Special Relativity, Lecture 7: Relativistic energy and momentum

Topics: Newtonian Force and Momentum; Nonconservation of Newtonian Momentum in Special Relativity; Relativistic Momentum; Relativistic Energy; Energy-Momentum Relation

Aims: To show how we must modify the Newtonian expressions for momentum and energy and the profound consequences of these changes

7.1 Newtonian Force and Momentum

According to Newton, objects change their state of motion in response to forces causing them to accelerate:

$$\boxed{\vec{F} = m\ddot{\vec{r}} = m\frac{d^2\vec{r}}{dt^2}} \quad \text{(Newton's 2nd law)} \quad \left[\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \vec{F} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}\right]$$

More generally, force is related to change in momentum

$$\boxed{\vec{p} = m\vec{v} = m\frac{d\vec{r}}{dt}} \quad \text{(Newtonian definition of momentum)}$$

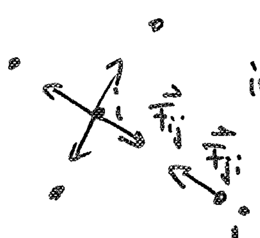
$$\rightarrow \boxed{\vec{F} = \frac{d\vec{p}}{dt}}$$

(generalized version of Newton's 2nd law, takes into account that mass can change over time, eg. if a rocket is burning fuel)

Note: for $m = \text{const}$: $\quad \frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} = m\frac{d^2\vec{r}}{dt^2}$

Internal and external forces:

System of N particles



internal forces

$\vec{F}_{ij} = -\vec{F}_{ji}$ (Newton's 3rd law)

total force on particle i:

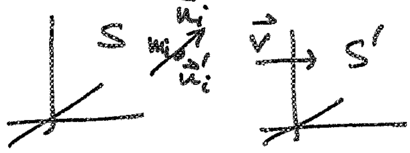
$$\vec{F}_i = \sum_{\substack{j=1 \\ (j\neq i)}}^{N} \vec{F}_{ij} + \vec{F}_{ext}$$

external force, same for all particles

$$\boxed{\text{If there is no external force } (\vec{F}_{ext}=0)\text{, the total momentum } \vec{P}_{total} = \sum_i \vec{p}_i \text{ is conserved } \left(\frac{d\vec{P}_{total}}{dt} = 0,\ \vec{P}_{total} = \text{const}\right).}$$

Proof:
$$\frac{d\vec{P}_{total}}{dt} = \sum_i \frac{d\vec{p}_i}{dt} = \sum_i \vec{F}_i = \sum_i \sum_{\substack{j \\ (j\neq i)}} \vec{F}_{ij} = \sum_{\substack{ij \\ (i\neq j)}} \vec{F}_{ij} = 0 \qquad \vec{F}_{ij} = -\vec{F}_{ji}$$

- This is certainly true experimentally in the nonrelativistic limit
- $\vec{P}_{total}$ is conserved from the point of view of all observers if we use the Galilean transformation



$$\vec{P}'_{total} = \sum_i \vec{p}_i' = \sum_i m_i \vec{u}_i'$$

$$\underset{G.T.}{=} \sum_i m_i (\vec{u}_i - \vec{v}) = \underset{\substack{\uparrow \\ \text{const}}}{\vec{P}_{total}} - \underset{\substack{\uparrow \\ \text{const}}}{M\vec{v}}$$

$$\Rightarrow \vec{P}'_{total} \text{ const}$$

7.2 Nonconservation of Newtonian Momentum in Special Relativity

Assume that the total momentum $\vec{P}_{total}$ defined as above is conserved in the inertial frame S.

We transform to another frame S′ using the relativistic velocity transformation derived in Lecture 6:

velocity x components of particle i: $u_{ix} = u_i$, $u'_{ix} = u_i'$ Trf: $u_i' = \dfrac{u_i - v}{1 - \frac{u_i v}{c^2}}$

$$P'_{total,x} = \sum_i m_i u_i' = \sum_i m_i \frac{u_i - v}{1 - \frac{u_i v}{c^2}} = \sum_i m_i \frac{u_i\Big(1 \underbrace{- \frac{u_i v}{c^2} + \frac{u_i v}{c^2}}_{=0 \text{ (trick)}}\Big) - v}{1 - \frac{u_i v}{c^2}}$$

$$= \underbrace{\sum_i m_i u_i}_{P_{total,x} = \text{const}} - v \sum_i m_i \frac{1 - u_i^2/c^2}{1 - u_i v/c^2}$$

For P'_x to be conserved, the last term has to be constant. But last term depends on individual values u_i of velocities which can change.

→ Either: a) momentum is not conserved in all frames
b) We have used the wrong definition of momentum

(a) is in contradiction to Einstein's first postulate!

7.3 Relativistic momentum

We will show that the appropriate definition of momentum in the relativistic context is

$$\boxed{\vec{p} = \gamma_u m \vec{u}} \quad (7.1) \qquad \left(\gamma_u = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \text{ with } u^2 = \vec{u}^2 = u_x^2 + u_y^2 + u_z^2 \right)$$

It will follow that the above definition will require a redefinition of energy, the other conserved quantity.

Conservation of relativistic momentum

Assume that $\vec{P}_{total} = \sum_i \vec{p}_i = \sum_i \gamma_{u_i} m_i \vec{u}_i$ is conserved in S, $\vec{P}_{total} = \text{const}$. We transform to the inertial frame S' by using the velocity transformation (6.4):

$$\vec{u}_i' = \begin{pmatrix} u'_{ix} \\ u'_{iy} \\ u'_{iz} \end{pmatrix} = \frac{1}{1-\frac{v u_{ix}}{c^2}} \begin{pmatrix} u_{ix} - v \\ u_{iy}/\gamma_v \\ u_{iz}/\gamma_v \end{pmatrix}$$

$$\rightarrow \vec{P}'_{total} = \sum_i \vec{p}'_i = \sum_i \gamma_{u_i'} m_i \vec{u}'_i = \sum_i \frac{\gamma_{u_i'} m_i}{1-\frac{v u_{ix}}{c^2}} \begin{pmatrix} u_{ix} - v \\ u_{iy}/\gamma_v \\ u_{iz}/\gamma_v \end{pmatrix}$$

In order to relate this to $\vec{P}_{total}$ we also have to transform $\gamma_{u_i'} = \frac{1}{\sqrt{1-\frac{u_i'^2}{c^2}}}$ using the velocity transformation (6.4)

After some exceedingly tedious algebra (not shown here):

$$\boxed{\gamma_{u_i'} = \gamma_{u_i}\gamma_v\left(1-\frac{u_{ix}v}{c^2}\right)} \qquad (7.2)$$

$$\Rightarrow \vec{P}_{total}' = \sum_i \gamma_{u_i} m_i \begin{pmatrix} \gamma_v(u_{ix}-v) \\ u_{iy} \\ u_{iz} \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_v P_{total,x} - \gamma_v v \sum_i \gamma_{u_i} m_i \\ P_{total,y} \\ P_{total,z} \end{pmatrix}$$

$\vec{P}_{total}$ is constant, therefore, $\vec{P}_{total}'$ is conserved exactly when $\sum_i \gamma_{u_i} m_i$ is a conserved quantity. We conclude that $\sum_i \gamma_{u_i} m_i$ must be proportional to the relativistic energy since this should be the other conserved quantity.

- Dimensional consideration: $\sum_i \gamma_{u_i} m_i$ has to be multiplied by velocity^2 to obtain an energy
- c is the only velocity which is constant for all observers

We conclude that the appropriate relativistic energy is given by:

$$\boxed{E = \gamma_u m c^2} \qquad (7.3)$$

Is the total relativistic energy indeed conserved?

$$E'_{total} = \sum_i \gamma_{u_i'} m_i c^2 \underset{(7.21)}{=} \sum_i \gamma_{u_i} \gamma_v \left(1 - \frac{u_{ix} v}{c^2}\right) m_i c^2$$

$$= \gamma_v \sum_i \gamma_{u_i} m_i c^2 - \gamma_v v \sum_i \gamma_{u_i} m_i u_{ix}$$

$$= \gamma_v E_{total} - \gamma_v v P_{total,x} = \text{constant} \checkmark$$

7.4 Relativistic Energy

total ~~mass~~ energy of a body moving with velocity $\vec{u}$ in an inertial frame S is

$$E = \gamma_u mc^2 \quad , \quad \gamma_u = 1/\sqrt{1-\frac{u^2}{c^2}}$$

This energy is nonzero at zero velocity, $u=0$!

Internal energy (rest energy)

for $u=0$: $\boxed{E_0 = mc^2}$ (7.41)

- correspondance between mass and internal energy of an object
- profound result! Classical physics had never produced a simple method to describe the internal energy of an object

kinetic energy:

should vanish for $u=0$.

kinetic energy = total energy − rest energy

$$\boxed{T} = E - E_0$$

$$= \gamma_u mc^2 - mc^2$$

$$= \boxed{(\gamma_u - 1) mc^2} \quad (7.5)$$

Small velocity approximation:

$u \ll c, \quad \frac{u}{c} \ll 1 \Rightarrow \frac{u^2}{c^2} \ll 1$

Taylor expansion:

$$\boxed{\gamma_u = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \underset{(*)}{=} 1 + \frac{1}{2}\frac{u^2}{c^2} + \frac{3}{8}\frac{u^4}{c^4} + \dots} \quad (7.6)$$

(*) $f(x) \underset{\text{small } x}{=} f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \dots \qquad x = \frac{u^2}{c^2}$

$f(x) = \frac{1}{\sqrt{1-x}} = (1-x)^{-1/2} \rightarrow f(0) = 1$

$f'(x) = -\frac{1}{2}(1-x)^{-3/2} \cdot (-1) \rightarrow f'(0) = 1/2$

$f''(x) = -3/4(1-x)^{-5/2} \cdot (-1) \rightarrow f''(0) = 3/4$

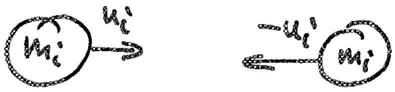
kinetic energy: $T = (\gamma_u - 1)mc^2$

$$= \underbrace{\frac{1}{2}mu^2}_{\text{classical kinetic energy}} + \underbrace{\frac{3}{8}m\frac{u^4}{c^2}}_{\text{1st relativistic correction to kinetic energy}} + \dots$$

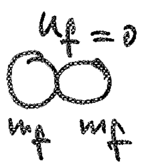
Example 1 : Inelastic Collision

Consider a head on, completely inelastic collision between two identical masses:

initially:



finally:



conservation of relativistic energy:

$E_i = E_f \Leftrightarrow 2\gamma_{u_i} m_i c^2 = 2\underset{\overset{\parallel}{1}}{\gamma_{u_f}} m_f c^2$

$$\Rightarrow m_f = \gamma_{u_i} m_i = \frac{1}{\sqrt{1-u_i^2/c^2}} m_i > m_i$$

- kinetic energy has been converted into mass
- the increase of mass ($m_f > m_i$) corresponds with an increase of internal energy

Small velocity approximation: $\gamma_{u_i} \simeq 1 + \frac{1}{2}\frac{u_i^2}{c^2} \Rightarrow \frac{m_f - m_i}{m_i} \simeq \frac{1}{2}\frac{u_i^2}{c^2}$

<u>Example 2:</u> Decay of Beryllium atom into two identical fragments

<u>initially:</u> $v = 0$ (at rest)

Be $\quad m_{Be} = 8{,}0031\,u$, $\quad u = 1{,}66 \cdot 10^{-27}\,kg$ atomic mass unit

<u>finally:</u> $\xleftarrow{v_2} \bigcirc \qquad \bigcirc \xrightarrow{v_1}$

$m = m_2 = m_1 = 4{,}0015\,u$

Q: What are the velocities of the emitted particles and how much kinetic energy do they have?

<u>Conservation of momentum:</u>

$$P_i = P_f \Leftrightarrow 0 = \gamma_1 m_1 v_1 + \gamma_2 m_2 v_2 \qquad \begin{pmatrix} \gamma_1 = \gamma_{v_1} \\ \gamma_2 = \gamma_{v_2} \end{pmatrix}$$

$$\underset{m_1 = m_2 = m}{\Longrightarrow} \quad \gamma_1 v_1 = -\gamma_2 v_2 \Leftrightarrow \frac{v_1}{\sqrt{1 - v_1^2/c^2}} = \frac{-v_2}{\sqrt{1 - v_2^2/c^2}}$$

$$\Rightarrow \boxed{v_1 = -v_2 =: v}$$

This is in a way obvious since $m_1 = m_2$. For $m_1 \neq m_2$ however, this step is nontrivial.

<u>Conservation of energy:</u>

$$E_i = E_f \Leftrightarrow m_{Be} c^2 = \gamma_1 m_1 c^2 + \gamma_2 m_2 c^2$$
$$= 2\gamma_v m c^2$$

$$\Rightarrow \underbrace{\frac{1}{\gamma_v}}_{= \sqrt{1 - \frac{v^2}{c^2}}} = \frac{2m}{m_{Be}} \Rightarrow \boxed{v = c\sqrt{1 - \left(\frac{2m}{m_{Be}}\right)^2}}$$

$$= c\sqrt{1 - \left(\frac{2 \cdot 4{,}0015}{8{,}0031}\right)^2}$$

$$= \underline{0{,}005\,c}$$

Kinetic energy of one fragment:

$$T = (\gamma_v - 1) mc^2 = \left(\frac{1}{\sqrt{1-0{,}005^2}} - 1\right) 4{,}0015 \cdot 1{,}66 \cdot 10^{-27}\,\text{kg} \cdot \left(2{,}998 \cdot 10^8 \tfrac{m}{s}\right)^2$$

$$= \boxed{0{,}75 \cdot 10^{-4}\,J}$$

- Gain of kinetic energy
- Loss of internal energy, total mass has gone down

7.5 The ultimate speed

Relativistically:

$\vec{p} = \gamma_u m \vec{u}$

$E = \gamma_u m c^2$

$T = (\gamma_u - 1) mc^2$

$\quad = E - E_0$

Newtonian / Gallilean:

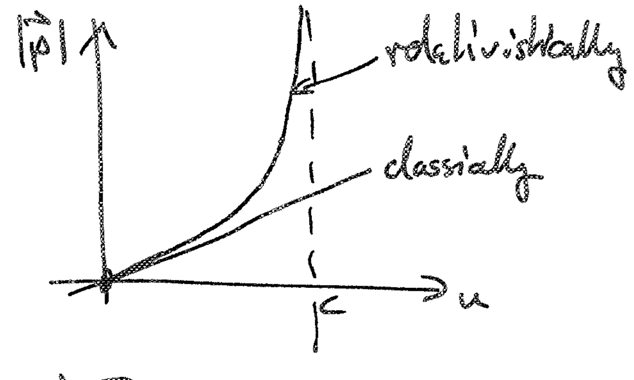
$\vec{p} = m\vec{u}$

$T = \frac{1}{2} m \vec{u}^2$

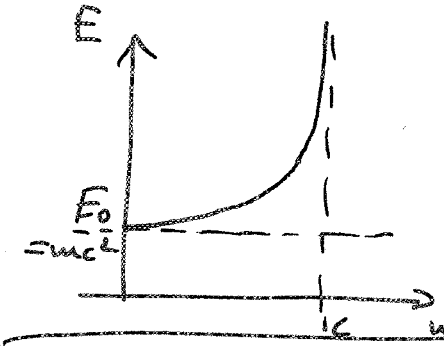
Q: What happens as $u \to c$?

$$\gamma_u = \frac{1}{\sqrt{1 - u^2/c^2}} \xrightarrow[u \to c]{} \infty$$

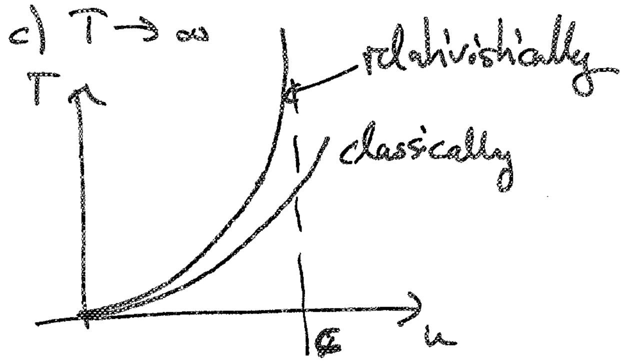
a) $|\vec{p}| \to \infty$



b) $E \to \infty$



c) $T \to \infty$



This implies that a material particle can never reach the speed of light as it would require an infinite energy to accelerate it there.

7.6 Relation between energy and momentum

Claim: Relativistic energy E and momentum $\vec{p}$ are related as

$$\boxed{E^2 - \vec{p}^{\,2} c^2 = m^2 c^4} = \text{const} \quad (7.6)$$

"rest mass"

- This holds for all observers. In S': $E'^2 - \vec{p}'^{\,2} c^2 = m^2 c^4$
- Recall the invariant interval considered previously

$$c^2 \Delta t^2 - \Delta x^2 = I$$ (same for two events observed in any frame)

Proof:

$$E^2 - \vec{p}^{\,2} c^2 = (\gamma_u m c^2)^2 - (\gamma_u m \vec{u})^2 c^2$$

$$= \gamma_u^2 m^2 c^2 (c^2 - u^2)$$

$$= m^2 c^4 \underbrace{(1 - u^2/c^2)\gamma_u^2}_{=1}$$

$$= m^2 c^4$$