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Quantile mechanics

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In both modern stochastic analysis and more traditional probability and statistics, one way of characterizing a static or dynamic probability distribution is through its quantile function. This paper is focused on obtaining a direct understanding of this function via the classical approach of establishing and then solving differential equations for the function. We establish ordinary differential equations and power series for the quantile functions of several common distributions. We then develop the partial differential equation for the evolution of the quantile function associated with the solution of a class of stochastic differential equations, by a transformation of the Fokker–Planck equation. We are able to utilize the static formulation to provide elementary time-dependent and equilibrium solutions.

Such a direct understanding is important because quantile functions find important uses in the simulation of physical and financial systems. The simplest way of simulating any non-uniform random variable is by applying its quantile function to uniform deviates. Modern methods of Monte–Carlo simulation, techniques based on low-discrepancy sequences and copula methods all call for the use of quantile functions of marginal distributions. We provide web resources for prototype implementations in computer code. These implementations may variously be used directly in live sampling models or in a high-precision benchmarking mode for developing fast rational approximations also for use in simulation.

1 Introduction

In both stochastic analysis and traditional probability and statistics, the quantile function offers a useful way of characterizing a static or dynamic distribution. An understanding of this function offers real benefits not available directly from the density or distribution function. For example, the simplest way of simulating any non-uniform random variable is by applying its quantile function to uniform deviates. This paper proposes to elevate quantile functions to the same level of management as many of the classical special functions of mathematical physics and applied analysis. That is, we form appropriate ordinary and partial differential equations, and proceed to the development of power series from the appropriate underlying ordinary differential equations. We shall consider a static analysis based on ordinary differential equations (ODEs) and a dynamic analysis based on a corresponding partial differential equation (PDE). In this paper the focus of the dynamic analysis will be a quantilized form of the integrated Fokker–Planck equation arising from a stochastic differential equation (SDE), but other dynamics may be considered in principle. We shall call this the QFPE. The management of the dynamic QFPE will require an understanding of the solutions of the corresponding ODEs, in much the same way as the management of, e.g., the Schröedinger equation requires an understanding of separation of variables and solutions of the corresponding (linear) ODEs. This will necessitate some new technology development for the static analysis. However, the static analysis has intrinsic interest, as well shall now discuss.

1.1 Background to the static analysis

Hitherto quantile functions, in the static context, have largely just been treated as inverses of often complicated distribution functions, and their use has largely been confined to (a) simple exact solutions, e.g., for the exponential distribution, (b) in terms of efficient rational approximations, usually for the normal case, and (c) as a root-finding exercise based on the distribution function. A treatment of some approaches to quantile functions and other sampling methods, such as rejection, is given in the classic text by Devroye [9]. An extensive discussion of the use of quantile functions in mainstream statistics is given in the book by Gilchrist [12], where several other examples of known simple forms for quantile functions are given. Several rational and related approximations have been given for the quantile function for normal case, sometimes called the *probit* function after the work of Bliss [4]. Perhaps the most detailed is 'algorithm AS241', developed by Wichura in 1988 [28] and is capable of machine precision in a double precision computing environment. Quantile functions are of course in widespread use in general statistics and often find representations in terms of lookup tables for key percentiles.

A detailed investigation of the quantile function for the Student case has been given by Shaw [22]. However, while power and tail series were developed there for the general case, the method relied on the inversion of series for the forward cumulative distribution function (CDF) using computer algebra and special computational environments. A prototype for many quantiles was given by the series solution of the non-linear ODE for the inverse error function given by Steinbrecher [24]. Here we extend this work to the normal distribution itself, the Student distribution, and gamma and beta distributions. In particular we are able to facilitate the implementation of the methods of [22] for the Student case in any computing environment to much higher accuracy. We shall argue that this *analytic* approach provides useful and practical techniques for the evaluation of such functions.

The value of quantile functions is increasingly appreciated in Monte–Carlo simulation. Quantile functions work very well with both copula methods and low-discrepancy sequences, in contrast to rejection methods based on sampling from a circle inside a square, e.g., for sampling of the Normal. These issues are discussed by Jäckel [17]. The use of quantile functions has however not taken root as much as it might due to the relative intractability of the functions, which this far have been considered as inverses of the often awkward CDFs of density functions, which are 'special functions' of some computational complexity. While such CDFs and their inverses are sometimes available in high-level mathematical computation languages such as *Mathematica*, these representations (e.g., using an inverse incomplete beta function in the case of the beta and

Student distributions) are less helpful in other languages. In the case of the normal distribution the quantile function has been managed by the use of composite rational and polynomial approximations based on decomposing the half-unit interval into two or more regions. The three-region approximation developed by Wichura in 1988 [28] and commonly known as 'AS241' is well-suited to machine-precision computations in a double-precision environment. Acklam's method [2] is a very useful method based on two levels of approximation. A simple two-region approximation provides relative accuracy of order 10^{-9} for the quantile, and a single refinement based on Newton–Raphson–Halley iteration produces machine-precision. An assessment of the merits of these schemes and others in common use has been given by one of us [23].

Such rational and related approximations provide useful and fast implementations in the case of the normal distribution, where much effort has been devoted to analyzing this single universal distribution with no embedded parameters, and a one-off composite rational approximation can be developed in detail. However, other cases of interest do involve other parameters, e.g., the degrees of freedom in the case of the Student, and two parameters in the case of the beta distribution, and so far as we are aware there is little known about rational or other approximations that have error properties that are managed uniformly in the distributional parameters. One outcome of this note is the capability to manage distributional parameters for several key distributions.

Furthermore, the emergence of 64-bit computers and the associated 96- and 128-bit standards of arithmetic also requires us to reconsider the use of standard existing approximations for quantiles, often based on rational approximations, as providing appropriate representations for these functions in high-precision environments. The methods we shall present here are capable of providing both arbitrary-precision benchmarks for new high-precision approximations, as well as components of real-time simulations.

We regard it as 'mathematically pleasing' because this aspect of probability theory can also be discussed in the same setting as the classical theory of special functions. Power series methods are at the heart of many aspects of applied mathematics. For example, the simple solutions of the Schröedinger equation rely for their interpretation on a discussion of their power series, and their termination to a polynomial or a certain type of convergence behaviour leads to an understanding of energy levels. Traditionally the interest in a power series approach to special functions has been focused on *linear* systems, and where one can derive explicit formulae for coefficients in terms of some combination of factorials. At this stage in our analysis our characterization of the the power series coefficients will be through a non-linear recursion associated with a non-linear ordinary differential equation. But this will allow numerical or symbolic computation to proceed.

The power series method is of course local in character. However, we shall show that for the cases of the Student and beta distributions, just two series will suffice for *global* applications. This work has the novel feature that we can treat as a non-linear ODE through almost entirely analytical methods, leaving a computer to solve the iteration as far as is needed and add up the series.

The use of differential equations and series methods for the analysis of quantile functions has its origins in the earlier work of Hill and Davis [13] and Abernathy and Smith [1]. This earlier work developed differential recursions with the emphasis on Cornish–Fisher expansions. In contrast, our own approach will obtain direct algebraic recursions for the quantile functions themselves, which are readily adapted for computation. Applications to Cornish–Fisher expansions do exist and will be given elsewhere.

1.2 Background to the dynamic analysis

There are many situations in modern applied mathematics where one is interested in a time-dependent probability distribution and many types of dynamics that might be postulated. This paper will give just one example based on a one-dimensional SDE. We shall derive the 'quantilized Fokker–Planck equation', or QPFE, by a transformation of the ordinary Fokker–Planck equation for a one-dimensional stochastic process. Then solutions will be developed.

The development of direct evolution equations for quantile functions seems to have a somewhat sparse presence in the literature. The authors are aware of work by Toscani and co-workers, and in particular a paper by Carrillo and Toscani [5]. Other discussions have been given by Toscani and co-workers in [27], and most recently, [3]. The motivation in that work is a one-dimensional approach to mass transportation problems. Dassios has studied temporal quantiles – see [8] and the references therein, following on from earlier work by Miura [19]. There is also the distinct concept of a *quantile process*, as discussed in [7]. The focus of our paper is, rather, to consider the *time evolution* of the *spatial quantile function*. The direct calculation of such spatial quantile functions would afford a new route to Monte–Carlo simulation, and given that quantiles may be robustly estimated from data, there is also the potential for estimating SDE structures from data, via the use of a QFPE.

1.3 Outline of the paper

The plan of this work is as follows. In section 2 we derive non-linear ordinary differential equations for some key distributions. In Section 3 we derive power series solutions and give a global solution for some distributions. Section 4 gives a brief discussion on classification aspects and the bivariate form. Section 5 gives an introduction to the dynamic analysis. Section 6 presents conclusions and further directions, and links to web resources. A sample detailed derivation of the non-linear recurrence for the power series coefficients is given in Appendix A. Matters to do with some practical implementations are deferred to in Appendix B and technical details on tails are given in Appendix C.

2 Quantile statics: ODEs for quantile functions

We will treat this very simply by working through different choices of density function. In each case all we do is first express the derivative of the quantile function w as a the reciprocal of the usual density function expressed in terms of w, then keep differentiating until we obtain a closed differential relation, which is generically non-linear. In each case, if $f(x)$ is the probability density function, the first order quantile ODE is just

$$\frac{dw}{du} = \frac{1}{f(w)}, \tag{1}$$

where $w(u)$ is the quantile function considered as a function of u, with $0 \leqslant u \leqslant 1$. We then differentiate again to find, at least for some key common cases, a simple-second order non-linear ODE.

2.1 The Normal distribution

In this case if $w(u)$ denotes the quantile function as a function of u, then

$$\frac{dw}{du} = \sqrt{2\pi}\exp\left(\frac{w^2}{2}\right). \tag{2}$$

Now we differentiate again, to obtain

$$\frac{d^2w}{du^2} = \sqrt{2\pi}\exp\left(\frac{w^2}{2}\right) w\frac{dw}{du} = w\left(\frac{dw}{du}\right)^2, \tag{3}$$

so we end up with an ODE and 'centre' conditions,

$$\begin{gathered}\frac{d^2w}{du^2} = w\left(\frac{dw}{du}\right)^2, \\ w(1/2) = 0, \quad w'(1/2) = \sqrt{2\pi}.\end{gathered} \tag{4}$$

This ODE is equivalent to that posted for inverse erf on the web [14].

2.2 The Student distribution—direct approach

In this case if $w(u)$ denotes the quantile function as a function of u, then

$$\frac{dw}{du} = \sqrt{n\pi}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}(1+w^2/n)^{\frac{n+1}{2}}. \tag{5}$$

Now we differentiate again, to obtain

$$\frac{d^2w}{du^2} = \sqrt{n\pi}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\frac{n+1}{n}(1+w^2/n)^{\frac{n-1}{2}} w\frac{dw}{du} = \frac{n+1}{n}\frac{1}{(1+w^2/n)} w\left(\frac{dw}{du}\right)^2, \tag{6}$$

so we end up with an ODE and 'centre' conditions,

$$\begin{gathered}\left(1+\frac{w^2}{n}\right)\frac{d^2w}{du^2} = \left(1+\frac{1}{n}\right) w\left(\frac{dw}{du}\right)^2, \\ w(1/2) = 0, \quad w'(1/2) = \sqrt{n\pi}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}.\end{gathered} \tag{7}$$

This ODE and the conditions reduce to those for the normal case as $n \to \infty$. Throughout this paper the degrees of freedom n is a real positive parameter and is not necessarily an integer.

2.3 The beta distribution and indirect Student

In the case of beta distribution with parameters a, b, if $w(u)$ denotes the quantile function as a function of u, then

$$\frac{dw}{du} = B(a,b)w^{1-a}(1-w)^{1-b}, \tag{8}$$

where $B(a, b)$ is the (complete) beta function. A further differentiation gives us

$$\frac{d^2w}{du^2} = \left[\left(\frac{1-a}{w}\right) - \left(\frac{1-b}{1-w}\right)\right]\left(\frac{dw}{du}\right)^2. \tag{9}$$

We differentiate again, reorganize and obtain the ODE

$$w(1-w)\frac{d^2w}{du^2} - (1-a+w(a+b-2))\left(\frac{dw}{du}\right)^2 = 0. \tag{10}$$

This ODE is the same as that posted for the inverse beta function on the web [15].

The boundary conditions may be stated as a combination of various conditions:

$$\begin{aligned} w(0) &= 0, \quad w(u) \sim [auB(a,b)]^{1/a} \quad \text{as } u \to 0, \\ w(1) &= 1, \quad w(u) \sim 1 - [b(1-u)B(a,b)]^{1/b} \quad \text{as } u \to 1. \end{aligned} \tag{11}$$

The choice $a = n/2, b = 1/2$ is special in that by a change of variables we may recover the Student distribution. The details of this will be given in the next section.

2.4 The Gamma distribution

In this case if $w(u)$ denotes the quantile function as a function of u, then, for $p > 0$,

$$\frac{dw}{du} = \Gamma(p)w^{1-p}\exp(w). \tag{12}$$

Now we differentiate again, reorganize as before, and obtain the ODE

$$w\frac{d^2w}{du^2} - (w+1-p)\left(\frac{dw}{du}\right)^2 = 0. \tag{13}$$

The boundary conditions at the origin take the form

$$w(0) = 0, \quad w(u) \sim [u\Gamma(p+1)]^{1/p} \text{ as } u \to 0. \tag{14}$$

3 Power Series Solutions of the ODEs

In this section we explore the solution of the non-linear ODEs by the standard method of power series. First we review the argument for the normal case, which appears to be special in that the recursion is reducible to a quadratic relation.

3.1 The power series for the normal quantile

There are two ways of approaching this. First we give the cubic recursion algorithm. Then we give the simplified quadratic method of [24]. First we set

$$v = \sqrt{2\pi}(u - 1/2), \tag{15}$$

and note the scale invariance of the independent variable in the ODE allowing us to write the problem as

$$\begin{gathered} \frac{d^2w}{dv^2} = w\left(\frac{dw}{dv}\right)^2, \\ w(0) = 0, \quad w'(0) = 1. \end{gathered} \tag{16}$$

We now assume that

$$w = \sum_{p=0}^{\infty} c_p v^{2p+1}, \tag{17}$$

and substitute into the differential equation. After some manipulations, summarized in Appendix A, we obtain an explicit forward cubic recurrence

$$c_p = \frac{1}{(2p)(2p+1)} \sum_{k=0}^{(p-1)} \sum_{l=0}^{(p-k-1)} (2l+1)(2p-2k-2l-1)c_k c_l c_{p-k-l-1}. \tag{18}$$

Next we review the optimized approach due to one of the [GS] noted in [24] for the error function. Here we give the argument normalized for the quantile function. The ODE may be recast as

$$\frac{d}{dv}\left(\frac{1}{w'(v)}\right) + w(v) = 0. \tag{19}$$

So now let

$$W(v) = \int_0^v w(t)\, dt. \tag{20}$$

Integrating and using the boundary conditions gives

$$\frac{1}{w'(v)} + W(v) = 1, \tag{21}$$

and so

$$w'(v) - w'(v)W(v) = 1. \tag{22}$$

Hence we now have a quadratic identity. We now assume that

$$w = \sum_{p=0}^{\infty} \frac{w_p}{(2p+1)} v^{2p+1}. \tag{23}$$

Simplification of these last two relations gives the quadratic recurrence

$$w_{p+1} = \frac{1}{2} \sum_{j=0}^{p} \frac{w_j w_{p-j}}{(j+1)(2j+1)}. \tag{24}$$

This is analogous to the original expression in [24], where some trivial relative factors of 2 arise from the consideration here of the normal quantile, whereas in [24] the inverse error function was considered.

3.2 The power series for the Student quantile

First of all we set

$$v = \sqrt{n\pi} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} (u - 1/2), \tag{25}$$

and exploit the scaling properties of our ODEs to write the problem as:

$$\begin{aligned} \left(1 + \frac{w^2}{n}\right) \frac{d^2 w}{dv^2} &= \left(1 + \frac{1}{n}\right) w \left(\frac{dw}{dv}\right)^2, \\ w(0) = 0, &\quad w'(0) = 1, \\ \frac{dw}{dv} &= (1 + w^2/n)^{\frac{n+1}{2}}. \end{aligned} \tag{26}$$

We now assume

$$w = \sum_{p=0}^{\infty} c_p v^{2p+1}, \tag{27}$$

and substitute into the differential equation. After some algebraic manipulations we obtain an explicit forward cubic recurrence:

$$\begin{aligned} &(2p)(2p+1)c_p = \\ &\sum_{k=0}^{(p-1)} \sum_{l=0}^{(p-k-1)} c_k c_l c_{p-k-l-1} \left(\left(1 + \frac{1}{n}\right) ((2l+1)(2p-2k-2l-1)) - \frac{2k(2k+1)}{n} \right). \end{aligned} \tag{28}$$

It is readily verified that the first few terms of the series obtained by this recursion correspond to those obtained by direct inversion of the CDF power series (see e.g., [22]). Now it is possible to generate many more terms with ease. Similarly, cases with closed-form solutions, $n = 1, 2, 4$ also confirm the recursion.

As with the normal case, the question arises as to whether there is a simpler, perhaps quadratic, representation. If we follow the route for the Gaussian case, some manipulations with equation (27) show that we can write[1] the system as

$$\frac{d}{dv}\left[\frac{1}{(w'(v))^{\alpha}}\right] + \frac{(n-1)}{n}w = 0, \tag{29}$$

where

$$\alpha = \frac{(n-1)}{(n+1)} < 1. \tag{30}$$

Introducing the integral W of w as before leads to

$$(w'(v))^{\alpha}\left[1 - \frac{(n-1)}{n}W(v)\right] = 1. \tag{31}$$

Unfortunately this equation now requires the manipulation of fractional powers of the power series. While this can theoretically be pursued, the cubic route is much more tractable, so we shall confine our analysis to that route. Another minor issue is that this representation becomes singular when $n = 1$, although this case of the Cauchy distribution is already well understood. We can also rewrite this representation in other ways, such as

$$W''(v) = \left(1 - \frac{(n-1)}{n}W(v)\right)^{-(n+1)/(n-1)}, \tag{32}$$

but this also makes power series more complicated than those arising from the cubic ODE.

3.3 The power series for the beta quantile

One can consider working from either boundary, $u = 0$ or $u = 1$. But we shall work from $u = 0$ and then use symmetry to infer the $u = 1$ case.

$$\begin{gathered} w(1-w)\frac{d^2w}{du^2} = (1 - a + w(a+b-2))\left(\frac{dw}{du}\right)^2, \\ w(0) = 0, \quad w(u) \sim [auB(a,b)]^{1/a} \quad \text{as } u \to 0. \end{gathered} \tag{33}$$

As before, rescaling invariance of the ODE allows us to set

$$v = auB(a,b), \tag{34}$$

so that the system is now

$$\begin{gathered} w(1-w)\frac{d^2w}{dv^2} = (1 - a + w(a+b-2))\left(\frac{dw}{dv}\right)^2, \\ w(0) = 0, \quad w(u) \sim v^{1/a} \quad \text{as } v \to 0. \end{gathered} \tag{35}$$

[1]We are grateful to an anonymous referee for pointing this out together with its motivation by considering the Student distribution as the equilibrium of a fast diffusion problem.

Some experimentation suggests that a power series in $x = v^{1/a}$ is useful, so we change independent variable to x. We then find that, with now $w = w(x)$,

$$w(1-w)\left(\frac{d^2w}{dx^2} + \frac{(1-a)}{x}\frac{dw}{dx}\right) = (1 - a + w(a+b-2))\left(\frac{dw}{dx}\right)^2, \qquad w(0) = 0, \quad w'(0) = 1. \tag{36}$$

Our experimentation suggests that we may now substitute

$$w(x) = \sum_{p=1}^{\infty} c_p x^p, \tag{37}$$

with $c_1 = 1$, and after some manipulations we find that there is a cubic recursion of the form, for $p \geqslant 2$:

$$\begin{aligned} c_p[p^2 + (a-2)p + (1-a)] = \\ (1-\delta_{p,2})\sum_{k=2}^{p-1} c_k c_{p+1-k}\,[(1-a)k(p-k) - k(k-1)] \\ + \sum_{k=1}^{p-1}\sum_{n=1}^{p-k} c_k c_n c_{p+1-k-n}\,[k(k-a) + (a+b-2)n(p+1-k-n)]\,. \end{aligned} \tag{38}$$

The quadratic term only contributes for $p \geqslant 3$. If we work out the first few we find that

$$\begin{aligned} c_1 = 1, \quad c_2 = \frac{b-1}{a+1}, \\ c_3 = \frac{(b-1)\left(a^2 + (3b-1)a + 5b - 4\right)}{2(a+1)^2(a+2)}, \end{aligned} \tag{39}$$

which is in accordance with known values [16].

3.4 The beta quantile and the Student distribution

Having established a power series for the Student quantile about $u = 1/2$, we consider how to establish a series about $u = 1$. This will manage sampling from the positive tail of the distribution, and by the use of symmetry, the negative tail. We achieve this by considering the beta distribution with $a = n/2$ and $b = 1/2$. This is an important case as it will allow us to generate the Student distribution with n degrees of freedom by another route, complete with suitable series for optimal use in the tails. We now explain how this works.

In the previous subsection we established the power series for the beta quantile function $w(u)$ with general parameters a, b. This, by construction, is a representation of the inverse beta function,

$$w(u) = \mathscr{I}_u^{-1}(a, b). \tag{40}$$

The quantile function for the Student distribution may be written, piecewise, in terms of this inverse beta function, as discussed in [22], by the use of the following formula[2]:

$$w_{T_n}(u) = \mathrm{sgn}\left(u - \frac{1}{2}\right)\sqrt{n\left(\frac{1}{\mathscr{I}^{-1}_{\mathrm{If}\left[u<\frac{1}{2},2u,2(1-u)\right]}\left(\frac{n}{2},\frac{1}{2}\right)} - 1\right)}. \tag{41}$$

So we see from this formula that the beta distribution covers the Student T in *two pieces*, with $a = n/2, b = 1/2$. If we focus on the region $u > 1/2$, then we have

$$w_{T_n}(u) = \sqrt{n\left(\frac{1}{\mathscr{I}^{-1}_{2(1-u)}\left(\frac{n}{2},\frac{1}{2}\right)} - 1\right)}, \tag{42}$$

so we see that the series for the Student around $u = 1$ is given by the expansion of the beta around $u = 0$, but with the substitution $u \to 2(1 - u)$. So, using the same scaling of the independent variables as in Section (3.3) we set

$$x = \left[n(1-u)B\left(\frac{n}{2},\frac{1}{2}\right)\right]^{2/n}, \tag{43}$$

and we form the series

$$w_\beta = \sum_{p=1}^{\infty} d_p x^p, \tag{44}$$

where $d_1 = 1$, and for $p \geqslant 2$:

$$\begin{aligned} d_p\left[p^2 + \left(\frac{n}{2} - 2\right)p + \left(1 - \frac{n}{2}\right)\right] = \\ (1-\delta_{p,2})\sum_{k=2}^{p-1} d_k d_{p+1-k}\left[\left(1-\frac{n}{2}\right)k(p-k) - k(k-1)\right] \\ + \sum_{k=1}^{p-1}\sum_{m=1}^{p-k} d_k d_m d_{p+1-k-m}\left[k\left(k-\frac{n}{2}\right) + \left(\frac{n}{2}-\frac{3}{2}\right)m(p+1-k-m)\right]. \end{aligned} \tag{45}$$

The quadratic term only contributes for $p \geqslant 3$. Then finally we calculate

$$w_T(u) = \sqrt{n(1/w_\beta - 1)}. \tag{46}$$

3.5 The power series for the gamma quantile

In this case the boundary conditions at the origin indicate the change of variable

$$v = (u\Gamma(p+1))^{1/p}, \tag{47}$$

[2]It was also noted in [22] that this representation is too slow for live Monte–Carlo use, but very useful for checking other representations.

and this maps the ODE to

$$w\left(\frac{d^2w}{dv^2}+\frac{(1-p)}{v}\frac{dw}{dv}\right)-(w+1-p)\left(\frac{dw}{dv}\right)^2=0. \tag{48}$$

The boundary conditions at the origin take the form

$$w(0)=0, \quad w'(0)=1. \tag{49}$$

We now try a series of the form

$$w(v)=\sum_{n=1}^{\infty} g_n v^n, \tag{50}$$

with $g_1 = 1$. The non-linear recurrence is found to be:

$$\begin{aligned} n(n+p)g_{n+1} &= \sum_{k=1}^{n}\sum_{l=1}^{n-k+1} g_k g_l g_{n-k-l+2}\, l(n-k-l+2) \\ &\quad - \Delta(n)\sum_{k=2}^{n} g_k g_{n-k+2}\, k[k-p-(1-p)(n+2-k)], \end{aligned} \tag{51}$$

where $\Delta(n)=0$ if $n<2$ and $\Delta(n)=1$, if $n \geqslant 2$. . The first few terms are

$$g_1=1, \quad g_2=\frac{1}{p+1}, \quad g_3=\frac{3p+5}{2(p+1)^2(p+2)}, \tag{52}$$

and some experimentation with many terms confirms that if $p = 1$ then $g_n = \frac{1}{n}$ in accordance with the exact solution, $w(u) = -\log(1-u)$, for this case. Another important case is that of $p = 1/2$, when we have the χ^2_1 distribution, with $v = \frac{\pi}{4}u^2$, and the coefficient sequence

$$g_n=\left\{1,\frac{2}{3},\frac{26}{45},\frac{176}{315},\frac{8138}{14175},\frac{286816}{467775},\frac{28631908}{42567525},\frac{480498944}{638512875},\ldots\right\}. \tag{53}$$

4 Classification and generalization

All of the cases considered here arise from ODEs of the form

$$\frac{d^2w}{du^2}=R(w)\left(\frac{dw}{du}\right)^2, \tag{54}$$

where $R(w)$ is a simple rational function. We shall say that the quantile is *rational* and of type (p,q) if $R = P/Q$ where P, Q are polynomials with no common factors of degree p, q. We can classify as follows:

- $p=q=0$: exponential distribution;
- $p=1, q=0$: normal distribution;
- $p=1, q=1$: gamma distribution;

- $p = 1, q = 2$: beta and Student distribution.

More generally, given a density $f(w)$, the associated quantile function satisfies a pair of first and second order differential equations

$$\frac{dw}{du} = \frac{1}{f(w)}, \quad \frac{d^2w}{du^2} = H(w)\left(\frac{dw}{du}\right)^2, \tag{55}$$

where

$$H(w) = -\frac{d}{dw}\log\{f(w)\}. \tag{56}$$

Notably[3], the cases where H is a rational function of type up to $(1, 2)$ are closely related to Pearson's system of distributions [20, 29]. Pearson's system, in our notation, is indeed characterized by the choices of a, b, c, m in the function

$$H(w) = \frac{w - m}{a + bw + cw^2}. \tag{57}$$

The distributions arising from this, Pearson's 'Types', are summarized on pp. 480–481 of [9] and discussed in detail by Johnson and Kotz [18]. The function H is not rational in general. An *algebraic* case is easily realized as Bagnold's hyperbolic distribution, and more general functions appear if we consider, for example, Barndorff–Nielsen's generalized hyperbolic distribution and special cases such as the variance gamma. Further work is needed to explore the optimal routes for solving such cases, and to establish when a power series method remains viable. These representations also open up avenues based on numerical differential equation solution techniques that are complementary to numerical root-finding methods based on the CDF.

4.1 Multivariate quantiles

We shall only make brief comment on the multivariate case. Chapter 13 of the book by Gilchrist contains an excellent summary of bivariate quantile distributions. Some clues as to how to proceed are also given in texts on copula theory in the context of copula simulation, e.g., Chapter 6 of the book by Cherubini *et al.* [6]. Working in two dimensions, the most natural approach would appear to be to consider the bivariate extension of equation (1), in the form suggested by Gilchrist. In our notation we proceed as follows. We form a vector quantile $(w_1(u_1, u_2), w_2(u_1, u_2))$. Then given a probability density function $f(w_1, w_2)$, the basic *first order* quantile ODE is then given by

$$\frac{\partial(w_1, w_2)}{\partial(u_1, u_2)} = \frac{1}{f(w_1, w_2)}, \tag{58}$$

subject to suitable boundary conditions. This of course is just the standard Jacobian transformation for a change of variables. It is an interesting question to consider the usefulness of taking further derivatives of this relationship. There are already two solutions

[3] We are grateful to Prof. Luc. Devroye for this observation.

to equation (58) in the case of the normal distribution that are implicit in well-known bivariate simulation methods. For example, setting

$$w_1(u_1, u_2) = w_0(u_1), \quad w_2(u_1, u_2) = w_0(u_2)\sqrt{1-\rho^2} + \rho w_0(u_1), \tag{59}$$

gives us a solution of equation (58), where w_0 is the quantile of the normal distribution, and f is given by

$$f(w_1, w_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(w_1^2 + w_2^2 - 2\rho w_1 w_2\right)\right\}. \tag{60}$$

The construction of the bivariate normal based on a pair of univariate quantiles with the Box–Muller simulation method, as discussed in Section 7.3.4 of [21], also gives a solution of equation (58) in the independent case. In our notation the Box–Muller formula can be considered as:

$$w_1(u_1, u_2) = \sqrt{-2\log(u_1)}\cos(2\pi u_2), \quad w_2(u_1, u_2) = \sqrt{-2\log(u_1)}\sin(2\pi u_2), \tag{61}$$

and correlated variables may be generated by superposition. So we can build two quite different mappings from the unit square to the bivariate normal. In general, multivariate and non-normal case, quantiles cannot of course be built up by such elementary superposition. We also refer the reader to [21] for an introduction to modern methods not employing quantiles, notably the 'ratio of uniforms' approach.

5 Quantile dynamics

There are many different ways in which the evolution of a time-dependent quantile function might be controlled. Given the importance of stochastic analysis in modern applied mathematics, we shall consider the case where the governing dynamics is defined by an elementary stochastic differential equation, or SDE. Let x_t be a stochastic process governed by the SDE

$$dx_t = \mu(x_t, t)dt + \Sigma(x_t, t)dW_t, \tag{62}$$

where μ, Σ are deterministic functions of x and t, and W_t is a standard Brownian motion. Let $f(x, t)$ denote the probability density function consistent with this SDE. The quantile function associated with this system is $Q(u, t)$, where $0 \leqslant u \leqslant 1$. It is defined by the integral condition that for each value of t,

$$\int_{-\infty}^{Q(u,t)} f(x, t)dx = u. \tag{63}$$

First we differentiate this relation with respect to u, and obtain the partial differential constraint

$$f(Q(u, t), t)\frac{\partial Q(u, t)}{\partial u} = 1\ . \tag{64}$$

We differentiate this relation again, and obtain the following second order PDE,

$$\frac{\partial f(Q(u,t),t)}{\partial Q}\left(\frac{\partial Q(u,t)}{\partial u}\right)^2 + f(Q(u,t),t)\frac{\partial^2 Q(u,t)}{\partial u^2} = 0. \tag{65}$$

This may be reorganized into the equation

$$\frac{\partial^2 Q(u,t)}{\partial u^2} = -\frac{\partial \log(f(Q(u,t),t))}{\partial Q}\left(\frac{\partial Q(u,t)}{\partial u}\right)^2. \tag{66}$$

In the case of no time dependence, this is readily recognized as the second order non-linear ODE derived and solved previously. In the time-dependent case we go back to the condition of equation (63) and now differentiate with respect to time. The fundamental theorem of calculus gives, first,

$$f(Q(u,t),t)\frac{\partial Q}{\partial t} + \int_{-\infty}^{Q(u,t)} \frac{\partial f(x,t)}{\partial t} dx = 0. \tag{67}$$

We use equation (64) to write this as:

$$\frac{\partial Q}{\partial t} = -\frac{\partial Q}{\partial u}\int_{-\infty}^{Q(u,t)} \frac{\partial f(x,t)}{\partial t} dx. \tag{68}$$

Next we use the Fokker–Planck (forward Kolmogorov) equation in the form [10]

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[-\mu f + \frac{1}{2}\frac{\partial}{\partial x}(\Sigma^2 f)\right], \tag{69}$$

where Σ is the volatility defined in the SDE of equation (62), and observe that we can do the integration directly, and obtain the following:

$$\frac{\partial Q}{\partial t} = -\frac{\partial Q}{\partial u}\left[-\mu(Q,t)f(Q,t) + \frac{1}{2}\left(\Sigma^2(Q,t)\frac{\partial f}{\partial Q} + f\frac{\partial \Sigma^2(Q,t)}{\partial Q}\right)\right]. \tag{70}$$

Finally we use our previous equations (64)–(65) to eliminate f and its derivative completely, leaving us with

$$\frac{\partial Q}{\partial t} = \mu(Q,t) - \frac{1}{2}\frac{\partial \Sigma^2}{\partial Q} + \frac{\Sigma^2(Q,t)}{2}\left(\frac{\partial Q}{\partial u}\right)^{-2}\frac{\partial^2 Q}{\partial u^2}. \tag{71}$$

This is the quantilized Fokker–Planck equation, or QFPE for short, associated with the SDE of equation (62). It is a non-linear PDE, and is the form of the Carrillo-Toscani equation [5] appropriate to solutions of an SDE.

5.1 Elementary solutions of the QFPE

Despite its non-linearities, we known that the QFPE of equation (71) must have some elementary Gaussian solutions that will now be verified. First we explore the QFPE for arithmetical Brownian motion. In this case we have μ and Σ constant and it is easy to check that

$$\frac{\partial Q}{\partial t} = \mu + \frac{\Sigma^2}{2}\left(\frac{\partial Q}{\partial u}\right)^{-2}\frac{\partial^2 Q}{\partial u^2} \tag{72}$$

has an obvious solution

$$Q(u,t) = c + \mu t + \Sigma\sqrt{t}w(u), \tag{73}$$

where $w(u)$ is the static normal quantile function we constructed in Section 3.1. Now consider $\Sigma(t)$ as independent of Q but time-dependent, and the class of quantiles of the form

$$Q(u,t) = a(t) + b(t)w(u). \tag{74}$$

Substitution into the QFPE and using equation (4) gives

$$\frac{da}{dt} + \frac{db}{dt}w(u) = \mu(a(t) + b(t)w(u), t) + \frac{\Sigma^2(t)}{2b(t)}w(u). \tag{75}$$

It is clear that this equation can be separated into two ODEs if μ is at most linear in Q. So we assume

$$\mu = m_1(t) - m_2(t)Q, \tag{76}$$

and have the two ordinary differential equations

$$\frac{da}{dt} + m_2(t)a = m_1(t), \quad \frac{db^2}{dt} + 2m_2(t)b^2 = \Sigma^2(t), \tag{77}$$

that are trivially solved by the use of integrating factors. If for example m_1 and m_2 are constant, we obtain the anticipated quantile solution for the OU process for time-dependent volatility in the form

$$Q = Q_0 e^{-m_2 t} + \frac{m_1}{m_2}(1 - e^{-m_2 t}) + w(u)e^{-m_2 t}\sqrt{\int_0^t ds\Sigma^2(s)e^{2m_2 s}}. \tag{78}$$

5.2 Geometric Brownian motion

This follows from the analysis of arithmetical Brownian motion by taking logarithms. If we consider a typical financial representation, and allow for deterministic time dependence, we have

$$\mu = \theta(t)Q, \quad \Sigma = \sigma(t)Q, \tag{79}$$

and so the QFPE equation (71) becomes

$$\frac{\partial Q}{\partial t} = (\theta(t) - \sigma^2(t))Q + \frac{\sigma^2(t)Q^2}{2}\left(\frac{\partial Q}{\partial u}\right)^{-2}\frac{\partial^2 Q}{\partial u^2}. \tag{80}$$

We make the assumption

$$Q(t,u) = Q_0 \exp\{a(t) + b(t)w(u)\}, \tag{81}$$

and are lead quickly to the condition

$$\frac{da}{dt} + \frac{db}{dt}w(u) = \theta(t) - \sigma^2(t) + \frac{\sigma^2(t)}{2}\left(\frac{w(u)}{b} + 1\right) = \theta(t) - \frac{\sigma^2(t)}{2} + \frac{\sigma^2(t)}{2b}w(u), \tag{82}$$

with solution

$$a(t) = \int_0^t \left(\theta(s) - \frac{\sigma^2(s)}{2}\right)ds, \quad b(t) = \sqrt{\int_0^t ds\sigma^2(s)}. \tag{83}$$

This solution, and the other Gaussian examples, are clearly what might have been expected from a knowledge of the nature of the solution of the associated SDEs. Our purpose here was simply to illustrate the direct solution of the QFPE using classical separation of variables methods utilizing a knowledge of the non-linear ODEs satisfied by the Gaussian quantile.

5.3 Equilibrium quantiles

If a solution to the SDE has reached an equilibrium distribution where $\partial Q/\partial t = 0$, then the QFPE reduces to

$$\frac{1}{2}\frac{\partial \Sigma^2}{\partial Q} - \mu(Q,t) = \frac{\Sigma^2(Q,t)}{2}\left(\frac{\partial Q}{\partial u}\right)^{-2}\frac{\partial^2 Q}{\partial u^2}. \tag{84}$$

We can consider Pearson's differential classification in relation to this equilibrium structure. We leave the reader to explore some of the better-known special cases, and consider here the general Pearson category and a special case leading to a Student t-distribution as the equilibrium. We consider an SDE containing independent sources of additive and multiplicative noise, in the form

$$dx_t = (\mu_1 - \mu_2 x_t)dt + \sigma_1 dW_{1t} + \sigma_2 x_t dW_{2t}, \tag{85}$$

where the W_i are correlated such that $E[dW_{1t}dW_{2t}] = \rho dt$. Then equation (85) is equivalent to a one-dimensional system in the form

$$dx_t = (\mu_1 - \mu_2 x_t)dt + \left(\sigma_1^2 + \sigma_2^2 x_t^2 + 2\rho\sigma_1\sigma_2 x_t\right)^{\frac{1}{2}} dW_t. \tag{86}$$

From this we can read off the functions in the QFPE as

$$\mu = \mu_1 - \mu_2 Q, \quad \Sigma^2 = \sigma_1^2 + \sigma_2^2 Q^2 + 2\rho\sigma_1\sigma_2 Q. \tag{87}$$

The equilibrium QPFE then gives the non-linear ODE

$$\frac{\partial^2 Q}{\partial u^2}\left(\frac{\partial Q}{\partial u}\right)^{-2} = \frac{2[(\rho\sigma_1\sigma_2 - \mu_1) + \left(\sigma_2^2 + \mu_2\right)Q]}{\sigma_1^2 + \sigma_2^2 Q^2 + 2\rho\sigma_1\sigma_2 Q}, \tag{88}$$

which corresponds to an interesting subset of the Pearson system embodied by equations (55) and (57). A particular case of interest is obtained by considering $\mu_1 = 0 = \rho$, so that we obtain the SDE

$$dx_t = -\mu_2 x_t dt + \sigma_1 dW_{1t} + \sigma_2 x_t dW_{2t}, \tag{89}$$

where the W_{it} are independent, and the quantile ODE reduces to

$$\frac{\partial^2 Q}{\partial u^2}\left(\frac{\partial Q}{\partial u}\right)^{-2} = \frac{2(\sigma_2^2 + \mu_2)Q}{\sigma_1^2 + \sigma_2^2 Q^2}. \tag{90}$$

If we compare this with equation (7), we see that we have a Student quantile with

$$Q = \frac{\sigma_1}{\sqrt{\sigma_2^2 + 2\mu_2}} w(u), \tag{91}$$

where $w(u)$ is the standard Student quantile satisfying equation (8) with degrees of freedom

$$n = 1 + 2\frac{\mu_2}{\sigma_2^2}. \tag{92}$$

So it is clear that we need $\mu_2 > 0$ for this to be a Student distribution. This of course corresponds to the requirement that the underlying SDE mean-revert to the origin, and this mean-reversion condition in turn allows an equilibrium to establish. This equilibrium origination of the standard Student distribution arises naturally in the study of the simplest linear and stochastic approximation of very complex physical systems, with manifest turbulence and self-organized criticality character. In particular the Student distribution arises as the equilibrium distribution, in special limiting cases of the stochastic processes studied in [25, 26]. The faster the mean-reversion rate is compared to the multiplicative volatility, the closer the system is to the normally distributed limit. The Student distribution also arises naturally in the modelling of asset returns [11, 22].

6 Conclusions and resources

We have shown how to characterize the key quantile functions of mathematical statistics through non-linear ordinary differential equations and how they may be solved via a power series whose coefficients are determined by a non-linear, usually cubic, recurrence relation. In the case of the Student and beta distributions, we are able to give both central and tail series, and for these distributions the pair of relevant series provide a good basis for numerical implementation. In the case of the normal and gamma, we have the start of a representation valid in arbitrary precision environments. Any of these methods may be used for precision benchmarking and for the test of rational approximations, and indeed to augment rational methods with polynomials in the central zone.

In contrast with rational approximation theory, these power series methods allow a meaningful mathematical extension of the concept of the quantile to the complex plane. Whether the ‘complex quantile function’ has a useful application is an open question.

We have also considered the non-linear integrated form of the Fokker–Planck equation, as introduced by Carrillo and Toscani [5], and demonstrated how the resulting non-linear PDE may be solved in some cases of interest.

6.1 Internet resources

Our intention is to provide a growing list of links to on line computational resources for the implementation of these and related methods. Some links are provided in the references. Specific discussions are also available as follows in Mathematica, C++ and Fortran 95. These will evolve as our knowledge about optimal implementations grows.

- A Mathematica file with investigations of some these methods is at: `http://www.mth.kcl.ac.uk/~shaww/web_page/papers/quantiles/QuantileDemos.nb`
- C++ code for the central normal power series is at:

  `http://www.mth.kcl.ac.uk/~shaww/web_page/papers/quantiles/normalquantile.cpp`
- Quad precision F95 code for the central normal power series is at:

  `http://www.mth.kcl.ac.uk/~shaww/web_page/papers/quantiles/normalquantile.f95`
- Quad precision F95 code for the Student case (central & tail) is at:

  `http://www.mth.kcl.ac.uk/~shaww/web_page/papers/quantiles/studentquantile.f95`

In addition a web page is being maintained with further examples and related article, with the entry point at `http://www.mth.kcl.ac.uk/~shaww/web_page/papers/quantiles/Quantiles.htm`

Appendix A: Derivation of a non-linear recurrence

The derivations of the various non-linear recurrence identities in Section 3 are a lengthy but straightforward matter just requiring care of the fine detail. In this appendix we shall give the details that establish the cubic Gaussian identity of equation (18). The ODE and initial conditions are

$$\frac{d^2w}{dv^2} = w\left(\frac{dw}{dv}\right)^2, \quad w(0) = 0, \quad w'(0) = 1. \tag{93}$$

Bearing in mind the symmetry and initial conditions we assume that

$$w = \sum_{p=0}^{\infty} c_p v^{2p+1}, \tag{94}$$

where $c_1 = 1$. Substitution of the series into the ODE gives the following identity:

$$\sum_{p=0}^{p=\infty}(2p+1)(2p)c_p v^{2p-1} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} c_k c_l c_m (2l+1)(2m+1)v^{2k+1+2l+2m}. \tag{95}$$

We now equate powers of v^{2p-1} on both sides of this equation, leading to

$$(2p+1)(2p)c_p = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} c_k c_l c_m (2l+1)(2m+1)\delta_{(2p-1),(2k+1+2l+2m)}, \tag{96}$$

where $\delta_{i,j}$ is the Kronecker delta. The terms in the infinite triple sum on the right vanish unless $2p-1 = 2k+1+2l+2m$, i.e., unless $p = k+l+m+1$. We also note that p, k, l, m are all non-negative integers. We dispose of the innermost sum by solving for m, i.e., $m = p-k-l-1$. With the outermost sum, the maximum possible value of k is $p-1$, and with the middle sum the maximum value of l for any given k is $p-k-1$. This gives the equation

$$(2p)(2p+1)c_p = \sum_{k=0}^{(p-1)}\sum_{l=0}^{(p-k-1)} (2l+1)(2p-2k-2l-1)c_k c_l c_{p-k-l-1} \tag{97}$$

leading immediately to equation (18). The quadratic Gaussian case of Section 3.2 is rather easier and left to the reader to reproduce. The cubic identities for other distributions may be obtained by careful variation of the manipulations given here. In all cases the calculation proceeds most straightforwardly by employing the δ-function, rather than trying to 'spot' the relevant powers in the non-linear sum. This is in contrast to the classical linear systems, where powers may be readily gathered by inspection.

Appendix B: Practicalities and implementations

In this section we discuss some issues related to implementation. We can consider both symbolic and exact implementations, and implementations in traditional floating-point arithmetic with various levels of precision.

The normal case

The raw recursions given here and in [24] will overflow or underflow floating point number bounds eventually. Fortunately there are simple rescalings that may be used to cure this problem. If we work in terms of the variable

$$\hat{v} = 2u - 1, \tag{98}$$

then the power series will exhibit a *unit* radius of convergence. We know it will blow up at $u = 0$ and $u = 1$. Then we form the power series

$$w = \sqrt{\frac{\pi}{2}}\sum_{p=0}^{\infty}\frac{d_p}{(2p+1)}\hat{v}^{2p+1}, \tag{99}$$

with $d_0 = 1$. In terms of these variables the quadratic recurrence becomes

$$d_{p+1} = \frac{\pi}{4} \sum_{j=0}^{p} \frac{d_j d_{p-j}}{(j+1)(2j+1)}. \tag{100}$$

With such a scheme the ratio $d_{p+1}/d_p \to 1$ as $p \to \infty$, and numerical implementations confirm this. The limits of how far the series can be taken are then just a matter of computation time. The second observation is that the sum involves the product $d_j d_{p-j}$ twice, so that one can optimize by summing half-way and taking care to note whether p is odd or even. The recurrence can be written

$$d_{p+1} = \frac{\pi}{4} \sum_{j=0}^{\lfloor \frac{p-1}{2} \rfloor} d_j d_{p-j} \left(\frac{1}{(j+1)(2j+1)} + \frac{1}{(p-j+1)(2p-2j+1)} \right)$$
$$+ \mathrm{If}\left(p \text{ is even}, \frac{\pi}{4} \frac{d_{\frac{p}{2}}^2}{\left(\frac{p}{2}+1\right)(p+1)}, \text{ else } 0\right). \tag{101}$$

In our initial implementations of the normal case in C++ and F95, attention has been confined to the central power series, and with very simple series termination criteria, involving termination when the next term to be computed falls below a certain level. While this is appropriate in a rigorous sense for alternating series, it is a temporary solution for monotone series such as those that are being considered here. The resulting error characteristics are nevertheless encouraging, and we can cover a large proportion of the unit interval with a precision that is more dependent on hardware and compiler choices than on the algorithm itself. In C++ the level of precision achievable depends on the quality of the implementation of the 'long double' data type. This is not, regrettably, rigorously specified in the C++ standard. The demonstration C++ code cited in Section 6.1 works on $0.0005 < u < 0.9995$ with maximum relative error less than 1.7×10^{-15} and on $0.0035 < u < 0.9965$ with maximum relative error less than 2.5×10^{-16}, when using the Bloodshed Dev C++ under Windows XP on a Pentium M. Results with other hardware and/or compilers will vary, especially if a poor long double implementation is used. For a more controlled implementation, we turn to quadruple precision Fortran 95. Using the Absoft compiler for MacOS on Intel, we obtained a relative error less than 10^{-29} on the interval $0.0007 < u < 0.9993$. All precision calculations were assessed by reference to the inverse error function built in to Mathematica 5.2, using 40 significant figures for the calculations. We stress that such code is not necessarily fast enough to use in live Monte–Carlo simulation, but, at this stage in the development, is more appropriately viewed as a means for benchmarking a new class of fast rational approximations.

The Student case

In this case we have a pair of power series for central and tail use and we may proceed directly to an implementation. One issue that has to be dealt with is the computation of the rescaling factor in the formula of equation (25), which we write as

$$v = \frac{\sqrt{n\pi}}{2} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}(2u-1) = \gamma_n(2u-1). \tag{102}$$

We use the identity

$$\gamma_{n+2} = \frac{\sqrt{n(n+2)}}{n+1}\gamma_n \tag{103}$$

to simplify the high-precision computation of this quantity. When n is an integer with $n \leqslant 10$ we use a precomputed value of γ_n. When n is a larger integer we recurse down to 9 or 10. When n is non-integer we recurse *up* to a value larger than 1000 and then apply the result that as $n \to \infty$, $\gamma_n \sqrt{\frac{2}{\pi}}$ is asymptotic to

$$1 + \frac{1}{4n} + \frac{1}{32n^2} - \frac{5}{128n^3} - \frac{21}{2048n^4} + \frac{399}{8192n^5} + \frac{869}{65536n^6} - \frac{39325}{262144n^7} - \frac{334477}{8388608n^8} + \frac{28717403}{33554432n^9} + \frac{59697183}{268435456n^{10}} - \frac{8400372435}{1073741824n^{11}} - \frac{34429291905}{17179869184n^{12}}. \tag{104}$$

Appendix C: Tail issues for the normal and gamma

It is evident that we have found power series about central points (normal and Student distributions) and about left end points (gamma and beta) and right end points (beta). The left end-point series for the beta also gives us the tails for the Student case. This leaves the issue of tail series for the normal and gamma. While we can use pure computational force to figure out many terms of the power series, a better treatment of the tails is needed. The most important case is that of the normal distribution, or, by taking its square, the gamma distribution with $p = 1/2$. There are various ways of thinking about this, based on what we have so far, but the neatest seems to be to exploit the fact that the gamma quantile ODE is already set up for an asymptotic analysis. The ODE for the gamma quantile with $p = 1/2$ is

$$\frac{d^2w}{du^2} = \left(1 + \frac{1}{2w}\right)\left(\frac{dw}{du}\right)^2, \tag{105}$$

so we can see right away that as $u \to 1$, $w \to +\infty$ and the asymptotic behaviour must be such that

$$\frac{d^2w}{du^2} \sim \left(\frac{dw}{du}\right)^2, \tag{106}$$

with a solution that we can infer from the exact solution for the exponential case, i.e.,

$$w \sim -\log[k(1-u)], k > 0. \tag{107}$$

Before proceeding further we look at the known asymptotic properties of the forward gamma CDF, for further insight. For large w the forward gamma CDF with $p = 1/2$ gives us

$$u \sim 1 - \frac{e^{-w}}{\sqrt{\pi w}}, \tag{108}$$

or, in other words,

$$\sqrt{\pi}(1-u) \sim \frac{e^{-w}}{\sqrt{w}}, \tag{109}$$

and taking logs we find that

$$-\log(\sqrt{\pi}(1-u)) \sim w + \frac{1}{2}\log(w), \tag{110}$$

and we see that $k = \sqrt{\pi}$. We now change independent variables to

$$v = -\log[\sqrt{\pi}(1-u)] \tag{111}$$

and establish the differential equation

$$\frac{d^2w}{dv^2} + \frac{dw}{dv} = \left(1 + \frac{1}{2w}\right)\left(\frac{dw}{dv}\right)^2, \tag{112}$$

with the asymptotic condition $w \sim v$ as $v \to \infty$, i.e., as $u \to 1$. It is possible to find asymptotic solutions of these relations. After some experimentation we find that asymptotic solutions exist in the form

$$w \sim v - \frac{1}{2}\log(v) + \sum_{k=1}^{k_{max}} \sum_{l=1}^{k+1} a_{kl}(\log(v))^{l-1} v^{-k}, \tag{113}$$

with a sequentially solvable set of conditions relating the coefficients a_{kl}. While it is difficult to give an explicit formula for the recursion, the coefficients are easily determined via symbolic computation. For example, with $k_{max} = 3$ we obtain the first few terms as

$$\begin{aligned} w \sim v &- \frac{\log(v)}{2} + \frac{1}{v}\left(\frac{\log(v)}{4} - \frac{1}{2}\right) + \frac{1}{v^2}\left(\frac{\log^2(v)}{16} - \frac{3\log(v)}{8} + \frac{7}{8}\right) \\ &+ \frac{1}{v^3}\left(\frac{\log^3(v)}{48} - \frac{7\log^2(v)}{32} + \frac{17\log(v)}{16} - \frac{107}{48}\right), \end{aligned} \tag{114}$$

with $v = -\log[\sqrt{\pi}(1-u)]$. This is the asymptotic form of the quantile function for the gamma distribution with $p = 1/2$. If we want the normal quantile for the positive tail, a small change of variables is needed. In the positive normal tail, the normal quantiles w_Φ is given in terms of the gamma quantile $w_{\Gamma_{1/2}}$, that we have just computed, by

$$w_\Phi(u) = \sqrt{2w_{\Gamma_{1/2}}(2u-1)}. \tag{115}$$

So to summarize for the normal case in the positive tail, the calculation is:

1. let $v = -\log[2\sqrt{\pi}(1-u)]$;
2. Work about w as above;
3. calculate $\sqrt{2w}$ for the normal quantile.

k	Order 3 asymp	Mathematica	AS241	Excel
3	3.089631766	3.090232306	3.090232306	3.090232306
4	3.718896182	3.719016485	3.719016485	3.719016485
5	4.264854901	4.264890794	4.264890794	4.264890794
6	4.753410708	4.753424309	4.753424309	4.753424341
7	5.199331535	5.199337582	5.199337582	5.199337618
8	5.611998229	5.612001244	5.612001244	5.612001258
9	5.997805376	5.997807015	5.997807015	5.997807018
10	6.361339949	6.361340902	6.361340902	6.361340808
11	6.706022570	6.706023155	6.706023155	6.706022335
12	7.034483450	7.034483825	7.034483825	7.034479171
13	7.348795853	7.348796103	7.348796103	7.348680374
14	7.650627921	7.650628093	7.650628093	7.650018522
15	7.941345205	7.941345326	7.941345326	7.934736689
16	8.222082128	8.222082216	8.222082216	NA
17	8.493793159	8.493793224	8.493793224	NA
18	8.757290300	8.757290349	8.757290349	NA
19	9.013271116	9.013271153	9.013271153	NA
20	9.262340061	9.262340090	9.262340090	NA
21	9.505024960	9.505024983	9.505024983	NA
22	9.741789925	9.741789943	9.741789943	NA
23	9.973045605	9.973045620	9.973045620	NA
24	10.19915741	10.19915742	10.19915742	NA
25	10.42045219	10.42045220	10.42045220	NA
26	10.63722367	10.63722368	10.63722368	NA
27	10.84973699	10.84973700	10.84973700	NA
28	11.05823241	11.05823241	11.05823241	NA
29	11.26292848	11.26292848	11.26292848	NA
30	11.46402468	11.46402469	11.46402469	NA
31	11.66170368	11.66170368	11.66170368	NA
32	11.85613321	11.85613322	11.85613322	NA
33	12.04746778	12.04746779	12.04746779	NA
34	12.23585004	12.23585005	12.23585005	NA
35	12.42141204	12.42141204	12.42141204	NA

We stress again that these are *asymptotic* formulae. They will turn out to be extraordinarily accurate in the deep tail, but less practical in the near tail zone. In the following table we look at the deep tail in more detail. Consider the quantiles associated with the points $u = 1 - 10^{-k}$, where $k = 3, 4, \ldots$. We consider the results from the simple third order formula above, the result from Mathematica's internal high precision error function (Mathematica V5.2), Wichura's algorithm AS241 [28] and Microsoft Excel (Excel 2004 for Mac). Note that in the last case Excel did not accept numbers as close to unity as supplied to the other algorithms. It is not designed to treat such numbers, though use of reflection symmetry can help to improve matters. AS241 was implemented in Mathematica in uncompiled form. The 'deep tail' quality of the simple asymptote becomes clear from this table. It does better than the simple spreadsheet implementation for values of u closer to unity that about 10^{-11}.

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