2. Ising chain in a transversal field

2.1. Quantum Ising model

$$\boxed{\hat{H} = -J\sum_{\langle ij\rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z - Jg\sum_i \hat{\sigma}_i^x}$$

- $\langle ij\rangle$ denotes summation over nearest-neighbor bonds
- Can be defined on different lattices in different spatial dimensions d

  Here: $d=1$ dimensional chain •–•–•–•–•
- $\hat{\sigma}_i^\alpha$ spin-½ operators on site i ; $\hat{S}_i^\alpha = \frac{\hbar}{2}\hat{\sigma}_i^\alpha$
- Commutator relations :

  same site: $[\hat{S}_i^\alpha, \hat{S}_i^\beta] = i\hbar\,\epsilon_{\alpha\beta\gamma}\hat{S}^\gamma$

  $\Rightarrow \boxed{[\hat{\sigma}_i^\alpha, \hat{\sigma}_i^\beta] = 2i\,\epsilon_{\alpha\beta\gamma}\,\hat{\sigma}^\gamma}$

  different sites $i\neq j$: $\boxed{[\hat{\sigma}_i^\alpha, \hat{\sigma}_j^\beta] = 0 \quad \text{for } i\neq j}$

  in the basis $|\uparrow\rangle, |\downarrow\rangle$ of $\hat{S}^z$:

  $\underline{\underline{\sigma}}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

  $\underline{\underline{\sigma}}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

  $\underline{\underline{\sigma}}^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$
- $J>0$ exchange constant which sets the microscopic energy scale
- $g>0$ dimensionless coupling constant

  → used to tune $\hat{H}$ across quantum phase transition
- Limiting behavior :

  $g\to 0$: ferromagnetic ground state , $|0\rangle = \prod_i |\uparrow\rangle_i$ or $|0\rangle = \prod_i |\downarrow\rangle_i$

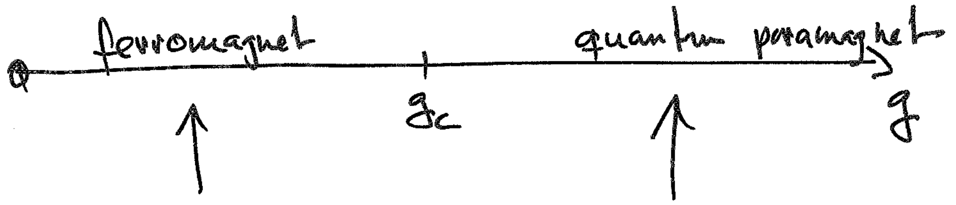
  Spontaneous breaking of $\mathbb{Z}_2$ symmetry

  $g\to\infty$: spins point along the field direction

  → non-degenerate ground state $|0\rangle = \prod_i |\rightarrow\rangle_i$

  $|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$

- We expect a <u>quantum-phase transition</u>

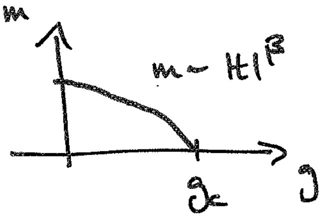


$$t := \frac{g-g_c}{g_c}$$

$\langle \hat{\sigma}_i^z \rangle = m \neq 0$

$\langle \hat{\sigma}_i^z \hat{\sigma}_j^z \rangle \xrightarrow[|r_i - r_j| \to \infty]{} m^2$

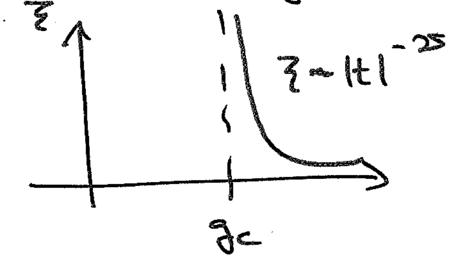
(long-range order)



$\langle \hat{\sigma}_i^z \rangle = 0$

$\langle \hat{\sigma}_i^z \hat{\sigma}_j^z \rangle \underset{|r_i - r_j| \to \infty}{\sim} e^{-|r_i - r_j|/\xi}$

(short range order)



- This is one of the very few examples where we can calculate critical exponents analytically
- Before we do so, we will try to understand the nature of the excitation in the two limiting cases ($g \to \infty$, $g \to 0$)

2.2. Strong Coupling, $g \gg 1$

- for $g = \infty$ ($1/g = 0$) groundstate $|0\rangle = \prod_i |\rightarrow\rangle_i$
- what are the "<u>propagating degrees of freedom</u>" at $1/g \ll 1$?
- Look at the excited states of $H_B = -Jg \sum_i \hat{\sigma}_i^x$

first excited state $|i\rangle := |\leftarrow\rangle_i \prod_{j(\neq i)} |\rightarrow\rangle_j$

this is like a quasiparticle on site i !

Excitation energy : $\Delta E = \langle i | H_B | i \rangle = 2Jg$

- These states are not eigenstates of $H_f = -J\sum_{\langle ij\rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z$ but we can express H_f (and H_B) in the „1-particle sector"

$$|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$$
$$|\leftarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$$

$$|i\rangle = |\cdots \rightarrow \rightarrow \cdots \rightarrow \overset{i}{\leftarrow} \rightarrow \cdots\rangle$$

$$\hat{\sigma}_z|\rightarrow\rangle = |\leftarrow\rangle$$
$$\hat{\sigma}_z|\leftarrow\rangle = |\rightarrow\rangle$$

$$H_f|i\rangle = -J\sum_l \hat{\sigma}_l^z \hat{\sigma}_{l+1}^z |i\rangle$$

$$= -J\Big(\underbrace{|\cdots \rightarrow \rightarrow \underset{i}{\rightarrow} \underset{i+1}{\leftarrow} \rightarrow \cdots\rangle}_{l=i} + \underbrace{|\cdots \rightarrow \underset{i-1}{\leftarrow} \underset{i}{\rightarrow} \rightarrow \cdots\rangle}_{l=i+1}$$
$$+ \text{ states in 3-particle sector}\Big)$$

$$\simeq -J(|i+1\rangle + |i-1\rangle) \qquad \langle j|H_f|i\rangle = -J(\delta_{j,i+1} + \delta_{j,i-1})$$

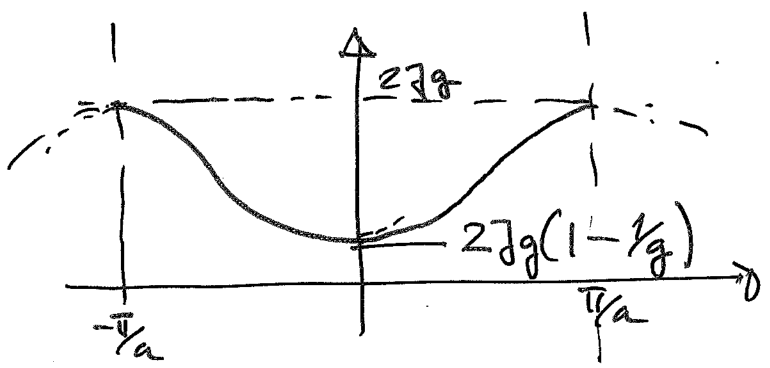
$$\rightarrow \hat{H} \simeq Jg\sum_i \left\{ 2|i\rangle\langle i| - \frac{1}{g}\big(|i\rangle\langle i+1| + |i\rangle\langle i-1|\big)\right\}$$

- Hamiltonian can be diagonalized by going to momentum space:

$$|k\rangle = \frac{1}{\sqrt{N}}\sum_i e^{ikx_i}|i\rangle$$

$$\rightarrow \hat{H} = \sum_k \epsilon_k |k\rangle\langle k| \quad \text{with} \quad \boxed{\epsilon_k = Jg\left(2 - \tfrac{2}{g}\cos(ka) + \mathcal{O}(1/g^2)\right)}$$

a lattice constant



2.3. Weak coupling, $g \ll 1$

- To identify the propagating degrees of freedom for $g \ll 1$, we have to look at the first excited states of $H_f = -J \sum_{\langle ij \rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z$
- First excited states are the ones where exactly one bond is unhappy:

$$|i\rangle = |\uparrow \cdots \underset{i}{\uparrow} \,|\, \underset{i+1}{\downarrow} \downarrow \cdots \downarrow\rangle$$

– quasiparticle picture
– domain walls or kinks in the magnetic order

Excitation energy $\Delta E = \langle i | H_f | i \rangle = 2J$

- We can also express $H_B = -Jg \sum_i \hat{\sigma}_i^x$ in this basis: $\hat{\sigma}^x |\uparrow\rangle = |\downarrow\rangle$, $\hat{\sigma}^x |\downarrow\rangle = |\uparrow\rangle$
  ($\hat{\sigma}^x$ spin-flip operator)

$$H_B |i\rangle = -Jg \sum_l \hat{\sigma}_l^x |i\rangle$$

$$= -Jg \Big(\underbrace{|\uparrow \cdots \underset{i-1}{\uparrow} \,|\, \underset{i}{\downarrow} \underset{i+1}{\downarrow} \cdots \downarrow\rangle}_{l=i} + \underbrace{|\uparrow \cdots \underset{i-1}{\uparrow} \underset{i}{\uparrow} \underset{i+1}{\uparrow} \,|\, \underset{i+2}{\downarrow} \cdots \downarrow\rangle}_{l=i+1}$$

$$+ \text{ states in the 3-particle sector} \Big)$$

$$\simeq -Jg \left(|i-1\rangle + |i+1\rangle \right)$$

$$\rightarrow H \simeq J \sum_i \left\{ 2 |i\rangle\langle i| - g\left(|i\rangle\langle i+1| + |i\rangle\langle i-1| \right) \right\}$$

- Diagonalization by momentum transformation

$$\boxed{\epsilon(k) = J(2 - 2g\cos(ka)) + \mathcal{O}(g^2)}$$

2.4. Exact Spectrum / Jordan – Wigner Transformation

- Qualitative considerations of previous chapters useful to develop intuitive physical picture
- Now we will calculate the exact spectrum by using the Jordan-Wigner transformation (JWT)

- Analytical solution possible for three reasons (crucial ingredients of JWT)
  - 1d problem
  - Hilbert space of $S=1/2$ spin can be identified with the one of a spinless fermion
  - resulting fermion system in non-interacting!

- Consider fermi operators c_i^+, c_i creating and annihilating a spinless fermion on site i

$$\boxed{\{c_i, c_j\} = \{c_i^+, c_j^+\} = 0 \quad , \quad \{c_i, c_j^+\} = c_i c_j^+ + c_j^+ c_i = \delta_{ij}}$$

Hilbert space on site i: $|0\rangle_i$ = no fermion
$|1\rangle_i$ = 1 fermion

We make the following identification:

spin 1/2	spinless fermion
$\|\uparrow\rangle_i$	$\|0\rangle_i$
$\|\downarrow\rangle_i$	$\|1\rangle_i$

$$\rightarrow \boxed{\hat{\sigma}_i^z = 1 - 2c_i^+ c_i}$$

Note: This relation is consistent with the fact that spin operators on different site commute

c_i operators anticommute but $c_i^+ c_i = \hat{n}_i$ commutes:

$i \neq j$: $\hat{n}_i \hat{n}_j = c_i^+ c_i c_j^+ c_j = (-1)^2 c_j^+ c_i^+ c_i c_j = (-1)^2 (-1)^2 c_j^+ c_j \hat{n}_i = \hat{n}_j \hat{n}_i$

- How to express spin operators $\hat{\sigma}_i^x, \hat{\sigma}_i^y$ by c_i^+ and c_i?

$$\hat{\sigma}_i^+ = \frac{\hat{\sigma}_i^x + i\hat{\sigma}_i^y}{2} \qquad \hat{\sigma}_i^- = \frac{\hat{\sigma}_i^x - i\hat{\sigma}_i^y}{2}$$

$$\hat{\sigma}_i^+ |\downarrow\rangle_i = |\uparrow\rangle_i \qquad \hat{\sigma}_i^- |\downarrow\rangle_i = 0$$

$$\hat{\sigma}_i^+ |\uparrow\rangle_i = 0 \qquad \hat{\sigma}_i^- |\uparrow\rangle_i = |\downarrow\rangle_i$$

looks like what fermi operator are doing

$$c_i |1\rangle_i = |0\rangle_i \qquad c_i^+ |1\rangle_i = 0$$

$$c_i |0\rangle_i = 0 \qquad c_i^+ |0\rangle_i = |1\rangle_i$$

$$\rightarrow \begin{cases} \sigma_i^+ = c_i \\ \sigma_i^- = c_i^+ \end{cases} \quad ??$$

Works on the same site:

$$[\hat{\sigma}^+, \hat{\sigma}^-] = [c, c^+] = cc^+ - c^+c = 1 - 2c^+c = \hat{\sigma}^z$$

consistent with spin commutator relation

Does not work if operator are on different sites:

$$[\hat{\sigma}_i^+, \hat{\sigma}_j^-] = 0 \neq [c_i, c_j^+] \quad , \quad \{c_i, c_j^+\} = 0$$

- Unfortunately, there is no local transformation. The solution to this problem was found by Jordan and Wigner [Z. Phys. 47, 631 (1928)]:

$$\boxed{\begin{aligned} \hat{\sigma}_i^+ &= \prod_{j<i} (1 - 2c_j^+ c_j)\, c_i \\ \hat{\sigma}_i^- &= \prod_{j<i} (1 - 2c_j^+ c_j)\, c_i^+ \end{aligned}}$$

- non-local transformation
- string operator
- works only in 1d

In the following we show that the operators $\hat\sigma_i^+, \hat\sigma_i^-$ defined as above, together with $\hat\sigma_i^z = 1-2c_i^+c_i$ obey the spin-commutator relations:

$$(1)\quad [\hat\sigma_i^+,\hat\sigma_j^-] = \delta_{ij}\,\hat\sigma_i^z \qquad \text{and} \qquad (2)\quad [\hat\sigma_i^z,\hat\sigma_j^\pm] = \pm 2\delta_{ij}\,\hat\sigma_i^\pm$$

Eq.(1): define: $D_i := \prod_{l<i}(1-2c_l^+c_l)$

i<j:

$$D_iD_j = \prod_{i\le l<j}(1-2c_l^+c_l) = D_jD_i \qquad D_i^2 = 1$$

$i\neq j$: $[\hat n_i,\hat n_j]=0$

$$(1-2c^+c)^2 = 1-4\hat n+4\hat n^2 \underset{\hat n^2=\hat n}{=} 1$$

$$\hat\sigma_i^+\hat\sigma_j^- = D_ic_iD_jc_j^+ = c_iD_iD_jc_j^+ = c_i\prod_{i\le l<j}(1-2c_l^+c_l)c_j^+$$

$$\underset{\left(\begin{array}{c}c(1-2c^+c)\\ = -(1-2c^+c)c\end{array}\right)}{=} -\prod_{i\le l<j}(1-2c_l^+c_l)\,c_ic_j^+ = \prod_{i\le l<j}(1-2c_l^+c_l)\,c_j^+c_i$$

$$\hat\sigma_j^-\hat\sigma_i^+ = D_jc_j^+D_ic_i \underset{i<j}{=} D_jD_i\,c_j^+c_i = \hat\sigma_i^+\hat\sigma_j^-$$

$$\rightarrow\ [\hat\sigma_i^+,\hat\sigma_j^-] = 0$$

i>j:

$$[\hat\sigma_i^+,\hat\sigma_j^-] = [\hat\sigma_j^+,\hat\sigma_i^-]^+ \underset{\substack{j<i\\ \text{(previous case)}}}{=} 0^+ = 0$$

i=j:

$$\left.\begin{array}{l}\hat\sigma_i^+\hat\sigma_i^- \underset{D_i^2=1}{=} c_ic_i^+\\ \hat\sigma_i^-\hat\sigma_i^+ = c_i^+c_i\end{array}\right\} \Rightarrow [\hat\sigma_i^+,\hat\sigma_i^-] = c_ic_i^+ - c_i^+c_i$$

$$= 1-2c_i^+c_i$$

$$= \hat\sigma_i^z$$

Eq. (2): $[\hat{\sigma}_i^z, \hat{\sigma}_j^+] = [1-2c_i^+c_i, D_j c_j] = -2[c_i^+c_i, D_j c_j]$

$$= -2D_j \underbrace{[c_i^+c_i, c_j]} - 2\underbrace{[c_i^+c_i, D_j]}_{=0} c_j$$

$$= c_i^+c_i c_j - c_j c_i^+ c_i$$
$$= -c_i^+ c_j c_i - c_j c_i^+ c_i$$
$$= -(\delta_{ij} - c_j c_i^+) c_i - c_j c_i^+ c_i$$
$$= -\delta_{ij} c_i$$

$$= 2\delta_{ij} D_i c_i = 2\delta_{ij} \hat{\sigma}_i^+$$

$$[\hat{\sigma}_i^z, \hat{\sigma}_j^-] = [\hat{\sigma}_j^+, \hat{\sigma}_i^z]^+ = -[\hat{\sigma}_i^z, \hat{\sigma}_j^+]^+$$
$$= -(2\delta_{ij} \hat{\sigma}_i^+)^+ = -2\delta_{ij} \hat{\sigma}_i^-$$

The transformation introduced here is the JWT in its conventional form. In the analysis of the Ising model, it is convenient to rotate the spins by 90° about the y-axis, so that

$$\hat{\sigma}^z \to \hat{\sigma}^x, \quad \underset{\hat{\sigma}^+ + \hat{\sigma}^-}{\overset{\hat{\sigma}^x}{\shortparallel}} \to -\hat{\sigma}^z$$

The mapping becomes:

$$\boxed{\begin{aligned} \hat{\sigma}_i^x &= 1 - 2c_i^+ c_i \\ \hat{\sigma}_i^z &= -\prod_{j<i} (1-2c_i^+c_i)(c_i + c_i^+) \end{aligned}}$$

We can now express the Ising model in terms of fermi operators:

$$H_B = -Jg \sum_i \hat{\sigma}_i^x = -JgN + 2Jg \sum_i c_i^+ c_i$$

$$H_f = -J \sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z$$

$$\hat{\sigma}_i^z \hat{\sigma}_{i+1}^z = D_i (c_i + c_i^+) D_{i+1} (c_{i+1} + c_{i+1}^+)$$

$$\underset{D_i D_{i+1} = 1 - 2c_i^+ c_i}{=} (c_i + c_i^+)(1 - 2c_i^+ c_i)(c_{i+1} + c_{i+1}^+)$$

$$\underset{(*)}{=} (-c_i + c_i^+)(c_{i+1} + c_{i+1}^+)$$

$$= c_i^+ c_{i+1} + c_{i+1}^+ c_i + c_i^+ c_{i+1}^+ + c_{i+1} c_i$$

(*)
$$c(1-2c^+c) = c - 2cc^+c = c - 2c + \underbrace{2c^+cc}_{=0} = -c$$
$$c^+(1-2c^+c) = c^+ - \underbrace{2c^+c^+c}_{=0} = c^+$$

$$\rightarrow \boxed{H = -J \sum_i \{ c_i^+ c_{i+1} + c_{i+1}^+ c_i + c_i^+ c_{i+1}^+ + c_{i+1} c_i - 2g c_i^+ c_i \}}$$

<u>momentum space</u>: $c_i = \int_k e^{-ikr_i} c_k$, $c_i^+ = \int_k e^{ikr_i} c_k^+$ $\left(\int_k = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \right)$

$a = 1$ lattice constant

$$\rightarrow \boxed{H = J \int_k \{ 2(g - \cos k) c_k^+ c_k + i \sin k (c_k^+ c_{-k}^+ + c_k c_{-k}) \}}$$

Next, we perform a <u>Bogolinbov transformation</u> to a new set of fermionic operators γ_k, γ_k^+ whose number is conserved.

$$\boxed{\gamma_k = u_k c_k - i v_k c_{-k}^+ \quad \text{with } u_k, v_k \in \mathbb{R},\ u_{-k} = u_k,\ v_{-k} = -v_k,\ u_k^2 + v_k^2 = 1}$$

Note: If we define the transformation as above, it is guaranteed that γ_k, γ_k^+ obey the canonical anticommutator relations

$$\boxed{\{\gamma_k, \gamma_{k'}\} = \{\gamma_k^+, \gamma_{k'}^+\} = 0 \quad , \quad \{\gamma_k, \gamma_{k'}^+\} = \delta_{kk'}}$$

Check this:

$$\{\gamma_k, \gamma_{k'}^+\} = \{u_k c_k - i v_k c_{-k}^+ , u_{k'} c_{k'}^+ + i v_{k'} c_{-k'}\}$$
$$= u_k u_{k'} \{c_k, c_{k'}^+\} + v_k v_{k'} \{c_{-k}^+, c_{-k'}\}$$
$$= (u_k^2 + v_k^2)\,\delta_{kk'} = \delta_{kk'}$$

$$\{\gamma_k, \gamma_{k'}\} = \{u_k c_k - i v_k c_{-k}^+ , u_{k'} c_{k'} - i v_{k'} c_{-k'}^+\}$$
$$= -i u_k v_{k'} \{c_k, c_{-k'}^+\} - i v_k u_{k'} \{c_{-k}^+, c_{k'}\}$$
$$= -i u_k v_{k'}\,\delta_{k',-k} - i v_k u_{k'}\,\delta_{k',-k}$$
$$= (-i u_k v_{-k} - i v_k u_{-k})\,\delta_{k',-k}$$
$$= \underbrace{(+i u_k v_k - i v_k u_k)}_{=0}\,\delta_{k',-k}$$
$$= 0$$

$$\{\gamma_k^+, \gamma_{k'}^+\} = \{\gamma_k, \gamma_{k'}\}^+ = 0$$

Determine u_k, v_k such that H is diagonal in the operators γ_k, γ_k^+:

$$\boxed{H = \int_k \epsilon_k \gamma_k^+ \gamma_k}$$

$$\gamma_k^+ \gamma_k = (u_k c_k^+ + i v_k c_{-k})(u_k c_k - i v_k c_{-k}^+)$$
$$= u_k^2 c_k^+ c_k + v_k^2 c_{-k} c_{-k}^+ - i u_k v_k c_k^+ c_{-k}^+ + i u_k v_k c_{-k} c_k$$
$$\underset{k \to -k}{=} (u_k^2 - v_k^2) c_k^+ c_k + i u_k v_k (c_{-k}^+ c_k^+ + c_{-k} c_k)$$

$$\Rightarrow \begin{cases} \text{I.}\ \epsilon_k (u_k^2 - v_k^2) = 2J(g - \cos k) \\ \text{II.}\ i\epsilon_k u_k v_k = -iJ \sin k \end{cases}$$

Since $u_k^2 + v_k^2 = 1$ and $u_{-k} = u_k$, $v_{-k} = -v_k$ we can parametrize as

$$u_k = \cos\phi_k \qquad \text{with} \quad \phi_{-k} = -\phi_k$$
$$v_k = \sin\phi_k$$

$$\rightarrow \begin{cases} \text{I.}\ \epsilon_k \overbrace{(\cos^2\phi_k - \sin^2\phi_k)}^{\cos(2\phi_k)} = 2J(g - \cos k) \\ \text{II.}\ \epsilon_k \underbrace{\cos\phi_k \sin\phi_k}_{\sin(2\phi_k)/2} = -J \sin k \end{cases}$$

$$\Rightarrow \boxed{\tan(2\phi_k) = \frac{\sin k}{\cos k - g}}$$

Dispersion: $$\epsilon_k = 2J \frac{g - \cos k}{\cos(2\phi_k)}$$

$$\frac{1}{\cos^2(2\phi_k)} = \frac{\cos^2(2\phi_k) + \sin^2(2\phi_k)}{\cos^2(2\phi_k)} = 1 + \tan^2(2\phi_k)$$

$$= 1 + \frac{\sin^2 k}{(\cos k - g)^2} = \frac{1 - 2g\cos k + g^2}{(\cos k - g)^2}$$

$$\Rightarrow \boxed{\epsilon_k = 2J\sqrt{1 - 2g\cos k + g^2}}$$

- From the exact spectrum, we can check the limiting cases:

a) weak coupling ($g \ll 1$): $\sqrt{1+\epsilon} = 1 + \frac{1}{2}\epsilon + \mathcal{O}(\epsilon^2)$

$$\rightarrow \epsilon_k = 2J(1 - g\cos k + \mathcal{O}(g^2))$$

b) strong coupling ($1/g \ll 1$):

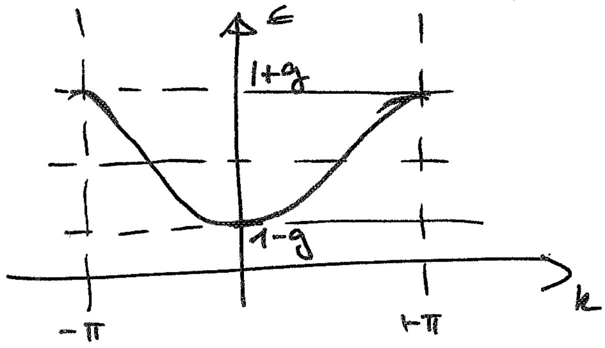
$$\epsilon_k = 2Jg\sqrt{1 - \frac{2}{g}\cos k + \mathcal{O}(1/g^2)} \simeq 2Jg\left(1 - \frac{1}{g}\cos k\right)$$

- How does the exact spectrum look like?

$$\epsilon_0 = 2J\sqrt{1-2g+g^2} = 2J\,|1-g|$$

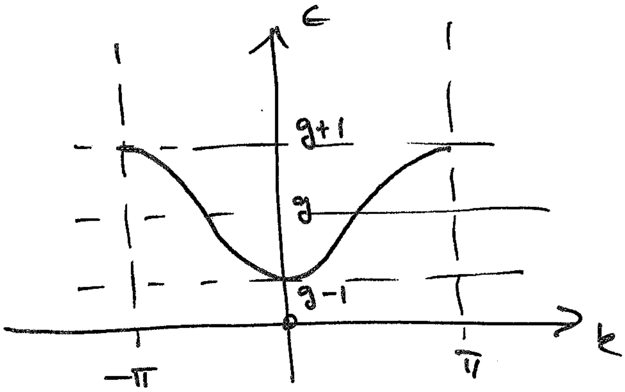
$$\epsilon_{\pi/2} = \epsilon_{-\pi/2} = 2J\sqrt{1+2g+g^2} = 2J\,|1+g|$$

<u>$g<1$:</u>
(ordered)



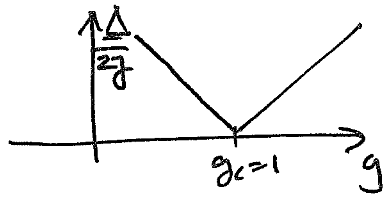
in the magnetically ordered phase there are no gapless excitations because we have broken a discrete symmetry ($\mathbb{Z}_2$)

<u>$g>1$:</u>



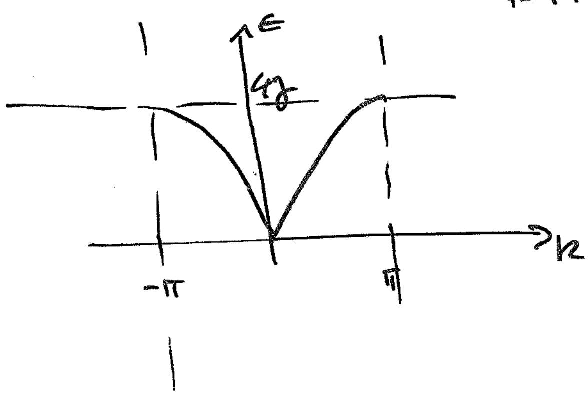
<u>gap:</u>

$$|\Delta| = 2J\,|1-g|$$



<u>At $g = g_c = 1$:</u>

$$\epsilon_k = 2J\sqrt{2-2\cos k} = 2\sqrt{2}\,\underbrace{\sqrt{1-\cos k}}_{1-1+\frac{1}{2}k^2} \simeq 2J\,|k|$$



2.5. Equal time correlations of the order parameter

We calculate the equal time correlation function $G(x_n, 0) = \langle \hat{\sigma}_i^z \hat{\sigma}_{i+n}^z \rangle$ at finite temperature using the JWT in the rotated spin basis:

$$G(x_n, 0) = \langle \hat{\sigma}_i^z \hat{\sigma}_{i+n}^z \rangle \qquad \hat{\sigma}_i^z = -\prod_{j<i}(1-2c_j^+c_j)(c_i+c_i^+)$$

$$= \langle \prod_{j<i}(1-2c_j^+c_j)(c_i+c_i^+) \prod_{j<i+n}(1-2c_j^+c_j)(c_{i+n}+c_{i+n}^+) \rangle$$

$$= \langle (c_i+c_i^+) \prod_{j=i}^{i+n-1} \underbrace{(1-2c_j^+c_j)}_{(c_j+c_j^+)(c_j^+-c_j)} (c_{i+n}+c_{i+n}^+) \rangle$$

$$\underset{\substack{(c+c^+)^2 \\ = 1}}{=} \langle (c_i^+-c_i) \prod_{j=i+1}^{i+n-1} (c_j^++c_j)(c_j^+-c_j) (c_{i+n}^++c_{i+n}) \rangle$$

$$= \langle B_i (A_{i+1} B_{i+1})(A_{i+2} B_{i+2}) \cdots (A_{i+n-1} B_{i+n-1}) A_{i+n} \rangle$$

.) define

$$\boxed{\begin{aligned} A_i &:= c_i^+ + c_i \\ B_i &:= c_i^+ - c_i \end{aligned}}$$

- Expectation value has to be taken with respect to a free Fermi theory
- Therefore, we can use Wick's Theorem
  → Decomposition in sums over products of expectation values of pairs of operators, $\langle A_i A_j \rangle$, $\langle B_i B_j \rangle$, $\langle A_i B_j \rangle$

- It turns out that only $\langle B_i A_j \rangle$ and $\langle A_i B_j \rangle$ are non-zero for $i \neq j$

$$\langle B_i A_j \rangle = \langle (c_i^+ - c_i)(c_j^+ + c_j) \rangle = \langle c_i^+ c_j \rangle - \langle c_i c_j^+ \rangle + \langle c_i^+ c_j^+ \rangle - \langle c_i c_j \rangle$$

$$= \langle c_i^+ c_j \rangle + \langle c_i^+ c_j^+ \rangle + c.c.$$

$$\langle c_i^+ c_j \rangle = \int\!\!\int_{k\,k'} e^{ikr_i} e^{-ik'r_j} \langle (u_k \gamma_k^+ - i v_k \gamma_{-k})(u_{k'} \gamma_{k'} + i v_{k'} \gamma_{-k'}^+) \rangle$$

$$= \int\!\!\int_{k\,k'} e^{ikr_i} e^{-ik'r_j} \left(u_k u_{k'} \langle \gamma_k^+ \gamma_{k'} \rangle + v_k v_{k'} \langle \gamma_{-k} \gamma_{-k'}^+ \rangle \right)$$

$$= \int_k e^{ik(r_i - r_j)} (u_k^2 - v_k^2) \underbrace{f_k}_{=\frac{1}{e^{\beta \epsilon_k}+1}}$$

Fermi function

$$\langle c_i^+ c_j^+ \rangle = \int\!\!\int_{k\,k'} e^{ikr_i} e^{ik'r_j} \langle (u_k \gamma_k^+ - i v_k \gamma_{-k})(u_{k'} \gamma_{k'}^+ - i v_{k'} \gamma_{-k'}) \rangle$$

$$= \int\!\!\int_{k\,k'} e^{ikr_i} e^{ik'r_j} \left(-i u_k v_{k'} \langle \gamma_k^+ \gamma_{-k'} \rangle - i v_k u_{k'} \langle \gamma_{-k} \gamma_{k'}^+ \rangle \right)$$

$$= \int_k e^{ik(r_i - r_j)} \, 2i\, u_k v_k f_k$$

$$\rightarrow \boxed{\langle B_i A_j \rangle} = \int_k \left[2\cos(k(r_i - r_j)) (u_k^2 - v_k^2) - 4 \sin(k(r_i - r_j)) u_k v_k \right] f_k$$

$$\boxed{= 4J \int_k \left[\cancel{2} \cos(k(r_i - r_j))(g - \cos k) + \sin(k(r_i - r_j)) \sin k \right] \frac{f_k}{\epsilon_k}}$$

Now we use Wick's Theorem to decompose

$$G(x_n, 0) = \langle B_i (A_{i+1} B_{i+1})(A_{i+2} B_{i+2}) \cdots (A_{i+n-1} B_{i+n-1}) A_{i+n} \rangle$$

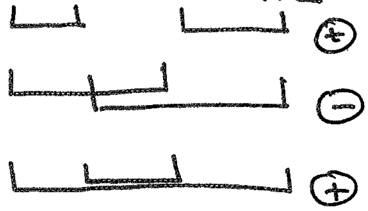
n = 1:

$$G(x_1) = \langle B_i A_{i+1} \rangle = D_0$$

$$\boxed{\text{define: } \langle B_i A_j \rangle =: D_{i-j+1}}$$

n = 2:

$$G(x_2) = \langle B_i (A_{i+1} B_{i+1}) A_{i+2} \rangle$$



$$= \langle B_i A_{i+1} \rangle \langle B_{i+1} A_{i+2} \rangle - \underbrace{\langle B_i B_{i+1} \rangle}_{=0} \underbrace{\langle A_{i+1} A_{i+2} \rangle}_{=0} + \langle B_i A_{i+2} \rangle \langle A_{i+1} B_{i+2} \rangle$$

$$= \langle B_i A_{i+1} \rangle \langle B_{i+1} A_{i+2} \rangle + \langle B_i A_{i+2} \rangle \langle A_{i+1} B_{i+2} \rangle$$

$$= \langle B_i A_{i+1} \rangle \langle B_{i+1} A_{i+2} \rangle - \langle B_i A_{i+2} \rangle \langle B_{i+2} A_{i+1} \rangle$$

$$= D_0^2 - D_{-1} D_{+1} = \det \begin{pmatrix} D_0 & D_{-1} \\ D_{+1} & D_0 \end{pmatrix}$$

for general n (show by complete induction over n if you want)

$$G(x_n) = \det \begin{pmatrix} D_0 & D_{-1} & D_{-2} & \cdots & D_{-n+1} \\ D_1 & \ddots & \ddots & \ddots & \vdots \\ D_2 & \ddots & \ddots & \ddots & D_{-2} \\ \vdots & \ddots & \ddots & \ddots & D_{-1} \\ D_{n-1} & \cdots & D_2 & D_1 & D_0 \end{pmatrix} =: T_n$$

- Mathematical problem to evaluate the determinant T_n
- To determine the <u>universal scaling limit</u>, we need to take the limit $n \to \infty$ while keeping the system close to its <u>critical point</u>
- T_n belongs to the class of Toeplitz determinants
- Using a fairly sophisticated mathematical theory for such determinants, the limit $T_{n\to\infty}$ can indeed be evaluated in closed form [see e.g. B.M. McCoy, Phys. Rev. 173, 531 (1968)]

$$\boxed{G(x) \underset{|x|\to\infty}{\simeq} Z \cdot T^{1/4} \cdot \tilde{G}(\Delta/T)\, e^{-\frac{T}{c}\tilde{F}(\Delta/T)\,|x|}}$$

Z nonuniversal constant, $\tilde{G}, \tilde{F}$ universal <u>scaling functions</u>

$$\boxed{\Delta = 2J(g_c - g)} \qquad \boxed{c = 2Ja}$$

explicit form of $\tilde{F}$ and $\tilde{G}$:

$$\tilde{F}(s) = |s|\,\Theta(-s) + \frac{1}{\pi}\int_0^\infty dy\, \ln\coth\frac{(y^2+s^2)^{1/2}}{2}$$

$$\ln\tilde{G}(s) = \int_s^1 \frac{dy}{y}\left[\left(\frac{dF(y)}{dy}\right)^2 - \frac{1}{4}\right] + \int_1^\infty \frac{dy}{y}\left(\frac{dF(y)}{dy}\right)^2$$

- Despite its appearance, the function $F(s)$ is smooth for all real s and is <u>analytic at $s=0$</u>

Note: Analyticity is required by the absence of any thermodynamic singularity at finite T for $\Delta = 0$

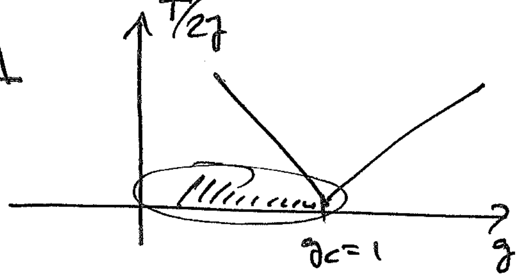
- <u>Important result</u>: The correlation function $G(x)$ always decays exponentially a $T>0 \Rightarrow$ No long-range order at finite temperature

inverse correlation length: $\boxed{\xi^{-1} = \frac{T}{c}\tilde{F}(\Delta/T)}$

2.6. Finite Temperature Crossovers

I. Low T on the magnetically ordered site ($g < g_c = 1$):

$$\Delta = 2J(1-g) > 0 \quad ; \quad T \ll \Delta$$



$$\rightsquigarrow \quad s = \frac{\Delta}{T} \gg 1$$

$$\tilde{F}(s) \underset{s>0}{=} \frac{1}{\pi}\int_0^\infty dy \; \ln \coth \underbrace{\frac{\sqrt{y^2+s^2}}{2}}_{=:x \;\gg 1}$$

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{1+e^{-2x}}{1-e^{-2x}} \approx 1 + 2e^{-2x}$$

$$\approx \frac{1}{\pi}\int_0^\infty dy \; \ln\Big(1 + 2\underbrace{e^{-\sqrt{y^2+s^2}}}_{\ll 1}\Big)$$

$$\approx \frac{2}{\pi}\int_0^\infty dy \; \underbrace{e^{-\sqrt{y^2+s^2}}}_{e^{-s\sqrt{1+y^2/s^2}}}$$

Integral dominated ~~by~~ $y \ll s$

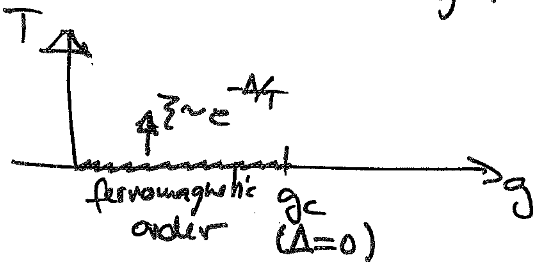
$$\simeq \frac{2}{\pi} e^{-s}\int_0^\infty dy \; e^{-\frac{y^2}{2s}} = \frac{2}{\pi} e^{-s} \frac{\sqrt{2\pi s}}{2}$$

$$= \sqrt{\frac{2}{\pi}s}\; e^{-s} \quad \Rightarrow \quad \boxed{\xi^{-1} = \frac{T}{c}\tilde{F}(s)}$$

$$= \frac{T}{c}\sqrt{\frac{2}{\pi}s}\; e^{-s}$$

$$\boxed{= \frac{1}{c}\sqrt{\frac{2\Delta T}{\pi}}\; e^{-\Delta/T}}$$

- As $T \to 0$, $\xi \to \infty$: Long range order at $T=0$



What happens at $T=0$?
Can we extract how the magnetization vanishes as $g \to g_c$?

For $T \to 0$: $s = \frac{\Delta}{T} \to \infty$, $\lim_{s\to\infty} \tilde{F}(s) = 0$

What is the asymptotic behavior of $\tilde{G}(s)$?

$$\ln \tilde{G}(s) = \int_s^1 \frac{dy}{y}\left(-\frac{1}{4}\right) = \frac{1}{4}\ln s \Rightarrow \tilde{G}(s) \simeq s^{1/4}$$

$$\Rightarrow G(x) \underset{\substack{|x|\to\infty \\ T=0}}{\approx} Z\,\Delta^{1/4} \overset{!}{=} m^2$$

reminder: $\frac{\Delta}{2J} = (g_c - g)$

$$\Rightarrow \boxed{m \sim (g_c - g)^{1/8}}$$

$\Rightarrow$ $\boxed{\beta = 1/8}$ critical exponent for the order parameter

II. Low T on the paramagnetic side : $\Delta<0$ $(g>1)$, $T \ll |\Delta|$:

$$\Delta = 2J(1-g) < 0, \quad s = \frac{\Delta}{T} < 0, \quad |s| \gg 1$$

$$F(s) = |s| + \underbrace{\sqrt{\frac{2}{\pi}|s|}\,e^{-|s|}}_{\text{as for previous case}}$$

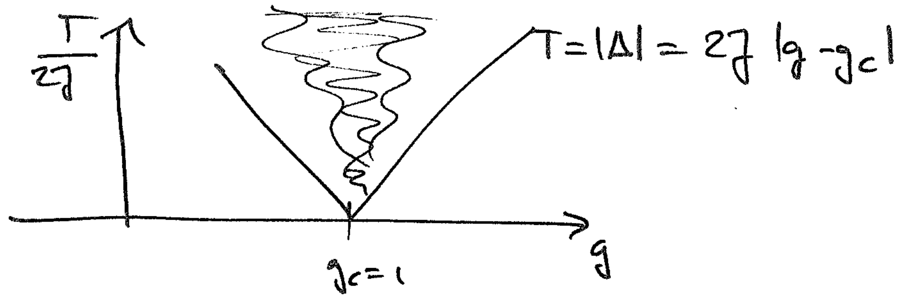
$$\Rightarrow \xi^{-1} = \frac{T}{c}\tilde{F}(s) = \frac{|\Delta|}{c} + \frac{1}{c}\sqrt{\frac{2|\Delta|T}{\pi}}\,e^{-|\Delta|/T} \underset{T\to0}{\simeq} \frac{|\Delta|}{c}$$

$$\Rightarrow \boxed{\xi \simeq \frac{c}{|\Delta|}} \quad \text{for } T \ll |\Delta|$$

(as it should on the paramagnetic side, ξ remains finite as $T\to0$)

- Since $|\Delta| = 2J|g_c - g|$, the correlation length diverges as $\boxed{\xi \sim (g-g_c)^{-\nu}}$ with $\boxed{\nu = 1}$ as $g \to g_c^+$

III. Quantum critical (continuum high T) : $T \gg |\Delta|$



$|s| = \frac{|\Delta|}{T} \ll 1$ $\qquad$ $\tilde{F}, \tilde{G}$ analytic at $s=0$

$\rightarrow \tilde{F}(s=\frac{\Delta}{T}) = \tilde{F}(0) + \tilde{F}'(0)\frac{\Delta}{T} + \dots$

$\tilde{G}(s) = \underbrace{\tilde{G}(0)}_{\simeq 0.8.6} + \tilde{G}'(0)\frac{\Delta}{T} + \dots$

$$\tilde{F}(0) = \frac{1}{\pi}\int_0^\infty dy \ \ln \coth y/2 = \pi/4$$

$$\rightarrow \quad G(x) \underset{|x|\to\infty}{\simeq} Z T^{1/4} G(0) \, e^{-\frac{T}{c}F(0)|x|}$$

$$= Z G(0) T^{1/4} e^{-\frac{\pi T}{4c}|x|}$$

$\rightarrow$ correlation length : $\boxed{\xi = \frac{4c}{\pi T}}$

How should the correlation function at the quantum critical point look like?

- We have to be very careful when taking the limits $T \to 0$, $\Delta \to 0$ $(g \to g_c)$ and $|x| \to \infty$
- At the QCP, the system is scale invariant $(T \to 0, \xi \to \infty)$

  $\Rightarrow$ Correlation function has to be a power law

  $$G(x) \underset{|x|\to\infty}{\sim} |x|^{-\delta}$$

- To obtain a power law, we have to take the limits $T \to 0$, $|x| \to \infty$ such that $T|x| = \text{const}$

  (Note that this implies $\underline{\underline{z=1}}$)

$$\rightarrow \quad \boxed{G(x) \underset{\substack{|x|\to\infty \\ T\to 0 \\ T|x| = \text{const}}}{\simeq} |x|^{-1/4}}$$

- From this result we can extract the anomalous dimension η :

At the critical point, the correlation function decays as $G(r) \sim \frac{1}{r^{D-2+\eta}}$ (see Chapter 1)

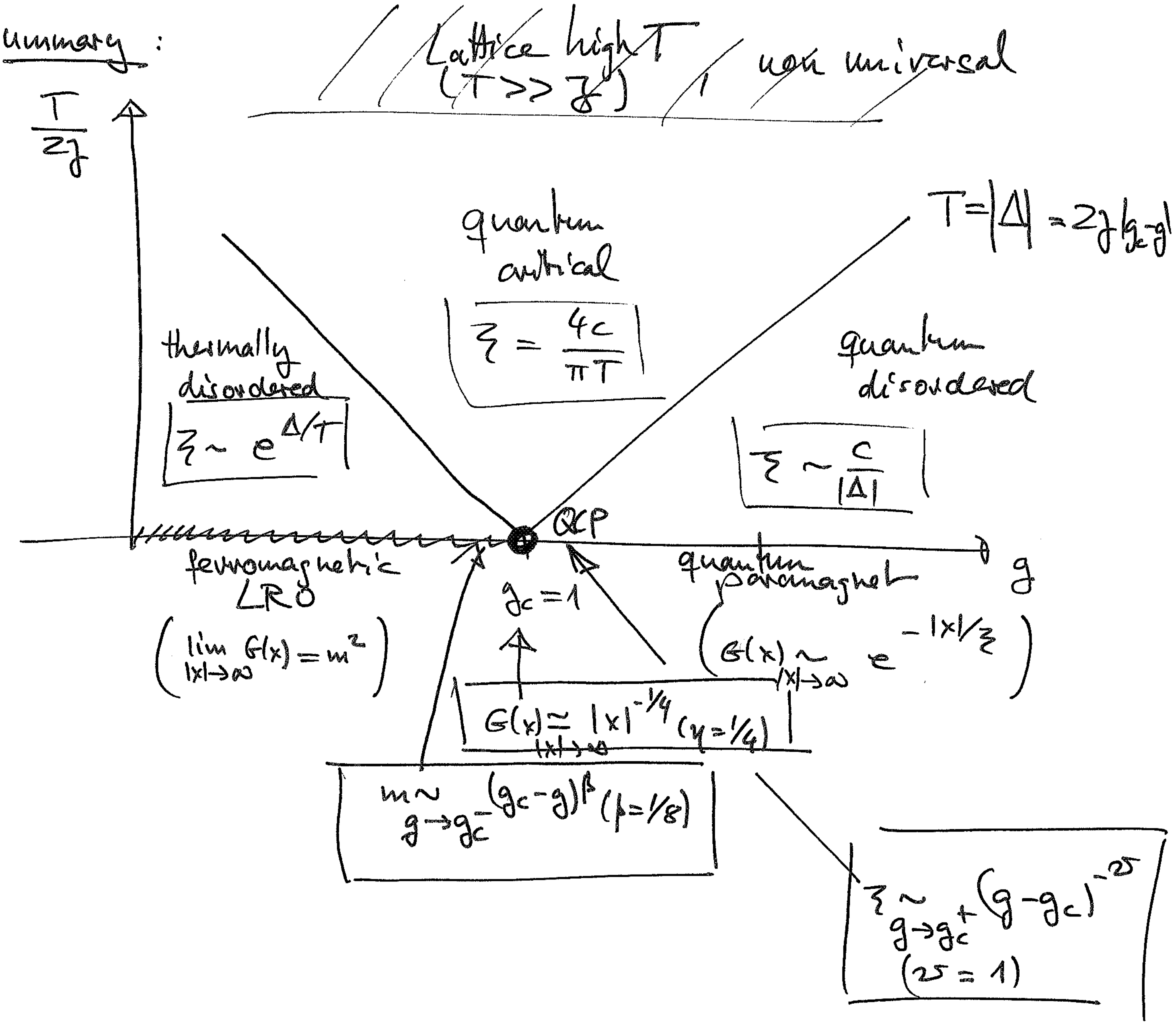
$$D = \underset{\substack{\uparrow \\ \text{dimension of the quantum system}}}{d} + \underset{\substack{\uparrow \\ \text{dynamical exponent}}}{z}$$

$\left.\begin{array}{l} d=1 \\ z=1 \end{array}\right\} \Rightarrow D=2$

$\Rightarrow G(r) \sim r^{-\eta}$

$\Rightarrow \boxed{\eta = 1/4}$

Summary :



2.7. Quantum-to-Classical Mapping

- In the previous sections, we have explicitly solved the quantum problem of the 1-dim. Ising chain in a transversal field
  - → Exact spectrum, critical exponents, finite temperature crossover of correlation length
- Exact solution (based on JWT) is very specific to the present problem (spin 1/2 operators, 1d problem)
- We are interested in a general approach which not only works for a particular problem!

d-dim. quantum problem	$\Longleftrightarrow$	D-dim. classical statistical physics ensemble

- classical phase transitions
- spontaneous symmetry breaking
- scaling
- renormalization group

I. Warm up exercise: Single Ising spin in a transversal field

$$\hat{H} = -h_x \hat{\sigma}_x - h_z \hat{\sigma}_z$$

This is a $\boxed{d=0}$ dimensional quantum problem

partition function: $Z = \mathrm{Tr}\, e^{-\beta \hat{H}} = \sum_{m=\pm 1} \langle m | e^{-\beta \hat{H}} | m \rangle$

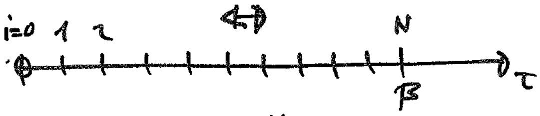
$m = \pm 1$, $|m=+1\rangle = |\uparrow\rangle$
$|m=-1\rangle = |\downarrow\rangle$

$\hat{\sigma}^z |m\rangle = m|m\rangle$
$\hat{\sigma}^x |m\rangle = |-m\rangle$

we use the <u>Suzuki-Trotter trick</u> to map the 0-dim. quantum problem onto a 1-dim. classical statistical physics problem.

$$e^{-\beta \hat{H}} = \left(e^{-\beta/N \hat{H}} \right)^N$$

$\boxed{\Delta\tau = \beta/N}$



D = 1 dim. lattice in "imaginary time"
($e^{-\tau \hat{H}}$ analog to unitary time evolution $e^{-i\hat{H}t}$, $\boxed{\tau = it}$)

$$Z = \sum_m \langle m | e^{-\Delta\tau \hat{H}} \cdot e^{-\Delta\tau \hat{H}} \cdot e^{-\Delta\tau \hat{H}} \cdots e^{-\Delta\tau \hat{H}} | m \rangle$$

$$1 = \sum_{m_1 = \pm 1} |m_1\rangle\langle m_1| \qquad 1 = \sum_{m_2 = \pm 1} |m_2\rangle\langle m_2| \qquad \ldots$$

$$= \sum_m \sum_{m_1 \ldots m_{N-1}} \langle m | e^{-\Delta\tau \hat{H}} | m_1 \rangle \langle m_1 | e^{-\Delta\tau \hat{H}} | m_2 \rangle \cdots \langle m_{N-1} | e^{-\Delta\tau \hat{H}} | m \rangle$$

$$\underset{\substack{m_0 := m \\ m_N = m_0}}{=} \sum_{\{m_0, \ldots, m_N\}} \prod_{i=0}^{N-1} \langle m_i | e^{-\Delta\tau \hat{H}} | m_{i+1} \rangle \, \delta_{m_N, m_0}$$

<u>Evaluation of the matrix element:</u>

$$A_{i,i+1} = \langle m_i | e^{-\Delta\tau \hat{H}} | m_{i+1} \rangle \approx \langle m_i | e^{+\Delta\tau h_x \hat{\sigma}_x} e^{\Delta\tau h_z \hat{\sigma}_z} | m_{i+1} \rangle$$

$$e^{A+B} = e^A e^B e^{-[A,B]/2}, \quad [A,B] = \mathcal{O}(\Delta\tau^2)$$

we can further use the identity

$$e^{\Delta\tau h_x \hat{\sigma}_x} = \cosh(\Delta\tau h_x) + \hat{\sigma}_x \sinh(\Delta\tau h_x)$$

<u>Proof:</u>

$$e^{\alpha \hat{\sigma}_x} = \sum_{n=0}^{\infty} \frac{1}{n!} (\alpha \hat{\sigma}_x)^n = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \hat{\sigma}_x^n \underset{\hat{\sigma}_x^2 = 1}{=} \sum_{k=0}^{\infty} \frac{\alpha^{2k}}{(2k)!} + \hat{\sigma}_x \sum_{n=0}^{\infty} \frac{\alpha^{2k+1}}{(2k+1)!}$$

$$\Delta_{i,i+1} = \langle m_i | \left(\cosh(\Delta\tau h_x) + \hat{\sigma}_x \sinh(\Delta\tau h_x)\right) e^{\Delta\tau h_z \hat{\sigma}_z} | m_{i+1}\rangle$$

$$= e^{\Delta\tau h_z m_{i+1}} \underbrace{\left(\cosh(\Delta\tau h_x)\, \delta_{m_i, m_{i+1}} + \sinh(\Delta\tau h_x)\, \delta_{m_{i+1}, -m_i} \right)}_{=: \gamma e^{\tilde{K} m_i m_{i+1}}}$$

$m_i = m_{i+1}$: I. $\quad \cosh(\Delta\tau h_x) = \gamma e^{\tilde{K}}$

$m_i = -m_{i+1}$: II. $\quad \sinh(\Delta\tau h_x) = \gamma e^{-\tilde{K}}$

$$\Rightarrow \left\{ \boxed{\begin{aligned} \gamma^2 &= \cosh(\Delta\tau h_x)\sinh(\Delta\tau h_x) = \Delta\tau h_x + \mathcal{O}(\Delta\tau^2) \\ \tilde{K} &= -\tfrac{1}{2}\ln\tanh(\Delta\tau h_x) \approx -\tfrac{1}{2}\ln(\Delta\tau h_x) \underset{\Delta\tau \ll 1}{>} 0 \end{aligned}} \right.$$

$$\Rightarrow \boxed{\gamma = e^{-\tilde{K}}}$$

$$\Rightarrow \quad Z = \gamma^N \sum_{m_0, \dots, m_{N-1}}^{(m_N = m_0)} e^{-\left(-\tilde{K}\sum_i m_i m_{i+1} - \tilde{h}\sum_i m_i\right)}$$

$$\boxed{\tilde{h} := \Delta\tau h_z}$$

Note: $\gamma^N = e^{-N\tilde{K}} = e^{\tilde{E}_0}$ with $\tilde{E}_0 = -N\tilde{K}$ the ground state energy for $\tilde{h} = 0$

Quantum model	Classical model
$\hat{H} = -h_x\hat{\sigma}_x - h_z\hat{\sigma}_z$	$\tilde{\mathcal{H}}_{cl} = \mathcal{H}_{cl}/T_{cl} = -\tilde{K}\sum_i m_i m_{i+1} - \tilde{h}\sum_i m_i - E$
$\hat{\sigma}_x, \hat{\sigma}_z$ spin 1/2 quantum operators	$m_i = \pm 1$ classical Ising variables
$d = 0$ (single-site model)	$D = d+1 = 1$ (one-dimensional chain)
temperature $T = 1/\beta$	lattice constant $\circ\ \circ\ \circ\ \circ\ \circ$ N sites, $a_\tau = \Delta\tau = \beta/N$

- Temperature T in the quantum model has nothing to do with effective temperature T_{cl} of the classical model ($T \neq T_{cl}$)
- We can only deduce ratios of classical coupling constants over classical temperature:

$$\tilde{K} = \frac{K}{T_{cl}} \simeq -\frac{1}{2}\ln(\Delta\tau\, h_x)$$

$$\tilde{h} = \frac{h}{T_{cl}} \simeq \Delta\tau\, h_x$$

$\Delta\tau = \frac{\beta}{N}, \ \beta = \frac{1}{T}$

non trivial dependence on temperature T of the quantum model!

II. Ising chain in transversal field

quantum model: $\hat{H} = -\underbrace{Jg}_{h_x}\sum_i \hat{\sigma}_i^x - J\sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z$

$\underline{d=1}$ $\quad \overset{i}{\circ}\ \overset{i+1}{\circ}\ \circ\ \circ\ \circ\ \circ$

quantum states of the chain:

$$|\{m_i\}\rangle = \prod_i \underbrace{|m_i\rangle_i}_{} = |m_0 \ldots m_{N_x-1}\rangle$$

Hilbert space on site i: $|\uparrow\rangle_i, |\downarrow\rangle_i$

We again use the Trotter formula/trick:

$$Z = \mathrm{Tr}\, e^{-\beta\hat{H}} = \sum_{\{m_i\}} \langle m_0 \ldots m_{N_x-1}| e^{-\beta\hat{H}} |m_0 \ldots m_{N_x-1}\rangle$$

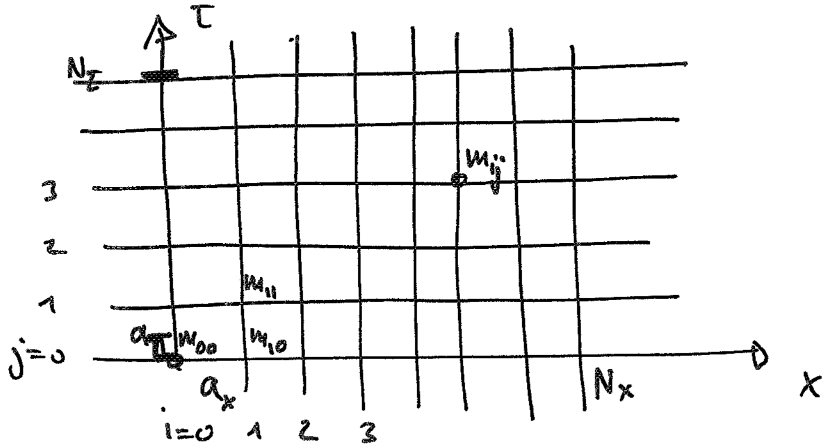
$$= \sum_{\{m_i\}} \langle m_0, \ldots, m_{N_x-1}| e^{-\Delta\tau\hat{H}} \cdot e^{-\Delta\tau\hat{H}} \cdot e^{-\Delta\tau\hat{H}} \ldots e^{-\Delta\tau\hat{H}} |m_0 \ldots m_{N_x-1}\rangle$$

$$1 = \sum_{m_{0,2}, \ldots m_{N_x-1,2}} |m_{0,2} \ldots m_{N_x-1,2}\rangle\langle m_{0,2} \ldots m_{N_x-1,2}|$$

$$1 = \sum_{m_{0,1} \ldots m_{N_x-1,1}} |m_{0,1} \ldots m_{N_x-1,1}\rangle\langle m_{0,1} \ldots m_{N_x-1,1}|$$

$$\boxed{\Delta\tau = \frac{\beta}{N_\tau}}$$

$$= \sum_{\{m_{ij}\}} \langle m_{0,0} \cdots m_{N_x-1,0} | e^{-\Delta\tau H} | m_{0,1} \cdots m_{N_x-1,1} \rangle \cdots \langle m_{0,N_\tau-1} \cdots m_{N_x-1,N_\tau-1} | e^{-\Delta\tau H} | m_{0,N_\tau} \cdots m_{N_x-1,N_\tau} \rangle$$



$$\boxed{\begin{aligned} a_\tau &= \Delta\tau = \beta / N_\tau \\ a_x &= L_x / N_x \end{aligned}}$$

- m_{ij} classical Ising variables on a 2-dimensional lattice, $m_{ij} = \pm 1$

Evaluation of matrix elements:

$$\langle m_{0,j} \cdots m_{N_x-1,j} | \underbrace{e^{-\Delta\tau \hat{H}}}_{\approx\, e^{\Delta\tau h_x \sum_i \hat{\sigma}_i^x} e^{\Delta\tau J \sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z}} | m_{0,j+1} \cdots m_{N_x-1,j+1} \rangle$$

$$= e^{\Delta\tau J \sum_i m_{i,j+1} \cdot m_{i+1,j+1}} \underbrace{\langle m_{0,j} \cdots m_{N_x-1,j} | e^{\Delta\tau h_x \sum_i \hat{\sigma}_i^x} | m_{0,j+1} \cdots m_{N_x-1,j+1} \rangle}_{= \prod_{i=0}^{N_x-1} \underbrace{\langle m_{ij} | e^{\Delta\tau h_x \hat{\sigma}_i^x} | m_{i,j+1} \rangle}_{\overset{(I)}{=} \gamma\, e^{\tilde{J}_\tau m_{ij} \cdot m_{i,j+1}}}}$$

with

$$\boxed{\begin{aligned} \tilde{J}_\tau &\approx -\tfrac{1}{2} \ln(\Delta\tau\, h_x) \\ \gamma &= \sqrt{\Delta\tau \cdot h_x} \end{aligned}}$$

$$\overset{\text{drop } \gamma^{N_x} \to \text{constant}}{=} e^{\Delta\tau J \sum_i m_{i,j+1} m_{i+1,j+1} + \tilde{J}_\tau \sum_i m_{ij} m_{i,j+1}}$$

$$\left(\underline{\text{or}}: \quad \begin{aligned} \tilde{J}_\tau &= -\tfrac{1}{2} \ln \tanh(\Delta\tau\, h_x) \\ \gamma^2 &= \cosh(\Delta\tau h_x) \sinh(\Delta\tau h_x) \end{aligned} \right)$$

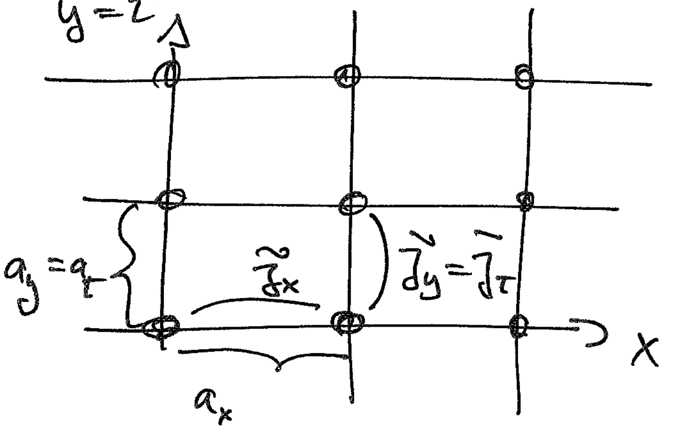
The partition function reads:

$$Z = \sum_{\{m_{ij}\}} e^{-\tilde{\mathcal{H}}_{cl}}$$

$$\boxed{\tilde{\mathcal{H}}_{cl} = \frac{\mathcal{H}_{cl}}{T_{cl}} = -\tilde{J}_x \sum_{ij} m_{ij}\, m_{i+1,j} - \tilde{J}_y \sum_{ij} m_{ij}\, m_{i+1,j}}$$

- $D = d+1 = 2$ dimensional classical Ising model

$$\boxed{\tilde{J}_x = J_x/T_{cl} = \Delta\tau J}\,, \quad \boxed{\tilde{J}_y = J_y/T_{cl} = \tilde{J}_\tau = -\tfrac{1}{2}\ln\tanh(\Delta\tau h_x)}$$



- model is anisotropic: $a_x \neq a_\tau$, $\tilde{J}_x \neq \tilde{J}_\tau$
- classical Ising model in the absence of external magnetic field can be solved exactly
  [L. Onsager, Phys. Rev. 65, 117 (1944)]

(Not surprising, since the corresponding 1d quantum model can also be solved exactly)

- In the thermodynamic limit, $N_x \to \infty$, $N_y = N_\tau \to \infty$, the system has a critical point given by the relation

$$\boxed{\sinh(2\underbrace{J_x \beta^c_{cl}}_{\tilde{J}_x}) \cdot \sinh(2\underbrace{J_y \beta^c_{cl}}_{\tilde{J}_y}) = 1}$$

where $\beta^c_{cl} = 1/T^c_{cl}$ is the critical inverse temperature for the classical system. For $T_{cl} < T^c_{cl}$ the system is magnetized, for $T_{cl} > T^c_{cl}$ disordered.

- The value of T_{cl}^c depends on the anisotropy J_x/J_y, the critical exponents do not!
  The critical points for all J_x/J_y are in the same universality class.

- What does the above relation imply for the quantum system?

$$\tilde{J}_x = J_x \beta_{cl} = \Delta\tau J$$
$$\tilde{J}_y = \tilde{J}_\tau = J_y \beta_{cl} = -\frac{1}{2} \ln(\tanh(\Delta\tau h_x))$$
$$= -\frac{1}{2} \ln \tanh(\Delta\tau J g)$$

→ The relation reads

$$\boxed{\sinh(2\Delta\tau J) \cdot \sinh\left(2\cdot\left(-\frac{1}{2}\right)\ln \tanh(\Delta\tau J g_c)\right) = 1}$$

Note: Keeping the lattice constants $a_x, a_\tau = \Delta\tau$ fixed, the thermodynamic limit $N_x \to \infty$, $N_y = N_\tau \to \infty$ means: $L_x \to \infty$, $L_\tau = \beta \to 0 \Leftrightarrow T \to 0$!

Thermal phase transition at T_{cl}^c in the classical 2D Ising model corresponds with quantum phase transition ($T=0$) in the $d=1$ dimensional transversal field Ising model.

$$1 = \sinh(2\Delta\tau J) \cdot \sinh\left(\ln \tanh^{-1}(\Delta\tau J g_c)\right)$$
$$\underset{\sinh x = \frac{e^x - e^{-x}}{2}}{=} \sinh(2\Delta\tau J)\frac{1}{2}\left(\tanh^{-1}(\Delta\tau J g_c) - \tanh(\Delta\tau J g_c)\right)$$
$$= \sinh(2\Delta\tau J) \cdot \frac{\cosh^2(\Delta\tau J g_c) - \sinh^2(\Delta\tau J g_c)}{2\cosh(\Delta\tau J g_c)\sinh(\Delta\tau J g_c)}$$
$$= \frac{\sinh(2\Delta\tau J)}{\sinh(2\Delta\tau J g_c)} \Rightarrow \boxed{g_c = 1}$$

2.8. Quantum Field Theory

- Exact solutions are possible only in very rare cases and usually only in $d=1$ (e.g. for the transversal field Ising model in $d=1$)
- At continuous (quantum) phase transitions, time and length scales of critical fluctuations diverge
- <u>Universality</u>: Nature of transitions should depend only on dimensionality and symmetries but not on microscopic details
- If we are not interested in the precise location of the phase transition but rather in the universality (set of critical exponents), we should be able to work with an <u>effective continuum quantum-field theory</u>

<u>Recipe:</u>

$\boxed{\text{quantum theory in } d \text{ dimensions}}$

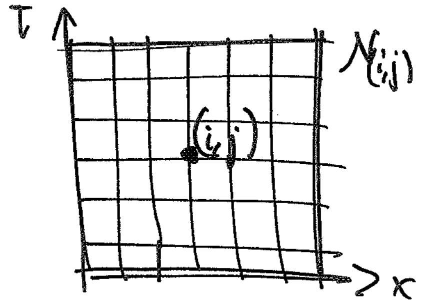
$\Downarrow$ quantum-to-classical mapping

$\boxed{D = d+\cancel{z}1 \text{ dimensional classical problem}}$

$\Downarrow$ coarse graining, continuum limit, expansion in terms of spatial and temporal gradients

$\boxed{\text{Effective continuum quantum-field theory}}$

Coarse graining



$$\mathcal{N} := \sum_{(i,j)\in \mathcal{N}_{(i,j)}} 1$$

$$\phi_{ij} := \frac{1}{\mathcal{N}} \sum_{(i,j)\in \mathcal{N}_{(i,j)}} m_{i,j}$$

If $\mathcal{N}_{(i,j)}$ contains a large number of sites, ϕ_{ij} is practically a continuous valued function, $\phi_{ij} \in \mathbb{R}$

$$\rightarrow \quad Z = \prod_{(i,j)} \int_{-\infty}^{\infty} d\phi_{ij} \; e^{-\sum_{ij} \left\{ -\gamma_x \phi_{ij}\phi_{i+1,j} - \gamma_\tau \phi_{ij}\phi_{i,j+1} + \tilde{V}(\phi_{ij}^2) \right\}}$$

- We have to add potential V that controls fluctuations of ϕ^2 and prevents it from becoming to large. Note that the coarse grained action respects the global $\mathbb{Z}_2$ symmetry $\phi_{ij} \to -\phi_{ij}$ for all (i,j).
- Now we can take the continuum limit and perform a spatial and temporal gradient expansion:

$$\phi_{ij} \to \phi(x,\tau) \qquad \phi: \mathbb{R}\times[0,\beta] \to \mathbb{R}$$

$$\sum_i a_x \to \int dx \qquad \sum_i a_\tau \to \int_0^\beta d\tau$$

$$-\frac{\phi_{ij}\phi_{i+1,j}}{a_x^2} = \frac{1}{2}\left(\frac{\phi_{ij}-\phi_{i+1,j}}{a_x}\right)^2 - \frac{\phi_{ij}^2+\phi_{i+1,j}^2}{2a_x^2} \to \frac{1}{2}(\partial_x\phi)^2 - \phi^2/a_x^2$$

$$-\frac{\phi_{ij}\phi_{i,j+1}}{a_\tau^2} \to \frac{1}{2}(\partial_\tau\phi)^2 - \phi^2/a_\tau^2$$

$$\rightarrow \quad Z = \int \mathcal{D}\phi(x,\tau)\, e^{-S[\phi]}$$

$$\boxed{S[\phi] = \int dx \int_0^\beta d\tau \left\{ \frac{1}{2}\left[(\partial_\tau\phi)^2 + c^2(\partial_x\phi)^2 \right] + \underbrace{V(\phi^2)}_{\frac{r}{2}\phi^2 + \frac{u}{4!}\phi^4} \right\}}$$

$$= \int dx \int_0^\beta d\tau \left\{ \frac{1}{2}\left[(\partial_\tau\phi)^2 + c^2(\partial_x\phi)^2 + r\phi^2 \right] + \frac{u}{4!}\phi^4 \right\}$$

Landau-Ginzburg Wilson action for the transversal field Ising model

- This quantum field theory will produce exactly the same set of critical exponents as calculated explicitly for the Ising ~~model~~ chain in a transversal field!

- We can generalize to an N component vector field $\underline{\phi} \in \mathbb{R}^N$ in d spatial dimensions:

$$S = \int d^d x \int_0^\beta d\tau \left\{ \frac{1}{2} \left[(\partial_\tau \underline{\phi})^2 + c^2 (\nabla \underline{\phi})^2 + r \underline{\phi}^2 \right] + \frac{u}{4!} \underline{\phi}^4 \right\}$$

critical exponents will depend on N and d.

- What is special about the above quantum-field theories is that time enters in exactly the same way as the spatial ~~components~~ coordinates (note that the prefactor c^2 can easily be absorbed in a rescaling of length)
For such theories, the scaling in space and time directions is the same: $z=1$

- In the next chapter, we will derive an effective quantum field theory with $z=2$.