# 4.7. Renormalization Group for the Quantum Non-Linear Sigma Model



- In this section, we work out the renormalization group equations for the Quantum NLσM, ignoring the residual Berry phase term $S_B$

$$\boxed{\begin{aligned} Z &= \int \mathcal{D}\underline{n}\; \delta(\underline{n}^2 - 1)\, e^{-S[\underline{n}]} \\ S[\underline{n}] &= \frac{\rho_s}{2} \int_0^\beta d\tau \int d^d r \left\{ (\nabla_r \underline{n})^2 + \frac{1}{c^2}\left(\frac{\partial n}{\partial \tau}\right)^2 \right\} \end{aligned}}$$

( $\rho_s$ spin stiffness, $c$ spin-wave velocity
$\beta$ inverse temperature $\Lambda$ momentum cut-off corresponding to a small distance cut-off )

- We first transform to a dimensionless form

$$\boxed{\begin{aligned} \underline{x} &:= \Lambda \underline{r} \quad &\text{space} \\ x_0 &:= \Lambda c \tau \quad &\text{time} \end{aligned}}$$

$$\to \quad S = \frac{\rho_s}{2} \int_0^{\Lambda c \beta} \frac{dx_0}{\Lambda c} \int \frac{d^d x}{\Lambda^d} \left\{ \Lambda^2 (\nabla_x \underline{n})^2 + \frac{(\Lambda c)^2}{c^2} (\partial_{x_0} \underline{n})^2 \right\}$$

$$= \frac{\rho_s}{2} \frac{1}{c\Lambda^{d-1}} \int_0^{\Lambda c \beta} dx_0 \int d^d x \sum_{i=0}^{d} (\partial_i \underline{n})^2$$

$$= \boxed{\frac{1}{2g} \int_0^u dx_0 \int d^d x \sum_i (\partial_i \underline{n})^2} \qquad \boxed{g = \frac{c\Lambda^{d-1}}{\rho_s} = \frac{\sqrt{2d}\,(\Lambda a)^{d-1}}{S\sqrt{1-2(d-1)\gamma}}}$$

$$\boxed{u = \Lambda c \beta}$$

$T \to \infty$: ($u \to 0$) classical limit
$S \to S_d^{cl}$

$T \to 0$: ($u \to \infty$) one-to-one correspondence to $(d+1)$ dim. classical model, $S \to S_{d+1}^{cl}$

- To derive the RG equation, we proceed in the same way as in the classical case

- Note that due to the rescaling to dimensionless units, the momentum cut-off is 1, $|\underline{k}| \le 1$

- Using the Polyakov decomposition into slow and fast fields,

$$\boxed{\vec{n}(\underline{x}, x_0) = \underline{n}_0(\underline{x}, x_0)\sqrt{1 - \sum_{\alpha=1}^{N} \phi_\alpha^2(\underline{x}, x_0)} + \sum_{\alpha=1}^{N} \phi_\alpha(\underline{x}, x_0)\underline{e}_\alpha(\underline{x}, x_0)}$$

slow — fast — slow


we obtain as before:

$$\boxed{\begin{aligned} S = &\underbrace{\frac{1}{2g}\int_0^u dx_0 \int d^dx \sum_i (\partial_i \underline{n}_0)^2}_{S_0^<} + \underbrace{\frac{1}{2g}\int_0^u dx_0 \int d^dx \sum_{i\alpha} (\partial_i \phi_\alpha)^2}_{S_0^>} \\ &+ \underbrace{\frac{1}{2g}\int_0^u dx_0 \int d^dx \sum_{i\alpha\beta} \left(B_{i\alpha} B_{i\beta} \phi_\alpha \phi_\beta - B_{i\alpha}^2 \phi_\beta^2\right)}_{S'} \end{aligned}}$$

* $\partial_i \vec{\underline{n}}_0 = \sum_\alpha B_{i\alpha} \underline{e}_\alpha$, $\quad B_{i\alpha} = \underline{e}_\alpha \cdot \partial_i \underline{n}_0$

* We have dropped the term $B_{i\alpha} \cdot \partial_i \phi_\alpha$ since it does not contribute to the renormalization

- As in the classical case, the correction at 1-loop order to $S_0^<$ is given by

$$\boxed{\underset{\langle S' \rangle_{0>}}{\overset{\parallel}{\delta S}} = \frac{1}{2g}\int_0^u dx_0 \int d^d x \sum_{i\alpha\beta}\left(B_{i\alpha}B_{i\beta}\langle\phi_\alpha\phi_\beta\rangle_{0>} - B_{i\alpha}^2\langle\phi_\beta^2\rangle_{0>}\right)}$$

- Correlation function in frequency and momentum space

Fourier transform: $\boxed{\phi_\alpha(\underline{x}, x_0) = \sum_{n=-\infty}^{\infty}\int_{\underline{k}} \tilde{\phi}_\alpha(\underline{k}, \omega_n) e^{i(\underline{k}\underline{x} + \omega_n x_0)}}$

discrete frequencies since $\phi$ is periodic, $\phi(\underline{x}, 0) = \phi(\underline{x}, u)$ $\boxed{\omega_n = \frac{2\pi n}{u}}$ Matsubara frequencies

$$\rightarrow \boxed{S_0^< = \frac{u}{2g}\sum_{n=-\infty}^{\infty}\int_{\underline{k}}\sum_{i\alpha}(k^2+\omega_n^2)\,\underbrace{\tilde{\phi}^*(\underline{k},\omega_n)}_{\tilde{\phi}(-\underline{k},-\omega_n)}\tilde{\phi}(\underline{k},\omega_n)}$$

$\rightarrow$ correlator from inverse of kernel

$$\boxed{\langle\tilde{\phi}_\alpha(\underline{k},\omega_n)\tilde{\phi}_\beta(\underline{k}',\omega_{n'})\rangle_0 = \frac{g}{u}\,\frac{\delta_{\alpha\beta}\,\delta_{n,-n'}\,\delta(\underline{k}+\underline{k}')}{k^2+\omega_n^2}}$$

- We use this to calculate the average $\langle\phi_\alpha(\underline{x},x_0)\phi_\beta(\underline{x},x_0)\rangle_{0>}$ over the outer momentum shell ($e^{-dl} \leq |\underline{k}| \leq 1$):

$$\langle\phi_\alpha(\underline{x},x_0)\phi_\beta(\underline{x},x_0)\rangle_{0>} = \sum_{nn'}\int_{\underline{k}\underline{k}'\in\text{shell}} e^{i(\omega_n+\omega_{n'})x_0}\, e^{i(\underline{k}+\underline{k}')\underline{x}} \times \langle\tilde{\phi}_\alpha(\underline{k},\omega_n)\tilde{\phi}_\beta(\underline{k}',\omega_{n'})\rangle_0$$

$$= \frac{g}{u}\,\delta_{\alpha\beta}\sum_{n=-\infty}^{\infty}\int_{\underline{k}\in\text{shell}}\frac{1}{\omega_n^2+k^2}$$

$$= \frac{g}{u} \delta_{\alpha\beta} \sum_{n=-\infty}^{\infty} \frac{S_d}{(2\pi)^d} \underbrace{\int_{e^{-dl}}^{1} dk\, k^{d-1} \frac{1}{k^2+\omega_n^2}}_{\approx \frac{1}{1+\omega_n^2}(1-e^{-dl}) \approx \frac{1}{1+\omega_n^2} dl}$$

$$= \frac{g}{u} \delta_{\alpha\beta} \frac{S_d}{(2\pi)^d} dl \underbrace{\sum_{n=-\infty}^{\infty} \frac{1}{\left(\frac{2\pi}{u}\right)^2 n^2 + 1}}_{= \frac{u}{2}\coth\left(\frac{u}{2}\right)}$$

$$= \boxed{\frac{g}{2} \delta_{\alpha\beta} \frac{S_d}{(2\pi)^d} dl \coth\left(\frac{u}{2}\right)}$$

- We use this result to calculate the correction $\langle S' \rangle_{0>} = \delta S$ to $S^0_<$ :

$$\boxed{\delta S} = \frac{1}{2g} \int_0^u dx_0 \int d^d\underline{x} \sum_{i\alpha\beta} \left( B_{i\alpha} B_{i\beta} \langle \phi_\alpha \phi_\beta \rangle_{0>} - B_{i\alpha}^2 \langle \phi_\beta^2 \rangle_{0>} \right)$$

$$= \frac{1}{4} \frac{S_d}{(2\pi)^d} \coth\left(\frac{u}{2}\right) dl \int_0^u dx_0 \int d^d\underline{x} \sum_{i\alpha} \left( B_{i\alpha}^2 - (N-1) B_{i\alpha}^2 \right)$$

$$= \boxed{-\frac{1}{4}(N-2) \coth\left(\frac{u}{2}\right) dl \frac{S_d}{(2\pi)^d} \int_0^u dx_0 \int d^d\underline{x} \sum_i (\partial_i \underline{u}_0)^2}$$

$$S_0^< + \delta S = \frac{1}{2g}\left(1 - \frac{g}{2}\frac{S_d}{(2\pi)^d}(N-2)\coth\left(\frac{u}{2}\right) dl\right)\int_0^u dx_0 \int d^d x \sum_i (\partial_i n_0)^2$$

$$\underset{\text{rescaling}}{=} \frac{1}{2g}\left(1 - \frac{g}{2}\frac{S_d}{(2\pi)^d}(N-2)\coth\left(\frac{u}{2}\right) dl\right) e^{(d+1)dl} e^{-2dl} \int_0^{u'} dx_0' \int d^d x' \sum_i (\partial_i' n_0)^2$$

$x_i' = e^{-dl} x_i \quad (i = 1, \ldots, d)$

$x_0' = e^{-z\,dl} x_0 \quad (\text{time})$

$\underset{z=1}{=} e^{-dl} x_0$

$$\Rightarrow \quad g^{-1}(l+dl) = g^{-1}(l)\left(1 + (d-1)dl - \frac{g(l)}{2}(N-2)\coth\left(\frac{u}{2}\right)\frac{S_d}{(2\pi)^d} dl\right)$$

$$\Rightarrow \quad \frac{dg^{-1}}{dl} = (d-1)g^{-1} - \frac{N-2}{2}\frac{S_d}{(2\pi)^d}\coth\left(\frac{u}{2}\right)$$

$$\Rightarrow \quad \boxed{\frac{dg}{dl} = (1-d)g + \frac{N-2}{2}\frac{S_d}{(2\pi)^d}\coth\left(\frac{u}{2}\right) g^2} \quad (1)$$

- $u$ is just trivially rescaled, $u = \Lambda c \beta$ "inverse temperature"

$u' = e^{-dl} u$

$= (1 - dl) u$

$$u(l+dl) = u(l) - u(l)\,dl$$

$$\Rightarrow \quad \boxed{\frac{du}{dl} = -u} \quad (2)$$

- We introduce the dimensionless temperature

$$\boxed{t := \frac{g}{u} = \frac{\Lambda^{d-1} c / \rho_s}{\Lambda c \beta} = \Lambda^{d-2} \frac{T}{\rho_s}}$$

This is the ration we have analyzed in the classical RG calculation

- RG equation for $t$:

$$\frac{dt}{dl} = \frac{d}{dl}\left(\frac{g}{u}\right) = \frac{1}{u}\frac{dg}{dl} - \frac{g}{u^2}\frac{du}{dl}$$

$$\underset{(1)(2)}{=} \frac{1}{u}\left[(1-d)g + \frac{N-2}{2}\frac{S_d}{(2\pi)^d}\coth\left(\frac{u}{2}\right) g^2\right]$$

$$+ \frac{g}{u^2} u$$

$$\Rightarrow \boxed{\frac{dt}{dl} = (2-d)t + \frac{N-2}{2}\frac{S_d}{(2\pi)^d}\coth\left(\frac{g}{2t}\right) gt}$$

$$\boxed{\frac{d}{dl}\left(\frac{g}{t}\right) = -\frac{g}{t}}$$

$$\boxed{\frac{dg}{dl} = (1-d)g + \frac{N-2}{2}\frac{S_d}{(2\pi)^d}\coth\left(\frac{g}{2t}\right) g^2}$$

- The two RG equations control the scale dependent flow of coupling constant $g$ and temperature $t$

- $g$ controls strength of quantum, $t$ of thermal fluctuations

# 4.7. Phase Diagrams for d=2 and d=3

- In this section we will determine the phase diagrams in the parameter space of $t$ (dimensionless temperature) and $g$ (controlling strength of quantum fluctuations) by integrating the RG flow equations ($N=3$):

$$\boxed{\begin{aligned} \frac{dt}{dl} &= (2-d)t + \frac{1}{2}\frac{S_d}{(2\pi)^d}\coth\left(\frac{g}{2t}\right)gt \\ \frac{dg}{dl} &= (1-d)g + \frac{1}{2}\frac{S_d}{(2\pi)^d}\coth\left(\frac{g}{2t}\right)g^2 \end{aligned}}$$

initial conditions: $t_0 = t(0) = \Lambda^{d-2}\frac{T}{\rho_s}$

$$g_0 = g(0) = \Lambda^{d-1}\frac{c}{\rho_s}$$

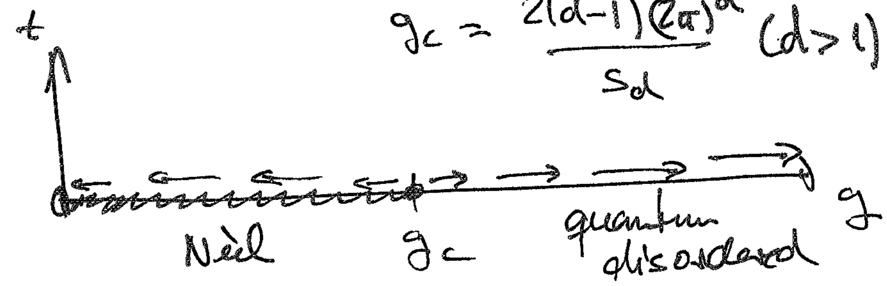
- Zero temperature, $T=0$

$\Rightarrow t(0) = 0 \Rightarrow t(l) \equiv 0$ Note: $\coth(x) \xrightarrow[x\to\infty]{} 1$

$$\rightarrow \boxed{\frac{dg}{dl} = (1-d)g + \frac{1}{2}\frac{S_d}{(2\pi)^d}g^2}$$

Fixed points: $g=0$ : no quantum fluctuations

$g_c = \frac{2(d-1)(2\pi)^d}{S_d}$ $(d>1)$ : quantum critical point

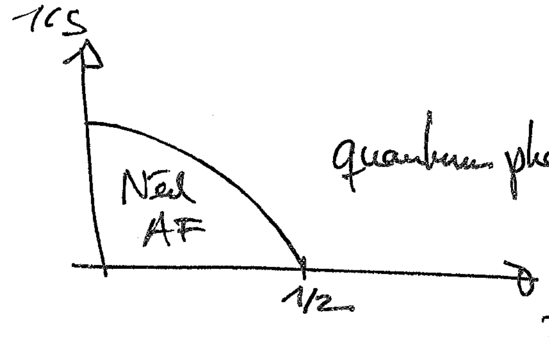


$g_0 < g_c$: on large scales, system is perfectly ordered, long-range Néel order

$g_0 > g_c$: quantum fluctuations become stronger on larger scales and diverge
→ system is quantum disordered

- The quantum critical point $g_c = \frac{2(d-1)(2\pi)^d}{S_d}$ translates into a phase boundary in the $1/S - \gamma = J_2/J_1$ - plane

e.g. $d=2$: $g_c = 4\pi \sim \frac{1}{S\sqrt{1-2\gamma}}$




quantum phase diagram

- No quantum fluctuations ($g=0$), finite temperature

$g=0 \Rightarrow u = g/t \rightarrow$ $(d+1)$-dim. quantum NLσM reduces to d-dim. classical NLσM

We should recover the classical RG equation if we take the limit $g \to 0$

$$\coth(x) \underset{x\to 0}{\sim} \frac{1}{x} \Rightarrow \coth\left(\frac{g}{2t}\right) gt \underset{g\to 0}{\longrightarrow} 2t^2$$

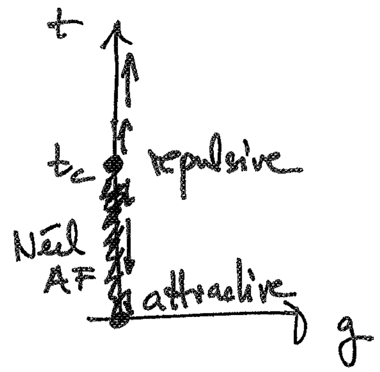
$$\rightarrow \quad \boxed{\frac{dt}{dl} = (2-d)t + \frac{S_d}{(2\pi)^d} t^2}$$

as derived previously for the classical NLσM

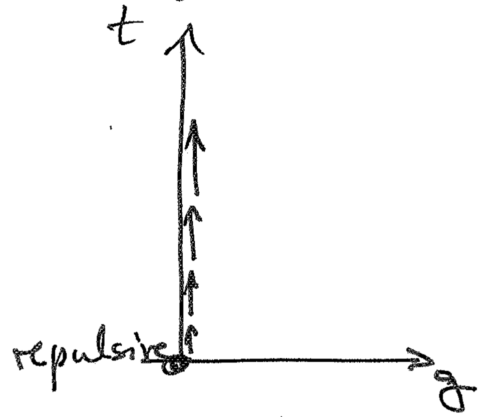
<u>fixed points</u>: $t = 0$

$$t_c = \frac{(d-2)(2\pi)^d}{S_d} \quad (d>2)$$

<u>d>2</u>:




<u>d=2</u>:



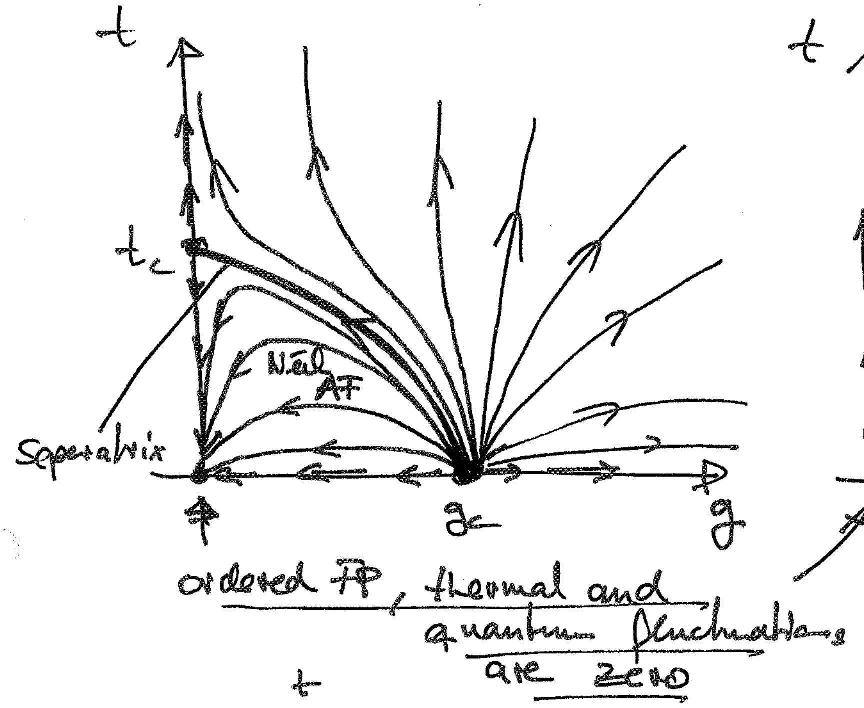

<u>for $t<t_c$</u>: thermal fluctuations renormalize to zero on large scales
$\Rightarrow$ <u>long-ranged AF order</u>

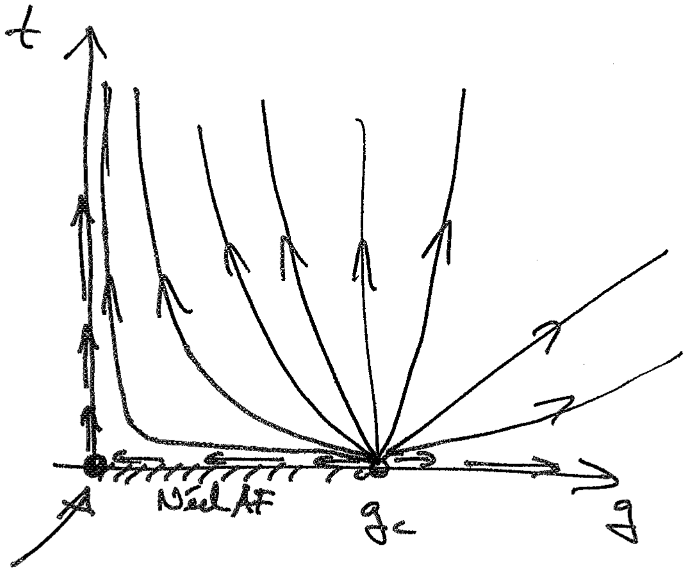
<u>for $t>t_c$</u>: fluctuation diverge on large scales $\Rightarrow$ no long range order, <u>thermally disordered</u>

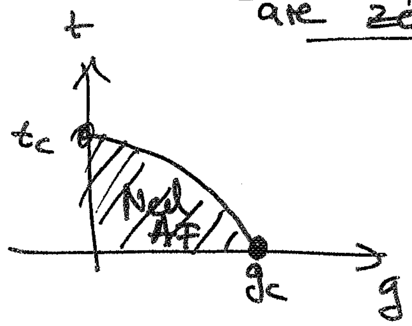
in $d=2$ long-ranged magnetic order is not possible at $t>0$ !

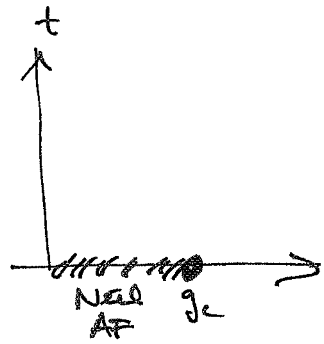
- Flow diagram in the g-t plane

d > 2: d = 2:



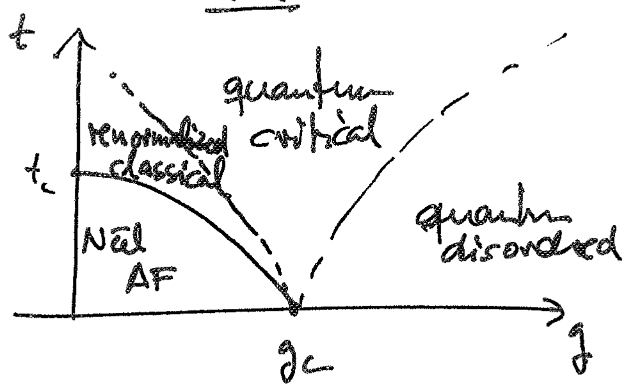



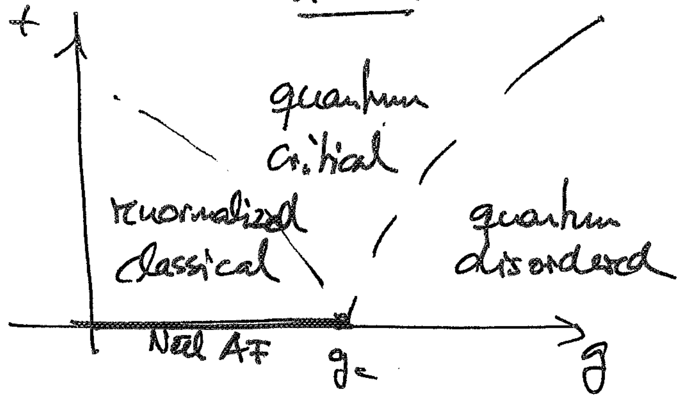




## 4.8. Finite Temperature Crossovers

d > 2 d = 2:




In the following, we will extract the finite temperature crossovers from the RG flow. We will focus on the case $d=2$.

- RG equations for $\underline{N=3}$ and $\underline{d=2}$

$$\frac{dg}{dl} = -g + \frac{1}{4\pi} \coth\left(\frac{g}{2t}\right) g^2$$

$$\frac{dt}{dl} = \frac{1}{4\pi} \coth\left(\frac{g}{2t}\right) gt$$

- Rescaling to absorb prefactors:

$$\boxed{\tilde{g} := \frac{g}{g_c}} = \boxed{\frac{g}{4\pi}} \qquad (\Rightarrow \tilde{g}_c = 1)$$

$$\boxed{\tilde{t} := \frac{t}{2\pi}}$$

$$\rightarrow \boxed{\begin{aligned} \frac{d\tilde{t}}{dl} &= \coth\left(\frac{\tilde{g}}{\tilde{t}}\right) \tilde{g}\tilde{t} \\ \frac{d\tilde{g}}{dl} &= -\tilde{g} + \coth\left(\frac{\tilde{g}}{\tilde{t}}\right) \tilde{g}^2 \end{aligned}}$$

- The RG equations can be solved analytically:

Step 1: Solve RG equation for $\frac{\tilde{g}}{\tilde{t}}$:

$$\frac{d}{dl} \frac{\tilde{g}}{\tilde{t}} = \frac{1}{\tilde{t}} \frac{d\tilde{g}}{dl} - \frac{\tilde{g}}{\tilde{t}^2} \frac{d\tilde{t}}{dl} = -\frac{\tilde{g}}{\tilde{t}}$$

$$\Rightarrow \boxed{\tilde{g}/\tilde{t} = \tilde{g}_0/\tilde{t}_0 \, e^{-l}}$$

Step 2: Plug into differential equation for $\tilde{t}$ and solve by separation of variables

$$\frac{d\tilde{t}}{dl} = \coth\left(\frac{\tilde{g}_0}{\tilde{t}_0} e^{-l}\right) \cdot \frac{\tilde{g}_0}{\tilde{t}_0} e^{-l} \, \tilde{t}^2$$

$$\underbrace{\int_{\tilde{t}_0}^{\tilde{t}} \frac{d\tilde{t}'}{\tilde{t}'^2}}_{= -\frac{1}{\tilde{t}} + \frac{1}{\tilde{t}_0}} = \underbrace{\frac{\tilde{g}_0}{\tilde{t}_0} \int_0^l dl' \, e^{-l'} \coth\left(\frac{\tilde{g}_0}{\tilde{t}_0} e^{-l'}\right)}_{= -\int_{\tilde{g}_0/\tilde{t}_0}^{\tilde{g}_0/\tilde{t}_0 \, e^{-l}} dz \, \coth(z)}$$

$$= -\ln(\sinh z) \Big|_{\tilde{g}_0/\tilde{t}_0}^{\tilde{g}_0/\tilde{t}_0 \, e^{-l}}$$

$$\Rightarrow \boxed{\tilde{t}(l) = \left[\frac{1}{\tilde{t}_0} + \ln \frac{\sinh\left(\frac{\tilde{g}_0}{\tilde{t}_0} e^{-l}\right)}{\sinh\left(\frac{\tilde{g}_0}{\tilde{t}_0}\right)}\right]^{-1}}$$

Step 3: From $\tilde{g}/\tilde{t} = \frac{\tilde{g}_0}{\tilde{t}_0} e^{-l}$ we obtain

$$\boxed{\tilde{g}(l) = e^{-l} \left[\frac{1}{\tilde{g}_0} + \frac{\tilde{t}_0}{\tilde{g}_0} \ln \frac{\sinh\left(\frac{\tilde{g}_0}{\tilde{t}_0} e^{-l}\right)}{\sinh \frac{\tilde{g}_0}{\tilde{t}_0}}\right]^{-1}}$$

- Only for $\tilde{t}_0 = 0$ and $\tilde{g}_0 < 1$ the system is ordered corresponding to an RG flow towards the ordered fixed point $(\tilde{g}, \tilde{t}) = (0, 0)$ where thermal and quantum fluctuations are suppressed.

- For $\tilde{g}_0 > 1$ or $\tilde{t}_0 > 0$ the correlation length $\xi$ is finite and the magnetic order is short ranged.

- Absence of long-range order corresponds to a divergence of $\tilde{g}(l)$ and/or $\tilde{t}(l)$

- If $\tilde{g}$ or $\tilde{t}$ becomes large, RG equations are no longer valid but we can still identify the correlation length

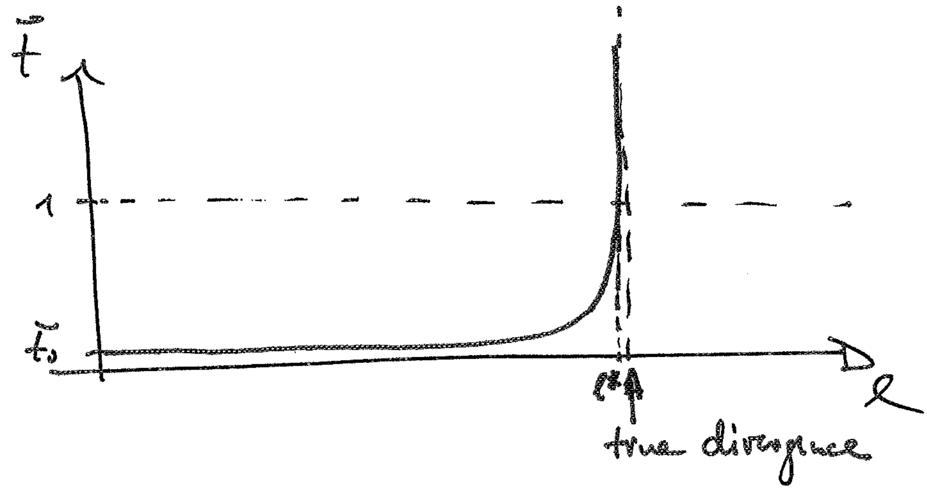
$$\boxed{\xi = a \cdot e^{l^*}}$$

from the scale $l^*$ where $\tilde{g}$ or $\tilde{t}$ shoots up:

Determine $l^*$ by $\boxed{\tilde{g}(l^*) = 2 \quad \text{or} \quad \tilde{t}(l^*) = 1,}$

whatever comes first

- Note that close to the transition, the divergence is very sharp that it doesn't matter which constant we use to determine $l^*$:



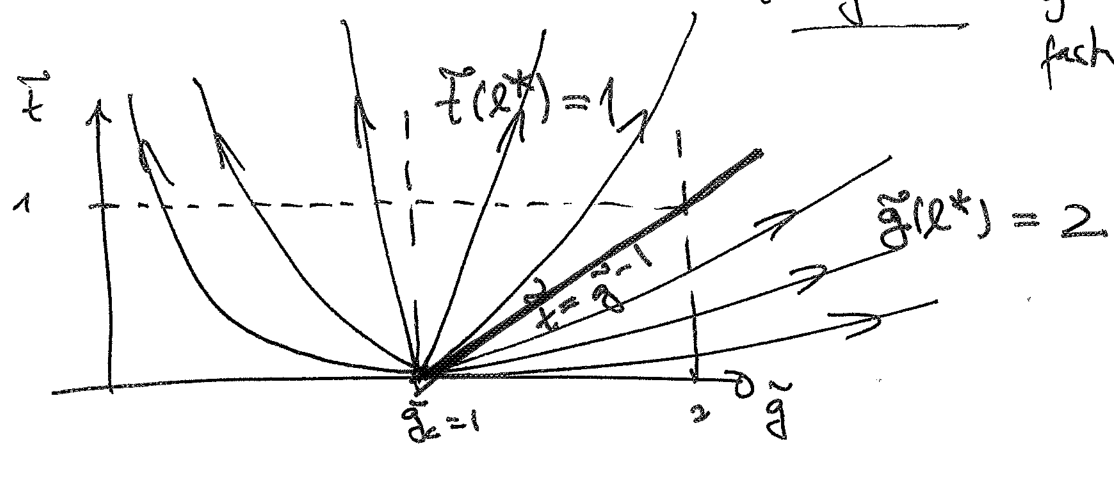

- If $\tilde{g}(l^*) = 2$ happens first, the order is destroyed predominantly by quantum fluctuations, whereas $\tilde{t}(l^*) = 1$ indicates the dominance of thermal fluctuations.

- Look at the low-temperature limit: $\left(\coth\left(\frac{\tilde{g}}{\tilde{t}}\right) \simeq 1\right)$

$$\left.\begin{aligned} \frac{d\tilde{t}}{dl} &\simeq \tilde{g}\tilde{t} \\ \frac{d\tilde{g}}{dl} &\simeq \tilde{g}(\tilde{g}-1) \end{aligned}\right\} \Rightarrow$$

$\underline{\tilde{t} > \tilde{g} - 1}$ : $\tilde{t}$ grows faster than $\tilde{g}$

$\underline{\tilde{t} < \tilde{g} - 1}$ : $\tilde{g}$ grows faster than $\tilde{t}$




- <u>Order destroyed by thermal fluctuations: $\tilde{t} > \tilde{g} - 1$</u>

$\tilde{t}(l^*) = 1$ , $\xi = a e^{l^*}$

$$1 = \frac{1}{\tilde{t}_0} + \ln \frac{\sinh\left(\frac{\tilde{g}_0}{\tilde{t}_0} \frac{a}{\xi}\right)}{\sinh\left(\frac{\tilde{g}_0}{\tilde{t}_0}\right)}$$

$$\Rightarrow \quad \exp\left(1 - \frac{1}{\tilde{t}_0}\right) = \frac{\sinh\left(\frac{\tilde{g}_0}{\tilde{t}_0} \frac{a}{\xi}\right)}{\sinh\left(\frac{\tilde{g}_0}{\tilde{t}_0}\right)}$$

$$\Rightarrow \quad \boxed{\frac{1}{\xi} = \frac{\tilde{t}_0}{a\tilde{g}_0} \sinh^{-1}\left(\exp\left(1 - \frac{1}{\tilde{t}_0}\right) \sinh\left(\frac{\tilde{g}_0}{\tilde{t}_0}\right)\right)}$$

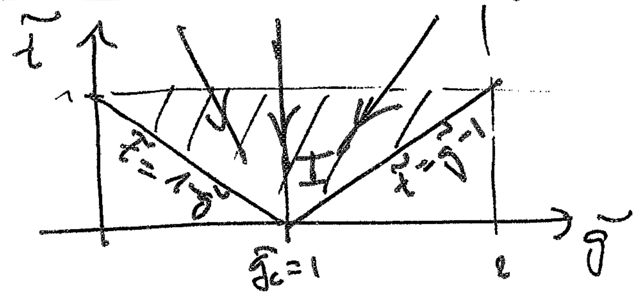
$$\boxed{\frac{\tilde{t}_0}{a\tilde{g}_0} = \frac{t_0/2\pi}{a g_0/4\pi} = \frac{2t_0}{g_0} = \frac{2T}{(a\Lambda)c} \sim \frac{k_B T}{\hbar c}}$$

- In the present case where thermal fluctuation dominate, we can distinguish between two regimes (I. quantum critical and II. renormalized classical regime)

$$\exp\left(1 - \frac{1}{\tilde{t}_0}\right)\sinh\left(\frac{\tilde{g}_0}{\tilde{t}_0}\right) = \frac{1}{2}\exp\left(1 - \frac{1}{\tilde{t}_0}\right)\left(\exp\left(\frac{\tilde{g}_0}{\tilde{t}_0}\right) - \exp\left(-\frac{\tilde{g}_0}{\tilde{t}_0}\right)\right)$$

$$= \frac{1}{2}\left(\exp\left(1 - \frac{1-\tilde{g}_0}{\tilde{t}_0}\right) - \underbrace{\exp\left(1 - \frac{\tilde{g}_0+1}{\tilde{t}_0}\right)}_{\xrightarrow{\tilde{t}_0 \to 0}\ 0}\right)$$

I. Quantum critical Region

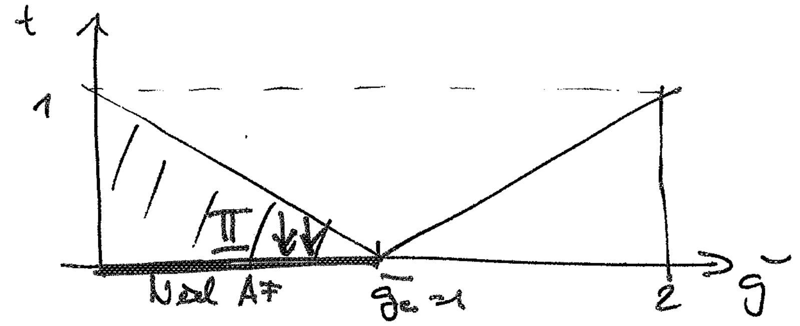



$$\boxed{1 - \tilde{t} < \tilde{g} < 1 + \tilde{t}}$$

$$\boxed{\xi = c\,\frac{\hbar c}{k_B T}}$$

At $\tilde{g}_0 = \tilde{g}_c = 1$ the system behaves like a 3dim classical spin system at its critical point for length scales less than the effective "slab thickness" $u = \hbar c \beta = C \frac{\hbar c}{k_B T}$

II. Renormalized classical Region $\boxed{\tilde{t} < 1 - \tilde{g}}$




$$\xi^{-1} \underset{T \to 0}{\approx} \frac{\tilde{t}_0}{a \tilde{g}_0} \sinh^{-1}\left( \frac{1}{2} \exp\left( - \frac{1 - \tilde{g}_0}{\tilde{t}_0} \right) \right)$$

$$\approx \frac{\tilde{t}_0}{2 a \tilde{g}_0} \exp\left( - \frac{1 - \tilde{g}_0}{\tilde{t}_0} \right)$$

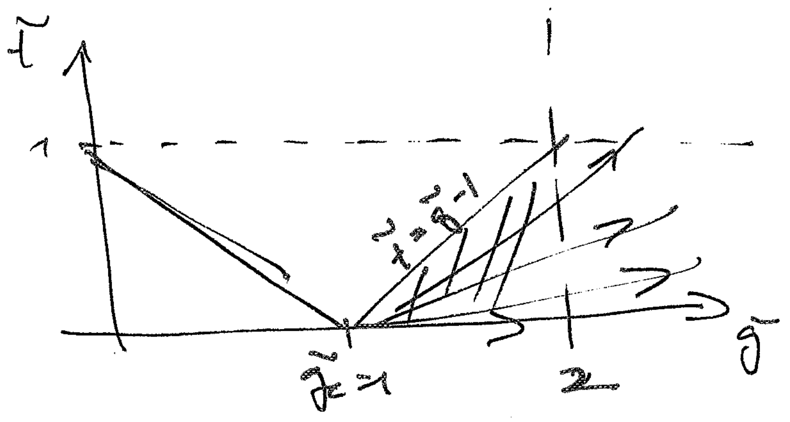
$$\Rightarrow \boxed{\xi = C \cdot \frac{\hbar c}{k_B T} \exp\left( 2\pi (1 - \tilde{g}_0) \frac{\rho_s}{k_B T} \right)}$$

prefactor changes if higher loop corrections are included

compare with classical d=2 case: $\xi_{cl} = 0.36 a \exp\left( \frac{2\pi \rho_s}{k_B T} \right)$

- $\xi$ diverges as in the classical model
- spin-stiffness is renormalized by quantum fluctuations: $\rho_s = (1 - \tilde{g}_0) \rho_s^{cl}$
- short wavelength replaced by thermal de Broglie wavelength $\hbar c / k_B T$

## III. Quantum Disordered Regime $(\tilde{t} < \tilde{g}-1)$



order destroyed by quantum fluctuations (even at $\tilde{t}=0$)

$\xi = a e^{\ell^*}$ , $\ell^*$ determined by $\tilde{g}(\ell^*) = 2$

$$\Rightarrow \quad \boxed{\xi = \frac{a\tilde{g}_0/2}{\tilde{g}_0 - 1 + \tilde{t}_0 \exp\left(-4\frac{\tilde{g}_0-1}{\tilde{t}_0}\right)} \simeq \frac{a\tilde{g}_0/2}{\tilde{g}_0-1}}$$

- correlation length is finite even at $T=0$

$$\xi_{T=0} = \frac{a\tilde{g}_0/2}{\tilde{g}_0-1} \sim \frac{1}{\Delta}$$

- corrections to the zero-temperature $(\tilde{t}_0=0)$ value are exponentially small at low temperatures
- Exponential temperature dependence is signature of a gap $\Delta \sim \tilde{g}_0 - 1$

- $\xi \sim (\tilde{g}_0-1)^{-1} \Rightarrow$ correlation-length exponent $\nu = 1$

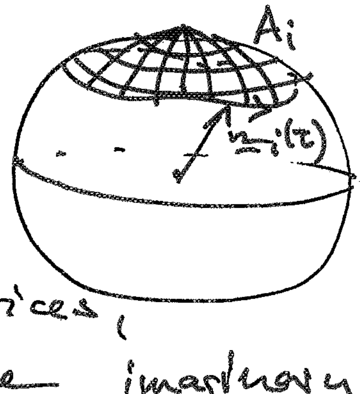
  value will change if we include higher-loop order correction
- Exponent should be the same as for the classical $D=3=2+1$ dim. Heisenberg model, $\nu = 0.7048 \pm 0.0030$ from Monte-Carlo

## 4.9 Crucial Role of Berry Phases

- So far, we have neglected the Berry phases

$$S_B' = iS \sum_i \epsilon_i \int_0^\beta d\tau \int_0^1 du \; \underline{n}_i \cdot \left( \frac{\partial n_i}{\partial \tau} \times \frac{\partial n_i}{\partial u} \right) = iS \sum_i \epsilon_i A_i$$

$$\epsilon_i = \begin{cases} +1 & i \in \text{sublattice } A \\ -1 & i \in \text{sublattice } B \end{cases}$$




- No continuum limit possible,
- alternates between the two sublattices,
- no classical interpretation since imaginary (phase)

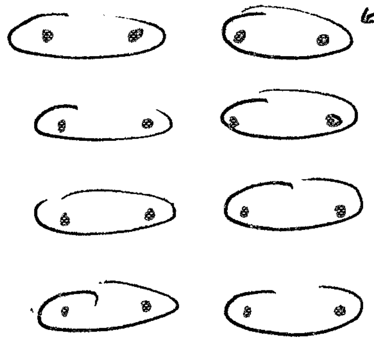
- In the Néel ordered phase, Berry phases are irrelevant and interfere destructively since $\underline{n}$ is smooth on the scale of the lattice and phases on the two sublattices sum up to zero since they have opposite sign
- In the paramagnetic phase, they can interfere constructively
  - + F.D.M. Haldane, Phys. Rev. Lett. 61, 1029 (1988)
  - + N. Read, S. Sachdev, Phys. Rev. Lett. 62, 1694 (1989)
  - + N. Read, S. Sachdev, Phys. Rev. B 42, 4568 (1990)
  - + E. Fradkin, M. Stone, Phys. Rev. B 38, R7215 (1988)

→ Breaking of $D_4$ lattice symmetry (90° rotations) by the formation of VBS phases

VBS = „Valence Bond Solid"

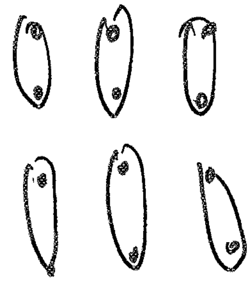
- Symmetry breaking in the paramagnet depends on the spin value $S$

$S = 1/2$ : $D_4$ symmetry completely broken



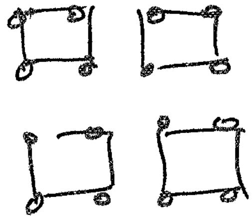
$= \frac{1}{\sqrt{2}} (\uparrow - \downarrow \;-\; \downarrow - \uparrow)$ singlet

or:



Columnar VBS

or superposition



plaquette VBS

$S = 1$ : chain-like VBS state

$D_4$ symmetry broken to $D_2$

$S = 2$ : (or other even integer spin)
no breaking of lattice symmetry, Berry phases are always irrelevant

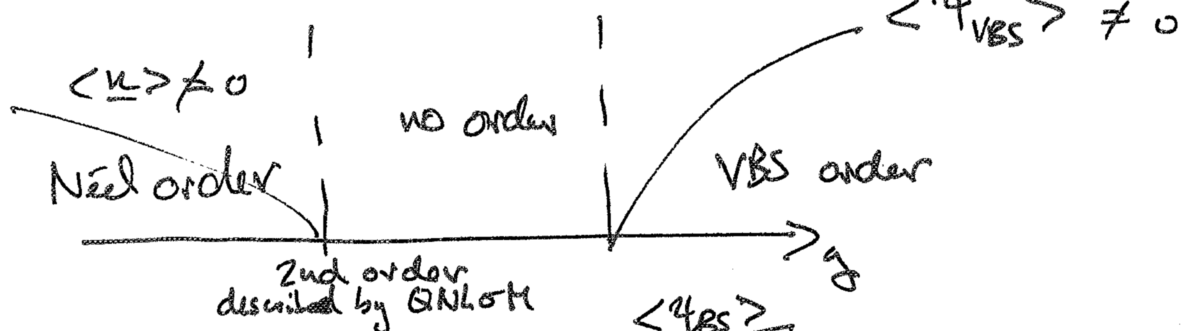
- Two phases with breakings of very different symmetry :

Néel AF : spin-rotation symmetry is broken

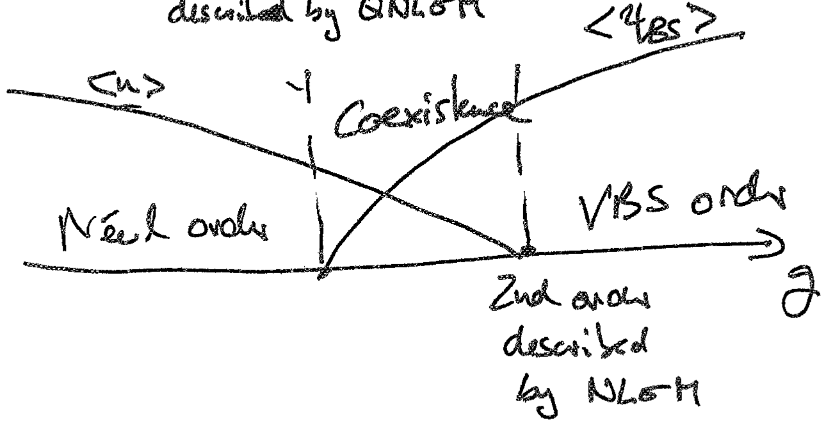
VBS : lattice rotation symmetry is broken

- From the standard Landau-Ginzburg-Wilson paradigm, we expect one of the following scenarios

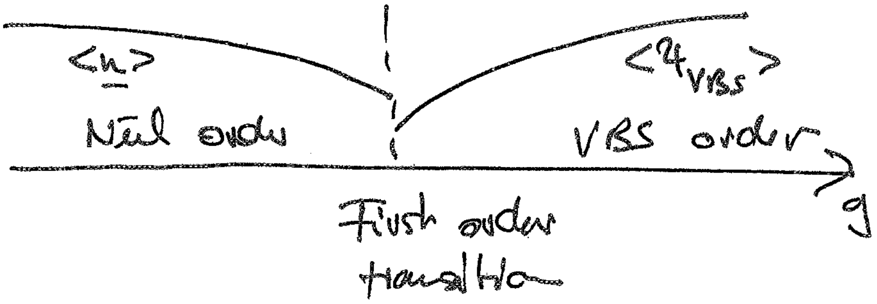
a)




b)




c)




A single 2nd order transition between the Néel AF and the VBS solid is extremely unlikely and would require fine-tuning.

- Theory of the „<u>Deconfined Quantum Critical Point</u>" predicts a direct 2nd order transition!

(T. Senthil, A. Vishwanath, L. Balents, S. Sachdev, M.P.A. Fisher, Phys. Rev. B 70, 144407 (2004), Science 303, 1490 (2004))

+ fractionalized spinor field $z$, $\underline{n} = z^{\dagger} \underline{\sigma} z$ coupled to compact U(1) gauge field, $z \to e^{i\gamma(\underline{r},t)} z$
+ Deconfinement of fractionalized excitation at and only at QCP ($\to$ different universality)