1. Equations of Motion

Newtonian Mechanics: Equations that determine the properties of a system in the next instant given their values at the preceding instant (causality)

- based on differential equations + initial conditions

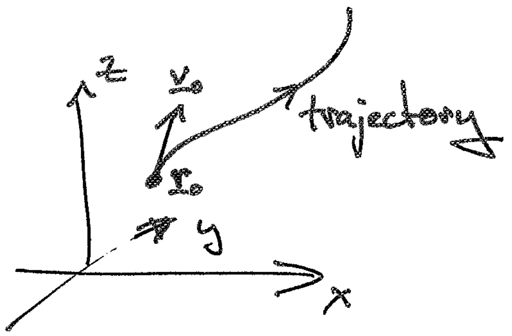
$\underline{r} = (x, y, z)$ position of a particle

$\underline{v} = \dot{\underline{r}} = \frac{d\underline{r}}{dt}$ velocity

$\underline{a} = \ddot{\underline{r}} = \frac{d^2\underline{r}}{dt^2}$ acceleration

$$\boxed{\underbrace{m}_{\text{mass}}\ddot{\underline{r}} = \underbrace{\underline{F}}_{\text{Force}}}$$ Newton's 2nd law

- 2nd order differential equation requires two initial conditions, e.g. $\boxed{\underline{r}(t_0) = \underline{r}_0}$, $\boxed{\underline{v}(t_0) = \underline{v}_0}$



- trajectory of the particle is completely and uniquely determined by differential equation (Newton's 2nd law) and initial conditions

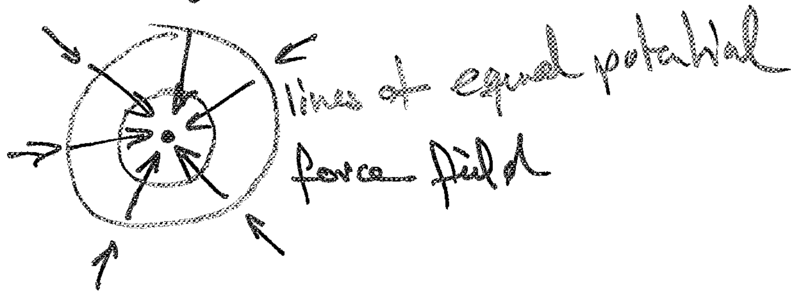
- conservative forces: $\boxed{\underline{F} = -\nabla U}$

  $U = U(\underline{r}, t)$ <u>potential energy</u>

<u>example</u>: Coulomb potential: $U(r) = -\frac{\alpha}{r}$

$\left(r = |\underline{r}| = \sqrt{\sum_i x_i^2} \right)$

$$\underline{F} = -\underline{\nabla} U = \underline{\nabla} \frac{\alpha}{r} = -\frac{\alpha}{r^2} \underline{e}_r \qquad \left(\underline{e}_r = \underline{r}/|\underline{r}| \right)$$



- Conservation of energy?

$$\frac{dE}{dt} = \frac{d}{dt}(T + U) = \frac{d}{dt}\left(\frac{m}{2}\dot{\underline{r}}^2 + U(\underline{r}, t) \right)$$

$$= m \dot{\underline{r}} \cdot \ddot{\underline{r}} + \underbrace{\frac{\partial U}{\partial \underline{r}}}_{= \nabla U} \cdot \dot{\underline{r}} + \frac{\partial U}{\partial t}$$

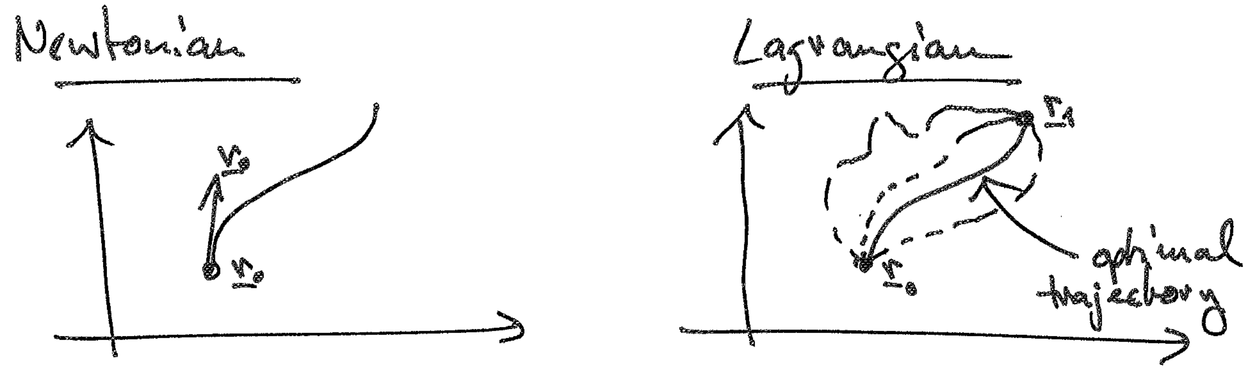
$$= \dot{\underline{r}} \underbrace{(m \ddot{\underline{r}} + \underline{\nabla} U)}_{= 0 \text{ (Newton's 2nd law)}} + \frac{\partial U}{\partial t}$$

energy conserved $\Leftrightarrow$ $\frac{\partial U}{\partial t} = 0$

$\rightarrow$ symmetries $\leftrightarrow$ conservation laws
(e.g. invariance
under time translation)

Lagrangian Mechanics:

- Mathematically equivalent but based on a different philosophy: destiny
- Looks at global properties of trajectories and finds the optimal one



- Connection to quantum mechanics (summation over all possible trajectories with certain weights; classical trajectory has maximum weight (highest probability))

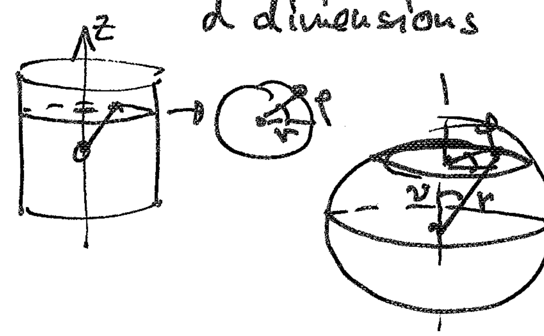
1.1. The Principle of Least Action

- Trajectory in the space of generalized coordinates $q = (q_1, q_2, \dots, q_N)$

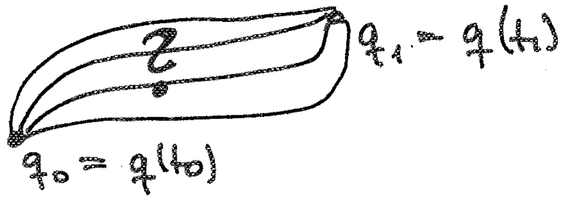
$q = (x, y, z)$ cartesian coordinates in $d=3$, one particle

$q = (\underbrace{x_1^{(1)}, \dots, x_d^{(1)}}, \underbrace{x_1^{(2)}, \dots, x_d^{(2)}}, \dots, \underbrace{x_1^{(n)}, \dots, x_d^{(n)}})$ cartesian coordinates of n particles in d dimensions

$$\left.\begin{array}{ll} q = (r, \varphi, z) & \text{cylindrical} \\ q = (r, \vartheta, \varphi) & \text{spherical} \end{array}\right\} \begin{array}{l} n=1 \\ d=3 \end{array}$$



- Suppose the particle is destined to travel from $q_0 = q(t_0)$ to $q_1 = q(t_1)$.
  What is the „optimal" trajectory ?



- Trajectory that is optimal is the one that <u>minimizes</u> the functional

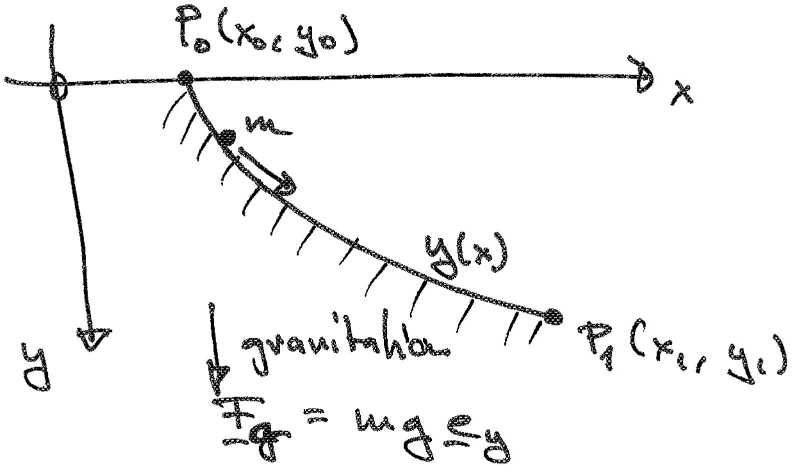
$$\boxed{S = \int_{t_0}^{t_1} dt \; L(q(t), \dot{q}(t), t)}$$

$$\underset{\text{action}}{\uparrow} \qquad \underset{\text{Lagrangian}}{\uparrow}$$

- This is a teleological principle rather than a causal/mechanical one (telos (greek) = end, purpose)
- At first glance it seems to contradict causality but it doesn't !
- Later we will see which form the Lagrangian L has to take that the optimal trajectory is consistent with Newton's 2nd law
- Deeper reason: quantum mechanics (later)

<u>Historical example of variational problem:</u>

Brachistochrone problem (Bernoulli 1696)
(brachis = short ; chronos = time)



- slide connecting P_0 and P_1
- no friction

Q: For which slide $y(x)$ is the sliding time T minimal?

- solution to the problem is a function $y(x)$
- We have to determine the functional $T(y, y')$ and minimize it to find the optimal slide

$$\boxed{T} = \int_{P_0 \to P_1} dt = \int_{P_0 \to P_1} \frac{ds}{v}$$

$$= \int_{x_0}^{x_1} dx \, \frac{\sqrt{1+y'^2}}{v}$$

$$\boxed{= \frac{1}{\sqrt{2g}} \int_{x_0}^{x_1} dx \, \sqrt{\frac{1+y'^2}{y}}}$$

$$ds = \sqrt{(dx)^2 + (dy)^2} = dx\sqrt{1+y'^2}$$

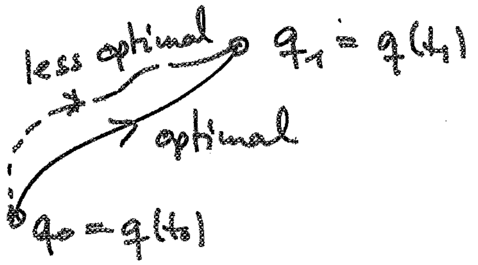
energy conservation:

$$\frac{1}{2}mv^2 = mgy \Rightarrow v = \sqrt{2gy}$$

- Functional $T(y, y')$ takes different values for different functions $y(x)$
- We will determine the optimal function later on using the techniques we will develop now

1.2. Euler – Lagrange Equations

- Consider infinitesimal variations of trajectories



$$q(t) \to q(t) + \delta q(t)$$

$\delta q(t)$ small

$$\delta q(t_0) = \delta q(t_1) = 0$$

change in action:

$$\delta S = \int_{t_0}^{t_1} dt \left\{ L(q+\delta q, \dot q + \delta \dot q, t) - L(q, \dot q, t) \right\}$$

$$= \int_{t_0}^{t_1} dt \left\{ \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot q} \delta \dot q + \mathcal{O}(\delta q^2, \delta \dot q^2) \right\}$$

$$= \int_{t_0}^{t_1} dt \left\{ \frac{\partial L}{\partial q} \delta q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q} \delta q \right) - \left(\frac{d}{dt} \frac{\partial L}{\partial \dot q} \right) \delta q \right\}$$

$$= \int_{t_0}^{t_1} dt \; \delta q \left\{ \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot q} \right\} + \underbrace{\left. \frac{\partial L}{\partial \dot q} \delta q \right|_{t_0}^{t_1}}_{\substack{=0 \text{ since} \\ \delta q(t_0) = \delta q(t_1) = 0}}$$

For optimal trajectory, $\delta S = 0$ for all infinitesimal variations δq

$$\Rightarrow \boxed{\frac{d}{dt} \frac{\partial L}{\partial \dot q} = \frac{\partial L}{\partial q}}$$

Euler – Lagrange equations

- For $q=(q_1,\dots,q_N)$ we have a set of N differential equations, 2nd. order, coupled
  $\forall \alpha=1,\dots,N$: $\frac{d}{dt}\frac{\partial L}{\partial \dot q_\alpha} = \frac{\partial L}{\partial q_\alpha}$,
- Determine trajectory $q(t)$ in configuration space for given $q(t_0)$, $\dot q(t_0)$
  $\rightarrow$ causality restored

Example: Brachistochrone problem

$$S(q,\dot q) = \int_{t_0}^{t_1} dt\, L(q,\dot q,t) \quad \Bigg| \quad T(y,y') = \int_{x_0}^{x_1} dx \underbrace{\sqrt{\frac{1+y'^2}{y}}}_{=: f(y,y',x)}$$

$$\rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot q} = \frac{\partial L}{\partial q} \quad \Bigg| \quad \rightarrow \boxed{\frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial y}} \quad (*)$$

We have to solve the 2nd order differential equation $(*)$. In the case that the Lagrangian L does not depend on t (here $f(y,y',x)$ does not depend on x) we get one integration for free! (We will give a deeper reason in Chapter 2)

Look at

$$\frac{df}{dx} = \frac{d}{dx} f(y,y') = \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''$$

$$\underset{EL.}{=} \left(\frac{d}{dx}\frac{\partial f}{\partial y'}\right) y' + \frac{\partial f}{\partial y'} y'' = \frac{d}{dx}\left(\frac{\partial f}{\partial y'} y'\right)$$

$$\Rightarrow \frac{d}{dx}\left(f - \frac{\partial f}{\partial y'} y'\right) = 0$$

$$\Rightarrow \boxed{f - \frac{\partial f}{\partial y'} y' = c_1}$$

one integration performed
$\Rightarrow$ differential equation 1st order

Explicitly: $\sqrt{\frac{1+y'^2}{y}} - \sqrt{\frac{1+y'^2}{y}}^{-1} \frac{y'^2}{y} = c_1$

$$\Rightarrow \frac{1+y'^2}{y} - \frac{y'^2}{y} = c_1 \sqrt{\frac{1+y'^2}{y}}$$

$$\Rightarrow \frac{1}{c_1^2 y^2} = \frac{1+y'^2}{y} \Rightarrow \boxed{y' = \sqrt{\frac{1}{c_1^2 y} - 1} = \sqrt{\frac{2r_0 - y}{y}}}$$

(we have redefined the integration constant, $r_0 := \frac{1}{2c_1^2}$)

We solve this differential equation by separation of variables.

$$\int dy \sqrt{\frac{y}{2r_0 - y}} = \int dx = x - c_2$$

Substitution: $y = 2r_0 \sin^2\frac{\varphi}{2} = r_0(1-\cos\varphi)$

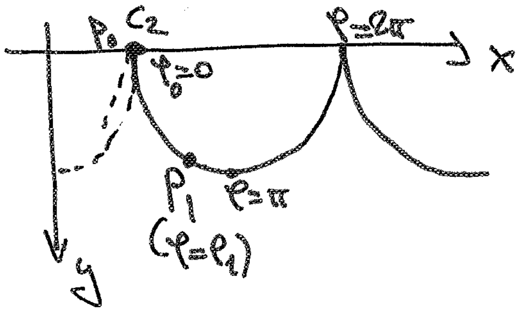
$$dy = 2r_0 \sin\varphi/2 \cos\varphi/2 \, d\varphi$$

$$\rightarrow x - c_2 = 2r_0 \int d\varphi \sqrt{\frac{\sin^2\varphi/2}{\cos^2\varphi/2}} \sin\varphi/2 \cos\varphi/2$$

$$= 2r_0 \int d\varphi \sin^2\varphi/2 = r_0 \int d\varphi (1-\cos\varphi)$$

$$= r_0(\varphi - \sin\varphi)$$

Solution: $\boxed{\underline{r}(\varphi) = \begin{pmatrix} x(\varphi) \\ y(\varphi) \end{pmatrix} = \begin{pmatrix} c_2 \\ 0 \end{pmatrix} + r_0 \begin{pmatrix} \varphi - \sin\varphi \\ 1-\cos\varphi \end{pmatrix}}$



parametrizes a cycloid

initial conditions:

$$\underline{r}(0) = \begin{pmatrix} c_2 \\ 0 \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \end{pmatrix} \Rightarrow \boxed{c_2 = x_0}$$

$$\underline{r}(\varphi_1) = \begin{pmatrix} x_0 \\ 0 \end{pmatrix} + r_0 \begin{pmatrix} \varphi_1 - \sin\varphi_1 \\ 1-\cos\varphi_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$\frac{x_1 - x_0}{y_1} = \frac{\varphi_1 - \sin\varphi_1}{1-\cos\varphi_1}$$ determines φ_1

$\rightarrow$ $$r_0 = \frac{y_1}{1-\cos\varphi_1}$$ determines r_0

1.3. Connection to Newtonian Mechanics

- We want to engineer Lagrangian L such that Euler-Lagrange equation is identical to Newton's 2nd law of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q} \overset{?}{\Longleftrightarrow} m\ddot{\underline{r}} = -\underline{\nabla} U = \underline{F}$$

$\rightarrow$ $$\boxed{L = \frac{m}{2}\dot{\underline{r}}^2 - U(\underline{r})}$$

each coordinate, e.g. x : ($\dot{\underline{r}}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$$

$$m\ddot{x} = -\frac{\partial U}{\partial x} = F_x$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\underline{r}}} = \frac{d}{dt} m\dot{\underline{r}} = m\ddot{\underline{r}}$$

$$\frac{\partial L}{\partial \underline{r}} = -\frac{\partial U}{\partial \underline{r}} = -\underline{\nabla} U = \underline{F}$$

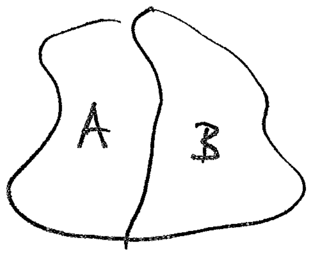
- We can express Newton's 2nd law as a principle of least action!

Minimizing $$S = \int_{t_0}^{t_1} dt \left\{ \frac{m}{2}\dot{\underline{r}}(t)^2 - U(\underline{r}(t)) \right\}$$
gives Newton's law of motion

1.4. General Properties of the Action

• Two independent systems

System A : $q_A = (q_{A,1}, \dots, q_{A,N})$
System B : $q_B = (q_{B,1}, \dots, q_{B,M})$



Addition of Lagrangians :

$$L(q_A, q_B, \dot{q}_A, \dot{q}_B, t) = L_A(q_A, \dot{q}_A, t) + L_B(q_B, \dot{q}_B, t)$$

leads to independent equations of motion

$$\frac{d}{dt}\frac{\partial L_A}{\partial \dot{q}_A} = \frac{\partial L_A}{\partial q_A} \quad \text{and} \quad \frac{d}{dt}\frac{\partial L_B}{\partial \dot{q}_B} = \frac{\partial L_B}{\partial q_B}$$

• Interacting subsystems

$$L = L_A(q_A, \dot{q}_A, t) + L_B(q_B, \dot{q}_B, t) + L_{AB}(q_A, q_B, \dot{q}_A, \dot{q}_B, t)$$

$\rightarrow$ Differential equations for A and B are no longer decoupled due to L_{AB}

• Invariance under multiplication with constants

$L' = \lambda L$, $\lambda = \text{const}$ $\Rightarrow$ same equation of motion

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}} = \lambda \frac{\partial L}{\partial \dot{q}}, \quad \frac{\partial L'}{\partial q} = \lambda \frac{\partial L}{\partial q}$$

$$\rightarrow \quad \frac{d}{dt}\frac{\partial L'}{\partial \dot{q}} = \frac{\partial L'}{\partial q} \quad \Leftrightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$$

- Invariance under adding total derivatives

$$L' = L(q,\dot q,t) + \frac{d}{dt} f(q,t)$$

why?

$$S' = \int_{t_0}^{t_1} dt\; L'(q,\dot q,t) = S + \underbrace{f(q(t_1),t_1) - f(q(t_0),t_0)}_{\substack{\text{unchanged by a variation} \\ \text{with } \delta q(t_0) = \delta q(t_1) = 0}}$$

$\Rightarrow$ identical equations of motion

Invariance can also be seen by explicit derivations of equations of motion:

$$L' = L + \frac{d}{dt} f(q,t) = L + \frac{\partial f}{\partial q}\dot q + \frac{\partial f}{\partial t}$$

Euler - Lagrange equations:

$$0 = \frac{d}{dt}\frac{\partial L'}{\partial \dot q} - \frac{\partial L'}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} + \frac{d}{dt}\frac{\partial}{\partial \dot q}\left(\frac{\partial f}{\partial q}\dot q + \frac{\partial f}{\partial q}\right)$$

$$- \frac{\partial}{\partial q}\left(\frac{\partial f}{\partial q}\dot q + \frac{\partial f}{\partial t}\right)$$

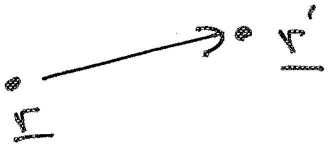
$$= \frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} + \underbrace{\frac{d}{dt}\frac{\partial f}{\partial q} - \left(\frac{\partial^2 f}{\partial q^2}\dot q + \frac{\partial^2 f}{\partial q\,\partial t}\right)}_{=0}$$

$\rightarrow$ same equation of motion

1.5. Galilean Invariance

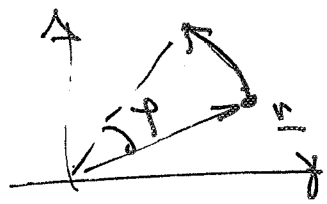
- In Section 1.3. we engineered Lagrangian L such that Euler-Lagrange equation gives us Newton's 2nd law of motion
- Now we derive L from first principle
- free particle in homogeneous and isotropic space (+ time homogeneous)

homogeneous:



invariance with respect to spatial shifts

isotropic:



invariance with respect to rotations

Invariance: equation of motion remains unchanged

$$\Rightarrow L = L(v^2) = \alpha_0 + \frac{\alpha_2}{2!} v^2 + \frac{\alpha_4}{4!} v^4 + \dots$$

only possibility!

We can set $\alpha_0 = 0$ without loss of generality as α_0 drops out in the Euler-Lagrange equations

E.L. equations: $\frac{d}{dt} \frac{\partial L}{\partial \dot{\underline{r}}} = 0$

$$\Rightarrow \forall i = 1, \dots, d: \quad \frac{\partial L}{\partial v_i} = \frac{\partial L}{\partial v^2} \underbrace{\frac{\partial v^2}{\partial v_i}}_{= 2 v_i} = \text{const.}$$

$\Rightarrow \underline{v} = \text{const} \rightarrow$ „inertial frame of reference"

- Galilean principle of relativity:

$\underline{r}' = \underline{r} + \underline{u}t \;, \quad t' = t$ (Galilean trf.) should leave the equations of motion invariant (valid for $u \ll c$)

$$\underline{v}' = \frac{d\underline{r}'}{dt'} = \frac{d\underline{r}'}{dt} = \frac{d\underline{r}}{dt} + \underline{u} = \underline{v} + \underline{u}$$

$\underline{v}' = \underline{v} + \underline{u}$, consider infinitesimally small $\underline{u}$

$$L' = L(v'^2) = L((\underline{v}+\underline{u})^2) = L(\underline{v}^2 + 2\underline{u}\,\underline{v} + \underline{u}^2)$$

$$= L(v^2) + \frac{\partial L}{\partial v^2} 2\underline{u}\,\underline{v} + \mathcal{O}(\underline{u}^2)$$

From invariance of equations of motion we require that

$$L' = L + \frac{df(\underline{r},t)}{dt}$$

$$\Rightarrow \frac{df(\underline{r},t)}{dt} = 2\underline{u}\,\underline{v}\frac{\partial L}{\partial v^2}$$

$$\| $$

$$\frac{\partial f}{\partial \underline{r}}\underline{v} + \frac{\partial f}{\partial t}$$

$$\Rightarrow \frac{\partial f}{\partial t} = 0 \quad \text{and} \quad \frac{\partial f}{\partial \underline{r}} = 2\frac{\partial L}{\partial v^2}\underline{u}$$

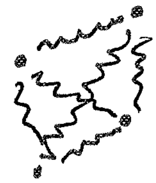
$\Rightarrow$ $\frac{\partial f}{\partial \underline{r}}$ independent of $\underline{v}$

$$\boxed{L = \frac{\alpha_2}{2} v^2 =: \frac{m}{2} v^2} \;, \quad f = m\underline{u}\cdot\underline{r}$$

m: mass

• Interaction

add $L_I = -U(\underline{r}_1, \dots, \underline{r}_N)$



$$L = \sum_i L_i(\underline{r}_i, \dot{\underline{r}}_i, t) + L_I$$

require:
instantaneous
+ only position dependent

$$\rightarrow \quad \boxed{L = \sum_i \frac{m_i}{2}\dot{\underline{r}}_i^2 - U(\underline{r}_1, \dots, \underline{r}_N)}$$

$$= T - U \quad \text{(kinetic energy} - \text{potential energy)}$$

We derived Newton's 2nd law:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\underline{r}}_i} = \frac{\partial L}{\partial \underline{r}_i} \quad \Leftrightarrow \quad m_i \ddot{\underline{r}}_i = -\underline{\nabla}_i U(\underline{r}_1, \dots, \underline{r}_N)$$

$(i = 1, \dots N)$

Coupled set of 2nd order differential equations

• Note: In general, potential consists of external potential and interaction potential:

$$U(\underline{r}_1, \dots, \underline{r}_N) = \sum_i U_{ext}(\underline{r}_i) + \underbrace{\frac{1}{2}\sum_{ij}^{(i\neq j)\,\prime} U_{int}(\underline{r}_i - \underline{r}_j)}_{\text{pair interactions}}$$