University of St Andrews
SUPA, School of Physics & Astronomy
North Haugh
St Andrews KY16 9SS

Dr. Frank Kruger
Room 243, phone: 01334 463120
frank.kruger@st-andrews.ac.uk
http://www.st-andrews.ac.uk/ fk20/

Quantum Phase Transitions
Problem Set 1

☞ *Continuous assessment is based on three homework sets.*
☞ *Please hand in your solutions to problem set 1 on or before Wednesday, May 8.*
☞ *Please scan your solutions and send them in pdf format to frank.kruger@st-andrews.ac.uk (or upload to MY.SUPA).*

The Classical Ising Model in Mean-Field Approximation [4+3+3+4+4+4+2+2=26 points]

Consider the classical Ising model $\mathcal{H} = -J\sum_{\langle i,j\rangle} s_i s_j - h\sum_i s_i$ with $J > 0$ the ferromagnetic coupling between nearest-neighbor Ising spins $s_i = \pm 1$ on a D-dimensional hypercubic lattice and h a homogeneous magnetic field. In the following, we analyze this model in the Curie-Weiss mean-field approximation. This amounts to the replacement of the coupling term $\sum_{i,j} s_i s_j$ by $\sum_{i,j}\langle s_i\rangle s_j$ where the average magnetization per spin $m = \langle s_i\rangle$ has to be determined self-consistently.

(a) By explicitly calculating the magnetization m after the mean-field decoupling, show that the self-consistency condition is given by the equation

$$m = \tanh\left(\frac{zmJ + h}{k_\mathrm{B}T}\right),$$

with $z = 2D$ the coordination number of the lattice. Show that the transition temperature at zero field is given by $T_c = zJ/k_\mathrm{B}$. (Hint: When is a non-trivial solution $m \neq 0$ possible?)

(b) Show that near and below T_c the magnetization at $h = 0$ is given by

$$m = \frac{T}{T_c}\sqrt{3\left(\frac{T_c - T}{T_c}\right)}$$

yielding the order-parameter exponent $\beta = 1/2$.

(c) Show that for small h and right at $T = T_c$,

$$m = \left(\frac{3h}{k_\mathrm{B}T_c}\right)^{1/3}$$

and hence $\delta = 3$.

(d) Show that the magnetic susceptibility χ diverges as $\chi \sim |T - T_c|^{-\gamma}$ with $\gamma = 1$ for both, $T \to T_c^+$ and $T \to T_c^-$ and that

$$\frac{\chi(T \to T_c^+)}{\chi(T \to T_c^-)} = 2.$$

(e) Show that the specific heat is discontinuous at $T = T_c$ corresponding to $\alpha = 0$.

(f) Assuming that for $t = (T - T_c)/T_c > 0$ the Fourier transform of the correlation function in $D = 3$ has the form $G(\mathbf{q}) = 1/(\mathbf{q}^2 + t)$, show that the correlation-length exponent and the anomalous dimension are given by $\nu = 1/2$ and $\eta = 0$, respectively (Hint: Fourier transform and complex integration).

(g) Combining the exponents α, β, γ, and δ calculated in the Curie-Weiss mean-field approximation with the result obtained in (f), check the validity of the four scaling relations.

(h) For the quantum Ising chain in a transversal field, we will demonstrate in the lecture that the magnetization vanishes with an exponent $\beta = 1/8$ as we approach the quantum critical point from the magnetically ordered side. What does this tell us about the upper critical dimension of the classical Ising model and the validity of the Curie-Weiss mean-field theory?