Volume integral equations on fractal domains and the Koch snowflake transmission problem

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Abstract

We study volume integral equation (VIE) reformulations of the inhomogeneous Helmholtz equation with piecewise constant refractive index on inhomogeneities with rough, possibly fractal, interface. We focus in particular on the case of scattering by the Koch snowflake. To obtain a numerical approximation we replace the inhomogeneity by a smoother "prefractal" approximation, and solve the VIE on the prefractal using piecewise constants on a suitable triangulation. Using the concept of Mosco convergence we prove the convergence of the corresponding Galerkin approximations on the prefractals to the true solution of the VIE on the fractal in the joint limit as the pre-fractal level $j \to \infty$ and mesh width $h \to 0$. We discuss the relationship between the convergence rate of the method and the Hausdorff dimension of the fractal boundary and present supporting numerical results.

Keywords: Helmholtz transmission problem, Volume integral equation, Mosco convergence

1 Motivation

We recall the classical Helmholtz transmission problem in $\mathbb{R}^d$, $d = 2, 3$: given wavenumbers $k_i, k_e > 0$, an incident plane wave $u^{inc} = e^{ik_e\hat{d}\cdot x}$, $|\hat{d}| = 1$, and a bounded Lipschitz open set $D \subset \mathbb{R}^d$, find an interior field $u_i \in H^1(D)$ and an exterior field $u_e \in H^{1,loc}(\mathbb{R}^d \setminus \overline{D})$ such that

$$\Delta u_i + k_i^2 u = 0 \quad \text{in } D, \tag{1}$$

$$\Delta u_e + k_e^2 u = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{D}, \tag{2}$$

$$u_e = u_i \quad \text{on } \partial D, \tag{3}$$

$$\partial_n u_e = a\partial_n u_i \quad \text{on } \partial D, \tag{4}$$

where $a > 0$ is a coupling constant, and the scattered field $u_e - u^{inc}$ satisfies the Sommerfeld radiation condition (SRC) at infinity. For bounded Lipschitz D, the problem (1)-(4) is well-posed, and can be recast as a system of boundary integral equations (BIEs) on ∂D [1], which, when ∂D is piecewise smooth, can be solved numerically using the boundary element method.

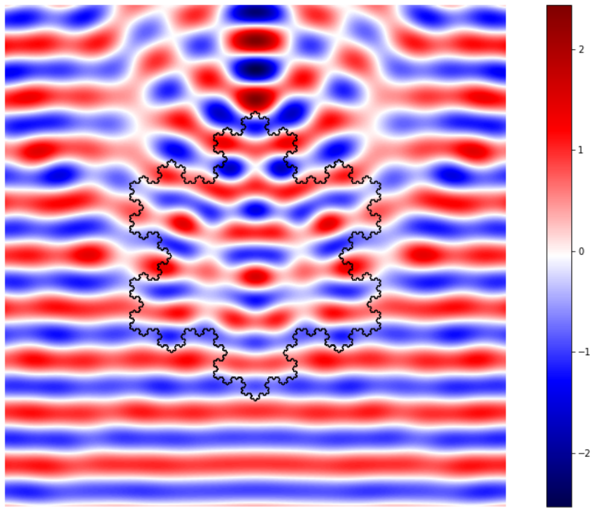

Figure 1: Scattering of a plane wave by the Koch snowflake, with $k_e = 30$ and $k_i = 45$.

However, for general non-Lipschitz D, such as the Koch snowflake, which has fractal boundary, the transmission problem (1)-(4) and its BIE reformulation no longer make sense, because one does not in general have well-defined Dirichlet and Neumann trace operators onto ∂D. The question we investigate in this work is: how can we pose "transmission problems" on such sets, and how can we efficiently approximate their solutions numerically?

2 Inhomogeneous Helmholtz equation and VIE reformulation

For general bounded open $D \subset \mathbb{R}^d$ with $|\partial D| = 0$, in place of the transmission problem (1)-(4) we seek a solution $u \in H^{1,loc}(\mathbb{R}^d)$ of the inhomogeneous Helmholtz equation

$$\Delta u + k_e^2 n(x) u = 0, \tag{5}$$

with refractive index

$$n(x) = \begin{cases} k_i^2/k_e^2, & x \in D, \\ 1, & x \in \mathbb{R}^d \setminus \overline{D}, \end{cases} \tag{6}$$

such that the scattered wave $u^{sca} = u - u^{inc}$, which should satisfy the equation

$$\Delta u^{sca} + k_e^2 n(x) u^{sca} = k_e^2 f,$$

where $f = (1-n)u^{inc}$, satisfies the SRC at infinity. The problem is well-posed (by a standard Riesz-Fredholm theory argument), and can be reformulated equivalently as a VIE on D via the Lippmann-Schwinger equation (LSE) (cf. [3–5]):

$$(I + k_e^2 V(1-n))u^{sca} = F, \quad \text{on } D, \qquad (7)$$

where V is the Newton potential, defined by

$$V\phi(x) = \int_D \Phi(x,y)\phi(y)\,dy, \quad \phi \in L^2(D),$$

with Φ denoting the fundamental solution of $(\Delta + k_e^2)u = 0$, and $F = -k_e^2 V f$. It is well known that $V : L^2(D) \to H^2(G)$ is bounded, and hence $V : L^2(D) \to L^2(G)$ is compact, for any bounded open $D, G \subset \mathbb{R}^n$. Hence the LSE operator is a compact perturbation of the identity operator on the space $L^2(D)$, and by the well-posedness of (5) and the equivalence of the latter with (7), equation (7) is well-posed.

3 Discretization and numerical analysis

To discretize (7) we replace D by a polygonal or polyhedral "prefractal" approximation D_j, and then compute an approximate solution u_j^{sca} of (7), with D replaced by D_j, using a Galerkin piecewise constant approximation on a convex mesh of D_j. To prove convergence of u_j^{sca} to u^{sca} we use the notion of Mosco convergence, which implies convergence of Galerkin solutions for operators that are compact perturbations of coercive operators [2].

Definition 1 *Let W and W_j, for $j \in \mathbb{N}$, be closed subspaces of a Hilbert space H. We say W_j Mosco converges to W if*

(i) $\forall w \in W$, $\exists w_j \in W_j$, $j \in \mathbb{N}$, *s.t* $\|w_j - w\|_H \to 0$ *as* $j \to \infty$;

(ii) if $\{W_{j_m}\}$ *is a subsequence of* $\{W_j\}$ *and* $w_m \in W_{j_m}$ *with* $w_m \rightharpoonup w$, *then* $w \in W$.

Applying this definition with $H = L^2(B)$ for some ball B containing D and D_j, we can prove Mosco convergence of $L^2(D_j)$ to $L^2(D)$, and hence of u_j^{sca} to u^{sca}, under quite general conditions on the prefractal approximations D_j.

When $D_j \subset D$ for each j we also have asymptotic quasi-optimality, so that for sufficiently large j the Galerkin error can be controlled in terms of the best approximation error of $u \in L^2(D)$ by elements v_h, for which we have

$$\|u - v_h\|_{L^2(D)} \leq \|u\|_{L^2(D \setminus \overline{D_j})} + \|u - v_h\|_{L^2(D_j)}.$$

Hence the best approximation error comprises two parts - the first due to the approximation of D by D_j, and the second due to the approximation of u on D_j by a piecewise constant function. Since $u \in H^1(D)$, we have the standard bound

$$\|u - v_h\|_{L^2(D_j)} \leq \frac{h_j}{\pi}\|u\|_{H^1(D)},$$

where h_j is the maximum mesh width on D_j, and since $u \in H^2(D) \subset C(\overline{D})$ we have

$$\|u\|_{L^2(D \setminus \overline{D_j})} \leq |D \setminus \overline{D_j}|^{1/2}\|u\|_{C(\overline{D})}.$$

The magnitude of $|D \setminus \overline{D_j}|$ depends on the fractal dimension of the boundary ∂D, which affects how well it can be approximated by polygons/polyhedra. For the particular case of the Koch snowflake we investigate the corresponding errors for different choices of D_j and h_j, including the standard prefractal approximations (unions of triangles), and the "pixellations" commonly used in VIE solvers e.g. VINES [6].

References

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