Special Relativity, Lecture 5: The Doppler Effect

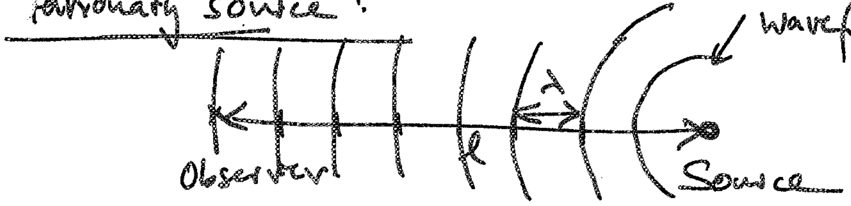
Topics: Classical Doppler Effect for sound; Doppler Shift for Light; Small velocity Approximation

Aims: To understand the frequency shift of a moving light source in special relativity

5.1 Classical Doppler Effect for sound

Soundwaves propagate in a medium, e.g. air. Sound velocity relative to air v_s.

Stationary source:



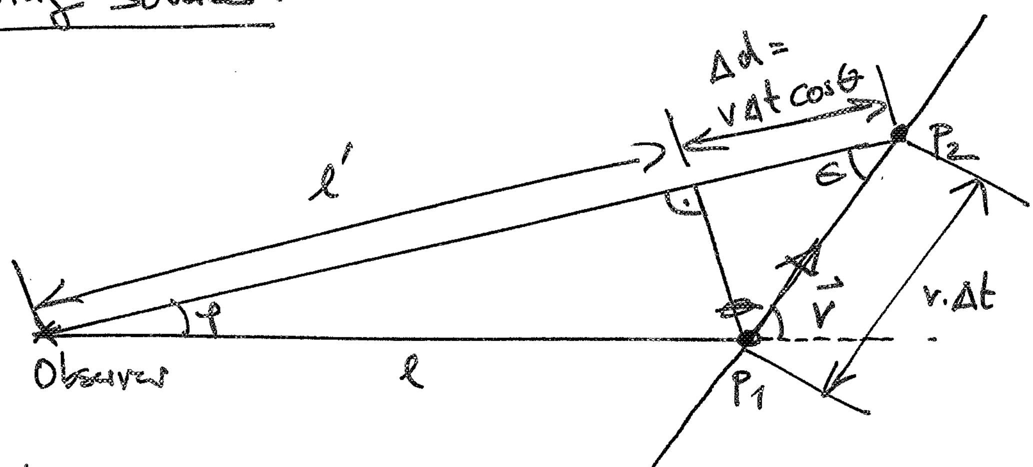
speed of sound v_s
wavelength λ

- wavefronts take time $t = \frac{\ell}{v_s}$ to reach observer
- in time t, ℓ/λ wavefronts arrive at observer

$$\Rightarrow \text{frequency } f_{\text{observer}} = \frac{\ell/\lambda}{t} = \frac{v_s}{\lambda} = f_{\text{source}} = \frac{1}{\Delta t}$$

time between emission of 2 consecutive wavefronts

Moving source:



- At time $t=0$, wavefront emitted at P_1
- At time Δt, wavefront emitted at P_2
- If source is far away, φ is very small $\Rightarrow \ell' \simeq \ell$

$\Rightarrow$ difference of distances OP_2 and OP_1 can be approximated by Δd

wavefront 1 reaches observer at $t_1 = l/v_s$

wavefront 2 reaches observer at $t_2 = \Delta t + \dfrac{l' + \Delta d}{v_s}$

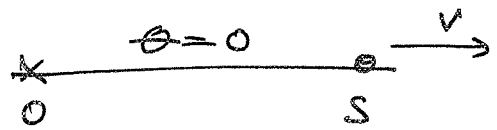
$$\approx \Delta t + \frac{l}{v_s} + \frac{\Delta d}{v_s}$$

$$\Rightarrow \Delta t_{observer} = t_2 - t_1 = \Delta t + \frac{\Delta d}{v_s} = \Delta t + \frac{v\cos\theta\, \Delta t}{v_s}$$

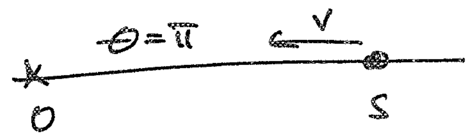
$$= \left(1 + \cos\theta \frac{v}{v_s}\right) \Delta t$$

$$\Rightarrow \boxed{f_{observer} = \frac{1}{\Delta t_{observer}}} = \boxed{\frac{f_{source}}{1 + \cos\theta \frac{v}{v_s}}}$$

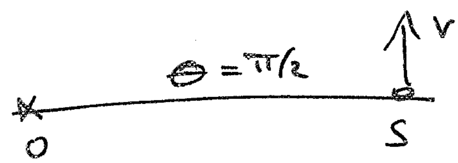
Special cases:

1) 

$$f_{observer} = \frac{f_{source}}{1 + v/v_s}$$

2) 

$$f_{observer} = \frac{f_{source}}{1 - v/v_s}$$

3) 

$$f_{observer} = f_{source}$$

Note that this result is a consequence of the approximation $l = l'$

5.2. Derivation of Doppler Shift for Light

Crucial differences to classical Doppler effect for sound:

- Light requires no medium
- All observers perceive the velocity of the wave as c
- time interval between the consecutive production of wavefronts is time dilated in special relativity

Work in the observer frame:

as before $\quad \Delta t_{observer} = \Delta t + \frac{\Delta d}{c}$

with Δt the time difference between the emission of wavefronts at P_1 and P_2 as measured by the observer and

$\Delta d = v \Delta t \cos\theta = OP_2 - OP_1$ in the observer frame

We have to take into account that the time interval Δt measured by the observer is dilated as compared to the proper time interval Δt_{source} in the frame of the source

$$\Delta t = \gamma \Delta t_{source}$$

$$\Rightarrow \quad \Delta t_{observer} = \left(1 + \frac{v}{c}\cos\theta\right)\Delta t$$

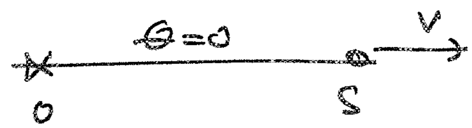
$$= \left(1 + \frac{v}{c}\cos\theta\right)\gamma \Delta t_{source}$$

$$= \frac{1 + \frac{v}{c}\cos\theta}{\sqrt{1 - v^2/c^2}} \Delta t_{source}$$

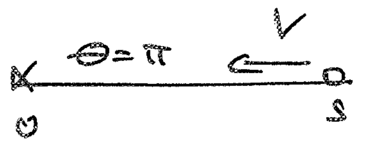
for frequencies:

$$\boxed{f_{observer} = \frac{\sqrt{1-\frac{v^2}{c^2}}}{1+\frac{v}{c}\cos\theta}\, f_{source}} \qquad (5.1)$$

Special cases:

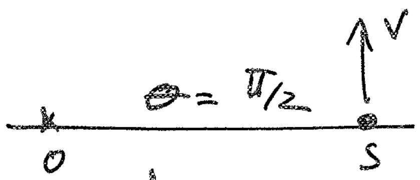
1) 

$$f_{observer} = \sqrt{\frac{1-\frac{v}{c}}{1+v/c}}\, f_{source}$$

2) 

$$f_{observer} = \sqrt{\frac{1+v/c}{1-v/c}}\, \cancel{f_{obser}}\ f_{source}$$

Again, we have the symmetry $f_{observer} \leftrightarrow f_{source}$ and $v \to -v$.

3) 

$$f_{observer} = \sqrt{1-\frac{v^2}{c^2}}\, f_{source}$$

„transverse Doppler Effect"

5.3. Small velocity approximation

In the limit of small velocities ($v \ll c$), we can expand the relativistic Doppler shift equation (6.1) in powers of $\beta := \frac{v}{c}$ by using a Taylor series expansion.

$$f_{observer} = g(\beta) \cdot f_{source}$$

$$g(\beta) = g(0) + g'(0)\beta + \frac{1}{2} g''(0)\beta^2 + \ldots = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!}\beta^n$$

$$g(\beta) = \frac{\sqrt{1-\beta^2}}{1+\cos\theta\cdot\beta} \rightarrow g(0) = 1$$

$$g'(\beta) = \frac{1/2\,(1-\beta^2)^{-1/2}\cdot(-2\beta)\,(1+\cos\theta\,\beta) - (1-\beta^2)^{1/2}\cos\theta}{(1+\cos\theta\,\beta)^2}$$

$$\rightarrow g'(0) = -\cos\theta$$

$$g''(\beta) = \dots \rightarrow g''(0) = 2\cos^2\theta - 1$$

$$\Rightarrow \frac{f_{observer}}{f_{source}} = 1 - \cos\theta\,\frac{v}{c} + \frac{2\cos^2\theta - 1}{2}\left(\frac{v}{c}\right)^2 + \dots$$

Special Cases:

1) $\theta = 0$: $\frac{f_{observer}}{f_{source}} = 1 - \frac{v}{c} + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \dots$

2) $\theta = \pi$: $\frac{f_{observer}}{f_{source}} = 1 + \frac{v}{c} + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \dots$

3) $\theta = \pi/2$: $\frac{f_{observer}}{f_{source}} = 1 - \frac{1}{2}\left(\frac{v}{c}\right)^2 + \dots$