Chapter 1

Differential Vector Operators

1.1 Scalar and Vector Fields

A field is a physical quantity having a value at every point within some region (domain) D of space. In this course we will define fields in two and three spatial dimensions, $D \subset \mathbb{R}^2$ or $D \subset \mathbb{R}^3$. If the domain is not specified we assume that it is equal to the entire space. Examples of fields are air pressure, temperature, velocity of water in the ocean, force, electric and magnetic fields. One distinguishes between *scalar fields* and *vector fields*:

- A *scalar field* is a physical quantity having a value but no direction at every point $\mathbf{r} = (x, y, x)$ in the domain. Examples are pressure and temperature.

$$\begin{aligned} \phi : D \subset \mathbb{R}^3 &\longrightarrow \mathbb{R} \\ \mathbf{r} &\longrightarrow \phi(\mathbf{r}). \end{aligned} \tag{1.1}$$

- A *vector field* is a physical quantity having both direction and magnitude at every point $\mathbf{r} = (x, y, x)$ in the domain. Examples include velocity, force, and electric and magnetic fields.

$$\begin{aligned} \mathbf{A} : D \subset \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ \mathbf{r} &\longrightarrow \mathbf{A}(\mathbf{r}) = \begin{pmatrix} A_x(\mathbf{r}) \\ A_y(\mathbf{r}) \\ A_z(\mathbf{r}) \end{pmatrix}. \end{aligned} \tag{1.2}$$

The main focus of Physics is on understanding how fields vary in space and time. This is equally the case for classical mechanics, quantum mechanics, electromagnetism, fluid dynamics, and quantum field theory. Local variations of fields will be described by *differential vector operators* which we will introduce in this chapter.

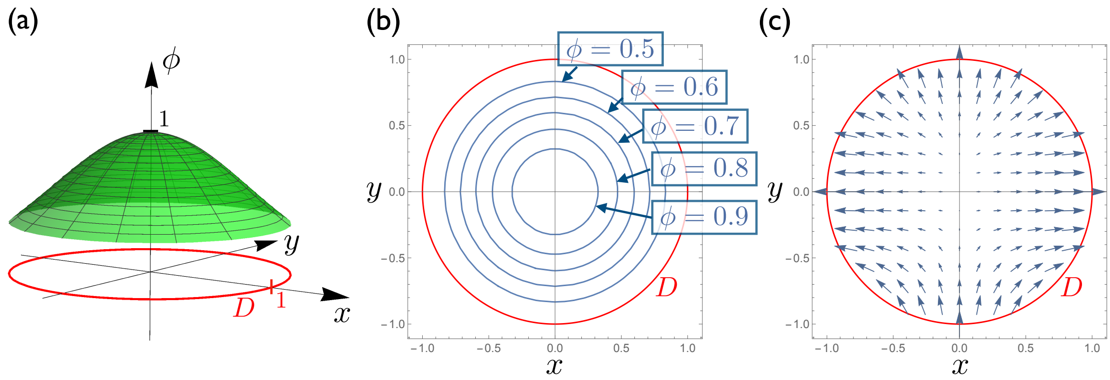

Figure 1.1: (a) An example of a two-dimensional scalar field, $\phi(\mathbf{r}) = \phi(x, y) = e^{-(x^2+y^2)}$, defined in the domain $D : x^2 + y^2 \leq 1$. The value of the field is shown along the z-axis. (b) Contour lines of the field ϕ. These are lines along which the field is constant. (c) The vector field $\mathbf{A}(\mathbf{r}) = \frac{1}{10}(x, y^2)^T$ over the same domain D.

1.2 Partial Derivatives of Fields

We can take the derivative of a scalar field $\phi(\mathbf{r}) = \phi(x, y, z)$ with respect to one of the coordinates, treating the other variables as constants. This operation is called *partial derivative*. E.g. the partial derivative with respect to x is denoted by $\frac{\partial \phi}{\partial x}$ (we say 'del' for ∂) or short $\partial_x \phi$. It measures the rate of change of ϕ as we change x, keeping y and z fixed.

Example: The partial derivatives of the scalar field $\phi(\mathbf{r}) = \phi(x, y) = x^2 + y + xye^{-x}$ are given by

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= 2x + ye^{-x}(1 - x), \\ \frac{\partial \phi}{\partial y} &= 1 + xe^{-x}. \end{aligned}$$

The partial derivative of a vector field is calculated by taking the partial derivative of each component, e.g.

$$\frac{\partial \mathbf{A}}{\partial x} = \partial_x \mathbf{A} = \begin{pmatrix} \partial_x A_x \\ \partial_x A_y \\ \partial_x A_z \end{pmatrix}.$$

Example: The partial derivatives of the vector field $\mathbf{A}(\mathbf{r}) = \begin{pmatrix} 2x - z \\ x^2 y \\ e^{xz^2} \end{pmatrix}$ are given by

$$\partial_x \mathbf{A} = \begin{pmatrix} 2 \\ 2xy \\ z^2 e^{xz^2} \end{pmatrix}, \quad \partial_y \mathbf{A} = \begin{pmatrix} 0 \\ x^2 \\ 0 \end{pmatrix}, \quad \partial_z \mathbf{A} = \begin{pmatrix} -1 \\ 0 \\ 2xze^{xz^2} \end{pmatrix}.$$

1.3 Directional Derivative and Gradient

The partial derivatives measure the change of ϕ along the directions of x, y and z. How to calculate the derivative of ϕ along a general direction, given by a unit vector [1]

$$\hat{\mathbf{u}} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}?$$

Let us define the auxiliary function

$$g(s) = \phi(\mathbf{r} + s\hat{\mathbf{u}}) = \phi(x + su_x, y + su_y, z + su_z)$$

of the single variable s. The derivative of ϕ along the direction $\hat{\mathbf{u}}$ is given by $g'(0) = \frac{dg}{ds}\big|_{s=0}$, which can be calculated using the chain rule,

$$g'(0) = u_x \frac{\partial \phi}{\partial x} + u_y \frac{\partial \phi}{\partial y} + u_z \frac{\partial \phi}{\partial z}.$$

Defining the *gradient operator*

$$\nabla = \hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}, \tag{1.3}$$

where

$$\hat{\mathbf{e}}_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{\mathbf{e}}_y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \hat{\mathbf{e}}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

denote the Cartesian basis vectors, we can write the *directional derivative* of ϕ along $\hat{\mathbf{u}}$ as

$$g'(0) = \hat{\mathbf{u}} \cdot \nabla \phi.$$

The vector field

$$\nabla \phi = \mathrm{grad}\phi = \hat{\mathbf{e}}_x \frac{\partial \phi}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial \phi}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial \phi}{\partial z} = \begin{pmatrix} \partial_x \phi \\ \partial_y \phi \\ \partial_z \phi \end{pmatrix} \tag{1.4}$$

is called the *gradient of* ϕ.[2] We can write the directional derivative of ϕ along $\hat{\mathbf{u}}$ as

$$\hat{\mathbf{u}} \cdot \nabla \phi = |\hat{\mathbf{u}}|\, |\nabla \phi| \cos \alpha = |\nabla \phi| \cos \alpha,$$

where $|\nabla \phi| = \sqrt{(\partial_x \phi)^2 + (\partial_y \phi)^2 + (\partial_z \phi)^2}$ is the length of the vector $\nabla \phi$ and α the angle between $\hat{\mathbf{u}}$ and $\nabla \phi$. Hence, the directional derivative is maximal when $\alpha = 0$ ($\hat{\mathbf{u}}$ parallel to $\nabla \phi$). This implies that

[1] A unit vector has length 1, $|\hat{\mathbf{u}}| = \sqrt{u_x^2 + u_y^2 + u_z^2} = 1$, and the hat symbol is used to indicate that the vector is a unit vector.

[2] In two dimensions the gradient of the field $\phi(\mathbf{r}) = \phi(x, y)$ is defined as $\nabla \phi = \mathrm{grad}\phi = \begin{pmatrix} \partial_x \phi \\ \partial_y \phi \end{pmatrix}$.

The gradient $\nabla\phi$ of a scalar field ϕ always points toward the direction of maximum increase of ϕ.

The directional derivative is zero along directions $\hat{\mathbf{u}}$ *tangential* to the *hypersurfaces* given by $\phi(\mathbf{r}) = C$ ($C \in \mathbb{R}$). In three dimensions the hypersurfaces are two-dimensional surfaces, in two dimensions contour lines. In general, a hypersurface has dimension $d-1$ where d is the dimension of the embedding space. A vanishing directional derivative along tangential directions, $\hat{\mathbf{u}} \cdot \nabla\phi = 0$, implies that:

The gradient $\nabla\phi$ of a scalar field ϕ is perpendicular (normal) to the hypersurfaces defined by $\phi(\mathbf{r}) = C$ (surfaces in $d = 3$, contour lines in $d = 2$).

Example: Consider the two-dimensional scalar field $\phi(\mathbf{r}) = e^{-(x^2+y^2)} = e^{-r^2}$.

$$\nabla\phi = \begin{pmatrix} \partial_x\phi \\ \partial_y\phi \end{pmatrix} = \begin{pmatrix} -2xe^{-(x^2+y^2)} \\ -2ye^{-(x^2+y^2)} \end{pmatrix} = -2e^{-(x^2+y^2)} \begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{-2e^{-r^2}}_{<0} \mathbf{r}.$$

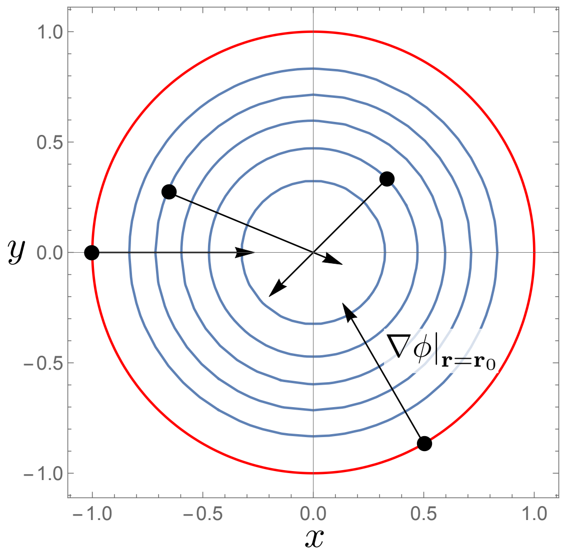

Figure 1.2: Gradients $\nabla\phi$ of the vector field $\phi(\mathbf{r}) = e^{-(x^2+y^2)}$ at a few selected points $\mathbf{r} = \mathbf{r}_0$. The gradients are perpendicular to the contour lines and point toward the direction of the steepest increase of ϕ.

Note that the field $\phi(\mathbf{r}) = e^{-(x^2+y^2)}$ just depends on the distance $r = \sqrt{x^2+y^2}$ of the point $\mathbf{r} = (x, y)$ to the origin, $\phi(\mathbf{r}) = e^{-r^2}$. The contour lines are therefore circles around the origin. Such scalar fields are called *rotationally symmetric*. It is possible to derive a general expression for the gradient of such fields:

The gradient of a *rotationally symmetric* scalar field $\phi(\mathbf{r}) = f(r)$ with $r = |\mathbf{r}|$ and $f(r)$ a differentiable function of r is given by

$$\nabla\phi = f'(r)\,\hat{\mathbf{e}}_r, \tag{1.5}$$

where $\hat{\mathbf{e}}_r := \mathbf{r}/r$ is the unit vector along the direction of $\mathbf{r}$.

Let us prove this equation for a two-dimensional scalar field. The proof in $d = 3$ and in fact general dimension is completely analogous.

$$\begin{aligned}\nabla\phi &= \begin{pmatrix}\partial_x\phi\\ \partial_y\phi\end{pmatrix} = \begin{pmatrix}\partial_x f(\sqrt{x^2+y^2})\\ \partial_y f(\sqrt{x^2+y^2})\end{pmatrix} = \begin{pmatrix}f'(\sqrt{x^2+y^2})\frac{1}{2\sqrt{x^2+y^2}}2x\\ f'(\sqrt{x^2+y^2})\frac{1}{2\sqrt{x^2+y^2}}2y\end{pmatrix}\\ &= f'(r)\frac{1}{r}\begin{pmatrix}x\\ y\end{pmatrix} = f'(r)\hat{\mathbf{e}}_r.\end{aligned}$$

1.4 The Total Differential of Fields

The *total differential* of a scalar field $\phi(\mathbf{r}) = \phi(x, y, z)$ is defined as

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz. \tag{1.6}$$

It measures the infinitesimal change of ϕ as we change x, y and z by infinitesimal amounts dx, dy and dz. Defining the *vectorial line element*

$$d\mathbf{r} = \begin{pmatrix}dx\\ dy\\ dz\end{pmatrix},$$

we can re-write the total differential in the compact form

$$d\phi = \nabla\phi \cdot d\mathbf{r}. \tag{1.7}$$

The *total differential* of a vector field $\mathbf{A}(\mathbf{r})$ is given by

$$d\mathbf{A} = \frac{\partial\mathbf{A}}{\partial x}dx + \frac{\partial\mathbf{A}}{\partial y}dy + \frac{\partial\mathbf{A}}{\partial z}dz. \tag{1.8}$$

Changing x by dx and keeping y and z constant, the vector $\mathbf{A}$ changes by the infinitesimal vector $d\mathbf{A}$ parallel to $\partial_x\mathbf{A} = \frac{\partial\mathbf{A}}{\partial x}$.

1.5 Divergence and Curl of Vector Fields

Consider a three-dimensional vector field

$$\mathbf{A}(\mathbf{r}) = \begin{pmatrix} A_x(\mathbf{r}) \\ A_y(\mathbf{r}) \\ A_z(\mathbf{r}) \end{pmatrix}.$$

The *divergence* of the vector field is defined as

$$\mathrm{div}\mathbf{A} = \nabla \cdot \mathbf{A} = \partial_x A_x + \partial_y A_y + \partial_z A_z. \tag{1.9}$$

In two dimensions, $\mathbf{A}(\mathbf{r}) = \begin{pmatrix} A_x(\mathbf{r}) \\ A_y(\mathbf{r}) \end{pmatrix}$, $\mathrm{div}\mathbf{A} = \nabla \cdot \mathbf{A} = \partial_x A_x + \partial_y A_y$. The divergence of a vector field is a scalar field.

Note that $\nabla \cdot \mathbf{A}$ is a common short-hand notation for the divergence of $\mathbf{A}$. It reflects the close analogy to the 'dot' product (scalar product) of two vectors. But be careful! ∇ is a vector operator that acts on the vector field $\mathbf{A}$ and therefore has to be "multiplied" from the left. While the divergence $\nabla \cdot \mathbf{A}$ is a scalar field,

$$\mathbf{A} \cdot \nabla = A_x \partial_x + A_y \partial_y + A_z \partial_z$$

is a differential operator that gives the directional derivative when acted on a scalar field. Hence $\nabla \cdot \mathbf{A} \neq \mathbf{A} \cdot \nabla$.

The *curl* of a three-dimensional vector field is defined as

$$\mathrm{curl}\mathbf{A} = \nabla \times \mathbf{A} = \begin{pmatrix} \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix}. \tag{1.10}$$

The curl of a vector field is a vector field. The short-hand notation $\nabla \times \mathbf{A}$ reflects the close analogy with the cross product (vector product) between two vectors. However, as in the case of the divergence, ∇ acts on the field and has to be "multiplied" from the left. The anti-commutation rule for the cross product of vectors, $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ does *not* apply in this case.

Example: Calculate the divergence and curl of the vector field

$$\mathbf{A} = \begin{pmatrix} x^2 yz \\ x + z \\ y^2 \end{pmatrix}.$$

$$\begin{aligned} \mathrm{div}\mathbf{A} &= \nabla \cdot \mathbf{A} = \partial_x A_x + \partial_y A_y + \partial_z A_z = 2xyz + 0 + 0 = 2xyz, \\ \mathrm{curl}\mathbf{A} &= \nabla \times \mathbf{A} = \begin{pmatrix} \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix} = \begin{pmatrix} 2y - 1 \\ x^2 y \\ 1 - x^2 z \end{pmatrix}. \end{aligned}$$

What do the divergence and curl tell us about a vector field? To find an answer to this question, let us consider the following very simple vector fields,

$$\mathbf{A}(\mathbf{r}) = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B}(\mathbf{r}) = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}.$$

In both cases the vectors lie in the xy-plane. We obtain

$$\nabla \cdot \mathbf{A} = 2, \;\; \nabla \times \mathbf{A} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; \quad \nabla \cdot \mathbf{B} = 0, \;\; \nabla \times \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.$$

Hence the vector field **A** has a constant, non-zero divergence and zero curl while **A** has a vanishing divergence and a constant curl which points along the z-axis.

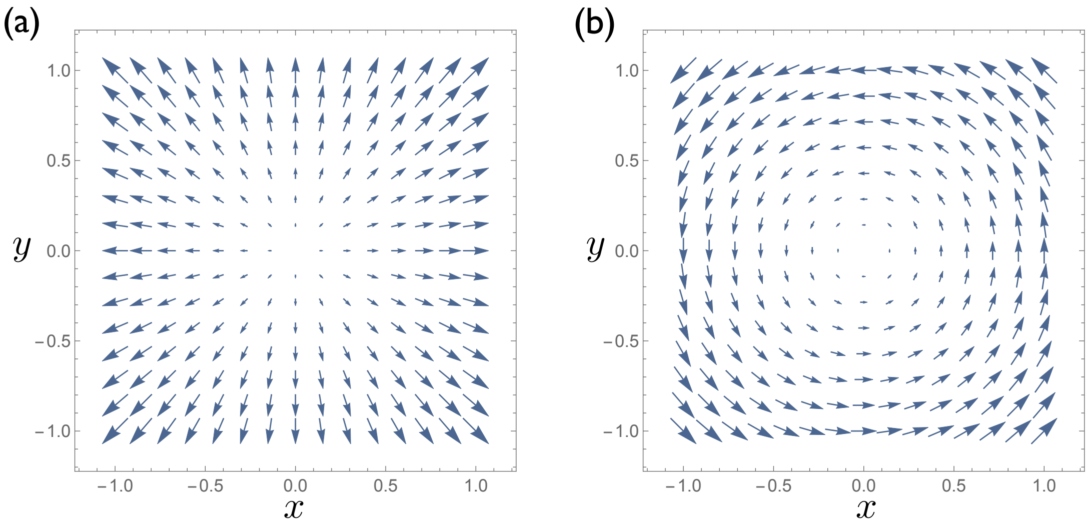

Figure 1.3: Plots of the vector fields $\mathbf{A}(\mathbf{r})$ (a) and $\mathbf{B}(\mathbf{r})$ (b). In both cases the vectors have been rescaled by a constant factor to a smaller length for greater clarity.

Let us assume that the fields describe the flow of the particles or molecules in a liquid. For the field **A** the origin acts like a *source* of particles while for **B** the flow is rotational around the origin, or in the three dimensional space around the z-axis. The curl points along the positive z-axis which, according to the "right-hand rule", corresponds with an anti-clockwise rotation. The divergence of a vector field indicates sinks and sources, the curl rotational flow.[3]

[3]For that reason the curl is sometimes also called rotation, or "rot" in short.

1.6 Product Rules

Let $\phi(\mathbf{r})$, $\rho(\mathbf{r})$ be scalar fields and $\mathbf{A}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ vector fields. Then the following product rules apply:

$$\nabla(\phi\rho) = (\nabla\phi)\,\rho + \phi\,(\nabla\rho) \tag{1.11a}$$
$$\nabla\cdot(\phi\mathbf{A}) = \phi\,(\nabla\cdot\mathbf{A}) + (\nabla\phi)\cdot\mathbf{A} \tag{1.11b}$$
$$\nabla\times(\phi\mathbf{A}) = \phi\,(\nabla\times\mathbf{A}) + (\nabla\phi)\times\mathbf{A} \tag{1.11c}$$
$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = (\nabla\times\mathbf{A})\cdot\mathbf{B} - \mathbf{A}\cdot(\nabla\times\mathbf{B}) \tag{1.11d}$$
$$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\nabla\cdot\mathbf{B} + \mathbf{B}\cdot\nabla)\mathbf{A} - (\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla)\mathbf{B} \tag{1.11e}$$

For the equations that involve curl operators and cross products between vector fields we require that $\mathbf{A}$ and $\mathbf{B}$ are three-dimensional vector fields. Equations (a) and (b) are valid in any dimension.

It is important to check that the equations involving differential vector operators are well defined, e.g. the divergence operator needs to act on a vector field. Let us inspect equation (b), for example. $\phi\mathbf{A}$ is a product between a scalar field and a vector field which results in a vector field. Hence taking the divergence of this field is a well defined operation and produces a scalar field. The terms on the right-hand side therefore need to be scalar fields. The first term is a product between two scalar fields, ϕ and div$\mathbf{A}$, resulting in a scalar field. The second term is a dot-product between two vector fields, $\nabla\phi$ and $\mathbf{A}$, which also results in a scalar field.

Let us prove equations (a), (b), and (e). The proof of the remaining equations (c) and (d) we leave for the problem-solving tutorial.

<u>Proof:</u>

(a) The idea is to use the conventional product rule of differentiation in each component of the gradient,

$$\begin{aligned}\nabla(\phi\rho) &= \begin{pmatrix}\partial_x(\phi\rho)\\ \partial_y(\phi\rho)\\ \partial_z(\phi\rho)\end{pmatrix} = \begin{pmatrix}(\partial_x\phi)\,\rho + \phi\,(\partial_x\rho)\\ (\partial_y\phi)\,\rho + \phi\,(\partial_y\rho)\\ (\partial_z\phi)\,\rho + \phi\,(\partial_z\rho)\end{pmatrix} = \begin{pmatrix}\partial_x\phi\\ \partial_y\phi\\ \partial_z\phi\end{pmatrix}\rho + \phi\begin{pmatrix}\partial_x\rho\\ \partial_y\rho\\ \partial_z\rho\end{pmatrix}\\ &= (\nabla\phi)\rho + \phi(\nabla\rho).\end{aligned}$$

(b)

$$\begin{aligned}\nabla\cdot(\phi\mathbf{A}) &= \nabla\cdot\begin{pmatrix}\phi A_x\\ \phi A_y\\ \phi A_z\end{pmatrix} = \partial_x(\phi A_x) + \partial_y(\phi A_y) + \partial_z(\phi A_z)\\ &= (\partial_x\phi)\,A_x + \phi\,(\partial_x A_x) + (\partial_y\phi)\,A_y + \phi\,(\partial_y A_y) + (\partial_z\phi)\,A_z + \phi\,(\partial_z A_z)\\ &= (\partial_x\phi)\,A_x + (\partial_y\phi)\,A_y + (\partial_z\phi)\,A_z + \phi\,(\partial_x A_x + \partial_y A_y + \partial_z A_z)\\ &= (\nabla\phi)\cdot\mathbf{A} + \phi\,(\nabla\cdot\mathbf{A}).\end{aligned}$$

Proof (continued):

(e) This equation is an identity between vectors. Here we just show that the x-components of the vector fields on both sides are equal. The proof for the other components is completely analogous. Let us first compute the x-component of the vector field on the left-hand side of the equation:

$$
\begin{aligned}
[\nabla \times (\mathbf{A} \times \mathbf{B})]_x &= \left[\nabla \times \begin{pmatrix} A_y B_z - A_z B_y \\ A_z B_x - A_x B_z \\ A_x B_y - A_y B_x \end{pmatrix}\right]_x \\
&= \partial_y (A_x B_y - A_y B_x) - \partial_z (A_z B_x - A_x B_z) \\
&= (\partial_y A_x) B_y + A_x (\partial_y B_y) - (\partial_y A_y) B_x - A_y (\partial_y B_x) \\
&\quad -(\partial_z A_z) B_x - A_z (\partial_z B_x) + (\partial_z A_x) B_z + A_x (\partial_z B_z)
\end{aligned}
$$

We show that we get the same expression by evaluating the x component of the right-hand side of the equation,

$$
\begin{aligned}
&[(\nabla \cdot \mathbf{B} + \mathbf{B} \cdot \nabla)\mathbf{A} - (\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla)\mathbf{B}]_x \\
&= (\partial_x B_x + \partial_y B_y + \partial_z B_z + B_x \partial_x + B_y \partial_y + B_z \partial_z) A_x \\
&\quad -(\partial_x A_x + \partial_y A_y + \partial_z A_z + A_x \partial_x + A_y \partial_y + A_z \partial_z) B_x \\
&= \cancel{(\partial_x B_x) A_x} + (\partial_y B_y) A_x + (\partial_z B_z) A_x + \cancel{B_x (\partial_x A_x)} + B_y (\partial_y A_x) + B_z (\partial_z A_x) \\
&\quad -\cancel{(\partial_x A_x) B_x} - (\partial_y A_y) B_x - (\partial_z A_z) B_x - \cancel{A_x (\partial_x B_x)} - A_y (\partial_y B_x) - A_z (\partial_z B_x) \\
&= A_x (\partial_y B_y) + A_x (\partial_z B_z) - A_y (\partial_y B_x) - A_z (\partial_z B_x) \\
&\quad +(\partial_y A_x) B_y + (\partial_z A_x) B_z - (\partial_y A_y) B_x - (\partial_z A_z) B_x \\
&= [\nabla \times (\mathbf{A} \times \mathbf{B})]_x \,.
\end{aligned}
$$

1.7 2nd Order Variations of Fields, Laplace Operator

Consider a scalar field $\phi(\mathbf{r})$. The gradient $\nabla\phi$ is a vector field. It is possible to compute the divergence of $\nabla\phi$ in both two and three dimensions. In $d = 3$:

$$
\text{div grad}\phi = \nabla \cdot \nabla\phi = \nabla \cdot \begin{pmatrix} \partial_x \phi \\ \partial_y \phi \\ \partial_z \phi \end{pmatrix} = \partial_x^2 \phi + \partial_y^2 \phi + \partial_z^2 \phi,
$$

where ∂_x^2 stands for $\frac{\partial^2}{\partial x^2}$. In $d = 2$ we simply obtain $\nabla \cdot \nabla\phi = \partial_x^2 \phi + \partial_y^2 \phi$.

$$\Delta\phi := \nabla^2\phi = \text{div grad}\phi = \partial_x^2\phi + \partial_y^2\phi + \partial_z^2\phi \tag{1.12}$$

is called the *Laplacian* of ϕ. The second order differential operator

$$\Delta = \nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 \tag{1.13}$$

is called the *Laplace operator*.

The Laplacian $\Delta\phi$ is a scalar field. As an example, let us compute the Laplacian of the field $\phi(\mathbf{r}) = e^{-(x^2+y^2)}$:

$$\begin{aligned}
\partial_x\phi &= -2xe^{-(x^2+y^2)} \\
\partial_x^2\phi &= -2e^{-(x^2+y^2)} + 4x^2e^{-(x^2+y^2)} = (4x^2-2)e^{-(x^2+y^2)} \\
\partial_y^2\phi &= (4y^2-2)e^{-(x^2+y^2)} \\
\Delta\phi &= \nabla^2\phi = \partial_x^2\phi + \partial_y^2\phi = (4x^2+4y^2-4)e^{-(x^2+y^2)} = 4(r^2-1)e^{-r^2}
\end{aligned}$$

We can also calculate the curl of a three-dimensional vector field $\nabla\phi$:

$$\nabla \times \nabla\phi = \nabla \times \begin{pmatrix} \partial_x\phi \\ \partial_y\phi \\ \partial_z\phi \end{pmatrix} = \begin{pmatrix} \partial_y\partial_z\phi - \partial_z\partial_y\phi \\ \partial_z\partial_x\phi - \partial_x\partial_z\phi \\ \partial_x\partial_y\phi - \partial_y\partial_x\phi \end{pmatrix} = \mathbf{0}.$$

In the last step we have used that the order of partial derivatives does not matter (partial derivatives commute). Hence, for any scalar field ϕ in three dimensional space

$$\text{curl grad}\phi = \nabla \times \nabla\phi = \mathbf{0}. \tag{1.14}$$

Another important relation is that the curl of a three-dimensional vector field $\mathbf{A}$ has zero divergence,

$$\text{div curl}\mathbf{A} = \nabla \cdot (\nabla \times \mathbf{A}) = 0. \tag{1.15}$$

You will prove this equation in the problem-solving tutorial.