THE DYNAMIC VICKREY AUCTION

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ABSTRACT. We study the efficient allocation of a single object over a finite time horizon. Buyers arrive randomly over time, are long-lived and have independent private values. The valuation of a buyer may depend on the time of the allocation in an arbitrary way. We construct an incentive compatible mechanism in which (A) there is a single financial transaction (with the buyer), (B) ex-post participation constraints are fulfilled, (C) there is no positive transfer to any agent and (D) payments are determined online. We exploit that under the efficient allocation rule, there is a unique potential winning period for each buyer. This reduces the multidimensional type to one dimension and the payment of the winner can be defined as the lowest valuation for the potential winning period, with which the buyer would have won the object. In a static model, this payment rule coincides with the payment rule of the Vickrey Auction.

KEYWORDS: Dynamic allocation problem; Efficiency; Auction; Multidimensional types
JEL-Codes: D44,D82

1. INTRODUCTION

In many situations, the allocation of an object is a dynamic problem. Buyers arrive over time and have preferences regarding the time of the allocation. In addition to the winner determination, an allocation mechanism also has to determine the time of the allocation. Think for example of a government that wants to privatize an asset or a state-owned company—"the object"—in order to reduce its budget deficit, and wants to achieve an efficient allocation of the object. Potential buyers arrive over time and have preferences as to when they would like to acquire the object. For example, there may be investors whose business plan requires to start using the object immediately, while others are more patient. Others again may need some time before they can buy the asset and start using it, for example because complementary investments have not been made yet, or because they require some time to secure funding of the acquisition.

In this paper, a mechanism is constructed that implements the efficient allocation rule in such a dynamic environment and satisfies the following properties:

(A) There is a *single monetary transaction* which takes place between the buyer of the object and the mechanism.
(B) *Ex-post participation constraints* are fulfilled for all agents.
(C) There is *no positive transfer payment* to any participant in any period.

Date: July 1, 2013, (first draft: October 31, 2007).
A previous version of this paper circulated under the title "An Efficient Intertemporal Auction". I would like to thank Dirk Bergemann, Olivier Bos, Olivier Compte, Philippe Jehiel, Benny Moldovanu, Paul Heidhues, Larry Samuelson, Nicolas Schutz, Rakesh Vohra, Alexander Westkamp, an anonymous referee, the associate editor, and seminar audiences at Paris School of Economics, Bonn, SMYE Lille 2008, ESEM Milan 2008 and the EDP Jamboree Paris 2007 for helpful comments and discussions. I would like to thank the European Research Council (ERC Advanced Investigator Grant, ESEI-249433) for financial support.
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(D) All *transactions are made online*—that is, all information that is needed to determine the payment must be available at the time of the allocation.

These properties are desirable from a practical point of view. Financial transactions with other agents than the winning bidder entail higher transaction costs. A mechanism that satisfies Property A therefore reduces transaction costs to a minimum. Properties B and C ensure that the mechanism can be implemented even if it is difficult or costly to prevent participants from reneging on their bids. If voluntary participation constraints are violated ex-post, the outcome, in particular payments required from the participants may be harder to enforce. For example, if loosing bidders have to make positive payments to the mechanism, they will try to withdraw their participation even if this entails a small cost. Similarly, a mechanism that regularly makes positive transfer payments will attract bidders who are not interested in the object and plan to withdraw their participation unless the outcome is that they receive a positive payment. A mechanism that satisfies Properties A to C therefore minimizes transaction costs or costs associated with the enforcement of the outcome of the mechanism. The last property (D) is indispensable in a dynamic allocation problem. If payments cannot be determined online, this means that additional information has to be collected from future buyers after the winner has already been determined. Incentives for reporting such information cannot be provided, however, because the allocation decision has already been made. Furthermore, online payments allow to match payments with delivery, which makes it easier to enforce payments.[1]

This paper analyzes the dynamic allocation of a single object over a finite time horizon. The model generalizes the standard independent private values framework. Potential buyers arrive randomly over time, they are long-lived, and their valuations for the object may depend on the time of the allocation in an arbitrary way. If the valuations did not depend on the timing of the allocation, the seller could wait for the last period and run a standard auction such as the Vickrey Auction (or the English Auction). In this auction format, the highest bidder wins and pays an amount equal to the second highest bid. If all bidders use their dominant strategies and place a bid equal to their valuation, an efficient allocation is achieved and moreover, Properties A to D are fulfilled.

In a dynamic framework, the existence of efficient mechanisms that satisfy these properties is not obvious. The static Vickrey-Clarke-Groves (Vickrey, 1961; Clarke, 1971; Groves, 1973) mechanism has been extended to various, quite general, dynamic settings (Athey and Segal, 2007; Bergemann and Välimäki, 2010; Cavallo et al., 2010; Parkes and Singh, 2003). The underlying principle of these dynamic versions of the VCG mechanism is that an agent who reports information that influences current or future allocation decisions, is required to pay an amount equal (in expectation) to the externality that her report imposes on all other agents. This implies that transactions are not limited to eventual buyers of the object. For example, a buyer A who reports that she is willing to pay a high price tomorrow, may induce the mechanism to store the object instead of selling it today to some impatient buyer B. Buyer A, however, will not get the object tomorrow if a third buyer C with a higher willingness to pay arrives. This is an example of a situation where,

[1]For a discussion of the advantages of property (D) see also Cole et al. (2008) who distinguish *prompt mechanisms* that determine payments at the time of the allocation, and *tardy mechanisms* that defer payments to later times (see also Babaioff et al., 2010).

in contrast to a static allocation problem, a buyer's report influences the allocation decision even though she does not win the object. Without A's report, the object would not be allocated to C because the option value of storing the object would be lower than B's valuation for an allocation today. In the dynamic variants of the VCG mechanism proposed in the literature, such *pivotal* but non-winning buyers have to be compensated or charged for the externality, respectively, in order to ensure incentive compatibility. Parkes and Singh (2003) and Bergemann and Välimäki (2010) construct mechanisms in which payments are made online, but require transactions with multiple agents. The mechanisms also transfer positive amounts to participants, or violate ex-post participation constraints, respectively (see Section 4.4). For the independent private values framework, little is known about the existence of efficient mechanisms that satisfy all Properties A to D.[2]

In the mechanism proposed in this paper, only the winning bidder has to make a payment, which is equal to the lowest valuation with which she could have won the object. This valuation is called the *critical valuation* of the winner. In the static Vickrey Auction, the critical valuation is equal to the second highest valuation and coincides with the externality that the winner imposes on the other bidders.[3] In the dynamic framework this is not the case. The critical valuation is the maximum of (a) the second highest valuation for the period in which the bidder wins, (b) the option value of storing the object for future allocations, and (c) the lowest valuation for the winning period for which it is efficient to store the object instead of allocating earlier. The discussion in Section 4.4 will highlight the difference between the critical valuation and the imposed externality.

The next section introduces the model. In the framework considered in this paper, the definition of a critical valuation is not straightforward because types are multidimensional. In Section 3, it is shown that the information about a buyer's type, which is relevant for determining the efficient allocation, is essentially one-dimensional. For each type, there is a unique period in which she can possibly win the object. Therefore, only the valuation for this period matters for the efficient allocation rule. This reduction to one dimension allows to define an order on the type-space that makes the notion of a critical valuation

[2]Gershkov and Moldovanu (2009) and Gershkov et al. (2013) show that in dynamic environments where valuations or arrivals are correlated over time, the efficient allocation rule cannot always be implemented by a mechanism that satisfies all Properties A to D. Correlated types or arrivals lead to an informational externality of a buyer's report on the seller's option value of storing the object for future allocations. In general, this informational externality can only be reflected in the payments made by the buyer if the designer can condition payments on the realized types and arrivals of future buyers, which are drawn from the correct distribution , but such a scheme violates Property D (see also Mezzetti, 2004; Athey and Segal, 2007). In the case of correlated valuations, Gershkov and Moldovanu (2009) demonstrate that online payments are insufficient to implement the efficient allocation rule if the informational externalities are too strong. In the case of correlated arrivals Gershkov et al. (2013) show that Property D can be fulfilled at the expense of Property C.

[3]In models with one-dimensional private information, such as the standard static auction model, critical valuation payment schemes arise naturally when a deterministic allocation rule is implemented in dominant strategies. The Vickrey Auction as an implementation of the efficient allocation rule is the classic example. The emergence of such payment schemes can also be observed in more general settings, in particular in dynamic models where additional dimensions of private information such as an arrival time or an exit time are added (see Hajiaghayi et al., 2005). Note, however, that the arrival time and exit time are special because (a) they enter the utility function only as a constraint and (b) misreports are restricted to one direction. In the present model, agents have fully multi-dimensional valuations so that critical value payment schemes are not straightforwardly defined. Second, the present paper uses periodic ex-post equilibrium rather than dominant strategies as a solution concept so that other payment schemes could be used even in one dimensional models or the restricted generalizations discussed before.

meaningful. In Section 4, these crucial properties are used to define the payment rule of the Dynamic Vickrey Auction and to prove that it is incentive compatible. All proofs can be found in the Appendix.

2. The Model

We consider a seller who owns a single indivisible object that she wants to sell within $T < \infty$ time periods in order to maximize welfare. In each period, a random number of potential buyers arrive. The seller can allocate the object to one of the buyers who have already arrived, or she can store the object for one period. Alternatively, she can dispose of the object. In this case, no future allocation is possible.

Buyers. The number of buyers who arrive in period $t = 1, \dots, T$ is denoted by $n_t \in \mathbb{N}_0$, and the probability that k buyers arrive is denoted ρ_t^k. The set of buyers who have arrived in or before period t is denoted by $I_t = \{1, \dots, \sum_{\tau=1}^{t} n_t\}$. If a buyer $i \in I_t$ gets the object in period t, then her instantaneous payoff is $v_t^i \in [0, \bar{v}]$. A buyer is therefore characterized by a vector of valuations $(v_t^i, \dots, v_T^i) \in S_t := [0, \bar{v}]^{T-t+1}$, where t is the arrival time. For $i \in I_t \setminus I_{t-1}$—that is, for a buyer who arrives in period t, the valuations are drawn from a joint distribution with c.d.f. $\Phi_t : S_t \to [0, 1]$. The whole vector of valuations is drawn and revealed in the buyer's arrival period. For the moment, we assume that the arrival of buyers and their types are revealed publicly. Private information will be introduced in Section 4. The valuations of different buyers are stochastically independent. Also, the number of buyers n_t who arrive in period t is independent of $n_1, \dots, n_{t-1}$ and of $(v^i)_{i \in I_T}$.

Utility is quasi-linear, buyers are risk neutral, and discount future payoffs. The expected utility of a buyer who arrives in period t is $\sum_{s=t}^{T} \delta^{s-t} \left\{ q_s^i v_s^i - p_s^i \right\}$, where q_s^i denotes the probability of winning the object in period s, p_s^i denotes the expected payment in period s, and $\delta \in (0, 1]$ denotes the discount factor. The discount factor is common to all buyers (and the seller) and can be equal to one. The valuation v_s^i can be interpreted as the present value of i's payoff stream if she gets the object in period s. For example, suppose that $T = 4$ and consider a buyer who arrives in period one. Suppose that she can store the object for one period at a cost of c, and that she can use the object in periods three and four to earn a net profit of Π per period. For this buyer, the vector of valuations is given by $(-c - \delta c + \delta^2 \Pi + \delta^3 \Pi, -c + \delta \Pi + \delta^2 \Pi, \Pi + \delta \Pi, \Pi)$.

The Seller. If the seller stores the object for one period, she incurs an instantaneous cost of $c_t^0 \in \mathbb{R}$.[4] To simplify notation, the cost is written as a negative payoff $v_t^0 = -c_t^0$. The seller can avoid the storage cost by disposing of the object. After disposal, the object cannot be allocated to any buyer in the future. The possibility of free disposal is modeled by a dummy buyer $i = -1$ with $v_t^{-1} = 0$ for all t, to who the seller can always allocate.[5] The seller's utility is quasi-linear, she is risk neutral, and discounts future payoffs with the same discount factor as the buyers.

[4] If $c_t^0 < 0$, this can be interpreted as the seller's payoff from using the object during period t.
[5] Note that $I_t \cap \{-1, 0\} = \emptyset$.

Histories and Order Statistics. A **history** $h_t = (I_t, (v^i)_{i\in I_t})$ describes the buyers who have arrived before or in period t. The history in period t, excluding one buyer $i \in I_t$, is written as h_t^{-i}. Note that for all $s \leq t$, $I_s = \{i \in I_t | v^i \in S_s\}$ and $n_s = |I_s \setminus I_{s-1}|$, are determined by the history h_t. For $s > t$, $\tilde{h}_s = (\tilde{I}_s, (\tilde{v}^i)_{i\in\tilde{I}_s})$ is called a **continuation of** h_t if $\tilde{h}_t = h_t$, where $\tilde{h}_t = (\tilde{I}_t, (\tilde{v}^i)_{i\in\tilde{I}_t})$ is the history for period t derived from $\tilde{h}_s$. For $s \geq t$, the highest valuation for period s, among the buyers $i \in I_t \cup \{-1\}$, is denoted by $\eta_s(I_t) := \max_{i\in I_t\cup\{-1\}} v_s^i$. Since v_s^i is maximized over $I_t \cup \{-1\}$ in this definition, we have $\eta_s(\emptyset) = 0$. Finally, the vectors of the highest order statistics of valuations for (the current and) all future periods for the set of buyers I_t are denoted by $\eta(I_t) = (\eta_t(I_t), \ldots, \eta_T(I_t))$ and $\eta_{>t}(I_t) = (\eta_{t+1}(I_t), \ldots, \eta_T(I_t))$, respectively.

3. The Ex-Ante Efficient Allocation Rule

In this section, we consider the pure allocation problem without private information about the arrival times and the valuations of the buyers. We define the ex-ante efficient allocation rule and derive several properties. The ex-ante efficient allocation rule is considered because allocation decisions in period t can only depend on the history h_t. This lack of foresight renders an ex-post efficient allocation rule infeasible.

3.1. **Definition of the Ex-Ante Efficient Allocation Rule.**

A deterministic allocation rule $x = (x_1, \ldots, x_T)$ specifies a decision $x(h_t, a_t) \in \{-1, 0\} \cup I_t$ for each state (h_t, a_t) in each period t.[6] The **state** (h_t, a_t) consists of the history of buyer arrivals h_t, and the availability of the object $a_t \in \{0, 1\}$. If $a_t = 1$, the seller has stored the object in all previous periods and any allocation decision $x_t(h_t, a_t) \in \{-1, 0\} \cup I_t$ is possible. $x_t = i \in I_t$ means that the object is allocated to buyer i, $x_t = 0$ means that the objects is stored for one period, and $x_t = -1$ means that the seller disposes of the object. If $a_t = 0$, the object has been allocated to a buyer, or has been disposed of in a previous period. In this case, no decision is possible. Therefore, we set $x_t(h_t, 0) = -1$ for all states $(h_t, 0)$.

The **ex-ante efficient allocation rule**, denoted by x^*, maximizes the ex-ante expected gains from trade minus the expected storage costs of the seller:[7]

$$\max_x E\left[\sum_{t=1}^{T} \delta^{t-1} v_t^{x_t(h_t,a_t)}\right].$$

For any state $(h_t, 1)$—that is, if the object is still available, the **option value of storing the object** is given by

$$V_t(h_t) := v_t^0 + E\left[\sum_{s=t+1}^{T} \delta^{s-t} v_s^{x_s^*(h_s,a_s)} \,\middle|\, h_t,\, a_t = 1,\, x_t = 0\right] \qquad \text{for } t < T.$$

For $t = T$, we set $V_T(h_T) = 0$ for all h_T.

If the ex-ante efficient allocation rule x^* allocates the object to buyer $i \in I_t$ in period t, then this buyer must have the highest valuation for period t—that is, $v_t^i = \eta_t(I_t)$. Since the seller has only one object, we can therefore write the option value as a function of the

[6] Ex-ante efficiency can be achieved by a deterministic allocation rule. Randomized allocations are therefore excluded to simplify the notation.

[7] Note that we have $v_t^{x_t(h_t,a_t)} = v_t^0 = -c_t^0$ if the seller stores the object in period t. If the object is disposed of in period t or if $a_t = 0$, we have $v_t^{x_t(h_t,a_t)} = v_t^{-1} = 0$.

first order statistics $\eta_{>t}(I_t) = (\eta_{t+1}(I_t), \ldots, \eta_T(I_t))$: $V_t(h_t) = V_t(\eta_{>t}(I_t))$. With this slight abuse of notation, we can write the ex-ante efficient allocation rule as $x_t^*(h_t, 0) = -1$ and

$$x_t^*(h_t, 1) = \begin{cases} \min\left\{\arg\max_{i \in I_t \cup \{-1\}} v_t^i\right\}, & \text{if } \eta_t(I_t) \geq V_t(\eta_{>t}(I_t)), \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

The object is stored if the option value exceeds $\eta_t(I_t)$. Otherwise the object is allocated to the buyer with the lowest index among all buyers who have $v_t^i = \eta_t(I_t)$. If $\eta_t(I_t) = 0$, the lowest index is $i = -1$ and the object is disposed of.[8]

3.2. **Properties of the Ex-Ante Efficient Allocation Rule.** In this subsection, several properties of the ex-ante efficient allocation rule are derived. We start by considering the allocation to individual buyers and observe that each buyer has a unique potential winning period. Next, we extend this observation to groups of buyers, and observe that there is a unique tentative winner for each state. Finally, we show that the underlying reason for these observations is that the ex-ante efficient allocation rule defines an order on the space of vectors of valuations that allows to define a (one-dimensional) score by which buyers can be compared.

Properties of the Allocation to Individual Buyers. If at least one buyer has arrived up until period t—that is, $I_t \neq \emptyset$, then the object is allocated to a buyer with the highest valuation in the current period whenever $\eta_t(I_t) \geq V_t(\eta_{>t}(I_t))$ and $\eta_t(I_t) > 0$. In this decision rule, we can replace $\eta_t(I_t)$ and $\eta_{>t}(I_t)$ by v_t^i and $v_{>t}^i = (v_{t+1}^i, \ldots, v_T^i)$, respectively, to obtain the optimal decision for the case that $I_t = \{i\}$—that is, the decision that is optimal in period t if only buyer i has arrived so far. If we use this decision rule in every period after buyer i's arrival, then we obtain the period in which buyer i would win if no other buyers arrived in any future period.

Definition 1. Let $i \in I_t \setminus I_{t-1}$—that is, i arrived in period t. If for buyer i, there exists a period $\theta^i \in \{t, \ldots, T\}$ such that

$$v_{\theta^i}^i \geq V_{\theta^i}(v_{>\theta^i}^i) \text{ and } v_{\theta^i}^i > 0, \text{ and } v_s^i < V_s(v_{>s}^i) \text{ for all } s \in \{t, \ldots \theta^i - 1\}, \tag{3.2}$$

then θ^i is called **the potential winning period of buyer** i. If no such period exists, the potential winning period of buyer i is set to $\theta^i = 0$.

For example, if there is no discounting ($\delta = 1$), and the seller has no storage cost ($v_t^0 = 0$, for all t), then a buyer with a constant valuation $v^i = (v, \ldots, v)$, $v > 0$, has potential winning period T. Since the valuations of this buyer do not decline over time, it is always better to store the object for one period, rather than to allocate to buyer i, because other buyers with higher valuations may arrive in the future. With discounting or storage costs, however, it may be optimal to allocate before the last period because the losses due to discounting and storage costs can be larger than the expected gains from

[8]The ex-ante efficient allocation rule is unique up to the tie-breaking rule. In (3.1), ties are broken in order to finish the allocation as early as possible. The seller only stores the object if this leads to a strictly higher payoff than allocating or disposing of the object in the current period. Moreover, if the object is not stored, and no buyer has a strictly positive valuation for the current period, the seller disposes of the object.

allocating to a buyer with a higher valuation. A potential winning period $\theta^i = 0$ signifies that buyer i can never get the object, even if no other buyers arrive.

In general, the potential winning period of a buyer depends on her vector of valuations, the discount factor, the seller's storage costs, the arrival process, and the distributions of valuations. It does not depend on the realized number of arrivals or the realized valuations of other buyers. Surprisingly, the potential winning period is the only period in which a buyer can win the object under the ex-ante efficient allocation rule.

Theorem 2. *Let $i \in I_t \setminus I_{t-1}$. If $x_s^*(\tilde{h}_s, a_s) = i$ for some $s \geq t$, and some continuation $\tilde{h}_s$ of h_t, then $s = \theta^i$.*

To get an intuition for this result, suppose that $T = 2$ and consider a buyer $i \in I_1$ who arrives in the first period. If the ex-ante efficient allocation rule allocates to this buyer in the first period, three conditions must be fulfilled. First, her valuation for period one must be strictly positive $v_1^i > 0$. Second, she must have the highest valuation for the first period—that is, $v_1^i = \eta_1(I_1) > 0$. Third, the option value of storing the object must not exceed her valuation for period one—that is, $v_1^i = \eta_1(I_1) \geq V_1(\eta_2(I_1))$. Since $i \in I_1$, and hence $v_2^i \leq \eta_2(I_1)$, the third condition implies that

$$v_1^i \geq V_1(v_2^i).$$

Therefore, the potential winning period of buyer i must be $\theta^i = 1$ if she gets the object in the first period.

Conversely, suppose that buyer i gets the object in the second period. Now her valuation for the second period must be strictly positive and equal to the highest valuation among all buyers—that is, $v_2^i = \eta_2(I_2) > 0$. Secondly, the option value of storing the object must be strictly greater than the highest valuation for period one—that is, $\eta_1(I_1) < V_1(\eta_2(I_1))$. The first condition and $i \in I_1$ imply that $\eta_2(I_1) = \eta_2(I_2) = v_2^i$, and hence $V_1(\eta_2(I_1)) = V_1(v_2^i)$. Using this and $v_1^i \leq \eta_1(I_1)$, the second condition yields

$$v_1^i < V_1(v_2^i).$$

Therefore, the potential winning period of buyer i must be $\theta^i = 2$ if she gets the object in the second period.

Properties of the Allocation to Groups of Buyers. The potential winning period of a single buyer has been defined by applying the ex-ante efficient allocation rule to her vector of valuations. Similarly, one could also apply the same decision rule to a fictitious buyer i who arrives in period t and has valuations $v^i = \eta(I_t)$. This leads to the following definition:

Definition 3. (i) For any history h_t, let θ^{I_t} be the potential winning period of a fictitious buyer with valuations $\eta(I_t)$ and arrival time t. We call θ^{I_t} the **potential winning period of the set** I_t.

(ii) If $\theta^{I_t} \neq 0$, then $i_t^* := \min\left\{i \in I_t \,\middle|\, v_{\theta^{I_t}}^i = \eta_{\theta^{I_t}}(I_t)\right\}$ is the **tentative winner in state** $(h_t, 1)$. If $\theta^{I_t} = 0$, then **no tentative winner exists** in state $(h_t, 1)$.

The potential winning period of the set I_t can be equal to zero. This is the case if the set of buyers is empty ($I_t = \emptyset$), so that $\eta_s(I_t) = 0$ for all $s \geq t$. It can also occur if $I_t \neq \emptyset$ but $\max\{\eta_t(I_t), V_t(\eta_{>t}(I_t))\} = 0$, so that $x^*(h_t, 1) = -1$. In the latter case, the efficient allocation rule disposes of the object because no surplus can be generated by allocating in the current period, and the storage costs are higher than the expected surplus from future allocations. Clearly, if $\theta^{I_t} \neq 0$, a buyer $i \in I_t$ can only get the object if $v^i_{\theta^{I_t}} = \eta_{\theta^{I_t}}(I_t)$. Among these buyers, the tie-breaking rule selects i^*_t. Hence, the tentative winner in state $(h_t, 1)$ is the unique buyer who can win the object among all buyers in I_t. Formally, we have

Corollary 4. *(i) Let i^*_t be the tentative winner in a state $(h_t, 1)$ with $\theta^{I_t} \neq 0$. Then $x_s(\tilde{h}_s, 1) \neq i$ for all $s \geq t$, all $i \in I_t \setminus \{i^*_t\}$, and all states $(\tilde{h}_s, 1)$, where $\tilde{h}_s$ is a continuation of h_t for which the object is stored in all periods $t, \dots, s-1$.*
If $\theta^{I_t} = 0$, $x_s(\tilde{h}_s, 1) \neq i$ holds for all $i \in I_t$.
(ii) The optimal allocation rule is given by $x^*(h_t, 0) = -1$ and

$$x^*(h_t,1) = \begin{cases} i^*_t, & \text{if a tentative winner exists in state } (h_t,1) \text{ and } \theta^{i^*_t} = t, \\ 0, & \text{if a tentative winner exists in state } (h_t,1) \text{ and } \theta^{i^*_t} > t \\ & \text{or if no tentative winner exists and } V_t(\eta_{>t}(I_t)) > 0, \\ -1, & \text{if no tentative winner exists and } V_t(\eta_{>t}(I_t)) \leq 0. \end{cases}$$

If we add additional buyers to the set I_t—that is, $\tilde{I}_t = I_t \cup A$, then the tentative winner for the larger set cannot be in $I_t \setminus \{i^*_t\}$. Formally, we have

Corollary 5. *Let $(h_t, 1)$ be a state with tentative winner i^*_t and let $\tilde{i}^*_t$ be the tentative winner in state $(\tilde{h}_s, 1)$, where $\tilde{I}_t = I_t \cup A$ and $\tilde{v}^i = v^i$ for all $i \in I_t$. Then $\tilde{i}^*_t \in A \cup \{i^*_t\}$.*

An immediate implication of this is that the tentative winner in period $t+1$ either remains the same as in period t if the object is stored ($x^*(h_t, 1) = 0$) or one of the new buyers who arrive in period $t+1$ becomes the tentative winner. Formally, we have

Corollary 6. *For $t < T$, consider a state $(h_t, 1)$ with $\theta^{I_T} \neq 0$ and $x^*(h_t, 1) = 0$. Let be $(\tilde{h}_{t+1}, 1)$ be a state for the next period, where $\tilde{h}_{t+1}$ is a continuation of h_t. Then $i^*_{t+1} \in (\tilde{I}_{t+1} \setminus I_t) \cup \{i^*_t\}$.*

Theorem 2 implies that the only information about a buyer that is relevant for the ex-ante efficient allocation rule is the potential winning period of the buyer and her valuation for that period. The existence of a tentative winner and the observations in Corollaries 5 and 6 imply that the ex-ante efficient allocation rule needs to keep track of only one buyer—the tentative winner. In states where a tentative winner exists, the object is allocated to her if $\theta^{i^*_t} = t$; otherwise the object is stored. In states where no tentative winner exists, the object is disposed of if $V_t(\eta_{>t}(\emptyset)) \leq 0$, and it is stored if $V_t(\eta_{>t}(\emptyset)) > 0$. In future periods, the stored object is allocated either to a buyer who arrives in the future, or to the current tentative winner.

Comparisons between Buyers. The fundamental insight behind Corollary 5 is that the ex-ante efficient allocation rule defines a complete order on the space of vectors of valuations S_t for each period t. The buyer who ranks highest in this order is the tentative winner. Corollary 6 implies that the ranking of buyers is time consistent—that is, if two buyers $i, j \in I_t$ have potential winning periods $\theta^i, \theta^j > t$ and i ranks higher than j in period t, then this ranking is not reversed in period $t+1$. If $\theta^i = \theta^j$, this is obvious because buyer i ranks higher than buyer j if and only if either $v^i_{\theta^i} > v^j_{\theta^i}$, or $v^i_{\theta^i} = v^j_{\theta^i}$ and $i < j$. The following definition defines a score for each buyer that (i) makes explicit the ranking of buyers with different potential winning periods and (ii) allows to rank buyers with different arrival times.

Definition 7. For a buyer i, the **score** π^i is defined by

$$v^i_{\theta^i} = V_{\theta^i}(0,\ldots,0,\pi^i) \qquad \text{if } 0 < \theta^i < T,$$

by $\pi^i = v^i_T$ if $\theta^i = T$, and by $\pi^i = 0$ if $\theta^i = 0$.

The score of a buyer with $\theta^i \neq 0$ is defined such that in the potential winning period θ^i, the expected surplus of the efficient allocation rule in a state $(h_t, 1)$ with $I_t = \{1\}$ and $v^1 = (0,\ldots,0,\pi^i)$ is equal to the valuation of buyer i. The following Theorem shows that (i) a similar property holds more generally in all periods until the potential winning period; and (ii) that the buyer with the highest score is the tentative winner.

Theorem 8. *(i) For $t < T$, and all $i \in I_t$ with $\theta^i \geq t$,*[9]

$$\max\left\{v^i_t, V_t(v^i_{>t})\right\} = V_t(0,\ldots,0,\pi^i). \tag{3.3}$$

(ii) Let h_t be a history for which $x^*_s(h_s,1) = 0$ for all $s < t$—that is, with arrivals given by h_t, the state in period t is $(h_t, 1)$. If a tentative winner exists in state $(h_t, 1)$, then it satisfies

$$i^*_t \in I^*_t := \arg\max_{i \in I_t} \pi^i.$$

The tie-breaking rule from (3.1) selects $i^*_t = \min\left\{i \in I^*_t \,\middle|\, \theta^i \leq \theta^j, \forall j \in I^*_t\right\}$.

Part (i) of this Theorem states that for any buyer i, whose potential winning period is the current period or lies in the future, the expected welfare with $i^*_t = i$ is the same as the expected welfare in a hypothetical state $(\tilde{h}_t, 1)$ with $\tilde{I}_t = \{1\}$ and $\tilde{v}^1 = (0,\ldots,0,\pi^i)$. This allows to compare buyers with potential winning periods in terms of their scores. If the potential winning periods of all buyers $i \in I_t$ are t or later, then the tentative winner in period t must be the buyer with the highest score. If, on the other hand, the potential winning period of the buyer with the highest score is in the past, then (3.3) does not apply.[10] In this case, however, the object must have been allocated in the past, because the buyer already had the highest score in her potential winning period. This leads to part (ii) of the Theorem which states that whenever the object is still available and a tentative winner exists, then she must have the highest score among all buyers.

4. Implementation of the Ex-Ante Efficient Allocation Rule

The goal of this section is to design a mechanism that implements the efficient allocation rule under asymmetric information and satisfies Properties A to D. The main focus is on

[9] If $\theta^i = 0$, we have $\max\left\{v^i_t, V_t(v^i_{>t})\right\} = \max\{0, V_t(0,\ldots,0)\} \geq 0$ which is strictly greater than $V_t(0,\ldots,0,\pi^i) = V_t(0,\ldots,0)$ if $V_t(0,\ldots,0) < 0$.

[10] If $0 < \theta^i < t$, then the score does not reflect the expected value of the efficient allocation if only buyer i is present because π^i is calculated based on i's valuation for the potential winning period, which lies in the past.

the construction of a *direct mechanism* that achieves this goal (Section 4.3). Section 4.5 informally discusses an indirect implementation via a sequence of ascending clock auctions.

4.1. **The Informational Environment.** So far, we have analyzed the efficient allocation rule without regard to asymmetric information. The analysis in Section 3 only imposed the informational constraint that allocation decisions cannot rely on foresight about the realized number of buyers that arrive in the future, and the realized valuations of these buyers. Now we will consider the case of asymmetric information about the arrival times and the valuations of the buyers. Each buyer privately learns her vector of valuations—her type—when she arrives. The seller does not observe the number of buyers that arrive in a period and does not know their types. Except for the knowledge about her own type and arrival time, no buyer has an informational advantage over the seller. In particular, buyers do not observe the arrival times of other buyers or the types of the other buyers. The distributions ρ_t, and Φ_t are common knowledge.

4.2. **Incentive Compatible Direct Mechanisms.** A **direct mechanism** (S, x, y) consists of signal spaces $S = (S_t)_{t\in\{1,\ldots,T\}}$, an allocation rule x, and a payment rule y. The signal space for period t is given by the type-space of buyers that arrive in period t—that is, $S_t = [0, \overline{v}]^{T-t+1}$. In period t, all buyers $i \in I_t$ can report a type $\hat{v}^i \in S_t$. For simplicity, we assume that each buyer cannot make multiple reports in different periods.[11] In each period, the reports made by the buyers in the current and all previous periods form a history $\hat{h}_t$. For all t, the allocation rule $x_t(\hat{h}_t, a_t)$ determines an allocation decision for all possible histories of reports. The payment rule $y_t(\hat{h}_t, a_t) = (y^i_t(\hat{h}_t, a_t))_{i\in I_t}$ determines the payment that each buyer $i \in I_t$ has to make to the seller in period t.

Now consider a buyer i, who arrives in period t and plans to make a report $\hat{v}^i \in S_s$ in period $s \geq t$. Assuming that all other buyers report truthfully, the *winning probability* of buyer i for period $\tau \geq s$, given her planned report $\hat{v}^i \in S_s$ in period s, and conditional on h^{-i}_t and $a_t = 1$ is given by

$$q^i_\tau(\hat{v}^i|h^{-i}_t) := \text{Prob}\left[x_\tau(h_\tau, a_\tau) = i \;\middle|\; h^{-i}_t, v^i = \hat{v}^i, a_t = 1\right].$$

In this definition, a_t is omitted as an argument of the winning probability because $q^i_\tau(\hat{v}^i|h^{-i}_t)$ will only be used in states with $a_t = 1$. The *expected payment* in period τ is given by

$$p^i_\tau(\hat{v}^i|h^{-i}_t) := E\left[y^i_\tau(h_\tau, a_\tau) \;\middle|\; h^{-i}_t, v^i = \hat{v}^i, a_t = 1\right].$$

With these definitions, the *expected utility* from participating in the mechanism with a reported type $\hat{v}^i \in S_s$ and true type $v^i \in S_t$ is

$$U(v^i, \hat{v}^i|h^{-i}_t) := \sum_{\tau=s}^{T} \delta^{\tau-t}\left[v^i_\tau q^i_\tau(\hat{v}^i|h^{-i}_t) - p^i_\tau(\hat{v}^i|h^{-i}_t)\right].$$

The expected utility from truth-telling is abbreviated as $U(v^i|h^{-i}_t) := U(v^i, v^i|h^{-i}_t)$.

Definition 9. (i) A direct mechanism (S, x, y) is **(periodic ex-post) incentive compatible** if for all $1 \leq t \leq s \leq T$, all possible histories of true types h_t for which

[11] This assumption can be dropped in the context of this paper.

$x(h_\tau, 1) = 0$ for all $\tau < t$, and all reports $\hat{v}^i \in S_s$,

$$U(v^i | h_t^{-i}) \geq U(v^i, \hat{v}^i | h_t^{-i}). \tag{4.1}$$

(i) An incentive compatible direct mechanism (S, x, y) is called **efficient** if $x = x^*$.

Periodic ex-post incentive compatibility is a hybrid concept that reflects the lack of foresight of the dynamic model. Expectations are taken with respect to the types of future buyers. In this sense, it resembles Bayes-Nash incentive compatibility. With respect to past and current buyers, incentive compatibility constraints must hold for every profile of types. Therefore, ex-post incentive compatibility is required only conditional on information that is already realized at the time when a buyer has arrived.[12]

The existence of an efficient, incentive compatible direct mechanism has been shown by Parkes and Singh (2003) and by Bergemann and Välimäki (2010).[13]

Theorem 10 (Parkes and Singh, 2003; Bergemann and Välimäki, 2010)**.** *There exists an efficient incentive compatible direct mechanism* (S, x^*, y).

4.3. **The Dynamic Vickrey Auction.** The mechanisms proposed by Parkes and Singh (2003) and Bergemann and Välimäki (2010) are defined for more general models than the allocation problem studied here. Applied to the present problem, these mechanisms do not satisfy all Properties A to D (see Section 4.4). The properties of the ex-ante efficient allocation rule, however, suggest a natural definition of a simple payment rule: Only the winning buyer has to make a payment, and her payment is equal to the lowest valuation for her potential winning period that would suffice for her to win. In what follows, this valuation will be called the critical valuation of the winner. In a static environment—that is if $T = 1$, this payment rule together with the efficient allocation rule defines the Vickrey Auction. The critical valuation of the winner is equal to the second highest bid. In the dynamic model studied in this paper, the critical valuation of a buyer is given by the valuation for her potential winning period for which the score of the winner is equal to the second highest score, or a reservation score if the second highest score is to low to induce the allocation rule to store the object until the winning period.

In order to define the critical valuation formally, we first define the **reservation score for period** t by

$$r_t := \inf \{ \pi \geq 0 \,|\, \forall s \in \{1, \ldots, t-1\} : \, V_s(0, \ldots, 0, \pi) > 0 \}, \qquad \text{for } t > 1$$

and $r_1 := 0$. A buyer with score $\pi^i \leq r_t$ cannot get the object in period t: even if she has the highest score in period t, then for some $s < t$, either $V_s(0, \ldots, 0, \pi^{i_s^*}) \leq 0$, or $\theta^{I_s} = 0$

[12] Note that Definition 9 also rules out profitable deviations in which a buyer delays her report and reports different types in later periods, conditional on the valuations of buyers who have arrived in the meantime. For period T, (4.1) ensures that it is optimal to report v_T^i truthfully, for every history h_T^{-i}. This applies to buyers who arrived in period T as well as to buyers who delayed their report because the mechanism cannot distinguish between them. In period $T-1$, (4.1) rules out that a delayed but truthful report of v_T^i is a profitable deviation. Therefore in period $T-1$, it is optimal to report (v_{T-1}^i, v_T^i) truthfully and without delay. Working backwards in time it follows inductively, that (4.1) rules out all feasible reporting strategies except a truthful report in the arrival period.

[13] Athey and Segal (2007) also show implementability of the efficient allocation rule. Their model focuses on incentives in teams and budget balance. The proposed mechanism requires all agents to be available in all periods.

and $V_s(\eta_{>s}(I_s)) \leq 0$, and hence $x_s^*(h_s, a_s) = -1$ so that $a_t = 0$. Next, we define the **critical score** for history h_t, of a buyer $i \in I_t$ with $\theta^i = t$, as the minimal score that i must have in order to win in period t:

$$\underline{\pi}(h_t^{-i}) := \max\left\{r_t, \max_{j \in I_t \setminus \{i\}} \pi^j\right\}.$$

Finally, for a given critical score π, the critical valuation for period t is given by

$$c_t(\pi) := \min\left\{v_t \geq 0 \,|\, v_t \geq V_t(0, \ldots, 0, \pi)\right\}.$$

With this notation, the **critical valuation of the winner** i in state $(h_t, 1)$ can be written as $c_t(\underline{\pi}(h_t^{-i}))$ and the payment rule of the Dynamic Vickrey Auction is defined as

$$y_t^{*i}(h_t, a_t) := \begin{cases} 0, & \text{if } x_t^*(h_t, a_t) \neq i, \\ c_t(\underline{\pi}(h_t^{-i})), & \text{if } x_t^*(h_t, a_t) = i. \end{cases} \tag{4.2}$$

From Theorem 2 we know that $x_t^*(h_t, a_t) = 1$ implies $t = \theta_i$. Therefore, by Theorem 8.(ii), $c_t(\underline{\pi}(h_t^{-i})) \in [0, v_t^i]$ if $x_t^*(h_t, a_t) = 1$. Since Properties A and D are obviously fulfilled, this shows that (S, x^*, y^*) satisfies all Properties A to D. The main result of the paper is that this mechanism is also incentive compatible.

Theorem 11. *The Dynamic Vickrey Auction (S, x^*, y^*) is periodic ex-post incentive compatible and satisfies Properties A to D.*

The proof consists of two parts. In the first part, it is shown that no buyer can gain by making a report after her arrival period, instead of reporting immediately. This shows that the incentive compatibility constraint for each buyer is a static incentive compatibility constraint. To show the first part, it is used that the ex-ante efficient allocation rule never allocates to a buyer with valuation zero so that reporting $(0, \ldots, 0, \hat{v}_s^i, \ldots, \hat{v}_T^i)$ in period $t < s$ cannot lead to an earlier allocation than reporting $(\hat{v}_s^i, \ldots, \hat{v}_t^i)$.[14] It follows that the expected utility from reporting $(0, \ldots, 0, \hat{v}_s^i, \ldots, \hat{v}_T^i)$ in the arrival period t yields an expected payoff at least as high as reporting $(\hat{v}_s^i, \ldots, \hat{v}_t^i)$ in period $s > t$. The second part of the proof uses payoff equivalence for multidimensional mechanisms to show that the Dynamic Vickrey Auction is incentive compatible.

4.4. **Comparison with Dynamic Versions of the VCG Mechanism.** The idea behind dynamic versions of the VCG mechanism is that the expected discounted payments of a buyer who makes a report $\hat{v}_s \in S_s$ is equal to

$$E\left[\sum_{\tau=s}^{T} \delta^{\tau-s} y_\tau^i(h_\tau, a_\tau) \,\middle|\, h_s = (\hat{h}_s^{-i}, \hat{v}^i), a_s\right] = E\left[\sum_{\tau=s}^{T} \delta^{\tau-s} v_\tau^{x_\tau^*(h_\tau, a_\tau)} \,\middle|\, h_s = (\hat{h}_s^{-i}, v^i = (0, \ldots, 0)), a_s\right]$$
$$- E\left[\sum_{\tau=s}^{T} \delta^{\tau-s} \left\{\mathbf{1}_{\{x_\tau^*(h_\tau, a_\tau) \neq i\}} v_\tau^{x_\tau^*(h_\tau, a_\tau)}\right\} \,\middle|\, h_s = (\hat{h}_s^{-i}, \hat{v}^i), a_s\right]. \tag{4.3}$$

In the first expression on the right-hand side, welfare is maximized under the assumption that i does not value the object. This is equal to the expected surplus from the allocation

[14] This argument relies on the tie-breaking rule. With a different tie-breaking rule, the same argument can be used if, for example, the seller's storage costs are low so that for all $t < T$, $V_t(0, \ldots, 0) > 0$—that is, the seller never disposes of the object in the first $T - 1$ periods.

to the other agents if i does not make a report. The second expression is the expected surplus from the allocation to the other agents if i reports $\hat{v}^i$. The expected discounted payments must therefore be equal to the expected externality that i's report imposes on the other agents.

Since (4.3) only pins down the expected payments, there are potentially many ways to define a payment rule that satisfies (4.3). If we apply the *online VCG mechanism* proposed by Parkes and Singh (2003) to the model of this paper, only tentative winners have non-zero payments. (The others do not create an externality.) A tentative winner only has to make a payment in the period in which she wins the object or in the period where she is replaced by a new tentative winner. This period is called the commitment period. The payment equals to the expected externality that the buyer has imposed on the other agents, where the expectation is taken conditional on the state in the commitment period. By the law of iterated expectations, this payment rule satisfies (4.3). Note however, that the online VCG mechanism does not satisfy all Properties A to D. In particular, there are transactions with tentative winners who do not win the object, which violates Property A. Moreover, a tentative winner who is replaced by a new tentative winner often created a positive externality which calls for a positive transfer to that bidder. This violates Property C.

If we apply the *dynamic pivot mechanism* proposed by Bergemann and Välimäki (2010), each buyer has to make a transfer equal to the expected externality of her report at the time of the report. Therefore, as in the online VCG mechanism, only tentative winners have non-zero transfers. Again this violates Property A. The difference between the online VCG mechanism and the dynamic pivot mechanism is that in the latter, the transfer is determined in the period when a buyer makes a report and becomes tentative winner. Therefore the payment is independent of whether the buyer wins the object. Since the expected externality must be negative, buyers who first become tentative winners and are replaced later, have a negative ex-post payoff. This violates Property B.

In contrast to these mechanisms, the Dynamic Vickrey Auction proposed here deviates from the idea that every agent has to make a payment equal to her externality at some state after making a report. For example, bidders who first become tentative winners but do not win the object often impose an externality on the other agents because their reports influence whether the object is stored or not. Nevertheless, there is no transfer between such a buyer and the mechanism. Similarly, the payment of the winning bidder does not coincide with the externality that she imposes. For example, suppose that buyer i wins in period θ^i and buyer j with potential winning period $\theta^j < \theta^i$ has the second highest score which determines i's payment. Then, by Theorem 8.(i), i has to pay

$$c_{\theta^i}(\underline{\pi}(h_{\theta^i}^{-i})) = \min\left\{\hat{v}^i_{\theta^i} \geq 0 \,\middle|\, v^j_{\theta^j} \leq V_{\theta^j}(0,\dots,0,\hat{v}^i_{\theta^i},0,\dots,0)\right\}.$$

If there are no storage costs and $\delta = 1$, this payment is obviously lower than $v^j_{\theta^j}$ but the externality is equal to $v^j_{\theta^j}$. In expectation, however, the payment in the Dynamic Vickrey Auction is equal to the expected externality and (4.3) is satisfied.

4.5. **Indirect Implementations.** The efficient allocation rule can also be implemented by an ascending clock auction. The main idea is that the auctioneer has a clock that starts at a score of zero in the first period. In each period t, bidders can drop out while the clock ascends continuously and the clock stops if only one active bidder remains.[15] Then, the last active bidder is offered to buy the object for a price of $c_t(\pi^a)$, where π^a is the score at which the clock has stopped and t is the current period. If she rejects this option and $\pi^a > \min\{r_{t+1}, \dots, r_T\}$, the auction proceeds to the next period. Otherwise, the clock jumps to $\min\{r_{t+1}, \dots, r_T\}$. After the jump, the last active bidder can either remain active in which case the object is stored for one period and the auction proceeds to the next period; or the bidder can decide to drop out, in which case the object is only stored if $V_t(\eta_{>t}(\emptyset)) > 0$. This auction has an equilibrium that implements the ex-ante efficient allocation rule. Payments in the equilibrium outcome are the same as in the Dynamic Vickrey Auction.

5. Conclusion

The main contribution of this paper is to show how the payments in a dynamic mechanism can be distributed over different states of the world such that (i) expected payments ensure incentive compatibility, and (ii) the resulting mechanism satisfies Properties A to D. The construction relies on the specific properties of the ex-ante efficient allocation rule that have been derived for a specific allocation problem.

The model is restrictive in that the allocation of a single object is studied. If more than one object is at sale, it is possible to construct simple examples where bidders can win in different periods depending on the history of arrivals. Therefore, future research will have to concentrate on the generalization of the weaker results of Corollaries 4 to 6. Also, the model assumes that the type of a buyer is fixed (no learning) and that buyers do not exit the mechanism. Finally, more general allocation problems than the allocation of a private good could be studied. These questions are left for future research.

Appendix

A.1. **Proof of Theorem 2.** Theorem 2 is proven by induction over T. For $T = 1$ the Theorem is obvious. The induction argument uses Corollaries 4 and 5 which are first proven under the assumption that Theorem 2 holds for T. It is then shown that Theorem 2 also holds for $T + 1$.

Proof of Corollary 4. (i) Fix $t \in \{1, \dots, T\}$ and a state $(h_t, 1)$ with $\theta^{I_t} \neq 0$. This implies that $I_t \neq \emptyset$. Consider an alternative fictitious state $(\hat{h}_t, 1)$ with a single fictitious buyer $\hat{I}_t = \{1\}$ with valuations $\hat{v}^1 = \eta(I_t)$. By Theorem 2, the fictitious buyer has a unique potential winning period which coincides with θ^{I_t}. Following state $(\hat{h}_t, 1)$, the ex-ante efficient allocation rule either allocates to buyer 1 in period θ^{I_t}, or it allocates to some buyer who arrives in a future period. Since the decision when to allocate only depends on the first order statistics, for each continuation, the allocation following state $(\hat{h}_t, 1)$ takes place in the same period as the allocation following state $(h_t, 1)$. Therefore, the only period in which a buyer from I_t can get the object is θ^{I_t}. Hence, if a buyer from I_t wins, it must be $i_t^* = \min\left\{i \in I_t \,\middle|\, v^i_{\theta^{I_t}} = \eta_{\theta^{I_t}}(I_t)\right\}$.

[15]Ties are ignored in this description.

If $\theta^{I_t} = 0$ and $I_t \neq 0$, the efficient allocation rule will never allocate to the fictitious buyer by Theorem 2. Hence, no buyer in I_t can get the object in or after state $(h_t, 1)$.

(ii) If a tentative winner exists in state $(h_t, 1)$ then Theorem 2 implies that $\max\{v_t^{i_t^*}, V_t(v_{>t}^{i_t^*})\} = \max\{\eta_t(I_t), V_t(\eta_{>t}(I_t))\}$. (To see this, note that $\max\{\eta_t(I_t), V_t(\eta_{>t}(I_t))\}$ cannot depend on the valuations of buyers $j \in I_t$, $j \neq i_t^*$, since these buyers will never get the object in or after state $(h_t, 1)$.) If $\theta^{i_t^*} = t$, $v_t^{i_t^*} \geq V_t(v_{>t}^{i_t^*})$ by the definition of $\theta^{i_t^*}$ and hence $\eta_t(I_t) = v_t^{i_t^*} \geq V_t(\eta_{>t}(I_t))$. Finally, $\theta^{i_t^*} \neq 0$ implies $v_{\theta^{i_t^*}}^{i_t^*} > 0$ and hence $x^*(h_t, 1) = i_t^*$. Similar arguments lead to the allocations stated in the Corollary for the other cases. □

Proof of Corollary 5. If a tentative winner exists in state $(h_t, 1)$, the expected surplus achieved by the ex-ante efficient allocation rule conditional on state $(h_t, 1)$, is $\max\{\eta_t(I_t), V_t(\eta_{>t}(I_t)\} = \max\{v_t^{i_t^*}, V_t(v_{>t}^{i_t^*})\}$. Define $B := \{i \in I_t \mid \max\{v_t^i, V_t(v_{>t}^i)\} = \max\{\eta_t(I_t), V_t(\eta_{>t}(I_t))\}\}$. The expected surplus achieved in state $(\tilde{h}_t, 1)$ must be greater or equal than in state $(h_t, 1)$. Therefore, $\tilde{i}_t^* \in B \cup A$. But if $\tilde{i}_t^* \in B$, then the efficient allocation rule selects the buyer in B with the earliest potential winning period and the lowest index—that is, $\tilde{i}_t^* = i_t^*$. □

Proof of Corollary 6. If $\tilde{I}_{t+1} = I_t$—that is, if no new buyers arrive in period t, then $i_{t+1}^* = i_t^*$. By Corollary 5, this implies that $i_{t+1}^* \in (\tilde{I}_{t+1} \setminus I_t) \cup \{i_t^*\}$ for all $(\tilde{h}_t, 1)$ where $\tilde{h}_t$ is a continuation of h_t. □

Proof of Theorem 2. The result is proven by induction over T. For $T = 1$, the result is trivial. Assume that the Theorem is true for allocation problems with $T-1$ periods. The statement is shown for T in four steps.

Step 1: If $x^*(h_1, 1) = 0$, then the state in period two is $(h_2, 1)$ where h_2 is a continuation of h_1. The allocation problem starting at state $(h_2, 1)$ is a problem with $T-1$ periods. Therefore, by the induction hypothesis, the Theorem holds for all buyers $i \notin I_1$.

Step 2: If a buyer $i \in I_1$ gets the object in period one, then $v_1^i = \eta_1(I_1) \geq V_1(\eta_{>1}(I_1)) \geq V_1(v_{>1}^i)$ and $v_1^i > 0$. Therefore $\theta^i = 1$.

Step 3: If a buyer $i \in I_1$ gets the object in some period $t > 1$, then $x^*(h_1, 1) = 0$. The allocation problem starting at state $(h_2, 1)$ is a problem with $T-1$ periods. Suppose that no tentative winner exists in state $(h_2, 1)$ and hence $\theta^{I_2} = 0$. By Corollary 4.(i), this is a contradiction to the assumption that i gets the object in some period $t > 1$. Therefore, we can assume that there exists a tentative winner $i_2^* \in I_2$ in period two. If i gets the object in some period $t > 1$, then $i_2^* = i$. Moreover, if $i_2^* = i$ for some state $(h_2, 1)$, then by Corollary 5, $i_2^* = i$ also if $I_2 \setminus I_1 = \emptyset$. Invoking Corollary 5 again, we have $i_2^* \in (I_2 \setminus I_1) \cup \{i\}$ for all states $(h_2, 1)$ where h_2 is a continuation of h_1. This implies that if i can get the object in some continuation of h_1, then i is the only buyer from I_1 who can possibly get the object for any continuation of h_1. Therefore $V_1(\eta_{>1}(I_1)) = V_1(v_{>1}^i)$. From $x^*(h_1, 1) = 0$ we can conclude that $\eta_1(I_1) < V_1(\eta_{>1}(I_1))$ and since $v_1^i \leq \eta_1(I_1)$, we have $v_1^i < V_1(v_{>1}^i)$. This implies that $\theta^i \neq 1$.

Step 4: Let θ^j be the potential winning period of a fictitious buyer j who arrives in period two and has the same valuations for periods $2, \ldots, T$ as i—that is, $v^j = (v_2^i, \ldots, v_T^i)$. By the induction hypothesis, j can only win in period θ^j. But if $x^*(h_1, 1) = 0$, the only period in which i can win is also θ^j because both buyers have the same valuations for the periods $2,\dots,T$. Finally, $v^i_1 < V_1(v^i_{>1})$, and the fact that θ^j is the potential winning period of j imply that $\theta^i = \theta^j$. Hence, if $i \in I_1$ gets the object in some period $t > 1$, then $t = \theta^i$. □

A.2. **Proof of Theorem 8.**

Proof of Theorem 8. (i) The result is shown by induction over t. For $t = T-1$ we have to show that $\max\{v^i_{T-1}, V_{T-1}(v^i_T)\} = V_{T-1}(\pi^i)$ if $\theta^i \geq T-1$. If $v^i_{T-1} \geq V_{T-1}(v^i_T)$, the potential winning period is $\theta^i = T-1$, and hence $\max\{v^i_{T-1}, V_{T-1}(v^i_T)\} = v^i_{T-1} = V_{T-1}(\pi^i)$ by the definition of the score. If $v^i_{T-1} < V_{T-1}(v^i_T)$, the potential winning period is $\theta^i = T$, and hence $v^i_T = \pi^i$ by the definition of the score. This shows the desired result for $t = T-1$.

Now suppose that (i) holds for $t+1,\dots,T-1$. We show that (i) also holds for t. If $v^i_t \geq V_t(v^i_{>t})$, then $\theta^i = t$ and $v^i_t = V(0,\dots,0,\pi^i)$ by definition. If $v^i_t < V_t(v^i_{>t})$, then $\theta^i > t$. Since there is a unique tentative winner for period $t+1$, we have

$$V_t(v^i_{>t}) = E\left[\max\left\{\max_{j\in I_{t+1}\setminus I_t}\left(\max\{v^j_{t+1}, V_{t+1}(v^j_{>t+1})\}\right), \max\{v^i_{t+1}, V_{t+1}(v^i_{>t+1})\}\right\}\right]$$

Since $\theta^i > t$, the induction hypothesis implies that $\max\{v^i_{t+1}, V_{t+1}(v^i_{>t+1})\} = V_{t+1}(0,\dots 0,\pi^i) = \max\{0, V_{t+1}(0,\dots 0,\pi^i)\}$. Plugging this into the above equation we get

$$\begin{aligned} V_t(v^i_{>t}) &= E\left[\max\left\{\max_{j\in I_{t+1}\setminus I_t}\left(\max\{v^j_{t+1}, V_{t+1}(v^j_{>t+1})\}\right), \max\{0, V_{t+1}(0,\dots 0,\pi^i)\}\right\}\right] \\ &= V_t(0,\dots 0,\pi^i) \end{aligned}$$

This shows the desired result for t.

(ii) Since $a_t = 1$, Corollary 4.(ii) implies that for all periods $s < t$ in which a tentative winner exists, we must have $\theta^{i^*_s} > s$. Let $\underline{s}$ be the first period in which a tentative winner exists. Then $\pi^i = 0$ for all $i \in I_{\underline{s}-1}$, and a tentative winner exists for all $s \geq \underline{s}$. Moreover, for $s \geq \underline{s}$, we have $\max\{v^{i^*_s}_s, V_s(v^{i^*_s}_{>s})\} \geq \max\{v^i_s, V_s(v^i_{>s})\}$ for all $i \in (I_s \setminus I_{s-1}) \cup \{i^*_{s-1}\}$ where we set $\{i^*_{s-1}\} = \emptyset$ if $s = \underline{s}$. This and part (i) imply that $V_s(0,\dots,0,\pi^{i^*_s}) \geq V_s(0,\dots,0,\pi^i)$ and hence $\pi^{i^*_s} \geq \pi^i$ for all $i \in (I_s \setminus I_{s-1}) \cup \{i^*_{s-1}\}$, since $\theta^i \geq s$ for all $i \in (I_s \setminus I_{s-1}) \cup \{i^*_{s-1}\}$. This also implies that $\pi^{i^*_t} \geq \pi^{i^*_{t-1}} \geq \dots \geq \pi^{i^*_{\underline{s}}}$ and hence we have $\pi^{i^*_t} = \max_{i\in I_t} \pi^i$. Among the buyers with the highest scores—that is, the set I^*_t—the tie-breaking rule assumed in (3.1) first selects the buyers with the lowest potential winning period. Among these, it selects the buyer with the lowest index. □

A.3. **Proof of Theorem 11.**

Proof of Theorem 11. The proof consists of two parts. The first part shows that in the Dynamic Vickrey Auction, no buyer can gain by delaying her report after her arrival period. The second part shows that in the arrival period, no buyer has an incentive to deviate from truth-telling.

For the first part, fix a period t and suppose that $x^*_s(h_s, 1) = 0$ for all $s < t$ so that the state in period t is $(h_t, 1)$. Fix $i \in I_t \setminus I_{t-1}$. Suppose that all other buyers report their types truthfully in their respective arrival periods.

Claim 12. Buyer i cannot gain by reporting a valuation greater than the true valuation for any period: For all $s \geq t$,

$$\max_{\substack{\hat{v}^i \in S_s \\ \forall \sigma \geq s:\, \hat{v}^i_\sigma \leq v^i_\sigma}} U(v^i, \hat{v}^i | h_t^{-i}) = \max_{\hat{v}^i \in S_s} U(v^i, \hat{v}^i | h_t^{-i})$$

Proof. Suppose by contradiction that the left-hand side is strictly smaller than the right-hand side. Let $\hat{v}^i$ be a maximizer for the right maximization problem. By Theorem 2, we can take $\hat{v}^i$ to be of the form $(0, \ldots, 0, \hat{v}^i_{\hat{\theta}^i}, 0, \ldots, 0) \in S_s$, where $\hat{\theta}^i$ is the potential winning period of $\hat{v}^i$. We can assume that $\hat{\theta}^i \neq 0$ because if $\hat{\theta} = 0$, $U(v^i, \hat{v}^i | h_t^{-i}) = 0$. Let $\hat{\pi}^i$ be the score of $\hat{v}^i$, and let $\breve{\pi}^i$ be the score of $(0, \ldots, 0, v^i_{\hat{\theta}^i}, 0, \ldots, 0) \in S_s$. By assumption, we have $\hat{v}^i_{\hat{\theta}^i} > v^i_{\hat{\theta}^i}$ and hence $\hat{\pi}^i \geq \breve{\pi}^i$. With both reports, i can only win in period $\hat{\theta}^i$, so we compare the payoffs of the two reports for continuations $\tilde{h}_{\hat{\theta}^i}$ of h_t. If $\breve{\pi}^i \leq \underline{\pi}(\tilde{h}^{-i}_{\hat{\theta}^i}) \leq \hat{\pi}^i$, then reporting $(0, \ldots, 0, v^i_{\hat{\theta}^i}, 0, \ldots, 0) \in S_s$ leads to a payoff of zero for i (she does not get the object and makes no payment), and reporting $\hat{v}^i$ leads to a non-positive payoff (if she gets the object, then the payment is larger than the true valuation). If $\underline{\pi}(\tilde{h}^{-i}_{\hat{\theta}^i}) < \breve{\pi}^i \leq \hat{\pi}^i$, then with both reports, i gets the object and the payment is the same. Finally, if $\breve{\pi}^i \leq \hat{\pi}^i < \underline{\pi}(\tilde{h}^{-i}_{\hat{\theta}^i})$, both reports lead to a payoff of zero. Therefore, $U(v^i, \hat{v}^i | h_t^{-i}) \leq U(v^i, (0, \ldots, 0, v^i_{\hat{\theta}^i}, 0, \ldots, 0) | h_t^{-i})$, which is a contradiction, and the claim is valid. □

Next, fix $t < T$, h_t, $i \in I_t \setminus I_{t-1}$, $s > t$, and $\hat{v}^i \in S_s$ where $\hat{v}^i_\sigma \leq v^i_\sigma$ for all $\sigma \geq s$. We show that $U(v^i, \tilde{v}^i, h_t^{-i}) \geq U(v^i, \hat{v}^i, h_t^{-i})$ for $\tilde{v}^i = (0, \ldots, 0, \hat{v}^i_s, \ldots, \hat{v}^i_T) \in S_t$. To see this, note first that with a report $\tilde{v}^i$ (made in period t!) buyer i cannot get the object in any of the periods $t, \ldots, s-1$, because the efficient allocation rule never allocates to a buyer with zero valuation for the period in which the object is allocated. Now we distinguish two cases according to the state in period s. *Case 1: If i does not make a report before period s, the object is not available in period s ($a_s = 0$).* In this case, the payoff of i will be zero if she reports $\hat{v}^i$ and non-negative if she reports $\tilde{v}^i$ since $\tilde{v}^i_\sigma \leq v^i_\sigma$, so that the payment will not exceed her true valuation v^i_σ if she gets the object in some period $\sigma \geq s$. *Case 2: The object is still available in period s if i does not make a report before period s.* In this case, the object would also be available if i reported $\tilde{v}^i$ in period t. Moreover, from period s onwards, the mechanism makes exactly the same decisions with the two reports (except for ties with other buyers in which case payoffs are zero for both reports). Therefore, the payoff will be the same for the two reports. To summarize, we have shown that for every possible continuation of the history h_t, the payoff from reporting $\tilde{v}^i$ in the Dynamic Vickrey Auction is at least as high as the payoff from reporting $\hat{v}^i$. This implies that $U(v^i, \tilde{v}^i | h_t^{-i}) \geq U(v^i, \hat{v}^i | h_t^{-i})$ and we have completed the first part of the proof.

For the second part, it remains to show that $v^i \in \arg\max_{\hat{v}^i \in S_t} U(v^i, \hat{v}^i | h_t^{-i})$ for all t, h_t and $i \in I_t \setminus I_{t-1}$, where h_t is a history of arrivals for which $x^*_s(h_s, 1) = 0$ for all $s < t$ so that the state in period t is $(h_t, 1)$. By Theorem 10 the ex-ante efficient allocation rule is implementable. Therefore, Proposition 1 in Jehiel et al. (1999) implies that the Dynamic Vickrey Auction is incentive compatible if for all t, all h_t and all $i \in I_t \setminus I_{t-1}$, the expected payoff from participation with a truthful report in the arrival period is given by

$$U(v^i|h_t^{-i}) = \int_0^1 \sum_{\tau=t}^{T} \delta^{\tau-t} \left[q_\tau^i(\gamma(s)|h_t^{-i}) \, \frac{d\gamma_\tau(s)}{ds} \right] ds \tag{A.1}$$

where $\gamma : [0,1] \to [0,\overline{v}]^{T-t+1}$ parametrizes a piecewise smooth curve that connects the origin with v^i.

Implementability of the ex-ante efficient allocation rule implies that $q^i(\cdot|h_t^{-i}) = (q_t^i(\cdot|h_t^{-i}), \ldots, q_T^i(\cdot|h_t^{-i}))$ is a conservative vector field (Jehiel et al., 1999). Therefore, γ can be chosen such that it is composed of two straight lines. The first connects the origin with $(0, \ldots, 0, v_{\theta_i}^i, 0, \ldots, 0)$ and the second line connects $(0, \ldots, 0, v_{\theta_i}^i, 0, \ldots, 0)$ with v^i:

$$\gamma(s) := \begin{cases} 2s(0, \ldots, 0, v_{\theta^i}^i, 0, \ldots, 0), & \text{if } 0 \le s \le \frac{1}{2}, \\ (2-2s)(0, \ldots, 0, v_{\theta^i}^i, 0, \ldots, 0) + (2s-1)v^i, & \text{if } \frac{1}{2} < s \le 1. \end{cases}$$

If $\theta^i = 0$, (A.1) holds because $U(v^i|h_t^{-i}) = 0$ and $q^i(\gamma(s)|h_t^{-i}) = 0$ so that the right-hand side of (A.1) also equals zero. Similarly, if $\theta^i \ge t$ but $v_{\theta^i}^i < c_{\theta^i}(r_{\theta^i})$, then $q^i(\gamma(s)|h_t^{-i}) = 0$ and $U(v^i|h_t^{-i}) = 0$ so that (A.1) is satisfied.

Next, we consider the case that $\theta^i \ge t$ and $v_{\theta^i}^i \ge c_{\theta^i}(r_{\theta^i})$. For $s \le 1/2$ we have $\frac{d\gamma_\tau(s)}{ds} = 0$ for $\tau \ne \theta^i$, and for $s > 1/2$ we have $\frac{d\gamma_{\theta^i}(s)}{ds} = 0$ and $q_\tau^i(\gamma(s)|h_t^{-i}) = 0$ for $\tau \ne \theta^i$ because the potential winning period of the type $\gamma(s)$ is θ^i. Hence (A.1) becomes

$$\begin{aligned} U(v^i|h_t^{-i}) &= \delta^{\theta_i - t} \int_0^{\frac{1}{2}} q_{\theta^i}^i(\gamma(s)|h_t^{-i}) \, 2v_{\theta^i}^i \, ds \\ &= \delta^{\theta_i - t} \int_0^{v_{\theta^i}^i} q_{\theta^i}^i((0, \ldots, 0, w_{\theta^i}^i, 0, \ldots, 0)|h_t^{-i}) dw_{\theta^i}^i, \end{aligned}$$

where we have used a change of variables to obtain the second line.

Next, we write the score of $(0, \ldots, 0, w_{\theta^i}^i, 0, \ldots, 0)$ as $\pi_{\theta^i}(w_{\theta^i}^i)$ and denote the cumulative distribution function of the highest score of the other buyers in period θ^i, conditional on h_t^{-i}, by $G_{\theta^i}^i(\cdot \mid h_t^{-i})$. With this notation we can further rearrange (A.1):[16]

$$\begin{aligned} U(v^i|h_t^{-i}) &= \delta^{\theta_i - t} \int_{c_{\theta_i}(r_{\theta_i})}^{v_{\theta^i}^i} G_{\theta^i}^i(\pi_{\theta^i}(w_{\theta^i}^i) \mid h_t^{-i}) \, dw_{\theta^i}^i, \\ &= \delta^{\theta_i - t} \int_{c_{\theta^i}(r_{\theta_i})}^{v_{\theta^i}^i} \int_0^{\pi_{\theta^i}(w_{\theta^i}^i)} dG_{\theta^i}^i(\pi \mid h_t^{-i}) \, dw_{\theta^i}^i, \\ &= \delta^{\theta_i - t} \int_{c_{\theta^i}(r_{\theta_i})}^{v_{\theta^i}^i} \left\{ \int_0^{\pi^i} \mathbf{1}_{\left\{\pi_{\theta^i}(w_{\theta^i}^i) \ge \pi\right\}} dG_{\theta^i}^i(\pi \mid h_t^{-i}) \right\} dw_{\theta^i}^i, \\ &= \delta^{\theta_i - t} \int_0^{\pi^i} \left\{ \int_{c_{\theta^i}(r_{\theta_i})}^{v_{\theta^i}^i} \mathbf{1}_{\left\{\pi_{\theta^i}(w_{\theta^i}^i) \ge \pi\right\}} \, dw_{\theta^i}^i \right\} dG_{\theta^i}^i(\pi \mid h_t^{-i}), \\ &= \delta^{\theta_i - t} \int_0^{\pi^i} \left\{ v_{\theta^i}^i - c_{\theta^i}^i \left(\max\{r_{\theta_i}, \pi\}\right) \right\} dG_{\theta^i}^i(\pi \mid h_t^{-i}), \\ &= v_{\theta^i}^i \, \delta^{\theta_i - t} \int_0^{\pi^i} dG_{\theta^i}^i(\pi \mid h_t^{-i}) - \delta^{\theta_i - t} \int_0^{\pi^i} c_{\theta^i}^i(\max\{r_{\theta_i}, \pi\}) \, dG_{\theta^i}^i(\pi \mid h_t^{-i}) \end{aligned}$$

[16]Ties are ignored in this derivation to simplify notation.

$$= \delta^{\theta_i - t} \left[q^i_{\theta^i}(v^i_{\theta^i} | h^{-i}_t)\, v^i_{\theta^i} - p^i_{\theta^i}(v^i_{\theta^i} | h^{-i}_t) \right]$$

In the third equality the upper bound of integration in the inner integral is extended to $\pi^i = \pi^{\theta^i}(v^i_{\theta^i})$. In the forth equality the order of integration is changed. Since $v^i_{\theta^i} \geq c^i_{\theta^i}(r_{\theta^i})$, and $\pi^{\theta^i}(w^i_{\theta^i}) \geq \pi$ is equivalent to $w^i_{\theta^i} \geq c^i_{\theta^i}(\pi)$, the inner integral can be computed easily which yields the fifth equality. Notice that the last line is the expected payoff from a truthful report in the arrival period in the Dynamic Vickrey Auction which completes the proof. □

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