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Quantum Phase Transitions
Problem Set 3

☞ *Continuous assessment is based on three homework sets.*
☞ *Please hand in your solutions to problem set 3 on or before Friday, May 31.*
☞ *Please scan your solutions and send them in pdf format to frank.kruger@st-andrews.ac.uk (or upload to MY.SUPA).*

Renormalization Group and ϵ expansion [8+4+6+6=24 points]

Consider the field theory described by the partition function $\mathcal{Z} = \int \mathcal{D}[\phi^*, \phi] \exp(-\mathcal{S})$ over the complex field $\phi(\mathbf{r})$ with the action

$$\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_h = \int \mathrm{d}^D\mathbf{r} \left\{ |\nabla\phi(\mathbf{r})|^2 + r|\phi(\mathbf{r})|^2 + h|\phi(\mathbf{r})|^4 \right\}.$$

In the following, assume that $h > 0$ and that we have rescaled to dimensionless length such that the momentum cut-off of the theory is $\Lambda = 1$ ($|\mathbf{k}| \leq 1$).

*(a) Show that the renormalization-group equation for the mass coefficient at 1-loop order is given by

$$\frac{\mathrm{d}r}{\mathrm{d}l} = 2r + 4\frac{S_D}{(2\pi)^D}\frac{h}{1+r}$$

with S_D the surface of the D dimensional unit sphere. Hints: (a) Decompose the fields into slow and fast components, $\phi(\mathbf{r}) = \phi_<(\mathbf{r}) + \phi_>(\mathbf{r})$, with $\phi_>$ depending only on momenta from the outer shell $e^{-dl} \leq |\mathbf{k}| \leq 1$. (b) Show that the correction to the $|\phi|^2$ term is given by $4h\langle\phi_>^*\phi_>\rangle_0 \int \mathrm{d}^D\mathbf{r}|\phi_<|^2$ where the factor 4 arrises from simple combinatorics. (c) Calculate the average over fast fields by using (without proof) that $\langle\phi^*(\mathbf{k})\phi(\mathbf{k}')\rangle_0 = \delta(\mathbf{k}-\mathbf{k}')(k^2+r)^{-1}$. (d) Determine the scaling dimension of the fields such that after rescaling of momenta the spatial gradient term is not renormalized.

(b) The renormalization group equation for the quartic vertex is given by

$$\frac{\mathrm{d}h}{\mathrm{d}l} = \epsilon h - 10\frac{S_D}{(2\pi)^D}\frac{h^2}{(1+r)^2},$$

with $\epsilon = 4 - D$, which you don't have to derive in this exercise. Show that after an appropriate rescaling $\tilde{h} = ch$, the RG equations up to quadratic order in r and $\tilde{h}$ take the form

$$
\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}l} &= 2r + 4\tilde{h} - 4r\tilde{h}, \\
\frac{\mathrm{d}\tilde{h}}{\mathrm{d}l} &= \epsilon\tilde{h} - 10\tilde{h}^2.
\end{aligned}
$$

(c) Determine the fixed point(s) of the RG equations in the cases $\epsilon < 0$ and $\epsilon > 0$ and sketch the RG flows in the r-$\tilde{h}$ parameter plane. (Hint: Analyze the stability of the fixed points by linearizing around them.) Interpret the different RG flow diagrams.

(d) Determine the correlation length exponents for $\epsilon < 0$ and $\epsilon > 0$. In the latter case, consider only corrections linear in ϵ (expansion slightly below the upper critical dimension $D = 4$). (Hint: Linearize around the critical fixed points, integrate the linearized differential equations and determine the correlation length as $\xi \sim e^{l^*}$ with l^* the scale where r becomes of order 1, $r(l^*) = 1$.)