A Hausdorff-Measure Boundary Element Method for Scattering by Fractal Screens II: Numerical Quadrature

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Abstract

In a related talk [1], the Boundary Element Method (BEM) is generalised to the case of scattering by fractal obstacles. Implementation requires evaluating integrals of singular Green's kernels over fractal domains, with respect to *Hausdorff measure*. This motivated the development of new quadrature rules, which are discussed here.

Keywords: Quadrature, BEM, Fractals

1 Introduction

We will study numerical quadrature rules for the evaluation of integrals of the form

$$I_{\Gamma,\Gamma'}[\Phi] := \int_\Gamma \int_{\Gamma'} \Phi(x,y)\mathrm{d}\mathcal{H}^{d'}(y)\mathrm{d}\mathcal{H}^{d}(x), \quad (1)$$

where Γ and Γ' are compact subsets of $\mathbb{R}^2$ of Hausdorff dimension $d>0$ and $d'>0$ respectively, $\mathcal{H}^d$ and $\mathcal{H}^{d'}$ are the corresponding Hausdorff measures, and $\Phi(x,y)=\frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{4\pi|x-y|}$ is the fundamental solution for the Helmholtz equation with wavenumber $k>0$ in $\mathbb{R}^3$. (In what follows, similar results hold for the analogous problem posed in $\mathbb{R}^2$.)

Our motivation for approximating (1) is the *Hausdorff BEM*, which is introduced and analysed in the talk [1]. Such BEMs can model scattering by planar screens with non-integer (fractal) dimension, i.e. $d \in (1,2)$.

2 Attractors of Iterated Function Systems

Now we describe in detail the class of fractal scatterers that we consider. An iterated function system (IFS) is a set of $2 \le M \in \mathbb{N}$ *contracting similarities* $s_m(x) = \rho_m A_m x + \delta_m$, with contraction factors $\rho_m \in (0,1)$, rotation matrices $A_m \in \mathbb{R}^{n\times n}$ and translations $\delta_m \in \mathbb{R}^n$, for $m = 1,\dots,M$. Saying that Γ is the *attractor* of the IFS means that Γ is the unique non-empty compact set satisfying $\Gamma = s(\Gamma)$, where $s(E) := \bigcup_{m=1}^M s_m(E), \quad E \subset \mathbb{R}^n$.

Our quadrature rules are based on splitting Γ into sub-components, using the IFS structure.

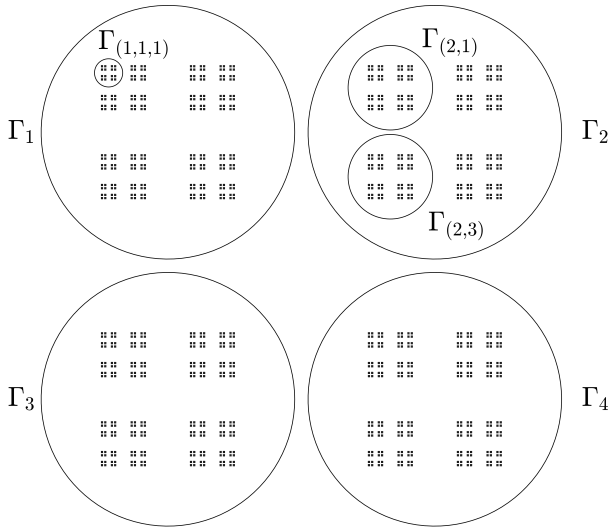

Figure 1: Vector indices on Cantor Dust.

To describe these sub-components we adopt vector index notation. For $\ell \in \mathbb{N}$ let $I_\ell := \{1,\dots,M\}^\ell$. Then for $E \subset \mathbb{R}^n$ let $E_0 := E$, and for $\mathbf{m} = (m_1,\dots,m_\ell) \in I_\ell$ define $E_{\mathbf{m}} := s_{\mathbf{m}}(E)$ and $s_{\mathbf{m}} := s_{m_1} \circ \ldots \circ s_{m_\ell}$. For an illustration of this notation in the case of the middle-third Cantor dust see Figure 1. We say Γ is *hull-disjoint* if

$$\mathcal{R} := \min_{m\neq m'}\{\mathrm{dist}(\mathrm{Hull}(\Gamma_m),\mathrm{Hull}(\Gamma_{m'}))\} > 0.$$

A key ingredient is the set of vector indices

$$L_h(\Gamma) := \big\{\mathbf{m} = (m_1,\dots,m_\ell) \in \cup_{\ell'\in\mathbb{N}} I_{\ell'} : \mathrm{diam}(\Gamma_{\mathbf{m}}) \le h \text{ and } \mathrm{diam}(\Gamma_{(m_1,\dots,m_{\ell-1})}) > h\big\}.$$

Heuristically, these indices correspond to a partition of Γ, where we have subdivided *just enough* so that all components have diameter no more than h. This is depicted in Figure 2.

3 The barycentre rule

We define the *barycentre rule* for double integrals:

$$Q^h_{\Gamma,\Gamma'}[f] := \sum_{\mathbf{m}\in L_h(\Gamma)} \sum_{\mathbf{m}'\in L_h(\Gamma')} w_{\mathbf{m}} w'_{\mathbf{m}'} f(x_{\mathbf{m}}, x'_{\mathbf{m}'}), \quad (2)$$

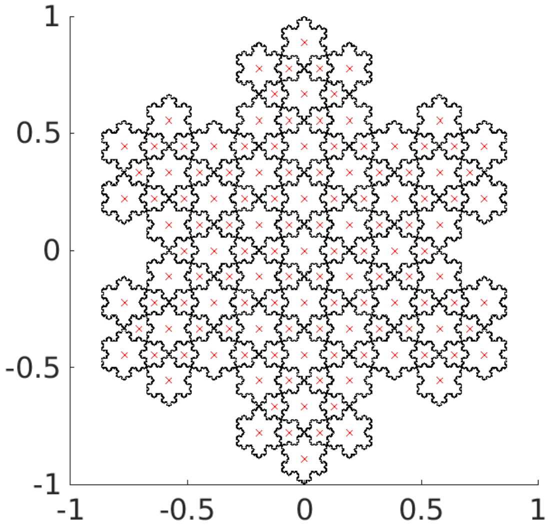

Figure 2: Partitioning Koch snowflake by $L_{0.3}(\Gamma)$. Barcentres $x_{\mathbf{m}}$ are represented by $\times$.

where the weights and nodes are given by $w_{\mathbf{m}} := \mathcal{H}^d(\Gamma_{\mathbf{m}})$ and $x_{\mathbf{m}} := \int_{\Gamma_{\mathbf{m}}} x \,\mathrm{d}\mathcal{H}^d(x)/\mathcal{H}^d(\Gamma_{\mathbf{m}})$ for $\mathbf{m} \in L_h(\Gamma)$, with analogous definitions for Γ'. The weights and nodes can be easily computed in terms of the IFS parameters, see [2, (27-29)]. For the single integral version of (2), see [2, §3.1]. In all estimates that follow, C denotes a constant which depends only on Γ.

Theorem 1 (Lipschitz integrands) *[2, Theorem 3.7] If $L_0[f]$ and $L_1[f]$ are the Lipschitz constants of f and ∇f respectively in* $\mathrm{Hull}(\Gamma) \times \mathrm{Hull}(\Gamma')$,

$$\left| I_{\Gamma,\Gamma'}[f] - Q^h_{\Gamma,\Gamma'}[f] \right| \le C L_p[f] h^{p+1} \text{ for } p \in \{0,1\}.$$

A result for non-diagonal entries of Hausdorff BEM matrices follows immediately:

Corollary 2 (Smooth Galerkin integrals) *[2, Proposition 5.2]*
If $R := \mathrm{dist}(\mathrm{Hull}(\Gamma), \mathrm{Hull}(\Gamma')) > 0$, then

$$\left| I_{\Gamma,\Gamma'}[\Phi] - Q^h_{\Gamma,\Gamma'}[\Phi] \right| \le C h^2 \frac{1 + (kR)^{n/2+1}}{R^{n+1}}.$$

4 Singular integrals of Laplace kernels

In Hausdorff BEM, the diagonal matrix elements correspond to (1) with $\Gamma = \Gamma'$. Because $|\Phi(x,y)| \to \infty$ as $|x-y| \to 0$, the rule (2) cannot be directly applied to (1) in this case. We will derive a new method for evaluating the singular (Laplace) component of (2), denoted $\Phi_0(x,y) := |x-y|^{-1}$. Then, to evaluate (1) with $\Gamma = \Gamma'$, we use a *singularity subtraction* technique, by considering the Lipschitz continuous function $\Phi_* := \Phi - \Phi_0$, and splitting the integral as follows

$$I_{\Gamma,\Gamma}[\Phi] = I_{\Gamma,\Gamma}[\Phi_0] + I_{\Gamma,\Gamma}[\Phi_*], \qquad (3)$$

and evaluating both components separately.

By exploiting the self-similarity of Γ, we can express $I_{\Gamma,\Gamma}[\Phi_0]$ as a linear function of $I_{\Gamma_m,\Gamma_m}[\Phi_0]$ for $m = 1, \dots, M$, which leads to

$$I_{\Gamma,\Gamma}[\Phi_0] = \frac{\sum_{m=1}^{M} \sum_{m' \neq m}^{M} I_{\Gamma_m,\Gamma_{m'}}[\Phi_0]}{1 - \sum_{m=1}^{M} \rho_m^{2d-1}}, \qquad (4)$$

representing a singular integral as a linear combination of smooth integrals. The smooth integrals of (4) can be approximated using (2); we denote this approximation by $Q^h_{\Gamma,\Gamma,0}$.

Theorem 3 (Singular Laplace-type integrals) *[2, Corollary 4.7] If Γ is Hull-disjoint, then*

$$\left| I_{\Gamma,\Gamma}[\Phi_0] - Q^h_{\Gamma,\Gamma,0} \right| \le C h^2 \mathcal{R}^{-3} \left(1 - \sum_{m=1}^{M} \rho_m^{2d-1} \right)^{-1}.$$

5 Approximating (1)

Noting the decomposition (3), Theorem 3 states that $I_{\Gamma,\Gamma}[\Phi_0]$ can be estimated with $O(h^2)$ error, provided Γ is hull-disjoint.

Since $\Phi_* \in C^{0,1}(\mathbb{R}^n) \setminus C^{1,1}(\mathbb{R}^n)$, Theorem 1 suggests $|I_{\Gamma,\Gamma}[\Phi_*] - Q^h_{\Gamma,\Gamma}[\Phi^*]| = O(h)$. With further work it can be shown that this is actually $O(h^2)$, when (i) Γ is hull-disjoint and (ii) $\rho_1 = \ldots = \rho_M$. Hence using (3) we can approximate $I_{\Gamma,\Gamma}[\Phi]$ with $O(h^2)$ accuracy (see [2, §5] for details).

Furthermore, numerical experiments [2, §6] suggest $O(h^2)$ convergence for fractals which violate either or both of the conditions (i)–(ii).

References

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