Special Relativity, Lecture 8: Spacetime, Causality and Light Cones

Topics: 4-vectors; Minkowski Metric; Spacetime Diagrams; Absolute Causality; Light Cones

8.1 Space-time 4-vectors

- We have learned that Special Relativity is about transforming coordinates of events with respect of different inertial frames of reference
- Since events are characterized by their time and position, it is tempting to unify space and time in a <u>four-dimensional vector space</u>

Space-time vector: $\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} =: (x^\mu)$

Notation:
$x^0 := ct$
$x^1 := x$
$x^2 := y$
$x^3 := z$

Lorentz transformation:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{=: \underline{\underline{\Lambda}}} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

4-vector notation $\quad x^{\mu\prime} = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu \, x^\nu$

To further simplify the notation, Einstein introduced the sum convention:

in products, indices that appear in „up-down" combinations are summed over

Using this convention, the Lorentz transformation is written as

$$\boxed{x^{\mu\prime} = \Lambda^\mu{}_\nu \, x^\nu} \quad (8.1)$$

Similarily, we can introduce an energy-momentum 4-vector:

$$\begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix} =: (p^\mu)$$

Notation:

$$p^0 = E/c$$
$$p^1 = p_x$$
$$p^2 = p_y$$
$$p^3 = p_z$$

In Lecture 8, we saw that

$$\left.\begin{cases} E' = \gamma_v E - \gamma_v v p_x \\ p_x' = -\gamma_v \frac{v}{c^2} E + \gamma_v p_x \\ p_y' = p_y \\ p_z' = p_z \end{cases}\right\} \Leftrightarrow \begin{pmatrix} E'/c \\ p_x' \\ p_y' \\ p_z' \end{pmatrix} = \underline{\underline{\Lambda}} \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

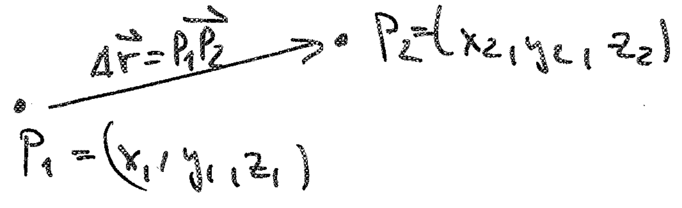
Therefore, the energy-momentum 4-vector transforms under the Lorentz transformation. In compact 4-vector notation:

$$\boxed{p^{\mu\,\prime} = \Lambda^\mu_{\ \nu}\, p^\nu} \quad (8.2)$$

8.2 Minkowski Metric

How do we measure distances in the 4-vector space of space-time?

Consider 3dim. space:



$$\Delta\vec{r} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix} = \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

Distance P_1P_2 = length of vector $\overrightarrow{P_1P_2}$

$$(\Delta r)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = (\Delta x, \Delta y, \Delta z) \cdot \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

- Doing exactly the same in 4-dim. space time is not a good idea since

$$(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 = (c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

is not the same for different observers

- We have learned that the interval

$I = (\Delta s)^2 = (c\Delta t)^2 - (\Delta r)^2$ is invariant under the Lorentz transformation

→ unique way of measuring distances in the 4 dim vector space of space-time

in matrix form:

$$I = (\Delta s)^2 = \underbrace{(c\Delta t, \Delta x, \Delta y, \Delta z)}_{(x_\mu) = (x^\mu)^T} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}_{=: \underline{g} = (g^\mu{}_\nu)} \underbrace{\begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}}_{(x^\nu)}$$

$$\Rightarrow \boxed{I = (\Delta s)^2 = x_\mu g^\mu{}_\nu x^\nu} \qquad (8.3)$$

- This structure is fundamental to the modern way of formulating special and general relativity
- The matrix $\underline{g} = (g^\mu{}_\nu)$ defines the metric in 4-dim spacetime (Minkowski metric)
- Metric tensor plays a central role in general relativity. In GR, the components of $(g^\mu{}_\nu)$ are no longer constants as in SR but rather functions of the space-time coordinates x^μ.

Using the Minkowski metric, we can express the energy-momentum relation in a similar way as the invariant interval:

$$E^2 - \vec{p}^{\,2} c^2 = m^2 c^4$$

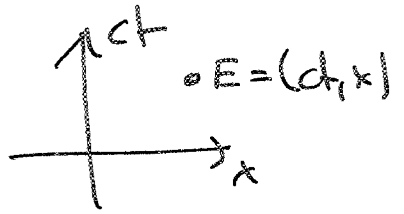
$$\Rightarrow (E/c)^2 - \vec{p}^{\,2} = m^2 c^2$$

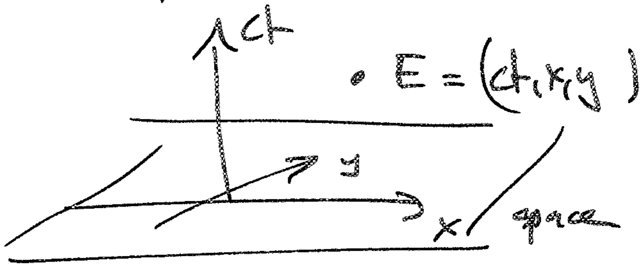
$$\Rightarrow (E/c, p_x, p_y, p_z)\ \underline{\underline{g}} \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix} = m^2 c^2$$

$$\Rightarrow \boxed{p_\mu g^\mu{}_\nu p^\nu = m^2 c^2} \quad (9.4)$$

9.3 Space-Time (Minkowski) Diagrams

Unfortunately, we cannot draw 4-dim. coordinate systems. In the case space is one- or two-dimensional.

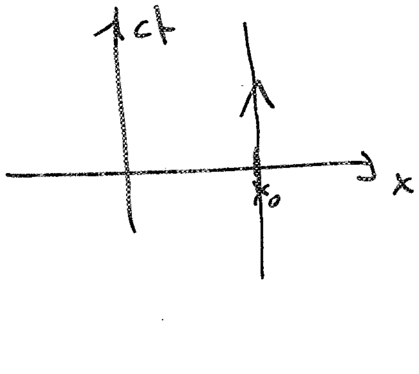




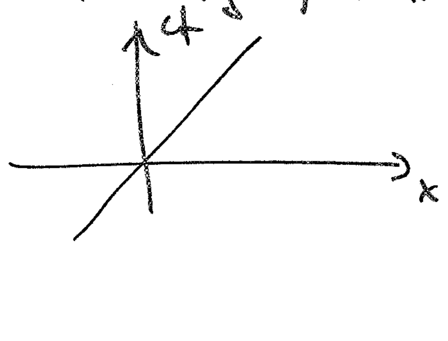
point in space-time uniquely defines time and space coordinates of an event.

How do particle worldlines look in space-time diagrams?

a) Particle at rest at $x = x_0$

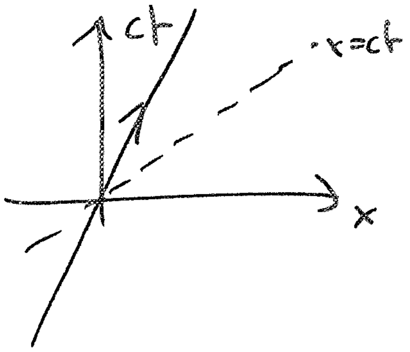


b) Light beam (moving with velocity c) : $x = ct$

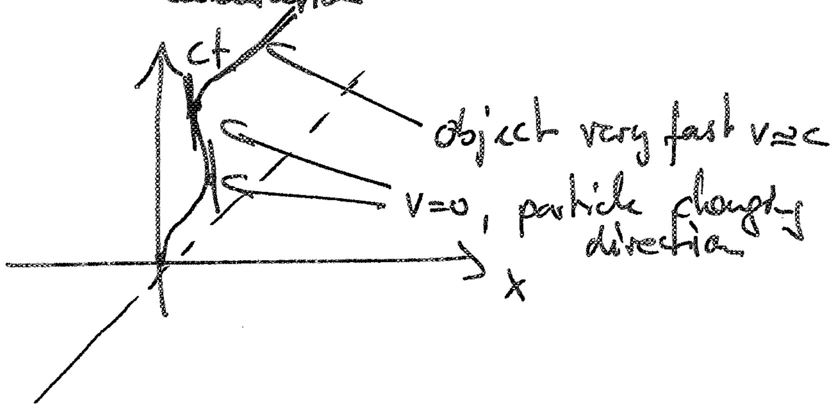


c) Particle moving with constant velocity $v<c$:

$x = vt < ct$

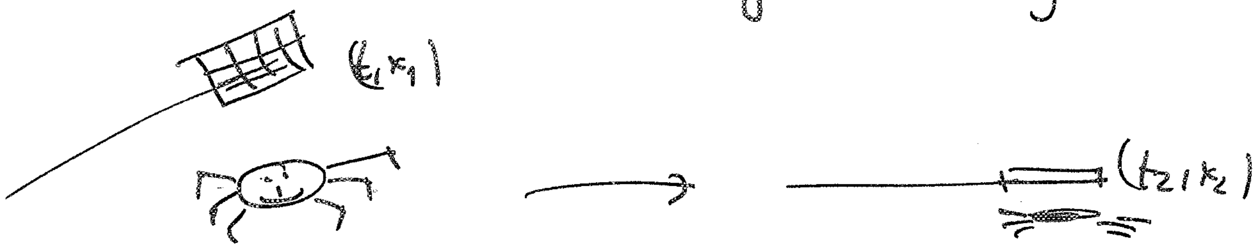


d) complicated worldline of an object that undergoes acceleration



8.4 Absolute causality

Q: Events can occur in different order in different frames. What if one event is the cause of another? (We call such events <u>causally connected</u>)



Suppose in S: $\Delta t = t_2 - t_1 > 0$, $\Delta x = x_2 - x_1$

in S': $\Delta t' = t_2' - t_1' < 0$

Lorentz transformation:
$$\underbrace{\Delta t'}_{<0} = \gamma\left(-\frac{v}{c^2}\Delta x + \Delta t\right)$$
$$= \gamma \underbrace{\Delta t}_{>0}\left(1 - \frac{v}{c^2}\frac{\Delta x}{\Delta t}\right)$$

$$\Rightarrow \quad 1 - \frac{v}{c^2}\frac{\Delta x}{\Delta t} < 0 \quad \Rightarrow \quad \frac{\Delta x}{\Delta t} > \frac{c^2}{v} \underset{c>v}{>} c$$

$$\Rightarrow \Delta x > c\Delta t$$

- The distance Δx between the events is further than light could travel in the time interval Δt
- Therefore the events cannot be causally connected (information transfer is not possible at speeds faster than c)

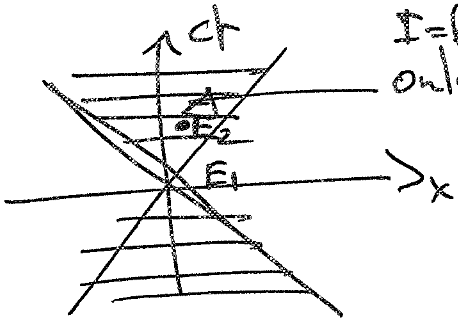
- time Order reversal can happen only for events that are not causally connected

8.5 Light cones

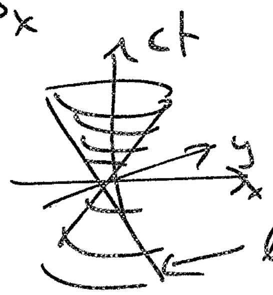
Assume that two events $E_1 = (t_1, \vec{r}_1)$ and $E_2 = (t_2, \vec{r}_2)$ with $t_1 < t_2$ are causally connected.

$$\Rightarrow \quad |\Delta\vec{r}| = |\vec{r}_2 - \vec{r}_1| < c\,\Delta t$$

$$\Rightarrow I = (\Delta s)^2 = c^2(\Delta t)^2 - (\Delta\vec{r})^2 > 0 \qquad (\text{time like interval})$$



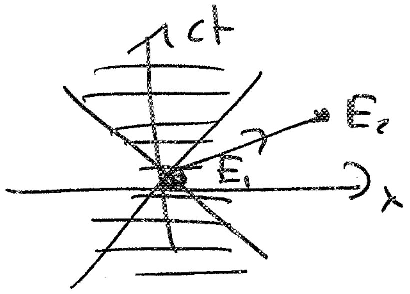
$I = (\Delta s)^2 > 0$
Only in this region, events E_2 can be causally connected to $E_1 = (0, \underline{0})$



light cone

b) space like interval

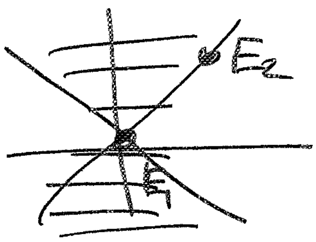
$$I = \Delta s^2 = c^2(\Delta t)^2 - |\Delta\vec{r}|^2 < 0$$



- exchange of information would require speed $> c$
- E_1 and E_2 cannot be causally connected

c) light like intervals

$$I = (\Delta s)^2 = 0$$



- marginal case, only light can process information fast enough to communicate between E_1 and E_2
- $E_1 \to E_2$: worldline of a light-beam