Special Relativity, Lecture 9: Resolving Paradoxes

Topics: The twin Paradox; The outwards Journey; The Return Trip; Resolution of paradox

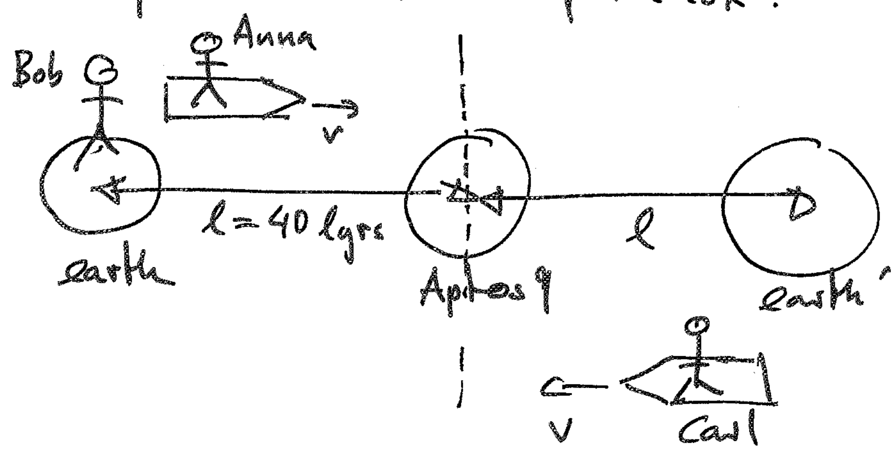
Aim: To show how apparent paradoxes can be resolved by a careful consideration of length contraction, time dilation and relative simultaneity.

9.1 The Twin Paradox

One twin stays on earth, the second travels away at high speed. Because of time dilation each sees other aging more slowly than himself. When the "moving" twin returns, each would claim that the other is younger. Contradiction!

In fact, there is no paradox. The situation is not symmetrical: the moving twin must turn around. This is an acceleration so the moving twin does not remain in the same inertial frame.

Our setup to resolve the paradox:



Carl is introduced to help resolve the paradox (3rd frame of reference)

Events:
1) Anna + Bob born, Anna leaves earth
2) Anna + Carl arrive at Aptos 9, Anna jumps on Carl's ship
3) Anna + Carl arrive back at earth

9.2. The outwards Journey

Anna born on earth, leaves for Aptos9

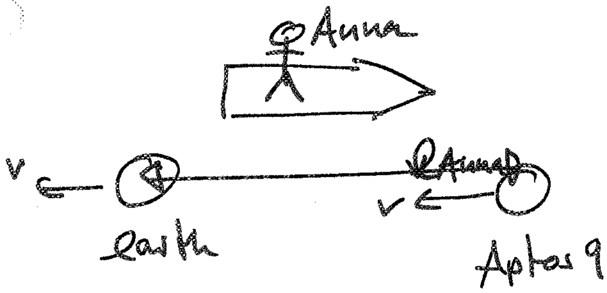
Q: Can she arrive as she turns 30?

Might seem that if it takes light 40 yrs at velocity c and Anna travels at less than she cannot do it.

Problem is essentially the same as the muon example.

- Length contraction according to Anna
- Time dilation according to Bob

Anna's point of view:



distance contracted to

$$l_{Anna} = \frac{l_{Bob}}{\gamma_v} = \frac{l}{\gamma_v}$$

At what speed does Anna have to travel to reach as she turns 30?

$\Delta t_{Anna} = 30$ yrs $\qquad \Delta t_{Anna} = \frac{l_{Anna}}{v} = \frac{l}{v\gamma_v} = \frac{l}{v}\sqrt{1-\frac{v^2}{c^2}}$

$$\Rightarrow \left(\frac{\Delta t_{Anna}}{l}\right)^2 v^2 = 1 - \frac{v^2}{c^2} \quad \Leftrightarrow \quad \frac{9}{16}\frac{v^2}{c^2} = 1 - \frac{v^2}{c^2}$$

$$\left(\frac{\Delta t_{Anna}}{l}\right)^2 = \left(\frac{30\text{ yrs}}{40\, c\text{ yrs}}\right)^2$$

$$\Rightarrow \boxed{v = \frac{4}{5}c}$$

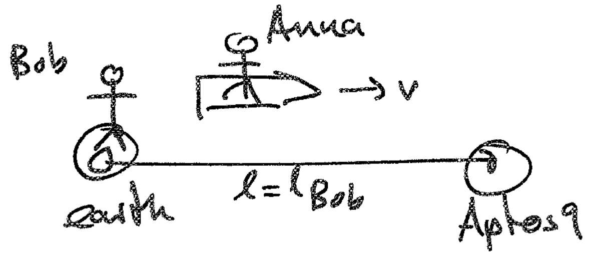
Therefore $\boxed{\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{16}{25}}} = \frac{5}{3}}$

In her frame, the distance between the earth and Aptos9 is $\boxed{l_{Anna} = \frac{40\text{ lyrs}}{5/3} = 24\text{ lyrs}}$

Bob's point of view:

Anna measures that it takes her $\Delta t_{Anna} = 30\,yrs$ to reach Aptos9.

How long does Bob think she takes?



$$\boxed{\Delta t_{Bob} = \frac{l}{v} = \frac{40\,lyrs}{4/5\,c} = 50\,yrs}$$

Check consistency: According to observers in Bob's frame, Anna's clock ticks more slowly because of time dilation

$$\boxed{\Delta t_{Anna\ to\ Bob} = \frac{\Delta t_{Bob}}{\gamma} = \frac{50\,yrs}{5/3} = 30\,yrs}$$

Anna and Bob agree on event: Anna arrives at age 30

Time dilation for Bob vs. Length Contraction for Anna

Back to Anna:

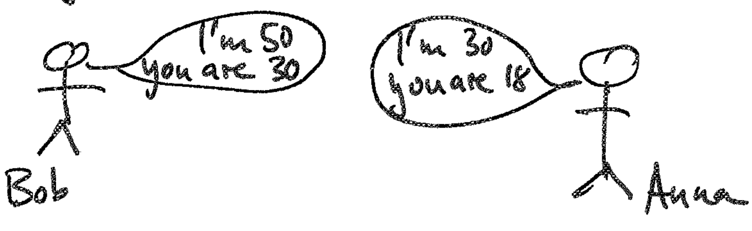
Q: What age does Anna measure Bob to be when she arrives at Aptos9?

By symmetry, Anna sees Bob's clock to move more slowly than hers. She measures Bob to be

$$\boxed{\Delta t_{Bob\ to\ Anna} = \frac{\Delta t_{Anna}}{\gamma} = \frac{30\,yrs}{5/3} = 18\,yrs}$$

This looks like a paradox!

If they could meet



But they can't meet without one of them changing direction!

Summary so far:

Anna's point of view:

Outward journey

$l_{Anna} = 24$ lyrs

$\Delta t_{Anna} = 30$ yrs

Anna sees Bob age to

$\Delta t_{Bob\ to\ Anna} = 18$ yrs

Bob's point of view:

Outward journey

$l_{Bob} = 40$ lyrs

$\Delta t_{Bob} = 50$ yrs

Bob concludes that Anna's age is

$\Delta t_{Anna\ to\ Bob} = 30$ yrs

(they agree over this: $\Delta t_{Anna} = 30$ yrs ↔ $\Delta t_{Anna\ to\ Bob} = 30$ yrs)

($\Delta t_{Bob\ to\ Anna} = 18$ yrs vs. $\Delta t_{Bob} = 50$ yrs:) no contradiction since they can't meet to compare notes without one of them changing frame

9.3 The Return Journey

In order to make clear exactly what happens, we will use the trick of considering Carl.
Carl reaches Aptos9 at the same time as Anna, travelling at $v = \frac{4}{5}c$ towards the earth.
We will work backwards from the point where Carl (with Anna on board) arrives at the earth — an event that he and Bob will agree about — to deduce what happens as Carl passes Aptos9

Bob's point of view:

Anna took 50 yrs to reach Aptos9 as viewed in Bob's frame

⇒ Carl takes 50 yrs to reach earth from Aptos9 as viewed in Bob's frame

⇒ Bob is 100 yrs old when Carl (+ Anna) arrives

> Note: This is a property of the event that Carl arrives and must be agreed on by Carl and Bob

Carl's point of view:

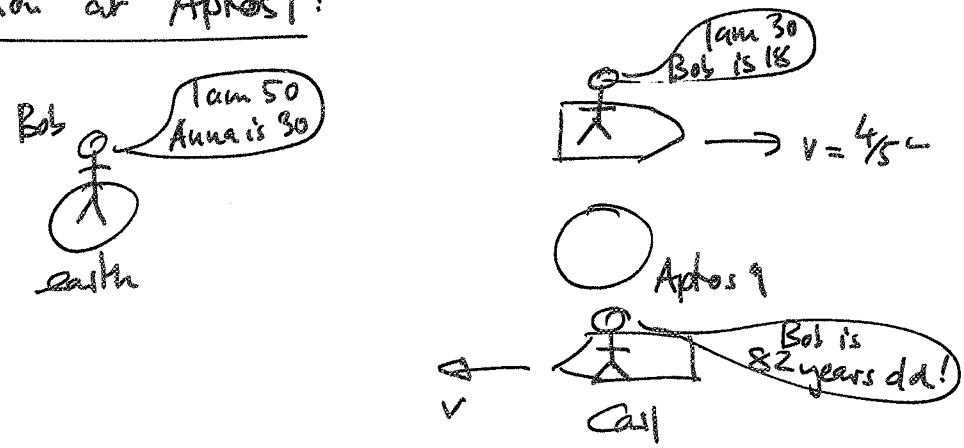
In Carl's frame, it takes 30 yrs to travel from Aptos 9 to earth (same as for Anna to travel from earth to Aptos 9)

From Carl's point of view, Bob moves at $v = \frac{4}{5}c$ so Bob's clock is time dilated and advances more slowly by a factor of $1/\gamma = 3/5$

$\Rightarrow$ Carl deduces that the journey from Aptos 9 to earth in Bob's frame took $3/5 \cdot 30$ yrs $= 18$ yrs

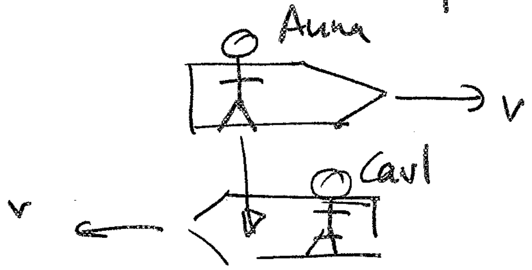
$\Rightarrow$ In order to agree upon Bob's age on arrival (100 yrs) Carl concludes that Bob has been $100 - 18 = 82$ yrs old when Carl left Aptos 9

Situation at Aptos 9:



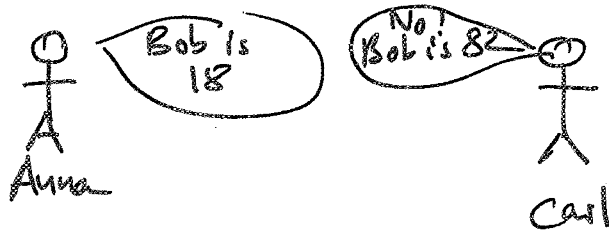
All are correct since all are in different frames of reference

Assume that Anna swaps ships at Aptos 9



- On returning to earth, Bob really is older (100) than Anna (60). This is not a paradox since the problem is not symmetric. Anna has switched inertial frames by jumping onto Carl's ship. This is an acceleration. This resolves the twin paradox.

- During her acceleration, Anna changes from a frame of reference in which Bob is deduced to be 18 to a frame in which he is deduced to be 82!



Is there a paradox?

No, this can be easily understood in terms of relative simultaneity. Resolve it!

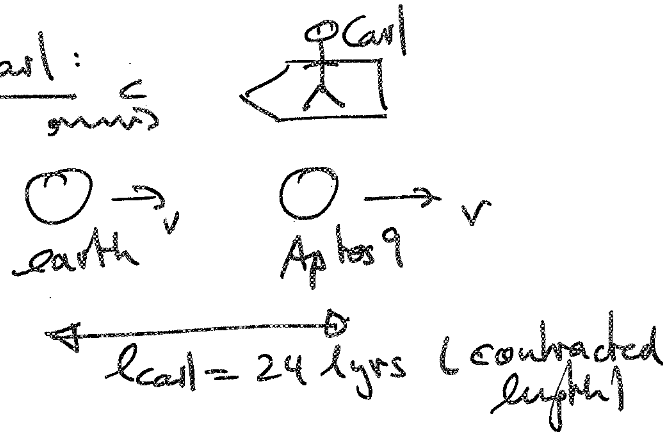
- Suppose that Bob sends a light beam timed to arrive at Aptos9 at the same time as Anna + Carl
- Light beam takes 40 yrs in Bob's frame and therefore it must have been sent when Bob was 10
- Let's say the beam encodes a picture of Bob when he was 10
- At Aptos9 we have an event with the following features:

  1: Anna arrives at age 30
  2: Carl arrives simultaneously
  3: light beam arrive simultaneously
  4: light beam encodes picture of Bob aged 10

All must agree on the properties of the event, in particular that Bob is 10 in the picture.

They disagree on the age that Bob must have at the time of the event at Aptos9!

for Carl:



$$\Delta t_{carl} = \frac{l_{carl}}{c-v}$$

(light and Aptos9 approach at relative speed $c-v$)

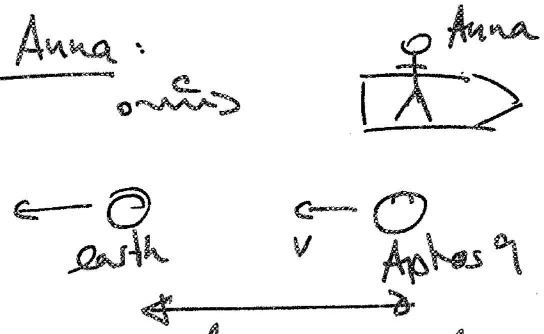
$$= \frac{24c \text{ yrs}}{c - 4/5\,c} = 120 \text{ yrs}$$

Time dilation: Carl perceives Bob's clock to run slowly by a factor of $1/\gamma = 3/5$

$$\Rightarrow \Delta t_{Bob} = \Delta t_{Carl}/\gamma = 120\,\text{yrs} \cdot 3/5 = 72\,\text{yrs}$$

$\Rightarrow$ Bob is deduced to be $10 + 72 = 82$ yrs

for Anna:



$l_{Anna} = 24$ lyrs

$$\Delta t_{Anna} = \frac{l_{Anna}}{c+v} = \frac{24c\,\text{yrs}}{c + 4/5\,c} = 13\,1/3\,\text{yrs}$$

(light and Aptos 9 approach at relative speed c+v)

Time dilation:

$$\Delta t_{Bob} = \Delta t_{Anna}/\gamma = 13\,1/3\,\text{yrs} \cdot 3/5 = 8\,\text{yrs}$$

$\Rightarrow$ Bob is deduced to be $10+8 = 18$ yrs

Note:
- consistent with previous results
- Two distinct events on earth (Bob 18 and Bob 82)
- Anna and Carl disagree about what event on earth is simultaneous with them arriving at Aptos 9
- For Anna Bob turning 18 and her arrival are simultaneous
  In Carl's frame (moving with 2v relative to Anna) these events are not simultaneous

This is the phenomenon of relative simultaneity