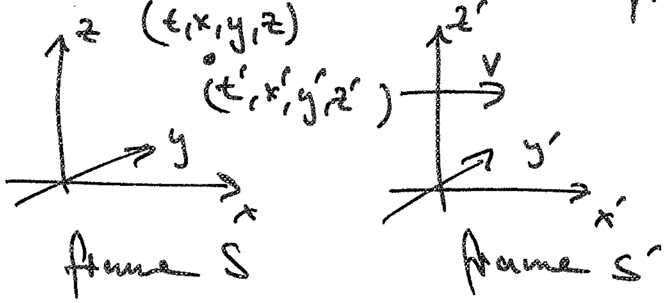
Special Relativity, Lecture 3: The Lorentz Transformation

Topics: Derivation of the Lorentz Transformation; Galileo's mistake; Transformations in matrix forms

3.1. Conventions

Special Relativity is about position and time of events measured by different observers, i.e. it is about assigning coordinates to the same events in different inertial frames of reference

We will often think about 2 inertial frames related as follows:



1) S' moves with velocity v relative to S, v parallel to x axis (S moves at −v in the x-direction relative to S')
2) x and x' parallel to each other
3) origins coincide at t = 0 and t' = 0
4) Events: anything with unique space and time coordinates

$$\begin{array}{ccc} S & & S' \\ (t, x, y, z) & \xleftrightarrow[\text{law}]{} & (t', x', y', z') \\ & \text{transformation} & \end{array}$$

- Newtonian Relativity: Galilean trf.
- Special Relativity: Lorentz trf.

3.2. Derivation of the Lorentz Transformation

Aim: To derive the transformation between coordinates that is consistent with Einstein's postulates

Long derivation: important to have the key points and general sweep clear before start

Broad Sweep:

- Since v is along x-direction, $y' = y$ and $z' = z$
- Consider free object – constant velocity in each frame

  At $t = t' = 0$ both origins at position of object, object moves along x-direction

$$\boxed{\begin{aligned} x' &= u' t' \\ x &= u t \end{aligned}} \quad (3.1)$$

- Newton's law the same in all inertial frames of reference (E1) ⇒

$$\boxed{\begin{aligned} x' &= Ax + Bt \\ t' &= Cx + Dt \end{aligned}} \quad (3.2)$$

  (transformation has to be linear in x and t)

- Deduce A, B, C and D by considering
  1) object at origin in S
  2) object at origin in S'
  3) light beam in S and S' (constancy of speed of light, (E2))

Drop y and z coordinates from now on.

Linearity of transformation:

in general, we can write the transformation as

$$\begin{aligned} x' &= f(x,t) \\ t' &= g(x,t) \end{aligned} \qquad \text{with functions } f, g : \underset{\substack{\uparrow \\ \text{space} \\ (x,\text{coordinate})}}{\mathbb{R}^{\cancel{3}}} \times \underset{\substack{\uparrow \\ \text{time}}}{\mathbb{R}} \to \mathbb{R}$$

in both frames, particle moves with constant velocity

$$\rightarrow \quad \frac{d^2 x}{dt^2} = 0 \,, \qquad \frac{d^2 x'}{dt'^2} = 0$$

$$0 = \frac{d^2x'}{dt'^2} = \frac{d^2}{dt'^2} f(x,t)$$

$$= \frac{d}{dt'}\left(\frac{\partial f}{\partial x}\frac{dx}{dt'} + \frac{\partial f}{\partial t}\frac{dt}{dt'}\right)$$

$$= \frac{\partial^2 f}{\partial x^2}\left(\frac{dx}{dt'}\right)^2 + \frac{\partial^2 f}{\partial t\,\partial x}\frac{dx}{dt'}\frac{dt}{dt'} + \frac{\partial^2 f}{\partial x\,\partial t}\frac{dx}{dt'} + \frac{dt}{dt'}$$

$$+ \frac{\partial^2 f}{\partial t^2}\left(\frac{dt}{dt'}\right)^2$$

$$= \left(\frac{dt}{dt'}\right)^2\left(u^2\frac{\partial^2 f}{\partial x^2} + 2u\frac{\partial^2 f}{\partial t\,\partial x} + \frac{\partial^2 f}{\partial t^2}\right)$$

$\frac{\partial^2 f}{\partial x\,\partial t} = \frac{\partial^2 f}{\partial t\,\partial x}$

$x = ut \Rightarrow \frac{dx}{dt'} = u\frac{dt}{dt'}$

Polynomial in u; has to vanish for all values of u

$$\Rightarrow \quad \frac{\partial^2 f}{\partial x^2} = 0, \quad \frac{\partial^2 f}{\partial x\,\partial t} = 0, \quad \frac{\partial^2 f}{\partial t^2} = 0$$

$\Rightarrow$ $f(x,t) = Ax + Bt$ with $A, B \in \mathbb{R}$, A,B constant

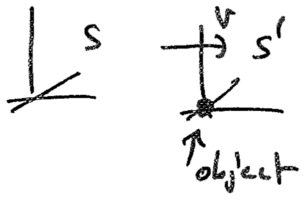
$x' = u't' \Rightarrow f(x,t) = u' g(x,t) \Rightarrow g$ is also linear function in x and t

$$\rightarrow \boxed{\begin{aligned} x' &= Ax + Bt \\ t' &= Cx + Dt \end{aligned}}$$

Deduce coefficients A, B, C and D:

1) Consider object fixed at origin in S'

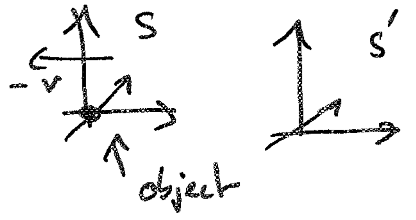
$x' = 0, \quad u = v$



$$0 = x' = Ax + Bt \underset{x=ut=vt}{=} Avt + Bt \Rightarrow \boxed{B = -Av} \quad (3.3)$$

2) object fixed at origin in S

$x = 0, \quad u' = -v$



$$x' = A\underset{0}{\underset{\|}{x}} + Bt = Bt \quad , \quad x' = u't' = -vt'$$

$$\Longrightarrow \quad Bt = -vt' \quad (1)$$

Further we have $t' = C\underset{0}{\underset{\|}{x}} + Dt = Dt$ (2)

Combining (1) and (2) we get: $Bt = -vDt \Rightarrow \boxed{-v = \frac{B}{D}}$ (3.4)

Combining (3.3) and (3.4) we get: $B = A\frac{B}{D} \underset{B \neq 0}{\Longrightarrow} \boxed{A = D}$ (3.5)

Now the transformation reads:

$$\begin{aligned} x' &= Ax + Bt \overset{(3.3)}{=} A(x - vt) \\ t' &= Cx + Dt \overset{(3.5)}{=} Cx + At \end{aligned} \qquad (3.6)$$

3) Consider now light beam in S and S'

(E2): Light beam propagates with c in both frames

$$\Longrightarrow \quad \begin{aligned} x &= ct \quad (3.7) \\ x' &= ct' \quad (3.8) \end{aligned}$$

Plug (3.6) into (3.8) : $A(x - vt) = c(Cx + At)$

$$\overset{(3.7)}{\Longrightarrow} A(ct - vt) = c(Cct + At)$$

$$\Longleftrightarrow -Avt = Cc^2t$$

$$\Longrightarrow \boxed{C = -A\frac{v}{c^2}} \quad (3.9)$$

We now have B, C and D in terms of A. The transformatio reads

$$\boxed{\begin{aligned} x' &= A(x - vt) \\ t' &= A\left(-\frac{v}{c^2}x + t\right) \end{aligned}} \quad (3.10)$$

Now we can make use of the first postulate (E1):

S' moving with v relative to S is equivalent to
S moving with $-v$ relative to S'

Exchange $(x,t) \leftrightarrow (x',t')$ and replace v by $-v$:

$$\boxed{\begin{aligned} x &= A(x' + vt') \\ t &= A\left(\frac{v}{c^2}x' + t'\right) \end{aligned}} \quad (3.11)$$

We can invert this transformation (solve for x' and t')

$$\frac{v}{c^2}x - t = A\frac{v^2}{c^2}t' - At' = A\left(\frac{v^2}{c^2} - 1\right)t'$$

$$\Leftrightarrow \boxed{t' = \frac{v/c^2}{A\left(\frac{v^2}{c^2} - 1\right)}x - \frac{1}{A\left(\frac{v^2}{c^2} - 1\right)}t} \quad (3.12)$$

(3.12) has to be identical to 2nd equation in (3.10)

$$A\left(-\frac{v}{c^2}x + t\right) = \frac{v/c^2}{A\left(\frac{v^2}{c^2} - 1\right)}x - \frac{1}{A\left(\frac{v^2}{c^2} - 1\right)}t$$

$$\Rightarrow -A = \frac{1}{A\left(\frac{v^2}{c^2} - 1\right)} \Rightarrow A^2 = \frac{1}{1 - \frac{v^2}{c^2}}$$

$$\Rightarrow \boxed{A = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}} \quad (3.13)$$

This is exactly the factor we encountered in our thought experiments on time dilation and length contraction

From now on:

$$\boxed{\gamma := \gamma_v := \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}} \quad (3.14)$$

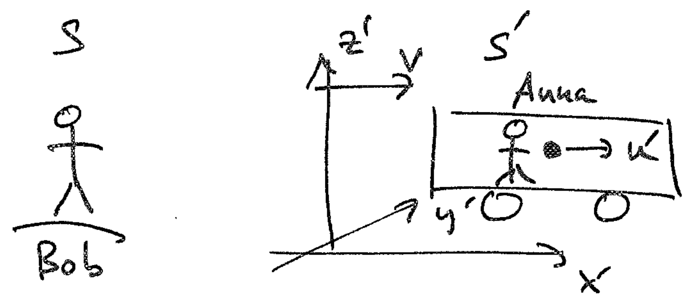
The transformation finally reads (plug $A = \gamma_v$ into (3.10))

$$\boxed{\begin{aligned} x' &= \gamma_v (x - vt) \\ t' &= \gamma_v \left(-\frac{v}{c^2} x + t\right) \end{aligned}} \quad (3.15)$$

- This transformation has been found previously by Lorentz (1904). Maxwell equations are invariant under this transformation
- Einstein derived the Lorentz Transformation only by assuming properties of space and time in form of his two postulates. This puts the Lorentz transformation on solid grounds!

3.3. Galileo's mistake

Aim: to try to deduce where Galileo went wrong



- Anna throws ball with velocity u' in her frame
- S and S' coincide when ball is thrown

in S': $x' = u't'$

in S: $x = \underset{\substack{\uparrow \\ \text{Anna} \\ \leftrightarrow \text{ball}}}{u't} + \underset{\substack{\uparrow \\ \text{Anna} \\ \leftrightarrow \text{Bob}}}{vt}$

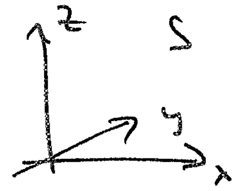
$$\boxed{\text{Galileo's big mistake: Assume } t = t' \text{ (time is absolute!)}}$$

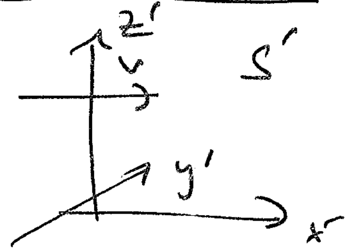
$$\Longrightarrow \quad x = x' + vt$$

Using the Galilean relativity principle and the Galilean transformation, velocities are additive. In the previous example: $u = u' + v$

We will see later that this is not the case in special relativity where the Galilean transformation is replaced by the Lorentz transformation.

3.4. Transformations in matrix form





Galilean trf.	Lorentz trf.
$t' = t$	$t' = \gamma(t - \frac{v}{c^2}x)$
$x' = x - vt$	$x' = \gamma(x - vt)$
$y' = y$	$y' = y$
$z' = z$	$z' = z$

We introduce a four dimensional space-time vector:

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \leftarrow$$

multiply by c to have all components in the units of length

Q: Why c and not other velocity?

A: c is the only velocity that is the same in all inertial frames of reference

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{v}{c} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \qquad \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Galilean trf.

Lorentz trf

(easier to remember in this form)