for $n \geq 0$ and duality,

$$\mathcal{W}_k^n = \begin{cases} \mathcal{Z}_k^n & \text{if } \ n \geq 0, \\ \mathcal{H}_k^n & \text{if } n \leq 0, \end{cases} \qquad \text{and} \qquad \mathcal{Y}_k^n = \begin{cases} \mathcal{H}_k^n & \text{if } \ n \geq 0, \\ \mathcal{Z}_k^n & \text{if } n \leq 0. \end{cases} \tag{5.7}$$

Thus, since $\mathcal{Z}_k^{-n} = (\mathcal{Z}_k^n)^*$,

$$\mathcal{Y}_k^{-n} = (\mathcal{W}_k^n)^*, \qquad \text{for all } n \in \mathbb{Z}.$$

One may think of $\mathcal{W}_k^n$ (respectively $\mathcal{Y}_k^n$) as the space “with” (respectively “without”) boundary conditions, and the application of P_k “removes” the boundary conditions.

Proposition 5.15 (P_k, P_k^* and $\mathcal{P}_k$ map $\mathcal{W}_k^{n+2}$ to $\mathcal{Y}_k^n$ continuously) *For all* $n \in \mathbb{Z}$ *and* $k_0 > 0$*, there exists* $C > 0$ *such that for all* $k \geq k_0$ *and all* $u \in \mathcal{W}_k^n$*,*

$$\|P_k u\|_{\mathcal{Y}_k^n} + \|P_k^* u\|_{\mathcal{Y}_k^n} + \|\mathcal{P}_k u\|_{\mathcal{Y}_k^n} \leq C\|u\|_{\mathcal{W}_k^{n+2}}$$

Proof. This is Assumption 5.13 for $n \geq 0$ and follows from it by duality for $n \leq -2$. Finally, Proposition 5.9 gives the result for $n = -1$ ■

Assumption 5.16 (Elliptic regularity) *Let* Q *equal either* P_k, P_k^* *or* $\mathcal{P}_k$ *and let* $n \in \mathbb{N}$*. If*

$$u \in \mathcal{H} \quad \text{and} \quad Qu \in \mathcal{H}_k^n,$$

then $u \in \mathcal{Z}_k^{n+2}$*, and for all* $k_0 > 0$ *and* $n \in \mathbb{N}$*, there exists* $C_{\text{ell}}(k_0, n) > 0$ *such that for all* $k \geq k_0$ *and* $u \in \mathcal{Z}_k^2$*,*

$$\|u\|_{\mathcal{Z}_k^{n+2}} \leq C_{\text{ell}}(k_0, n)\Big(\|u\|_{\mathcal{H}} + \|Qu\|_{\mathcal{H}_k^n}\Big).$$

Proposition 5.17 (Norms of P_k^{-1} and $(P_k^*)^{-1}$ from $\mathcal{W}_k^n$ to $\mathcal{Y}_k^{n+2}$) *Suppose that* $P_k : \mathcal{Z}_k \to \mathcal{Z}_k^*$ *is invertible. Then for all* $n \in \mathbb{Z}$*,* $P_k^{-1} : \mathcal{Y}_k^n \to \mathcal{W}_k^{n+2}$ *and* $(P_k^*)^{-1} : \mathcal{Y}_k^n \to \mathcal{W}_k^{n+2}$ *are continuous and for all* $k_0 > 0$*, there exists* $C > 0$ *such that for all* $k \geq k_0$*,*

$$\|P_k^{-1} u\|_{\mathcal{W}_k^{n+2}} + \|(P_k^*)^{-1} u\|_{\mathcal{W}_k^{n+2}} \leq C\big(1 + \rho(k)\big)\|u\|_{\mathcal{Y}_k^n} \tag{5.8}$$

where $\rho(k)$ *is defined by* (5.3)*.*

Proof. The case $n = -1$ is Proposition 5.9. Hence it remains to prove (5.8) for $n \geq 0$, since the case $n \leq -2$ follows by duality. We proceed by induction. First, for $n = 0$, let $u \in \mathcal{H}$. Then $P_k^{-1} u \in \mathcal{H}$ and thus by elliptic regularity, $u \in \mathcal{Z}_k^2$ with

$$\|u\|_{\mathcal{Z}_k^2} \leq C_{\text{ell}}(k_0, 0)(\|P_k^{-1} u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}}) \leq C_{\text{ell}}(k_0, 0)(1 + \rho(k))\|u\|_{\mathcal{H}} \leq C\rho(k)\|u\|_{\mathcal{H}}$$

by definition of $\rho(k)$, where C depends only on k_0. Next, let $n \geq 0$ and suppose that there exists $C > 0$ such that

$$\|P_k^{-1} u\|_{\mathcal{Z}_k^{n+2}} \leq C\rho(k)\|u\|_{\mathcal{H}_k^n}.$$

Let $u \in \mathcal{H}_k^{n+1}$. Then by elliptic regularity and using the continuous embeddings $\mathcal{Z}_k^{n+2} \subset \mathcal{H}$ and $\mathcal{H}_k^{n+1} \subset \mathcal{H}_k^n$,

$$\begin{aligned}\|P_k^{-1}u\|_{\mathcal{Z}_k^{n+3}} &\leq C_\ell(k_0, n+1)(\|P_k^{-1}u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}_k^{n+1}}) \\ &\leq C_\ell(k_0, n+1)(\|P_k^{-1}u\|_{\mathcal{Z}_k^{n+2}} + \|u\|_{\mathcal{H}_k^{n+1}}) \\ &\leq C_\ell(k_0, n+1)(C\rho(k)\|u\|_{\mathcal{H}_k^n} + \|u\|_{\mathcal{H}_k^{n+1}}) \\ &\leq C'\rho(k)\|u\|_{\mathcal{H}_k^{n+1}}\end{aligned}$$

where C' depends only on k_0 and n. ∎

Proposition 5.18 (Resolvent norm from $\mathcal{Y}_k^{n-1}$ to $\mathcal{W}_k^{n+1}$) *Let $k_0 > 0$ and $n \in \mathbb{Z}$. There exists $C(k_0, n) > 0$ such that for all $k \geq k_0$ and all $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$\|(\mathcal{P}_k - z)^{-1}\|_{\mathcal{Y}_k^{n-1} \to \mathcal{W}_k^{n+1}} \leq C(k_0, n) \frac{\langle z \rangle^{1+\lfloor |n|/2 \rfloor}}{|\Im(z)|}.$$

Proof. The result for $n = 0$ is Proposition 5.11. For $n = 1$, we use that

$$\|(\mathcal{P}_k - z)^{-1}u\|_{\mathcal{H}} \leq \frac{\|u\|_{\mathcal{H}}}{|\Im(z)|}.$$

and the elliptic regularity (Assumption 5.16) to write

$$\begin{aligned}\|(\mathcal{P}_k - z)^{-1}u\|_{\mathcal{Z}_k^2} &\leq C(\|\mathcal{P}_k(\mathcal{P}_k - z)^{-1}u\|_{\mathcal{H}} + \|(\mathcal{P}_k - z)^{-1}u\|_{\mathcal{H}}) \\ &\leq C\Big(\|u\|_{\mathcal{H}} + |z| \cdot \|(\mathcal{P}_k - z)^{-1}u\|_{\mathcal{H}} + \|(\mathcal{P}_k - z)^{-1}u\|_{\mathcal{H}}\Big) \\ &\leq C\frac{\langle z \rangle}{|\Im(z)|}\|u\|_{\mathcal{H}}.\end{aligned}$$

Next let $n \geq 0$ and suppose that

$$\|(\mathcal{P}_k - z)^{-1}u\|_{\mathcal{Z}_k^{n+1}} \leq C\frac{\langle z \rangle^m}{|\Im(z)|}\|u\|_{\mathcal{Z}_k^{n-1}}$$

for all $u \in \mathcal{Z}_k^{n-1}$. Then elliptic regularity gives

$$\begin{aligned}\|(\mathcal{P}_k - z)^{-1}u\|_{\mathcal{Z}_k^{n+3}} &\leq C\|(\mathcal{P}_k - z)^{-1}u\|_{\mathcal{H}} + \|\mathcal{P}_k(\mathcal{P}_k - z)^{-1}u\|_{\mathcal{Z}_k^{n+1}} \\ &\leq C\Big(|\Im(z)|^{-1}\|u\|_{\mathcal{H}} + \|u\|_{\mathcal{Z}_k^{n+1}} + |z| \cdot \|(\mathcal{P}_k - z)^{-1}u\|_{\mathcal{Z}_k^{n+1}}\Big) \\ &\leq C\Big(|\Im(z)|^{-1}\|u\|_{\mathcal{H}} + \|u\|_{\mathcal{Z}_k^{n+1}} + \frac{\langle z \rangle^{m+1}}{|\Im(z)|}\|u\|_{\mathcal{Z}_k^{n-1}}\Big) \\ &\leq C\frac{\langle z \rangle^{m+1}}{|\Im(z)|}\|u\|_{\mathcal{Z}_k^{n+1}}.\end{aligned}$$

Thus by induction, for all $n \geq 0$,

$$\|(\mathcal{P}_k - z)^{-1}\|_{\mathcal{Y}_k^{n-1} \to \mathcal{W}_k^{n+1}} \leq \frac{\langle z \rangle^{1+\lfloor |n|/2 \rfloor}}{|\Im(z)|}.$$

The result for $n \leq 0$ follows by duality. ∎

5.2 The spaces $\mathcal{D}_k^s$

Since $\lambda \mapsto (\lambda + C_{\mathrm{Ga}}(k_0) + 1)^{s/2}$, $s \geq 0$, is finite for $\lambda \in \sigma(\mathcal{P}_k)$, the functional calculus of unbounded self-adjoint operators (see, e.g., [Sch12, Section 5.3]) allows us to define the self-adjoint operator

$$\mathcal{X}_{k,s} := (\mathcal{P}_k + (C_{\mathrm{Ga}}(k_0) + 1)\,\mathrm{I})^{s/2} \tag{5.9}$$

with domain

$$\mathcal{D}_k^s := \mathcal{D}(\mathcal{X}_{k,s}) \subset \mathcal{H}$$

where the inclusion is dense in the $\mathcal{H}$ norm. Since the functional calculus is an algebra homomorphism, $\mathcal{X}_{k,s} = \mathcal{X}_k^s$, where $\mathcal{X}_k := \mathcal{X}_{k,1}$. Since $\mathcal{X}_k^s$ is self-adjoint, it is, in particular, a closed operator, so the space $\mathcal{D}_k^s$ is a Hilbert space for the graph norm

$$\|u\|^2_{\mathcal{X}_k^s} := \|u\|^2_{\mathcal{H}} + \|\mathcal{X}_k^s u\|^2_{\mathcal{H}}.$$

Moreover, $\sigma(\mathcal{X}_k^s) \subset [1, +\infty)$, hence

$$\|u\|^2_{\mathcal{H}} \leq (u, \mathcal{X}_k^s u)_{\mathcal{H}} \leq \|u\|_{\mathcal{H}} \|\mathcal{X}_k^s u\|_{\mathcal{H}},$$

so the graph norm associated of $\mathcal{X}_k^s$ is equivalent to the norm

$$\|u\|^2_{\mathcal{D}_k^s} := \|\mathcal{X}_k^s u\|^2_{\mathcal{H}}. \tag{5.10}$$

This way, the operator $\mathcal{X}_k^t$ induces an isometry from $\mathcal{D}_k^s$ to $\mathcal{D}_k^{s-t}$ for all $s \geq t \geq 0$.

Proposition 5.19 *$\mathcal{Z}_k^2 = \mathcal{D}_k^2$ with equal norms. Furthermore $\mathcal{Z}_k = \mathcal{D}_k^1$ with equivalent norms; more precisely, for all $k_0 > 0$, there exist constant $C(k_0) > 0$ such that for all $k \geq 0$ and for all $u \in \mathcal{D}^2$*

$$\frac{1}{C(k_0)} \|u\|_{\mathcal{Z}_k} \leq \|u\|^2_{\mathcal{D}_k^1} \leq C(k_0) \|u\|_{\mathcal{Z}_k}.$$

Proof. The first statement follows from the fact that $\mathcal{X}_k^2$ and $\mathcal{P}_k$ differ by a multiple of identity, and by the definition of the norm of $\mathcal{Z}_k^2$ (compare (5.2) with the combination of (5.9) and (5.10)). On the other hand, $\mathcal{X}_k^2$ is the operator associated to the lower semi-bounded form $a_k^+ : \mathcal{Z}_k \times \mathcal{Z}_k \to \mathbb{C}$ defined by

$$a_k^+(u, v) := \Re a_k(u, v) + (C_{\mathrm{Ga}}(k_0) + 1)(u, v)_{\mathcal{H}}$$

in the sense of [Sch12, Definition 10.4]. In particular, by Theorem 10.7 and Proposition 10.5 in the latter reference,

$$\mathcal{Z}_k = \mathcal{D}(|\mathcal{X}_k^2|^{1/2}) = \mathcal{D}(\mathcal{X}_k) = \mathcal{D}_k^1.$$

The equivalence of the norms follows from the continuity of $\Re a_k$ and the Gårding inequality. ∎

Corollary 5.20 *For all $n \in \mathbb{N}$, $\mathcal{D}_k^n \subset \mathcal{Z}_k^n$ and the embedding is continuous.*

Proof. The result is immediate if $n = 0$ and is Proposition 5.19 above for $n = 1, 2$. Finally, if $u \in \mathcal{D}_k^{n+2}$, then

$$\mathcal{P}_k u = (\mathcal{X}_k^2 - C_{\mathrm{Ga}}(k_0) + 1)u \in \mathcal{D}_k^n$$

so the result follows by induction using elliptic regularity (Assumption 5.16). ∎

We also define $\mathcal{D}_k^{-s} := (\mathcal{D}_k^s)^*$. Since $\mathcal{D}_k^s$ is dense in $\mathcal{H}_k$ for $s \geq 0$, $\mathcal{H}$ can be identified as a subspace of $\mathcal{D}_k^{-s}$, so that $\mathcal{D}_k^t \subset \mathcal{D}_k^s$ for all real $s \leq t$. We can extend $\mathcal{X}_k$ uniquely into a linear map from $\mathcal{D}_k^s$ to $\mathcal{D}_k^{s-1}$ for all $s \in [0, 1]$ by putting

$$\langle \mathcal{X}_k u, v \rangle := \langle \mathcal{X}_k^s u, \mathcal{X}_k^{1-s} v \rangle \quad \text{for all } (u, v) \in \mathcal{D}_k^s \times \mathcal{D}_k^{1-s}.$$

This way, $\mathcal{X}_k : \mathcal{D}_k^s \to \mathcal{D}_k^{s-1}$ for all $s \in [0, +\infty)$ is an isometry and this is extended to $s \leq 0$ by duality. In turn, this allows us to view $\mathcal{P}_k$ as a map $\mathcal{P}_k : \mathcal{D}_k^s \to \mathcal{D}_k^{s-2}$ for all $s \in \mathbb{R}$ by $\mathcal{P}_k := \mathcal{X}_k^2 - (C_{\mathrm{Ga}}(k_0) + 1)\,\mathrm{I}$, with

$$\|\mathcal{P}_k u\|_{\mathcal{D}_k^s} \leq (C_{\mathrm{Ga}}(k_0) + 1) \|u\|_{\mathcal{D}_k^{s+2}}.$$

Proposition 5.21 (Resolvent estimates in the $(\mathcal{D}_k^s)$ scale) *Let $k_0 > 0$ and $s \in \mathbb{R}$. There exists $C > 0$ such that for all $k > 0$ and all $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$\|(\mathcal{P}_k - z)^{-1}\|_{\mathcal{D}_k^s \to \mathcal{D}_k^s} \leq |\Im(z)|^{-1},$$

$$\|(\mathcal{P}_k - z)^{-1}\|_{\mathcal{D}_k^s \to \mathcal{D}_k^{s+2}} \leq C\langle z \rangle |\Im(z)|^{-1},$$

where $\langle z \rangle := 1 + |z|$.

Proof. Using the fact that $\mathcal{X}_k^s : \mathcal{D}_k^s \to \mathcal{H}$ is an isometry, and functional calculus,

$$\|(\mathcal{P}_k - z)^{-1}\|_{\mathcal{D}_k^s \to \mathcal{D}_k^s} \leq \|\mathcal{X}_k^s(\mathcal{P}_k - z)^{-1}\mathcal{X}_k^{-s}\|_{\mathcal{H}\to\mathcal{H}} = \|(\mathcal{P}_k - z)^{-1}\|_{\mathcal{H}\to\mathcal{H}} \leq |\Im(z)|^{-1}.$$

Similarly,

$$\|(\mathcal{P}_k - z)^{-1}\|_{\mathcal{D}_k^s \to \mathcal{D}_k^{s+2}} = \|\mathcal{X}_k^{s+2}\big(\mathcal{X}_k^2 - (C_{\mathrm{Ga}} + z)\big)^{-1}\mathcal{X}_k^{-s}\|_{\mathcal{H}\to\mathcal{H}} = \|g(\mathcal{X}_k^2)\|_{\mathcal{H}\to\mathcal{H}} \leq \sup_{x\in\mathbb{R}} |g(x)|$$

where $g(x) := \frac{x}{x-(z+C_{\mathrm{Ga}})}$. Since for all $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\sup_{x\in\mathbb{R}} \left|\frac{x}{x-z}\right| = \frac{|z|}{|\Im(z)|},$$

we conclude that

$$\sup_{x\in\mathbb{R}} |g(x)| \leq (1 + C_{\mathrm{Ga}})\langle z\rangle |\Im(z)|^{-1},$$

completing the proof. ∎

Proposition 5.22 (Functions of $\mathcal{P}_k$) *Let $k_0 > 0$ and $s \geq 0$. There exists $C > 0$ such that, for all $k \geq k_0$ and for any function $f : \mathbb{R} \to \mathbb{C}$ satisfying*

$$\|f\|_{\infty,s} := \sup_{x\in\mathbb{R}} (1 + |x|^s)|f(x)| < \infty,$$

the operator $f(\mathcal{P}_k) : \mathcal{H} \to \mathcal{H}$ defined by the functional calculus extends uniquely into a continuous map from $\mathcal{D}_k^{-s}$ to $\mathcal{D}_k^s$, with

$$\|f(\mathcal{P}_k)\|_{\mathcal{D}_k^{-s}\to\mathcal{D}_k^s} \leq C\|f\|_{\infty,s}.$$

In particular (by Corollary 5.20 and the definitions of $\mathcal{Y}_k^n$ and $\mathcal{W}_k^n$ (5.6)) for any $n \in \mathbb{N}$, $f(\mathcal{P}_k) : \mathcal{Y}_k^{-n} \to \mathcal{W}_k^n$ is continuous.

Proof. By functional calculus and using that $\mathcal{X}_k^t = (\mathcal{P}_k + C_{\mathrm{Ga}}(k_0) + 1)^{t/2} : \mathcal{D}_k^t \to \mathcal{H}$ is an isometry for all $t \in \mathbb{R}$,

$$\|f(\mathcal{P}_k)\|_{\mathcal{D}_k^{-s}\to\mathcal{D}_k^s} \leq \|(\mathcal{P}_k + C_{\mathrm{Ga}}(k_0) + 1)^{s/2} f(\mathcal{P}_k)(\mathcal{P}_k + C_{\mathrm{Ga}}(k_0) + 1)^{s/2}\|_{\mathcal{H}\to\mathcal{H}} = \|g(\mathcal{P}_k)\|_{\mathcal{H}\to\mathcal{H}}$$

where $g(x) = (C_{\mathrm{Ga}} + 1 + x)^s f(x)$ satisfies

$$|g(x)| \leq 2^s(C_{\mathrm{Ga}} + 1)^s(1 + |x|^s)|f(x)|$$

for all $x \in \sigma(\mathcal{P}_k)$. Hence, $\|g(\mathcal{P}_k)\|_{\mathcal{H}\to\mathcal{H}} \leq C\|f\|_{\infty,s}$ and the claim follows. ∎

5.3 Elliptic perturbation of P_k

By Proposition 5.6, for every $k_0 > 0$, there exists a real-valued, compactly supported function $\psi^\sharp \in C_c^\infty(\mathbb{R})$ such that

$$\psi^\sharp(x) \geq \frac{-x + C_{\mathrm{Ga}}}{2} \quad \text{for all } x \in \sigma(\mathcal{P}_k). \tag{5.11}$$

Following [GS25, Lemma 2.1], define

$$S_k := \psi^\sharp(\mathcal{P}_k) \tag{5.12}$$

by the functional calculus. Since $\psi^\sharp$ has compact support,

$$S_k : \mathcal{D}_k^{-n} \to \mathcal{D}_k^n$$

is continuous for all $n \in \mathbb{N}$ by Proposition 5.22. In what follows, the *elliptic perturbation* of P_k is defined by

$$P_k^\sharp := P_k + S_k. \tag{5.13}$$

The associated sesquilinear form, denoted by $a_k^\sharp : \mathcal{Z}_k \times \mathcal{Z}_k \to \mathbb{C}$, is thus given by

$$a_k^\sharp(u, v) := a_k(u, v) + (S_k u, v)_{\mathcal{H}}. \tag{5.14}$$

Proposition 5.23 (Properties of $P_k^\sharp$) *For every $k_0 > 0$ and any integer n, there exists a positive constant $C_\sharp(k_0, n)$ such that, for all $k \geq k_0$,*

$$\Re(a_k^\sharp(u, u)) \geq C_\sharp(k_0, 1)\|u\|^2_{\mathcal{Z}_k} \quad \textit{for all } u \in \mathcal{Z}_k, \tag{5.15}$$

the operator $P_k^\sharp : \mathcal{W}_k^{n+2} \to \mathcal{Y}_k^n$ is an isomorphism, and

$$\|(P_k^\sharp)^{-1}u\|_{\mathcal{W}_k^{n+2}} \leq C_\sharp(k_0, n)\|u\|_{\mathcal{Y}_k^n} \quad \textit{for all } u \in \mathcal{Y}_k^n.$$

Proof. By (5.11),

$$x + \psi^\sharp(x) \geq \frac{1}{2}(x + C_{\rm Ga}) \quad \text{for all } x \in \sigma(\mathcal{P}_k).$$

Therefore, by the functional calculus, for all $u \in \mathcal{Z}_k^1$,

$$\Re(a_k^\sharp(u, u)) = \Re a_k(u, u) + (\psi^\sharp(\mathcal{P}_k)u, u)_{\mathcal{H}} = \big(\mathcal{P}_k + \psi^\sharp(\mathcal{P}_k)u, u\big)_{\mathcal{H}} \geq \frac{1}{2}\big((\mathcal{P}_k + C_{\rm Ga}\,\mathrm{I})u, u\big)_{\mathcal{H}}.$$

Hence, by the Gårding inequality,

$$\Re(a_k^\sharp(u, u)) \geq \frac{c_{\rm Ga}}{2}\|u\|^2_{\mathcal{Z}_k} \quad \text{for all } u \in \mathcal{Z}_k^2,$$

and the same inequality holds for all $u \in \mathcal{Z}_k$ by the density of $\mathcal{Z}_k^2$ in $\mathcal{Z}_k$ and continuity of $a_k^\sharp$. Thus, $a_k^\sharp$ is coercive, and the Lax-Milgram lemma implies that $P_k^\sharp : \mathcal{Z}_k \to (\mathcal{Z}_k)^*$ is boundedly invertible; this is the required result for $n = 1$. With $n \geq 2$, let $u \in \mathcal{H}_k^{n-2}$ and suppose that $v \in \mathcal{Z}_k$ satisfies $P_k^\sharp v = u$. Then $P_k v = u - S_k v \in \mathcal{H}_k^{n-2}$ (by the smoothing property of S_k from Proposition 5.22), so that $v \in \mathcal{Z}_k^n$ by elliptic regularity (Assumption 5.16). Moreover, since $\|v\|_{\mathcal{H}} \leq \|v\|_{\mathcal{Z}_k} \leq \|u\|_{\mathcal{Z}_k^*}$ (again by the Lax-Milgram lemma),

$$\|v\|_{\mathcal{Z}_k^n} \leq C\big(\|v\|_{\mathcal{H}} + \|u - S_k v\|_{\mathcal{H}_k^{n-2}}\big) \leq C\big(\|u\|_{(\mathcal{Z}_k)^*} + \|u\|_{\mathcal{H}_k^{n-2}}\big) \leq C\|u\|_{\mathcal{H}_k^{n-2}},$$

which proves the result for $n \geq 2$. The same reasoning applied to $P_k^* + S_k^* = P_k^* + S_k$ followed by a duality argument (recalling that the dual of $\mathcal{Y}_k^n$ is $\mathcal{W}_k^{-n}$) gives the result for $n \leq 0$. ■

5.4 Order notation

Let

$$\mathcal{W}_k^\infty := \bigcap_{n\in\mathbb{Z}} \mathcal{W}_k^n, \quad \mathcal{W}_k^{-\infty} := \bigcup_{n\in\mathbb{Z}} \mathcal{W}_k^n,$$

and define $\mathcal{Y}_k^{\pm\infty}$ and $\mathcal{D}_k^{\pm\infty}$ similarly.

Definition 5.24 (Order notation) *Let $(\eta_k)_{k>0}$ be a family of real numbers. Let $m, n \in \mathbb{Z}$ and let $L : \mathcal{W}_k^\infty \to \mathcal{Y}_k^\infty$ be a linear operator. Then*

$$L = O_m(\eta^n; \mathcal{W}_k \to \mathcal{Y}_k)$$

if, for all $k_0 > 0$ and for all $j \in \mathbb{Z}$, there exists a real number $C(k_0, j) > 0$ such that for all $k \geq k_0$ and all $u \in \mathcal{W}_k^j$,

$$\|Lu\|_{\mathcal{Y}_k^{j-m}} \leq C(k_0, j)\eta_k^n\|u\|_{\mathcal{W}_k^j}. \tag{5.16}$$

The notations $L = O_m(\eta^n; \mathcal{Y}_k \to \mathcal{Y}_k)$, $L = O_m(\eta^n; \mathcal{D}_k \to \mathcal{D}_k)$ are defined similarly.

Observe that these order relations can then be combined multiplicatively; e.g.,

$$L_1 = O_m(\eta^n; \mathcal{W}_k \to \mathcal{Y}_k)\,,\ L_2 = O_{m'}(\eta^{n'}; \mathcal{Y}_k \to \mathcal{W}_k) \implies L_1L_2 = O_{m+m'}(\eta^{n+n'}; \mathcal{Y}_k \to \mathcal{Y}_k).$$

5.5 Spatial pseudolocality

The statement and proof of the main result use both frequency cut-offs in the form $f(\mathcal{P}_k)$ for $f \in \mathcal{S}(\mathbb{R})$ and spatial cut-offs coming from smooth compactly-supported functions. The following assumption encapsulates the properties of these spatial cut-offs that are required in this section.

Definition 5.25 (Abstract "spatial cutoffs") *Let $(\eta_k)_{k>0}$ be a family of real numbers. We say that a linear operator is a* spatial cutoff *of order m* with parameter *η if $R = O_m(1; \mathcal{Y}_k \to \mathcal{Y}_k)$,*

$$\mathrm{ad}_R^N Q = O_{m-N+2}(\eta^{-N}; \mathcal{W}_k \to \mathcal{Y}_k), \quad \text{and} \quad \mathrm{ad}_{R^*}^N Q = O_{m-N+2}(\eta^{-N}; \mathcal{W}_k \to \mathcal{Y}_k),$$

where Q is any one of the operators P_k, P_k^* and $\mathcal{P}_k$. The set of spatial cutoffs of order m and parameter is denoted by $\mathcal{L}_{\mathrm{sc}}^m(\eta)$, and we write $\mathcal{L}_{\mathrm{sc}}(\eta) := \mathcal{L}_{\mathrm{sc}}^0(\eta)$. We omit the η from the notation when it will not lead to confusion.

Remark 5.26 *Recall from Remark 5.8 that in the model settings of Table 5.1, the operator Q above is not a differential operator. Therefore the commutators $\mathrm{ad}_R^N Q$ and $\mathrm{ad}_{R^*}^N Q$ a priori contain boundary terms, hence some care must be taken to check the continuity properties above. In §6, we show that if R is given by the multiplication with a smooth cut-off function χ* with vanishing normal derivative on $\partial\Omega_-$, *then it satisfies the commutator estimates above.*

Let $A, B = O_0(1, \mathcal{W}_k \to \mathcal{W}_k) \cap O_0(1, \mathcal{Y}_k \to \mathcal{Y}_k)$, where the intersection notation is used to denote that the equation holds with either term on the right-hand side, and and let $R \in \mathcal{L}_{\mathrm{sc}}$. We say that A and B are *separated by R* if both

$$A(\mathrm{I} - R) = O_0(\eta^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k) \cap O_0(\eta^{-\infty}; \mathcal{W}_k \to \mathcal{W}_k),$$

$$RB = O_0(\eta^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k) \cap O_0(\eta^{-\infty}; \mathcal{W}_k \to \mathcal{W}_k),$$

We say that A and B are *separated* if they are separated by R for some $R \in \mathcal{L}_{\mathrm{sc}}$.

The main result on spatial pseudolocality is as follows.

Theorem 5.27 (Pseudolocality of abstract Helmholtz operators) *Let $f \in \mathcal{S}(\mathbb{R})$, let A, B be separated, and let Q be one of the operators P_k, P_k^* or $\mathcal{P}_k$. Then*

$$Af(\mathcal{P}_k)B = O_{-\infty}(\eta^{-\infty}; \mathcal{Y}_k \to \mathcal{W}_k), \tag{5.17}$$

$$AQB = O_2(\eta^{-\infty}; \mathcal{W}_k \to \mathcal{Y}_k) \tag{5.18}$$

$$A(P_k^\sharp)^{-1}B = O_{-2}(\eta^{-\infty}; \mathcal{Y}_k \to \mathcal{W}_k). \tag{5.19}$$

Remark 5.28 (The constants implicit in Theorem 5.27) *Both the assumptions and the conclusions of Theorem 5.27 involve implicit constants coming from the order notation Definition 5.24 used to denote bounds of the form (5.16). It is clear from the proof of Theorem 5.27 (but cumbersome to write precisely) that given $f \in \mathcal{S}(\mathbb{R})$ and a list of constants $\mathscr{C}$, there is another list of constants $\mathscr{C}'$ such that for all $(\eta_k)_{k>0}$, A, B and R satisfying the assumptions of the theorem with the constants $\mathscr{C}$, the conclusions of Theorem 5.27 hold with the constants $\mathscr{C}'$. The same is true of all results in the remainder of this section.*

We first reduce the proof of Theorem 5.27 to the proof of various commutator estimates.

Proposition 5.29 *Suppose that for all $R \in \mathcal{L}_{\mathrm{sc}}$, for all $f \in \mathcal{S}(\mathbb{R})$, and for every $N \in \mathbb{N}$,*

$$\mathrm{ad}_R^N f(\mathcal{P}_k) = O_{-\infty}(\eta^{-N}; \mathcal{Y}_k \to \mathcal{W}_k), \tag{5.20}$$

$$\mathrm{ad}_R^N (P_k^\sharp)^{-1} = O_{-2}(\eta^{-N}; \mathcal{Y}_k \to \mathcal{W}_k). \tag{5.21}$$

Then the results of Theorem 5.27 hold.

Proof. The structure of the argument is the same for all three results, albeit with different spaces. We therefore omit the spaces from the notation for brevity.

Since A and B are assumed to be separated, there exists $R \in \mathcal{L}_{\rm sc}$ such that $A(\mathrm{I}-R) = O_0(\eta^{-\infty})$ and $RB = O_0(\eta^{-\infty})$. Hence, for any operator X such that $X = O_m(1)$, since $A, B \in \mathcal{L}_{\rm sc} \subset \mathcal{L}^0$,

$$\begin{aligned} AXB &= ARXB + A(\mathrm{I}-R)XB \\ &= A(\mathrm{ad}_R X)B + AXRB + A(\mathrm{I}-R)XB \\ &= A(\mathrm{ad}_R X)B + O_m(\eta^{-\infty}). \end{aligned}$$

Repeating this argument N times gives

$$AXB = A(\mathrm{ad}_R^N X)B + O_m(\eta^{-\infty}), \tag{5.22}$$

since $\mathrm{ad}_R^i X = O_m(1)$ for any $i \in \mathbb{N}$ (as can be easily checked by induction). By (5.22) applied with $X = f(\mathcal{P}_k)$ (and $m = -\infty$ by Proposition 5.22), $X = Q$ (and $m = 2$), and $X = (P_k^\sharp)^{-1}$ (and $m = -2$ by Proposition 5.23), for all $N \in \mathbb{N}$,

$$Af(\mathcal{P}_k)B = A(\mathrm{ad}_R^N f(\mathcal{P}_k))B + O_{-\infty}(\eta^{-\infty})$$

$$AQB = A(\mathrm{ad}_R^N Q)B + O_2(\eta^{-\infty})$$

$$A(P_k^\sharp)^{-1}B = A(\mathrm{ad}_R^N (P_k^\sharp)^{-1})B + O_{-2}(\eta^{-\infty}).$$

Hence (5.18) follows immediately from the middle equality and Definition 5.25, while the first and third equality show that (5.17) and (5.19) follow from (5.20) and (5.21), respectively.

■

We now prove that the assumptions (5.20)-(5.21) of Proposition 5.29 hold true. In order to do this, the main tool the Helffer-Sjöstrand formula, which allows to express $f(\mathcal{P}_k)$ in terms of $(\mathcal{P}_k - z)^{-1}$. This formula is recalled below (for a proof, see, e.g., [Zwo12, Theorem 14.8]).

Proposition 5.30 (Helffer-Sjöstrand formula) *For all $f \in \mathcal{S}(\mathbb{R})$, there exists a continuous function $w : \mathbb{C} \to \mathbb{C}$ such that if $\mathcal{A}$ is a self-adjoint operator on a Hilbert space, then*

$$f(\mathcal{A}) = \int_{\mathbb{C}} w(z)(\mathcal{A} - z)^{-1} dm_{\mathbb{C}}(z)$$

where $dm_{\mathbb{C}}(x+iy) = dx\,dy$ and for every $M \in \mathbb{N}$, there exists κ_M such that

$$|w(z)| \leq \kappa_M \langle z\rangle^{-2M} |\Im(z)|^M \qquad \text{for all } z \in \mathbb{C}. \tag{5.23}$$

The function w in Proposition 5.30 is obtained (up to a constant factor) via a so-called "quasi-analytic extension" of f; see, e.g., [Zwo12, Theorem 3.6].

By the Helffer-Sjöstrand formula,

$$\mathrm{ad}_R^N f(\mathcal{P}_k) = \int_{\mathbb{C}} w(z)\, \mathrm{ad}_R^N (\mathcal{P}_k - z)^{-1} dm_{\mathbb{C}}(z)$$

Therefore, to prove (5.20), (5.21) we need to bound $\mathrm{ad}_R^N Y^{-1}$ with $Y = P_k^\sharp$ or $Y = (\mathcal{P}_k - z)^{-1}$, in terms of $\mathrm{ad}_R^N Y$ and Y^{-1}.

The next result uses the notation that $A = O_m\big(f(\eta, n, z); \mathcal{Y}_k \to \mathcal{W}_k\big)$ if $\|Au\|_{\mathcal{W}_k^{n-m}} \leq f(\eta_k, n, z)\|u\|_{\mathcal{Y}_k^n}$.

Proposition 5.31 *Let Ω a subset of $\mathbb{C}$. Suppose that $X = O_m(1; \mathcal{Y}_k \to \mathcal{Y}_k) \cap O_m(1; \mathcal{W}_k \to \mathcal{W}_k)$ and for every $z \in \Omega$, let $Y_z, Y_z^* : \mathcal{W}_k^{n+2} \to \mathcal{Y}_k^n$ be invertible. Furthermore, suppose that there are $L_n \geq 0$ such that*

(a) for all $z \in \Omega$,

$$Y_z^{-1} = C_1(z) O_{-2}\big(\langle z\rangle^{L_n}; \mathcal{Y}_k \to \mathcal{W}_k\big), \quad (Y_z^*)^{-1} = C_1(z) O_{-2}\big(\langle z\rangle^{L_n}; \mathcal{Y}_k \to \mathcal{W}_k\big)$$

(b) for all $z \in \Omega$,

$$\mathrm{ad}_X^N Y_z = C_2(z) O_{2+N(m-1)}\big(\eta^{-N}; \mathcal{W}_k \to \mathcal{Y}_k\big), \quad \mathrm{ad}_{X^*}^N Y_z^* = C_2(z) O_{2+N(m-1)}\big(\eta^{-N}; \mathcal{W}_k \to \mathcal{Y}_k\big)$$

for some functions $C_1, C_2 : \Omega \to \mathbb{R}_+$. Then for all $N \in \mathbb{N}$, $n \in \mathbb{Z}$, there is M_n such that, for all $z \in \Omega$,

$$\operatorname{ad}_X^N Y_z^{-1} = (1 + C_1(z))^{N+1}(1 + C_2(z))^N O_{-2+N(m-1)}\big(\eta^{-N}\langle z\rangle^{M_n}; \mathcal{Y}_k \to \mathcal{W}_k\big).$$

Proof. The main idea is that $\operatorname{ad}_X^N Y_z^{-1}$ is equal to a linear combination of terms of the form $Y_z^{-1}(\operatorname{ad}_X^{i_1} Y_z)Y_z^{-1}(\operatorname{ad}_X^{i_2} Y_z)Y_z^{-1} \dots Y_z^{-1}(\operatorname{ad}_X^{i_M} Y_z)Y_z^{-1}$, and the next definitions formalize this more precisely.[4]

We will prove the lemma by showing the estimate for $\operatorname{ad}_X^N Y_z^{-1}$ acting on elements of $\mathcal{H}$ and then (using the second parts of assumptions (a) and (b)) argue by duality to act on $\mathcal{H}_k^{-n}$.

An operator $a_N : \mathcal{H} \to \mathcal{H}$ is called an (N, z)-*atom* if either

(i) $N = 0$ and $a_N = 1$, or

(ii) $a_N = (\operatorname{ad}_X^N Y_z)Y_z^{-1}$, or

(iii) $a_N = a_i a_j$ where a_i is an (i, z)-atom and a_j is an (j, z)-atom with $i+j = N$ and $1 \le i, j \le N-1$.

An operator t_N is called an (N, z)-*term* if it is of the form

$$t_N = \sum_{j=1}^{J} \sigma_j Y_z^{-1} a_{N,j}$$

where $J \in \mathbb{N}$, σ_j are real coefficients and $a_{N,j}$ are (N, z)-atoms. For example,

$$t_5 = Y_z^{-1}(\operatorname{ad}_X^5 Y_z)Y_z^{-1} - Y_z^{-1}(\operatorname{ad}_X^2 Y_z)Y_z^{-1}(\operatorname{ad}_X^3 Y_z)Y_z^{-1}$$

is a $(5, z)$-term. Notice that if t_i and t_j are (i, z)- and (j, z)-terms, then $t_i Y_z t_j$ is an $(i + j, z)$-term.

It follows immediately from assumptions (a) and (b), by induction on N, that if $t_N(z)$ is a (N, z)-term for all $z \in \Omega$, then

$$t_N(z) = (1 + C_1(z))^{N+1}(1 + C_2(z))^N O_{-2+N(m-1)}(\eta^{-N}\langle z\rangle^{M_n}; \mathcal{Y}_k \to \mathcal{W}_k).$$

Thus it remains to show that for all $z \in \Omega$, $\operatorname{ad}_X^N Y_z^{-1} : \mathcal{H} \to \mathcal{H}$ is an (N, z)-term. For this, it suffices to prove that, for all $N \in \mathbb{N}$,

$$t_N \text{ is an } (N, z)\text{-term} \quad \Longrightarrow \quad \operatorname{ad}_X t_N \text{ is an } (N + 1, z)\text{-term}. \tag{5.24}$$

By linearity, it is enough to prove (5.24) in the case where $t_N = Y_z^{-1} a_N$ for some (N, z)-atom a_N. We consider separately the three cases (i), (ii), (iii) above in the definition of an (N, z)-atom.

Case (i): If $a_N = 1$, then

$$\operatorname{ad}_X t_N = \operatorname{ad}_X Y_z^{-1} = XY_z^{-1} - Y_z^{-1}X = Y_z^{-1}Y_z XY_z^{-1} - Y_z^{-1}XY_z Y_z^{-1} = -Y_z^{-1}(\operatorname{ad}_X Y_z)Y_z^{-1}$$

which is a $(1, z)$-term acting on u. This shows the implication (5.24) for $N = 0$, and in the following cases, we fix $N \ge 1$ and proceed by induction assuming that it holds for all $i \le N - 1$.

Case (ii): If $a_N = (\operatorname{ad}_X^N Y_z)Y_z^{-1}$, then

$$\operatorname{ad}_X t_N = \quad (\operatorname{ad}_X Y_z^{-1})(\operatorname{ad}_X^N Y_z)Y_z^{-1} + Y_z^{-1}(\operatorname{ad}_X^{N+1} Y_z)Y_z^{-1} + Y_z^{-1}(\operatorname{ad}_X^N Y_z)(\operatorname{ad}_X Y_z^{-1}).$$

The second term on the right-hand side is an $(N+1, z)$-term. The first term on the right-hand side can be rewritten as

$$-Y_z^{-1}\underbrace{(\operatorname{ad}_X Y_z)Y_z^{-1}}_{(1,z)\text{-atom}}\underbrace{(\operatorname{ad}_X^N Y_z)Y_z^{-1}}_{(N,z)\text{-atom}}.$$

[4] It is in fact possible to give a full closed-form expression for $\operatorname{ad}_X^N Y^{-1}$ involving sums of compositions of quantities of the form $(\operatorname{ad}_X^i Y)$ and Y^{-1}. However, the fomula and its proof, involving sums over all possible ordered partitions of $\{1, \dots, N\}$, are slightly cumbersome and for the present purposes, this would be more information than actually needed.

This is thus an $(N+1,z)$-term. Similarly, the third term is an $(N+1,z)$-term, and thus $\operatorname{ad}_X t_N$ is an $(N+1,z)$-term.

Case (iii): If $a_N = a_i a_j$ then, since $a_j : \mathcal{H} \to \mathcal{H}$,

$$t_N = Y_z^{-1} a_i a_j = Y_z^{-1} a_i Y_z Y_z^{-1} a_j = t_i Y_z t_j$$

where $t_i := Y_z^{-1} a_i$ and $t_j := Y_z^{-1} a_j$ are (i,z)- and (j,z)-terms, respectively, with $i+j=n$. Thus

$$\operatorname{ad}_X t_N = (\operatorname{ad}_X t_i) Y_z t_j + t_i (\operatorname{ad}_X Y_z) t_j + t_i Y_z (\operatorname{ad}_X t_j).$$

The first term is an $(N+1,z)$-term by the induction hypothesis. Similarly, the last term is an $(N+1,z)$-term. The middle term can be rewritten as

$$t_i (\operatorname{ad}_X Y_z) t_j = \overbrace{t_i Y_z \underbrace{\underbrace{Y_z^{-1} (\operatorname{ad}_X Y_z) Y_z^{-1}}_{(1,z)\text{-term}} Y_z t_j}_{(j+1,z)\text{-term}}}^{(i+(j+1),z)\text{-term}}$$

which is an $(N+1,z)$-term. This concludes the proof. ∎

We can now prove the estimate (5.20).

Proposition 5.32 *For any $N \in \mathbb{N}$, $f \in \mathcal{S}(\mathbb{R})$ and $R \in \mathcal{L}^m_{\mathrm{sc}}$,*

$$\operatorname{ad}_R^N f(\mathcal{P}_k) = O_{-\infty}(\eta^{-N}; \mathcal{Y}_k \to \mathcal{W}_k).$$

Proof. By the Helffer–Sjöstrand formula,

$$\operatorname{ad}_R^N f(\mathcal{P}_k) = \int_{\mathbb{C}} w(z) \operatorname{ad}_R^N (\mathcal{P}_k - z)^{-1} \, dm_{\mathbb{C}}(z).$$

By Proposition 5.31 with $X = R$, $\Omega = \mathbb{C} \setminus \mathbb{R}$, $Y_z = (\mathcal{P}_k - z)$, the first commutator property of spatial cutoffs, and the resolvent estimate of Proposition 5.18,

$$\operatorname{ad}_R^N (\mathcal{P}_k - z)^{-1} = \left(1 + \frac{\langle z \rangle}{|\Im(z)|}\right)^N O_{-2+N(m-1)}(\eta^{-N} \langle z \rangle^{M_n}; \mathcal{Y}_k \to \mathcal{W}_k).$$

Therefore,

$$\operatorname{ad}_R^N f(\mathcal{P}_k) = O_{-2+N(m-1)}\left(\eta^{-N} \int_{\mathbb{C}^N} w(z) \langle z \rangle^{M_n} \left(1 + \frac{\langle z \rangle}{|\Im(z)|}\right)^N dm_{\mathbb{C}}(z); \mathcal{Y}_k \to \mathcal{W}_k\right).$$

The bound (5.23) on w implies that the integral is finite, and thus, for all $f \in \mathcal{S}(\mathbb{R})$,

$$\operatorname{ad}_R^N f(\mathcal{P}_k) = O_{-2+N(m-1)}(\eta^{-N}; \mathcal{Y}_k \to \mathcal{W}_k). \tag{5.25}$$

We now upgrade the regularity index from $-2+N(m-1)$ to $-\infty$ by induction on N. For $N=0$,

$$\operatorname{ad}_R^0 f(\mathcal{P}_k) = f(\mathcal{P}_k) = O_{-\infty}(1; \mathcal{Y}_k \to \mathcal{W}_k) \tag{5.26}$$

by Proposition 5.22. Next fix an integer $N \geq 1$ and suppose that for all $i \leq N-1$ and all $g \in \mathcal{S}(\mathbb{R})$,

$$\operatorname{ad}_R^i g(\mathcal{P}_k) = O_{-\infty}(\eta^{-i}; \mathcal{Y}_k \to \mathcal{W}_k).$$

By, e.g., [Voi84, Theorem 3.2], given a Schwartz function f, there exist two Schwartz functions f_1 and f_2 such that $f = f_1 f_2$. Thus $f(\mathcal{P}_k) = f_1(\mathcal{P}_k) f_2(\mathcal{P}_k)$ with $f_1, f_2 \in \mathcal{S}(\mathbb{R})$. Furthermore, by the Leibniz identity

$$\operatorname{ad}_X^N (YZ) = \sum_{i=0}^{N} \binom{N}{i} (\operatorname{ad}_X^i Y)(\operatorname{ad}_X^{N-i} Z).$$

Thus,

$$\begin{aligned}&\mathrm{ad}_R^N(f_1(\mathcal{P}_k)f_2(\mathcal{P}_k))\\&\quad= f_1(\mathcal{P}_k)(\mathrm{ad}_R^N f_2(\mathcal{P}_k)) + (\mathrm{ad}_R^N f_1(\mathcal{P}_k))f_2(\mathcal{P}_k) + \sum_{i=1}^{N-1}\binom{N}{i}\mathrm{ad}_R^i f_1(\mathcal{P}_k)\,\mathrm{ad}_R^{N-i} f_2(\mathcal{P}_k).\end{aligned}$$

Bounding the first two terms on the right-hand side by (5.26) and (5.25), and bounding the third term by the induction hypothesis, we obtain that

$$\mathrm{ad}_R^N(f_1(\mathcal{P}_k)f_2(\mathcal{P}_k)) = O_{-\infty}(1)O_{-2+N(m-1)}(\eta^{-N}) + \sum_{i=1}^{N-1} O_{-\infty}(k^{-i})O_{-\infty}(\eta^{-N+i}),$$

where all the operators are $\mathcal{Y}_k \to \mathcal{W}_k$. Since $\mathcal{W}_k^n \subset \mathcal{Y}_k^n$ for all n, $\mathrm{ad}_R^N(f_1(\mathcal{P}_k)f_2(\mathcal{P}_k)) = O_{-\infty}(\eta^{-N};\mathcal{Y}_k \to \mathcal{W}_k)$, and the proof is complete. ∎

We now record a variant of the previous result.

Proposition 5.33 *Let $N\in\mathbb{N}$, $f\in\mathcal{S}(\mathbb{R})$ and let $R = O_m(1;\mathcal{D}_k\to\mathcal{D}_k)$ be such that*

$$\mathrm{ad}_R^N \mathcal{P}_k = O_{m-N+2}(\eta^{-N};\mathcal{D}_k\to\mathcal{D}_k).$$

Then

$$\mathrm{ad}_R^N f(\mathcal{P}_k) = O_{-\infty}(\eta^{-N};\mathcal{D}_k\to\mathcal{D}_k)$$

The proof is the same as that of Proposition 5.32, using a variant of Proposition 5.31 in the scale $(\mathcal{D}_k^s)_{s\in\mathbb{R}}$, and using the mapping properties of $(\mathcal{P}_k - z), (\mathcal{P}_k - z)^{-1}$ and $f(\mathcal{P}_k)$ in this scale (with the latter two mapping properties coming from Propositions 5.21 and 5.22).

Proposition 5.34 *For all $N\in\mathbb{N}$ and $R\in\mathcal{L}_{\mathrm{sc}}^m$,*

$$\mathrm{ad}_R^N(P_k^\sharp)^{-1} = O_{-2+N(m-1)}(\eta^{-N};\mathcal{Y}_k\to\mathcal{W}_k).$$

Proof. This follows from Proposition 5.31 applied with $X = R$, $\Omega = \{1\}$ and $Y_1 = P_k^\sharp$. Indeed, for assumption (a), the required estimate is given by Proposition 5.23, while for assumption (b),

$$\mathrm{ad}_R^N P_k^\sharp = \mathrm{ad}_R^N P_k + \mathrm{ad}_R^N \psi(\mathcal{P}_k) = O_{2+N(m-1)}(\eta^{-N}) + O_{-\infty}(\eta^{-N})$$

by the definition of spatial cutoffs (i.e., Definition 5.25) and by Proposition 5.32. ∎

5.6 Boundary compatible operators and pseudolocality in frequency

In some cases, we will need to use pseudolocality in the *frequency space* in addition to the physical space. To this end, we introduce the set of *boundary compatible operators*, $\mathcal{L}_{\mathrm{b}}^m$.

Definition 5.35 (Boundary compatible operators) $X = O_m(1;\mathcal{D}_k\to\mathcal{D}_k)$ *is a* boundary compatible operator *of order m, $X\in\mathcal{L}_{\mathrm{b}}^m$, if, for all integers $N\geq 0$,*

$$\mathrm{ad}_{\mathcal{P}_k}^N X = O_{N+m}(\eta^{-N};\mathcal{D}_k\to\mathcal{D}_k) \quad \textit{and} \quad \mathrm{ad}_{\mathcal{P}_k}^N X^* = O_{N+m}(\eta^{-N};\mathcal{D}_k\to\mathcal{D}_k).$$

We write $\mathcal{L}_{\mathrm{b}} := \mathcal{L}_{\mathrm{b}}^0$.

Remark 5.36 *We highlight that the requirement that an operator is boundary compatible is much more stringent that the requirement that it is an abstract spatial cut-off, essentially because a function $u\in\mathcal{H}$ must satisfy many "boundary conditions" to belong to the spaces $\mathcal{D}_k^n$ for large n (see Table 5.1). In the setting of the Helmholtz PML problem of §6.3 below, we show that if R is given by the multiplication with a smooth cut-off function* that is locally constant near the obstacle boundary, and vanishes near in a neighbourhood of the truncation boundary*, then it satisfies the commutator estimates above.*

Theorem 5.37 (Frequency pseudolocality) *Let $X \in \mathcal{L}_{\mathrm{b}}^{m}$, $f, g \in C^{\infty}(\mathbb{R})$ be polynomially bounded, such that one of f and g is in $\mathcal{S}(\mathbb{R})$, and* $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$. *Then*

$$f(\mathcal{P}_k)Xg(\mathcal{P}_k) = O_{-\infty}(\eta^{-\infty}; \mathcal{D}_k \to \mathcal{D}_k). \tag{5.27}$$

As in the case of spatial pseudolocality, the proof of Theorem 5.37 can be reduced to certain commutator estimates.

Proposition 5.38 *Suppose that for all $X \in \mathcal{L}_{\mathrm{b}}^{m}$,*

$$\operatorname{ad}_{f(\mathcal{P}_k)}^{N} X = O_{-\infty}(\eta^{-N}; \mathcal{D}_k \to \mathcal{D}_k). \tag{5.28}$$

Then the results of Theorem 5.37 hold.

Proof. Without loss of generality, assume that f is Schwartz (the proof when g is Schwartz is analogous). Since the sets $\operatorname{supp} f$ and $\operatorname{supp} g$ are disjoint, there exists $f_1 \in \mathcal{S}(\mathbb{R})$ such that $\operatorname{supp} f_1 \cap \operatorname{supp} g = \emptyset$ and $\operatorname{supp}(1 - f_1) \cap \operatorname{supp} f = 0$. Therefore,

$$f(\mathcal{P}_k)Xg(\mathcal{P}_k) = f(\mathcal{P}_k)(\operatorname{ad}_{f_1(\mathcal{P}_k)}^{N} X)g(\mathcal{P}_k).$$

Therefore, (5.27), follows from (5.28) and the mapping properties of $f(\mathcal{P}_k)$ and $g(\mathcal{P}_k)$ from Proposition 5.22. ∎

Proposition 5.39 *Given $f \in \mathcal{S}(\mathbb{R})$, $m \in \mathbb{R}$, and $X \in \mathcal{L}_{\mathrm{b}}^{m}$,*

$$\operatorname{ad}_{f(\mathcal{P}_k)}^{N} X = O_{-\infty}(\eta^{-N}; \mathcal{D}_k \to \mathcal{D}_k)$$

(i.e., the bound (5.28) holds).

Proof. Using the Helffer-Sjöstrand formula, commutators with $f(\mathcal{P}_k)$ can be expressed in terms of commutators with $(\mathcal{P}_k - z)^{-1}$: for all $f_j \in \mathcal{S}(\mathbb{R})$, $j = 1, \ldots, N$, all $X \in \mathcal{L}_{\mathrm{b}}^{m}$, and for every integer $N \in \mathbb{N}$,

$$\begin{aligned} &\operatorname{ad}_{f_N(\mathcal{P}_k)} \operatorname{ad}_{f_{N-1}(\mathcal{P}_k)} \ldots \operatorname{ad}_{f_1(\mathcal{P}_k)} X \\ &\qquad = \int_{\mathbb{C}^N} w_1(z_1) \ldots w_N(z_N) (\operatorname{ad}_{(\mathcal{P}_k - z_N)^{-1}} \ldots \operatorname{ad}_{(\mathcal{P}_k - z_1)^{-1}} X) \, dm_{\mathbb{C}^N}(z_1, \ldots, z_N), \end{aligned} \tag{5.29}$$

where w_i is as in Proposition 5.30 with $f = f_i$. Using the identities $\operatorname{ad}_{(\mathcal{P}_k - z)^{-1}} X = -(\mathcal{P}_k - z)^{-1} \operatorname{ad}_{\mathcal{P}_k} X (\mathcal{P}_k - z)^{-1}$, $\operatorname{ad}_X(YZ) = (\operatorname{ad}_X Y)Z + Y(\operatorname{ad}_X Z)$, and the fact that $\operatorname{ad}_{(\mathcal{P}_k - z)^{-1}} (\mathcal{P}_k - z')^{-1} = 0$, one obtains the formula

$$\operatorname{ad}_{(\mathcal{P}_k - z_N)^{-1}} \ldots \operatorname{ad}_{(\mathcal{P}_k - z_1)^{-1}} X = (-1)^N \prod_{i=1}^{N} (\mathcal{P}_k - z_i)^{-1} (\operatorname{ad}_{\mathcal{P}_k}^{N} X) \prod_{i=1}^{N} (\mathcal{P}_k - z_i)^{-1}.$$

Therefore, by the resolvent estimate in Proposition 5.21 and the commutator assumption for frequency cutoffs (in Definition 5.35),

$$\operatorname{ad}_{(\mathcal{P}_k - z_N)^{-1}} \ldots \operatorname{ad}_{(\mathcal{P}_k - z_1)^{-1}} X = \prod_{j=1}^{N} \langle z_j \rangle^2 |\Im(z_j)|^{-2} O_{-3N+m}(\eta^{-N}; \mathcal{D}_k \to \mathcal{D}_k).$$

Using this in (5.29) and recalling the decay properties of the w_j (5.23), we obtain that, for any $f_1, \ldots, f_N \in \mathcal{S}(\mathbb{R})$,

$$\operatorname{ad}_{f_N(\mathcal{P}_k)} \ldots \operatorname{ad}_{f_1(\mathcal{P}_k)} X = O_{-3N+m}(\eta^{-N}; \mathcal{D}_k \to \mathcal{D}_k).$$

In particular, if $f_j = f$, then $\operatorname{ad}_{f(\mathcal{P}_k)}^{N} X = O_{-3N+m}(\eta^{-N}; \mathcal{D}_k \to \mathcal{D}_k)$.

To upgrade the regularity index from $-3N + m$ to $-\infty$, we use again that, given a Schwartz f, there exist two Schwartz functions f_1 and f_2 such that $f = f_1 f_2$. By the functional calculus, $f(\mathcal{P}_k) = f_1(\mathcal{P}_k) f_2(\mathcal{P}_k)$. Furthermore, $\operatorname{ad}_{WY} Z = (\operatorname{ad}_W Z)Y + W(\operatorname{ad}_Y Z)$. Thus,

$$\begin{aligned} \operatorname{ad}_{f(\mathcal{P}_k)}^{N} X = \operatorname{ad}_{f_1(\mathcal{P}_k) f_2(\mathcal{P}_k)} \operatorname{ad}_{f(\mathcal{P}_k)}^{N-1} X &= \underbrace{(\operatorname{ad}_{f_1(\mathcal{P}_k)} \operatorname{ad}_{f(\mathcal{P}_k)}^{N-1} X)}_{O_{-3N}(\eta^{-N})} \underbrace{f_2(\mathcal{P}_k)}_{O_{-\infty}(1)} + \underbrace{f_1(\mathcal{P}_k)}_{O_{-\infty}(1)} \underbrace{(\operatorname{ad}_{f_2(\mathcal{P}_k)} \operatorname{ad}_{f(\mathcal{P}_k)}^{N-1} X)}_{O_{-3N}(\eta^{-N})} \\ &= O_{-\infty}(\eta^{-N}; \mathcal{D}_k \to \mathcal{D}_k) \end{aligned}$$

which completes the proof. ∎

6 Pseudolocality results applied to the Helmholtz problem

In this section, we specialise the results of the previous section to the Helmholtz PML operator, that is, to the case where $\mathcal{H}_k^n = H_k^n$, a_k is defined by (3.1) and $\mathcal{Z}_k$ is defined by (3.2). With these definitions fixed, we keep the rest of the notation from Section 5 (indeed, once we check that Assumptions 5.1, 5.2, 5.3, 5.13 and 5.16 hold, all other objects appearing in this section are then defined in terms of $\mathcal{H}_k^n$, $\mathcal{Z}_k$ and a_k). Observe that $\mathcal{Z}_k$ is a subspace of H_k^1 with Dirichlet conditions on either $\partial\Omega$ (Dirichlet setting) or just $\Gamma_{\rm tr}$ (Neumann setting), and for all $j \geq 0$, $\mathcal{Y}_k^j = H_k^j$ and the inclusion $H_k^{-j} \subset \mathcal{Y}_k^{-j}$ is continuous. In particular, if $R = O_{-\infty}(\eta^n; \mathcal{Y}_k \to \mathcal{Y}_k)$, then

$$\|Ru\|_{H_k^N} \leq C\eta^n \|u\|_{H_k^{-N}} \quad \text{for all } N \in \mathbb{N}.$$

In this particular setting, we give sufficient conditions for smooth cut-off functions $\chi \in C^\infty(\overline{\Omega})$ to fulfil the conditions of Definition 5.25 or Definition 5.35. We also identify some boundary-compatible operators in the sense of Definition 5.35 that are used for the proof of Theorem 3.11.

In the remainder of this paper, we adopt the following notation.

Definition 6.1 *Given two cutoff functions $\varphi, \widetilde{\varphi}, \psi \in C^\infty(\overline{\Omega})$ and a real number $d > 0$, we write*

$$\begin{aligned} \varphi \perp_d \psi \quad &\Longleftrightarrow \quad \operatorname{dist}(\operatorname{supp}\varphi, \operatorname{supp}\psi) > d \\ \varphi \prec_d \widetilde{\varphi} \quad &\Longleftrightarrow \quad \varphi \perp_d (1 - \widetilde{\varphi}). \end{aligned}$$

We abbreviate $\perp_0$ and $\prec_0$ by $\perp$ and $\prec$.

6.1 Verifying the assumptions

We start by showing that the assumptions of Section 5 hold for the PML problem.

With the PDE coefficients in A_θ, b_θ, and n_θ defined in §A, that section shows that there exists $\omega \in \mathbb{R}$ such that $e^{i\omega} a(\cdot,\cdot)$ satisfies Assumption 5.1 and 5.2 (with $\omega = 0$ for the most commonly-used radial PML construction). Since

$$a_k(u - u_h, v_h) = 0 \quad \text{if and only if} \quad e^{i\omega} a_k(u - u_h, v_h) = 0,$$

without loss of generality we can assume that $\omega = 0$.

It is standard that Assumption 5.3 holds with $\mathcal{Z}_k^2$ defined in (3.7). Moreover, for all $u \in \mathcal{Z}_k^2$,

$$P_k u = -k^{-2} \operatorname{div}(A_\theta \nabla u) + k^{-2} \langle b_\theta(x), \nabla u \rangle - n_\theta u.$$

Thus, $P_k : \mathcal{Z}_k^n \to \mathcal{H}_k^{n-2}$ is continuous for $n \geq 2$, thus Assumption 5.13 holds. The smoothness of $\partial\Omega$, A_θ, b_θ, and n_θ ensure Assumption 5.16 (elliptic regularity) holds

Since $\mathcal{D}_k^0 = \mathcal{H}$, $\mathcal{D}_k^1 = \mathcal{Z}_k$, and $\mathcal{X}_k^{-2} := (\mathcal{P}_k + (C_{\rm Ga}(k_0) + 1)\,\mathrm{I})^{-1}$ is an isomorphism from $\mathcal{D}_k^n$ to $\mathcal{D}_k^{n+2}$, it follows by induction that

$$\mathcal{D}_{k,d}^n = \left\{ u \in H^n(\Omega) : u, \mathcal{P}_k u, \ldots, \mathcal{P}_k^{\lceil n/2 \rceil - 1} u \in H_0^1(\Omega) \right\}$$

and

$$\begin{aligned} \mathcal{D}_{k,n}^n = \Big\{ u \in H^n(\Omega) \ : u = \mathcal{P}_k u = \ldots = \mathcal{P}_k^{\lceil n/2 \rceil - 1} u = 0 \text{ on } \Gamma_{\rm tr} \\ \text{and } \partial_{\nu, A_\theta} u = \partial_{\nu, A_\theta}(\mathcal{P}_k u) = \ldots = \partial_{\nu, A_\theta}(\mathcal{P}_k^{\lceil n/2 \rceil - 1} u) = 0 \text{ on } \partial\Omega_- \Big\}. \end{aligned}$$

Now

$$\|u\|_{\mathcal{D}_k^n} = \begin{cases} \|\mathcal{X}_k^{2m} u\|_{\mathcal{H}_k}, & n = 2m, \\ \|\mathcal{X}_k^{2m} u\|_{\mathcal{Z}_k}, & n = 2m + 1, \end{cases}$$

Since $\mathcal{X}_k^2 = \mathcal{P}_k + (C_{\rm Ga}(k_0) + 1)\,\mathrm{I}$ so that, by induction, for $m \in \mathbb{N}$, $\mathcal{X}_k^{2m}$ coincides on $\mathcal{D}_k^{2m}$ (and thus on $\mathcal{D}_k^{2m+1}$) with a differential operator of order $2m$. Thus

$$\|u\|_{\mathcal{D}_k^n} \leq C \|u\|_{\mathcal{H}_k^n} \quad \text{for } u \in \mathcal{D}_k^n. \tag{6.1}$$

6.2 Pseudolocality results with smooth cut-off functions

The main result of this section is the following.

Theorem 6.2 *Suppose that $\chi_1, \chi_2 \in C^\infty(\overline{\Omega})$ with $\operatorname{supp}\chi_1 \cap \operatorname{supp}\chi_2 = \emptyset$. Then, for all $f \in \mathcal{S}(\mathbb{R})$,*

$$\chi_1 f(\mathcal{P}_k)\chi_2 = O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k), \qquad \chi_1 R_k^\sharp \chi_2 = O_{-2}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k).$$

This result is proved by using Theorem 5.27 and showing, first, that if $\chi \in C^\infty(\overline{\Omega})$ and $\partial_\nu \chi|_{\partial\Omega_-} = 0$ then $\chi \in \mathcal{L}_{\rm sc}$ (see Lemma 6.3 below) and, second, that given $\chi_1, \chi_2 \in C^\infty(\overline{\Omega})$, there exist $\widetilde{\chi}_j \in C^\infty(\overline{\Omega})$ with $\partial_\nu \widetilde{\chi}_j|_{\partial\Omega_-} = 0$, $j = 1, 2$, such that $\chi_j \prec \widetilde{\chi}_j$, and $\widetilde{\chi}_1 \perp \widetilde{\chi}_2$ (see Lemma 6.6).

Lemma 6.3 *Given $\{C_N\}_{N>0} \subset \mathbb{R}_+$, there exist $\{C'_{N,n}\}_{N,n} \subset \mathbb{R}_+$ such that the following is true. For all $\varepsilon > 0$, if $\chi \in C^\infty(\overline{\Omega})$ satisfies $\partial_\nu \chi|_{\partial\Omega_-} = 0$ and*

$$\max_{|\alpha| \le N} \varepsilon^{|\alpha|} |\partial^\alpha \chi| \le C_N \quad \textit{for all } N \in \mathbb{N},$$

then

$$\|\operatorname{ad}_\chi^N Q\|_{\mathcal{W}_k^n \to \mathcal{Y}_k^{n+2-N}} \le C'_{N,n}(\varepsilon k)^{-N}, \tag{6.2}$$

where Q is any one of the operators P_k, P_k^* or $\mathcal{P}_k$. In particular $\chi \in \mathcal{L}_{\rm sc}$.

Proof. We start by considering $\operatorname{ad}_\chi P_k$ acting in $\mathcal{Z}_k^1$. Indeed, suppose that $u, v \in \mathcal{Z}_k^1$. Then, $\chi u, \bar{\chi} v \in \mathcal{Z}_k^1$ and hence

$$\begin{aligned}
\langle \operatorname{ad}_\chi P_k u, v\rangle &= a_k(u, \bar{\chi} v) - a_k(\chi u, v) \\
&= k^{-2}\Big(\langle A_\theta \nabla u, \nabla(\bar{\chi} v)\rangle - \langle A_\theta(\nabla \chi u), \nabla v\rangle\Big) + k^{-2}\big\langle \chi b_\theta \cdot \nabla u - b_\theta \cdot \nabla(\chi u), v\big\rangle \\
&= k^{-2}\Big(\langle A_\theta \nabla u, v \nabla \bar{\chi}\rangle - \langle u A_\theta \nabla \chi, \nabla v\rangle\Big) - k^{-2}\langle (b_\theta \cdot \nabla \chi) u, v\rangle \\
&= k^{-2}\big\langle (A_\theta \nabla u) \cdot \nabla \chi + \nabla \cdot (u A_\theta \nabla \chi), v\big\rangle - k^{-2}\langle (b_\theta \cdot \nabla \chi) u, v\rangle,
\end{aligned} \tag{6.3}$$

where in the last line we use that $\partial_\nu \chi|_{\partial\Omega_-} = 0$ and $u|_{\Gamma_{\rm tr}} = 0$. In particular, for $n \ge 1$, $\|\operatorname{ad}_\chi P_k u\|_{\mathcal{H}_k^{n-1}} \le C(\varepsilon k)^{-1}\|u\|_{\mathcal{Z}_k^n}$. Next, since

$$\langle \operatorname{ad}_\chi^2 P_k u, v\rangle = \langle \operatorname{ad}_\chi P_k u, \overline{\chi} v\rangle - \langle \operatorname{ad}_\chi P_k(\chi u), v\rangle,$$

a short calculation using (6.3) implies that, for $u, v \in \mathcal{Z}_k^1$,

$$\langle \operatorname{ad}_\chi^2 P_k u, v\rangle = -k^{-2}\big\langle 2u(A_\theta \nabla \chi) \cdot \nabla \chi, v\big\rangle. \tag{6.4}$$

Thus, for $n \ge 0$, $\|\operatorname{ad}_\chi^2 P_k u\|_{\mathcal{H}_k^n} \le C(\varepsilon k)^{-2}\|u\|_{\mathcal{Z}_k^n}$. Since there are no derivatives of u on the right-hand side of (6.4), a similar calculation shows that $u \in \mathcal{Z}_k^1$, $(\operatorname{ad}_\chi^N P_k)u = 0$ for $N \ge 3$. Thus

$$\|\operatorname{ad}_\chi^N P_k\|_{\mathcal{Z}_k^n \to \mathcal{H}_k^{n-2+N}} = \|\operatorname{ad}_\chi^N P_k\|_{\mathcal{W}_k^n \to \mathcal{Y}_k^{n-2+N}} \le C(\varepsilon k)^{-N} \quad \text{for } n \ge 0, \tag{6.5}$$

and, by identical arguments,

$$\|\operatorname{ad}_\chi^N P_k^*\|_{\mathcal{Z}_k^n \to \mathcal{H}_k^{n-2+N}} = \|\operatorname{ad}_\chi^N P_k^*\|_{\mathcal{W}_k^n \to \mathcal{Y}_k^{n-2+N}} \le C(\varepsilon k)^{-N} \quad \text{for } n \ge 0. \tag{6.6}$$

Now, for $\ell \ge 0$, $u \in \mathcal{H}_k^{-\ell}$ and $v \in \mathcal{Z}_k^{\ell+2}$, by the fact that $(\operatorname{ad}_A^N B)^* = (-1)^N \operatorname{ad}_{A^*}^N B^*$ (which one can prove by induction),

$$\big|\big\langle (\operatorname{ad}_\chi^N P_k)u, v\big\rangle\big| = \big|\big\langle u, (-1)^N(\operatorname{ad}_{\bar{\chi}}^N P_k^*)v\big\rangle\big| \le \|u\|_{\mathcal{H}_k^{-\ell}}\|(\operatorname{ad}_{\bar{\chi}}^N P_k^*)v\|_{\mathcal{H}_k^\ell}$$

If $N \ge 3$, then the right-hand side of the last inequality is zero. Otherwise, (6.6) with $n = \ell + 2 - N \ge 0$ implies that

$$\big|\big\langle (\operatorname{ad}_\chi^N P_k)u, v\big\rangle\big| \le C(\varepsilon k)^{-N}\|u\|_{\mathcal{H}_k^{-\ell}}\|v\|_{\mathcal{Z}_k^{\ell+2-N}}.$$

Since $\mathcal{Z}_k^{\ell+2}$ is dense in $\mathcal{Z}_k^{\ell+2-N}$, the previous inequality and similar arguments for P_k^* imply that

$$\| \operatorname{ad}_\chi^N P_k^* \|_{\mathcal{H}_k^{-\ell} \to \mathcal{Z}_k^{-\ell-2+N}} + \| \operatorname{ad}_\chi^N P_k \|_{\mathcal{H}_k^{-\ell} \to \mathcal{Z}_k^{-\ell-2+N}} \le C(\varepsilon k)^{-N}, \qquad \ell \ge 0. \tag{6.7}$$

The combination of (6.5), (6.6), and (6.7) are then (6.2). ∎

We first prove the existence of a cut-off in $\mathcal{L}_{\rm sc}$ between Ω_1 and Ω_2, under the assumption that Ω_1 is sufficiently small – this assumption allows us to use Fermi normal coordinates defined by $\partial\Omega$ on Ω_1.

Lemma 6.4 *There exists $\epsilon_0 > 0$ such that for all $N > 0$, there exists $C_N > 0$ such that for all $0 < \epsilon < \epsilon_0$ the following is true. If $\Omega_1, \Omega_2 \subset \Omega$ are such that $d(\Omega_1, \Omega_2) > \epsilon$ and if there exists $y \in \Omega$ such that $\Omega_1 \subset B(y, \epsilon_0)$, then there exists $\chi \in C^\infty(\overline{\Omega}; [0,1])$ satisfying*

$$\Omega_1 \cap \operatorname{supp}(1-\chi) = \emptyset, \qquad \operatorname{supp}\chi \cap \Omega_2 = \emptyset,$$
$$|\partial^\alpha \chi| \le C_N \epsilon^{-|\alpha|} \text{ for } |\alpha| \le N, \qquad \text{and} \qquad \partial_\nu \chi|_{\partial\Omega} = 0.$$

Proof. Let $U_{\rm Fermi}$ be a tubular neighbourhood of $\partial\Omega$ in which there exists a Fermi normal coordinate chart; we denote these coordinates below by $(x_1, x')_F$ (where the subscript F emphasises that these are not Euclidean coordinates). Let $\delta_0 > 0$ be such that $U_{\rm Fermi} \supset B(\partial\Omega, 20\delta_0) := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < 20\delta_0\}$ and let $\epsilon_0 = 9\delta_0$. If $\Omega_1 \not\subset U_{\rm Fermi}$ then $\operatorname{dist}(\Omega_1, \partial\Omega) \ge 2\delta_0 = (2/9)\epsilon_0$, and the existence of χ follows immediately (e.g., by mollification of the indicator function of Ω_1). We therefore assume that $\Omega_1 \subset U_{\rm Fermi}$. Now, there exists $c_F > 0$ (depending only on Ω) such that, for all $r > 0$ and for all $(0, x') \in \partial\Omega$,

$$\big\{(0, y')_F : |y' - x'| \le c_F r\big\} \subset B\big((0, x')_F, r\big) \cap \partial\Omega. \tag{6.8}$$

We now define some mollifiers and cut-off functions. Fix $\psi_m \in C_c^\infty(B_{\mathbb{R}^m}(0,1))$ such that $\int \psi_m = 1$, $m = d-1, d$. Fix also $\widetilde{\psi}_1 \in C_c^\infty(-2,2)$ with $(-1,1) \cap \operatorname{supp}(1-\widetilde{\psi}_1) = \emptyset$. Then, for $\delta > 0$, set $\psi_{m,\delta}(x) := \delta^{-m}\psi_m(\delta^{-1}x)$, and $\widetilde{\psi}_{1,\delta}(x) := \widetilde{\psi}_1(\delta^{-1}x)$.

Let $\delta = \epsilon/10$ and $\widetilde{\Omega}_1 := B(\Omega_1, 4\delta)$; note that $d(\widetilde{\Omega}_1, \Omega_2) > 6\epsilon/10$. Let $\widetilde{\Omega}_1^\partial := \widetilde{\Omega}_1 \cap \partial\Omega$; note that this set may be empty.

Now let

$$\chi(x_1, x') := \big(1_{\widetilde{\Omega}_1^\partial} * \psi_{d-1, c_F\delta}\big)(x')\widetilde{\psi}_{1,\delta}(x_1) + \big(1_{\widetilde{\Omega}_1} * \psi_{d,\delta}\big)(x)\big(1 - \widetilde{\psi}_{1,\delta}(x_1)\big) =: \chi_{\rm near} + \chi_{\rm far}, \tag{6.9}$$

We now check that χ has the required properties. First,

$$\|\partial^\alpha(u * v)\|_{L^\infty} = \|(\partial^\alpha u) * v\|_{L^\infty} \le \|\partial^\alpha u\|_{L^1}\|v\|_{L^\infty},$$

so that

$$\|\big(\partial^\alpha(u*v)\big)w\|_{L^\infty} \le \|w\|_{L^\infty}\, \|(\partial^\alpha u) * v\|_{L^\infty} \le \|w\|_{L^\infty}\, \|\partial^\alpha u\|_{L^1}\|v\|_{L^\infty}.$$

Combining this with the product rule and

$$\|\partial^\alpha \widetilde{\psi}_{1,\delta}\|_{L^\infty} \le C_\alpha \delta^{-\alpha_1} \qquad \text{and} \qquad \|\partial^\alpha \psi_{m,\delta}\|_{L^1} \le C_\alpha \delta^{-|\alpha'|},\ m = d-1, d,$$

implies the required derivative estimates on χ. Next, since $\chi_{\rm far} \equiv 0$ near $\partial\Omega$ and $\partial_{x_1}^\alpha \chi_{\rm near}|_{x_1=0} = 0$ for any α, it follows that $\partial_\nu \chi|_{\partial\Omega} = 0$.

We now show that $\Omega_1 \cap \operatorname{supp}(1-\chi) = \emptyset$; we do this by showing that

$$\chi(x) = 1 \quad \text{when} \quad x = (x_1, x') \in B(\Omega_1, \delta) \cap \overline{\Omega}. \tag{6.10}$$

First observe that, for such x, $\big(1_{\widetilde{\Omega}_1} * \psi_{d,\delta}\big)(x) = 1$. Then, since $\widetilde{\psi}_{1,\delta}(x_1) = 0$ when $x_1 \ge 2\delta$, (6.10) then follows from (6.9) if we can show that

$$\big(1_{\widetilde{\Omega}_1^\partial} * \psi_{d-1, c_F\delta}\big)(x') = 1 \quad \text{when } x_1 \le 2\delta \tag{6.11}$$

(i.e., on the support of $\widetilde{\psi}_{1,\delta}$). To prove (6.11), observe that, by the triangle inequality,

$$d((0,x')_F, \Omega_1) \le x_1 + d(x,\Omega_1) < 2\delta + d(x,\Omega_1) < 3\delta.$$

Since $\widetilde{\Omega}_1 := B(\Omega_1, 4\delta)$, $B((0,x')_F,\delta) \subset \widetilde{\Omega}_1$. Thus $B((0,x')_F,\delta)\cap\partial\Omega \subset \widetilde{\Omega}_1 \cap \partial\Omega = \widetilde{\Omega}_1^{\partial}$. and (6.11) follows by (6.8).

Finally, we show that $\Omega_2 \cap \operatorname{supp}\chi = \emptyset$, again by showing that

$$\chi(x) = 0 \quad \text{when} \quad x = (x_1, x')_F \in B(\Omega_2, \delta) \cap \overline{\Omega}.$$

Then, $d(x, \widetilde{\Omega}_1) > \epsilon - 5\delta = 5\delta$; thus $(1_{\widetilde{\Omega}_1} * \psi_{d,\delta})(x) = 0$ and $\chi_{\text{far}} = 0$. If $|x_1| > 2\delta$, $\chi_{\text{near}} = 0$. Otherwise, we claim that

$$d((0,x')_F, \widetilde{\Omega}_1^{\partial}) \ge \epsilon - 7\delta = 3\delta; \tag{6.12}$$

then, by (6.8),

$$\big\{(0,y')_F : |y'-x'| \le c_F\delta\big\} \cap \widetilde{\Omega}_1^{\partial} \subset B((0,x')_F,\delta) \cap \widetilde{\Omega}_1^{\partial} = \emptyset$$

so that $(1_{\widetilde{\Omega}_1^{\partial}} * \psi_{d-1,c_F\delta})(x) = 0$ (and hence $\chi_{\text{near}} = 0$). The inequality (6.12) follows by the triangle inequality:

$$\begin{aligned}\epsilon \le d(\Omega_1,\Omega_2) &\le d(\Omega_1, \widetilde{\Omega}_1^{\partial}) + d((0,x')_F, \widetilde{\Omega}_1^{\partial}) + d((0,x')_F, x) + d(x,\Omega_2),\\ &\le 4\delta + d((0,x')_F, \widetilde{\Omega}_1^{\partial}) + 2\delta + \delta,\end{aligned}$$

concluding the proof. ∎

Lemma 6.5 (Partition of unity satisfying Neumann boundary conditions) *Let $\Omega_i \subset \Omega$, $i = 1,\dots,N$ be open with $\Omega \subset \cup_i \Omega_i$. Then there exist $\varphi_i \in C^\infty(\overline{\Omega})$ satisfying*

$$\operatorname{supp}\varphi_i \subset \Omega_i \cup \partial\Omega, \quad \partial_\nu \varphi_i = 0 \ \ \text{on } \partial\Omega, \quad \text{and} \quad \sum_{i=1}^N \varphi_i \equiv 1.$$

Proof. Let $U_i \Subset \Omega_i$ be open sets such that $\Omega \subset \cup_i U_i$. Then, for $\epsilon > 0$ small enough, and $y \in \overline{U}_i$, $B(y, 2\epsilon) \subset \Omega_i$. In addition, since $\overline{U}_i$ is compact, there are $\{y_{ij}\}_{j=1}^{N_i} \subset \overline{U}_i$ such that $U_i \subset \cup_{j=1}^{N_i} B(y_{ij},\epsilon)$.

By Lemma 6.4 (applied with $\Omega_1 = B(y_{ij},\epsilon)$ and $\Omega_2 = \Omega \setminus \overline{\Omega_i}$), for $\epsilon > 0$ small enough, there are $\widetilde{\varphi}_{ij} \in C^\infty(\overline{\Omega};[0,1])$ such that $\operatorname{supp}(1-\widetilde{\varphi}_{ij}) \cap B(y_{ij},\epsilon) = \emptyset$, $\operatorname{supp}\widetilde{\varphi}_{ij} \subset \Omega_i \cup \partial\Omega$, and $\partial_\nu \widetilde{\varphi}_{ij} = 0$ on $\partial\Omega$. Noting that $\sum_{i=1}^N \sum_{j=1}^{N_i} \widetilde{\varphi}_{ij} \ge 1$, we define

$$\varphi_i := \frac{\sum_{j=1}^{N_i} \widetilde{\varphi}_{ij}}{\sum_{i=1}^N \sum_{j=1}^{N_i} \widetilde{\varphi}_{ij}},$$

which has the required properties. ∎

We now remove the assumption from Lemma 6.4 that Ω_1 is sufficiently small.

Lemma 6.6 *There exists $\epsilon_0 > 0$ such that for all $N > 0$, there exists $C_N > 0$ such that for all $0 < \epsilon < \epsilon_0$ and $\Omega_1, \Omega_2 \subset \Omega$ and $d(\Omega_1,\Omega_2) > \epsilon$, there exists $\chi \in C^\infty(\overline{\Omega})$ satisfying*

$$\begin{aligned} &\Omega_1 \cap \operatorname{supp}(1-\chi) = \emptyset, \qquad \operatorname{supp}\chi \cap \Omega_2 = \emptyset,\\ &|\partial^\alpha \chi| \le C_N \epsilon^{-|\alpha|}, \qquad |\alpha| \le N, \qquad \partial_\nu \chi|_{\partial\Omega} = 0, \end{aligned} \tag{6.13}$$

Proof. Let ϵ_0 be as in Lemma 6.4. Since $\overline{\Omega}$ is compact, there exist $\{x_i\}_{i=1}^M$ such that $\overline{\Omega} \subset \cup_{i=1}^M B(x_i,\epsilon_0)$. Then, by Lemma 6.5, there exist $\{\varphi_i\}_{i=1}^M$ a partition of unity subordinate to $\{B(x_i,\epsilon_0)\}_{i=1}^M$ satisfying $\partial_\nu\varphi_i = 0$ on $\partial\Omega$. Then, let $\Omega_{1,i} := \Omega_1 \cap B(x_i,\epsilon_0)$. By Lemma 6.4, there exists χ_i such that the conditions in (6.13) hold with Ω_1 replaced by $\Omega_{1,i}$. Define $\chi := \sum_{i=1}^M \chi_i\varphi_i$. The derivative estimates in (6.13) then follow from the product rule and the fact that the derivatives of φ_i are independent of ϵ (but depend on ϵ_0). The condition that $\partial_\nu\chi|_{\partial\Omega} = 0$ follows since both

$\partial_\nu \chi_i = 0$ and $\partial_\nu \varphi_i = 0$. Next, since $\operatorname{supp} \chi_i \cap \Omega_2 = \emptyset$, $\operatorname{supp} \chi \cap \Omega_2 \subset \cup_i \operatorname{supp}(\chi_i \varphi_i) \cap \Omega_2 = \emptyset$. Finally, since $\chi_i \equiv 1$ on $\Omega_1 \cap B(x_i, \epsilon_0)$ and $\operatorname{supp} \varphi_i \subset B(x_i, \epsilon_0)$, $(1-\chi_i)\varphi_i = 0$ on Ω_1, and thus

$$\operatorname{supp}(1-\chi) \cap \Omega_1 = \operatorname{supp}\Big(\sum_{i=1}^N (1-\chi_i)\varphi_i\Big) \cap \Omega_1 = \emptyset.$$

■

Proof of Theorem 6.2. Since $\operatorname{supp} \chi_1 \cap \operatorname{supp} \chi_2 = \emptyset$, there exist Ω_i neighbourhoods of $\operatorname{supp} \chi_i$ with $d(\Omega_1, \Omega_2) > 0$. Therefore, by Lemma 6.6 there are $\widetilde{\chi}_i$ with $\operatorname{supp} \chi_i \cap \operatorname{supp}(1-\widetilde{\chi}_i) = \emptyset$, $\operatorname{supp} \chi_i \cap \operatorname{supp} \widetilde{\chi}_j = \emptyset$, $i \neq j$, and $\partial_\nu \widetilde{\chi}_i = 0$. Hence by Lemma 6.3, $\widetilde{\chi}_i \in \mathcal{L}_{\rm sc}$ and using Theorem 5.27, we have

$$\chi_1 f(\mathcal{P}_k)\chi_2 = \chi_1 \widetilde{\chi}_1 f(\mathcal{P}_k)\widetilde{\chi}_2 \chi_2 = \chi_1 O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{W}_k)\chi_2 = O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k),$$

since $\chi_j = O_0(1; \mathcal{Y}_k \to \mathcal{Y}_k)$. The proof for $R_k^\sharp$ is identical. ■

6.3 Some boundary compatible operators

Lemma 6.7 *If* $\varphi \in C^\infty(\overline{\Omega})$, $\operatorname{supp} \nabla\varphi \cap \partial\Omega = \emptyset$, *and* $\operatorname{supp} \varphi \cap \Gamma_{\rm tr} = \emptyset$, *then* $\varphi \in \mathcal{L}_{\rm b}$ *in the sense of Definition 5.35.*

Proof. By Definition 5.35, we need to show that $\operatorname{ad}^N_{\mathcal{P}_k} \varphi = O_N(k^{-N}; \mathcal{D}_k \to \mathcal{D}_k)$. Let $\widetilde{\varphi} \in C_c^\infty(\Omega)$ be such that $\widetilde{\varphi} \equiv 1$ on $\operatorname{supp} \nabla\varphi$. We first show that $\operatorname{ad}^N_{\mathcal{P}_k} \varphi = (\operatorname{ad}^N_{L_k} \varphi)\widetilde{\varphi}$ on $C^\infty(\overline{\Omega})$ where L_k is a second-order differential operator. By (6.3), for $u, v \in \mathcal{Z}_k$,

$$\langle (\operatorname{ad}_{\mathcal{P}_k} \varphi)u, v\rangle = -k^{-2}\big\langle (\Re A_\theta \nabla u)\cdot \nabla\varphi + \nabla\cdot(u(\Re A_\theta)\nabla\varphi), v\big\rangle. \tag{6.14}$$

By (6.14),

$$\operatorname{ad}_{\mathcal{P}_k} \varphi = \widetilde{\varphi}(\operatorname{ad}_{\mathcal{P}_k} \varphi)\widetilde{\varphi} = \widetilde{\varphi}(\operatorname{ad}_{L_k} \varphi)\widetilde{\varphi}.$$

Thus

$$\operatorname{ad}^2_{\mathcal{P}_k} \varphi = \mathcal{P}_k \widetilde{\varphi}(\operatorname{ad}_{L_k} \varphi)\widetilde{\varphi} - \widetilde{\varphi}(\operatorname{ad}_{L_k} \varphi)\widetilde{\varphi}\mathcal{P}_k.$$

Since $\mathcal{P}_k\widetilde{\varphi} = L_k\widetilde{\varphi}$ and $\widetilde{\varphi}\mathcal{P}_k = \widetilde{\varphi}L_k$,

$$\operatorname{ad}^2_{\mathcal{P}_k} \varphi = L_k \widetilde{\varphi}(\operatorname{ad}_{L_k} \varphi)\widetilde{\varphi} - \widetilde{\varphi}(\operatorname{ad}_{L_k} \varphi)\widetilde{\varphi}L_k = \operatorname{ad}^2_{L_k} \varphi = (\operatorname{ad}^2_{L_k} \varphi)\widetilde{\varphi};$$

the fact that $\operatorname{ad}^N_{\mathcal{P}_k} \varphi = (\operatorname{ad}^N_{L_k} \varphi)\widetilde{\varphi}$ can be proved similarly by induction. Therefore, given $u \in \mathcal{D}_k^{n+N}$, $\operatorname{ad}^N_{\mathcal{P}_k} \varphi u = (\operatorname{ad}^N_{L_k} \varphi)\widetilde{\varphi} u \in \mathcal{H}_k^n$ with compact support in Ω, and thus, when $n \in \mathbb{N}$, $(\operatorname{ad}^N_{\mathcal{P}_k} \varphi)u \in \mathcal{D}_k^n$. Thus, by (6.1), commutator results for differential operators, and Corollary 5.20, for $n \in \mathbb{N}$,

$$\|(\operatorname{ad}^N_{\mathcal{P}_k} \varphi)u\|_{\mathcal{D}_k^n} \le \|(\operatorname{ad}^N_{\mathcal{P}_k} \varphi)u\|_{\mathcal{H}_k^n} = \|(\operatorname{ad}^N_{L_k} \varphi)u\|_{\mathcal{H}_k^n} \le Ck^{-N}\|u\|_{\mathcal{H}_k^{n+N}} \le C'k^{-N}\|u\|_{\mathcal{D}_k^{n+N}}.$$

By the spectral theorem, $(\mathcal{D}_k^s)_s$ is an interpolation scale, and the result for general n follows by duality and interpolation. ■

Lemma 6.8 ($\varphi P_k \varphi \in \mathcal{L}^2_{\rm b}$ for suitable φ) *Suppose that* $\varphi \in C^\infty(\overline{\Omega})$, $\operatorname{supp} \nabla\varphi \cap \partial\Omega = \emptyset$, *and* $\operatorname{supp} \varphi \cap \Gamma_{\rm tr} = \emptyset$. *Then,*

$$\operatorname{ad}^N_{\varphi\mathcal{P}_k\varphi} P_k = O_{N+2}(k^{-N}; \mathcal{D}_k \to \mathcal{D}_k), \quad \operatorname{ad}^N_{\varphi\mathcal{P}_k^*\varphi} P_k = O_{N+2}(k^{-N}; \mathcal{D}_k \to \mathcal{D}_k), \tag{6.15}$$

and

$$\operatorname{ad}^N_{\mathcal{P}_k} \varphi P_k \varphi = O_{N+2}(k^{-N}; \mathcal{D}_k \to \mathcal{D}_k), \quad \operatorname{ad}^N_{\mathcal{P}_k^*} \varphi P_k \varphi = O_{N+2}(k^{-N}; \mathcal{D}_k \to \mathcal{D}_k)\,; \tag{6.16}$$

in particular, $\varphi P_k \varphi \in \mathcal{L}^2_{\rm b}$ *in the sense of Definition 5.35.*

Proof of Lemma 6.8. We prove (6.15); the proof of (6.16) is very similar. We write

$$\operatorname{ad}^N_{\varphi\mathcal{P}_k\varphi} P_k = \psi_0 \operatorname{ad}^N_{\varphi\mathcal{P}_k\varphi} P_k + (1-\psi_0)\operatorname{ad}^N_{\varphi\mathcal{P}_k\varphi} P_k,$$

where ψ_0 is supported in a region close to $\partial\Omega$ where φ is constant. More precisely, let $\psi_i \in C^\infty(\overline{\Omega})$, $i=-1,0,1$ with $\operatorname{supp}(1-\psi_i)\cap\partial\Omega_- = \emptyset$, $\operatorname{supp}\psi_i \cap \operatorname{supp}(c-\varphi)=\emptyset$, $\operatorname{supp}(1-\psi_i)\cap\operatorname{supp}\psi_{i-1}=\emptyset$, and $\psi_2 P_k = \psi_2 P_k^*$ (note that such functions exist since $\operatorname{supp}\nabla\varphi\cap\partial\Omega_-=\emptyset$).

By locality of $\mathcal{P}_k$ and P_k and the fact that $\mathcal{P}_k = P_k$ on $\operatorname{supp}\psi_2 \supset \operatorname{supp}\psi_0$,

$$\psi_0 \operatorname{ad}^N_{\varphi\mathcal{P}_k\varphi} P_k = \psi_0 \operatorname{ad}^N_{cP_kc} P_k = 0.$$

Now, let $\tilde\varphi \in C^\infty(\overline{\Omega})$ with $\operatorname{supp}\tilde\varphi\cap\Gamma_{\rm tr}=\emptyset$ and $\operatorname{supp}\varphi\cap\operatorname{supp}(1-\tilde\varphi)=\emptyset$. Then

$$(1-\psi_0)\operatorname{ad}^N_{\varphi\mathcal{P}_k\varphi} P_k = (1-\psi_0)\operatorname{ad}^N_{\varphi(1-\psi_{-1})\mathcal{P}_k\varphi(1-\psi_{-1})}\big[(1-\psi_{-1})\tilde\varphi P_k(1-\psi_{-1})\tilde\varphi\big].$$

Since $\operatorname{supp}\tilde\varphi\cap\operatorname{supp}(1-\psi_{-1})\cap\partial\Omega=\emptyset$, integration by parts (with all the boundary terms vanishing) shows that $\varphi(1-\psi_{-1})\mathcal{P}_k\varphi(1-\psi_{-1})$ and $(1-\psi_{-1})\tilde\varphi P_k(1-\psi_{-1})\tilde\varphi$ coincide with differential operators on $C^\infty(\overline{\Omega})$. The result

$$\|\operatorname{ad}^N_{\varphi\mathcal{P}_k\varphi} P_k\|_{\mathcal{D}^n_k\to\mathcal{D}^{n-N-2}_k} \leq Ck^{-N}$$

then follows by direct differentiation (using the product rule) and then density of $C^\infty(\overline{\Omega})$ in $\mathcal{D}^n_k$. The proof of the analogous bound for P_k^* is identical. ■

Lemma 6.9 ($\varphi(P_k^\sharp)^{-1}\varphi, \varphi(P_k^{\sharp,*})^{-1}\varphi \in \mathcal{L}_{\rm b}^{-2}$) *Suppose that $\varphi\in C^\infty(\overline{\Omega})$ and $\operatorname{supp}\nabla\varphi\cap\partial\Omega=\emptyset$, and $\operatorname{supp}\varphi\cap\Gamma_{\rm tr}=\emptyset$. Then,*

$$\operatorname{ad}^N_{\mathcal{P}_k}\varphi(P_k^\sharp)^{-1}\varphi = O_{N-2}(k^{-N};\mathcal{D}_k\to\mathcal{D}_k),\qquad \operatorname{ad}^N_{\mathcal{P}_k}\varphi(P_k^{\sharp,*})^{-1}\varphi = O_{N-2}(k^{-N};\mathcal{D}_k\to\mathcal{D}_k)$$

and thus $\varphi(P_k^\sharp)^{-1}\varphi, \varphi(P_k^{\sharp,*})^{-1}\varphi\in\mathcal{L}_{\rm b}^{-2}$ in the sense of Definition 5.35.

Proof of Lemma 6.9. We prove the statement for $P_k^\sharp$. The proof for $(P_k^\sharp)^*$ is identical. Let $\tilde\varphi\in C^\infty(\overline{\Omega})$ with $\operatorname{supp}\nabla\tilde\varphi\cap\partial\Omega=\emptyset$ and $\operatorname{supp}(1-\tilde\varphi)\cap\operatorname{supp}\varphi=\emptyset$. Then, by locality of $\mathcal{P}_k$,

$$\operatorname{ad}^N_{\mathcal{P}_k}\varphi(P_k^\sharp)^{-1}\varphi = \operatorname{ad}^N_{\tilde\varphi\mathcal{P}_k\tilde\varphi}\varphi(P_k^\sharp)^{-1}\varphi, \tag{6.17}$$

By Lemma 6.7, $\operatorname{ad}^N_{\mathcal{P}_k}\varphi = O_N(k^{-N};\mathcal{D}_k\to\mathcal{D}_k)$, so that, by repeated use of the identity $\operatorname{ad}_{AB}C = A(\operatorname{ad}_B C)+(\operatorname{ad}_A C)B$,

$$\operatorname{ad}^N_{\tilde\varphi\mathcal{P}_k\tilde\varphi}\varphi = O_N(k^{-N};\mathcal{D}_k\to\mathcal{D}_k). \tag{6.18}$$

Therefore, by the combination of (6.18), (6.17), and repeated use of the identity $\operatorname{ad}_A BC = (\operatorname{ad}_A B)C + B(\operatorname{ad}_A C)$, it is enough to show that

$$\operatorname{ad}^N_{\tilde\varphi\mathcal{P}_k\tilde\varphi}(P_k^\sharp)^{-1} = O_{N-2}(k^{-N};\mathcal{D}_k\to\mathcal{D}_k). \tag{6.19}$$

To prove (6.19) we use Proposition 5.31. For this, observe that

$$\operatorname{ad}^N_{\tilde\varphi\mathcal{P}_k\tilde\varphi}P_k^\sharp = \operatorname{ad}^N_{\tilde\varphi\mathcal{P}_k\tilde\varphi}(P_k+\psi(\mathcal{P}_k)).$$

By Lemma 6.8,

$$\operatorname{ad}^N_{\tilde\varphi\mathcal{P}_k\tilde\varphi}P_k = O_{2+N}(k^{-N};\mathcal{D}_k\to\mathcal{D}_k)\quad\text{ and }\quad \operatorname{ad}^N_{\tilde\varphi\mathcal{P}_k\tilde\varphi}P_k^* = O_{2+N}(k^{-N};\mathcal{D}_k\to\mathcal{D}_k).$$

By induction, $\operatorname{ad}^N_A(B+C) = \operatorname{ad}^N_A B + \operatorname{ad}^N_A C$, and so

$$\operatorname{ad}^N_{\tilde\varphi\mathcal{P}_k\tilde\varphi}\mathcal{P}_k = O_{2+N}(k^{-N};\mathcal{D}_k\to\mathcal{D}_k). \tag{6.20}$$

Therefore, by Proposition 5.33,

$$\operatorname{ad}^N_{\tilde\varphi\mathcal{P}_k\tilde\varphi}\psi(\mathcal{P}_k) = O_{2+N}(k^{-N};\mathcal{D}_k\to\mathcal{D}_k).$$

We deduce that

$$\operatorname{ad}^N_{\tilde\varphi\mathcal{P}_k\tilde\varphi}P_k^\sharp = O_{2+N}(k^{-N};\mathcal{D}_k\to\mathcal{D}_k)$$

and (6.19) – and hence also the result – then follows from Proposition 5.31. ■

7 Pseudolocality of the elliptic projection

In Sections 4, 5, and 6, we studied pseudolocality properties at the continuous level. Another key tool required for the proof of Theorem 3.11 is of a discrete nature, namely, we need to establish the spatial pseudolocality of the Galerkin projection $\Pi_k^\sharp$ associated to (the adjoint of) the sesquilinear form $a_k^\sharp$ defined in Definition 5.13, see Theorem 7.2 below.

We keep the notation of Section 6. The operator $\Pi_k^\sharp$ is defined as follows.

Definition 7.1 (Elliptic projection) *Given $k > 0$ and a linear subspace $V_k \subset \mathcal{Z}_k$, the* elliptic projection *onto V_k is the linear operator $\Pi_k^\sharp : \mathcal{Z}_k \to V_k$ defined by*

$$a_k^\sharp(v, \Pi_k^\sharp u) = a_k^\sharp(v, u) \qquad \text{for all } v \in V_k,$$

where we recall that $\mathcal{Z}_k$ is defined by (3.2), $a_k^\sharp(u, v) = a_k(u, v) + (S_k u, v)_{\mathcal{H}}$ and S_k is defined by (5.12).

The operator $\Pi_k^\sharp$ is well-defined for all $k > 0$ by the Lax-Milgram theorem, since $a_k^\sharp$ is coercive (by Proposition 5.23).

Theorem 7.2 (Pseudolocality of $\mathrm{I} - \Pi_k^\sharp$) *Let $(V_k)_{k>0}$ be a well-behaved finite-element of order p in the sense of Definition 3.7, let $k_0 > 0$ and let $\mathfrak{c} > 0$. There exists $h_0 > 0$ such that for all $N > 0$, $\chi, \psi \in C^\infty(\overline{\Omega})$ satisfying $\chi \perp_{\mathfrak{c}} \psi$, there exists $C > 0$ such that for all $k \geq k_0$, $h \leq h_0$, and $u \in \mathcal{Z}_k$*

$$\|\chi(\mathrm{I} - \Pi_k^\sharp)\psi u\|_{H_k^1} \leq C k^{-N} \|(\mathrm{I} - \Pi_k^\sharp)\psi u\|_{H_k^{-p}},$$

where $\Pi_k^\sharp$ is the elliptic projection onto V_k.

Remark 7.3 *Through the constants $\mathfrak{c}$ and h_0, the assumptions of Theorem 7.2 require a sufficient number of "layers" of elements separating the supports of χ and ψ.*

Theorem 7.2 is an immediate consequence of the following two lemmas.

Lemma 7.4 *Let $(V_k)_{k>0}$ be a well-behaved finite-element of order p in the sense of Definition 3.7, and let $\mathfrak{c} > 0$. Then, there exists $h_0 > 0$ such that the following holds. For any $k_0 > 0$, $N > 0$, and any $\chi_-, \chi_+, \psi \in C^\infty(\overline{\Omega})$ satisfying*

$$\chi_- \prec_{\mathfrak{c}} \chi_+ \qquad \text{and} \qquad \chi_+ \perp \psi,$$

there exists $C > 0$ such that, for all $k \geq k_0$, $h \leq h_0$, and $u \in \mathcal{Z}_k$,

$$\|\chi_-(\mathrm{I} - \Pi_k^\sharp)\psi u\|_{H_k^1} \leq C k^{-N} \Big(\|\chi_+(\mathrm{I} - \Pi_k^\sharp)\psi u\|_{L^2} + \|(\mathrm{I} - \Pi_k^\sharp)\psi u\|_{H_k^{-N}} \Big).$$

Lemma 7.5 *Let $(V_k)_{k>0}$ be a well-behaved finite-element of order p in the sense of Definition 3.7, let $k_0 > 0$ and $\mathfrak{c} > 0$. Then, there exists $h_0 > 0$ such that for all $N > 0$ and every χ, $\psi \in C^\infty(\overline{\Omega})$ satisfying $\chi \perp_{\mathfrak{c}} \psi$, there exists $C > 0$ such that, for all $k \geq k_0$, $h \leq h_0$, and $u \in \mathcal{Z}_k$,*

$$\|\chi(\mathrm{I} - \Pi_k^\sharp)\psi u\|_{L^2} \leq C \|(\mathrm{I} - \Pi_k^\sharp)\psi u\|_{H_k^{-p}}. \tag{7.1}$$

If S_k is (formally) set to zero, then Lemmas 7.4 and 7.5 are analogous to [AGS24, Lemmas 5.1 and 5.5], respectively.

Proof of Lemma 7.4. Let $\mathfrak{c} > 0$ be given and let $h_0 > 0$ be a sufficiently small constant depending only on $\mathfrak{c}$. Fix $k_0 > 0$, $N > 0$ and χ_-, χ_+ and ψ as in the statement. In what follows, we denote by C a generic constant depending only on the previous quantities.

We first claim that, without loss of generality, we can assume that $\partial_\nu \chi_- = 0$ and thus

$$\mathrm{ad}_{\chi_-} P_k = O_1(k^{-1}; \mathcal{W}_k \to \mathcal{Y}_k) \tag{7.2}$$

by Lemma 6.3 and Definition 5.25 and

$$\mathrm{ad}_{\chi_-} S_k = O_{-\infty}(k^{-1}; \mathcal{Y}_k \to \mathcal{W}_k) \tag{7.3}$$

by Proposition 5.32. Indeed, if $\partial_\nu \chi_- \neq 0$ we apply Lemma 6.6 with $\Omega_1 = \operatorname{supp} \chi_-$ and Ω_2 equal to $\operatorname{supp}(1 - \chi_+)$ enlarged by distance $\mathfrak{c}/2$. We then relabel the resulting cut-off function χ_- and replace $\mathfrak{c}$ by $\mathfrak{c}/2$.

Let $k \geq k_0$, $u \in \mathcal{Z}_k$ and suppose that $h \leq h_0$. It is sufficient to prove that

$$\|\chi_- \Pi_k^\sharp \psi u\|_{H_k^1} \leq C k^{-1/2} \|\chi_+ \Pi_k^\sharp \psi u\|_{L^2} + C k^{-N} \|(\mathrm{I} - \Pi_k^\sharp)\psi u\|_{H_k^{-N}} \tag{7.4}$$

since by iterating (7.4) $2N$ times, (changing the cutoffs χ_- and χ_+), one arrives at

$$\begin{aligned}\|\chi_- \Pi_k^\sharp \psi u\|_{H_k^1} &\leq C k^{-N} \|\chi_+ \Pi_k^\sharp \psi u\|_{L^2} + C k^{-N} \|(\mathrm{I} - \Pi_k^\sharp)\psi u\|_{H_k^{-N}} \\ &= C k^{-N} \|\chi_+ (\mathrm{I} - \Pi_k^\sharp)\psi u\|_{L^2} + C k^{-N} \|(\mathrm{I} - \Pi_k^\sharp)\psi u\|_{H_k^{-N}}\end{aligned}$$

using the fact that $\chi_+ \psi = 0$.

Let $\chi_0 \in C^\infty(\overline{\Omega})$ be such that $\chi_- \prec_{\mathfrak{c}/4} \chi_0 \prec_{\mathfrak{c}/4} \chi_+$. By the coercivity of $P_k^\sharp$ (cf. (5.15)), the definition of $\Pi_k^\sharp$ (Definition 7.1), locality of P_k and the fact that $\chi_- \psi = \chi_0 \psi = 0$,

$$\begin{aligned}\|\chi_- \Pi_k^\sharp \psi u\|_{H_k^1}^2 &\leq C \left| \left\langle P_k^\sharp \chi_- \Pi_k^\sharp \psi u, \chi_- \Pi_k^\sharp \psi u \right\rangle \right| \\ &= C \left| \left\langle P_k^\sharp \chi_- \Pi_k^\sharp \psi u, \chi_- (\mathrm{I} - \Pi_k^\sharp)\psi u \right\rangle \right| \\ &= C \Big(\left| \langle P_k^\sharp \chi_-^2 \Pi_k^\sharp \psi u, (\mathrm{I} - \Pi_k^\sharp)\psi u \rangle \right| + \langle [P_k^\sharp, \chi_-] \chi_- \Pi_k^\sharp \psi u, (\mathrm{I} - \Pi_k^\sharp)\psi u \rangle \Big) \\ &= C \Big(\left| \langle P_k^\sharp (\chi_-^2 \Pi_k^\sharp \psi u - w_h), (\mathrm{I} - \Pi_k^\sharp)\psi u \rangle \right| + \left| \langle [P_k + S_k, \chi_-] \chi_- \Pi_k^\sharp \psi u, (\mathrm{I} - \Pi_k^\sharp)\psi u \rangle \right| \Big) \\ &= C \big(\left| \left\langle P_k (\chi_-^2 \Pi_k^\sharp \psi u - w_h), (\mathrm{I} - \Pi_k^\sharp)\psi u \right\rangle \right| + \left| \left\langle [P_k, \chi_-] \chi_- \Pi_k^\sharp \psi u, \chi_0 (\mathrm{I} - \Pi_k^\sharp)\psi u \right\rangle \right| \big) + r,\end{aligned}$$

for all $w_h \in V_k$, where

$$r := \left| \left\langle S_k (\chi_-^2 \Pi_k^\sharp \psi u - w_h), (\mathrm{I} - \Pi_k^\sharp)\psi u \right\rangle \right| + \left| \left\langle [S_k, \chi_-] \chi_- \Pi_k^\sharp \psi u, (\mathrm{I} - \Pi_k^\sharp)\psi u \right\rangle \right| \tag{7.5}$$

is the "non-local" part. By (7.2) combined with the fact that $\chi_0 \psi = 0$,

$$\|\chi_- \Pi_k^\sharp \psi u\|_{H_k^1}^2 \leq C \left| \langle P_k (\chi_-^2 \Pi_k^\sharp \psi u - w_h), (\mathrm{I} - \Pi_k^\sharp)\psi u \rangle \right| + C k^{-1} \|\chi_- \Pi_k^\sharp \psi u\|_{H_k^1} \|\chi_0 \Pi_k^\sharp \psi u\|_{L^2} + r$$

which implies

$$\|\chi_- \Pi_k^\sharp \psi u\|_{H_k^1}^2 \leq C \left| \langle P_k^\sharp (\chi_-^2 \Pi_k^\sharp \psi u - w_h), (\mathrm{I} - \Pi_k^\sharp)\psi u \rangle \right| + C k^{-2} \|\chi_0 \Pi_k^\sharp \psi u\|_{L^2}^2 + r. \tag{7.6}$$

Let U_0 be a neighbourhood of $\operatorname{supp} \chi_-$, and U_1 a set contained in $\{\chi_0 \equiv 1\}$. Since $\chi_- \prec_{\mathfrak{c}/4} \chi_0 \prec_{\mathfrak{c}/4} \chi_+$, we can arrange that

$$d := \partial_<(U_0, U_1) \geq \mathfrak{c}/8.$$

Hence, taking $h_0 < \frac{\mathfrak{c}}{8\kappa}$, where κ is as in Assumption 3.5, we ensure that $d \geq \kappa h_0$. Thus, we can find a super-approximation w_h to $\chi_-^2 \Pi_k^\sharp \psi u$ with $\operatorname{supp} w_h \subset U_1$. Now, for all $\epsilon < 1$,

$$\begin{aligned}\left| \left\langle P_k (\chi_-^2 \Pi_k^\sharp \psi u - w_h), (\mathrm{I} - \Pi_k^\sharp)\psi u \right\rangle \right| &= \left| \left\langle P_k (\chi_-^2 \Pi_k^\sharp \psi u - w_h), \Pi_k^\sharp \psi u \right\rangle \right| \qquad \text{(by locality of } P_k) \\ &\leq \sum_{K \in \mathcal{T}_k : K \cap \operatorname{supp} \chi_0 \neq \emptyset} \|\chi_-^2 \Pi_k^\sharp \psi u - w_h\|_{H_k^1(K)} \|\Pi_k^\sharp \psi u\|_{H_k^1(K)} \quad \text{(by the definition of } a_k(\cdot,\cdot)) \\ &\leq \sum_{K \in \mathcal{T}_k : K \cap \operatorname{supp} \chi_0 \neq \emptyset} \|\Pi_k^\sharp \psi u\|_{H_k^1(K)} \frac{h_K}{d} \Big(\|\Pi_k^\sharp \psi u\|_{L^2(K)} + \|\chi_- \Pi_k^\sharp \psi u\|_{H_k^1(K)} \Big) \\ &\qquad \text{(by the super-approximation property, Assumption 3.5)} \\ &\leq C \sum_{K \in \mathcal{T} : K \cap \operatorname{supp} \chi_0 \neq \emptyset} \|\Pi_k^\sharp \psi u\|_{L^2(K)} \frac{1}{kd} \Big(\|\Pi_k^\sharp \psi u\|_{L^2(K)} + \|\chi_- \Pi_k^\sharp \psi u\|_{H_k^1(K)} \Big)\end{aligned}$$

(by the inverse inequality, Assumption 3.6)

$$\leq C \sum_{K\in\mathcal{T}_k: K\cap \mathrm{supp}(\chi_0)\neq\emptyset} \frac{(1+\epsilon^{-1})}{(kd)^2}\|\Pi_k^\sharp\psi u\|^2_{L^2(K)} + \epsilon\|\chi_-\Pi_k^\sharp\psi u\|^2_{H^1_k(K)}$$

$$\leq \frac{C(1+\epsilon^{-1})}{(kd)^2}\|\chi_+\Pi_k^\sharp\psi u\|^2_{L^2(\Omega)} + \epsilon\|\chi_-\Pi_k^\sharp\psi u\|^2_{\mathcal{Z}_k}. \tag{7.7}$$

Inputting (7.7) into (7.6) and recalling that $d \geq \mathfrak{c}/8$ and $k \geq k_0$, we find that

$$\|\chi_-\Pi_k^\sharp\psi u\|^2_{\mathcal{Z}_k} \leq Ck^{-2}\|\chi_+\Pi_k^\sharp\psi u\|^2_{L^2(\Omega)} + r, \tag{7.8}$$

To estimate r, since $\mathrm{dist}(\mathrm{supp}(w_h, \mathrm{supp}(1-\chi_0))) > 0$, pseudolocality of S_k (Theorem 6.2) implies that

$$\big|\langle S_k(\chi_-^2\Pi_k^\sharp\psi u - w_h), (1-\chi_0)(\mathrm{I}-\Pi_k^\sharp)\psi u\rangle\big| \leq Ck^{-N}\big\|\chi_-^2\Pi_k^\sharp\psi u - w_h\big\|_{\mathcal{Z}_k}\big\|(\mathrm{I}-\Pi_k^\sharp)\psi u\big\|_{H_k^{-N}}.$$

Arguing as in (7.7), but now using Assumption 3.2 where before we used Assumption 3.6, and also recalling that $d \geq \mathfrak{c}/8$, we find that

$$\|\chi_-^2\Pi_k^\sharp\psi u - w_h\|^2_{\mathcal{Z}_k} \leq \frac{C}{k^2}\Big(\|\chi_-\Pi_k^\sharp\psi u\|^2_{\mathcal{Z}_k} + \|\chi_+\Pi_k^\sharp\psi u\|^2_{L^2(\Omega)}\Big).$$

Therefore, for all $\epsilon < 1$,

$$\begin{aligned}&\big|\langle S_k(\chi_-^2\Pi_k^\sharp\psi u - w_h), (1-\chi_0)(\mathrm{I}-\Pi_k^\sharp)\psi u\rangle\big|\\ &\qquad = Ck^{-N}\Big(\epsilon\|\chi_-\Pi_k^\sharp\psi u\|^2_{\mathcal{Z}_k} + \epsilon\|\chi_+\Pi_k^\sharp\psi u\|^2_{L^2(\Omega)} + \epsilon^{-1}\big\|(\mathrm{I}-\Pi_k^\sharp)\psi u\big\|^2_{H_k^{-N}}\Big).\end{aligned} \tag{7.9}$$

Reasoning similarly and using the mapping properties of S_k, we find that

$$\begin{aligned}\big|\langle S_k(\chi_-^2\Pi_k^\sharp\psi u - w_h), \chi_0(\mathrm{I}-\Pi_k^\sharp)\psi u\rangle\big| &\leq Ck^{-1}\Big(\epsilon\|\chi_-\Pi_k^\sharp\psi u\|^2_{\mathcal{Z}_k} + \epsilon\|\chi_+\Pi_k^\sharp\psi u\|^2_{\mathcal{H}} + \epsilon^{-1}\big\|\chi_0\Pi_k^\sharp\psi u\big\|^2_{H_k^{-N}}\Big)\\ &\leq Ck^{-1}\Big(\epsilon\|\chi_-\Pi_k^\sharp\psi u\|^2_{\mathcal{Z}_k} + (\epsilon+\epsilon^{-1})\|\chi_+\Pi_k^\sharp\psi u\|^2_{L^2(\Omega)}\Big).\end{aligned}$$

Arguing similarly, we obtain

$$\begin{aligned}&\big|\big\langle[S_k,\chi_-]\chi_-\Pi_k^\sharp\psi u, (\mathrm{I}-\Pi_k^\sharp)\psi u\big\rangle\big|\\ &\leq C\Big(\epsilon k^{-2}\|\chi_-\Pi_k^\sharp\psi u\|^2_{\mathcal{Z}_k} + k^{-2}(\epsilon+\epsilon^{-1})\|\chi_+\Pi_k^\sharp\psi u\|^2_{L^2(\Omega)} + \epsilon^{-1}k^{-N}\|(\mathrm{I}-\Pi_k^\sharp)\psi u\|^2_{H_k^{-N}}\Big);\end{aligned} \tag{7.10}$$

indeed, $[\chi_-, S_k] = \chi_0[\chi_-, S_k] + (1-\chi_0)[\chi_-, S_k]$ and

$$\chi_0[\chi_-, S_k] = O_{-\infty}(k^{-1}; \mathcal{Y}_k \to \mathcal{W}_k) \quad\text{and}\quad (1-\chi_0)[\chi_-, S_k] = -(1-\chi_0)S_k\chi_- = O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k)$$

by, respectively, (7.3) and Theorem 6.2.

Combining (7.9)-(7.10) thus leads to

$$r \leq C\epsilon\|\chi_-\Pi_k^\sharp\psi u\|^2_{\mathcal{Z}_k} + C(1+\epsilon^{-1})k^{-1}\|\chi_+\Pi_k^\sharp\psi u\|^2_{L^2(\Omega)} + C\epsilon^{-1}k^{-N}\|(\mathrm{I}-\Pi_k^\sharp)\psi u\|^2_{H_k^{-N}}.$$

for all $\epsilon < 1$. Inserting this estimate in (7.8) and taking ϵ sufficiently small, we obtain (7.4) and hence the result. ∎

In the proof of Lemma 7.5, we need a variant of Lemma 7.4 where, roughly speaking, the contributions at distance $k^{-1}2^n$ from $\mathrm{supp}\,\chi$ are multiplied by a weight decaying exponentially in n. The main tool is the following lemma:

Lemma 7.6 (Dyadic decomposition for S_k) *Let $\phi_0 \in C_c^\infty(\mathbb{R}^d)$ with $\phi_0(x) = 1$ for $|x| \leq \frac{1}{2}$, $\phi_0(x) = 0$ for $|x| \geq 1$, and let $\phi_n(x) := \phi_0(x/2^n) - \phi_0(x/2^{n-1})$ for $n \geq 1$. Let $x_0 \in \overline{\Omega}$, and $R > 0$, and let $\varphi_{n,k} \in C^\infty(\overline{\Omega})$ be defined by*

$$\varphi_{n,k}(x) := \phi_n((x-x_0)/(Rk^{-1})).$$

Then, for any $k_0 > 0$ and $N \in \mathbb{N}$, there exists $C(N, k_0, R) > 0$ such that for all $k \geq k_0$ and $n \in \mathbb{N}$,

$$\|\varphi_{n,k}S_k\varphi_{0,k}\|_{L^2\to H_k^N} \leq C(N, k_0, R)2^{-nN}.$$

Proof. We start by observing that $\operatorname{supp}\phi_0 \subset B(0,1)$, while for $n \geq 1$,

$$\operatorname{supp}(\phi_n) \subset B(0,2^n) \setminus B(0,2^{n-2}),$$

and in particular, $\phi_n \perp \phi_0$ for $n \geq 2$. Let χ_n be given by Lemma 6.6 applied with $\Omega_1 := \operatorname{supp}\varphi_{k,n}$ and $\Omega_2 := \operatorname{supp}\varphi_{k,0}$ and $\varepsilon = O(k^{-1}2^n)$. Then $\varphi_{n,k}$ and $\varphi_{0,k}$ are separated by χ_n, and the result of the Lemma follows by the combination of Lemma 6.3, Theorem 5.27 and Remark 5.28. ∎

Lemma 7.7 *Let $(V_k)_{k>0}$ be a well-behaved finite-element of order k, let $k_0 > 0$, $C_\dagger > 0$, let $\psi \in C^\infty(\overline{\Omega})$ and let $N > 0$. Then there exists $C > 0$ and $\mu > 0$ such that the following is true. If $k \geq k_0$, $x_0 \in \overline{\Omega}$, $R > 0$ and $\chi_-, \chi_+ \in C^\infty(\overline{\Omega})$ satisfy*

1. $\operatorname{supp}\chi_- \subset \operatorname{supp}\chi_+ \subset B(x_0, Rk^{-1}/4)$
2. $d := \operatorname{dist}(\operatorname{supp}\chi_-, \operatorname{supp}(1-\chi_+)) > \mu k^{-1}$
3. $\max_{|\alpha|=n} \|\partial^\alpha \chi_-\|_{L^\infty} \leq C_\dagger k^n, \quad n = 0,\dots,p,$
4. $\operatorname{supp}\chi_+ \cap \operatorname{supp}\psi = \emptyset,$,

then for all $u \in \mathcal{Z}_k$,

$$\|\chi_- \Pi_k^\sharp \psi u\|_{\mathcal{Z}_k} \leq C\|\chi_+ \Pi_k^\sharp \psi u\|_{L^2(\Omega)} + C\sum_{n=0}^\infty 2^{-Nn}\|\varphi_{n,k}(\mathrm{I} - \Pi_k^\sharp)\psi u\|_{H_k^{-N}}$$

where $\varphi_{n,k}$ is as in Lemma 7.6.

Proof of Lemma 7.7. Let $k_0 > 0$ be given and let $\mu := C\kappa$ where C is as in Assumption 3.3 and κ is as in Assumption 3.5. Let $C_\dagger > 0$, $\psi \in C^\infty(\overline{\Omega})$, $N > 0$ and denote by $C > 0$ any generic constant depending only on the previous quantities. Let $k \geq k_0$, suppose that $h \leq h_0$ and let x_0, R and χ_-, χ_+ as in the statement. The choice of r implies that

$$d \geq \kappa h.$$

Therefore, one can proceed as in the proof of Lemma 7.4 using the super-approximation property (Assumption 3.5), but taking into account that, now, d scales as k^{-1} instead of 1, so that the analogue of (7.2) is

$$\|\operatorname{ad}_{\chi_-} P_k\|_{\mathcal{W}_k^n \to \mathcal{Y}_k^{n-1}} \leq C.$$

This leads to

$$\|\chi_- \Pi_k^\sharp \psi u\|_{\mathcal{Z}_k}^2 \leq C\|\chi_+ \Pi_k^\sharp \psi u\|_{L^2(\Omega)}^2 + r, \tag{7.11}$$

where, as in (7.5),

$$r := \left|\left\langle S_k(\chi_-^2 \Pi_k^\sharp \psi u - w_h), (\mathrm{I} - \Pi_k^\sharp)\psi u\right\rangle\right| + \left|\left\langle [S_k, \chi_-]\chi_- \Pi_k^\sharp \psi u, (\mathrm{I} - \Pi_k^\sharp)\psi u\right\rangle\right|. \tag{7.12}$$

We now use the property that for all $x \in \mathbb{R}^d$,

$$\sum_{n\in\mathbb{N}} \varphi_{n,k}^2(x) \geq 1/2$$

(see e.g. [AG07, Lemma 1.1.1]) to write, for any $f, g \in \mathcal{Z}_k$,

$$|\langle S_k f, g\rangle| \leq 2\sum_{n=0}^\infty |\langle \varphi_{n,k} S_k f, \varphi_{n,k} g\rangle|.$$

Taking $f = \chi_-^2 \Pi_k^\sharp \psi u - w_h$ and $g = (\mathrm{I} - \Pi_k^\sharp)\psi u$, and using the fact that $f = \varphi_{0,k} f$, the first term of (7.12) is estimated by

$$\begin{aligned}
&\Big|\big\langle S_k(\chi_-^2 \Pi_k^\sharp \psi u - w_h), (\mathrm{I} - \Pi_k^\sharp)\psi u\big\rangle\Big| \\
&\quad \le 2 \sum_{n=0}^{\infty} \big|\big\langle \varphi_{n,k} S_k \varphi_{0,k}(\chi_-^2 \Pi_k^\sharp \psi u - w_h), \varphi_{n,k}(\mathrm{I} - \Pi_k^\sharp)\psi u\big\rangle\big| \\
&\quad \le C \|\chi_-^2 \Pi_k^\sharp \psi u - w_h\|_{\mathcal{Z}_k} \sum_{n=0}^{\infty} \|\varphi_{n,k} S_k \varphi_{0,k}\|_{\mathcal{Z}_k \to H_k^N} \|\varphi_{n,k}(\mathrm{I} - \Pi_k^\sharp)\psi u\|_{H_k^{-N}} \\
&\quad \le C(\|\chi_- \Pi_k^\sharp u\|_{\mathcal{Z}_k} + \|\chi_+ \Pi_k^\sharp u\|_{L^2(\Omega)}) \sum_{n=0}^{\infty} \|\varphi_{n,k} S_k \varphi_{0,k}\|_{\mathcal{Z}_k \to H_k^N} \|\varphi_{n,k}(\mathrm{I} - \Pi_k^\sharp)\psi u\|_{H_k^{-N}},
\end{aligned}$$

where we have used that

$$\|\chi_-^2 \Pi_k^\sharp \psi u - w_h\|_{\mathcal{Z}_k} \le C \frac{1}{(kd)^2} \|\chi_- \Pi_k^\sharp u\|_{\mathcal{Z}_k} + \|\chi_+ \Pi_k^\sharp u\|_{L^2(\Omega)}$$

(obtained by reasoning as in (7.7)), and taken into account that $(kd)^{-1} \le M^{-1} \le C$. By Lemma 7.6,

$$\|\varphi_{n,k} S_k \varphi_{0,k}\|_{\mathcal{Z}_k \to H_k^N} \le C 2^{-Nn} \quad \text{for all } n \in \mathbb{N}.$$

Hence, for all $\epsilon < 1$,

$$\begin{aligned}
&\Big|\big\langle S_k(\chi_-^2 \Pi_k^\sharp \psi u - w_h), (\mathrm{I} - \Pi_k^\sharp)\psi u\big\rangle\Big| \\
&\quad \le C\epsilon \|\chi_- \Pi_k^\sharp \psi u\|_{\mathcal{Z}_k}^2 + C\epsilon \|\chi_+ \Pi_k^\sharp \psi u\|_{L^2(\Omega)}^2 + C\epsilon^{-1} \left(\sum_{n=0}^{\infty} 2^{-Nn} \|\varphi_{n,k}(\mathrm{I} - \Pi_k^\sharp)\psi u\|_{H_k^{-N}} \right)^2 \qquad (7.13)
\end{aligned}$$

Similarly, using that $\varphi_{n,k}\chi_- = 0$ for $n \ge 1$, we deduce that for all $\varepsilon < 1$,

$$\begin{aligned}
&\Big|\big\langle [\chi_-, S_k]\chi_- \Pi_k^\sharp \psi u, (\mathrm{I} - \Pi_k^\sharp)\psi u\big\rangle\Big| \\
&\quad \le 2\Big|\langle \varphi_{0,k}[S_k, \chi_-]\chi_- \Pi_k^\sharp \psi u, \varphi_{0,k}(\mathrm{I} - \Pi_k^\sharp)\psi u\rangle\Big| + \sum_{n=1}^{\infty} \Big|\big\langle \varphi_{n,k} S_k \chi_-^2 \Pi_k^\sharp \psi u, \varphi_{n,k}(\mathrm{I} - \Pi_k^\sharp)\psi u\big\rangle\Big| \\
&\quad \le C\big\|\chi_- \Pi_k^\sharp \psi u\big\|_{\mathcal{Z}_k} \big\|\varphi_{0,k}[S_k, \chi_-]\big\|_{\mathcal{Z}_k \to H_k^N} \big\|\varphi_{0,k}(\mathrm{I} - \Pi_k^\sharp)\psi u\big\|_{H_k^{-N}} \\
&\qquad + C\|\chi_- \Pi_k^\sharp \psi u\|_{\mathcal{Z}_k} \sum_{n=1}^{\infty} \|\varphi_{n,k} S_k \varphi_{0,k}\|_{\mathcal{Z}_k \to H_k^N} \|\varphi_{n,k}(\mathrm{I} - \Pi_k^\sharp)\psi u\|_{H_k^{-N}} \\
&\quad \le C\varepsilon \|\chi_- \Pi_k^\sharp \psi u\|_{\mathcal{Z}_k}^2 + C\varepsilon^{-1} \left(\sum_{n=0}^{\infty} 2^{-Nn} \|\varphi_{n,k}(\mathrm{I} - \Pi_k^\sharp)\psi u\|_{H_k^{-N}} \right)^2. \qquad (7.14)
\end{aligned}$$

Adding (7.13) and (7.14), inserting the result in (7.12) and then in (7.11), and letting ε be small enough, we obtain the result. ■

Proof of Lemma 7.5. We claim that, given $\mathfrak{c} > 0$, there exists $h_0 > 0$ such that for all $k_0 > 0$ and $\chi_-, \chi_+, \psi \in C^\infty(\overline{\Omega})$ satisfying

$$\chi_- \prec_{\mathfrak{c}} \chi_+ \quad \text{and} \quad \chi_- \perp \psi \qquad (7.15)$$

there exists $C > 0$ such that for all $k \ge k_0$, $h \le h_0$, $u \in \mathcal{Z}_k$ and $0 \le j \le p-1$,

$$\|\chi_- \Pi_k^\sharp \psi u\|_{H_k^{-j}} \le C\Big(\|\chi_+ \Pi_k^\sharp \psi u\|_{H_k^{-(j+1)}} + \big\|(\mathrm{I} - \Pi_k^\sharp)\psi u\big\|_{H_k^{-p}} \Big). \qquad (7.16)$$

If this is true, the lemma follows easily. Indeed, given $\mathfrak{c} > 0$, let $\mathfrak{c}' = \frac{\mathfrak{c}}{2p}$, and given χ, ψ as in the statement, let $\chi_1, \ldots, \chi_p$ be a sequence of nested cutoffs, i.e. such that

$$\chi_i \prec_{\mathfrak{c}'} \chi_{i+1} \qquad \text{for all } i \in \{1, \ldots, p-1\} \quad \text{and} \quad \chi_p \perp \psi.$$

One then applies (7.16) p times with $\chi_- = \chi_i$ and $\chi_+ = \chi_{i+1}$, using at the end that $\chi_p \Pi_k^\sharp \psi u = \chi_p (\mathrm{I} - \Pi_k^\sharp)\psi u$ to obtain (7.1).

It therefore remains to prove (7.16). Let $\mathfrak{c} > 0$ be given and let $h_0 > 0$ be a sufficiently small constant. Let $k_0 > 0$, $\chi_-, \chi_+, \psi \in C^\infty(\overline{\Omega})$ such that (7.15) holds and let $C > 0$ denote a generic constant depending only on the previous quantities. Let $k \geq k_0$, suppose that $h \leq h_0$, let $u \in \mathcal{Z}_k$ and let $0 \leq j \leq p-1$.

By arguing exactly as at the start of the proof of Lemma 7.4, without loss of generality, we can assume that $\partial_\nu \chi_- = 0$ and thus the commutator estimates (7.2) and (7.3) hold (the first one by Lemma 6.3 and Definition 5.25, and the second one by Proposition 5.32).

Let $\chi_0, \chi_1 \in C^\infty(\overline{\Omega})$ be such that

$$\chi_- \prec_{(\mathfrak{c}/3)} \chi_0 \prec_{(\mathfrak{c}/3)} \chi_1 \prec_{(\mathfrak{c}/3)} \chi_+.$$

To prove (7.16), it is sufficient to show that, for all $v \in H_k^j$,

$$\left|\langle v, \chi_- \Pi_k^\sharp \psi u\rangle\right| \leq C\Big(\|\chi_+ \Pi_k^\sharp \psi u\|_{H_k^{-(j+1)}} + \big\|(\mathrm{I} - \Pi_k^\sharp)\psi u\big\|_{H_k^{-p}}\Big)\|v\|_{H_k^j}. \tag{7.17}$$

By the relation $P_k^\sharp R_k^\sharp = \mathrm{I}$, the definition of $\Pi_k^\sharp$ (Definition 7.1), and the fact that $\chi_- \psi = 0$, for all $w_h \in V_k$,

$$\begin{aligned}
\langle v, \chi_- \Pi_k^\sharp \psi u\rangle &= \big\langle P_k^\sharp R_k^\sharp v, \chi_-(\mathrm{I} - \Pi_k^\sharp)\psi u\big\rangle \\
&= \big\langle \chi_- P_k^\sharp R_k^\sharp v, (\mathrm{I} - \Pi_k^\sharp)\psi u\big\rangle \\
&= \big\langle P_k^\sharp \chi_- R_k^\sharp v, (\mathrm{I} - \Pi_k^\sharp)\psi u\big\rangle + \big\langle [\chi_-, P_k^\sharp] R_k^\sharp v, (\mathrm{I} - \Pi_k^\sharp)\psi u\big\rangle \\
&= \big\langle P_k^\sharp\big(\chi_- R_k^\sharp v - w_h\big), (\mathrm{I} - \Pi_k^\sharp)\psi u\big\rangle + \big\langle [\chi_-, P_k^\sharp] R_k^\sharp v, (\mathrm{I} - \Pi_k^\sharp)\psi u\big\rangle.
\end{aligned} \tag{7.18}$$

For the second term on the right-hand side of (7.18), since $\chi_- = \chi_- \chi_+$, by locality of P_k, and by the mapping properties of $R_k^\sharp$ (Proposition 5.23), $[\chi_-, P_k]$ (from (7.2)), and $[\chi_-, S_k]$ (from (7.3)),

$$\begin{aligned}
\Big|\big\langle [\chi_-, P_k^\sharp] R_k^\sharp v, (\mathrm{I} - \Pi_k^\sharp)\psi u\big\rangle\Big| &\leq \Big|\big\langle \chi_+ [\chi_-, P_k] R_k^\sharp v, (\mathrm{I} - \Pi_k^\sharp)\psi u\big\rangle\Big| + \Big|\big\langle [\chi_-, S_k] R_k^\sharp v, (\mathrm{I} - \Pi_k^\sharp)\psi u\big\rangle\Big| \\
&\leq C k^{-1}\|v\|_{H_k^j}\Big(\|\chi_+(\mathrm{I} - \Pi_k^\sharp)\psi u\|_{H_k^{-j-1}} + \big\|(\mathrm{I} - \Pi_k^\sharp)\psi u\big\|_{H_k^{-p}}\Big).
\end{aligned}$$

Furthermore, by the mapping properties of S_k (Proposition 5.22),

$$\Big|\big\langle S_k(\chi_- R_k^\sharp v - w_h), (\mathrm{I} - \Pi_k^\sharp)\psi u\big\rangle\Big| \leq C\big\|\chi_- R_k^\sharp v - w_h\big\|_{\mathcal{Z}_k^1}\big\|(\mathrm{I} - \Pi_k^\sharp)\psi u\big\|_{H_k^{-p}}.$$

Using the approximation property of V_k (Assumption 3.4) with $m = 1$, and using that $j+2 \leq p+1$, we can choose w_h supported in $\operatorname{supp}\chi_0$ such that

$$\sum_{K \in \mathcal{T}_k} (h_K k)^{-2(j+1)}\|\chi_- R_k^\sharp v - w_h\|^2_{H_k^1(K)} \leq C\|\chi_- R_k^\sharp v\|^2_{H_k^{j+2}} \leq C\|v\|^2_{H_k^j}. \tag{7.19}$$

In particular,

$$\begin{aligned}
\big\|\chi_- R_k^\sharp v - w_h\big\|^2_{\mathcal{Z}_k^1} = \sum_{K \in \mathcal{T}_k} \|\chi_- R_k^\sharp v - w_h\|_{H_k^1(K)} &\leq (hk)^{2j}\sum_{K \in \mathcal{T}_k} h_K^{-2j}\|\chi_- R_k^\sharp v - w_h\|_{H_k^1(K)} \\
&\leq C(hk)^{2j}\,\|v\|^2_{H_k^j}\,,
\end{aligned}$$

where we recall that $h := \max_{K \in \mathcal{T}_k} h_K$. Hence, with this choice of w_h, since V_k satisfies $hk \leq C$,

$$\Big|\big\langle S_k(\chi_- R_k^\sharp v - w_h), (\mathrm{I} - \Pi_k^\sharp)\psi u\big\rangle\Big| \leq C\|v\|_{H_k^j}\big\|(\mathrm{I} - \Pi_k^\sharp)\psi u\big\|_{H_k^{-p}}.$$

Therefore, to prove (7.17), it remains to prove that for this choice of $w_h \in V_k$,

$$\left|\big\langle P_k\big(\chi_- R_k^\sharp v - w_h\big), (\mathrm{I} - \Pi_k^\sharp)\psi u\big\rangle\right| \leq C\Big(\|\chi_+ \Pi_k^\sharp \psi u\|_{H_k^{-(j+1)}} + \big\|(\mathrm{I} - \Pi_k^\sharp)\psi u\big\|_{H_k^{-p}}\Big)\|v\|_{H_k^j}. \tag{7.20}$$

By (in this order) the locality of P_k and the fact that $\chi_0 \equiv 1$ on the support of $\chi_- R_k^\sharp v - w_h$, continuity of P_k, and (7.19),

$$\begin{aligned}
&|\langle P_k(\chi_- R_k^\sharp v - w_h), (\mathrm{I} - \Pi_k^\sharp)\psi u\rangle| \\
&\quad = |\langle P_k(\chi_- R_k^\sharp v - w_h), \chi_0(\mathrm{I} - \Pi_k^\sharp)\psi u\rangle| \\
&\quad \le \sum_{K\in\mathcal{T}_k} \|\chi_0(\mathrm{I} - \Pi_k^\sharp)\psi u\|_{H_k^1(K)} \|\chi_- R_{P_k^\sharp} v - w_h\|_{H_k^1(K)} \\
&\quad \le \Big(\sum_{K\in\mathcal{T}_k} (h_K k)^{2(j+1)} \|\chi_0 \Pi_k^\sharp \psi u\|^2_{H_k^1(K)}\Big)^{\frac12} \Big(\sum_{K\in\mathcal{T}_k} (h_K k)^{-2(j+1)} \|\chi_- R_{P_k^\sharp} v - w_h\|^2_{H_k^1(K)}\Big)^{\frac12} \\
&\quad \le C\Big(\sum_{K\in\mathcal{T}_k} (h_K k)^{2(j+1)} \|\chi_0 \Pi_k^\sharp \psi u\|^2_{H_k^1(K)}\Big)^{\frac12} \|v\|_{H_k^j}. \qquad (7.21)
\end{aligned}$$

To apply the arguments from [AGS24, Lemma 5.5], and especially the wavelength-scale quasi-uniformity (Assumption 3.3), we now need to group the elements $K \in \mathcal{T}_k$ into sets lying within balls of radius $\approx k^{-1}$.

To this end, we choose a sufficiently large constant $\mu > 0$ depending only on h_0 and k_0 and let $\{x_\ell\}_{\ell=1}^{\mathcal{L}} \subset \Omega$ be a "maximal μk^{-1} separated set", constructed inductively by choosing an initial point $x_1 \in \Omega$, and if $x_1, \dots, x_\ell$ are constructed, choosing $x_{\ell+1} \in \Omega \setminus \cup_{m=1}^{\ell} B(x_\ell, \mu k^{-1})$ if this set is not empty, or finishing the construction with $\ell = \mathcal{L}$ otherwise. By construction,

$$\Omega \subset \bigcup_{\ell=1}^{\mathcal{L}} B(x_\ell, \mu k^{-1}),$$

and one can check that for all $M > 0$, there exists $\mathfrak{D}_M > 0$ depending solely on M and the space dimension d, and there exists a partition of $\{1, \dots, \mathcal{L}\}$ into $\mathfrak{D}_M$ sets $\mathcal{J}_1^M, \mathcal{J}_2^M, \dots, \mathcal{J}_{\mathfrak{D}_M}^M$, such that

$$(\ell_1, \ell_2 \in \mathcal{J}_m^M \ \text{ and } \ \ell_1 \neq \ell_2) \Rightarrow B(x_{\ell_1}, M\mu k^{-1}) \cap B(x_{\ell_2}, M\mu k^{-1}) = \emptyset,$$

i.e., the maximal number of overlaps between balls of radius $M\mu k^{-1}$ with centers in $\{x_\ell\}_{\ell=1}^{\mathcal{L}}$ is $\mathfrak{D}_M$. Define

$$h_\ell := \max\big\{h_K \,:\, K \cap B(x_\ell, k^{-1}) \neq \emptyset\big\} \le C_M \inf\big\{h_K \,:\, K \cap B(x_\ell, M\mu k^{-1}) \neq \emptyset\big\},$$

where the second inequality follows from Assumption 3.3. For all $1 \le \ell \le \mathcal{L}$ and for $m \ge 1$, let $\chi_{\ell,m} \in C^\infty(\overline{\Omega})$ be such that

$$\operatorname{supp} \chi_{\ell,m} \subset B(x_\ell, (m+1)\mu k^{-1}) \cap \overline{\Omega}, \quad \operatorname{supp}(1 - \chi_{\ell,m}) \cap B(x_\ell, m\mu k^{-1}) \cap \overline{\Omega} = \emptyset. \qquad (7.22)$$

Using a construction via scaling, one can arrange that there exists a universal constant $C_\dagger$ such that

$$\|\partial^\alpha \chi_{\ell,m}\|_\infty \le C_\dagger (\mu k^{-1})^{-|\alpha|} \quad \text{for all } |\alpha| \le p. \qquad (7.23)$$

By choosing μ large enough, one ensures that $\mu k^{-1} \ge 2h$, which implies that $K \cap B(x_\ell, \mu k^{-1}) \neq \emptyset \implies K \subset B(x_\ell, 2\mu k^{-1})$. Therefore,

$$\begin{aligned}
\sum_{K\in\mathcal{T}_k} (h_K k)^{2(j+1)} \|\chi_0 \Pi_k^\sharp \psi u\|^2_{H_k^1(K)} &\le C \sum_{\ell=1}^{\mathcal{L}} (h_\ell k)^{2(j+1)} \sum_{K \cap B(x_\ell, \mu k^{-1}) \neq \emptyset} \|\chi_0 \Pi_k^\sharp \psi u\|^2_{H_k^1(K)} \\
&\le C \sum_{\ell=1}^{\mathcal{L}} (h_\ell k)^{2(j+1)} \sum_{K \subset B(x_\ell, 2\mu k^{-1})} \|\chi_0 \Pi_k^\sharp \psi u\|^2_{H_k^1(K)} \\
&\le C \sum_{\ell=1}^{\mathcal{L}} (h_\ell k)^{2(j+1)} \|\chi_0 \chi_{\ell,2} \Pi_k^\sharp \psi u\|^2_{\mathcal{Z}_k}. \qquad (7.24)
\end{aligned}$$

Next we apply Lemma 7.7 to estimate the norms $\|\chi_0 \chi_{\ell,2} \Pi_k^\sharp \psi u\|_{\mathcal{Z}_k}$. Choosing $R = 20\mu$, one gets that for every $\ell = 1, \dots, \mathcal{L}$,

$$\operatorname{supp}(\chi_0 \chi_{\ell,2}) \subset \operatorname{supp}(\chi_1 \chi_{\ell,4}) \subset B(x_\ell, Rk^{-1}/4),$$

so that assumption (1) is satisfied. Moreover, by definition of $\chi_{\ell,m}$ (see (7.22)),

$$d := \mathrm{dist}(\mathrm{supp}(\chi_0\chi_{\ell,2}), \mathrm{supp}(1-\chi_1\chi_{\ell_4})) \geq \mathrm{dist}(B(x_\ell, 3\mu k^{-1}), B(x_\ell, 4\mu k^{-1})^c) \geq \mu k^{-1},$$

showing that assumption (2) is also satisfied by taking μ sufficiently large. Assumption (3) follows from this and the control on the derivatives of $\chi_{\ell,m}$ in (7.23). Finally, we have $\mathrm{supp}(\chi_1\chi_{\ell,4}) \subset \mathrm{supp}\,\chi_1 \subset (\mathrm{supp}\,\psi)^c$ so that assumption (4) holds. Therefore, by Lemma 7.7, for $N \geq p$,

$$\|\chi_0\chi_{\ell,2}\Pi_k^\sharp \psi u\|^2_{\mathcal{Z}_k} \leq C\|\chi_1\chi_{\ell,4}\Pi_k^\sharp\psi u\|^2_{L^2(\Omega)} + C\sum_{n=0}^{\infty} 2^{-Nn}\|\varphi_{n,\ell}(\mathrm{I}-\Pi_k^\sharp)\psi u\|^2_{\mathcal{H}_k^{-p}} \tag{7.25}$$

where

$$\varphi_{n,\ell} = \phi_n\big((x-x_\ell)/(Rk^{-1})\big) \text{ for } n \geq 0, \tag{7.26}$$

with ϕ_n as in Lemma 7.6. (note that here it is crucial that $\mathfrak{C}$ in Lemma 7.7 does not depend on χ_- and χ_+).

By (7.20), (7.21), (7.24), and (7.25), to prove (7.17), it is sufficient to prove that

$$\sum_{\ell=1}^{\mathcal{L}}(h_\ell k)^{2(j+1)}\|\chi_1\chi_{\ell,4}\Pi_k^\sharp\psi u\|^2_{L^2(\Omega)} \leq C\|\chi_+\Pi_k^\sharp\psi u\|^2_{H_k^{-(j+1)}}. \tag{7.27}$$

and

$$\sum_{\ell=1}^{\mathcal{L}}(h_\ell k)^{2(j+1)}\sum_{n=0}^{\infty}2^{-Nn}\|\varphi_{n,\ell}(\mathrm{I}-\Pi_k^\sharp)\psi u\|^2_{H_k^{-p}} \leq C\|(\mathrm{I}-\Pi_k^\sharp)\psi u\|^2_{H_k^{-p}}. \tag{7.28}$$

By the wavelength-scale quasi-uniformity (Assumption 3.3) and the inverse inequality (Assumption 3.6),

$$\begin{aligned}\sum_{\ell=1}^{\mathcal{L}}(h_\ell k)^{2j+2}\|\chi_1\chi_{\ell,4}\Pi_k^\sharp\psi u\|^2_{L^2(\Omega)} &\leq C\sum_{\ell=1}^{\mathcal{L}}\sum_{K\cap\mathrm{supp}\,\chi_1\chi_{\ell,4}\neq\emptyset}(h_K k)^{2j+2}\|\Pi_k^\sharp\psi u\|^2_{L^2(K)}\\ &\leq C\sum_{\ell=1}^{\mathcal{L}}\sum_{K\cap\mathrm{supp}\,\chi_1\chi_{\ell,4}\neq\emptyset}\|\Pi_k^\sharp\psi u\|^2_{H^{-(j+1)}(K)}.\end{aligned}$$

Applying [AGS24, Lemma 5.2] (a simple bound on sums of negative Sobolev norms on elements by a global dual norm) and Lemma 7.8 below, there exists $M > 0$ large enough such that

$$\sum_{l=1}^{\mathcal{L}}\sum_{K\cap\mathrm{supp}\,\chi_1\chi_{\ell,4}\neq\emptyset}\|\Pi_k^\sharp\psi u\|^2_{H^{-(j+1)}(K)} \leq C\sum_{l=1}^{\mathcal{L}}\big\|\chi_+\chi_{\ell,M}\Pi_k^\sharp\psi u\big\|^2_{H_k^{-(j+1)}} \leq C\mathfrak{D}_M\|\chi_+\Pi_k^\sharp\psi u\|^2_{H_k^{-(j+1)}},$$

and the combination of these last two displayed equations is (7.27).

On the other hand, by the definition of $\varphi_{n,\ell}$ (7.26), there exists $C_{p,R} > 0$ such that for every $x \in \mathbb{R}^d$,

$$\sum_{\ell=1}^{\mathcal{L}}\sum_{n=0}^{\infty}2^{-Nn}\Big(\max_{1\leq\alpha\leq p}k^{-|\alpha|}|\varphi_{n,\ell}(x)|\Big)^2 \leq C_{p,R}\sum_{n=0}^{\infty}2^{-Nn}\kappa_n(x) \tag{7.29}$$

where

$$\kappa_n(x) = \mathrm{Card}\Big(\big\{1\leq\ell\leq\mathcal{L} \,:\, \mathrm{dist}(x,x_\ell)\leq 2^{n+1}Rk^{-1}\big\}\Big).$$

To estimate $\kappa_n(x)$, we write

$$\kappa_n(x) = \sum_{m=1}^{\mathfrak{D}_1}\mathrm{Card}(K_m(x)), \quad \text{where} \quad K_m(x) := \big\{\ell\in\mathcal{J}_m^1 \,:\, \mathrm{dist}(x,x_\ell)\leq 2^{n+1}Rk^{-1}\big\}$$

and since the μk^{-1} balls centered at x_ℓ for $\ell \in \mathcal{J}_m^1$ are pairwise disjoint,

$$\sum_{\ell\in K_m(x)}\mu(B(x_\ell,k\mu^{-1})) \leq \mu(B(x,2^{n+1}Rk^{-1})) \quad\Longleftrightarrow\quad \mathrm{Card}(K_m(x)) \leq 2^{d(n+1)}\frac{R^d}{\mu^d}.$$

This proves that $\kappa_n(x) \le C2^{nd}$. Inserting this bound into (7.29) and choosing N large enough, we find that

$$\sum_{\ell=1}^{\mathcal{L}} \sum_{n=0}^{\infty} 2^{-Nn} \Big(\max_{1\le\alpha\le p} k^{-|\alpha|} |\varphi_{n,\ell}(x)| \Big)^2 \le CC_{p,R}.$$

We now use Lemma 7.8 with $\{\chi_n\}_n = \{2^{-nN/2}\varphi_{n,\ell}, : \ell = 1,\dots,\mathcal{L}, n = 0,\dots,\infty\}$ to obtain that

$$\sum_{\ell=1}^{\mathcal{L}} \sum_{n=0}^{\infty} 2^{-Nn} \|\varphi_{n,\ell}(\mathrm{I} - \Pi_k^\sharp)\psi u\|^2_{H_k^{-p}} \le CC_{p,R} \|(\mathrm{I} - \Pi_k^\sharp)\psi u\|^2_{H_k^{-p}},$$

which is (7.28), and the proof is complete. ∎

Finally, we prove the following technical lemma used in the proof of Lemma 7.5.

Lemma 7.8 *Given $N > 0$ there exists $C_N > 0$ such that the following is true. Suppose that $(\chi_n)_n$ is such that there exists $C_{\rm over} > 0$ such that for all $x \in \mathbb{R}^d$,*

$$\sum_{n=0}^{\infty} \Big(\max_{|\alpha|\le N} k^{-|\alpha|} |\partial^\alpha \chi_n(x)| \Big)^2 \le C_{\rm over}.$$

Then, for all $v \in H_k^{-N}$,

$$\sum_n \|\chi_n v\|^2_{H_k^{-N}} \le C_N C_{\rm over} \|v\|^2_{H_k^{-N}}. \tag{7.30}$$

Proof. For every n, let $\theta_n := \|\chi_n v\|_{H_k^{-N}}$, let $\varphi_n \in H_k^N$ with unit norm, and let

$$\varphi := \sum_n \chi_n \theta_n \varphi_n.$$

Then by assumption,

$$\|\varphi\|^2_{\mathcal{H}_k^N} \le C_N C_{\rm over} \sum_n \theta_n^2.$$

Therefore,

$$\|v\|_{H_k^{-N}} \ge \frac{|(v,\varphi)|^2}{\|\varphi\|^2_{H_k^N}} \ge \frac{\big|\sum_n (v, \theta_n \chi_n \varphi_n)\big|^2}{C_N C_{\rm over} \sum_n \theta_n^2} = \frac{\big|\sum_n \theta_n (\chi_n v, \varphi_n)\big|^2}{C_N C_{\rm over} \sum_n \theta_n^2}.$$

By taking the supremum over each φ_n,

$$\|v\|_{H_k^{-N}} \ge \frac{1}{C_N C_{\rm over}} \frac{\big|\sum_n \theta_n^2\big|^2}{\sum_n \theta_n^2} = \frac{1}{C_N C_{\rm over}} \sum_n \theta_n^2,$$

and the result (7.30) follows. ∎

8 Proof of the main result (Theorem 3.11)

We fix $a_k : H_k^1 \times H_k^1 \to \mathbb{R}$, $\mathcal{J} \subset \mathbb{R}_+$, p, $(V_k)_{k>0}$ as in the statement of Theorem 3.11, and keep the definitions and notations from Sections 3, 5, and 7. We denote by $h = h(k) := \max_{K\in\mathcal{T}_k} h_K$.

8.1 Outline of the proof

Let $X^-(\ell), X^+(\ell) \in \mathbb{R}^{M_{\rm I}}, X^{\rm P} \in \mathbb{R}^{M_{\rm P}}$ be the column vectors defined by

$$X_i^-(\ell) := \|\chi_i \Psi(u-u_h)\|_{H_k^\ell}, \quad X_i^+(\ell) := \|\chi_i(1-\Psi)(u-u_h)\|_{H_k^\ell}, \quad 1 \le i \le M_{\rm I},$$

$$\text{and} \quad X_i^{\rm P}(\ell) := \|\chi_{i+M_{\rm I}}(u-u_h)\|_{H_k^\ell} \quad 1 \le i \le M_{\rm P}$$

and let $Z \in \mathbb{R}^M$ be the column vector of local best approximation errors, i.e.,

$$Z_i = \|u - w_h\|_{H_k^1(\Omega_i)}$$

where w_h is an arbitrary, fixed element of V_k. The heart of the proof of Theorem 3.11 consists of forming a matrix system of inequalities for the vector $X(\ell) = \begin{pmatrix} X^-(\ell) \\ X^+(\ell) \\ X^{\mathrm{P}}(\ell) \end{pmatrix}$. We start by obtaining this system in the lowest possible norm, which is dictated by the polynomial order p, i.e., with $\ell = -p+1$. In Lemma 8.10, we show that

$$X(-p+1) \le C_\dagger W X(-p+1) + CBZ + R \tag{8.1}$$

where R is a superalgebraically small remainder term, where W and B are the matrices defined in (3.13) and (3.12), and where $C_\dagger, C$ are positive constants. Therefore, if $(I - C_\dagger W)^{-1}$ exists, then

$$X(-p+1) \le \mathfrak{C}(I - C_\dagger W)^{-1} BZ + R\,.$$

Each line of the inequality (8.1) is obtained by applying a localised version of the elliptic-projection argument, Lemma 8.1, and exploiting both the local behaviour of the mesh size and the microlocal behaviour of the solution operator of the continuous problem from §4 (and in particular, its improved behaviour on high-frequencies or in the PML region, leading to Lemmas 8.7 and 8.8).

We then use Theorem 8.13 to bound $(I - C_\dagger W)^{-1}$ in terms of the simple-path matrix $T^\star$ of $C_\dagger W$ (Definition 3.9), giving

$$X(-p+1) \le \mathfrak{C} T^\star BZ + R\,. \tag{8.2}$$

Next, we upgrade (8.2) to higher norms, i.e., we estimate $X(\ell)$ for $1-p \le \ell \le 1$. For this, we notice that, on the one hand, since Ψ is smoothing and pseudo-local,

$$X^-(\ell) \le CX^-(-p+1) + R \tag{8.3}$$

for all $1-p \le \ell \le 1$. (In fact, (8.3) should actually have an $\widetilde{X}^-$ on the right-hand side involving $\widetilde{\chi}_i$ such that $\chi_i \prec \widetilde{\chi}_i$, but we have neglected this in this outline for brevity.)

On the other hand, by the "improved" local duality arguments of Lemmas 8.7 and 8.8,

$$X^+(\ell) \lesssim (\mathcal{H}k)^{\ell+1} Z + (\mathcal{H}k)^{p+\ell+1} X^-(-p+1) + (\mathcal{H}^{\min}(N)k^N) X^+(-p+1) + R \tag{8.4}$$

$$X^{\mathrm{P}}(\ell) \lesssim (\mathcal{H}k)^{\ell+1} Z + (\mathcal{H}^{\min}(N)k^N)\big(X^+(-p+1) + X^-(-p+1)\big) + R \tag{8.5}$$

for all $1-p \le \ell \le 0$. Combined with (8.2), this gives the bounds in the second and third block rows of (3.17), up to the L^2 norm. Finally, to obtain (8.4) and (8.5) in the H_k^1 norm, i.e., for $\ell = 1$, we use Lemmas 8.15 and 8.17 which give

$$X^+(1) \lesssim Z + X^+(0) + (\mathcal{H}_{\mathrm{I,I}}k)^p X^-(-p+1) + R, \tag{8.6}$$

$$X^{\mathrm{P}}(1) \lesssim Z + X^{\mathrm{P}}(0) + R. \tag{8.7}$$

The estimates in the H_k^1 norm are then obtained by inserting (8.2), (8.4), (8.5) into (8.6) and (8.7).

8.2 Localised duality argument

The next result relates the Galerkin error in some region A of phase-space to (i) the set of local best-approximation errors in subdomains covering Ω and (ii) the set of local Galerkin errors in these subdomains, modulo a small global term. The subdomain contributions are weighted by "transfer coefficients" $\eta_{j\to A}$ that describe the corresponding local behavior of the Helmholtz solution operator. This result is applied several times later in the proof of Theorem 3.11, for special choices of the partition of unity $\{\phi_j\}$ and operators A.

Lemma 8.1 (Localised duality argument) *Given $\mathfrak{c}$, $k_0 > 0$, there exists h_0 such that the following holds. Let $N > 0$ and let $\{\phi_j\}_{j=1}^J$ and $\{\tilde{\phi}_j\}_{j=1}^J$ be such that*

$$\phi_j, \tilde{\phi}_j \in C_c^\infty(\mathbb{R}^d, [0,1]), \qquad \phi_j \prec_{\mathfrak{c}} \tilde{\phi}_j, \qquad j = 1, \ldots, J, \tag{8.8}$$

and $\sum_{j=1}^J \phi_j = 1$ on Ω. For any $k > 0$, define

$$h_j := \max\Big\{h_K \ : \ K \in \mathcal{T}_k \ \text{ s.t. } \ K \cap \operatorname{supp}\tilde{\phi}_j \neq \emptyset\Big\}.$$

Then there exists $C > 0$ such that for each $\ell \in \{0, \ldots, p-1\}$, for all $k \geq k_0$, $k \notin \mathcal{J}$, with $h \leq h_0$, for all $A : H_k^{-\ell} \to L^2(\Omega)$, for all $u - u_h$ satisfying (3.16), and for all $w_{h,j} \in V_k, j = 1, \ldots, J$,

$$\begin{aligned}\|A(u-u_h)\|_{L^2} \leq C \sum_{j=1}^J \eta_{j\to A}(h_j k)^p \Big((h_j k)^{-p} \|\tilde{\phi}_j(u - w_{h,j})\|_{H_k^1} + \|\tilde{\phi}_j(u-u_h)\|_{H_k^{-N}} \Big) \\ + Ck^{-N}(hk)^{p+\ell+1}\|A\|_{H_k^{-\ell}\to L^2} \Big(\sum_{j=1}^J (hk)^{-p}\|u - w_{h,j}\|_{H_k^1} + \|u-u_h\|_{H_k^{-N}}\Big)\end{aligned}$$

where for all $j \in \{1, \ldots, J\}$,

$$\eta_{j\to A} := (h_j k)^p \|\tilde{\phi}_j R_k^* A^*\|_{L^2\to L^2} + (h_j k)^{\ell+1}\|\tilde{\phi}_j (R_k^\sharp)^* A^*\|_{L^2 \to H_k^{\ell+2}}. \tag{8.9}$$

To prove Lemma 8.1 we use the following two lemmas; the first is a localised version of the classic Aubin-Nitsche duality argument applied to the operator $P_k^\sharp$ defined in (5.14), and the second is a localised version of the bound on the adjoint-approximability constant from [GS25, Theorem 1.7] (with similar bounds appearing in [MS10, MS11, CFN20, LSW22b, GLSW23, GLSW24, BCFM25]). Recall the definition of $\Pi_k^\sharp$ from Definition 7.1, and let

$$\Pi_k : H_k^1 \to V_k$$

be the H_k^1-orthogonal projection onto V_k.

Lemma 8.2 (Localised Aubin-Nitsche argument for $P_k^\sharp$) *For any $\mathfrak{c} > 0$, there exists $h_0 > 0$ such that the following holds. Let $k_0 > 0$, $N > 0$, $\chi \in C^\infty(\overline{\Omega})$ and $U \subset \overline{\Omega}$ be such that*

$$\operatorname{supp}\chi \subset U, \quad \partial_<(\operatorname{supp}\chi, U) \geq \mathfrak{c},$$

where the notation $\partial_<$ is defined by (3.4). For any $k > 0$, let

$$h_U := \max\big\{h_K \ : \ K \in \mathcal{T}_k, \ K \cap U \neq \emptyset\big\}.$$

Then, there exists $C > 0$ such that for all $\ell = 0 \ldots, p-1$, $k \geq k_0$, $h \leq h_0$, and $u \in H_k^1$,

$$\|\chi(\mathrm{I} - \Pi_k^\sharp)u\|_{H_k^{-\ell}} \leq C\left((h_U k)^{\ell+1} + k^{-N}(hk)^{\ell+1}\right)\|(\mathrm{I} - \Pi_k)u\|_{H_k^1}.$$

Proof. Fix $\mathfrak{c} > 0$, and let h_0 be such that $\mathfrak{c} \geq 2\kappa h_0$ where κ is as in Assumption 3.4. Let N, χ and U as in the statement, and let C denote a generic constant depending only on the previous quantities. Let $\widetilde{\chi} \in C^\infty(\overline{\Omega})$ be such that $\chi \prec_{\mathfrak{c}/2} \widetilde{\chi}$, $\operatorname{supp}\widetilde{\chi} \subset U$, and

$$\partial_<(\operatorname{supp}\widetilde{\chi}, U) \geq \mathfrak{c}/2.$$

Let $\ell \in \{0, \ldots, p-1\}$, and let $v \in H_k^\ell$ be such that $\|v\|_{H_k^\ell} = 1$.

By the Definition of $\Pi_k^\sharp$ (Definition 7.1), for all $w_{h,1}, w_{h,2} \in V_k$, letting $w_h := w_{h,1} + w_{h,2}$,

$$\begin{aligned}\big|\langle v, \chi(\mathrm{I} - \Pi_k^\sharp)u\rangle\big| &= \big|\langle \chi v, (\mathrm{I} - \Pi_k^\sharp)u\rangle\big| \\ &= \big|\langle P_k^\sharp(R_k^\sharp \chi v - w_h), (\mathrm{I} - \Pi_k^\sharp)u\rangle\big|\end{aligned}$$

$$\leq C\|R_k^\sharp \chi v - w_h\|_{H_k^1}\|(\mathrm{I}-\Pi_k^\sharp)u\|_{H_k^1}$$
$$\leq C\Big(\|\widetilde{\chi}R_k^\sharp\chi v - w_{h,1}\|_{H_k^1} + \|(1-\widetilde{\chi})R_k^\sharp\chi v - w_{h,2}\|_{H_k^1}\Big)\inf_{w_h\in V_k}\|u-w_h\|_{H_k^1}, \quad (8.10)$$

where we used Céa's lemma for the coercive operator $P_k^\sharp$ in the last step. By the approximation property of V_k (Assumption 3.4), $w_{h,1}, w_{h,2} \in V_k$ can be chosen such that

$$\sum_{K\in\mathcal{T}_k}(h_K k)^{2-2(\ell+2)}\|\widetilde{\chi}R_k^\sharp\chi v - w_{h,1}\|^2_{H_k^1(K)} \leq C\|\widetilde{\chi}R_k^\sharp\chi v\|^2_{H_k^{\ell+2}},$$

$$\sum_{K\in\mathcal{T}_k}(h_K k)^{2-2(\ell+2)}\|(1-\widetilde{\chi})R_k^\sharp\chi v - w_{h,2}\|^2_{H_k^1(K)} \leq C\|(1-\widetilde{\chi})R_k^\sharp\chi v\|^2_{H_k^{\ell+2}},$$

with in addition $\operatorname{supp} w_{h,1} \subset U$. In this case, by the definition of h_U and h,

$$\|\widetilde{\chi}R_k^\sharp\chi v - w_{h,1}\|_{H_k^1} \leq C(h_U k)^{\ell+1}\|\widetilde{\chi}R_k^\sharp\chi v\|_{H_k^{\ell+2}}, \quad \text{and} \quad (8.11)$$

$$\|(1-\widetilde{\chi})R_k^\sharp\chi v - w_{h,1}\|_{H_k^1} \leq C(hk)^{\ell+1}\|(1-\widetilde{\chi})R_k^\sharp\chi v\|_{H_k^{\ell+2}}. \quad (8.12)$$

Using (8.11) and (8.12) in (8.10) and the estimates

$$\|\widetilde{\chi}R_k^\sharp\chi v\|_{H_k^{\ell+2}} \leq C\|R_k^\sharp v\|_{H_k^{\ell+2}} \leq C\|v\|_{H_k^\ell},$$

(by the mapping properties of $R_k^\sharp$, Proposition 5.23) and

$$\|(1-\widetilde{\chi})R_k^\sharp\chi v\|_{H_k^{\ell+2}} \leq C\|R_k^\sharp v\|_{H_k^{\ell+2}} \leq Ck^{-N}\|v\|_{H_k^\ell},$$

(by pseudo-locality of $R_k^\sharp$, Theorem 6.2), we obtain

$$\big|\langle v, \chi(\mathrm{I}-\Pi_k^\sharp)u\rangle\big| \leq C\big((h_U k)^{\ell+1} + k^{-N}(hk)^{\ell+1}\big)\inf_{w_h\in V_k}\|u-w_h\|_{H_k^1}$$

and the conclusion follows by taking the supremum over v. ∎

Lemma 8.2 has the following special case when $\chi \equiv 1$ on $\overline{\Omega}$:

Corollary 8.3 *Given $k_0 > 0$, there exists $C > 0$ such that for all $\ell \in \{0, \dots, p-1\}$, and for all $u \in H_k^\ell \cap \mathcal{Z}_k$,*

$$\|(\mathrm{I}-\Pi_k^\sharp)u\|_{H_k^{-\ell}} \leq C(hk)^{\ell-1}\|(\mathrm{I}-\Pi_k)u\|_{H_k^1}.$$

Definition 8.4 (Localised adjoint-approximability constant) *For $A : H_k^{-\ell} \to L^2$ and $\phi \in C^\infty(\overline{\Omega})$, define the* localised adjoint-approximability constant *associated to ϕ and A as*

$$\eta(\phi \to A) := \|(\mathrm{I}-\Pi_k)\phi R_k^* A^*\|_{L^2\to H_k^1}.$$

Lemma 8.5 (Bound on $\eta(\phi \to A)$) *For all $k_0 > 0$ and $\mathfrak{c} > 0$, there exists $h_0 > 0$ such that, for all $N > 0$, $\phi, \widetilde{\phi} \in C^\infty(\overline{\Omega})$ with $\phi_j \prec_{\mathfrak{c}} \widetilde{\phi}$, there exists $C > 0$ such that for all $h \leq h_0$, for all $k \geq k_0$, and for all $A : H_k^{-\ell} \to L^2$,*

$$\eta(\phi \to A) \leq C\Big((h_{\tilde{\phi}}k)^p\|\tilde{\phi}R_k^* A^*\|_{L^2\to L^2} + (h_{\tilde{\phi}}k)^{\ell+1}\|\phi(R_k^\sharp)^* A^*\|_{L^2\to H_k^{\ell+2}} + (hk)^p k^{-N}\Big),$$

where $h_{\tilde{\phi}} := \max\Big\{h_K \;:\; K \in \mathcal{T}_k \;\text{ s.t. }\; K \cap \operatorname{supp}\tilde{\phi} \neq \emptyset\Big\}$.

Proof. Let $k_0, \mathfrak{c} > 0$ and let $h_0 > 0$ be such that $\mathfrak{c} \geq \kappa h_0$ where κ is as in Assumption 3.4. Let $N > 0$, $\phi, \widetilde{\phi}$ be as in the statement. Let $\check{\phi} \in C^\infty(\overline{\Omega})$ be such that $\phi \prec \check{\phi} \prec \widetilde{\phi}$. Let C denote a generic positive constant depending only on the previous quantities. Since $(P_k^\sharp)^* = P_k^* + S_k$, applying $(R_k^\sharp)^*$ to the left, and then R_k^* to the right, we obtain that

$$R_k^* = (R_k^\sharp)^* + (R_k^\sharp)^* S_k R_k^*. \quad (8.13)$$

Thus

$$\begin{aligned}\eta(\phi \to A) &\le \|(\mathrm{I}-\Pi_k)\phi(R_k^\sharp)^*A^*\|_{L^2\to H_k^1} + \|(\mathrm{I}-\Pi_k)\phi(R_k^\sharp)^*S_kR_k^*A^*\|_{L^2\to H_k^1}\\ &\le C(h_{\tilde\phi}k)^{\ell+1}\|\phi(R_k^\sharp)^*A^*\|_{L^2\to H_k^{\ell+2}} + C(h_{\tilde\phi}k)^p\|\phi(R_k^\sharp)^*S_kR_k^*A^*\|_{L^2\to H_k^{p+1}} \qquad (8.14)\end{aligned}$$

by the approximation property of V_k (Assumption 3.4), which can be applied since $\phi R_k^\sharp$ maps L^2 into $\mathcal{Z}_k$ (i.e., $\phi R_k^\sharp$ satisfies a zero Dirichlet boundary condition on $\Gamma_{\rm tr}$ and, if necessary, also on $\partial\Omega_-$). Finally, one can use pseudolocality of $(R_k^\sharp)^*$ and S_k (Theorem 6.2) to "move ϕ to the right of S_k" in the second term, as follows

$$\begin{aligned}\phi(R_k^\sharp)^*S_k &= \phi(R_k^\sharp)^*S_k\widetilde\phi + \phi(R_k^\sharp)^*S_k(1-\widetilde\phi)\\ &= \phi(R_k^\sharp)^*S_k\widetilde\phi + \phi(R_k^\sharp)^*[\check\phi S_k(1-\widetilde\phi)] + \phi[(R_k^\sharp)^*(1-\check\phi)]S_k(1-\widetilde\phi)\\ &= \phi(R_k^\sharp)^*S_k\widetilde\phi + \phi(R_k^\sharp)^*O_{-\infty}(k^{-\infty};\mathcal{Y}_k\to\mathcal{Y}_k) + O_{-\infty}(k^{-\infty};\mathcal{Y}_k\to\mathcal{Y}_k)S_k(1-\widetilde\phi)\\ &= \phi(R_k^\sharp)^*S_k\widetilde\phi + O_{-\infty}(k^{-\infty};\mathcal{Y}_k\to\mathcal{Y}_k), \qquad (8.15)\end{aligned}$$

using the mapping properties of S_k (Proposition 5.22) and of $R_k^\sharp$ (Proposition 5.23). Inserting (8.15) into (8.14) and using the continuity of $R_k^\sharp$ from H_k^{p-1} to H_k^{p+1} (Proposition 5.23) and of S_k from $L^2\to H_k^{p-1}$ (Proposition 5.22), the result follows. ∎

Proof of Lemma 8.1. Let $\mathfrak{c}, k_0 > 0$, and let $h_0 > 0$ be small enough to apply Theorem 7.2, Lemma 8.2 and Lemma 8.5. Fix $\{\phi_j\}_{j=1}^J$, $\{\widetilde\phi_j\}_{j=1}^J$ as in the statement, and let $N > 0$. Let C denote a generic constant (whose value may change from line to line) depending only on the previous quantities. Let $k \ge k_0$ with $k \notin \mathcal{J}$. By Assumption 3.1, there exists $N' > 0$ such that

$$k^{-N'}\rho(k) \le Ck^{-N}. \qquad (8.16)$$

Let $v \in L^2$ with $\|v\|_{L^2} = 1$. Arguing as in (1.29), we obtain that, for all $w_{h,j} \in V_k$, $j = 1,\ldots,J$,

$$\big\langle A(u-u_h), v\big\rangle = \sum_{j=1}^J \big\langle u - w_{h,j}, (P_k^\sharp)^*(\mathrm{I}-\Pi_k^\sharp)\phi_jR_k^*A^*v\big\rangle - \sum_{j=1}^J\big\langle u-u_h, S_k(\mathrm{I}-\Pi_k^\sharp)\phi_jR_k^*A^*v\big\rangle. \qquad (8.17)$$

The plan is to use the pseudo-locality properties of $(P_k^\sharp)$, S_k and $(\mathrm{I}-\Pi_k^\sharp)$ shown in Sections 5-7, to show that, up to small remainders,

$$P_k^\sharp(\mathrm{I}-\Pi_k^\sharp)\phi_j \approx \widetilde\phi_jP_k^\sharp\check\phi_j(\mathrm{I}-\Pi_k^\sharp)\phi_j \quad\text{and}\quad S_k(\mathrm{I}-\Pi_k^\sharp)\phi_j \approx \widetilde\phi_jS_k\check\phi_j(\mathrm{I}-\Pi_k^\sharp)\phi_j\,,$$

where $\check\phi_j \in C_c^\infty(\mathbb{R}^2,[0,1])$ is such that

$$\phi_j \prec_{\mathfrak{c}/4} \check\phi_j \prec_{\mathfrak{c}/4} \widetilde\phi_j \quad \text{for all } j\in\{1,\ldots,J\}.$$

To achieve this, we rewrite the difference as

$$X(\mathrm{I}-\Pi_k^\sharp)\phi_j - \widetilde\phi_jX\check\phi_j(\mathrm{I}-\Pi_k^\sharp)\phi_jv = X(1-\check\phi_j)(\mathrm{I}-\Pi_k^\sharp)\phi_j + (1-\widetilde\phi_j)X\check\phi_j(\mathrm{I}-\Pi_k^\sharp)\phi_j, \qquad (8.18)$$

where X is either $(P_k^\sharp)^*$ or S_k. First, when $X = S_k$, (8.18) gives, for all $w \in H_k^{\ell+2}\cap\mathcal{Z}_k$,

$$\begin{aligned}&\|S_k(\mathrm{I}-\Pi_k^\sharp)\phi_jw - \widetilde\phi_jS_k\check\phi_j(\mathrm{I}-\Pi_k^\sharp)\phi_jw\|_{H_k^N}\\ &\quad\le \|S_k\|_{H_k^1\to H_k^N}\|(1-\check\phi_j)(\mathrm{I}-\Pi_k^\sharp)\phi_jw\|_{H_k^1} + \|(1-\widetilde\phi_j)S_k\check\phi_j\|_{H_k^{-p}\to H_k^N}\|(\mathrm{I}-\Pi_k^\sharp)\phi_jw\|_{H_k^{-p}}\\ &\quad\le Ck^{-N'}\|(\mathrm{I}-\Pi_k^\sharp)\phi_jw\|_{H_k^{-p}} \qquad \text{(by Theorem 7.2 and (5.17) of Theorem 5.27)}\\ &\quad\le Ck^{-N'}\|(\mathrm{I}-\Pi_k^\sharp)\phi_jw\|_{H_k^{-p+1}}\\ &\quad\le Ck^{-N'}(hk)^{p+\ell+1}\|w\|_{H_k^{\ell+2}} \qquad \text{(by Corollary 8.3)}, \qquad (8.19)\end{aligned}$$

where the condition that $w \in \mathcal{Z}_k$ is need to apply Corollary 8.3. In particular, taking $w = R_k^* A^* v$, (8.19) gives

$$\begin{aligned}\Big\| \Big(S_k (\mathrm{I} - \Pi_k^\sharp)\phi_j - \widetilde{\phi}_j S_k \check{\phi}_j (\mathrm{I} - \Pi_k^\sharp)\phi_j \Big) R_k^* A^* v \Big\|_{H_k^N} \\ \leq C k^{-N'} (hk)^{p+\ell+1} \big(1+\rho(k)\big) \|A^*\|_{L^2 \to H_k^\ell} \|v\|_{L^2} \\ \leq C k^{-N} (hk)^{p+\ell+1} \|A^*\|_{L^2 \to H_k^\ell}\end{aligned} \tag{8.20}$$

using the mapping properties of R_k^* from Proposition 5.17 and the definition of N' in (8.16).

Similarly, when $X = P_k^\sharp$, (8.18) gives

$$\begin{aligned}&\|(P_k^\sharp)^* (\mathrm{I} - \Pi_k^\sharp)\phi_j w - \widetilde{\phi}_j (P_k^\sharp)^* \check{\phi}_j (\mathrm{I} - \Pi_k^\sharp)\phi_j w\|_{H_k^{-1}} \\ &\quad \leq \|P_k^\sharp\|_{H_k^1 \to H_k^{-1}} \|(1-\check{\phi}_j)(\mathrm{I} - \Pi_k^\sharp)\phi_j w\|_{H_k^1} + \|(1-\widetilde{\phi}_j)(P_k^\sharp)^*\check{\phi}_j\|_{H_k^1 \to H_k^{-1}} \|(\mathrm{I} - \Pi_k^\sharp)\phi_j w\|_{H_k^1} \\ &\quad \leq C k^{-N'} \|(\mathrm{I} - \Pi_k^\sharp)\phi_j w\|_{H_k^{-p}} + C k^{-N} (hk)^{\ell+1} \|w\|_{H_k^{\ell+2}} \\ &\qquad\qquad \text{(by Theorem 7.2, (5.17) of Theorem 5.37, and Assumption 3.4)} \\ &\quad \leq C k^{-N'} \big((hk)^{p+\ell+1} + (hk)^{\ell+1}\big) \|w\|_{H_k^{\ell+2}}, \qquad \text{(by Corollary 8.3)}.\end{aligned} \tag{8.21}$$

Choosing again $w = R_k^* A^* v$ in (8.21),

$$\begin{aligned}\Big\| \Big((P_k^\sharp)^* (\mathrm{I} - \Pi_k^\sharp)\phi_j - \widetilde{\phi}_j (P_k^\sharp)^* \check{\phi}_j (\mathrm{I} - \Pi_k^\sharp)\phi_j \Big) R_k^* A^* v \Big\|_{H_k^{-1}} \\ \leq C k^{-N} (hk)^{\ell+1} \|A^*\|_{L^2 \to H_k^\ell} \|v\|_{L^2}.\end{aligned} \tag{8.22}$$

Therefore, by the combination of (8.17), (8.20) and (8.22),

$$\begin{aligned}&\Big|\big\langle A(u-u_h), v\big\rangle\Big| \\ &\leq C\Bigg(\sum_{j=1}^J \Big|\big\langle u - w_{h,j}, \widetilde{\phi}_j (P_k^\sharp)^* \check{\phi}_j (\mathrm{I} - \Pi_k^\sharp)\phi_j R_k^* A^* v\big\rangle\Big| + \sum_{j=1}^J \Big|\big\langle u - u_h, \widetilde{\phi}_j S_k \check{\phi}_j (\mathrm{I} - \Pi_k^\sharp)\phi_j R_k^* A^* v\big\rangle\Big| + R \Bigg), \\ &= C\Bigg(\sum_{j=1}^J \Big|\big\langle P_k^\sharp \widetilde{\phi}_j (u - w_{h,j}), \check{\phi}_j (\mathrm{I} - \Pi_k^\sharp)\phi_j R_k^* A^* v\big\rangle\Big| + \sum_{j=1}^J \Big|\big\langle S_k \widetilde{\phi}_j (u - u_h), \check{\phi}_j (\mathrm{I} - \Pi_k^\sharp)\phi_j R_k^* A^* v\big\rangle\Big| + R \Bigg),\end{aligned} \tag{8.23}$$

where

$$R := k^{-N} (hk)^{\ell+1} \|A^*\|_{L^2 \to H_k^\ell} \Big(\sum_{j=1}^J \|u - w_{h,j}\|_{H_k^1} + (hk)^p \|u - u_h\|_{H_k^{-N}} \Big)$$

(where we have used that $\|v\|_{L^2} = 1$). Since $a_k^\sharp$ is coercive, Céa's lemma implies that

$$\|(\mathrm{I} - \Pi_k^\sharp) v\|_{H_k^1} \leq C \|(\mathrm{I} - \Pi_k) v\|_{H_k^1}.$$

Therefore, for each $j \in \{1, \ldots, J\}$,

$$\|(\mathrm{I} - \Pi_k^\sharp)\phi_j R_k^* A^* v\|_{H_k^1} \leq C \eta(\phi_j \to A) \|v\|_{L^2} = C\eta(\phi_j \to A), \tag{8.24}$$

where $\eta(\phi_j \to A)$ is the localised adjoint-approximability constant defined in Definition 8.4. Similarly, by Lemma 8.2 with $\ell = p-1$,

$$\|\check{\phi}_j (\mathrm{I} - \Pi_k^\sharp)\phi_j R_k^* A^* v\|_{H_k^{-p+1}} \leq C\big((h_j k)^p + k^{-N'}(hk)^p\big)\eta(\phi_j \to A), \tag{8.25}$$

By (8.24), (8.25) and the mapping properties of S_k (Proposition 5.22) in (8.23),

$$\Big|\big\langle A(u-u_h), v\big\rangle\Big| \leq C \sum_{j=1}^J \eta(\phi_j \to A) \Big(\|\widetilde{\phi}_j (u - w_{h,j})\|_{H_k^1} + (h_j k)^p \|\widetilde{\phi}_j (u - u_h)\|_{H_k^{-N}} \Big) + C(R + R')$$

where $R' = k^{-N}(hk)^p(\sum_{j=1}^J \|u-w_{h,j}\|_{H^1_k} + (hk)^p\|u-u_h\|_{H_k^{-N}})$, using the fact that R_k^*, and thus $\eta(\phi_j \to A)$, are polynomially bounded on $\mathbb{R}_+ \setminus \mathcal{J}$, thanks to Assumption 3.1 and Proposition 5.17 (while $R_k^\sharp$ is bounded by Proposition 5.23). The result then follows by using Lemma 8.5 to estimate the constants $\eta(\phi_j \to A)$, and taking the supremum over v. ■

8.3 Improvements at high frequency and in the PML region

We now use the improved behavior of the resolvent on (i) high-frequency functions and (ii) functions localised in the PML region to improve Lemma 8.1.

Lemma 8.6 (Improvement of R_k^* on high-frequencies) *Let $\psi \in C_c^\infty(\mathbb{R})$ satisfy $\psi^\sharp \prec \psi$ and let $\Psi := \psi(\mathcal{P}_k)$ and let $\varphi \in C^\infty(\overline{\Omega})$ be such that $\operatorname{supp}\varphi \cap \Gamma_{\rm tr} = \emptyset$. Then*

$$R_k^*(1-\Psi)\varphi = (R_k^\sharp)^*(1-\Psi)\varphi + O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k).$$

Proof. We use again the resolvent identity (8.13) to write

$$R_k^*(1-\Psi) = (R_k^\sharp)^*(1-\Psi) + R_k^* S_k (R_k^\sharp)^*(1-\Psi).$$

The idea is to now use pseudolocality of $(R_k^\sharp)^*$ to move $(1-\Psi)$ next to S_k, with this product then zero since $\psi^\sharp(1-\psi) = 0$. The issue is that we have only shown that $(R_k^\sharp)^*$ is pseudolocal with respect to frequency cut-offs when sandwiched by appropriate spatial cut offs – see Lemma 6.9 and Theorem 5.37.

To this end, let $\varphi_{\rm P,1}, \varphi_{\rm P,2} \in C^\infty(\overline{\Omega})$ be such that $\varphi \prec \varphi_{\rm P,1} \prec \varphi_{\rm P,2}$, and

$$\operatorname{supp}(\varphi_{\rm P,2}) \cap \Gamma_{\rm tr} = \emptyset \quad \text{and} \quad \operatorname{supp}(1-\varphi_{\rm P,1}) \cap \partial\Omega_- = \emptyset. \tag{8.26}$$

By Lemma 6.2 applied to both $(1-\Psi)$ and $(R_k^\sharp)^*$,

$$\begin{aligned}
R_k^* S_k (R_k^\sharp)^*(1-\Psi)\varphi &= R_k^* S_k (R_k^\sharp)^* \varphi_{\rm P,1}(1-\Psi)\varphi + O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k) \\
&= R_k^* S_k \varphi_{\rm P,2} (R_k^\sharp)^* \varphi_{\rm P,1}(1-\Psi)\varphi + O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k), \\
&= R_k^* S_k \varphi_{\rm P,2} (R_k^\sharp)^* \varphi_{\rm P,2}\varphi_{\rm P,1}(1-\Psi)\varphi + O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k), \\
&= R_k^* S_k \big(\varphi_{\rm P,2} (R_k^\sharp)^* \varphi_{\rm P,2}\big)(1-\Psi)\varphi + O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k).
\end{aligned}$$

By Lemma 6.9 (with $\varphi = \varphi_{\rm P,2}$) $\varphi_{\rm P,2}(R_k^\sharp)^*\varphi_{\rm P,2} \in \mathcal{L}^{\rm f}_{-2}$. Thus by Theorem 5.37, with $\psi^\sharp \prec \widetilde{\psi} \prec \psi$,

$$\begin{aligned}
R_k^* S (R_k^\sharp)^*(1-\Psi)\varphi &= R_k^* S \varphi_{\rm P,2}(R_k^\sharp)^*\varphi_{\rm P,2}(1-\Psi)\varphi + O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k) \\
&= R_k^* S (1-\widetilde{\Psi})\varphi_{\rm P,2}(R_k^\sharp)^*\varphi_{\rm P,2}(1-\Psi)\varphi + O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k) \\
&= O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k),
\end{aligned}$$

where we have used that

$$O_{-\infty}(k^{-\infty}; \mathcal{D}_k \to \mathcal{D}_k) = O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k),$$

since, for any $n \in \mathbb{Z}$, $\mathcal{D}_k^{|n|} \subset \mathcal{Y}_k^n \subset \mathcal{D}_k^{-|n|}$ with continuous inclusions (by Corollary 5.20). ■

Lemma 8.7 (High-frequency upgrade) *For any $\mathfrak{c}$, k_0, there exists h_0 such that the following is true. Let $N > 0$, let $\psi \in C_c^\infty(\mathbb{R})$ satisfy $\psi^\sharp \prec \psi$ and let $\Psi := \psi(\mathcal{P}_k)$. Let $\phi, \widetilde{\phi} \in C^\infty(\overline{\Omega})$ be such that $\phi \prec_{\mathfrak{c}} \widetilde{\phi}$ and $\operatorname{supp}\widetilde{\phi} \cap \Gamma_{\rm tr} = \emptyset$. Then there exists $C > 0$ such that for all $\ell \in \{0, \dots, p-1\}$, for all $k \geq k_0$, $k \notin \mathcal{J}$, $h \leq h_0$ and $w_h \in V_k$,*

$$\begin{aligned}
&\|\phi(1-\Psi)(u-u_h)\|_{H_k^{-\ell}} \\
&\leq C(h_{\widetilde{\phi}}k)^{p+\ell+1}\Big((h_{\widetilde{\phi}}k)^{-p}\|\widetilde{\phi}(u-w_h)\|_{H_k^1} + \|\widetilde{\phi}\Psi(u-u_h)\|_{H_k^{-N}} + (h_{\widetilde{\phi}}k)^N\|\widetilde{\phi}(1-\Psi)(u-u_h)\|_{H_k^{-N}}\Big) \\
&\qquad + Ck^{-N}(hk)^{p+\ell+1}\Big((hk)^{-p}\|u-w_h\|_{H_k^1} + \|u-u_h\|_{H_k^{-N}}\Big),
\end{aligned} \tag{8.27}$$

where $h_{\widetilde{\phi}} := \max\big\{\operatorname{diam}(K) \,:\, K \in \mathcal{T}_k \ \text{ s.t. } \ K \cap \operatorname{supp}\widetilde{\phi} \neq \emptyset\big\}$.

We highlight that the advantage of (8.27) over the bound in Lemma 8.1 is the arbitrary power N in the term $(h_{\tilde{\phi}}k)^N\|\widetilde{\phi}(1-\Psi)(u-u_h)\|_{H_k^{-N}}$.

Proof of Lemma 8.7. Let $\mathfrak{c}>0$, $k_0>0$. Let $c\in(0,\mathfrak{c})$ be arbitrary and let h_0 be small enough to apply Lemma 8.1 with $\mathfrak{c}=c$. Let N, ψ, ϕ and $\widetilde{\phi}$ as in the statement, and let $\chi,\widetilde{\chi}\in C^\infty(\overline{\Omega})$ be any cutoff functions chosen such that $\chi\prec_c\widetilde{\chi}$ and $\operatorname{supp}\widetilde{\chi}\cap\Gamma_{\rm tr}=\emptyset$. Denote by C any positive constant whose value depends only on the previous quantities. Then, given $k\geq k_0$, $k\notin\mathcal{J}$, $h\leq h_0$ and $w_h\in V_k$, it is enough to show that

$$\begin{aligned}&\|\chi(1-\Psi)(u-u_h)\|_{H_k^{-\ell}}\\&\leq C(h_{\tilde{\chi}}k)^{\ell+1}\Big(\|\widetilde{\chi}(u-w_h)\|_{H_k^1}+(h_{\tilde{\chi}}k)^p\|\widetilde{\chi}\Psi(u-u_h)\|_{H_k^{-N}}+(h_{\tilde{\chi}}k)^p\|\widetilde{\chi}(1-\Psi)(u-u_h)\|_{H_k^{-N}}\Big)\\&\quad+Ck^{-N}(hk)^{\ell+1}\Big(\|u-w_h\|_{H_k^1}+(hk)^p\|u-u_h\|_{H_k^{-N}}\Big).\end{aligned}\tag{8.28}$$

where $h_{\tilde{\chi}}$ is defined analogously to $h_{\tilde{\phi}}$. Indeed, one can then apply (8.28) iteratively with a sequence of cut-offs appropriately nested between ϕ and $\widetilde{\phi}$.

Let $\check{\chi},\hat{\chi}$ be such that $\chi\prec_{c/4}\check{\chi}\prec_{c/4}\hat{\chi}\prec_{c/4}\widetilde{\chi}$. We apply Lemma 8.1 with $A=E_\ell\chi(1-\Psi)$, where $E_\ell:H_k^{-\ell}\to L^2$ an isomorphism, with $\{\phi_j\}_{j=1}^2:=\{\hat{\chi},1-\hat{\chi}\}$ (i.e., only two functions in the partition of unity) and $\{\tilde{\phi}_j\}_{j=1}^2=\{\widetilde{\chi},1-\check{\chi}\}$. Then, by Lemma 8.6 (since $\operatorname{supp}\chi\cap\Gamma_{\rm tr}=\emptyset$),

$$\|\widetilde{\chi}R_k^*A^*\|_{L^2\to L^2}\leq\|\widetilde{\chi}(R_k^\sharp)^*A^*\|_{L^2\to L^2}+Ck^{-N}$$

and

$$\|(1-\check{\chi})R_k^*A^*\|_{L^2\to L^2}\leq\|(1-\check{\chi})(R_k^\sharp)^*A^*\|_{L^2\to L^2}+Ck^{-N}.$$

Moreover, by Theorem 6.2,

$$\|(1-\check{\chi})(R_k^\sharp)^*A^*\|_{L^2\to H_k^{\ell+2}}\leq Ck^{-N}$$

and by Proposition 5.23

$$\|\widetilde{\chi}(R_k^\sharp)^*A^*\|_{L^2\to H_k^{\ell+2}}\leq C.$$

Therefore, with $\eta_{j\to A}$ defined by (8.9),

$$\eta_{1\to A}\leq C\Big((h_{\tilde{\chi}}k)^p\|\widetilde{\chi}R_k^*A^*\|_{L^2\to L^2}+(h_{\tilde{\chi}}k)^{\ell+1}\|\widetilde{\chi}(R_k^\sharp)^*A^*\|_{L^2\to H_k^{\ell+2}}\Big)\leq C(h_{\tilde{\chi}}k)^{\ell+1},$$

and $\eta_{2\to A}\leq Ck^{-N}(hk)^{\ell+1}$. Lemma 8.1 thus gives

$$\begin{aligned}\|\chi(1-\Psi)(u-u_h)\|_{H_k^{-\ell}}\leq(h_{\tilde{\chi}}k)^{\ell+1}C\Big(\|\widetilde{\chi}(u-w_h)\|_{H_k^1}+(h_{\tilde{\chi}}k)^p\|\widetilde{\chi}(u-u_h)\|_{H_k^{-N}}\Big)\\+Ck^{-N}(hk)^{\ell+1}\Big(\|u-w_h\|_{H_k^1}+(hk)^p\|u-u_h\|_{H_k^{-N}}\Big),\end{aligned}$$

and (8.28) follows using $\|\widetilde{\chi}(u-u_h)\|_{H_k^{-N}}\leq\|\widetilde{\chi}\Psi(u-u_h)\|_{H_k^{-N}}+\|\widetilde{\chi}(1-\Psi)(u-u_h)\|_{H_k^{-N}}$. ■

Recall from §3 that $U_{\rm P}$ is a neighbourhood of $\Gamma_{\rm tr}$ such that Theorem 4.2 holds on $U_{\rm P}$.

Lemma 8.8 (PML upgrade) *For any $\mathfrak{c}$, k_0, there exists $h_0>0$ such that the following is true. Let $N>0$, let $\psi\in C_c^\infty(\mathbb{R})$, let $\Psi:=\psi(\mathcal{P}_k)$, and let $\phi,\widetilde{\phi}\in C^\infty(\overline{\Omega})$ be such that $\phi\prec_{\mathfrak{c}}\widetilde{\phi}$ and $\operatorname{supp}\widetilde{\phi}\subset U_{\rm P}$. Then there exists $C>0$ such that, for all $\ell\in\{0,\dots,p-1\}$, $k\geq k_0$, $k\notin\mathcal{J}$, $h\leq h_0$ and $w_h\in V_k$,*

$$\begin{aligned}&\|\phi\Psi(u-u_h)\|_{H_k^{-\ell}}+\|\phi(1-\Psi)(u-u_h)\|_{H_k^{-\ell}}\\&\qquad\leq C(h_{\tilde{\phi}}k)^{p+\ell+1}\Big((h_{\tilde{\phi}}k)^{-p}\|\widetilde{\phi}(u-w_h)\|_{H_k^1}+(h_{\tilde{\phi}}k)^N\|\widetilde{\phi}(u-u_h)\|_{H_k^{-N}}\Big)\\&\qquad\qquad+Ck^{-N}(hk)^{p+\ell+1}\Big((hk)^{-p}\|u-w_h\|_{H_k^1}+\|u-u_h\|_{H_k^{-N}}\Big),\end{aligned}$$

where $h_{\tilde{\phi}}:=\max\Big\{\operatorname{diam}(K)\;:\;K\in\mathcal{T}_k\;\;\text{s.t.}\;\;K\cap\operatorname{supp}\widetilde{\phi}\neq\emptyset\Big\}$.

Proof. We proceed as in the proof of Lemma 8.7, with $A = E_\ell \chi \Psi$ or $A = E_\ell \chi(1-\Psi)$, but this time, the terms $\eta_{j\to A}$ in Lemma 8.1 are bounded by first using pseudolocality of Ψ (or $1-\Psi$) to write

$$R_k^* \Psi \chi = R_k^* \underline{\chi} \Psi \chi + O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k),$$

where $\chi \prec \underline{\chi} \prec \check{\chi}$. The conclusion is then obtained from

$$\|(1-\check{\chi}) R_k^* \underline{\chi}\|_{L^2\to L^2} \le C k^{-N} \quad \text{and} \quad \|\widetilde{\chi} R_k^* \underline{\chi}\|_{L^2\to L^2} \le C,$$

with these bounds following from Theorem 4.2, since $1-\check{\chi} \perp \underline{\chi}$ and $\operatorname{supp} \widetilde{\chi} \subset U_{\rm P}$. ■

Remark 8.9 *The iteration in the proofs of Lemmas 8.7 and 8.8 is possible because the $\eta_{j\to A}$ are small, precisely because of the "good" behaviour of the solution operator on high frequencies/in the PML, respectively.*

8.4 Estimates in lowest regularity

In the remainder of this section, we fix a cover $\{\Omega_j\}_{1\le j\le M}$ satisfying (3.8).

Lemma 8.10 (The system of inequalities involving X) *Let $\{\chi_i\}_{i=1}^M$ be such that (3.14) holds, let $\psi \in C_c^\infty(\mathbb{R})$ with $\psi^\sharp \prec \psi$, let $\Psi = \psi(\mathcal{P}_k)$, and let $k_0, N > 0$. Then, there exists $h_0, C_\dagger, C > 0$ such that the following holds for all $k \ge k_0$, $k \notin \mathcal{J}$, $h \le h_0$, $u - u_h$ satisfying (3.16) and $w_{h,i} \in V_k$, $i \in \{1,\dots,M\}$. Letting X^-, X^+, $X^{\rm P}$ be the column vectors of local Galerkin errors defined by*

$$X_i^- = \|\chi_i \Psi(u-u_h)\|_{H_k^{-p+1}}, \quad X_i^+ = \|\chi_i (1-\Psi)(u-u_h)\|_{H_k^{-p+1}}, \quad i = 1,\dots,M_{\rm I},$$
$$X_i^{\rm P} := \|\chi_{M_{\rm I}+i}(u-u_h)\|_{H_k^{-p+1}}, \quad i = 1,\dots,M_{\rm P} \qquad X := \begin{pmatrix} X^- \\ X^+ \\ X^{\rm P}\end{pmatrix} \tag{8.29}$$

(with $\pm$ standing for high and low frequency), Z the column vector of local best approximation errors defined by

$$Z_i = \|u - w_{h,i}\|_{H_k^1(\Omega_i)},$$

and B, W the matrices defined by (3.12) and (3.13), the following system of inequalities

$$\big(I - C_\dagger W\big) X \le C \big(BZ + R\,\mathbf{1}\big) \tag{8.30}$$

holds in the component-wise sense, with $\mathbf{1} := \begin{pmatrix}1 & \cdots & 1\end{pmatrix}^T$ and $R := R_1 + R_2$, where

$$R_1 := k^{-N}(hk)^p \sum_{i=1}^M \|u - w_{h,i}\|_{H_k^1}, \qquad R_2 := k^{-N}(hk)^{2p}\|u-u_h\|_{H_k^{-N}}.$$

Let $\pi_{{\rm I},\pm} \in \mathbb{M}((2M_{\rm I}+M_{\rm P})\times M)$ and $\pi_{\rm P} \in \mathbb{M}(M_{\rm P}\times(2M_{\rm I}+M_{\rm P}))$ be defined by

$$\pi_{{\rm I},-} := \begin{pmatrix} I_{M_{\rm I}} & 0_{M_{\rm I}} & 0_{M_{\rm I}\times M_{\rm P}}\end{pmatrix}, \quad \pi_{{\rm I},+} := \begin{pmatrix} 0_{M_{\rm I}} & I_{M_{\rm I}} & 0_{M_{\rm I}\times M_{\rm P}}\end{pmatrix}, \tag{8.31}$$

$$\pi_{\rm P} = \begin{pmatrix} 0_{M_{\rm P}\times M_{\rm I}} & 0_{M_{\rm P}\times M_{\rm I}} & I_{M_{\rm P}}\end{pmatrix}. \tag{8.32}$$

Lemma 8.10 has the following corollary.

Corollary 8.11 *Let $\{\chi_i\}_{i=1}^M$, ψ be as in the statement of Lemma 8.10 and let $k_0, N > 0$. Then there exist $C_\dagger, h_0 > 0$ such that for every $M > 0$ and $C_M > 0$ there exists $C > 0$ such that the following holds. For all $k \ge k_0$, $k \notin \mathcal{J}$, $h \le h_0$, if*

$$\sum_{n=0}^\infty (C_\dagger W)^n \le C_M k^M,$$

and if $u-u_h$ satisfies (3.16), then

$$X \le C(I-C_\dagger W)^{-1}BZ + CR_1\mathbf{1}$$

for all $w_h \in V_k$, with X, Z, R_1, and $\mathbf{1}$ defined as in Lemma 8.10.

That is, for $1 \le i \le M_{\rm I}$,

$$\|\chi_i \Psi(u-u_h)\|_{H_k^{-p+1}} \le C\sum_{j=1}^{M}\left[\pi_{{\rm I},-}(I-C_\dagger W)^{-1}B\right]_{i,j}\|u-w_{h,j}\|_{H_k^1(\Omega_j)} + CR_1,$$

$$\|\chi_i(1-\Psi)(u-u_h)\|_{H_k^{-p+1}} \le C\sum_{j=1}^{M}\left[\pi_{{\rm I},+}(I-C_\dagger W)^{-1}B\right]_{i,j}\|u-w_{h,j}\|_{H_k^1(\Omega_j)} + CR_1,$$

and for $1 \le i \le M_{\rm P}$,

$$\|\chi_{M_{\rm I}+i}(u-u_h)\|_{H_k^{-p+1}} \le C\sum_{j=1}^{M}\left[\pi_{\rm P}(I-C_\dagger W)^{-1}B\right]_{i,j}\|u-w_h\|_{H_k^1(\Omega_j)} + CR_1,$$

where $\pi_{{\rm I},\pm}$ and $\pi_{\rm P}$ are defined by (8.31) and (8.32).

Corollary 8.11 follows from Lemma 8.10 using the following lemma.

Lemma 8.12 *For all $k_0>0$, there exist constants $C, h_0, N'>0$ such that for all $k \ge k_0$, $k \notin \mathcal{J}$, $h \le h_0$, $u-u_h$ satisfying (3.16) and $w_h \in V_k$,*

$$\|u-u_h\|_{H_k^{-p+1}} \le Ck^{N'}(hk)^p\|u-w_h\|_{H_k^1}.$$

Proof. We apply Lemma 8.10 with any cover $\{\chi_i\}_{i=1}^M$ satisfying (3.14) and with $N=2p$. By the definition of X (8.29) and the fact that for $k \notin \mathcal{J}$, all the elements of B (3.12) are bounded by $Ck^{N'}(hk)^p$ for some $N'>0$ (by Assumption 3.1)

$$\begin{aligned}\|u-u_h\|_{H_k^{-p+1}} \le \sum_{i=1}^{2M_{\rm I}+M_{\rm P}} X_i &\le Ck^{N'}(hk)^p\|u-w_h\|_{H_k^1} + Ck^{-2p}(hk)^{2p}\|u-u_h\|_{H_k^{-p+1}}\\ &= Ck^{N'}(hk)^p\|u-w_h\|_{H_k^1} + Ch^p\|u-u_h\|_{H_k^{-p+1}};\end{aligned}$$

the result then follows by choosing h_0 small enough. ∎

Outline of the proof of Lemma 8.10 The main idea of this proof – and, indeed, the heart of the paper – is that one can use the localised duality argument (Lemma 8.1) to obtain a system of inequalities (as in (8.30)) relating local Galerkin errors and local best approximation errors. By choosing $A=\chi_i\Psi$ or $\chi_i(1-\Psi)$ in Lemma 8.1, this allows to obtain bounds for X^- and X^+. However, it turns out that this idea is not quite sufficient to fully exploit the fact that the solution operator on either the PML or high frequencies is pseudolocal (via Theorem 4.2 and Lemma 8.6 below). Our method is to split the domains $\{\Omega_i\}$ more finely, use Lemma 8.1 on this finer cover and then gather back the errors on the original domains. The improvements over the straightforward application of Lemma 8.1 are that, thanks to pseudolocality, we obtain instances of h_{ij} instead of h_j, and we exploit the situations where the resolvent on $\Omega_i \cap \Omega_j$ behaves better than on Ω_j.

Proof of Lemma 8.10. Throughout this proof, let χ_i, ψ, Ψ, k_0 and N be as in the statement. Without loss in generality, we can assume $N \ge p-1$. Denote by C any positive constant, and by $h_0>0$ a small enough constant (to be specified in the proof), whose values only depends on the previous quantities. Now let $k \ge k_0$, $k \notin \mathcal{J}$, such that $h \le h_0$, $u-u_h$ satisfying (3.16), and $w_{h,j} \in V_k$, $j \in \{1,\dots,M\}$.

Definition of an expanded set of domains. We first define an expanded set of domains $\{\widehat{\Omega}_i\}_{1\le i\le \widehat{M}}$. These appear only in the proof, and allow us to exploit the fact that the intersection of an "interior domain" (i.e., an Ω_i for $1 \le i \le M_{\rm I}$) and a "PML domain" (i.e., an Ω_i for $M_{\rm I+1} \le i \le M$) occurs only in the PML.

Let

$$\Omega_{\rm P} := \bigcup_{j=M_{\rm I+1}}^{M} \Omega_j$$

be the union of all domains lying in the PML region. Recall that $\Omega_{\rm P} \Subset U_{\rm P}$ by assumption (3.8). Thus, for each $i \in \{1,\ldots,M_{\rm I}\}$, we may find two open sets V_i, W_i such that

$$\Omega_i \cap \Omega_{\rm P} \Subset V_i \Subset W_i \Subset U_{\rm P}.$$

Let

$$\Omega_i^\circ := \Omega_i \setminus \overline{W_i}\,, \qquad \text{and} \qquad \Omega_i^\times := \Omega_i \cap V_i$$

(where $\Omega_i^\times$ may be empty) and observe that

$$\Omega_i = \Omega_i^\circ \cup \Omega_i^\times\,, \quad \Omega_i^\circ \cap \overline{\Omega_{\rm P}} = \emptyset\,, \quad \text{and} \quad \Omega_i^\times \Subset U_{\rm P}$$

(the notation $\times$ is chosen because these domains "cross" the PML). Let $\varphi_i^\circ, \tilde\varphi_i^\circ, \varphi_i^\times \in C^\infty(\overline{\Omega})$ be such that

$$\varphi_i^\circ \prec \tilde\varphi_i^\circ, \tag{8.33}$$

$$\chi_i \prec \varphi_i^\circ + \varphi_i^\times\,, \quad \operatorname{supp}(\tilde\varphi_i^\circ) \subset \Omega_i^\circ \cup \partial\Omega. \tag{8.34}$$

Let

$$\chi_i^\circ := \chi_i \tilde\varphi_i^\circ \qquad \text{and} \qquad \chi_i^\times := \chi_i \varphi_i^\times,$$

so that, in particular,

$$\big\{\chi_i \equiv 1\big\} \subset \big\{\chi_i^\times \equiv 1\big\} \cup \big\{\chi_i^\circ \equiv 1\big\}, \quad i = 1,\ldots,M_{\rm I}. \tag{8.35}$$

To see (8.35), observe that if $\chi_i(x) = 1$ and $\varphi_i^\times(x) \neq 1$, then $\varphi_i^\circ(x) \neq 0$ (since $\varphi_i^\circ = 1 - \varphi_i^\times$ on $\operatorname{supp}\tilde\chi_i \supset \operatorname{supp}\chi$ by (8.34)), and thus $\tilde\varphi_i^\circ(x) = 1$ by (8.33).

We now renumber

$$\chi_1^\circ,\ldots,\chi_{M_{\rm I}}^\circ,\chi_1^\times,\ldots,\chi_{M_{\rm I}}^\times,\chi_{M_{\rm I+1}},\ldots,\chi_M \qquad \text{as } \{\varphi_{i,1}\}_{1\le i\le \widehat{M}},$$

with $\widehat{M} = 2M_{\rm I} + M_{\rm P}$, and

$$\Omega_1^\circ,\ldots,\Omega_{M_{\rm I}}^\circ,\Omega_1^\times,\ldots,\Omega_{M_{\rm I}}^\times,\Omega_{M_{\rm I+1}},\ldots,\Omega_M \qquad \text{as } \{\widehat{\Omega}_i\}_{1\le i\le \widehat{M}}.$$

The key properties of these domains and cutoffs that we use in the rest of the proof are that the condition (3.14) still holds, i.e.

$$\Omega \subset \bigcup_{i=1}^{\widehat{M}} \operatorname{int}\big(\{\varphi_{i,1} \equiv 1\}\big) \tag{8.36}$$

(by (8.35)) and, for $i = 1,\ldots,M_{\rm I}$,

$$\max\big\{\varphi_{i,1},\varphi_{i+M_{\rm I},1}\big\} \le \chi_i \le \varphi_{i,1} + \varphi_{i+M_{\rm I},1} \qquad \text{and} \qquad \Omega_i = \widehat{\Omega}_i \cup \widehat{\Omega}_{i+M_{\rm I}} \tag{8.37}$$

(by (8.34)). Let $\widehat{h}_j$ be upper bounds for the local meshwidth on $\widehat{\Omega}_j$ and define $\widehat{h}_{ij}$ analogously to (3.9).

Definition of suitable cut-off functions. Let $\{\varphi_{i,0}\}_{i=1}^{\widehat{M}}$ be a partition of unity subordinate to the cover (8.36) of Ω, and thus such that $\varphi_{i,0} \prec \varphi_{i,1}$.

Given $\{\varphi_{i,0}\}_{i=1}^{\widehat{M}}$ and $\{\varphi_{i,1}\}_{i=1}^{\widehat{M}}$, there exists $\mathfrak{c} > 0$ and sequences $\{\varphi_{i,\nu}\}$, $\nu = 2,3,4$, of elements of $C_c^\infty(\mathbb{R}^d)$ supported in Ω_i such that $\varphi_{i,\nu} \prec_{\mathfrak{c}} \varphi_{i,\nu+1}$, for $i = 1,\ldots,\widehat{M}$ and $\nu = 0,\ldots,3$ (the conditions involving $\prec_{\mathfrak{c}}$ are used below to apply Lemma 8.1). Let

$$\varphi_{i,\nu}^\circ := \varphi_{i,\nu}\,, \quad \varphi_{i,\nu}^\times := \varphi_{i+M_{\rm I},\nu}\,, \quad \varphi_{i,\nu}^{\rm P} := \varphi_{i+2M_{\rm I}}. \tag{8.38}$$

Bound on X^-. We first show that

$$X^- \le C\left(\mathcal{C}(\mathcal{H}_{\mathrm{I,I}}k)^{2p} \quad \mathcal{C}(\mathcal{H}_{\mathrm{I,I}}k)^{2p} \quad \mathcal{H}^{\min}_{\mathrm{I,P}}(N)k^N\right) X + C\left(\mathcal{C}(\mathcal{H}_{\mathrm{I,I}}k)^{p} \quad 0\right) Z + CR, \tag{8.39}$$

which gives the first block row of (8.30) (where here, and in the rest of the proof, we use the convention that a vector plus a scalar is the vector obtained by adding the scalar to every entry). To do this, we estimate X^- (defined by (8.29))

$$X^- \le X^{\circ,-} + X^{\times,-}$$

where $X^{-,\circ} := \|\chi_i^\circ \Psi(u-u_h)\|_{H_k^{-p+1}}$ and $X^{-,\times} := \|\chi_i^\times \Psi(u-u_h)\|_{H_k^{-p+1}}$.[5] We estimate $X^{-,\circ}$ and $X^{-,\times}$ separately.

Bound on $X^{\circ,-}$. The main work is to bound $X^{\circ,-}$. To this end, we fix $i \in \{1,\dots,M_{\mathrm{I}}\}$ and apply Lemma 8.1 with $A = A_i := \chi_i^\circ \Psi$ (observe that the smoothing property of Ψ, Proposition 5.22, implies that $A_i : H_k^{-p+1} \to L^2$) and the functions $\{\phi_j\}_{1\le j\le 2\widehat{M}}$, $\{\widetilde{\phi}_j\}_{1\le j\le 2\widehat{M}}$ defined by

$$\phi_j := \begin{cases} \varphi_{i,3}^\circ \varphi_{j,0}, & j = 1,\dots,\widehat{M},\\ (1-\varphi_{i,3}^\circ)\varphi_{j-\widehat{M},0}, & j = \widehat{M}+1,\dots,2\widehat{M},\end{cases}$$

and

$$\widetilde{\phi}_j := \begin{cases} \varphi_{i,4}^\circ \varphi_{j,1}, & j = 1,\dots,\widehat{M},\\ (1-\varphi_{i,2}^\circ)\varphi_{j-\widehat{M},1}, & j = \widehat{M}+1,\dots,2\widehat{M}.\end{cases}$$

With these definitions, $\{\phi_j\}_{1\le j\le 2\widehat{M}}$ is indeed a partition of unity on Ω and $\{\widetilde{\phi}_j\}_{1\le j\le 2\widehat{M}}$ satisfies the condition (8.8) by the definition of $\varphi_{j,\nu}$. Therefore, choosing h_0 small enough, Lemma 8.1 ensures that

$$X_i^{\circ,-} \le C\sum_{j=1}^{\widehat{M}} \left[(\widehat{h}_{ij}k)^{2p}\alpha_{j\to i} + (\widehat{h}_j k)^{2p}\alpha'_{j\to i}\right]\widehat{X}_j + \left[(\widehat{h}_{ij}k)^{p}\alpha_{j\to i} + (\widehat{h}_j k)^{p}\alpha'_{j\to i}\right]\widehat{Z}_j + CR, \tag{8.40}$$

where $R = k^{-N}\left((hk)^{2p}\|u-u_h\|_{H_k^{-N}} + (hk)^p \sum_{j=1}^M \|u-w_{h,j}\|_{H_k^1}\right)$,

$$\widehat{X}_j := \|\varphi_{j,1}(u-u_h)\|_{H_k^{-N}}, \quad \widehat{Z}_j := \|\varphi_{j,1}(u-w_{h,j})\|_{H_k^1}, \quad j = 1,\dots,\widehat{M}, \tag{8.41}$$

and

$$\alpha_{j\to i} := \|\varphi_{i,4}^\circ\varphi_{j,1}R_k^*\Psi\chi_i^\circ\|_{L^2\to L^2} + \|\varphi_{i,4}^\circ\varphi_{j,1}(R_k^\sharp)^*\Psi\chi_i^\circ\|_{L^2\to H_k^{p+1}},$$

$$\alpha'_{j\to i} := \|(1-\varphi_{i,2}^\circ)\varphi_{j,1}R_k^*\Psi\chi_i^\circ\|_{L^2\to L^2} + \|(1-\varphi_{i,2}^\circ)\varphi_{j,1}(R_k^\sharp)^*\Psi\chi_i^\circ\|_{L^2\to H_k^{p+1}}.$$

Since $N \ge p-1$, by (8.41), (8.37), and (8.29),

$$\widehat{X}_j \le \|\varphi_{j,1}^\circ\Psi(u-u_h)\|_{H_k^{-p+1}} + \|\varphi_{j,1}^\circ(1-\Psi)(u-u_h)\|_{H_k^{-p+1}} \le X_j^- + X_j^+, \qquad 1\le j\le M_{\mathrm{I}}, \tag{8.42}$$

$$\widehat{X}_{j+M_{\mathrm{I}}} \le \|\varphi_{j,1}^\times\Psi(u-u_h)\|_{H_k^{-p+1}} + \|\varphi_{j,1}^\times(1-\Psi)(u-u_h)\|_{H_k^{-p+1}} \le X_j^- + X_j^+, \qquad 1\le j\le M_{\mathrm{I}}, \tag{8.43}$$

$$\widehat{X}_{j+2M_{\mathrm{I}}} \le \|\varphi_{j,1}^{\mathrm{P}}(u-u_h)\|_{H_k^{-p+1}} = X_j^{\mathrm{P}}, \qquad 1\le j\le M_{\mathrm{P}}, \tag{8.44}$$

and similarly,

$$\widehat{Z}_j \le \begin{cases} Z_j & 1\le j\le M_{\mathrm{I}},\\ Z_{j-M_{\mathrm{I}}} & M_{\mathrm{I}}+1\le j\le 2M_{\mathrm{I}},\\ Z_{j-M_{\mathrm{I}}} & 2M_{\mathrm{I}}+1\le j\le \widehat{M}.\end{cases} \tag{8.45}$$

[5] Without this splitting, one only gets $\mathcal{H}^{\min}_{\mathrm{I,P}}(2p)k^{2p}$ instead $\mathcal{H}^{\min}_{\mathrm{I,P}}(N)k^N$ in the third block of the first matrix in the right-hand side of (8.39).

Let

$$\omega_{j\to i} := (\widehat{h}_{ij}k)^{2p}\alpha_{j\to i} + (\widehat{h}_j k)^{2p}\alpha'_{j\to i}\,, \quad \beta_{j\to i} := (\widehat{h}_{ij}k)^{p}\alpha_{j\to i} + (\widehat{h}_j k)^{p}\alpha'_{j\to i}$$

and

$$\omega^{\circ}_{j\to i} := \omega_{j\to i}\,, \quad \omega^{\times}_{j\to i} := \omega_{j+M_{\rm I}\to i}, \quad j=1,\ldots,M_{\rm I},$$
$$\omega^{\rm P}_{j\to i} := \omega_{j+2M_{\rm I}\to i}, \quad j=1,\ldots,M_{\rm P},$$

and define $\beta^{\circ}_{j\to i}, \beta^{\times}_{j\to i}$ and $\beta^{\rm P}_{j\to i}$ analogously. Then (8.40) can be written as

$$X^{\circ,-} \leq C\begin{pmatrix}\mathscr{A} & \mathscr{B} & \mathscr{C}\end{pmatrix} X + \begin{pmatrix}\mathscr{D} & \mathscr{E}\end{pmatrix} Z + R \tag{8.46}$$

where, for $1\leq i\leq M_{\rm I}$,

$$\mathscr{A}_{ij} = \mathscr{B}_{ij} = \omega^{\circ}_{j\to i} + \omega^{\times}_{j\to i}\,, \quad 1\leq j\leq M_{\rm I}, \qquad \mathscr{C}_{ij} = \omega^{\rm P}_{j\to i}\,, \quad 1\leq j\leq M_{\rm P}, \tag{8.47}$$

and

$$\mathscr{D}_{ij} = \beta^{\circ}_{j\to i} + \beta^{\times}_{j\to i}\,, \quad 1\leq j\leq M_{\rm I}, \qquad \mathscr{E}_{ij} = \beta^{\rm P}_{j\to i}\,, \quad 1\leq j\leq M_{\rm P}. \tag{8.48}$$

To proceed, we now bound the following four terms appearing in the definitions of $\alpha_{j\to i}$ and $\alpha'_{j\to i}$:

$$\|\varphi_{i,4}\varphi_{j,1}R_k^*\Psi\varphi^{\circ}_{i,1}\|_{L^2\to L^2}, \quad \|\varphi_{i,4}\varphi_{j,1}(R_k^{\sharp})^*\Psi\varphi^{\circ}_{i,1}\|_{L^2\to H_k^{p+1}},$$

$$\|(1-\varphi_{i,2})\varphi_{j,1}R_k^*\Psi\varphi^{\circ}_{i,1}\|_{L^2\to L^2}, \quad \text{and} \quad \|(1-\varphi_{i,2})\varphi_{j,1}(R_k^{\sharp})^*\Psi\varphi^{\circ}_{i,1}\|_{L^2\to H_k^{p+1}},$$

where we have used that $\chi^{\circ}_i = \varphi_{i,1} = \varphi^{\circ}_{i,1}$ for $i=1,\ldots,M_{\rm I}$ by (8.38).

First, by pseudolocality of Ψ (Lemma 6.2), polynomial boundedness of R_k^* (Assumption 3.1) and boundedness of R_k^* in $U_{\rm P}$ (estimate (4.3) in Theorem 4.2),

$$\|\varphi_{i,4}\varphi_{j,1}R_k^*\Psi\varphi^{\circ}_{i,1}\|_{L^2\to L^2} \leq C 1_{\{\widehat{\Omega}_j\cap\Omega^{\circ}_i\neq\emptyset\}} \begin{cases} k^{-N} + \|1_{\Omega^{\circ}_j}R_k^* 1_{\Omega^{\circ}_i}\|_{L^2\to L^2}, & 1\leq j\leq M_{\rm I}\\ 1, & M_{\rm I}+1\leq j\leq 2M_{\rm I}\\ 0, & 2M_{\rm I}+1\leq j\leq \widehat{M},\end{cases} \tag{8.49}$$

since by definition, for $j\in\{M_{\rm I}+1,\ldots,2M_{\rm I}\}$, $\widehat{\Omega}_j\subset U_{\rm P}$, and for $j\in\{2M_{\rm I}+1,\ldots,\widehat{M}\}$, $\widehat{\Omega}_j\subset\Omega_{\rm P}$, while $\Omega^{\circ}_j\cap\Omega_{\rm P}=\emptyset$.

Second, by the mapping properties of $R_k^{\sharp}$ (Proposition 5.23), boundedness of $\Psi : L^2\to H_k^{p-1}$ (Proposition 5.22), and similar arguments,

$$\|\varphi_{i,4}\varphi_{j,1}(R_k^{\sharp})^*\Psi\varphi^{\circ}_{i,1}\|_{L^2\to L^2} \leq C 1_{\{\widehat{\Omega}_j\cap\Omega^{\circ}_i\neq\emptyset\}} \begin{cases} 1, & 1\leq j\leq M_{\rm I}\\ 1, & M_{\rm I}+1\leq j\leq 2M_{\rm I}\\ 0, & 2M_{\rm I}+1\leq j\leq \widehat{M}.\end{cases}$$

Third, by pseudolocality of Ψ again, and of R_k in $U_{\rm P}$ (estimate (4.4) of Theorem 4.2), and since $\varphi^{\circ}_{i,2}\prec\varphi_{i,3}$,

$$\|(1-\varphi^{\circ}_{i,2})\varphi_{j,1}R_k^*\Psi\varphi^{\circ}_{i,1}\|_{L^2\to L^2} \leq Ck^{-N} + C\begin{cases}\|1_{\Omega^{\circ}_j}R_k^* 1_{\Omega^{\circ}_i}\|_{L^2\to L^2}, & 1\leq j\leq M_{\rm I}\\ 0, & M_{\rm I}+1\leq j\leq \widehat{M}.\end{cases}$$

Finally, by pseudolocality of $(R_k^{\sharp})^*$ and Ψ (Lemma 6.2), since $\varphi^{\circ}_{i,2}\prec_c\varphi^{\circ}_{i,3}$,

$$\|(1-\varphi^{\circ}_{i,2})\varphi_{j,1}(R_k^{\sharp})^*\Psi\varphi^{\circ}_{i,1}\|_{L^2\to H_k^{p+1}} \leq Ck^{-N}. \tag{8.50}$$

From the estimates (8.49)-(8.50), we deduce that

$$\omega^{\circ}_{j\to i} \leq C\Big((\widehat{h}_{ij}k)^{2p}1_{\{\widehat{\Omega}_j\cap\Omega^{\circ}_i\neq\emptyset\}} + (\widehat{h}_j k)^{2p}\Big)\Big(k^{-N} + \|1_{\Omega^{\circ}_j}R_k^* 1_{\Omega^{\circ}_i}\|_{L^2\to L^2}\Big) \leq C(h_j k)^{2p}\mathcal{C}_{ij} + C(hk)^{2p}k^{-N},$$

using the inclusions $\Omega_j^\circ \subset \Omega_j$, the fact that if $A' \subset A$, $B' \subset B$, then

$$\|1_{A'} R_k^* 1_{B'}\| = \|1_{A'} 1_A R_k^* 1_B 1_{B'}\| \leq \|1_{A'}\| \|1_A R_k^* 1_B\| \|1_{B'}\| \leq \|1_A R_k^* 1_B\|,$$

and the fact that $1 \leq \mathcal{C}_{ij}$ when $\Omega_i^\circ \cap \Omega_j^\circ \neq \emptyset$. Similarly,

$$\omega_{j\to i}^\times \leq 1_{\{\Omega_j^\times \cap \Omega_i^\circ \neq \emptyset\}} (\widehat{h}_{ij} k)^{2p} + C(\widehat{h}_j k)^{2p} k^{-N} \leq C(\mathcal{H}^{\min}(2p))_{ij} k^{2p} + C(hk)^{2p} k^{-N}.$$

Therefore, by (8.47), the following inequalities hold componentwise

$$\mathscr{A} \leq C\mathcal{C}(\mathcal{H}k)^{2p} + C(hk)^{2p} k^{-N}, \quad \mathscr{B} \leq C\mathcal{C}(\mathcal{H}k)^{2p} + C(hk)^{2p} k^{-N}.$$

One can check in a similar way that, by (8.48),

$$\mathscr{D} \leq C\mathcal{C}(\mathcal{H}k)^p + C(hk)^p k^{-N}.$$

Finally, the estimates (8.49)-(8.50) also imply that, for $j \in \{1, \ldots, M_{\mathrm{P}}\}$,

$$\omega_{j\to i}^{\mathrm{P}} \leq C(hk)^{2p} k^{-N}, \qquad \beta_{j\to i}^{\mathrm{P}} \leq C(hk)^p k^{-N},$$

and thus, componentwise, by (8.47) and (8.48),

$$\mathscr{C} \leq C(hk)^{2p} k^{-N}, \quad \mathscr{E} \leq C(hk)^p k^{-N}.$$

Taking into account the definition of R, (8.46) thus yields

$$X^{\circ,-} \leq C \begin{pmatrix} \mathcal{C}(\mathcal{H}_{\mathrm{I,I}}k)^{2p} & \mathcal{C}(\mathcal{H}_{\mathrm{I,I}}k)^{2p} & 0 \end{pmatrix} X + C \begin{pmatrix} \mathcal{C}(\mathcal{H}_{\mathrm{I,I}}k)^p & 0 \end{pmatrix} Z + CR \tag{8.51}$$

Bound on $X^{\times,-}$. To bound $X_i^{\times,-}$ for $i \in \{1, \ldots, M_{\mathrm{I}}\}$, we write

$$X_i^{\times,-} = \|\varphi_{i,1}^\times \Psi(u - u_h)\|_{H_k^{-p+1}} \leq \sum_{j=1}^{\widehat{M}} \|\varphi_{i,1}^\times \varphi_{j,0} \Psi(u - u_h)\|_{H_k^{-p+1}}$$

since $\{\varphi_{j,0}\}_{1\leq j\leq \widehat{M}}$ is a partition of unity. Applying Lemma 8.8 with $\ell = p-1$ to each term with $\varphi = \varphi_{i,1}^\times \varphi_{j,0}$, $\tilde{\varphi} = \varphi_{i,2}^\times \varphi_{j,1}$, we deduce that

$$X_i^{\times,-} \leq C \sum_{j=1}^{\widehat{M}} 1_{\{\Omega_i^\times \cap \widehat{\Omega}_j \neq \emptyset\}} \left((\widehat{h}_{ij} k)^N \widehat{X}_j + (\widehat{h}_{ij} k)^p \min(\widehat{Z}_j, Z_i) \right) + CR, \tag{8.52}$$

with $\widehat{Z}_j$ and $\widehat{X}_j$ given by (8.41) and Z_i given by (8.29). Estimating $\{\widehat{X}_j\}_{1\leq j\leq \widehat{M}}$ and $\{\widehat{Z}_j\}_{1\leq j\leq \widehat{M}}$ in terms of $\{Z_j\}_{1\leq j\leq M}$ and $\{X_j^\pm\}_{1\leq j\leq M}$ as in (8.42)-(8.45), we obtain

$$\begin{aligned} X^{\times,-} \leq C &\begin{pmatrix} \mathcal{H}_{\mathrm{I,I}}^{\min}(N) k^N & \mathcal{H}_{\mathrm{I,I}}^{\min}(N) k^N & \mathcal{H}_{\mathrm{I,P}}^{\min}(N) k^N \end{pmatrix} X \\ &+ C \begin{pmatrix} (\mathcal{H}_{\mathrm{I,I}} k)^p & 0 \end{pmatrix} Z + CR, \end{aligned} \tag{8.53}$$

where, in the minimum in (8.52), we always choose the Z_i. Summing the estimates (8.51) and (8.53) gives the claimed estimate (8.39) for X^-.

Bound on X^+. We now show that

$$\begin{aligned} X^+ \leq C &\begin{pmatrix} (\mathcal{H}_{\mathrm{I,I}}^{\min}(2p) k^{2p} & \mathcal{H}_{\mathrm{I,I}}^{\min}(N) k^N & \mathcal{H}_{\mathrm{I,P}}^{\min}(N) k^N \end{pmatrix} X \\ &+ C \begin{pmatrix} (\mathcal{H}_{\mathrm{I,I}} k)^p & 0 \end{pmatrix} Z + CR \end{aligned} \tag{8.54}$$

which gives the second block row of (8.30). As before, we write

$$X^+ \leq X^{\circ,+} + X^{\times,+}$$

where $X^{\circ,+} := \|\chi_i^\circ(1-\Psi)(u-u_h)\|_{H_k^{-p+1}}$ and $X^{\times,+} := \|\chi_i^\times(1-\Psi)(u-u_h)\|_{H_k^{-p+1}}$. Using exactly the same method as above for the bound on $X^{\times,-}$, we obtain

$$\begin{aligned} X^{\times,+} \leq C \begin{pmatrix} \mathcal{H}_{\mathrm{I,I}}^{\min}(N)k^N & \mathcal{H}_{\mathrm{I,I}}^{\min}(N)k^N & \mathcal{H}_{\mathrm{I,P}}^{\min}(N)k^N \end{pmatrix} X \\ + C \begin{pmatrix} (\mathcal{H}_{\mathrm{I,I}}k)^p & 0 \end{pmatrix} Z + CR. \end{aligned} \tag{8.55}$$

Using the same arguments, but applying Lemma 8.7 instead of Lemma 8.8, we obtain

$$\begin{aligned} X^{\circ,+} \leq C \begin{pmatrix} \mathcal{H}_{\mathrm{I,I}}^{\min}(2p)k^{2p} & \mathcal{H}_{\mathrm{I,I}}^{\min}(N)k^N & 0 \end{pmatrix} X \\ + C \begin{pmatrix} (\mathcal{H}_{\mathrm{I,I}}k)^p & 0 \end{pmatrix} Z + CR. \end{aligned} \tag{8.56}$$

The bound (8.54) then follows by adding (8.55) and (8.56).

Bound on X^{P}. Following the same method as for $X^{\times,-}$, we obtain

$$\begin{aligned} X^{\mathrm{P}} \leq C \begin{pmatrix} \mathcal{H}_{\mathrm{P,I}}^{\min}(N)k^N & \mathcal{H}_{\mathrm{P,I}}^{\min}(N)k^N & \mathcal{H}_{\mathrm{P,P}}^{\min}(N)k^N \end{pmatrix} X \\ + C \begin{pmatrix} 0 & (\mathcal{H}_{\mathrm{P,P}}k)^p \end{pmatrix} Z + CR. \end{aligned} \tag{8.57}$$

Gathering the estimates (8.39), (8.54) and (8.57) and taking into account the definitions of B and W in (3.12) and (3.13), we obtain (8.30), which concludes the proof of the lemma. ∎

8.5 A bound on $(I - C_\dagger W)^{-1}$ via graph arguments

We now state the result that allows to bound the matrix $(I - C_\dagger W)^{-1}$ coefficientwise by the simple-path matrix (see Definition 3.9) of $C_\dagger W$ in Corollary 8.11. The proof is deferred to Appendix B. Recall the graph notation from §3.6.

Theorem 8.13 (A bound on $(I-W)^{-1}$ by the simple-path matrix) *Let $M \in \mathbb{N}$, let $W \in \mathbb{M}(M \times M)$ be a matrix with non-negative coefficients.*

$$\text{If} \quad c := \sum_{L \in \mathbb{SL}} W_L < 1, \quad \text{then} \quad \sum_{n=0}^{\infty} W^n < \infty, \tag{8.58}$$

and

$$T^\star \leq \sum_{n=0}^{\infty} W^n \leq \frac{1}{1-c} T^\star \tag{8.59}$$

in the componentwise sense, where $T^\star$ is the simple-path matrix of W.

8.6 Estimates in higher norms and completion of the proof of Theorem 3.11

In this section, we complete the proof of Theorem 3.11. In view of Corollary 8.11 and Theorem 8.13, the main task is to obtain higher norm estimates for the Galerkin error.

We now fix $\{\chi\}_{i=1}^M$, k_0, N and ψ as in the statement of Theorem 3.11. For $i = 1, \ldots, M$, let $\chi_{i,\nu} \in C^\infty(\overline{\Omega})$, $\nu = 0, 1, 2, 3$, be such that

$$\chi_{i,0} \prec \chi_{i,1} \prec \chi_{i,2} \prec \chi_{i,3}$$

with $\chi_{i,0} = \chi_i$ and $\mathrm{supp}(\chi_{i,\nu}) \subset \Omega_i \cap \partial\Omega$. Let $C_\dagger > 0$ be as in Corollary 8.11 applied with $\{\chi_{i,2}\}_{i=1}^M$. Let h_0 be sufficiently small, depending on k_0 and the cutoff functions (this restriction that h_0 is sufficiently small comes from the applications below of Corollary 8.11 and Lemmas 8.7, 8.8, 8.12, 8.15, and 8.17). Let $c \in (0,1)$ and suppose that the simple loop condition (3.15) holds. Let C denote any positive constant whose value only depends on the previous quantities. Let $k \geq k_0$, $k \notin \mathcal{J}$, $u \in H_k^1$ and $w_h \in V_k$. To show that there exists a unique $u_h \in V_k$ such that (3.16) holds, by linearity and the fact that V_k is finite-dimensional, it suffices to show that when $u = 0$, the unique solution to (3.16) is $u_h = 0$. But since the latter is a consequence of (3.18), without loss of generality, we may assume that there exists $u_h \in V_k$ satisfying (3.16) and it remains to prove the bound (3.17).

By Theorem 8.13,

$$\sum_{n=0}^{\infty}(C_\dagger W)^n \leq \frac{1}{1-c}T^\star,$$

where $T^\star$ is the simple-path matrix of $C_\dagger W$. Since $\rho(k)$ is polynomially bounded on $\mathbb{R}_+ \setminus \mathcal{J}$, $T^\star \leq Ck^M$ coefficient-wise (since by definition, the coefficients of $T^\star$ are finite linear combination of finite products of coefficients of $C_\dagger W$). Thus, we can apply Corollary 8.11 to deduce that for $i = 1, \dots, M_{\rm I}$,

$$\|\chi_{i,2}\Psi(u-u_h)\|_{H_k^{-p+1}} \leq C\sum_{j=1}^{M}[\pi_{{\rm I},-}T^\star B]_{i,j}\,\|u-w_{h,j}\|_{H_k^1(\Omega_j)} + CR, \tag{8.60}$$

$$\|\chi_{i,2}(1-\Psi)(u-u_h)\|_{H_k^{-p+1}} \leq C\sum_{j=1}^{M}[\pi_{{\rm I},+}T^\star B]_{i,j}\,\|u-w_{h,j}\|_{H_k^1(\Omega_j)} + CR \tag{8.61}$$

and for $1 \leq i \leq M_{\rm P}$,

$$\|\chi_{M_{\rm I}+i,2}(u-u_h)\|_{H_k^{-p+1}} \leq C\sum_{j=1}^{M}[\pi_{\rm P}T^\star B]_{i,j}\,\|u-w_{h,j}\|_{H_k^1(\Omega_j)} + CR, \tag{8.62}$$

with $R = k^{-N}(hk)^p\sum_{j=1}^M \|u-w_{h,j}\|_{H_k^1}$.

Low-frequency bound in arbitrary norms

The first block row of (3.17), i.e., the low-frequency bound, follows from (8.60) by applying Lemma 8.14 below for each $i = 1, \dots, M_{\rm I}$, with $\phi = \chi_{i,0} = \chi_i$, $\tilde{\phi} := \chi_{i,2}$, $v = u - u_h$ and $N = N'$ large enough, and then using Lemma 8.12 (with $w_h = (\sum_{j=1}^J w_{h,j})/J$) to estimate $k^{-N'}\|u-u_h\|_{H_k^{-N'}}$ by R.

Lemma 8.14 (Low-frequency shift) *Let $\phi, \tilde{\phi}, \in C^\infty(\overline{\Omega})$ be such that $\phi \prec \tilde{\phi}$ and $\operatorname{supp}\tilde{\phi}\cap\Gamma_{\rm tr} = \emptyset$. Let $\psi \in C_c^\infty(\mathbb{R})$ and let $\Psi := \psi(\mathcal{P}_k)$. Then, for all $k_0 > 0$ and $N \in \mathbb{N}$, there exists $C > 0$ such that*

$$\|\phi\Psi v\|_{H_k^N} \leq C\|\tilde{\phi}\Psi v\|_{H_k^{-N}} + Ck^{-N}\|v\|_{H_k^{-N}} \quad \text{for all } k \geq k_0 \text{ and } v \in H_k^{-N}.$$

Proof. As in the proof of Lemma 8.6, the assumptions let us define $\varphi_{\rm P} \in C^\infty(\overline{\Omega})$ such that (i) $\varphi_{\rm P} \equiv 1$ near $\partial\Omega_-$, (ii) $\varphi_{\rm P} \equiv 0$ near $\Gamma_{\rm tr}$ and (iii) $\tilde{\phi} \prec \varphi_{\rm P}$. By Lemma 6.7, $\varphi_{\rm P}$ is a boundary compatible operator in the sense of Definition 5.35, and thus by Theorem 5.37,

$$\phi\Psi = \phi\varphi_{\rm P}\Psi = \phi\widetilde{\Psi}\varphi_{\rm P}\Psi + \phi O_{-\infty}(k^{-\infty};\mathcal{D}_k \to \mathcal{D}_k) = \phi\widetilde{\Psi}\varphi_{\rm P}\Psi + O_{-\infty}(k^{-\infty};\mathcal{Y}_k \to \mathcal{Y}_k)$$

where $\widetilde{\psi} \in \mathcal{S}(\mathbb{R})$ is such that $\psi \prec \widetilde{\psi}$ and $\widetilde{\Psi} := \widetilde{\psi}(\mathcal{P}_k)$, and where the last step uses the fact that for $n \geq 0$, $\mathcal{Y}_k^{-n} \subset \mathcal{D}_k^{-n}$, $\mathcal{D}_k^n \subset \mathcal{Y}_k^n$ are continuous inclusions and $\phi \in O_0(1, \mathcal{Y}_k \to \mathcal{Y}_k)$. Hence, since $\phi\widetilde{\Psi} = \phi\widetilde{\Psi}\tilde{\phi} + O_{-\infty}(k^{-\infty};\mathcal{Y}_k \to \mathcal{Y}_k)$ by Theorem 6.2,

$$\phi\Psi = \phi\widetilde{\Psi}\tilde{\phi}\Psi + O_{-\infty}(k^{-\infty};\mathcal{Y}_k \to \mathcal{Y}_k),$$

using that $\varphi_{\rm P}\tilde{\phi} = \tilde{\phi}$. Thus,

$$\|\phi\Psi u\|_{H_k^N} \leq \|\phi\|_{H_k^N \to H_k^N}\|\widetilde{\Psi}\|_{H_k^{-N}\to H_k^N}\|\tilde{\phi}\Psi u\|_{H_k^{-N}} + Ck^{-N}\|u\|_{H_k^{-N}}$$

and the conclusion follows using the mapping properties of $\widetilde{\Psi}$ from Proposition 5.22. ∎

High-frequency and PML bounds up to the L^2 norm.

The second and third block rows of (3.17) when $m \in \{1, \ldots, p\}$ (i.e., up to the L^2 norm) are obtained by using Lemma 8.7 and 8.8, and then using Lemma 8.12 (again with $w_h = (\sum_{j=1}^J w_{h,j})/J$) to estimate $\|u - u_h\|_{H_k^{-N}}$ in the remainder term.

Indeed, to prove the second block row of (3.17) for $m \in \{1, \ldots, p\}$, observe that from Lemma 8.7 (with $\ell = m - 1$) combined with Lemma 8.12 (with $w_h = w_{h,i}$), with $1 \le i \le M_{\rm I}$,

$$\begin{aligned} \|\chi_{i,1}(1-\Psi)(u-u_h)\|_{H_k^{1-m}} \le C(h_i k)^m \Big(&\|u - w_{h,i}\|_{H_k^1(\Omega_i)} + (h_i k)^p \|\chi_{i,2}\Psi(u-u_h)\|_{H_k^{-p+1}} \\ &+ (h_i k)^N \|\chi_{i,2}(1-\Psi)(u-u_h)\|_{H_k^{-p+1}} \Big) + Ck^{-N}(hk)^m \|u - w_{h,i}\|_{H_k^1}; \end{aligned} \tag{8.63}$$

the second block row of (3.17) then follows from (8.63) and (8.61), using that $\chi_{i,0} \prec \chi_{i,1}$ (this extra "layer" is used in the proof for $m = 0$ below).

The third block row of (3.17), i.e., the bound on the PML error, is proved in a similar way to the high-frequency bound, using Lemma 8.8 instead of Lemma 8.7. Indeed, Lemma 8.8 (with $\ell = m - 1$ and N sufficiently large) combined with Lemma 8.12 (with $w_h = w_{h,i}$) implies that, with $M_{\rm I} + 1 \le i \le M$,

$$\begin{aligned} \|\chi_{i,1}(u-u_h)\|_{H_k^{1-m}} \le C(h_i k)^m \Big(\|u - w_{h,i}\|_{H_k^1(\Omega_i)} + (h_i k)^N \|\chi_{i,2}(u-u_h)\|_{H_k^{-p+1}} \Big) \\ + Ck^{-N}(hk)^m \|u - w_{h,i}\|_{H_k^1}. \end{aligned} \tag{8.64}$$

The third block row of (3.17) then follows from (8.64) and (8.62), using again that $\chi_{i,0} \prec \chi_{i,1}$.

High-frequency and PML bound in the energy norm.

The second block row of (3.17) for $m = 0$ (i.e., in the H_k^1 norm) follows from (8.61), (8.63) with $m = 1$, and the following lemma applied with $\phi = \chi_{i,0}$, $\widetilde{\phi} := \chi_{i,1}$, $N := N'$ large enough and using Lemma 8.12 (with $w_h = w_{h,i}$) to estimate $k^{-N'}\|u - u_h\|_{H_k^{-N'}}$ by R.

Lemma 8.15 *For any $k_0 > 0$ and $\mathfrak{c} > 0$, there exists $h_0 > 0$ such that the following holds. Let $\phi, \widetilde{\phi} \in C^\infty(\overline{\Omega})$ be such that*

$$\phi \prec_{\mathfrak{c}} \widetilde{\phi} \quad \textit{and} \quad \operatorname{supp}(\widetilde{\phi}) \cap \Gamma_{\rm tr} = \emptyset.$$

Furthermore, let $\psi, \psi_0 \in C_c^\infty(\mathbb{R})$ be such that $\psi_0 \prec \psi$, let $\Psi := \psi(\mathcal{P}_k)$, $\Psi_0 := \psi_0(\mathcal{P}_k)$ and

$$A := \phi(1 - \Psi) \quad \textit{and} \quad \widetilde{A} := \widetilde{\phi}(1 - \Psi_0).$$

Then, for all $N > 0$, there exists $C > 0$ such that for all $k \ge k_0$, $h \le h_0$, $u - u_h$ satisfying (3.16) and for all $w_h \in V_k$,

$$\begin{aligned} \|A(u-u_h)\|_{H_k^1} \le C \Big(\|\widetilde{\phi}(u - w_h)\|_{H_k^1} + \|\widetilde{A}(u-u_h)\|_{L^2} + (h_{\tilde{\phi}} k)^p \|\widetilde{\phi}(u-u_h)\|_{H_k^{-N}} \Big) \\ + Ck^{-N}(\|u - w_h\|_{H_k^1} + \|u - u_h\|_{H_k^{-N}}). \end{aligned}$$

where $h_{\tilde{\phi}} := \max \big\{ h_K \ : \ K \in \mathcal{T}_k \ \text{ s.t. } \ K \cap \operatorname{supp}(\widetilde{\phi}) \neq \emptyset \big\}$.

The heart of the proof of Lemma 8.15 is that, by the Gårding inequality, Galerkin orthogonality, and the definition of $\Pi_k^\sharp$,

$$\begin{aligned} &\|A(u-u_h)\|_{H_k^1}^2 \le \Re\big\langle P_k A(u-u_h), A(u-u_h)\big\rangle + C\|A(u-u_h)\|_{L^2}^2 \\ &\le \big|\big\langle P_k(u-u_h), (I - \Pi_k^\sharp)A^* A(u-u_h)\big\rangle\big| + \big|\big\langle [P_k, A](u-u_h), A(u-u_h)\big\rangle\big| + C\|A(u-u_h)\|_{L^2}^2. \end{aligned} \tag{8.65}$$

The first term on the right-hand side of (8.65) is dealt with in a similar way to the proof of Lemma 8.1 (compare the first term on the right-hand side of (8.65) to the right-hand side of (8.17)). The following lemma deals with the second term on the right-hand side of (8.65) (i.e., the commutator term).

Lemma 8.16 *Let $\phi, \widetilde{\phi} \in C^\infty(\overline{\Omega})$ be such that $\partial_\nu \phi|_{\partial\Omega_-} = 0$, $\phi \prec \widetilde{\phi}$ and $\mathrm{supp}(\widetilde{\phi}) \cap \Gamma_{\rm tr} = \emptyset$, let $\psi, \psi_0 \in C^\infty_c(\mathbb{R})$ be such that $\psi_0 \prec \psi$, let $\Psi := \psi(\mathcal{P}_k)$ and $\Psi_0 := \psi_0(\mathcal{P}_k)$ and let*

$$A := \phi(1-\Psi) \quad \textit{and} \quad \widetilde{A} := \widetilde{\phi}(1-\Psi_0)$$

Then

$$\begin{aligned} A &= A\widetilde{A} + O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k) \\ [P_k, A] &= [P_k, A]\widetilde{A} + O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k) \end{aligned} \tag{8.66}$$

and for all $k_0 > 0$, there exists $C > 0$ such that for all $k \geq k_0$,

$$\|[P_k, A]\|_{L^2 \to (\mathcal{Z}_k)^*} \leq Ck^{-1}. \tag{8.67}$$

Proof. Similar to in the proof of Lemma 8.6, let $\varphi_{\rm P} \in C^\infty(\overline{\Omega})$, be such that $\widetilde{\phi} \prec \varphi_{\rm P}$ and that

$$\mathrm{supp}(\varphi_{\rm P}) \cap \Gamma_{\rm tr} = \emptyset \quad \text{and} \quad \mathrm{supp}(1-\varphi_{\rm P}) \cap \partial\Omega_- = \emptyset$$

(compare to (8.26)). We claim that

$$[P_k, A] = [\varphi_{\rm P} P_k \varphi_{\rm P}, A] + O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k) \tag{8.68}$$

$$A = A\widetilde{A} + O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k), \tag{8.69}$$

$$A(\varphi_{\rm P} P_k \varphi_{\rm P}) = A(\varphi_{\rm P} P_k \varphi_{\rm P})\widetilde{A} + O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k). \tag{8.70}$$

Once these three properties are shown, we obtain (8.66) and (8.67) as follows. First,

$$\begin{aligned} [P_k, A] &= [\varphi_{\rm P} P_k \varphi_{\rm P}, A] + O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k) && \text{(by (8.68))} \\ &= [\varphi_{\rm P} P_k \varphi_{\rm P}, A]\widetilde{A} + O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k) && \text{(by (8.69) and (8.70))} \\ &= [P_k, A]\widetilde{A} + O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k) && \text{(by (8.68))} \end{aligned}$$

which is (8.66). Second,

$$\begin{aligned} [P_k, A] &= [\varphi_{\rm P} P_k \varphi_{\rm P}, A] + O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k) \qquad \text{(by (8.68)} \\ &= [\varphi_{\rm P} P_k \varphi_{\rm P}, \phi](1-\Psi) + \phi[\varphi_{\rm P} P_k \varphi_{\rm P}, (1-\Psi)] + O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k) \\ &\qquad \text{(by definition of } A) \\ &= \varphi_{\rm P}[P_k, \phi]\varphi_{\rm P}(1-\Psi) + \phi[\varphi_{\rm P} P_k \varphi_{\rm P}, (1-\Psi)] \qquad \text{(since } [\varphi_{\rm P}, \phi] = 0) \end{aligned} \tag{8.71}$$

By Lemma 6.3, $\phi \in \mathcal{L}_{\rm sc}$, and by Lemma 6.8, $\varphi_{\rm P} P_k \varphi_{\rm P} \in \mathcal{L}_{\rm b}$. Thus, by the Definition of these spaces (Definitions 5.25 and 5.35), (8.71) gives (8.67).

We now prove (8.68)-(8.70). First, by locality of P_k,

$$P_k A = P_k \phi(1-\Psi) = \varphi_{\rm P} P_k \varphi_{\rm P} \phi(1-\Psi) = \varphi_{\rm P} P_k \varphi_{\rm P} A. \tag{8.72}$$

Moreover, by Theorem 6.2, the locality of P_k, the fact that $\widetilde{\phi} = \widetilde{\phi}\varphi_{\rm P}$, and then Theorem 6.2 again,

$$\begin{aligned} AP_k = \phi(1-\Psi)P_k &= \phi(1-\Psi)\widetilde{\phi}P_k + O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k) \\ &= \phi(1-\Psi)\widetilde{\phi}P_k\varphi_{\rm P} + O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k) \\ &= \phi(1-\Psi)\widetilde{\phi}\varphi_{\rm P} P_k \varphi_{\rm P} + O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k) \\ &= \phi(1-\Psi)\varphi_{\rm P} P_k \varphi_{\rm P} + O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k), \\ &= A\varphi_{\rm P} P_k \varphi_{\rm P} + O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k). \end{aligned} \tag{8.73}$$

Combining (8.72) and (8.73) gives (8.68). Second, by the fact that $(1-\psi) = (1-\psi)(1-\psi_0)$ and by Theorem 6.2,

$$\begin{aligned} A = \phi(1-\Psi) = \phi(1-\Psi)(1-\Psi_0) &= \phi(1-\Psi)\widetilde{\phi}(1-\Psi_0) + O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k), \\ &= A\widetilde{A} + O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k). \end{aligned}$$

which is (8.69). Finally, using again that $\varphi_{\mathrm{P}} P_k \varphi_{\mathrm{P}} \in \mathcal{L}_{\mathrm{b}}$, we obtain

$$(1-\Psi)(\varphi_{\mathrm{P}} P_k \varphi_{\mathrm{P}}) = (1-\Psi)(\varphi_{\mathrm{P}} P_k \varphi_{\mathrm{P}})(1-\Psi_0) + O_{-\infty}(k^{-\infty}, \mathcal{Y}_k \to \mathcal{Y}_k)$$

by Theorem 5.37. Left-multiplying by ϕ thus gives

$$\begin{aligned}
A(\varphi_{\mathrm{P}} P_k \varphi_{\mathrm{P}}) &= \phi(1-\Psi)(\varphi_{\mathrm{P}} P_k \varphi_{\mathrm{P}})(1-\Psi_0) + O_{-\infty}(k^{-\infty}, \mathcal{Y}_k \to \mathcal{Y}_k) \\
&= \phi(1-\Psi)(\varphi_{\mathrm{P}} P_k \varphi_{\mathrm{P}})\widetilde{\phi}(1-\Psi_0) \\
&\qquad + \phi(1-\Psi)(\varphi_{\mathrm{P}} P_k \varphi_{\mathrm{P}})(1-\widetilde{\phi})(1-\Psi_0) + O_{-\infty}(k^{-\infty}, \mathcal{Y}_k \to \mathcal{Y}_k) \\
&= A(\varphi_{\mathrm{P}} P_k \varphi_{\mathrm{P}})\widetilde{A} \\
&\qquad + \phi(1-\Psi)(\varphi_{\mathrm{P}} P_k \varphi_{\mathrm{P}})(1-\widetilde{\phi})(1-\Psi_0) + O_{-\infty}(k^{-\infty}, \mathcal{Y}_k \to \mathcal{Y}_k),
\end{aligned}$$

and (8.70) then follows from locality of $\varphi_{\mathrm{P}} P_k \varphi_{\mathrm{P}}$ and pseudolocality of Ψ (Theorem 6.2), since

$$\begin{aligned}
&\phi(1-\Psi)(\varphi_{\mathrm{P}} P_k \varphi_{\mathrm{P}})(1-\widetilde{\phi}) \\
&\quad = [\phi(1-\Psi)(1-\check{\phi})](\varphi_{\mathrm{P}} P_k \varphi_{\mathrm{P}})(1-\widetilde{\phi}) + \phi(1-\Psi)[\check{\phi}(\varphi_{\mathrm{P}} P_k \varphi_{\mathrm{P}})(1-\widetilde{\phi})] \\
&\quad = O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k)(\varphi_{\mathrm{P}} P_k \varphi_{\mathrm{P}})(1-\widetilde{\phi}) + 0 \\
&\quad = O_{-\infty}(k^{-\infty}; \mathcal{Y}_k \to \mathcal{Y}_k)
\end{aligned}$$

where $\check{\phi} \in C^\infty(\overline{\Omega})$ is such that $\phi \prec \check{\phi} \prec \widetilde{\phi}$. ■

Proof of Lemma 8.15. Let $k_0 > 0$ and $\mathfrak{c} > 0$, and let h_0 be small enough to apply Theorem 7.2 and Lemma 8.2. Let ϕ, $\widetilde{\phi}$, ψ, ψ_0 and N be as in the statement, and denote by C any positive constant depending only on the previous quantities. Let $k \geq k_0$, suppose that $h \leq h_0$ let $u - u_h$ be such that (3.16) holds. Let $\phi_1, \phi_2, \phi_3 \in C^\infty(\overline{\Omega})$ be such that

$$\phi \prec_{\mathfrak{c}/4} \phi_1 \prec_{\mathfrak{c}/4} \phi_2 \prec_{\mathfrak{c}/4} \phi_3 \prec_{\mathfrak{c}/4} \widetilde{\phi}$$

with, additionally, $\partial_\nu(\phi_1)|_{\partial\Omega_-} = 0$; such a ϕ_1 exists by Lemma 6.6. Since

$$\|\phi(1-\Psi)u\|_{H^1_k} \leq \|\phi_1(1-\Psi)u\|_{H^1_k},$$

it is enough to estimate the latter. Let $A = \check{\phi}(1-\Psi)$. By the Gårding inequality and Galerkin orthogonality (3.16),

$$\begin{aligned}
\|A(u-u_h)\|^2_{H^1_k} &\leq \Re\big\langle P_k A(u-u_h), A(u-u_h)\big\rangle + C\|A(u-u_h)\|^2_{L^2} \\
&\leq \big|\big\langle P_k(u-u_h), (\mathrm{I}-\Pi^\sharp_k)A^*A(u-u_h)\big\rangle\big| \\
&\qquad\qquad + \big|\big\langle [P_k, A](u-u_h), A(u-u_h)\big\rangle\big| + C\|A(u-u_h)\|^2_{L^2}.
\end{aligned} \tag{8.74}$$

With $\widetilde{A}$ as in the statement, Lemma 8.16 gives

$$\begin{aligned}
&\big|\big\langle [P_k, A](u-u_h), A(u-u_h)\big\rangle\big| \\
&\quad \leq \big|\big\langle [P_k, A]\widetilde{A}(u-u_h), A(u-u_h)\big\rangle\big| + Ck^{-N}\|u-u_h\|_{H_k^{-N}}\|A(u-u_h)\|_{H^1_k} \\
&\quad \leq C\left(k^{-1}\|\widetilde{A}(u-u_h)\|_{L^2} + Ck^{-N}\|u-u_h\|_{H_k^{-N}}\right)\|A(u-u_h)\|_{H^1_k},
\end{aligned} \tag{8.75}$$

and

$$\begin{aligned}
\|A(u-u_h)\|^2_{L^2} &\leq \|A(u-u_h)\|_{L^2}\|A(u-u_h)\|_{H^1_k} \\
&\leq C\left(\|\widetilde{A}(u-u_h)\|_{L^2} + Ck^{-N}\|u-u_h\|_{H_k^{-N}}\right)\|A(u-u_h)\|_{H^1_k}.
\end{aligned} \tag{8.76}$$

Combining (8.74), (8.75) and (8.76), we deduce that

$$\|A(u-u_h)\|^2_{H^1_k} \leq \big|\big\langle P_k(u-u_h), (\mathrm{I}-\Pi^\sharp_k)A^*A(u-u_h)\big\rangle\big| + C\|\widetilde{A}(u-u_h)\|^2_{L^2} + Ck^{-2N}\|u-u_h\|^2_{H_k^{-N}},$$

and to conclude the proof, it remains to show that

$$\left|\left\langle P_k(u-u_h), (\mathrm{I}-\Pi_k^\sharp)A^*A(u-u_h)\right\rangle\right| \\ \leq (\|\widetilde{\phi}(u-w_h)\|_{H_k^1} + (h_{\tilde{\phi}}k)^p\|\widetilde{\phi}(u-u_h)\|_{H_k^{-N}} + CR)\|A(u-u_h)\|_{H_k^1} \tag{8.77}$$

where $R := k^{-N}(\|u-u_h\|_{H_k^{-N}} + \|u-w_h\|_{H_k^1})$.

To establish (8.77), we use the identity

$$\left\langle P_k(u-u_h), (\mathrm{I}-\Pi_k^\sharp)A^*A(u-u_h)\right\rangle \\ = \left\langle u-w_h, (P_k^\sharp)^*(\mathrm{I}-\Pi_k^\sharp)A^*A(u-u_h)\right\rangle - \left\langle u-u_h, S_k(\mathrm{I}-\Pi_k^\sharp)A^*A(u-u_h)\right\rangle$$

(shown in the same manner as (8.17) in the proof of the localised duality argument, Lemma 8.1). Next, by pseudo-locality of $1-\Psi$, $A^* = \phi_2 A^* + O_{-\infty}(k^{-\infty};\mathcal{Y}_k \to \mathcal{Y}_k)$, so that

$$\left|\left\langle P_k(u-u_h), (\mathrm{I}-\Pi_k^\sharp)A^*A(u-u_h)\right\rangle\right| \\ \leq \left|\left\langle u-w_h, (P_k^\sharp)^*(\mathrm{I}-\Pi_k^\sharp)\phi_2 A^*A(u-u_h)\right\rangle\right| + \left|\left\langle (u-u_h), S_k(\mathrm{I}-\Pi_k^\sharp)\phi_2 A^*A(u-u_h)\right\rangle\right| \\ + CR\|A(u-u_h)\|_{H_k^1}.$$

Now, adapting the arguments in the proof of Lemma 8.1 (from (8.17) to (8.23)), we obtain

$$\left|\left\langle P_k(u-u_h), (\mathrm{I}-\Pi_k^\sharp)A^*A(u-u_h)\right\rangle\right| \\ \leq \left|\left\langle P_k^\sharp\widetilde{\phi}(u-w_h), \phi_3(\mathrm{I}-\Pi_k^\sharp)w\right\rangle\right| + \left|\left\langle S_k\widetilde{\phi}(u-u_h), \widetilde{S}_k\phi_3(\mathrm{I}-\Pi_k^\sharp)w\right\rangle\right| + CR\|A(u-u_h)\|_{H_k^1}, \tag{8.78}$$

where $w = \phi_2 A^*A(u-u_h)$ and $\widetilde{S}_k := \widetilde{\psi}(\mathcal{P}_k)$ where $\widetilde{\psi} \in C_c^\infty(\mathbb{R})$ is such that $\psi \prec \widetilde{\psi}$. Namely, we follow exactly the same steps as in (8.19) and (8.21) but with $\ell = -1$, and in (8.20) and (8.22), choosing $v = A^*A(u-u_h)$, we use the estimate $\|v\|_{H_k^1} \leq \|A^*\|_{H_k^1\to H_k^1}\|A(u-u_h)\|_{H^1}$, and the fact that $\|A^*\|_{H_k^1\to H_k^1} \leq C$. Finally, by the quasi-optimality of $\Pi_k^\sharp$ and the previous bound on $\|A^*\|_{H_k^1\to H_k^1}$,

$$\|(\mathrm{I}-\Pi_k^\sharp)w\|_{H_k^1} \leq C\|A(u-u_h)\|_{H_k^1},$$

and in turn, by Lemma 8.2,

$$\|\widetilde{S}_k\phi_3(\mathrm{I}-\Pi_k^\sharp)w\|_{L^2} \leq C\Big((h_{\tilde{\phi}}k)^p + k^{-N}(hk)^p\Big)\|A(u-u_h)\|_{H_k^1}.$$

Using these bounds in (8.78),

$$\left|\left\langle P_k(u-u_h), (\mathrm{I}-\Pi_k^\sharp)A^*A(u-u_h)\right\rangle\right| \\ \leq C\left(\|\widetilde{\phi}(u-w_h)\|_{H_k^1} + \big((\widetilde{h}_\phi k)^p + k^{-N}(hk)^p\big)\|S_k\widetilde{\phi}(u-u_h)\|_{L^2} + CR\right)\|A(u-u_h)\|_{H_k^1}$$

and (8.77) follows by using the mapping properties of S_k. ■

The proof of the third block row of (3.17) in the H_k^1 norm (i.e., $m=0$), is similar to that of the second block row, using Lemma 8.17 below instead of Lemma 8.15.

Lemma 8.17 *For any $k_0 > 0$ and $\mathfrak{c} > 0$, there exists $h_0 > 0$ such that the following holds. Let $\phi, \widetilde{\phi} \in C^\infty(\overline{\Omega})$ be such that $\phi \prec_{\mathfrak{c}} \widetilde{\phi}$. Then, for all $N > 0$, there exists $C > 0$ such that for all $k \geq k_0$, $h \leq h_0$, $u - u_h$ satisfying (3.16) and for all $w_h \in V_k$,*

$$\|\phi(u-u_h)\|_{H_k^1} \leq C\left(\|\widetilde{\phi}(u-w_h)\|_{H_k^1} + \|\widetilde{\phi}(u-u_h)\|_{L^2}\right) \\ + Ck^{-N}\big(\|u-w_h\|_{H_k^1} + \|u-u_h\|_{H_k^{-N}}\big).$$

where $h_{\tilde{\phi}} := \max\Big\{h_K \;:\; K \in \mathcal{T}_k \;\text{ s.t. }\; K \cap \operatorname{supp}(\widetilde{\phi}) \neq \emptyset\Big\}$.

Proof. Let $\phi_1, \phi_2 \in C^\infty(\overline{\Omega})$ be such that

$$\phi \prec \phi_1 \prec \phi_2 \prec \widetilde{\phi},$$

with in addition, $\partial_\nu(\phi_1)|_{\partial\Omega_-} = 0$. Then, observe that

$$\|\phi(u - u_h)\|_{H^1_k} \le C\|\phi_1(u - u_h)\|_{H^1_k}.$$

The proof is now identical to that of Lemma 8.15, with the following replacement for Lemma 8.16: (i) $[P_k, \phi_1] = [P_k, \phi_1]\phi_2$, which follows from locality of P_k and (ii), $\|[P_k, \phi_1]\|_{L^2\to\mathcal{Z}^*_k} \le Ck^{-1}$ is continuous, which follows from Lemma 6.3 and Definition 5.25. ■

9 Proof of Theorem 1.3

Under the assumptions on $(\mathcal{T}_k)_{k>0}$ in Theorem 1.3, the family $(V^p_{\mathcal{T}_k})_{k>0}$ is a well-behaved finite element of order p in the sense of Definition 3.7; we can therefore apply Theorem 3.11 (in particular, (3.18)). By (8.59), $T^\star \le \sum_{n=0}^\infty (C_\dagger W)^n$. To prove Theorem 1.3, it is therefore sufficient to show that, provided the mesh conditions (1.8) holds, the loop condition (3.15) holds and

$$\begin{pmatrix} \mathrm{I} & (\mathcal{H}_{\mathrm{I,I}}k)^N & 0 \\ 0 & 0 & (h_{\mathcal{P}}k)^N \end{pmatrix} \sum_{n=0}^\infty (C_\dagger W)^n B \le C(\mathscr{T}\mathscr{B} + \mathscr{R}) \tag{9.1}$$

where B, W are defined by (3.12), (3.13), while $\mathcal{H}_{\mathrm{I,I}}$, $\mathscr{T}$ and $\mathscr{B}$ and $\mathscr{R}$ are defined by (1.6). In fact, by Theorem 8.13, the loop condition (3.15) holds if and only if the sum $\sum_{n=0}^\infty W^n$ converges, and from the way the simple-path matrix $T^\star$ was used in the proof of Theorem 3.11 to bound $(I - C_\dagger W)^{-1}$, it suffices to show (9.1) with $T^\star$ replaced by $\sum_{n=0}^\infty W^n$. In addition, since, under the mesh conditions (1.8), $(\mathcal{H}_{\mathrm{I,I}}k)^{2pN'} \le k^{-N'}\,\mathrm{I}$ and $(h_{\mathrm{P}}k) \le c$, it suffices to show that

$$\begin{pmatrix} \mathrm{I} & 0 & 0 \\ 0 & 0 & \mathrm{I} \end{pmatrix} \sum_{n=0}^\infty (C_\dagger W)^n B \le C\mathscr{T}\mathscr{B}. \tag{9.2}$$

We obtain (9.2) by "forgetting" about the improvements on the high-frequency components of the Galerkin error. That is, we consider the directed graph $\mathcal{G}$ in Figure 9.1 – which describes the error propagation without any frequency splitting (where we have used the bounds on the solution operator from §4) – and let

$$\mathscr{W} := \begin{pmatrix} \mathcal{C}_{\mathrm{I,I}}(\mathcal{H}_{\mathrm{I,I}}k)^p & h_{\min}(N)k^N \\ h_{\min}(N)^T k^N & (h_{\mathcal{P}}k)^N \end{pmatrix}$$

be the associated weighted adjacency matrix. Here,

$$h_{\min}(N) = \begin{pmatrix} 0 & h^N_{\mathcal{V},\mathcal{P}} & h^N_{\mathcal{I},\mathcal{P}} & h^N_{\mathcal{P}} \end{pmatrix}^T.$$

The point is that $\mathscr{T}$ is, up to a constant, the simple-path matrix of $\mathscr{W}$. More precisely, the following result holds.

Lemma 9.1 *For all $C_\dagger > 0$, there exists $c, C > 0$ such that if (1.8) holds with c, then*

$$\sum_{n=0}^\infty C_\dagger^n \mathscr{W}^n \le C\mathscr{T}.$$

Proof. Observe that under (1.8), only one edge in this graph can possibly have a weight $> c$, namely,

$$\mathscr{W}_{1,2} = \sqrt{k\rho(k)}(h_{\mathcal{V}}k)^{2p}.$$

Moreover, any simple loop in this graph containing the edge $\mathscr{W}_{1,2}$ must also contain the edge

$$\mathscr{W}_{2,1} = (h_{\mathcal{K}}k)^{2p}\sqrt{k\rho(k)},$$

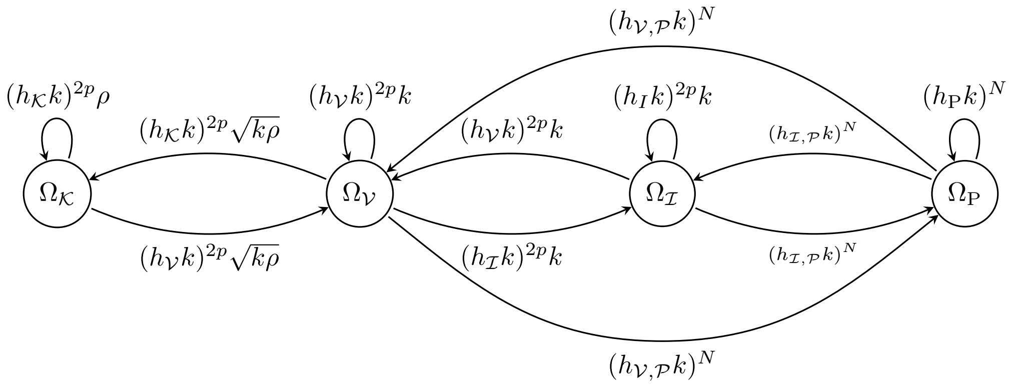


Figure 9.1: The graph showing propagation of errors for the decomposition into $\Omega_\mathcal{K}$, $\Omega_\mathcal{V}$, $\Omega_\mathcal{I}$. Recall that $h_{\mathcal{V},\mathcal{P}} = \min(h_\mathcal{V}, h_\mathcal{P})$ and $h_{\mathcal{I},\mathcal{P}} = \min(h_\mathcal{I}, h_\mathcal{P})$.

and the product of these two weights is

$$\mathscr{W}_{1,2}\mathscr{W}_{2,1} = \underbrace{(h_\mathcal{K}k)^{2p}\rho(k)}_{<c}\underbrace{(h_\mathcal{V}k)^{2p}k}_{<c} \leq c^2.$$

Therefore, provided that c is sufficiently small, the sum of the weights of all simple loops in $\mathcal{G}$ can be made < 1. The conclusion follows by remarking that, under the mesh conditions (1.8),

$$\sum_{p\in\mathbb{V}_{ij}} C_\dagger^{|p|}\mathscr{W}_p \leq C\mathscr{T}_{ij},$$

as can be checked by direct calculation. ■

By Lemma 9.1, it suffices to show that for all $C_\dagger > 0$, there exists $C'_\dagger > 0$ and $C > 0$ such that

$$\begin{pmatrix} \mathrm{I} & 0 & 0 \\ 0 & 0 & \mathrm{I} \end{pmatrix} (C_\dagger W)^n B \leq C C'^n_\dagger \mathscr{W}^n \mathscr{B}. \tag{9.3}$$

To prove this, we first observe that, with $\ell = p$ or $2p$,

$$\mathcal{H}^{\min}_{\mathrm{I},\mathrm{I}}(\ell)k^\ell \leq \mathcal{C}_{\mathrm{I},\mathrm{I}}(\mathcal{H}_{\mathrm{I},\mathrm{I}}k)^\ell.$$

Indeed, $(\mathcal{C}_{\mathrm{I},\mathrm{I}})_{ij} \geq 1$ for all i, j such that $(\mathcal{H}^{\min}_\mathrm{I}(\ell))_{ij} \neq 0$ (since when the domains overlap, the norm of the solution operator is ≥ 1). Since $\mathcal{H}$ is diagonal, it follows that for such pairs i, j and all $\ell \geq 0$,

$$(\mathcal{H}^{\min}_{\mathrm{I},\mathrm{I}}(\ell))_{ij} = \min(h_i, h_j)^\ell \leq (\mathcal{C}_{\mathrm{I},\mathrm{I}})_{ij} h_j^\ell = (\mathcal{C}_{\mathrm{I},\mathrm{I}}(\mathcal{H})^\ell)_{ij}.$$

Therefore, the matrix W associated to the full graph (Figure 3.1) and the matrix B can be estimated by blocks as

$$W \leq C \begin{pmatrix} \mathcal{C}_{\mathrm{I},\mathrm{I}}(\mathcal{H}_{\mathrm{I},\mathrm{I}}k)^{2p} & \mathcal{C}_{\mathrm{I},\mathrm{I}}(\mathcal{H}_{\mathrm{I},\mathrm{I}}k)^{2p} & h_{\min}(N)k^N \\ \mathcal{C}_{\mathrm{I},\mathrm{I}}(\mathcal{H}_{\mathrm{I},\mathrm{I}}k)^{2p} & \mathcal{C}_{\mathrm{I},\mathrm{I}}(\mathcal{H}_{\mathrm{I},\mathrm{I}}k)^{2p} & h_{\min}(N)k^N \\ h^T_{\min}(N)k^N & h^T_{\min}(N)k^N & (h_\mathcal{P}k)^N \end{pmatrix} = C \begin{pmatrix} \mathrm{diag}(A(2p))J & Kb(N) \\ b(N)^T K^T & (h_\mathcal{P}k)^N \end{pmatrix}$$

$$B \leq C \begin{pmatrix} \mathcal{C}_{\mathrm{I},\mathrm{I}}(\mathcal{H}_{\mathrm{I},\mathrm{I}}k)^p & 0 \\ \mathcal{C}_{\mathrm{I},\mathrm{I}}(\mathcal{H}_{\mathrm{I},\mathrm{I}}k)^p & 0 \\ 0 & (h_\mathcal{P}k)^p \end{pmatrix} = C \begin{pmatrix} \mathrm{diag}(A(p))K & 0 \\ 0 & (h_\mathcal{P}k)^p \end{pmatrix}$$

where

$$A(\ell) = \mathcal{C}_{\mathrm{I},\mathrm{I}}(\mathcal{H}_{\mathrm{I},\mathrm{I}}k)^\ell, \quad \mathrm{diag}(A) := \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad b(\ell) = h_{\min}(\ell)k^\ell, \quad J = \begin{pmatrix} I_3 & I_3 \\ I_3 & I_3 \end{pmatrix} \quad K = \begin{pmatrix} I_3 \\ I_3 \end{pmatrix}$$

with I_3 denoting the 3×3 identity matrix. With these definitions, observe that

$$\mathscr{W} = \begin{pmatrix} A(2p) & b(N) \\ b(N)^T & (h_{\mathcal{P}}k)^N \end{pmatrix}, \quad \mathscr{B} = \begin{pmatrix} A(p) & 0 \\ 0 & (h_{\mathcal{P}}k)^p \end{pmatrix};$$

thus the estimate (9.3) immediately follows from the next lemma.

Lemma 9.2 *Let A, B be $M\times M$ matrices with positive coefficients, $b, b' \in \mathbb{R}^M_+$, $c \in \mathbb{R}_+$. Then, for all n,*

$$\begin{pmatrix} \mathrm{I}_M & 0 & 0 \\ 0 & 0 & \mathrm{I}_M \end{pmatrix} \begin{pmatrix} \mathrm{diag}(A)J & Kb \\ b^T K^T & c \end{pmatrix}^n \begin{pmatrix} \mathrm{diag}(A')K & 0 \\ 0 & c' \end{pmatrix} \leq 2^{n+1} \begin{pmatrix} A & b \\ b^T & c \end{pmatrix}^n \begin{pmatrix} A' & 0 \\ 0 & c' \end{pmatrix}.$$

Proof. Let

$$\begin{pmatrix} A_n & b_n \\ b_n^T & c_n \end{pmatrix} := \begin{pmatrix} A & b \\ b^T & c \end{pmatrix}^n.$$

Using that $J^2 = 2J$, $JK = 2K$ and $K^T K = 2$, one can check by an easy induction that

$$\begin{pmatrix} \mathrm{diag}(A)J & Kb \\ b^T K^T & c \end{pmatrix}^n \leq 2^n \begin{pmatrix} \mathrm{diag}(A_n)J & Kb_n \\ b_n^T K^T & c_n. \end{pmatrix}$$

The result then follows using that

$$\begin{pmatrix} \mathrm{I}_M & 0 & 0 \\ 0 & 0 & \mathrm{I}_M \end{pmatrix} \begin{pmatrix} \mathrm{diag}(A_n)J & Kb_n \\ b_n^T K^T & c_n \end{pmatrix} \begin{pmatrix} \mathrm{diag}(A')K & Kb' \\ b'^T & c' \end{pmatrix} = \begin{pmatrix} A_n & A_n & b \\ b & b & c \end{pmatrix} \begin{pmatrix} A' & 0 \\ A' & 0 \\ 0 & c' \end{pmatrix}$$
$$\leq 2 \begin{pmatrix} A_n & b_n \\ b_n^T & c_n \end{pmatrix} \begin{pmatrix} A' & 0 \\ 0 & c' \end{pmatrix}.$$

■

A Definition of radial perfectly matched layers

Let $R_{\rm scat}$ be such that

$$\mathrm{supp}(A - I) \cup \mathrm{supp}(n-1) \cup \Omega \Subset B_{R_{\rm scat}}. \tag{A.1}$$

Let $R_{\rm PML,-} > R_{\rm scat}$ be such that $B_{R_{\rm PML,-}} \Subset \Omega_{\rm tr}$.

As in §1.1 and §3.1, let $\Omega := \Omega_+ \cap \Omega_{\rm tr}$ and $\Gamma_{\rm tr} := \partial\Omega_{\rm tr}$. For $0 \leq \theta < \pi/2$, let the PML scaling function $f_\theta \in C^\infty([0,\infty);\mathbb{R})$ be defined by $f_\theta(r) := f(r)\tan\theta$ for some f satisfying

$$\big\{f(r) = 0\big\} = \big\{f'(r) = 0\big\} = \big\{r \leq R_{\rm PML,-}\big\}, \quad f'(r) \geq 0; \tag{A.2}$$

i.e., the scaling "turns on" at $r = R_{\rm PML,-}$. Given $f_\theta(r)$, let

$$\alpha(r) := 1 + \mathrm{i}f'_\theta(r) \quad \text{and} \quad \beta(r) := 1 + \mathrm{i}f_\theta(r)/r.$$

We now define two possible PML problems (1.2); both are formed by first replacing Δ in (1.1) by

$$\begin{aligned}\Delta_\theta &= \Big(\frac{1}{1+\mathrm{i}f'_\theta(r)}\frac{\partial}{\partial r}\Big)^2 + \frac{d-1}{(r+\mathrm{i}f_\theta(r))(1+\mathrm{i}f'_\theta(r))}\frac{\partial}{\partial r} + \frac{1}{(r+\mathrm{i}f_\theta(r))^2}\Delta_\omega, \\ &= \frac{1}{(1+\mathrm{i}f'_\theta(r))(r+\mathrm{i}f_\theta(r))^{d-1}}\frac{\partial}{\partial r}\left(\frac{(r+\mathrm{i}f_\theta(r))^{d-1}}{1+\mathrm{i}f'_\theta(r)}\frac{\partial}{\partial r}\right) + \frac{1}{(r+\mathrm{i}f_\theta(r))^2}\Delta_\omega\end{aligned}$$

(with Δ_ω the surface Laplacian on S^{d-1}) and then either multiplying by $\alpha\beta^{d-1}$ or not – the coefficients A_θ, b_θ, and n_θ for both options are defined below.

Comparison of the two different formulations. The multiplication by $\alpha\beta^{d-1}$ has the advantage that the resulting operator is in divergence form; however, for P_k to satisfy (3.5), one requires additional assumptions. In particular, [GLSW24, Lemma 2.3] shows that (3.5) holds for any $f_\theta(r)$ satisfying the above assumptions in $d=2$ and holds in $d=3$ when $f_\theta(r)/r$ is, in addition, non-decreasing and [GLSW24, Remark 2.1] shows that such an additional assumption is needed.

If one instead integrates by parts the complex-scaled PDE directly (i.e., avoids the above multiplication), then the resulting sesquilinear form satisfies the Gårding inequality after multiplication by $\mathrm{e}^{\mathrm{i}\omega}$, for some suitable constant ω [GLS24, Lemma A.6].

We highlight that, in other papers on PMLs, the scaled variable, which in our case is $r+\mathrm{i}f_\theta(r)$, is often written as $r(1+\mathrm{i}\widetilde{\sigma}(r))$ with $\widetilde{\sigma}(r)=\sigma_0$ for r sufficiently large; see, e.g., [HSZ03, §4], [BP07, §2]. Therefore, to convert from our notation, set $\widetilde{\sigma}(r)=f_\theta(r)/r$ and $\sigma_0=\tan\theta$. In this alternative notation, the assumption that $f_\theta(r)/r$ is nondecreasing is therefore that $\widetilde{\sigma}$ is nondecreasing – see [BP07, §2].

The sesquilinear form after multiplication by $\alpha\beta^{d-1}$. Define $b_\theta:=0$,

$$A_\theta := \begin{cases} A & \text{in } \Omega, \\ HDH^T & \text{in } (B_{R_{\mathrm{PML},-}})^c \end{cases} \quad \text{and} \quad n_\theta := \begin{cases} n & \text{in } \Omega\cap B_{R_{\mathrm{PML},-}}, \\ \alpha(r)\beta(r)^{d-1} & \text{in } (B_{R_{\mathrm{PML},-}})^c, \end{cases} \tag{A.3}$$

where, in polar coordinates (r,φ),

$$D=\begin{pmatrix} \beta(r)\alpha(r)^{-1} & 0 \\ 0 & \alpha(r)\beta(r)^{-1} \end{pmatrix} \quad \text{and} \quad H=\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \text{ for } d=2,$$

and, in spherical polar coordinates (r,φ,ϕ),

$$D=\begin{pmatrix} \beta(r)^2\alpha(r)^{-1} & 0 & 0 \\ 0 & \alpha(r) & 0 \\ 0 & 0 & \alpha(r) \end{pmatrix} \text{ and } H=\begin{pmatrix} \sin\varphi\cos\phi & \cos\varphi\cos\phi & -\sin\phi \\ \sin\varphi\sin\phi & \cos\varphi\sin\phi & \cos\phi \\ \cos\varphi & -\sin\varphi & 0 \end{pmatrix}$$

for $d=3$. (observe that then $A=I$ and $n=1$ when $r=R_{\mathrm{PML},-}$ and thus A_θ and n_θ are continuous at $r=R_{\mathrm{PML},-}$).

Lemma A.1 ([GLSW24, Lemma 2.3]) *Let f_θ satisfy (A.2) and the additional assumption when $d=3$ that $f_\theta(r)/r$ is nondecreasing, given $\epsilon>0$ there exists $c>0$ such that, for all $\epsilon\le\theta\le\pi/2-\epsilon$, A_θ defined by (A.3) satisfies*

$$\Re\big(A_\theta(x)\xi,\xi\big)_2 \ge c\|\xi\|_2^2 \quad \text{for all } \xi\in\mathbb{C}^d \text{ and } x\in\Omega;$$

thus the Gårding inequality (3.5) holds.

The sesquilinear form without multiplication by $\alpha\beta^{d-1}$. Define

$$A_\theta := \begin{cases} A & \text{in } \Omega, \\ HDH^T & \text{in } (B_{R_{\mathrm{PML},-}})^c \end{cases} \text{ and } n_\theta := \begin{cases} n & \text{in } \Omega\cap B_{R_{\mathrm{PML},-}}, \\ 1 & \text{in } (B_{R_{\mathrm{PML},-}})^c, \end{cases} \tag{A.4}$$

where, in polar coordinates (r,φ),

$$D=\begin{pmatrix} \alpha(r)^{-2} & 0 \\ 0 & \beta(r)^{-2} \end{pmatrix} \quad \text{and} \quad H=\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \text{ for } d=2,$$

and, in spherical polar coordinates (r,φ,ϕ),

$$D=\begin{pmatrix} \alpha(r)^{-2} & 0 & 0 \\ 0 & \beta(r)^{-2} & 0 \\ 0 & 0 & \beta(r)^{-2} \end{pmatrix} \text{ and } H=\begin{pmatrix} \sin\varphi\cos\phi & \cos\varphi\cos\phi & -\sin\phi \\ \sin\varphi\sin\phi & \cos\varphi\sin\phi & \cos\phi \\ \cos\varphi & -\sin\varphi & 0 \end{pmatrix}$$

for $d = 3$ (observe that then $A_\theta = I$ and $n_\theta = 1$ when $r = R_{\mathrm{PML},-}$ and thus A and n are continuous at $r = R_{\mathrm{PML},-}$). In addition, for $d = 2$,

$$b_\theta(r) = \begin{cases} 0 & \Omega \cap B_{R_{\mathrm{PML},-}} \\ H \begin{pmatrix} \alpha^{-2}\big(\log(\alpha\beta)\big)' \\ 0 \end{pmatrix} & (B_{R_{\mathrm{PML},-}})^c, \end{cases}$$

and for $d = 3$

$$b_\theta(r) = \begin{cases} 0 & \Omega \cap B_{R_{\mathrm{PML},-}} \\ H \begin{pmatrix} \alpha^{-2}\big(\log(\alpha\beta^2)\big)' \\ 0 \\ 0 \end{pmatrix} & (B_{R_{\mathrm{PML},-}})^c. \end{cases}$$

Lemma A.2 ([GLS24, Lemma A.6]) *Let f_θ satisfy (A.2). Given $\epsilon > 0$ there exists $\omega \in \mathbb{R}, c > 0$ such that, for all $\epsilon \le \theta \le \pi/2 - \epsilon$, A_θ defined by (A.4) satisfies*

$$\Re\big(\mathrm{e}^{\mathrm{i}\omega} A_\theta(x)\xi, \xi\big)_2 \ge c\|\xi\|_2^2 \quad \textit{for all } \xi \in \mathbb{C}^d \textit{ and } x \in \Omega;$$

thus the Gårding inequality (3.5) holds for the sesquilinear form $\mathrm{e}^{\mathrm{i}\omega} a_k(\cdot,\cdot)$.

B Loop decompositions in directed graphs (Theorem 8.13)

Fix a matrix $W \in \mathbb{M}(M \times M)$, let $\mathcal{G}$ be the graph associated to W as defined in §3.6 and $T^\star$ the simple-path matrix of W. Denote by $\mathbb{P}_{ij}$ the set of paths from i to j. Recalling the classical identity

$$(W^n)_{ij} = \sum_{p \in \mathbb{P}_{ij} \text{ s.t. } |p| = n} W_p$$

and summing over n, one obtains that

$$\left[\sum_{n \in \mathbb{N}} W^n\right]_{ij} = \sum_{p \in \mathbb{P}_{ij}} W_p,$$

provided that the right-hand side converges. The first inequality in (8.59) then follows immediately.

To prove the implication in (8.58) and the second inequality in (8.59), it is sufficient to show that

$$\sum_{p \in \mathbb{P}_{ij}} W_p \le \frac{1}{1-c} T^\star_{ij} \tag{B.1}$$

B.1 Outline

We show (B.1) by constructing an injective map

$$\mathcal{D}ec : \mathbb{P}_{ij} \to \mathbb{V}_{ij} \times \mathbb{SL}^{(\mathbb{N})}$$

where for any set A, $A^{(\mathbb{N})}$ denotes the set of finite ordered sequences of elements of A (possibly of size 0). The map $\mathcal{D}ec$ is defined in Definition B.4 below, and its properties are stated in Lemma B.5. It corresponds to a decomposition of every path $p \in \mathbb{P}_{ij}$ into a non-intersecting segment $v \in \mathbb{V}_{ij}$ and a tuple of simple loops $(L_1, \ldots, L_Q) \in \mathbb{SL}^{(\mathbb{N})}$. The idea is that one obtains the decomposition by recursively removing loops from p until the remainder is non-intersecting. If one defines

$$W_{(v,(L_1,\ldots,L_Q))} := W_v W_{L_1} \ldots W_{L_Q}, \tag{B.2}$$

then it will be seen that $W_p = W_{\mathcal{D}ec(p)}$ for all $p \in \mathbb{P}_{ij}$. The proof of Lemma 8.13 is then obtained as follows:

$$\sum_{p \in \mathbb{P}_{ij}} W_p = \sum_{p \in \mathbb{P}_{ij}} W_{\mathcal{D}ec(p)} = \sum_{q \in \mathcal{D}ec(\mathbb{P}_{ij})} W_q \le \sum_{q \in \mathbb{V}_{ij} \times \mathbb{SL}^{(\mathbb{N})}} W_q$$

since $\mathcal{D}ec$ is injective. The last term can be rewritten as

$$\sum_{q\in\mathbb{V}_{ij}\times\mathbb{SL}^{(\mathbb{N})}} W_q = \sum_{v\in\mathbb{V}_{ij}} \sum_{Q=0}^{\infty} \sum_{L_1,\ldots,L_Q\in\mathbb{SL}} W_v W_{L_1}\ldots W_{L_Q} = \sum_{v\in\mathbb{V}_{ij}} W_v \sum_{Q=0}^{\infty} \left(\sum_{L\in\mathbb{SL}} W_L\right)^Q .$$

Therefore if $\left(\sum_{L\in\mathbb{SL}} W_L\right) \leq c < 1$, then

$$\sum_{p\in\mathbb{P}_{ij}} W_p \leq \frac{1}{1-c} \sum_{v\in\mathbb{V}_{ij}} W_v = T^\star_{ij}.$$

B.2 Construction of the map $\mathcal{D}$ec

For $1 \leq \ell \leq m \leq |p|+1$, the *splice* of p between ℓ and m, denoted by $p[\ell,m)$, is the path obtained from p by only keeping the edges from ℓ to $m-1$, that is

$$p[\ell,m) := e_\ell e_{\ell+1}\ldots e_{m-1},$$

with the convention that $p[\ell,\ell) = \mathbf{0}$. Given two paths $p = e_1e_2\ldots e_{|p|}$ and $q = f_1f_2\ldots f_{|q|}$ such that $p(|p|+1) = q(1)$, the *concatenation* of p and q is defined by

$$p\cdot q = e_1e_2\ldots e_{|p|}f_1f_2\ldots f_{|q|},$$

with the convention that for all paths p, $p\cdot\mathbf{0} = \mathbf{0}\cdot p = p$. In particular, for all $p,q\in\mathbb{P}$,

$$|p\cdot q| = |p| + |q|.$$

Furthermore, when $m > \ell$, $p[\ell,m)$ is a path from $p(\ell)$ to $p(m)$, and for all $1\leq \ell \leq |p|+1$,

$$p = p[1,\ell)\cdot p[\ell,|p|+1).$$

If $p(\ell) = i_0$ and L_{i_0} is either $\mathbf{0}$ or a loop through i_0, one can then define the *insertion of L_{i_0} in p at index ℓ* by

$$p \overset{\ell}{\angle} L_{i_0} := p[1,\ell)\cdot L_{i_0}\cdot p[\ell,|p|+1).$$

To extract the first loop of a self-intersecting path, one can "follow" the path until some vertex $i_\times$ occurs for the second time. One then backtracks to the first occurence of that vertex, and the splice in between those two occurences defines a simple loop that can be extracted from p. More precisely, let

$$\ell_\times(p) := \inf\Big\{\ell\in\{1,\ldots,|p|+1\} \,:\, p(\ell)\in\{p(1),\ldots,p(\ell-1)\}\Big\},$$

the *index of first crossing*. Note that $\ell_\times(p) = \infty$ if, and only if, p is non-intersecting. If $\ell_\times(p) < \infty$, define $i_\times(p) := p(\ell_\times(p))$ the *first crossing point* of p, and

$$\ell_0(p) := \inf\Big\{\ell\in\{0,\ldots,|p|+1\} \,:\, p(\ell) = i_\times(p)\Big\}$$

the first index at which p visits $i_\times(p)$. These definitions are illustrated in Figure B.1. Define the maps $L:\mathbb{P}\to\mathbb{P}$ and $E:\mathbb{P}\to\mathbb{P}$ by

- $L(p) := \begin{cases} p[\ell_0(p), \ell_\times(p)) & \text{if } \ell_\times(p) < \infty, \\ \mathbf{0} & \text{if } \ell_\times(p) = \infty. \end{cases}$

  the *first loop* in p, and

- $E(p) := \begin{cases} p[1, \ell_0(p)) \cdot p[\ell_\times(p), |p|+1) & \text{if } \ell_\times(p) < \infty, \\ p & \text{if } \ell_\times(p) = \infty, \end{cases}$

  the remainder after extracting the loop $L(p)$.

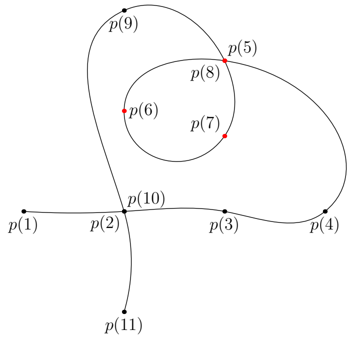


Figure B.1: Example of a self-intersecting path. Here, $\ell_0(p) = 5$ and $\ell_\times(p) = 8$. The vertices of $L(p)$ are highlighted in red.

The properties of L and E are summarized in the following lemma. The proof is immediate from the definitions.

Lemma B.1 *For all paths p,*

$$p = E(p) \overset{\ell_0(p)}{\angle} L(p),$$
$$|p| = |E(p)| + |L(p)| \quad \text{and} \quad W_p = W_{L(p)} W_{E(p)}.$$

The path p is non-intersecting if, and only if, $L(p) = \mathbf{0}$, in which case, $p = E(p)$. Otherwise, $L(p) \in \mathbb{SL}$, $|L(p)| \geq 1$ and

$$\ell_0(p) = \inf\Big\{\ell \in \{0, \dots, |E(p)|+1\} : \big(E(p)\big)(\ell) = \big(L(p)\big)(1)\Big\}.$$

If $p \in \mathbb{P}_{ij}$, then either

1. *$E(p) \in \mathbb{P}_{ij}$, or*
2. *$E(p) = \mathbf{0}$, and this can only occur if $i = j$.*

Define $E^n(p) := E(E^{n-1}(p))$, with $E^0(p) := p$.

Corollary B.2 *Let $p \in \mathbb{P}$. Then there exists a unique number $n_0 \in \mathbb{N}$, the* number of loops in p, *such that the following properties hold:*

- *either $n_0 = 0$ or $E^{n_0-1}(p) \neq E^{n_0}(p)$,*
- *for all $n \geq n_0$, $E^n(p) = E^{n_0}(p)$.*

Proof. If $E^{n+1}(p) \neq E^n(p)$, then by Lemma B.1, $|E^{n+1}(p)| \leq |E^n(p)| - 1$. Since infinite sequences of natural numbers cannot be strictly decreasing, the sequence $(E^n(p))_n$ must eventually stagnate. ■

Corollary B.3 *The map*

$$\begin{array}{rlcl} \mathrm{Dec}: & \mathbb{P} \times \mathbb{P}^{(\mathbb{N})} & \to & \mathbb{P} \times \mathbb{P}^{(\mathbb{N})} \\ & p, (L_1, \dots, L_Q) & \mapsto & E(p), (L_1, \dots, L_Q, L(p)) \end{array}$$

is injective. If $X \in \mathbb{P} \times \mathbb{P}^{(\mathbb{N})}$ and if W_X is defined as in (B.2), then

$$W_{\mathrm{Dec}(X)} = W_X. \tag{B.3}$$

Proof. Suppose that

$$(E(p),(L_1,\dots,L_Q,L(p)))=(E(p'),(L'_1,\dots,L'_Q,L(p')))$$

and let $E=E(p)=E(p')$ and $L=L(p)=L(p')$. To conclude, it suffices to show that $p=p'$ (since it is obvious that $L_i=L'_i$ for $1\le i\le Q$). There are two cases: either $|L|=0$ or $|L|\ge 1$. By Lemma B.1, in the first case, $E=p=p'$. In the second case, since $E(p)=E(p')$ and $L(p)=L(p')$,

$$\ell_0(p)=\ell_0(p')=\inf\Big\{\ell\in\{0,\dots,|E|+1\}\,:\,E(\ell)=L(1)\Big\}=:\ell_0$$

and $p=p'=E\overset{\ell_0}{\frown}L$. Thus in both cases, $p=p'$. The proof of (B.3) is immediate. ∎

Definition B.4 (Loop decomposition of a path) *Given $p\in\mathbb{P}$, the* loop-decomposition *of p, denoted by $\mathcal{D}\text{ec}(p)\in\mathbb{P}\times\mathbb{P}^{(\mathbb{N})}$, is defined by*

$$\mathcal{D}\text{ec}(p):=\text{Dec}^{n_0}(p,\varnothing)$$

where $\varnothing$ is the empty sequence of paths, and n_0 is the number of loops in p.

Lemma B.5 *For all $p\in\mathbb{P}_{ij}$, $\mathcal{D}\text{ec}(p)\in\mathbb{V}_{ij}\times\mathbb{SL}^{(\mathbb{N})}$, and*

$$W_{\mathcal{D}\text{ec}(p)}=W_p. \tag{B.4}$$

Furthermore, the map $\mathcal{D}\text{ec}:\mathbb{P}\to\mathbb{V}\times\mathbb{SL}^{(\mathbb{N})}$ is injective.

Proof. Write $\text{Dec}^{n_0}(p,\varnothing)=(v,(L_1,\dots,L_{n_0}))$, and observe that $v=E^{n_0}(p)$. If v were self-intersecting, then it would follow that $E^{n_0+1}(p)\neq E^{n_0}(p)$, contradicting Corollary B.2. Thus, $v\in\mathbb{V}$. If $p\in\mathbb{P}_{ij}$ then either $i\neq j$, in which case it follows by B.1 that $v\in\mathbb{V}_{ij}$, or $i=j$ in which case $v=\mathbf{0}$ (otherwise we would have an non-intersecting path in $\mathbb{P}_{ii}$, which is impossible). On the other hand, one can check easily by induction that

$$L_1=L(p),\quad L_2=L(E(p)),\quad \dots\quad L_{n_0}=L(E^{n_0-1}(p)),$$

and thus, by Lemma B.1, for $1\le i\le n_0$, L_i is either $\mathbf{0}$ or a simple loop. But L_i cannot be $\mathbf{0}$, since this would imply that $E^{i-1}(p)=E^i(p)$, contradicting again Corollary B.2. Thus $L_1,\dots,L_{n_0}\in\mathbb{SL}$.

The relation (B.4) follows immediately from (B.3).

Finally, suppose that $\mathcal{D}\text{ec}(p)=\mathcal{D}\text{ec}(p')$. Then p and p' have the same number of loops n_0 (otherwise, the list of loops in their loop-decomposition could not be the same) and thus $\mathcal{D}\text{ec}(p)=\text{Dec}^{n_0}(p,\varnothing)$, $\mathcal{D}\text{ec}(p')=\text{Dec}^{n_0}(p',\varnothing)$ and therefore

$$\text{Dec}^{n_0}(p,\varnothing)=\text{Dec}^{n_0}(p',\varnothing)$$

But since Dec is injective (by Corollary B.3), Dec^{n_0} is injective, and thus it must be that $p=p'$. ∎

C Proofs of the local bounds on the solution operator (Theorems 4.1 and 4.2)

In this section we prove Theorems 4.1 and 4.2. In fact, we prove a stronger analogue of Theorem 4.1 phrased using semiclassical pseudodifferential operators – i.e., pseudodifferential operators in a calculus where each derivative is weighted by k^{-1}. Furthermore, because we work on a bounded domain, we need a special class of pseudodifferential operators adapted to the boundary.

C.1 Pseudodifferential operators and b-pseudodifferential operators

C.1.1 Semiclassical pseudodifferential operators.

Semiclassical pseudodifferential operators are generalisations of Fourier multipliers acting as

$$\text{Op}(a)u(x):=\frac{1}{(2\pi h)^d}\int e^{\frac{i}{h}\langle x-y,\xi\rangle}a(x,\xi)u(y)dyd\xi,$$

where, for some $m \in \mathbb{R}$, a satisfies

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m-|\alpha|}.$$

In this case we write $a \in S^m(T^*\mathbb{R}^d)$ and $\mathrm{Op}(a) \in \Psi^m(\mathbb{R}^d)$. When $m = 0$ we write $S(T^*\mathbb{R}^d)$ and $\Psi(\mathbb{R}^d)$ respectively. This class of pseudodifferential operators is the natural class of operators generalising quantization of $b(x)(hD)^\alpha$ for some $b \in C^\infty(\mathbb{R}^d)$ and $\alpha \in \mathbb{N}^d$. For more details and information about the calculus of such operators see e.g. [DZ19, Appendix E] and [Zwo12].

The class of b-pseudodifferential operators that we work with is, instead, the natural class of operators quantizing differential operators that are tangential to the boundary of Ω_+. Away from $\partial\Omega_+$ they are pseudodifferential operators in the sense above, but near $\partial\Omega_+$ they have a different form. In particular, in coordinates (x_1, x') with $\partial\Omega_+ = \{x_1 = 0\}$, their symbols are functions on the b-cotangent bundle, ${}^bT^*\Omega_+$, whose sections are of the form

$$\sigma \frac{dx_1}{x_1} + \xi' dx'.$$

Notice that ${}^bT^*\Omega_+$ is the dual to sections of $T^*\Omega_+$ that are tangent to $\partial\Omega_+$. We also write ${}^b\overline{T^*\Omega_+}$ for the fiber radially compactified b-contangent bundle; i.e., ${}^bT^*\Omega_+$ with the sphere at infinity in (σ, ξ') attached.

In coordinates, b-pseudodifferential operators are of the form

$$\mathrm{Op}_b(a)(u)(x) = \frac{1}{(2\pi h)^d} \int e^{\frac{i}{h}((x_1-y_1)\xi_1 + (x'-y'),\xi')} \phi(x_1/y_1) a(x_1, x', x_1\xi_1, \xi') u(y) dy d\xi,$$

where $\phi \in C_c^\infty(1/2, 2)$ with $\phi \equiv 1$ near 1 and for some m

$$|D_x^\alpha D_\sigma^j D_{\xi'}^\beta a(x_1, x', \sigma, \xi')| \leq C_{j\alpha\beta} \langle (\sigma, \xi') \rangle^{m-j-|\beta|}.$$

In this case, we write $\mathrm{Op}_b(a) \in \Psi_b^m(\Omega_+)$ and $a \in S^m({}^bT^*\Omega_+)$. When $m = 0$ we write $S({}^bT^*\Omega_+)$ and $\Psi_b(\Omega_+)$ respectively. We also write $\Psi_b^{-\infty} = \cap_m \Psi_b^m$.

The class comes equipped with principal symbol map ${}^b\sigma : {}^b\Psi^m(\Omega_+) \to S^m({}^bT^*\Omega_+)/h^b S^{m-1}(T^*\Omega_+)$ such that if $A \in \Psi_b(\Omega_+)$ and $\sigma(A) = 0$ then $A \in h\Psi_b^{m-1}(\Omega_+)$. We now introduce two important sets for b-pseudodifferential operators. For $A \in \Psi_b^m(\Omega_+)$ and $q \in {}^b\overline{T^*\Omega_+}$, we say $q \in {}^b\,\mathrm{Ell}(A)$ if there is a neighbourhood, U of q such that

$$|\sigma(A)(q')| \langle (\sigma, \xi') \rangle^{-m} > c > 0, \qquad q' \in U \cap {}^bT^*\Omega_+.$$

Next, we say $q \notin {}^b\,\mathrm{WF}(A)$ if there is $E \in {}^b\Psi(\Omega_+)$ with $q \in {}^b\,\mathrm{Ell}(E)$ such that

$$EA \in h^\infty \Psi_b^{-\infty}.$$

For a more complete treatment of these operators, we refer the reader to [HV18, Appendix A] and the references therein.

C.1.2 The generalised bicharacteristic flow on ${}^bT^*\Omega_+$.

Let $p_\theta \in S^2(T^*\Omega)$ denote the semiclassical principal symbol of P_k and observe that on $B(0, R_{\mathrm{scat}})$, $p_\theta = \sum_{ij} g^{ij}(x)\xi_i\xi_j - 1$, where $g^{-1}(x) = A(x)/n(x)$. We then let $\varphi_t : \pi^b\{\Re p_\theta = 0\} \to \pi^b\{\Re p_\theta = 0\}$ be the generalised bicharacteristic flow for $\Re p_\theta$ in the sense of [Vas08, Definition 1.1].

We are now in a position to define the *forward and backward trapped sets* Γ_- and Γ_+, respectively,

$$\Gamma_\pm := \Big\{ q \in \pi^b(\{\Re p_\theta = 0\}) : \sup\{t > 0 : \varphi_{\mp t}(q) \in {}^bT^*\Omega\} = \infty \Big\},$$

as well as the trapped set,

$$K := \Gamma_+ \cap \Gamma_-.$$

One can show that $\Gamma_\pm$ and hence K are closed (see e.g. [DZ19, Proposition 6.3]).

Remark C.1 *For this flow to exist, we assume that $\partial\Omega$ has no infinite order tangency with the Hamiltonian bicharacteristics of $\Re p_\theta$. By [Hör85, Theorem 24.3.9] this suffices for the flow to be unique and by [Hör85, Example 24.3.11] uniqueness may fail in the opposite case.*

We highlight that the definitions of K and $\Gamma_\pm$, and the statements of Theorems C.3 and C.4 below can all be rewritten without the need for uniqueness of the flow (with the results of the theorems still true); however, for simplicity we do not do this.

C.2 Improved resolvent estimates

We can now state our improved estimates on the solution operator. Below, we use the notation ${}^b\Psi(\Omega)$ for elements of ${}^b\Psi(\Omega_+)$ whose kernels are supported away from $\Gamma_{\rm tr}$.

Theorem C.2 *Let $k_0 > 0$, and let $\mathcal{J}$ be such that Assumption 1.2 holds. Then for all $A \in {}^b\Psi(\Omega)$ with ${}^b\,\mathrm{WF}(A) \cap K = \emptyset$, there exists $C > 0$ such that, for all $k \in (k_0, \infty) \setminus \mathcal{J}$,*

$$\|AR_k\|_{L^2\to L^2} + \|R_kA\|_{L^2\to L^2} \le C\sqrt{\|R_k\|k}, \qquad \|AR_kA\|_{L^2\to L^2} \le Ck.$$

Theorem C.3 *Let $k_0 > 0$ and let $\mathcal{J}$ be such that Assumption 1.2 holds. Then for all $A, B \in {}^b\Psi(\Omega)$ with*

$$\overline{{}^b\,\mathrm{WF}(A) \cup \bigcup_{t\ge 0} \varphi_{-t}({}^b\,\mathrm{WF}(A) \cap \pi^b(\{\Re p_\theta = 0\}))} \cap {}^b\,\mathrm{WF}(B) = \emptyset, \qquad {}^b\,\mathrm{WF}(A) \cap \Gamma_+ = \emptyset$$

and all $N > 0$ there exists $C > 0$ such that, for all $k \in (k_0, \infty) \setminus \mathcal{J}$,

$$\|AR_kB\|_{L^2\to L^2} \le Ck^{-N}.$$

Theorem C.4 *Let $k_0 > 0$ and let $\mathcal{J}$ be such that Assumption 1.2 holds. Then for all $A, B \in {}^b\Psi(\Omega)$ with*

$$\overline{{}^b\,\mathrm{WF}(A) \cup \bigcup_{t\ge 0} \varphi_{t}({}^b\,\mathrm{WF}(A) \cap \pi^b(\{\Re p_\theta = 0\}))} \cap {}^b\,\mathrm{WF}(B) = \emptyset, \qquad {}^b\,\mathrm{WF}(A) \cap \Gamma_- = \emptyset$$

and all $N > 0$ there exists $C > 0$ such that, for all $k \in (k_0, \infty) \setminus \mathcal{J}$,

$$\|AR_k^*B\|_{L^2\to L^2} \le Ck^{-N}.$$

Proof of Theorem 4.1 using Theorems C.2-C.3. Part (i) of Theorem 4.1 follows immediately from Theorem C.2. Part (ii) of Theorem 4.1 follows from Theorems C.3 and C.3 by choosing $B = \psi$ (i.e., the cutoff in $\mathcal{K}$) and then noting that the choice $A = \chi$ (i.e., the cutoff in $\mathcal{I}$) satisfies the assumptions in Theorems C.3 and C.3. ■

C.3 Estimates away from the scatterer

We start by proving an estimate 'deep' in the PML region; i.e. near the truncation boundary.

Lemma C.5 *There exists $U \subset \Omega$ such that $\overline{U}$ is a neighbourhood of $\Gamma_{\rm tr}$ and for all $k_0 > 0$, $\psi, \widetilde{\psi} \in C^\infty(\Omega)$ with $\operatorname{supp}\psi, \operatorname{supp}\widetilde{\psi} \subset U$, $\psi \prec \widetilde{\psi}$, and $\operatorname{supp}(1-\psi) \cap \partial\Omega = \emptyset$, there exists $C > 0$ such that for $k > k_0$,*

$$\|\psi u\|_{H^1_k} \le C\Big(\|\widetilde{\psi}P_ku\|_{L^2} + Ck^{-N}\|u\|_{H_k^{-N}}\Big). \tag{C.1}$$

Proof. By [GLS23, Lemma 4.4], there is U with $\overline{U}$ a neighbourhood of $\partial\Omega_{\rm tr}$ such that for $v \in H^1_k(\Omega)$ with $v|_{\Gamma_{\rm tr}} = 0$ and $\operatorname{supp} v \subset U$,

$$\|v\|_{H^1_k} \le C\|P_kv\|_{L^2}. \tag{C.2}$$

Let $\psi_j \in C^\infty_c(U)$ be such that $\psi_1 \prec \psi_2 \prec \widetilde{\psi}$ and $\operatorname{supp}(1-\psi_1) \cap \operatorname{supp}\partial\psi = \emptyset$. Applying (C.2) with $v = \psi u$, we obtain

$$\|\psi u\|_{H^1_k} \le C\|P_k\psi u\|_{L^2} \le C\big(\|\psi P_ku\|_{L^2} + \big\|[P_k,\psi]u\big\|_{L^2}\big) \le C\big(\|\psi P_ku\|_{L^2} + Ck^{-1}\|\psi_1u\|_{H^1_k}\big) \tag{C.3}$$

Now, shrinking U if necessary so that

$$\{x \in U \ : \ \text{there exists } \xi \text{ such that } p_\theta(x,\xi) = 0\} = \emptyset,$$

the elliptic parametrix construction [DZ19, Proposition E.32] implies that

$$\|\psi_1u\|_{H^1_k} \le C\|\psi_2P_ku\|_{L^2} + Ck^{-N}\|u\|_{H_k^{-N}}, \tag{C.4}$$

and the result follows by combining (C.3) and (C.4). ■

We now prove Theorem 4.2.

Proof of Theorem 4.2. The bound $\|\chi R_k\|_{L^2\to L^2} \leq C$ follows immediately from (C.1), with the bound $\|R_k\chi\|_{L^2\to L^2} \leq C$ then following by applying the previous bound with P_k replaced by P_k^*.

The bound $\|\chi R_k\psi\|_{L^2\to H_k^1} \leq Ck^{-N}$ also follows immediately from (C.1), with then the bound $\|\chi R_k\psi\|_{L^2\to H_k^N} \leq Ck^{-N}$ following by elliptic regularity up to the boundary.

Finally, the bound $\|\psi R_k\chi\|_{L^2\to L^2} \leq Ck^{-N}$ follows by applying the bound $\|\chi R_k\psi\|_{L^2\to L^2} \leq Ck^{-N}$ with P_k replaced by P_k^*, and then the bound $\|\psi R_k\chi\|_{L^2\to H_k^N} \leq Ck^{-N}$ follows by elliptic regularity up to the boundary. ■

Next, we prove an estimate near incoming points away from the truncation boundary.

Lemma C.6 *Let $m \geq 2$, $\chi \in C_c^\infty(\Omega \setminus \overline{B(0, R_{\rm scat})})$ (where $R_{\rm scat}$ is defined by (A.1)). Then there is $\epsilon > 0$ such that for $A, B \in \Psi^0$ with*

$$\mathrm{WF}(A) \cap \Big\{\langle \tfrac{x}{|x|}, \xi\rangle \geq \epsilon,\, p_\theta(x,\xi) = 0\Big\} = \emptyset,$$

$$\mathrm{WF}(A) \cup \bigcap_{t\geq 0} \{(x - t\xi, \xi) \,:\, (x,\xi) \in \mathrm{WF}(A) \cap \{p_\theta = 0\}\} \cap \{p_\theta = 0\} \subset \mathrm{Ell}(B)$$

and $N > 0$, given $k_0 > 0$ there exists $C > 0$ such that for all $k \geq k_0$

$$\|A\chi u\|_{H_k^m} \leq Ck\|BP_k u\|_{H_k^{m-2}} + Ck^{-N}\|u\|_{H_k^{-N}}.$$

Proof. Since

$$\mathrm{WF}(A) \cap \Big\{\langle \tfrac{x}{|x|}, \xi\rangle \geq \epsilon,\, p_\theta(x,\xi) = 0\Big\},$$

there is a neighbourhood, V of $\{p_\theta = 0\}$ such that

$$\mathrm{WF}(A\chi) \cap V \subset \{\langle \tfrac{x}{|x|}, \xi\rangle < 2\epsilon\}.$$

In particular, for $(x,\xi) \in \mathrm{WF}(A\chi) \cap V$, and $t \geq 0$,

$$|x - t\xi|^2 = |x|^2 - 2t|x|\langle \tfrac{x}{|x|}, \xi\rangle + t|\xi|^2 \geq |x|^2 - 4t|x|\epsilon + t^2 \geq |x|^2(1 - 2\epsilon) + t^2(1 - 2\epsilon).$$

Therefore, there is $T > 0$ such that for all $(x,\xi) \in \mathrm{WF}(A\chi) \cap V$, there is $0 \leq t \leq T$ such that $(x - t\xi, \xi) \notin \{p_\theta = 0\}$. Using a microlocal partition of unity on $\mathrm{WF}(A\chi) \cap V$, $\{X_j\}_{j=1}^N$, there are $0 < T_j \leq T$ and $E_j \in \Psi^{\rm comp}$ with $\mathrm{WF}(E_j) \subset \{p_\theta \neq 0\} \cap \Omega_{\rm tr} \cap \mathrm{Ell}(B)$ such that

$$\{(x - t\xi, \xi) \,:\, (x,\xi) \in \mathrm{WF}(AX_j\chi) \cap V,\, 0 \leq t \leq T_j\} \subset \mathrm{Ell}(B)$$

and

$$\{(x - T_j\xi, \xi) \,:\, (x,\xi) \in \mathrm{WF}(AX_j\chi) \cap V\} \subset \mathrm{Ell}(E_j).$$

Now, let $X \in \Psi^{\rm comp}$ with $\mathrm{WF}(X) \subset V$ and $\mathrm{WF}(I - X) \cap \{p_\theta = 0\} = \emptyset$. Then, by the elliptic parametrix construction [DZ19, Proposition E.32]

$$\|(I - X)A\chi u\|_{H_k^m} \leq C\|BP_k u\|_{H_k^{m-2}} + Ck^{-N}\|u\|_{H_k^{-N}}. \tag{C.5}$$

On the other hand, using that $X \in \Psi^{\rm comp}$ and then that $\Im p_\theta \leq 0$ near $p_\theta = 0$, by [DZ19, Theorem E.47] we have

$$\|XAX_j\chi u\|_{H_k^m} \leq C\|XAX_j\chi u\|_{H_k^{-N}} + Ck^{-N}\|u\|_{H_k^{-N}} \leq Ck\|BP_k u\|_{L^2} + \|E_j u\|_{L^2} + Ck^{-N}\|u\|_{H_k^{-N}}. \tag{C.6}$$

Finally, since $\mathrm{WF}(E_j) \subset \{p_\theta \neq 0\} \cap \mathrm{Ell}(B)$, we have

$$\|E_j u\|_{L^2} \leq C\|BP_k u\|_{L^2} + Ck^{-N}\|u\|_{H_k^{-N}}. \tag{C.7}$$

Combining (C.5), (C.6), and (C.7), and summing in j,

$$\|A\chi u\|_{H_k^m} \leq Ck\|BP_k u\|_{H_k^{m-2}} + Ck^{-N}\|u\|_{H_k^{-N}}.$$

■

We now prove the key propagation lemma that allows us to improve resolvent estimates away from trapping. In particular, we estimate u in an annulus away from Ω_- but inside $B(0, R_{\mathrm{PML}_-})$

Lemma C.7 *Let $R_{\mathrm{PML}_+} > R_{\mathrm{PML}_-}$ with $B(0, R_{\mathrm{PML}_+}) \Subset \Omega_{\mathrm{tr}}$, $a \in C_c^\infty((R_{\mathrm{scat}}, R_{\mathrm{PML}_-}))$ and $b \in C_c^\infty(R_{\mathrm{scat}}, R_{\mathrm{PML}_+})$ with*

$$b \equiv 1 \text{ on } \{x \in (R_{\mathrm{scat}}, R_{\mathrm{PML}_-}) \,:\, x \geq \inf \operatorname{supp} a\}.$$

and define $A = a(|x|)$, $B = B(|x|)$. Then, for $X \in {}^b\Psi^0$ with ${}^b\operatorname{WF}(I - X) \cap {}^b\operatorname{WF}(P_k u) = \emptyset$, given $k_0 > 0$ there exists $C > 0$ such that for all $k \geq k_0$

$$\|Au\|_{L^2}^2 \leq Ck\|P_k u\|_{L^2}\|Xu\|_{L^2} + Ck^2\|BP_k u\|_{L^2}^2 + C_N k^{-N}\|u\|_{L^2}^2.$$

Proof. Let $a, b_1 \in C_c^\infty((R_{\mathrm{scat}}, R_{\mathrm{PML}_-}))$ with $a \prec b_1$, $\operatorname{supp} b_1 \cap \{b < \frac{1}{2}\} = \emptyset$. Let $b_2 \in C_c^\infty(R_{\mathrm{scat}}, R_{\mathrm{PML}_+})$ with $\operatorname{supp} b_2 \cap \{b < \frac{1}{2}\} = \emptyset$ and

$$b_2 \equiv 1 \text{ on } \big\{x \in (R_{\mathrm{scat}}, R_{\mathrm{PML}_-}) \,:\, x \geq \inf \operatorname{supp} b_1\big\}.$$

Let $A = a(|x|)$ and $B_j = b_j(|x|)$, $j = 1, 2$. We claim that

$$c\|Au\|_{H_k^s}^2 \leq Ck\|P_k u\|_{L^2}\|Xu\|_{L^2} + k^2\|B_2 P_k u\|_{L^2}^2 + Ck^{-1}\|B_1 u\|_{L^2}^2 + Ck^{-N}\|u\|_{L^2}^2. \tag{C.8}$$

To establish (C.8), first let $g \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} g \subset [0, R_{\mathrm{PML}_-})$, $g \geq 0$, $g' \leq 0$, $\operatorname{supp} g' \subset (R_{\mathrm{scat}}, R_{\mathrm{PML}_-})$ $g' \leq -1$ on $\operatorname{supp} a$, and $\operatorname{supp}(1 - b_1) \cap \operatorname{supp} g' = \emptyset$. Next, let $E \in \Psi^0$ with $0 \leq \sigma(E) \leq 1$, and

$$\operatorname{WF}(E) \subset \Big\{(x, \xi) \,:\, \langle \tfrac{x}{|x|}, \xi\rangle < 2\epsilon\langle\xi\rangle\Big\}, \qquad \operatorname{WF}(I - E) \cap \Big\{(x, \xi) \,:\, \langle \tfrac{x}{|x|}, \xi\rangle < \epsilon\langle\xi\rangle\Big\} = \emptyset.$$

Finally, let $b_0 \in C_c^\infty((R_{\mathrm{scat}}, R_{\mathrm{PML}_-})$ with $g' \prec b_0 \prec b_1$.

Put $G = g(|x|)$, $B_0 = b_0(|x|)$ and consider

$$k\Im\big\langle P_k u, G^2 u\big\rangle = \frac{k}{2i}\big\langle [P_k, G^2]u, u\big\rangle = \frac{k}{2i}\big\langle B_0[P_k, G^2]B_0 B_1 u, B_1 u\big\rangle.$$

Now, define $Z := \frac{k}{2i}B_0[P_k, G^2]B_0 \in \Psi^1$ and observe that

$$\begin{aligned}
\sigma(Z) = b_0^2 g(|x|)\langle \xi, \partial_x g(|x|)\rangle &= b_0^2 g(|x|)\langle \xi, \tfrac{x}{|x|}\rangle g'(|x|) \\
&= b_0^2\Big(g(|x|)\langle \xi, \tfrac{x}{|x|}\rangle g'(|x|)(1 - \sigma(E^2)) + g(|x|)\langle \xi, \tfrac{x}{|x|}\rangle g'(|x|)\sigma(E^2)\Big) \\
&\leq b_0^2\Big(- c\epsilon a^2\langle\xi\rangle(1 - \sigma(E^2)) + g(|x|)\langle \xi, \tfrac{x}{|x|}\rangle g'(|x|)\sigma(E^2)\Big) \\
&\leq b_0^2\Big(- c\epsilon a^2\langle\xi\rangle + C\sigma(E^2)|\xi|\Big) \\
&\leq b_0^2\Big(- c\epsilon a^2\langle\xi\rangle + C\sigma(E^2)\langle\xi\rangle + Cp_\theta^2\Big)
\end{aligned}$$

Therefore, by the microlocal Gårding inequality [DZ19, Proposition E.34],

$$\frac{k}{2i}\langle [P_k, G^2]B_1 u, B_1 u\rangle \leq -c_\epsilon\|B_0 A B_1 u\|_{H_k^1}^2 + C\|B_0 E B_1 u\|_{H_k^1}^2 + C\|B_0 P_k B_1 u\|_{L^2}^2 + Ck^{-1}\|B_1 u\|_{L^2}^2$$

Thus, since $a \prec b_0 \prec b_1$ and ${}^b\operatorname{WF}(I - X) \cap {}^b\operatorname{WF}(P_k u) = \emptyset$,

$$c_\epsilon\|Au\|_{L^2}^2 \leq Ck\|P_k u\|_{L^2}\|GXu\|_{L^2} + C\|EB_1 u\|_{H_k^1}^2 + C\|B_1 P_k u\|_{L^2}^2 + Ck^{-1}\|B_1 u\|_{L^2}^2 + Ck^{-N}\|u\|_{H_k^{-N}}^2$$

Then, by Lemma C.6,

$$c_\epsilon \|Au\|_{L^2}^2 \le Ck\|P_k u\|_{L^2}\|Xu\|_{L^2} + Ck^2\|B_2P_ku\|_{L^2}^2 + Ck^{-1}\|B_1u\|_{L^2}^2 + Ck^{-N}\|u\|_{H_k^{-N}}^2$$

as claimed in (C.8).

Now, suppose by induction that for $a, b_1 \in C_c^\infty((R_{\rm scat}, R_{\rm PML_-}))$ with $a \prec b_1$ and $b_2 \in C_c^\infty(R_{\rm scat}, R_{\rm PML_+})$ with $\operatorname{supp} b_2 \cap \{b < \frac{1}{2}\} = \emptyset$ and

$$b_2 \equiv 1 \text{ on } \Big\{x \in (R_{\rm scat}, R_{\rm PML_-}) \,:\, x \ge \inf \operatorname{supp} b_1\Big\},$$

we have, with $A = a(|x|)$ and $B_1 = b_1(|x|)$, $B_2 = b_2(|x|)$,

$$c\|Au\|_{L^2}^2 \le Ck\|P_ku\|_{L^2}\|Xu\|_{L^2} + Ck^2\|B_2P_ku\|_{L^2}^2 + Ck^{-L}\|B_1u\|_{L^2}^2 + Ck^{-N}\|u\|_{H_k^{-N}}^2. \tag{C.9}$$

Now, fix $a, b_1 \in C_c^\infty((R_{\rm scat}, R_{\rm PML_-}))$ with $a \prec b_1$, and $b_2 \in C_c^\infty(R_{\rm scat}, R_{\rm PML_+})$ with $\operatorname{supp} b_2 \cap \{b < \frac{1}{2}\} = \emptyset$ and

$$b_2 \equiv 1 \text{ on } \big\{x \in (R_{\rm scat}, R_{\rm PML_-}) \,:\, x \ge \inf \operatorname{supp} b_1\big\}.$$

Then, by (C.9), letting $\widetilde{b}_1 \in C_c^\infty((R_{\rm scat}, R_{\rm PML_-}))$ with $a \prec \widetilde{b}_1 \prec b_1$ and $\widetilde{b}_2 \in C_c^\infty(R_{\rm scat}, R_{\rm PML_+})$ with $\operatorname{supp} \widetilde{b}_2 \cap \{b_2 < \frac{1}{2}\} = \emptyset$ and

$$\widetilde{b}_2 \equiv 1 \text{ on } \{x \in (R_{\rm scat}, R_{\rm PML_-}) \,:\, x \ge \inf \operatorname{supp} \widetilde{b}_1\},$$

by (C.9) with $A = a(|x|)$ and $\widetilde{B}_1 = b_1(|x|)$, $\widetilde{B}_2 = b_2(|x|)$,

$$c\|Au\|_{L^2}^2 \le Ck\|P_ku\|_{L^2}\|Xu\|_{L^2} + Ck^2\|\widetilde{B}_2P_ku\|_{L^2}^2 + Ck^{-L}\|\widetilde{B}_1u\|_{L^2}^2 + Ck^{-N}\|u\|_{H_k^{-N}}^2.$$

Now, by (C.8) with A replaced by $\widetilde{B}_1$,

$$c\|\widetilde{B}_1u\|_{L^2}^2 \le Ck\|P_ku\|_{L^2}\|Xu\|_{L^2} + Ck^{-1}\|B_1u\|_{L^2}^2 + Ck^{-N}\|u\|_{H_k^{-N}}^2.$$

Hence,

$$\begin{aligned} c\|Au\|_{L^2}^2 &\le Ck\|P_ku\|_{L^2}\|Xu\|_{L^2} + Ck^2\|\widetilde{B}_2P_ku\|_{L^2}^2 + Ck^{-L-1}\|B_1u\|_{L^2}^2 + Ck^{-N}\|u\|_{H_k^{-N}}^2 \\ &\le Ck\|P_ku\|_{L^2}\|Xu\|_{L^2} + Ck^2\|B_2P_ku\|_{L^2}^2 + Ck^{-L-1}\|B_1u\|_{L^2}^2 + Ck^{-N}\|u\|_{H_k^{-N}}^2; \end{aligned}$$

we have therefore obtained (C.9) with L replaced by $L+1$, and the result then follows by induction. ■

C.4 Estimates near the scatterer and away from trapping

Before proceeding, we record the following consequences of [Vas08, Proposition 4.6, Theorems 8.1 and 8.5].

Theorem C.8 *Let $A, E \in {}^b\Psi(\Omega)$ with ${}^b\operatorname{WF}(A) \cup {}^b\operatorname{WF}(E) \subset \Omega_{\rm tr}$ with ${}^b\operatorname{WF}(A) \subset {}^b\operatorname{Ell}(E)$ and ${}^b\operatorname{WF}(A) \cap {}^b\pi(\{p_\theta = 0\}) = \emptyset$. Then, for all $k_0 > 0$ there exists $C > 0$ such that*

$$\|Au\|_{L^2} \le C\|EP_ku\|_{L^2} + C_Nk^{-N}\|u\|_{L^2}$$

Proof. The estimate follows from [Vas08, Proposition 4.6] when ${}^b\operatorname{WF}(A) \subset B(0, R_{\rm scat})$ and from the standard elliptic parametrix construction [DZ19, Proposition E.32], when ${}^b\operatorname{WF}(A) \cap T^*\partial\Omega_- = \emptyset$. ■

Theorem C.9 *Let $A, B, E \in {}^b\Psi(\Omega)$ such that*

$${}^b\operatorname{WF}(A) \cup \bigcup_{t=0}^{T} \varphi_{-t}({}^b\operatorname{WF}(A) \cap \pi^b(\{\Re p_\theta = 0\}) \subset {}^b\operatorname{Ell}(E),$$

$$\textit{and for all } q \in {}^b\mathrm{WF}(A) \cap \pi^b(\{\Re p_\theta = 0\}), \qquad \bigcup_{t=0}^{T} \varphi_{-t}(q) \cap {}^b\mathrm{Ell}(B) \neq \emptyset.$$

Then, for all $k_0 > 0$ *there exists* $C > 0$ *such that*

$$\|Au\|_{L^2} \leq Ck\|EP_k u\|_{L^2} + \|Bu\|_{L^2} + C_N k^{-N}\|u\|_{L^2}$$

Proof. The estimates follow from the combination of the propagation results in [Vas08, Theorem 8.1] (for Dirichlet boundary conditions on $\partial\Omega_-$) and [Vas08, Theorem 8.5] (for Neumann boundary conditions on $\partial\Omega_-$) applied near the $\partial\Omega$ and [DZ19, Theorem E.47] applied away from $\partial\Omega_-$. ■

Our next lemma shows that, to measure u away from trapping, we need only have an estimate for u in an annulus.

Lemma C.10 *Suppose that* $A \in {}^b\Psi(\Omega)$ *and* ${}^b\mathrm{WF}(A) \cap K = \emptyset$. *Then, for any* $R_{\rm scat} < R_1 < R_{\rm PML_-}$ *and* $B \in C_c^\infty(\Omega)$ *with* $\mathrm{supp}(1-B) \cap \{|x| = R_1\} = \emptyset$, *given* $k_0 > 0$ *there exists* $C > 0$ *such that for all* $k \geq k_0$

$$\|Au\|_{L^2} \leq Ck\|P_k u\|_{L^2} + \|Bu\|_{L^2} + C_N k^{-N}\|u\|_{L^2}^2. \tag{C.10}$$

Proof. First, by Lemma C.5 we may assume that

$${}^b\mathrm{WF}(A) \subset \Omega_{\rm tr}.$$

Next observe that if ${}^b\mathrm{WF}(A) \cap {}^b\pi(\{p_\theta = 0\}) = \emptyset$, then by the ellipticity results in Theorem C.8

$$\|Au\|_{L^2} \leq C\|P_k u\|_{L^2} + C_N k^{-N}\|u\|_{L^2}.$$

Therefore, we may assume that ${}^b\mathrm{WF}(A)$ is contained in a small neighbourhood of ${}^b\pi(\{p_\theta = 0\})$.

If ${}^b\mathrm{WF}(A) \subset \{B \equiv 1\}$ then, by the elliptic parametrix [DZ19, Proposition E.32],

$$\|Au\|_{L^2} \leq C\|Bu\|_{L^2} + Ck^{-N}\|u\|_{L^2}.$$

Therefore, using a partition of unity, we need only consider two cases: ${}^b\mathrm{WF}(A) \subset \{|x| > R_1\}$ and ${}^b\mathrm{WF}(A) \subset \{|x| < R_1\}$.

First, suppose that ${}^b\mathrm{WF}(A) \subset \{|x| > R_1\}$. Let U be as in Lemma C.5. Then, there exists $T > 0$ such that for all $(x, \xi) \in {}^b\mathrm{WF}(A)$, there is $t \in [0, T]$ such that

$$\varphi_{-t}(x, \xi) \in \{B \equiv 1\} \cup \{(x, \xi) \,:\, x \in U\}$$

(i.e., flowing backwards, one either hits $B \equiv 1$ or the PML). In particular, by the propagation results [DZ19, Theorem E.47] there is $\psi \in C^\infty(U)$ with $\psi \equiv 1$ near $\Gamma_{\rm tr}$ such that

$$\|Au\|_{L^2} \leq Ck\|P_k u\|_{L^2} + \|Bu\|_{L^2} + \|\psi u\|_{L^2} + Ck^{-N}\|u\|_{L^2}.$$

By Lemma C.5, we then obtain

$$\|Au\|_{L^2} \leq Ck\|P_k u\|_{L^2} + \|Bu\|_{L^2} + Ck^{-N}\|u\|_{L^2}$$

as required.

Next, suppose ${}^b\mathrm{WF}(A) \subset \{|x| < R_1\}$. Then, since ${}^b\mathrm{WF}(A) \cap K = \emptyset$, applying a partition of unity again, we may assume there exists $T > 0$ such that for all $(x, \xi) \in {}^b\mathrm{WF}(A)$, either there is $t \in [0, T]$ such that

$$\varphi_t(x, \xi) \in \Big\{(x, \xi) \,:\, B(x) > \tfrac{1}{2}\Big\}, \qquad \bigcup_{s \in [0,t]} \varphi_s(x, \xi) \in B(0, R_{\rm PML_-}),$$

(informally, one flows forwards from A, staying away from the PML region, and reaches where $B > 1/2$ at time t) or there is $t \in [0, T]$ such that

$$\varphi_{-t}(x, \xi) \in \Big\{(x, \xi) \,:\, B(x) > \tfrac{1}{2}\Big\}, \qquad \bigcup_{s \in [-t,0]} \varphi_s(x, \xi) \in B(0, R_{\rm PML_-}).$$

(informally, one flows backwards from A, staying away from the PML region, and reaches where $B > 1/2$ at time t). The result (C.10) then follows by the propagation results of Theorem C.9. ■

Finally, we combine the above lemmas to show that we may estimate u away from trapping by u near the wavefront set of $P_k u$. In particular, this will improve the resolvent estimate when the measurement is away from trapping.

Lemma C.11 *Suppose that $A \in {}^b\Psi(\Omega)$ and ${}^b\mathrm{WF}(A) \cap K = \emptyset$. Then, for any $X \in {}^b\Psi^0$ with ${}^b\mathrm{WF}(I - X) \cap {}^b\mathrm{WF}(P_k u) = \emptyset$, given $k_0 > 0$ there exists $C > 0$ such that for all $k \geq k_0$*

$$\|Au\|_{L^2}^2 \leq Ck\|P_k u\|_{L^2}\|Xu\|_{L^2} + Ck^2\|P_k u\|_{L^2}^2 + C_N k^{-N}\|u\|_{L^2}^2.$$

Proof. Let $R_{\mathrm{scat}} < R_1 < R_{\mathrm{PML}_-}$ and $b \in C_c^\infty(R_{\mathrm{scat}}, R_{\mathrm{PML}_-})$ with $\mathrm{supp}(1 - b) \cap \{|x| = R_1\} = \emptyset$. Then, by Lemma C.10

$$\|Au\|_{L^2}^2 \leq Ck^2\|P_k u\|_{L^2}^2 + C\|Bu\|_{L^2}^2 + C_N k^{-N}\|u\|_{L^2}^2,$$

and, by Lemma C.7,

$$\|Bu\|_{L^2}^2 \leq Ck\|P_k u\|_{L^2}\|Xu\|_{L^2} + Ck^2\|P_k u\|_{L^2}^2 + C_N k^{-N}\|u\|_{L^2}^2,$$

which completes the proof. ■

When the both the data and measurement are away from trapping, we can use the previous lemma to improve our estimates further– all the way to a non-trapping type bound.

Lemma C.12 *Suppose that ${}^b\mathrm{WF}(P_k u) \cap K = \emptyset$, then for any $A \in {}^b\Psi(\Omega)$ with ${}^b\mathrm{WF}(A) \cap K = \emptyset$ given $k_0 > 0$ there exists $C > 0$ such that for all $k \geq k_0$*

$$\|Au\|_{L^2} \leq Ck\|P_k u\|_{L^2} + Ck^{-N}\|u\|_{L^2}.$$

Proof. Let $\tilde{A}, X \in {}^b\Psi(\Omega)$ with ${}^b\mathrm{WF}(I - X) \cap {}^b\mathrm{WF}(P_k u) = \emptyset$, ${}^b\mathrm{WF}(\tilde{A}) \cap K = \emptyset$, and ${}^b\mathrm{WF}(I - \tilde{A}) \cap ({}^b\mathrm{WF}(A) \cup {}^b\mathrm{WF}(X)) = \emptyset$. By Lemma C.11,

$$\|\tilde{A}u\|_{L^2}^2 \leq Ck\|P_k u\|_{L^2}\|Xu\|_{L^2} + Ck^2\|P_k u\|_{L^2}^2 + Ck^{-N}\|u\|_{L^2}.$$

Then, by the elliptic parametrix construction in the b-calculus [GW23, Equation 3.11] (see also [HV18, Appendix A]),

$$\|Xu\|_{L^2} \leq C\|\tilde{A}u\|_{L^2} + Ck^{-N}\|u\|_{L^2}$$

Combining the last two inequalities and using the inequality (5.5), we obtain that

$$\|\tilde{A}u\|_{L^2}^2 \leq Ck^2\|P_k u\|_{L^2}^2 + Ck^{-N}\|u\|_{L^2}^2.$$

Finally, since $\tilde{A}$ is elliptic on $\mathrm{WF}(A)$,

$$\|Au\|_{L^2} \leq C\|\tilde{A}u\|_{L^2} + Ck^{-N}\|u\|_{L^2},$$

which completes the proof ■

C.5 Proof of Theorem C.2

To prove the first bound in Theorem C.2, let $u = R_k f$. Then, by Lemma C.11 with $X = I$,

$$\begin{aligned}\|Au\|_{L^2}^2 &\leq Ck\|P_k u\|_{L^2}\|u\|_{L^2} + Ck^2\|P_k u\|_{L^2}^2 + C_N k^{-N}\|u\|_{L^2}^2 \\ &\leq Ck\|R_k\|_{L^2\to L^2}\|f\|_{L^2}^2 + Ck^2\|f\|_{L^2}^2 + C_N k^{-N}\|R_k\|_{L^2\to L^2}^2\|f\|^2 \\ &\leq C(k\|R_k\|_{L^2\to L^2} + k^2)\|f\|_{L^2}^2.\end{aligned}$$

In particular,

$$\|AR_k\|_{L^2\to L^2} \leq C(\sqrt{k\|R_k\|_{L^2\to L^2}} + k) \leq C\sqrt{k\|R_k\|_{L^2\to L^2}},$$

where the last inequality follows since $\|R_k\|_{L^2\to L^2} \geq ck$.

Reversing the direction of the flow in all of the above lemmas (or, equivalently, applying the above results to $-P_k^*$), the proof of Lemma C.11 also yields

$$\|A^*u\|_{L^2}^2 \leq Ck\|P_k^*u\|_{L^2}\|u\|_{L^2} + Ck^2\|P_k^*u\|_{L^2}^2 + C_Nk^{-N}\|u\|_{L^2}^2.$$

Therefore, putting $u = R_k^*f$, and arguing in the same way as above, we obtain

$$\|R_kA\|_{L^2\to L^2} = \|A^*R_k^*\|_{L^2\to L^2} \leq C\sqrt{k\|R_k\|_{L^2\to L^2}}.$$

To prove the second bound in Theorem C.2, let $u = R_kAf$. Then, by Lemma C.12, since $P_ku = Af$, and ${}^b\,\mathrm{WF}(A) \cap K = \emptyset$,

$$\|Au\|_{L^2} \leq Ck\|Af\|_{L^2} + Ck^{-N}\|u\|_{L^2} \leq Ck\|f\|_{L^2} + Ck^{-N}\|R_k\|_{L^2\to L^2}\|f\|_{L^2}.$$

C.6 Proof of Theorems C.3 and C.4

Proof of Theorem C.3. Since ${}^b\,\mathrm{WF}(A) \cap \Gamma_+ = \emptyset$ there exists $T > 0$ and $B_1 \in \Psi(\Omega)$ such that $\mathrm{WF}(B_1) \subset ((T^*\Omega \setminus \overline{T^*B(0, R_{\mathrm{scat}})}) \cap \{p_\theta \neq 0\}) \setminus {}^b\,\mathrm{WF}(B)$ and for all $q \in {}^b\,\mathrm{WF}(A) \cap \pi^b(\{\Re p_\theta = 0\})$,

$$\bigcup_{t=0}^{T} \varphi_{-t}(q) \cap \mathrm{Ell}(B_1) \neq \emptyset$$

(informally, B_1 is supported in the PML region away from B, and flowing backwards from A one hits B_1). In addition, since

$${}^b\,\mathrm{WF}(A) \cup \bigcup_{t=0}^{T} \varphi_{-t}\big({}^b\,\mathrm{WF}(A) \cap \pi^b(\{\Re p_\theta = 0\})\big) \cap {}^b\,\mathrm{WF}(B) = \emptyset,$$

there exists $E \in {}^b\Psi(\Omega)$ such that

$${}^b\,\mathrm{WF}(A) \cup \bigcup_{t=0}^{T} \varphi_{-t}\big({}^b\,\mathrm{WF}(A) \cap \pi^b(\{\Re p_\theta = 0\})\big) \subset {}^b\,\mathrm{Ell}(E), \qquad {}^b\,\mathrm{WF}(E) \cap {}^b\,\mathrm{WF}(B) = \emptyset.$$

Therefore, applying Theorem C.9 with $u = R_kBf$, and then using both ${}^b\,\mathrm{WF}(E) \cap {}^b\,\mathrm{WF}(B) = \emptyset$ and Assumption 1.2, we obtain

$$\|Au\|_{L^2} \leq Ck\|EBf\|_{L^2} + C\|B_1u\|_{L^2} + C_Nk^{-N}\|u\|_{L^2} \leq \|B_1u\|_{L^2} + C_Nk^{-N}\|f\|_{L^2}. \tag{C.11}$$

The elliptic parametrix construction [DZ19, Proposition E.32] then implies that

$$\|B_1u\|_{L^2} \leq C\|B_1Bf\|_{L^2} + C_Nk^{-N}\|u\|_{L^2} \leq C_Nk^{-N}\|f\|_{L^2}. \tag{C.12}$$

Combining (C.11) and (C.12), we obtain that

$$\|Au\|_{L^2} \leq C_Nk^{-N}\|f\|_{L^2}$$

and the result $\|AR_kB\|_{L^2\to L^2} \leq C_Nk^{-N}$ follows. ∎

The proof of Theorem C.4 is nearly identical with P_k replaced by $-P_k^*$.

References

[AG07] S. Alinhac and P. Gérard. *Pseudo-differential operators and the Nash-Moser theorem*, volume 82 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2007. Translated from the 1991 French original by Stephen S. Wilson.

[AGS24] M. Averseng, J. Galkowski, and E. A Spence. Helmholtz FEM solutions are locally quasi-optimal modulo low frequencies. *Adv. Comp. Math.*, 50(112), 2024.

[BBR10] J.-F. Bony, N. Burq, and T. Ramond. Minoration de la résolvante dans le cas captif. *Comptes Rendus. Mathématique*, 348(23-24):1279–1282, 2010.

[BCFG17] H. Barucq, T. Chaumont-Frelet, and C. Gout. Stability analysis of heterogeneous Helmholtz problems and finite element solution based on propagation media approximation. *Math. Comp.*, 86(307):2129–2157, 2017.

[BCFM25] M. Bernkopf, T. Chaumont-Frelet, and J. M. Melenk. Wavenumber-explicit stability and convergence analysis of *hp* finite element discretizations of Helmholtz problems in piecewise smooth media. *Mathematics of Computation*, 94(351):73–122, 2025.

[BP07] J. Bramble and J. Pasciak. Analysis of a finite PML approximation for the three dimensional time-harmonic Maxwell and acoustic scattering problems. *Mathematics of Computation*, 76(258):597–614, 2007.

[Bre20] S. C. Brenner. A general superapproximation result. *Computational Methods in Applied Mathematics*, 20(4):763–767, 2020.

[BS00] I. M. Babuška and S. A. Sauter. Is the pollution effect of the FEM avoidable for the Helmholtz equation considering high wave numbers? *SIAM Review*, pages 451–484, 2000.

[BS08] S. C. Brenner and L. R. Scott. *The Mathematical Theory of Finite Element Methods*, volume 15 of *Texts in Applied Mathematics*. Springer, 3rd edition, 2008.

[CFGT22] T. Chaumont-Frelet, S. Gallistl, D.and Nicaise, and J. Tomezyk. Wavenumber explicit convergence analysis for finite element discretizations of time-harmonic wave propagation problems with perfectly matched layers. *Communications in Mathematical Sciences*, 20(1):1–52, 2022.

[CFN20] T. Chaumont-Frelet and S. Nicaise. Wavenumber explicit convergence analysis for finite element discretizations of general wave propagation problem. *IMA J. Numer. Anal.*, 40(2):1503–1543, 2020.

[CFS25] T. Chaumont-Frelet and E. A. Spence. The geometric error is less than the pollution error when solving the high-frequency Helmholtz equation with high-order FEM on curved domains. *IMA J. Numer. Anal.*, to appear, 2025.

[CW13] H. Christianson and J. Wunsch. Local smoothing for the Schrödinger equation with a prescribed loss. *American Journal of Mathematics*, 135(6):1601–1632, 2013.

[CWSGS20] S. N. Chandler-Wilde, E. A. Spence, A. Gibbs, and V. P. Smyshlyaev. High-frequency bounds for the helmholtz equation under parabolic trapping and applications in numerical analysis. *SIAM Journal on Mathematical Analysis*, 52(1):845–893, 2020.

[DG17] S. Dyatlov and J. Galkowski. Fractal Weyl laws and wave decay for general trapping. *Nonlinearity*, 30(12):4301, 2017.

[DGS11] A. Demlow, J. Guzmán, and A. H. Schatz. Local energy estimates for the finite element method on sharply varying grids. *Mathematics of Computation*, 80(273):1–9, 2011.

[DV12a] K. Datchev and A. Vasy. Propagation through trapped sets and semiclassical resolvent estimates. *Annales de l'Institut Fourier*, 62(6):2347–2377, 2012.

[DV12b] K. Datchev and A. Vasy. Semiclassical resolvent estimates at trapped sets. *Annales de l'Institut Fourier*, 62(6):2379–2384, 2012.

[DW15] Y. Du and H. Wu. Preasymptotic error analysis of higher order FEM and CIP-FEM for Helmholtz equation with high wave number. *SIAM J. Numer. Anal.*, 53(2):782–804, 2015.

[DZ19] S. Dyatlov and M. Zworski. *Mathematical theory of scattering resonances*, volume 200 of *Graduate Studies in Mathematics*. American Mathematical Society, 2019.

[EO16] C. L. Epstein and M. O'Neil. Smoothed corners and scattered waves. *SIAM J. Sci. Comput.*, 38(5):A2665–A2698, 2016.

[FW09] X. Feng and H. Wu. Discontinuous Galerkin methods for the Helmholtz equation with large wave number. *SIAM J. Numer. Anal.*, 47(4):2872–2896, 2009.

[FW11] X. Feng and H. Wu. *hp*-Discontinuous Galerkin methods for the Helmholtz equation with large wave number. *Math. Comp.*, 80(276):1997–2024, 2011.

[Gal25] J. Galkowski. Lower bounds for piecewise polynomial approximations of oscillatory functions. *Journal of Approximation Theory*, 305:106100, 2025.

[GLS23] J. Galkowski, D. Lafontaine, and E. A. Spence. Perfectly-matched-layer truncation is exponentially accurate at high frequency. *SIAM J. Math. Anal.*, 55(4):3344–3394, 2023.

[GLS24] J. Galkowski, D. Lafontaine, and E. A. Spence. Local absorbing boundary conditions on fixed domains give order-one errors for high-frequency waves. *IMA. J. Num. Anal.*, 44(4):1946–2069, 2024.

[GLSW23] J. Galkowski, D. Lafontaine, E. A. Spence, and J. Wunsch. Decompositions of high-frequency Helmholtz solutions via functional calculus, and application to the finite element method. *SIAM J. Math. Anal.*, 55(4):3903–3958, 2023.

[GLSW24] J. Galkowski, D. Lafontaine, E.A. Spence, and J. Wunsch. The *hp*-FEM applied to the Helmholtz equation with PML truncation does not suffer from the pollution effect. *Comm. Math. Sci.*, 22(7):1761–1816, 2024.

[GS25] J. Galkowski and E. A. Spence. Sharp preasymptotic error bounds for the Helmholtz h-FEM. *SIAM J. Numer. Anal.*, 63(1):1–22, 2025.

[GW23] O. Gannot and J. Wunsch. Semiclassical diffraction by conormal potential singularities. *Ann. Sci. ENS*, 56(3):713–800, 2023.

[Hec98] F. Hecht. BAMG: bidimensional anisotropic mesh generator. *User Guide. INRIA, Rocquencourt*, 17, 1998.

[Hör85] L. Hörmander. *The analysis of linear partial differential operators III: pseudo-differential operators*. Springer, 1985.

[HSZ03] T. Hohage, F. Schmidt, and L. Zschiedrich. Solving time-harmonic scattering problems based on the pole condition II: convergence of the PML method. *SIAM Journal on Mathematical Analysis*, 35(3):547–560, 2003.

[HV18] P. Hintz and A. Vasy. The global non-linear stability of the Kerr–de Sitter family of black holes. *Acta Math.*, 220(1):1–206, 2018.

[IB95] F. Ihlenburg and I. Babuška. Finite element solution of the Helmholtz equation with high wave number Part I: The h-version of the FEM. *Comput. Math. Appl.*, 30(9):9–37, 1995.

[IB97] F. Ihlenburg and I. Babuska. Finite element solution of the Helmholtz equation with high wave number part II: the *hp* version of the FEM. *SIAM J. Numer. Anal.*, 34(1):315–358, 1997.

[Ihl98] F. Ihlenburg. *Finite element analysis of acoustic scattering.* Springer Verlag, 1998.

[LSW21] D. Lafontaine, E. A. Spence, and J. Wunsch. For most frequencies, strong trapping has a weak effect in frequency-domain scattering. *Communications on Pure and Applied Mathematics*, 74(10):2025–2063, 2021.

[LSW22a] D. Lafontaine, E. A. Spence, and J. Wunsch. A sharp relative-error bound for the Helmholtz h-FEM at high frequency. *Numerische Mathematik*, 150:137–178, 2022.

[LSW22b] D. Lafontaine, E. A. Spence, and J. Wunsch. Wavenumber-explicit convergence of the *hp*-FEM for the full-space heterogeneous Helmholtz equation with smooth coefficients. *Comp. Math. Appl.*, 113:59–69, 2022.

[LW19] Y. Li and H. Wu. FEM and CIP-FEM for Helmholtz Equation with High Wave Number and Perfectly Matched Layer Truncation. *SIAM J. Numer. Anal.*, 57(1):96–126, 2019.

[MS10] J. M. Melenk and S. Sauter. Convergence analysis for finite element discretizations of the Helmholtz equation with Dirichlet-to-Neumann boundary conditions. *Math. Comp*, 79(272):1871–1914, 2010.

[MS11] J. M. Melenk and S. Sauter. Wavenumber explicit convergence analysis for Galerkin discretizations of the Helmholtz equation. *SIAM J. Numer. Anal.*, 49:1210–1243, 2011.

[NS74] J. A. Nitsche and A. H. Schatz. Interior estimates for Ritz-Galerkin methods. *Mathematics of Computation*, 28(128):937–958, 1974.

[Pem20] O. R. Pembery. *The Helmholtz Equation in Heterogeneous and Random Media: Analysis and Numerics*. PhD thesis, University of Bath, 2020.

[Sch12] K. Schmüdgen. *Unbounded self-adjoint operators on Hilbert space*, volume 265. Springer Science & Business Media, 2012.

[Vas08] A. Vasy. Propagation of singularities for the wave equation on manifolds with corners. *Annals of Mathematics*, 168:749–812, 2008.

[Voi84] J. Voigt. Factorization in some Fréchet algebras of differentiable functions. *Studia Mathematica*, 77(4):333–348, 1984.

[Wah91] L. B. Wahlbin. Local behavior in finite element methods. In *Handbook of numerical analysis, Vol. II*, pages 353–522. North-Holland, Amsterdam, 1991.

[Wu14] H. Wu. Pre-asymptotic error analysis of CIP-FEM and FEM for the Helmholtz equation with high wave number. Part I: linear version. *IMA J. Numer. Anal.*, 34(3):1266–1288, 2014.

[Zwo12] M. Zworski. *Semiclassical analysis*, volume 138 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2012.