3. Integration of Equation of motion

3.1. One-dimensional motion

$$\boxed{L = \frac{m}{2}\dot{x}^2 - U(x)}$$

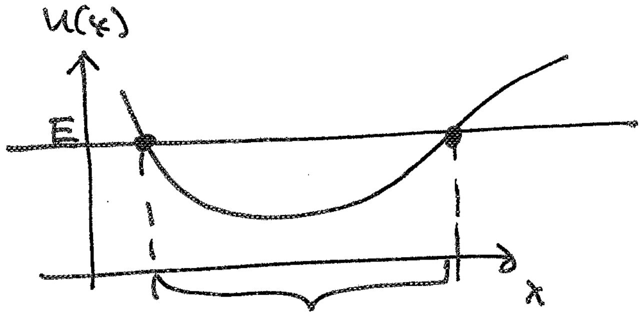
single particle in potential $U(x)$

- $L = L(x, \dot{x}) \Rightarrow$ Energy is conserved

$$E = \dot{x}\frac{\partial L}{\partial \dot{x}} - L = \frac{m}{2}\dot{x}^2 + U(x) = \text{const}$$

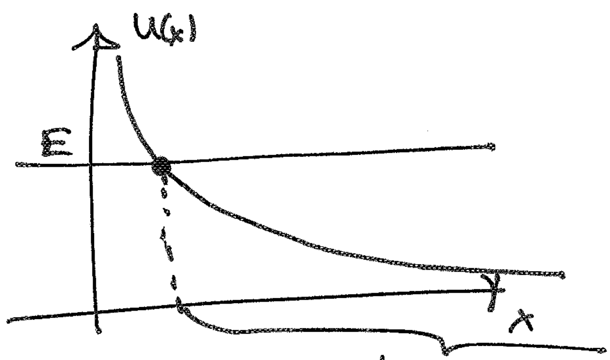
- The nature of the motion depends upon the value of E and the shape of the potential

bounded motion:



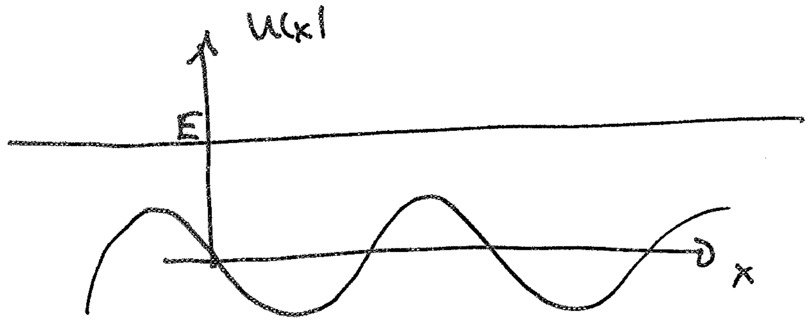
→ oscillation

semi-bounded motion:



one inflection point

unbounded motion



$\forall x: \; E > U(x)$

$$\boxed{\text{inflection (turning) points: } \; E = U(x_0), \; \dot{x} = 0}$$

- Instead of equation of motion (Euler-Lagrange equation) which is 2nd order equation we start from energy conservation (or other conserved quantity)

$\Rightarrow$ one integration for free! Remaining differential equation is 1st order!

$$E = \frac{m}{2}\dot{x}^2 + U(x) \Leftrightarrow \left(\frac{dx}{dt}\right)^2 = \frac{2}{m}\left(E - U(x)\right)$$

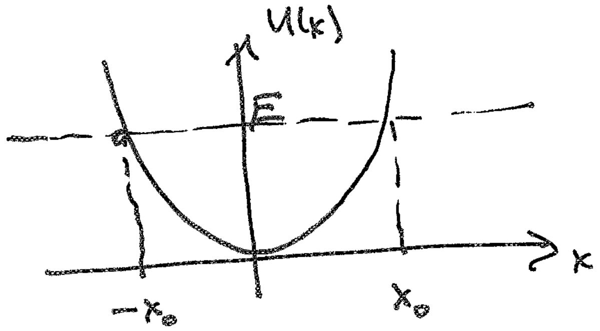
Can be solved by separation of variables (for $U(x) \leq E$)

$$\sqrt{\frac{m}{2}} \frac{dx}{\sqrt{E - U(x)}} = dt$$

$$\Rightarrow \boxed{t - t_0 = \sqrt{\frac{m}{2}} \int \frac{dx}{\sqrt{E - U(x)}}}$$

- From the above integration we obtain $t(x)$ and by inversion the trajectory $x(t)$

Example: Harmonic oscillator



$$U(x) = \frac{k}{2} x^2$$

turning points at $\pm x_0$ determine total energy

$$\boxed{E = \underbrace{T}_{=0} + U(x_0) = \frac{k}{2} x_0^2}$$

$$t-t_0 = \sqrt{\frac{m}{2}} \int \frac{dx}{\sqrt{\frac{k}{2}x_0^2 - \frac{k}{2}x^2}} = \sqrt{\frac{m}{k}} \int \frac{dx}{\sqrt{x_0^2-x^2}}$$

$$\underset{\substack{\xi = x/x_0 \\ d\xi = \frac{dx}{x_0}}}{=} \sqrt{\frac{m}{k}} \int \frac{d\xi}{\sqrt{1-\xi^2}} = \sqrt{\frac{m}{k}} \arcsin \xi$$

$$= \sqrt{\frac{m}{k}} \arcsin \frac{x}{x_0}$$

$$\Rightarrow \quad \boxed{x(t) = x_0 \sin[\omega(t-t_0)] \ , \quad \omega = \sqrt{\frac{k}{m}}}$$

3.2. Two-body Problem in 3D

Two point masses with interaction potential that depends upon the distance between them

$$\boxed{L = \frac{m_1}{2} \dot{\underline{r}}_1^2 + \frac{m_2}{2} \dot{\underline{r}}_2^2 - U(|\underline{r}_1 - \underline{r}_2|)}$$

- Center of mass and relative coordinates:

$$\boxed{\underline{R} = \frac{m_1 \underline{r}_1 + m_2 \underline{r}_2}{m_1 + m_2} \ , \quad \underline{r} = \underline{r}_2 - \underline{r}_1}$$

- inverse transformation:

$$\underline{r}_1 = \underline{R} - \frac{m_2}{M} \underline{r}$$

$$\underline{r}_2 = \underline{R} + \frac{m_1}{M} \underline{r}$$

$M = m_1 + m_2$ total mass

- Separation of center of mass and relative motion

$$\boxed{L = \frac{m_1}{2}\dot{\underline{r}}_1^2 + \frac{m_2}{2}\dot{\underline{r}}_2^2 - U(|\underline{r}_1 - \underline{r}_2|)}$$

$$= \frac{m_1}{2}\left(\dot{\underline{R}} - \frac{m_2}{M}\dot{\underline{r}}\right)^2 + \frac{m_2}{2}\left(\dot{\underline{R}} + \frac{m_1}{M}\dot{\underline{r}}\right)^2 - U(r)$$

$$= \frac{m_1+m_2}{2}\dot{\underline{R}}^2 + \frac{1}{2}\underbrace{\frac{m_1 m_2^2 + m_1^2 m_2}{M^2}}_{=\frac{m_1 m_2}{M}}\dot{\underline{r}}^2 - U(r)$$

$$= \boxed{L_R + L_r}$$

$$\boxed{L_R = \frac{M}{2}\dot{\underline{R}}^2}$$ free center of mass motion

$\boxed{M = m_1 + m_2}$ total mass

$$\boxed{L_r = \frac{\mu}{2}\dot{\underline{r}}^2 - U(r)}$$ relative motion, corresponds with Lagrangian for single particle with reduced mass (effective mass)

$$\boxed{\mu = \frac{m_1 m_2}{m_1 + m_2} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)^{-1}}$$

(harmonic average)

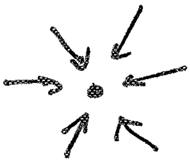
- We have decoupled the 2-body problem into two independent „single-body" problems

⇒ Energies of center of mass and relative motion are conserved

- angular momentum conservation

$$L_r = \frac{\mu}{2}\dot{\underline{r}}^2 - U(r)$$

potential $U(r)$ depends only on the absolute value of $\underline{r}$; central force problem



$$\underline{F} = -\underline{\nabla} U = -U'(r)\cdot \underline{e}_r$$

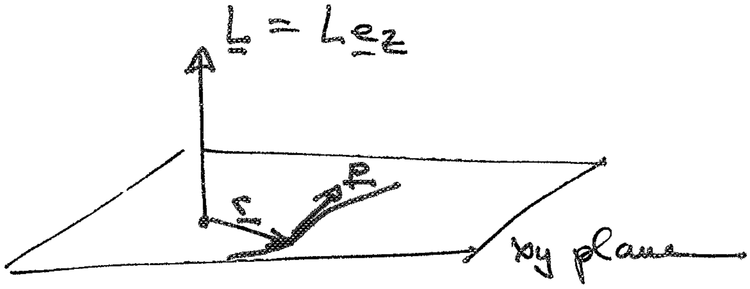
$$\underline{e}_r := \underline{r}/r$$

$\Rightarrow$ L_r is invariant under rotations (around the origin)

(2.3.) $\Rightarrow$ angular momentum $\boxed{\underline{L} = \underline{r}\times\underline{p}}$ is conserved,

$$\boxed{\frac{d\underline{L}}{dt} = 0 \;,\; \underline{L} = \text{const}} ,$$

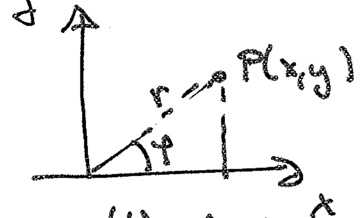
$\Rightarrow$ Motion is confined to a plane perpendicular to $\underline{L}$



convention: plane: $=0$

$$\underline{L} = L_z\,\underline{e}_z$$

- We introduce <u>polar coordinates</u> to describe this two-dimensional motion:

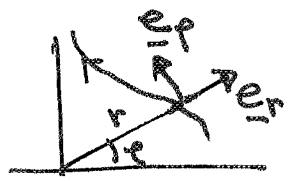


$$\boxed{\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \end{aligned}}$$

$r = r(t)$, $\varphi = \varphi(t)$

orthonormal basis vectors:

$$\boxed{\underline{e}_r = \underline{r}/r = \begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix} \qquad \underline{e}_\varphi = \begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix}}$$



moving coordinate frame:

$\underline{e}_r = \underline{e}_r(t)\,, \quad \underline{e}_\varphi = \underline{e}_\varphi(t)$

Volume (area) element from Jacobian determinant:

$$\boxed{dA} = \det\begin{pmatrix}\frac{\partial x}{\partial r} & \frac{\partial y}{\partial r}\\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi}\end{pmatrix} dr\, d\varphi = \det\begin{pmatrix}\cos\varphi & \sin\varphi\\ -r\sin\varphi & r\cos\varphi\end{pmatrix} dr\, d\varphi$$

$$= \boxed{r\, dr\, d\varphi}$$

- Express $\mathcal{L}_r$ in polar coordinates:

$$\dot{\underline{r}}^2 = \left(\frac{d}{dt}(r\underline{e}_r)\right)^2 = (\dot r \underline{e}_r + r\dot{\underline{e}}_r)^2$$

$$= (\dot r \underline{e}_r + r\dot\varphi \underline{e}_\varphi)^2 = \dot r^2 + (r\dot\varphi)^2$$

$$\rightarrow \boxed{\mathcal{L}_r = \frac{\mu}{2}\dot r^2 + \frac{\mu}{2}(r\dot\varphi)^2 - U(r)}$$

radial kinetic energy — rotational kinetic energy

- Look at Euler-Lagrange equations:

$$\text{I.} \quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot r} = \frac{\partial \mathcal{L}}{\partial r} \iff \frac{d}{dt}(\mu \dot r) = -U'(r)$$

$$\text{II.} \quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\varphi} = \frac{\partial \mathcal{L}}{\partial \varphi} \iff \frac{d}{dt}(\mu r^2 \dot\varphi) = 0$$

$$\frac{\partial L}{\partial \varphi} = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial \dot\varphi} = \mu r^2 \dot\varphi \quad \text{conserved quantity!}$$

This is indeed $L_z = |\underline{L}|$:

$$\boxed{L_z} = |\underline{r} \times \underline{p}| = \mu |\underline{r} \times \dot{\underline{r}}| \qquad \left[\begin{pmatrix} \underline{e}_r \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}, \dots \right]$$

$$= \mu \left| r \begin{pmatrix} \underline{e}_r \\ 0 \end{pmatrix} \times \frac{d}{dt}\left(r \begin{pmatrix} \underline{e}_r \\ 0 \end{pmatrix} \right) \right|$$

$$= \mu r \left| \dot r \underbrace{\begin{pmatrix} \underline{e}_r \\ 0 \end{pmatrix} \times \begin{pmatrix} \underline{e}_r \\ 0 \end{pmatrix}}_{=0} + r \dot\varphi \begin{pmatrix} \underline{e}_r \\ 0 \end{pmatrix} \times \begin{pmatrix} \underline{e}_\varphi \\ 0 \end{pmatrix} \right|$$

$$= \mu r^2 \dot\varphi \underbrace{\left| \begin{pmatrix} \underline{e}_r \\ 0 \end{pmatrix} \times \begin{pmatrix} \underline{e}_\varphi \\ 0 \end{pmatrix} \right|}_{=1} = \boxed{\mu r^2 \dot\varphi} \quad \text{constant of motion}$$

- Summary: 2 constant of motion for relative motion

$$E = \frac{\mu}{2}\left(\dot r^2 + (r\dot\varphi)^2\right) + U(r)$$

$$L_z = \mu r^2 \dot\varphi$$

- Effective potential

We have 2 first-order differential equation which can be decoupled:

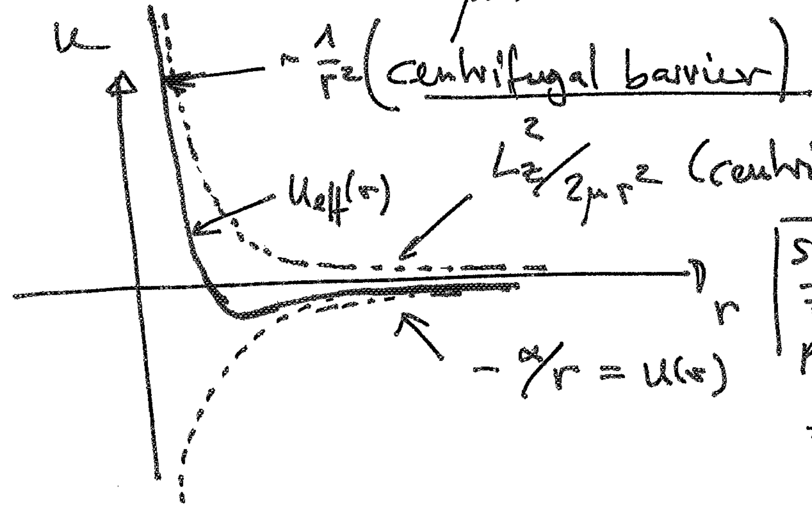
$$r\dot\varphi = \frac{L_z}{\mu r} \longrightarrow E = \frac{\mu}{2}\dot r^2 + \frac{L_z^2}{2\mu r^2} + U(r)$$

→ Equivalent to 1-dim. problem in effective potential:

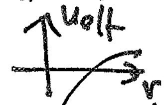
$$\boxed{E = \frac{\mu}{2}\dot r^2 + U_{eff}(r) \quad ; \quad U_{eff}(r) = U(r) + \underbrace{\frac{L_z^2}{2\mu r^2}}_{\text{centrifugal contribution}}}$$

Example: Coulomb potential $U(r) = -\alpha/r$

$$\rightarrow U_{eff}(r) = -\frac{\alpha}{r} + \frac{L_z^2}{2\mu r^2}$$



Side Remark:
For $U(r) = -\alpha/r^k$, $k>2$, potential overcomes centrifugal barrier
$\Rightarrow$ Irrisistable attraction (black hole)
$U \to -\infty$ as $r \to 0$



- We can deduce the trajectories in the same way as previously in the 1d case:

$$E = \frac{\mu}{2}\dot{r}^2 + U_{eff}(r) \Rightarrow \boxed{t - t_0 = \sqrt{\frac{\mu}{2}} \int \frac{dr}{\sqrt{E - U_{eff}(r)}}}$$

Integral might be difficult, but gives us $t(r)$ and by inversion $r(t)$.

- As a next step, we obtain $\varphi(t)$ by solving the first-order differential equation

$$\dot{\varphi} = \frac{d\varphi}{dt} = \frac{L_z}{\mu r^2(t)} \quad \text{(Separation of variables)}$$

$$\Rightarrow \boxed{\varphi - \varphi_0 = \frac{L_z}{\mu} \int \frac{dt}{r^2(t)}}$$

- We obtain $\varphi(r)$ (and by inversion $r(\varphi)$) as $\varphi(t(r))$

- Alternatively, we can obtain $\varphi(r)$ by solving the differential equation

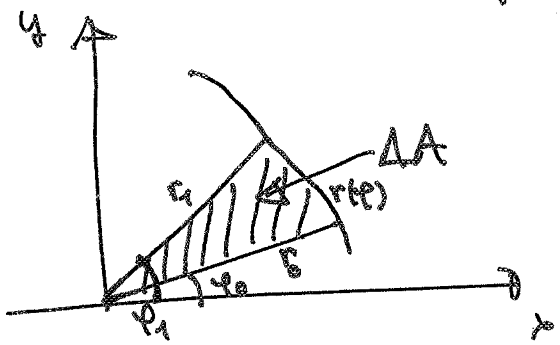
$$\frac{dr}{d\varphi} = \frac{dr/dt}{d\varphi/dt} = \frac{\sqrt{\frac{2}{\mu}}\sqrt{E-U_{eff}(r)}}{L_z/\mu r^2} = \frac{\sqrt{2\mu}}{L_z}\, r^2\sqrt{E-U_{eff}(r)}$$

$$\Rightarrow \boxed{\varphi-\varphi_0 = \frac{L_z}{\sqrt{2\mu}}\int\frac{dr}{r^2\sqrt{E-U_{eff}(r)}}}$$

- Area law (<u>Kepler's 2nd law</u>)

Consequence of angular momentum conservation

Area enclosed by trajectory $r(\varphi)$



$$\Delta A = \int_{\varphi_0}^{\varphi_1} d\varphi \int_0^{r(\varphi)} dr \;\; r\,dr\,d\varphi$$
$$= \frac{1}{2}\int_{\varphi_0}^{\varphi_1} d\varphi \;\; r^2(\varphi)$$

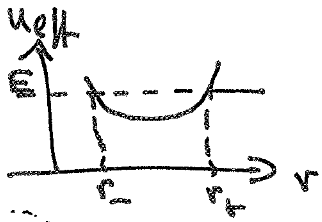
$$L_z = \mu r^2\dot{\varphi} \quad\Rightarrow\quad \frac{L_z}{\mu}\underbrace{(t_1-t_0)}_{\Delta t} = \int_{\varphi_0}^{\varphi} r^2(\varphi)\,d\varphi$$

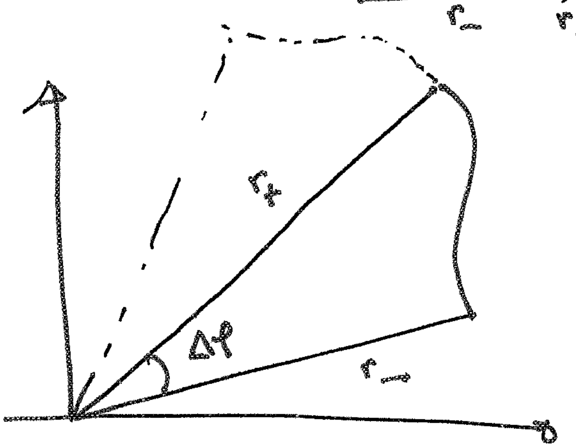
$\Rightarrow$ <u>Enclosed area</u> proportional to time interval,

$\boxed{\Delta A = \frac{L_z}{2\mu}\Delta t}$, trajectory covers equal areas in equal times

- Closed Orbits:

Consider bound motion between radial turning points r_+ and r_-



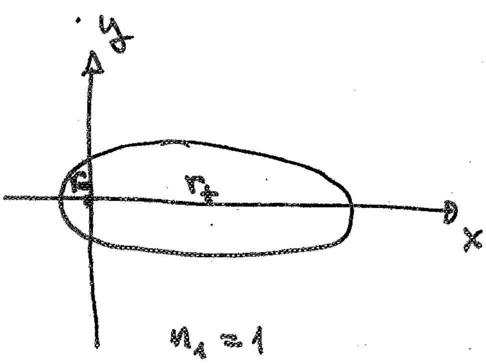


$$\Delta\varphi = \frac{L_z}{\sqrt{2\mu}} \int_{r_-}^{r_+} \frac{dr}{r^2 \sqrt{E - U_{eff}(r)}}$$

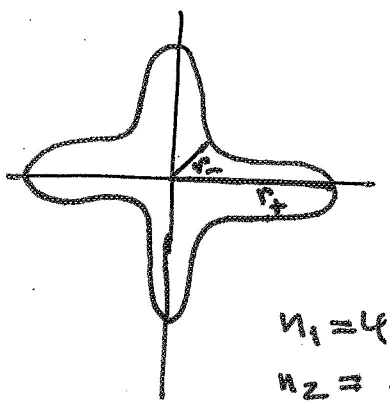
- Full radial oscillation cycle ($r_- \to r_+ \to r_-$) corresponds to angle $2\Delta\varphi$
- closed orbit if integer multiple of $2\Delta\varphi$ is equal to 2π or integer multiple of 2π

$$n_1 \cdot 2\Delta\varphi = n_2 \cdot 2\pi \qquad n_1, n_2 \in \mathbb{N}$$

$$\Leftrightarrow \quad \boxed{\frac{2L_z}{\sqrt{2\mu}} \int_{r_-}^{r_+} \frac{dr}{r^2 \sqrt{E - U_{eff}(r)}} = 2\pi \frac{n_2}{n_1}}$$



$n_1 = 1$
$n_2 = 1$



$n_1 = 4$
$n_2 = 1$

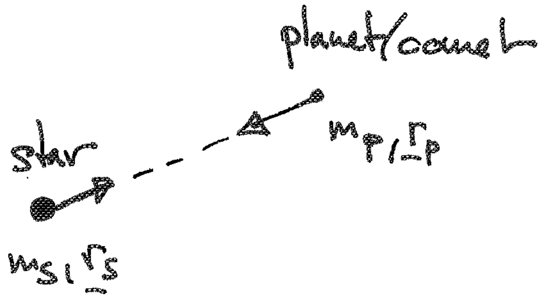
Bertrand's Theorem (1873)

$$\boxed{\begin{array}{l}\text{Only two potentials give rise to stable closed orbits}\\ (1)\ U(r) = -\frac{\alpha}{r} \quad \text{(Coulomb potential = gravitational or electrostatic potential)}\\ (2)\ U(r) = \frac{1}{2}kr^2 \quad \text{(radial harmonic oscillator potential)}\end{array}}$$

3.3. Kepler Problem / Planetary Motion

$$\boxed{U(r) = -\frac{\alpha}{r}}$$

gravitational potential of spherically symmetric object (star); $r >$ radius of star



$$F = G\frac{m_s m_p}{|\underline{r}_s - \underline{r}_p|^2}$$

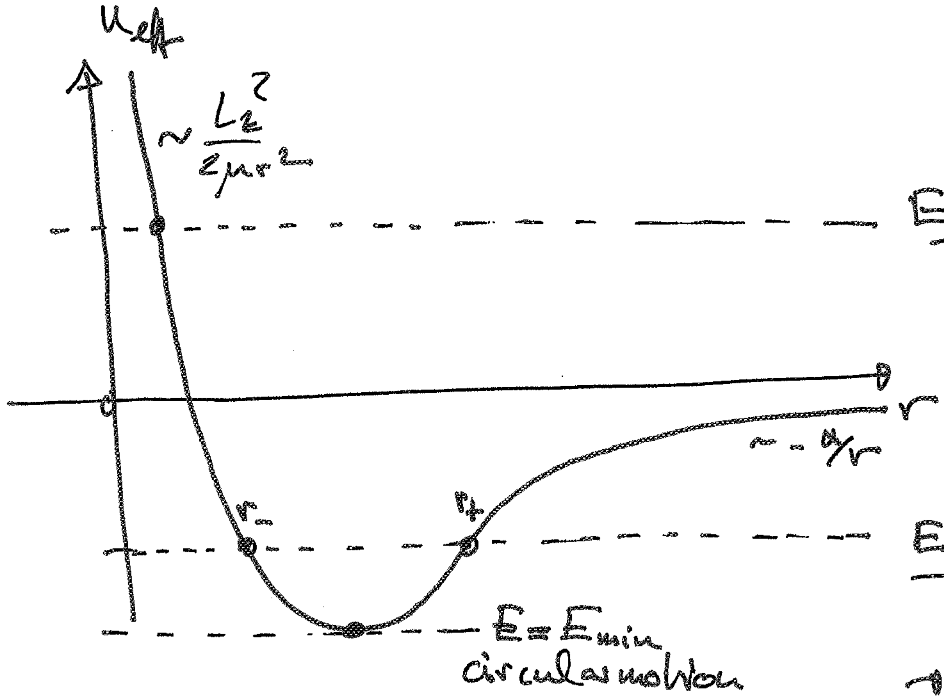
$$U(|\underline{r}_s - \underline{r}_p|) = -G\frac{m_s m_p}{|\underline{r}_s - \underline{r}_p|}$$

$$\boxed{\alpha = G m_s m_p}$$

reduced mass:

$$\boxed{\mu = \frac{m_s m_p}{m_s + m_p}} \underset{m_s \gg m_p}{\approx} m_p$$

$$\rightarrow \boxed{U_{eff}(r) = -\frac{\alpha}{r} + \frac{L_z^2}{2\mu r^2}} \quad ; r \in \mathbb{R}_+$$



$E > 0$: unbound motion
→ comets

$E < 0$: bound motion between r_- and r_+
→ planets (stable orbits)

- $L_z = 0 \Rightarrow$ direct hit

- Turning points

solutions of $E = U_{eff}(r) = -\frac{\alpha}{r} + \frac{L_z^2}{2\mu r^2}$ for $r \in \mathbb{R}_+$

$$\frac{1}{r^2} - \frac{2\mu\alpha}{L_z^2}\frac{1}{r} - \frac{2\mu E}{L_z^2} = 0$$

$$\Rightarrow \boxed{\frac{1}{r_\pm}} = \frac{\mu\alpha}{L_z^2} \pm \sqrt{\left(\frac{\mu\alpha}{L_z^2}\right)^2 + \frac{2\mu E}{L_z^2}}$$

$$= \frac{\mu\alpha}{L_z^2}\left(1 \pm \sqrt{1 + \frac{2EL_z^2}{\mu\alpha^2}}\right)$$

$$= \boxed{\frac{1}{p}(1 \pm \epsilon)}$$

$\boxed{p = \frac{L_z^2}{\mu\alpha}}$ parameter and $\boxed{\epsilon = \sqrt{1 + \frac{2EL_z^2}{\mu\alpha^2}}}$ eccentricity

$0 < \epsilon < 1$: ($E_{min} < E < 0$)

two positive solutions $0 < r_- < r_+$ $\Rightarrow$ planets

($\epsilon = 0 \Rightarrow E = -\frac{\mu\alpha^2}{2L_z^2} = E_{min}$: circular orbit)

$\epsilon > 1$: ($E > 0$)

only one positive solution $r_- > 0$

$\Rightarrow$ comets

$\epsilon = 1$ ($E = 0$) is marginal case , $\frac{1}{r_-} = \frac{2}{p}$, $\frac{1}{r_+} = 0$

for $\epsilon \to 1_-$, $r_+ \to \infty$

- Trajectories in polar coordinates

We start from the general solution for the trajectory from section 3.2. with the $U_{eff}(r)$ in the Kepler problem:

$$\varphi - \varphi_0 = \frac{L_z}{\sqrt{2\mu}} \int \frac{dr}{r^2 \sqrt{E + \frac{\alpha}{r} - \frac{L_z^2}{2\mu r^2}}}$$

$$\underset{(*)}{=} - \sqrt{\frac{\mu\alpha^2}{2L_z^2}} \int \frac{d\xi}{\sqrt{E + \frac{\mu\alpha^2}{L_z^2}\xi - \frac{\mu\alpha^2}{2L_z^2}\xi^2}}$$

(*) Substitution $\xi = \frac{L_z^2}{\mu\alpha r} = \frac{p}{r}$

$$d\xi = -\frac{L_z^2}{\mu\alpha r^2} dr \Leftrightarrow \frac{dr}{r^2} = -\frac{\mu\alpha}{L_z^2} d\xi$$

$$= - \int \frac{d\xi}{\sqrt{\frac{2EL_z^2}{\mu\alpha^2} + 2\xi - \xi^2}}$$

$$= - \int \frac{d\xi}{\sqrt{\frac{2EL_z^2}{\mu\alpha^2} + 1 - (\xi-1)^2}}$$

$$= - \int \frac{d\xi}{\sqrt{\epsilon^2 - (\xi-1)^2}} = \arccos\left(\frac{\xi-1}{\epsilon}\right)$$

invert function:

$$\epsilon \cos(\varphi - \varphi_0) = \xi - 1 = p/r - 1$$

$$\Rightarrow \boxed{r(\varphi) = \frac{p}{1+\epsilon\cos(\varphi-\varphi_0)}}$$ trajectories in polar coordinates

- Transform to cartesian coordinates

rotate frame of reference such that $\varphi_0 = 0$ and use transformation

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$

$$r = \frac{p}{1+\epsilon\cos\varphi} \Leftrightarrow r = \sqrt{x^2+y^2} = \frac{p}{1+\epsilon/r\, x}$$

$$\Leftrightarrow r + \epsilon x = p \Rightarrow r^2 = x^2+y^2 = (p-\epsilon x)^2$$

$$\Leftrightarrow \quad (1-\epsilon^2)x^2 + 2\epsilon p x + y^2 = p^2$$ „quadratic form"

$$\underset{\epsilon\neq 1}{\Leftrightarrow} \quad (1-\epsilon^2)\left(x^2 + 2\frac{\epsilon p}{1-\epsilon^2}x\right) + y^2 = p^2$$

$$\Leftrightarrow \quad (1-\epsilon^2)\left[\left(x+\frac{\epsilon p}{1-\epsilon^2}\right)^2 - \frac{\epsilon^2 p^2}{(1-\epsilon^2)^2}\right] + y^2 = p^2$$

$$\Leftrightarrow \quad (1-\epsilon^2)\left(x+\frac{\epsilon p}{1-\epsilon^2}\right)^2 + y^2 = p^2 + \frac{\epsilon^2 p^2}{1-\epsilon^2} = \frac{p^2}{1-\epsilon^2}$$

$$\underset{\epsilon\neq 1}{\Leftrightarrow} \quad \boxed{\frac{(1-\epsilon^2)^2}{p^2}\left(x+\frac{\epsilon p}{1-\epsilon^2}\right)^2 + \frac{1-\epsilon^2}{p^2}y^2 = 1} \quad (*)$$

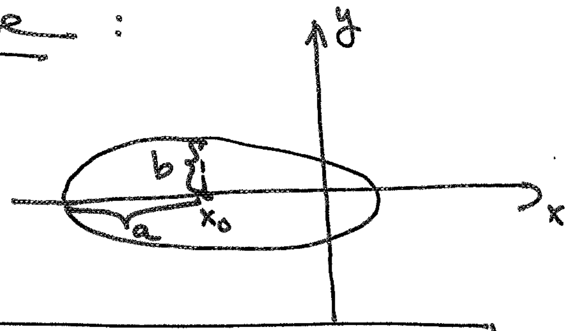
for $\underline{\epsilon = 1}$: $\boxed{2px + y^2 = p^2}$

I. Closed Orbits (Planets): $0 \leq \epsilon < 1$

$$\frac{1-\epsilon^2}{p^2} > 0$$

$\Rightarrow$ Equation (*) describes an ellipse:

$$\boxed{\frac{(x-x_0)^2}{a^2} + \frac{y^2}{b^2} = 1}$$



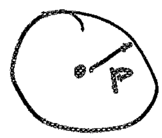
$$\boxed{x_0 = -\frac{\epsilon p}{1-\epsilon^2}, \quad a = \frac{p}{1-\epsilon^2}, \quad b = \frac{p}{\sqrt{1-\epsilon^2}} < a}$$

a, b major and minor half axes of ellipse

It follows that $|x_0| = \sqrt{a^2-b^2}$

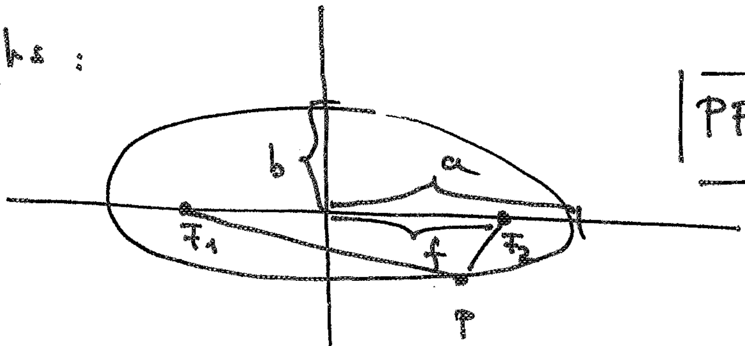
- $\epsilon = 0$ $(E = E_{min})$: $x_0 = 0$, $a = b = p$

  circular motion



- the origin for the relative coordinate $\underline{r} = \underline{r}_p - \underline{r}_s$ is a focal point of the ellipse

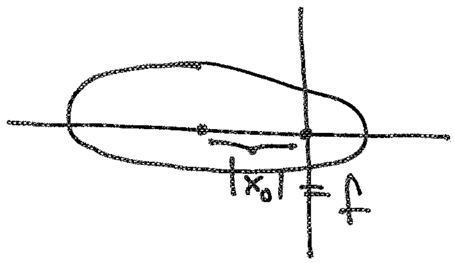
focal points:



$$\boxed{PF_1 + PF_2 = 2a}$$

$$\boxed{f = a\epsilon}$$

in our case: $|x_0| = \frac{\epsilon p}{1-\epsilon^2} = \epsilon \frac{p}{1-\epsilon^2} = \epsilon a = f$



$\Rightarrow$ origin coincides with focal point

Note: Origin for relative coordinate corresponds to center of mass

$$\underline{R} = \frac{m_s \underline{r}_s + m_p \underline{r}_p}{m_s + m_p} \underset{m_s \gg m_p}{\approx} \underline{r}_s$$

- Kepler's laws of planetary motion

$$\boxed{\text{(K1) The orbit of every planet is an ellipse with the sun at one of the foci.}}$$

Our calculation confirms that the orbits are ellipses. For $m_s \gg m_p$, the sun is indeed close to one of the foci since $\underline{R} \approx \underline{r}_s$.

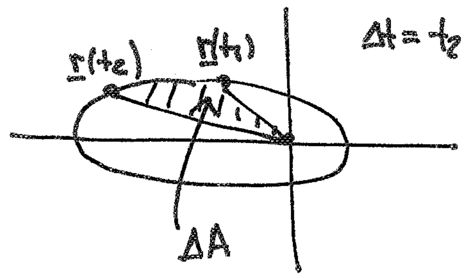
(K2) A line joining a planet and the sun sweeps out equal areas during equal time intervals.

We derived Kepler's 2nd law in section 3.2. Note that this law is true for any radially symmetric potential $V(r)$ and a consequence of angular momentum conservation.

(K3) The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.

Proof: Use 2nd law:

$\Delta t = t_2 - t_1$



$$\Delta A = \frac{L_z}{2\mu} \Delta t$$

$$\Rightarrow \quad \frac{L_z}{2\mu} T = A = \pi a b$$

$$\rightarrow \quad T = 2\pi \frac{\mu}{L_z} a b \underset{p = \frac{L_z^2}{\mu\alpha}}{=} 2\pi \sqrt{\frac{\mu}{\alpha}}\, p^{-1/2} a b$$

$$\underset{a = \frac{p}{1-\epsilon^2}}{=} 2\pi \sqrt{\mu/\alpha}\; a^{3/2}$$

$$b = \frac{p}{\sqrt{1-\epsilon^2}} = \sqrt{ap}$$

$$\Rightarrow \quad \boxed{T^2 = 4\pi^2 \frac{\mu}{\alpha} a^3}$$

For planets in our solar system:

$$\alpha = G m_s m_p \;, \quad \mu = \frac{m_s m_p}{m_s + m_p} \simeq m_p$$

$$\rightarrow \quad T^2 \simeq \underbrace{\frac{4\pi^2}{G m_s}}_{\text{same for all planets}} a^3 \quad \Longrightarrow \quad \boxed{\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}}$$

$T_{1,2}$ and $a_{1,2}$ are orbital periods and major half axes for 2 different planets

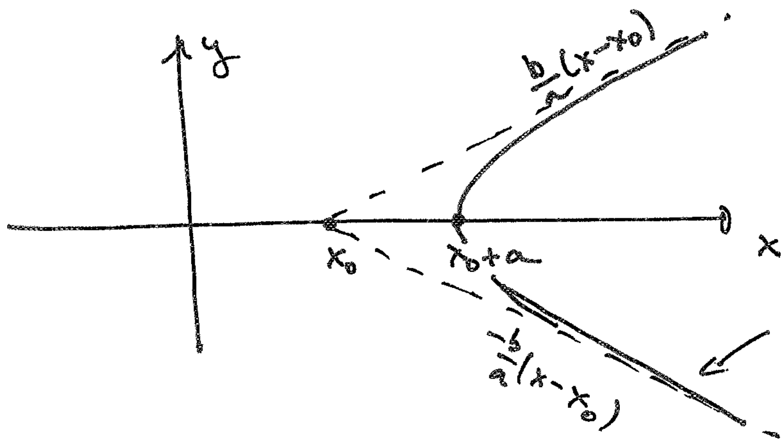
II. Unbounded motion: comets : $\epsilon > 1$

$$\underbrace{\frac{(1-\epsilon^2)^2}{p^2}}_{>0} \left(x + \frac{\epsilon p}{1-\epsilon^2} \right)^2 + \underbrace{\frac{1-\epsilon^2}{p^2}}_{<0} y^2 = 1$$

$$\boxed{\frac{(x-x_0)^2}{a^2} - \frac{y^2}{b^2} = 1}$$

Equation determines hyperbola

$$\boxed{x_0 = \frac{\epsilon p}{\epsilon^2 - 1} \;, \quad a = \frac{p}{\epsilon^2 - 1} \;, \quad b = \frac{p}{\sqrt{\epsilon^2 - 1}}}$$



$y = 0 \Rightarrow x = x_0 + a$

Asymptotics:

$x - x_0 \gg a$:

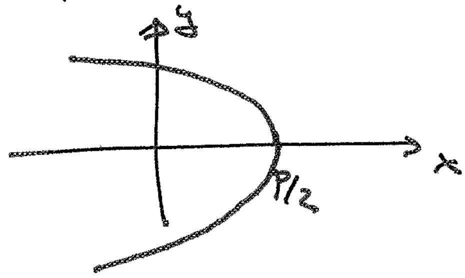
$$\frac{(x-x_0)^2}{a^2} \simeq \frac{y^2}{b^2}$$

$$\Rightarrow y = \pm \, b/a \, (x - x_0)$$

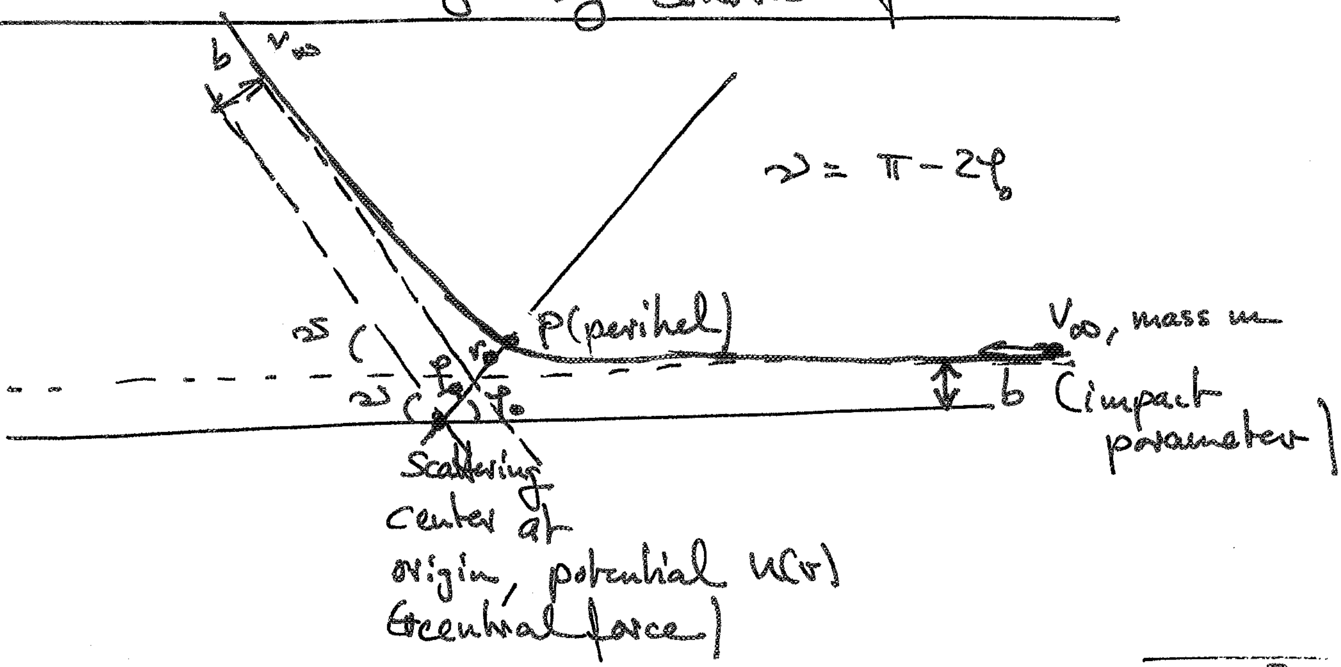
III. Marginal case : $\epsilon = 1$ $(E=0)$

$$2px + y^2 = p^2 \Leftrightarrow x = -\frac{1}{2p}y^2 + \frac{p}{2}$$

parabola



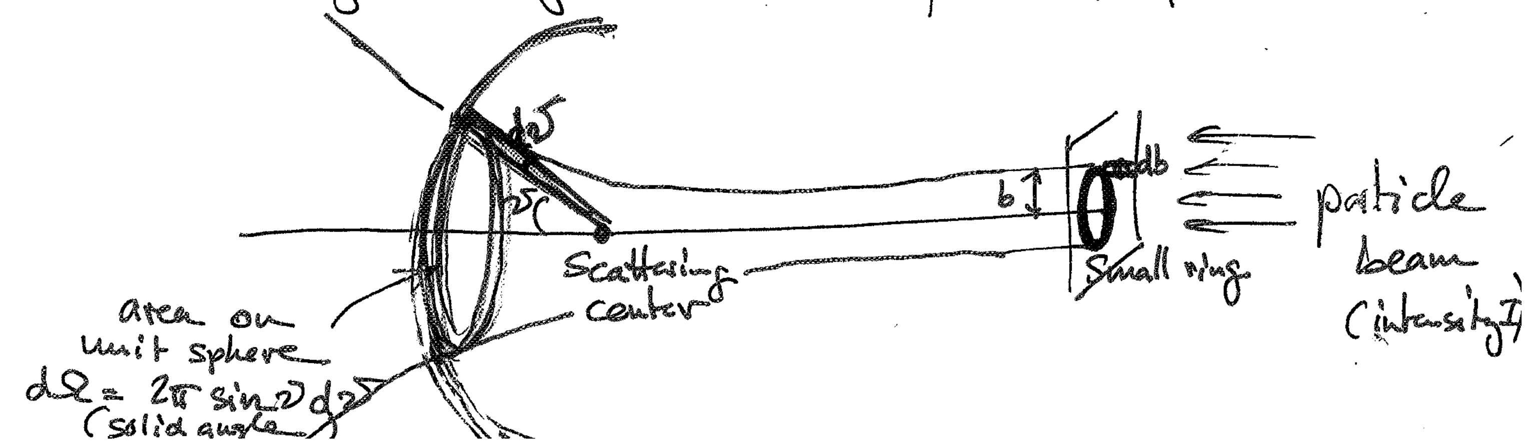
3.4. Elastic scattering by central forces



$$\varphi(r) - \varphi_0 = \frac{L_z}{\sqrt{2m}} \int_{r_0}^{r} \frac{d\tilde{r}}{\tilde{r}^2 \sqrt{E - U(\tilde{r}) - \frac{L_z^2}{2m\tilde{r}^2}}}$$

$$\boxed{E = \frac{m}{2}v_\infty^2 \qquad L_z = m b v_\infty}$$

- Scattering theory deals with flux of particles



- Differential scattering cross section σ:

$$\sigma\, d\Omega := \frac{\#\text{ particles scattered into } d\Omega \text{ per second}}{\text{intensity } I \text{ of particle beam}} = \frac{dN}{I}$$

$$\rightarrow \sigma \cdot 2\pi \sin\vartheta\, d\vartheta = \frac{dN}{I} = \frac{I \cdot \overbrace{2\pi b\,|db|}^{\text{area of annulus}}}{I}$$

$$\Rightarrow \boxed{\sigma = \frac{b}{\sin\vartheta}\left|\frac{db}{d\vartheta}\right|}$$

Example: Rutherford scattering

$U(r) = \frac{\alpha}{r}$ $(\alpha > 0)$ (electrostatic repulsion)

$$\varphi - \varphi_0 = \frac{L_z}{\sqrt{2m}} \int \frac{dr}{r^2\sqrt{E - \frac{\alpha}{r} - \frac{L_z^2}{2mr^2}}}$$

$$\xi = \frac{L_z^2}{m\alpha r} = p/r \qquad - \int \frac{d\xi}{\sqrt{\epsilon^2 - (\xi+1)^2}}$$

different sign for repulsive potential

$$= \arccos \frac{\xi+1}{\epsilon}$$

$$\rightarrow \boxed{\frac{1}{r(\varphi)} = \frac{1}{p}\left(-1 + \epsilon\cos(\varphi - \varphi_0)\right)}$$

$$\boxed{p = \frac{L_z^2}{m\alpha} = \frac{m v_\infty^2 b^2}{\alpha}}$$

$$\boxed{\epsilon^2 = 1 + \frac{2EL_z^2}{m\alpha^2} = 1 + \frac{m^2 v_\infty^4}{\alpha^2} b^2} > 1$$

Minimum separation (perihel) for $\varphi = \varphi_0$:

$$r_0 = \frac{p}{-1+\epsilon}$$

For $\varphi = 0$: $\frac{1}{r} = 0 \Rightarrow -1 + \epsilon \cos\varphi_0 = 0$

$$\Leftrightarrow \cos\varphi_0 = \frac{1}{\epsilon}$$

Using that $\vartheta = \pi - 2\varphi_0$, we obtain

$$\cos\varphi_0 = \cos\left(\frac{\pi}{2} - \frac{\vartheta}{2}\right) = \sin\frac{\vartheta}{2} = \frac{1}{\epsilon} = \frac{1}{\sqrt{1 + \frac{m^2 v_\infty^4}{\alpha^2} b^2}}$$

$$\Rightarrow 1 + \frac{m^2 v_\infty^4}{\alpha^2} b^2 = \frac{1}{\sin^2\frac{\vartheta}{2}}$$

$$\Rightarrow \boxed{b(\vartheta) = \left[\frac{\alpha^2}{m^2 v_\infty^4}\left(\frac{1}{\sin^2\frac{\vartheta}{2}} - 1\right)\right]^{1/2} = \frac{\alpha}{m v_\infty^2} \cot\frac{\vartheta}{2}}$$

$$\boxed{\sigma = \frac{b}{\sin\vartheta}\left|\frac{db}{d\vartheta}\right| = \frac{1}{\sin\vartheta}\left(\frac{\alpha}{m v_\infty^2}\right)^2 \cot\frac{\vartheta}{2} \frac{d}{d\vartheta}\cot\frac{\vartheta}{2} = \frac{1}{4}\left(\frac{\alpha}{m v_\infty^2}\right)^2 \frac{1}{\sin^4\frac{\vartheta}{2}}}$$