Construction of surface measures for Brownian motion

N. SIDOROVA
University of Bath
Department of Mathematical Sciences
Bath, BA2 7AY
United Kingdom.

O. WITTICH
Eindhoven University of Technology
Department of Mathematics and Computer Science
P.O. Box 513, 5600 MB Eindhoven
The Netherlands

1 Introduction

As a general setup for the notion of surface measure, we consider a measured metric space (Ω, d, μ) consisting of a metric space (Ω, d) and a Borel probability measure μ on Ω. A subset $A \subset \Omega$ is called *Minkowski-regular* if, as ε tends to zero, the sequence

$$\mu_\varepsilon := \frac{1}{\mu(A(\varepsilon))}\mu|_{A(\varepsilon)}$$

of probability measures supported by the ε-neighbourhoods

$$A(\varepsilon) := \{\omega \in \Omega \,:\, d(\omega, A) < \varepsilon\}$$

converges weakly to a probability measure μ_0 supported by A. The measure μ_0 is called the *induced Minkowski-* or *surface measure*. In this paper, we investigate Minkowski-regularity and surface measures in the case where the measured metric space is the space of continuous paths in a Riemannian manifold equipped with Wiener measure, the law of Brownian motion on this manifold. The subsets we are interested in are path spaces of regularly embedded closed submanifolds.

If we consider path spaces with finite time horizon T, the measures μ_ε are immediately identified with the laws of Brownian motion on M conditioned to the event that it stays within the ε-neighbourhood up to time T. This suggests looking at the problem from a probabilistic (or measure-theoretic) point of view, trying to construct the limit measure by considering the processes as limits of solutions of stochastic differential equations or by successively pinning the ambient Brownian motion to the submanifold. We will consider these ideas in the second part of the paper.

In the first part of the paper, we consider a different, more analytical approach. It is based on the observation that the law of conditioned Brownian motion is intimately connected to the law of Brownian motion *absorbed* at the boundary of the tube. This is explained in Section 3. But the main point with this approach is that it reveals a connection between two a priori different concepts, the construction of surface measures for Brownian motion and the effective dynamics of a quantum particle confined to a small tubular neighbourhood.

Consider a free particle in the Euclidean space $\mathbb{R}^n$ whose motion is driven by the Hamiltonian $-\Delta/2$ on $L^2(\mathbb{R}^n, dV_{\mathbb{R}^n})$, where $dV_{\mathbb{R}^n}$ is the Lebesgue measure on $\mathbb{R}^n$. We are interested in the limit behaviour of this particle when it is forced to stay increasingly close to some Riemannian manifold L isometrically embedded into $\mathbb{R}^n$.

The motion of a free particle on L is believed to be driven by the Hamiltonian $-\Delta_L/2$, where Δ_L is the Laplace–Beltrami operator on $L^2(L, dV_L)$ and dV_L denotes the induced Lebesgue measure on L. Since the Laplace–Beltrami operator is determined by the metric on the manifold L, the free motion on L is completely intrinsic and does not depend on the embedding of L into $\mathbb{R}^n$. The free motion on L corresponds to an *idealised concept of constraining* when the particle is forced to lie in the manifold exactly.

However, in reality the constraining is actually performed by introducing stronger and stronger forces pushing the particle to the manifold. For each $\varepsilon > 0$, one would consider the motion driven by a Hamiltonian $H_\varepsilon = -\Delta/2 + U_\varepsilon$, where U_ε is a family of non-negative potentials such that $U_\varepsilon|_L = 0$ and $U_\varepsilon(x) \to \infty$ for $x \notin L$, and study the limiting dynamics as $\varepsilon \to 0$. Surprisingly, in the most-natural cases this limit behavior will be different from the motion of the free (that is, *ideally constrained*) particle on L. More precisely, the new dynamics will be determined by the Hamiltonian $-\Delta_L/2 + W$, where $W \in C^\infty(L)$ is a smooth effective potential, which depends both on the intrinsic geometry of the manifold and of the embedding.

In this paper, we consider the hard-wall potential defined as

$$U_\varepsilon(x) = \begin{cases} 0, & x \in L(\varepsilon), \\ \infty, & \text{otherwise}, \end{cases}$$

where $L(\varepsilon)$ denotes the tubular ε-neighbourhood of L in $\mathbb{R}^n$. The soft (quadratic) potentials have been studied in [5]. Considering the hard-wall potential is equivalent to imposing *Dirichlet boundary conditions* on the boundary $\partial L(\varepsilon)$ of the tube.

The paper is organised as follows:

In Section 2 we study the dynamics corresponding to the absorbed motions. To make the essential steps more transparent, we consider the example of a curve in $\mathbb{R}^2$. In this way, lengthy differential geometric calculations are kept short.

Since Dirichlet boundary conditions cause a loss of mass as time goes on, one needs to renormalise the generators to avoid degeneration. Then one can study their strong convergence with the help of epiconvergence of the corresponding quadratic forms.

In Section 3 we discuss the relation between the two approaches. In particular, we show that the conditioned Brownian motion is a Brownian motion with a time-dependent drift, and we compute its transition probabilities in terms of the Dirichlet laplacian in the tube $L(\varepsilon)$.

In Section 4 we state the results for a more general situation, where we isometrically embed L into another Riemannian manifold M instead of $\mathbb{R}^n$. One can observe the same effect here: the limit dynamics of the conditioned Brownian motions is no longer intrinsic and it is described by an effective potential W, which is given in terms of both intrinsic (such as the scalar curvature) and extrinsic (such as mean and sectional curvatures) characteristics of L.

In Section 5 we explain the main idea of the probabilistic approach to the surface measures. We also discuss the original approach to Wiener surface measures, which was suggested in [14]. We show that it leads to the same surface measure as the conditioned Brownian motions and can be treated using the same technique.

So far, all subsets considered were Minkowski-regular and the surface measures were equivalent to the intrinsic measures on the subsets. Therefore, in Section 6, we finally discuss two related open problems, one about conditioning to tubes of variable diameter which is related to considering *soft constraints*, and one about conditioning to singular submanifolds, where we expect a completely different behavior. The path spaces of these subsets are not even Minkowski-regular in general, if one restricts oneself to continuous paths.

2 Example: the ground state of an electron confined to a curved planar wire

One possible application of surface measures is the description of quantum particles such as electrons which are confined to move within small spatial structures such as thin layers. As an example, we consider an electron that is confined to a planar wire. The wire is described as the tubular neighbourhood of a real line which is isometrically embedded into $\mathbb{R}^2$. The electron is forced to stay within the wire by introducing Dirichlet boundary conditions for the free particle Hamiltonian on the tube. If the diameter $\varepsilon > 0$ of the tube is small enough, the asymptotic dynamic letting ε tend to zero provides an effective description of the actual behavior of the electron. Somewhat surprisingly, it turns out that the motion of the electron is influenced by the geometry of the embedding and thus differs considerably from the behavior of a free electron on the real line.

The asymptotic result presented at the end of this section is valid in much greater generality. However, considering tubular neighbourhoods around submanifolds of Riemannian manifolds results in a huge bookkeeping problem, keeping track of Fermi coordinates, second fundamental form, metric and so forth. We decided to discuss the problem at hand basically for two reasons. On the one hand, the geometric situation is sufficiently involved to show most of the complications that are present in the general situation. On the other hand, the analysis of the problem leading to the necessity of rescaling and renormalisation, and the general line of the proof using epiconvergence and weak compactness in the boundary Sobolev space becomes more visible since it is no longer hidden behind lengthy differential geometric calculations. Besides, the underlying physical intuition is also helpful to grasp the structure of the problem.

One last remark: We mainly want to investigate the notion of surface measure. Thus we consider the case of semigroups taking the interpretation of the Dirichlet problem as the confinement of a quantum particle merely as a motivation. But although electrons may certainly not be interpreted as Brownian particles, the surface measure contains valuable information about the ground state of the quantum particle (see e.g. [13], p. 57) and the stationary Gibbs measure induced by it.

2.1 Geometry: an embedded curved wire

Let $\phi : \mathbb{R} \to \mathbb{R}^2$ be a smooth curve parametrised by arc-length. That means we consider an isometric embedding of the line into the plane. We denote the embedded line by $L := \{\phi(s) : s \in \mathbb{R}\}$. Let

$$e_1(s) := \dot{\phi}(s), \quad e_2(t) := \ddot{\phi}(s)/\|\ddot{\phi}(s)\|$$

be the 2-frame for the curve. We assume that the embedding is *regular* in the sense that the curvature

$$\kappa(s) := \langle \ddot{\phi}, e_2 \rangle(s)$$

is uniformly bounded, i.e. there is some $K > 0$ with $\|\ddot{\phi}\| < K$ for all $s \in \mathbb{R}$.

To confine the electron to the vicinity of the curve, we introduce an infinite hard-wall potential where the walls are given by the boundary components of the *tubular ε-neighbourhood*

$$L(\varepsilon) := \{x \in \mathbb{R}^2 : d(x, L) < \varepsilon\},$$

$\varepsilon > 0$, of the embedded line L. That means the confining potential is given by

$$U_C(x) := \begin{cases} 0, & x \in L(\varepsilon), \\ \infty, & \text{else.} \end{cases}$$

We now assume that $\varepsilon < K^{-1}$ is smaller than the *radius of curvature*. That implies that the ε-neighbourhood is diffeomorphic to the product of the real line and a small interval, i.e. $L(\varepsilon) = \mathbb{R}\times(-\varepsilon,\varepsilon)$. A diffeomorphism $\Phi : \mathbb{R}\times(-\varepsilon,\varepsilon) \to L(\varepsilon)$ is explicitly given by so called *Fermi coordinates*

$$\Phi(s, w) := \phi(s) + we_2(s)$$

for the tube. In these local coordinates, the metric is given by

$$g(s,w) := \begin{pmatrix} \langle\Phi_s,\Phi_s\rangle & \langle\Phi_s,\Phi_w\rangle \\ \langle\Phi_w,\Phi_s\rangle & \langle\Phi_w,\Phi_w\rangle \end{pmatrix} = \begin{pmatrix} (1-w\kappa(s))^2 & 0 \\ 0 & 1 \end{pmatrix}$$

where Φ_s, Φ_w denote the partial derivatives with respect to s, w. The associated Riemannian volume is, in the same local coordinates, given by

$$dV(s,w) = \sqrt{\det g}(s,w)ds\,dw = (1-w\kappa(s))\,ds\,dw$$

and using this measure, we can define the space $L^2(L(\varepsilon), g)$ and the *boundary Sobolev space* ${\rm H_0}^1(L(\varepsilon), g)$ (cf. [16]).

The laplacian with Dirichlet boundary conditions on the tube is the unique self-adjoint operator on $L^2(L(\varepsilon), g)$ which is associated to the quadratic form

$$q_\varepsilon(f) := \int_{L(\varepsilon)} dV\, g(df, df)$$

with domain $\mathcal{D}(q_\varepsilon) = {\rm H_0}^1(L(\varepsilon), g)$. As ε tends to zero, this expression will simply tend to zero. This is already obvious from the fact that $L \subset \mathbb{R}^2$ is a zero set. Therefore, to understand the details of the dynamic behaviour of an electron or a Brownian particle in a small tube around L, we have to apply some normalising transformation of the actual situation which is reminiscent of looking at the particles on the tube using some special kind of microscope.

2.2 Rescaling

From now on, we will always assume for simplicity that the curvature radius K^{-1} is greater than 1 so that $L(1) = \mathbb{R}\times(-1,1)$ via Φ. Now we want to make precise what we mean by a normalising transformation. To do so, we first use the fact that $\Phi : \mathbb{R}\times(-1,1) \to L(1)$ yields a global map of the 1-tube and that we can therefore use these local coordinates to define other Riemannian structures on and maps between the tubes. First of all, we define another metric on $L(1)$ which, in general, is different from the metric induced by the embedding.

Definition 1 *The* reference metric *on $L(1)$ is the metric which is given in local Fermi coordinates by*

$$g(s,w) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The associated Riemannian volume is $dV_0 = ds\,dw$.

The second component that is necessary to construct the desired microscope is the *rescaling map*.

Definition 2 *Let* $0 < \varepsilon < 1$. *The* rescaling map $\sigma_\varepsilon : L(\varepsilon) \to L(1)$ *is the diffeomorphism given in local Fermi coordinates by*

$$\sigma_\varepsilon(s, w) := (s, w/\varepsilon).$$

The induced map $\sigma_\varepsilon^* : C(L(1)) \to C(L(\varepsilon))$ *on the respective spaces of continuous functions is given by* $\sigma_\varepsilon^*(f) := f \circ \sigma_\varepsilon$.

With the help of the rescaling map, we will now transform the family $Q_{\varepsilon,\varepsilon>0}$ of quadratic forms on different domains into a family of quadratic forms on a fixed domain. Let

$$\rho(s, w) := \frac{dV}{dV_0} = 1 - w\kappa(s)$$

be the *Radon–Nikodym density* of the two Riemannian volume measures. On the one hand, the map σ_ε, $\varepsilon > 0$, can be extended to $L^2(L(\varepsilon), g)$ and

$$\Sigma_\varepsilon : L^2(L(1), g) \to L^2(L(\varepsilon), g_0) \tag{1}$$

given by $\Sigma_\varepsilon(f) := \{\varepsilon\,\rho\}^{-1/2}\sigma_\varepsilon^*(f)$ is actually a *unitary* map. This is the microscope under which we want to consider the dynamics on the small tubes. A crucial observation is that since $\rho > 1 - K > 0$ is smooth, Σ_ε also yields a homeomorphism

$$\Sigma_\varepsilon : \mathrm{H_0}^1(L(1), g) \to \mathrm{H_0}^1(L(\varepsilon), g_0).$$

That means we can use the maps $\Sigma_{\varepsilon,\varepsilon>0}$ to transform the family $q_{\varepsilon,\varepsilon>0}$ to a family of quadratic forms on a fixed Hilbert space.

Definition 3 *The* rescaled family *associated to the family* $q_{\varepsilon,\varepsilon>0}$ *of quadratic forms is the family* $Q_{\varepsilon,\varepsilon>0}$ *given by*

$$Q_\varepsilon := q_\varepsilon \circ \Sigma_\varepsilon,$$

with common domain $\mathcal{D}(Q_\varepsilon) = \mathrm{H_0}^1(L(1), g_0)$.

To summarise: The rescaling map works like a microscope that just enlarges the direction perpendicular to the embedded wire. We use it to transform the perturbation problem for the forms q_ε into a corresponding perturbation problem for a family of forms on a fixed Hilbert space.

It will turn out that we still obtain a singular perturbation problem. The reason is that the eigenvalues of the associated operators tend to infinity as ε tends to zero. That means that there is no reasonable limit dynamic. To avoid this kind of degeneration, one has to suitably *renormalize*.

2.3 Renormalisation

First of all, let us see what the rescaled family looks like. A short calculation yields for the exterior derivative of the transformed function

$$d\Sigma_\varepsilon(f) = (\varepsilon\rho)^{-1/2}\left(d\sigma_\varepsilon^*(f) - \frac{1}{2}\sigma_\varepsilon^*(f)\, d\log\rho\right).$$

That implies

$$\begin{aligned}
Q_\varepsilon \circ \Sigma_\varepsilon(f) &= \int_{L(\varepsilon)} dV\, g(d\Sigma_\varepsilon(f), d\Sigma_\varepsilon(f)) \\
&= \frac{1}{\varepsilon}\int_{L(\varepsilon)} dV_0\, (g(d\sigma_\varepsilon^*(f), d\sigma_\varepsilon^*(f)) + \sigma_\varepsilon^*(f)^2\, g(d\log\rho, d\log\rho)/4) \\
&\quad -\frac{1}{\varepsilon}\int_{L(\varepsilon)} dV\, \rho^{-1} g(\sigma_\varepsilon^*(f)\, d\log\rho, d\sigma_\varepsilon^*(f)).
\end{aligned}$$

The first two terms already yield a quadratic form that one would expect for an elliptic operator with a zero order term that can be interpreted as a potential. After partial integration, the third term will fit into that picture. To do so, we use

$$\sigma_\varepsilon^*(f)^2 = \sigma_\varepsilon^*(f^2), \quad \sigma_\varepsilon^*(f)\, d\sigma_\varepsilon^*(f) = \tfrac{1}{2} d\sigma_\varepsilon^*(f^2), \quad \rho^{-1}\, d\log\rho = d\rho^{-1},$$

and the fact that $\sigma_\varepsilon^*(f) \in \mathrm{H_0}^1(L(\varepsilon), g)$ has generalised boundary value zero. That implies for the third term that the boundary contribution to Green's formula vanishes and we obtain

$$\begin{aligned}
\int_{L(\varepsilon)} dV\, \rho^{-1} g(\sigma_\varepsilon^*(f)\, d\log\rho, d\sigma_\varepsilon^*(f)) &= -\frac{1}{2}\int_{L(\varepsilon)} dV\;\, g(d\rho^{-1}, d\sigma_\varepsilon^*(f^2)) \\
&= \frac{1}{2}\int_{L(\varepsilon)} dV\, \Delta\rho^{-1}\, \sigma_\varepsilon^*(f^2) \\
&= \frac{1}{2}\int_{L(\varepsilon)} dV_0\, \rho\, \Delta\rho^{-1}\, \sigma_\varepsilon^*(f^2) \\
&= \frac{1}{2}\int_{L(\varepsilon)} dV_0\, (\|d\log\rho\|^2 - \Delta\log\rho)\, \sigma_\varepsilon^*(f^2).
\end{aligned}$$

In the last step we used

$$\rho\,\Delta\rho^{-1} = \|d\log\rho\|^2 - \Delta\log\rho$$

where Δ denotes the (non-negative) Laplace–Beltrami operator and $\| - \|$ the norm on the cotangent bundle, both associated to the metric g. Applying the transformation formula for integrals to the expression for Q_ε established so far yields as a first major step the following description of the rescaled family.

Proposition 1 *Let $W : L(1) \to \mathbb{R}$ be the smooth potential*

$$W(x) := \frac{1}{2}\Delta \log \rho - \frac{1}{4}\|d \log \rho\|^2 \tag{2}$$

and $W_\varepsilon := W \circ \sigma_\varepsilon$ the rescaled potential. *Furthermore, let g_ε denote the* rescaled metric *on $L(1)$, i.e.*

$$g_\varepsilon(df, dh) := g(d\sigma_\varepsilon^*(f), d\sigma_\varepsilon^*(h)).$$

Then, the rescaled family is given by

$$Q_\varepsilon(f) = \int_{L(1)} dV_0 \, \left(g_\varepsilon(df, df) + W_\varepsilon f^2\right).$$

Thus the transformed perturbation problem on $L(1)$ yields a family of quadratic forms that consists of the form associated to the Dirichlet laplacian on $L(1)$ which is associated to the rescaled metric g_ε together with a potential that also depends on the geometry of the configuration. To see this, it might be helpful to consider the rescaled family in our special situation in local coordinates. First of all, the rescaled metric is given by (in the sequel $\kappa \equiv \kappa(s)$)

$$\begin{aligned} g_\varepsilon(s, w) &= \begin{pmatrix} (1 - \varepsilon w\kappa)^2 & 0 \\ 0 & \varepsilon^{-2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^{-2} \end{pmatrix} - \varepsilon \begin{pmatrix} 2w\kappa & 0 \\ 0 & 0 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} w^2\kappa^2 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Denoting by $g_{0,\varepsilon}(df, dh) := g_0(d\sigma_\varepsilon^*(f), d\sigma_\varepsilon^*(h))$ the rescaled reference metric, we can write

$$g_\varepsilon(s, w) - g_{0,\varepsilon}(s, w) = \varepsilon\, w \begin{pmatrix} 2\kappa & 0 \\ 0 & 0 \end{pmatrix} + \varepsilon^2 w^2 \begin{pmatrix} \kappa^2 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3}$$

Hence, the rescaled metric degenerates as ε tends to zero. The degeneration is the same as for the rescaled reference metric and the difference between the two families tends to zero.

This observation and the fact that the laplacian is naturally associated to the metric make us believe that such a perturbation ansatz is also successful for the operators. Thus we will later consider the laplacian to the rescaled metric as a perturbation of the laplacian to the reference metric and deduce its asymptotic behaviour from that.

The rescaled potential is given by

$$\begin{aligned} W_\varepsilon(s, w) &= -\frac{1}{2}\varepsilon(1 - \varepsilon\, w\kappa)w\ddot{\kappa} + \frac{3}{4}\varepsilon^2 w^2 \dot{\kappa}^2 - \frac{1}{4}\frac{\kappa^2}{(1 - \varepsilon w\kappa)^2} \\ &= -\frac{1}{4}\kappa^2 + R_\varepsilon(s, w), \end{aligned}$$

where

$$R_\varepsilon(s,w) = \frac{\varepsilon\, w}{2}\left(\frac{\kappa^3}{(1-\varepsilon\, w\kappa)^2} - \ddot{\kappa}\right) + \frac{\varepsilon^2 w^2}{4}\left(2\kappa\ddot{\kappa} + 3\dot{\kappa}^2 - \frac{\kappa^4}{(1-\varepsilon\, w\kappa)^2}\right).$$

For compact submanifolds, i.e. an embedding of the one-sphere $S^1 \subset \mathbb{R}^2$ it would be clear that the remainder is actually $O(\varepsilon)$ with respect to the supremum norm. But for the embedding of the real line, we have to assume that it is *geometrically finite* meaning that the absolute values of first and second derivatives of the curvature are bounded, i.e.

$$|\dot{\kappa}|, |\ddot{\kappa}| < C. \tag{4}$$

In that case, $R_\varepsilon(s,w) = O(\varepsilon)$ with respect to the supremum norm on the tube such that the potential W_ε tends to the potential

$$W_0(s,w) = -\frac{1}{4}\kappa^2(s) \tag{5}$$

which is constant on the fibres $F_s := \{(s,w) : w \in (-1,1)\}$ of the tubular neighbourhood.

By equation (3), the difference of the rescaled metric and the rescaled reference metric tends to zero as ε tends to zero. On the other hand, the potential also converges uniformly to W_0 as ε tends to zero. It is therefore quite a natural idea to first solve the asymptotic problem for the reference family and then to treat the induced family by perturbation-theoretical methods. Let us thus consider the family of quadratic forms

$$Q_{0,\varepsilon}(f) := \int_{L(1)} dV_0\, g_{0,\varepsilon}(df, df),$$

$\varepsilon > 0$, with domain $\mathcal{D}(Q_{0,\varepsilon}) = \mathrm{H_0}^1(L(1), g_0)$. In local Fermi coordinates, we may use the fact that the tubular neighbourhood is *trivial* in the sense that it is globally diffeomorphic to the product $\mathbb{R} \times (-1,1)$ which implies that

$$\mathrm{H_0}^1(L(1), g_0) = \mathrm{H}^1(\mathbb{R}) \otimes \mathrm{H_0}^1(-1,1)$$

and with respect to that decomposition we may write

$$Q_{0,\varepsilon}(f) = \int_{\mathbb{R}} ds \int_{-1}^{1} ds\, |\partial_s \otimes 1(f)|^2 + \frac{1}{\varepsilon^2}\int_{\mathbb{R}} ds \int_{-1}^{1} dw\, |1 \otimes \partial_w(f)|^2.$$

From this expression, it is obvious that $Q_{0,\varepsilon}(f)$ will tend to infinity as ε tends to zero as long as

$$\int_{\mathbb{R}} ds \int_{-1}^{1} dw\, |1 \otimes \partial_w(f)|^2 > 0.$$

To investigate when this type of degeneration happens and how it can be avoided, we consider the family of quadratic forms

$$Q^+_{0,\varepsilon}(f) := \frac{1}{\varepsilon^2}\int_{\mathbb{R}} ds \int_{-1}^{1} dw\, |\partial_w(f)|^2,$$

$\varepsilon > 0$, with natural domain $\mathcal{D}(Q^+_{0,\varepsilon}) = L^2(\mathbb{R}) \otimes {\mathrm{H}_0}^1(-1,1)$. The main observation is now that this quadratic form is *decomposable* with respect to the *direct integral decomposition* ([8])

$$\mathcal{D}(Q^+_{0,\varepsilon}) = \int_{s\in\mathbb{R}}^{\oplus} ds\, {\mathrm{H}_0}^1(F_s) := \left\{ (f_s)_{s\in\mathbb{R}} \,:\, f_s \in {\mathrm{H}_0}^1(F_s), \int_{\mathbb{R}} ds\, \|f_s\|^2_{{\mathrm{H}_0}^1(F_s)} < \infty \right\}$$

where, in addition, $s \mapsto f_s$ is supposed to be measurable. If we use

$$\|f\|^2_{{\mathrm{H}_0}^1(F_s)} := \int_{-1}^{1} dw\, |\partial_w f|^2(w,s)$$

as Sobolev-norm on the space ${\mathrm{H}_0}^1(F_s)$, the decomposed quadratic form is hence given by

$$Q^+_{0,\varepsilon}(f) := \frac{1}{\varepsilon^2} \int_{\mathbb{R}} ds\, \|f_s\|^2_{{\mathrm{H}_0}^1(F_s)}. \tag{6}$$

Since the Sobolev-norm is strictly positive definite, the first conclusion that we can draw from this representation of the quadratic form is that for all $f \neq 0$ we have

$$\lim_{\varepsilon\to 0} Q^+_{0,\varepsilon}(f) = \infty,$$

i.e. the family $Q_{0,\varepsilon}$ of quadratic forms degenerates in the most dramatic way. Thus, the perturbation problem obtained by considering the rescaled families will still not yield a sensible answer.

To overcome this difficulty, we have to *renormalise* the rescaled families. To see how this can be done, we need the following two basic observations:

1. The operator associated to the quadratic form $q_s(f) := \|f_s\|^2_{{\mathrm{H}_0}^1(F_s)}$ with domain $\mathcal{D}(q_s) = {\mathrm{H}_0}^1(F_s)$ is the *Dirichlet laplacian* on F_s, i.e. the Laplace–Beltrami operator $-\Delta_s$ on F_s with respect to the metric induced by g_0 with domain $\mathcal{D}(\Delta_s) = {\mathrm{H}_0}^1 \cap \mathrm{H}^2(F_s)$,

2. In Fermi coordinates, we see that the induced metric on F_s is simply the flat metric such that all fibres F_s are isometric to the interval $(-1,1) \subset \mathbb{R}$ equipped with the flat metric. In particular, the associated Dirichlet laplacians are all *isospectral.*

These two statements together imply by standard results [8] about decomposable operators

Proposition 2 *The operator associated to the quadratic form $Q^+_{0,\varepsilon}$ is the decomposable operator*

$$D := -\int_{L}^{\oplus} ds\, \Delta_s$$

with domain $\mathcal{D}(D) := \int_L^\oplus ds\, \mathrm{H_0}^1 \cap \mathrm{H}^2(F_s)$. *The operator is unbounded on* $L^2(L(1), g_0)$ *and if we denote the spectral decomposition of the Dirichlet laplacian* $-\Delta_{(-1,1)}$ *on* $(-1,1) \subset \mathbb{R}$ *by*

$$-\Delta_{(-1,1)} f = \sum_{k\geq 0} \lambda_k u_k \langle u_k, f\rangle_{L^2(-1,1)},$$

then the spectral decomposition of D *is given by*

$$Df(s,w) = \sum_{k\geq 0} \lambda_k \langle u_k, f_s\rangle_{L^2(F_s)} u_k(w) \tag{7}$$

where we identify a function $f \in \mathcal{D}(D)$ *with its decomposition* $(f_s)_{s\in\mathbb{R}}$.

By this result, we can now introduce a *renormalisation* to the effect that the family of forms $Q^+_{0,\varepsilon}$ degenerates everywhere except on a non-trivial subspace. From the spectral decomposition of the associated Laplace operators, we obtain

$$\begin{aligned}
Q^+_{0,\varepsilon}(f) - \frac{\lambda_0}{\varepsilon^2}\langle f, f\rangle_{L^2(L(1),g_0)} &= \frac{1}{\varepsilon^2}\int_{\mathbb{R}} ds\, \|f_s\|^2_{\mathrm{H_0}^1(F_s)} - \frac{\lambda_0}{\varepsilon^2}\int_{\mathbb{R}} ds \int_{-1}^{1} dw\, |f_s|^2 \\
&= \frac{1}{\varepsilon^2}\int_{\mathbb{R}} ds\, \left[\|f_s\|^2_{\mathrm{H_0}^1(F_s)} - \lambda_0 \|f_s\|^2_{L^2(F_s)}\right] \\
&= \sum_{k\geq 1} \frac{\lambda_k - \lambda_0}{\varepsilon^2} \int_{\mathbb{R}} ds\, |\langle u_k, f_s\rangle_{L^2(F_s)}|^2.
\end{aligned}$$

That implies that the quadratic forms $Q^+_{0,\varepsilon}$, $\varepsilon > 0$ have a non-trivial common *null space*

$$\mathcal{N} := \{f \in \mathcal{D}(Q^+_{0,\varepsilon}) \,:\, E_0 f = f\} \tag{8}$$

where the $L^2(L(1), g_0)$-orthogonal projection E_0 onto the null space is given by

$$E_0 f(s,w) := \langle u_0, f_s\rangle_{L^2(F_s)} u_0(w).$$

As ε tends to zero, we have thus

$$\lim_{\varepsilon\to 0} Q^+_{0,\varepsilon}(f) - \frac{\lambda_0}{\varepsilon^2}\langle f, f\rangle_{L^2(L(1),g_0)} = \begin{cases} 0, & f \in \mathcal{N}, \\ \infty, & \text{else.} \end{cases}$$

pointwise. This is the reason to expect that — after renormalisation — we will obtain some non-trivial limit dynamic on the subspace $\mathcal{N}$. To obtain a proper asymptotic problem we therefore introduce the following two modified families of quadratic forms.

Definition 4 *The* renormalised rescaled *families of quadratic forms with parameter* $\varepsilon > 0$ *are given by*

$$Q^R_{0,\varepsilon}(f) := \int_{L(1)} dV_0\, g_{0,\varepsilon}(df, df) - \frac{\lambda_0}{\varepsilon^2}\langle f, f\rangle_{L^2(L(1),g_0)}$$

for the reference metric and by

$$Q^R_\varepsilon(f) := \int_{L(1)} dV_0\,(g_\varepsilon(df,df) + W_\varepsilon f^2) - \frac{\lambda_0}{\varepsilon^2}\langle f, f\rangle_{L^2(L(1),g_0)}$$

for the induced metric. The common domain of both families is $\mathrm{H_0}^1(L(1), g_0)$.

Starting from the idea of looking at electrons on small tubes by a special kind of microscope, we thus arrived at a problem that we finally can solve: We have to calculate a renormalised limit dynamic that concentrates on the subspace $\mathcal{N} \subset L^2(L(1), g_0)$.

2.4 Epiconvergence

In the preceding subsection, we computed that the modified families $Q^+_{0,\varepsilon}$ converge pointwise to a non-trivial limit as ε tends to zero. However, pointwise convergence of the quadratic forms is useless if one strives to say something about convergence of the associated operators. The proper notion for this is *epiconvergence* ([1], [3]).

To explain this notion, let H be a Hilbert space and q_n, $q_\infty : H \to \mathbb{R}$, $n \geq 1$ be non-negative but not necessarily densely defined quadratic forms with domains $\mathcal{D}(q_n)$, $\mathcal{D}(q_\infty)$ respectively. Denote by H_n, $H_\infty \subset H$ the closures of the domains with respect to the norm in H.

Definition 5 *The sequence q_n of non-negative closed quadratic forms on H* epi-converges *to the closed quadratic form q with respect to the* weak topology *on H iff*

1. *For all $u \in \mathcal{D}(q_\infty)$ there is a weakly convergent sequence $u_n \to u$ such that*
$$\lim_n q_n(u_n) = q_\infty(u).$$
2. *For all $u \in \mathcal{D}(q_\infty)$ and for all weakly convergent sequences $u_n \to u$ we have*
$$\liminf_n q_n(u_n) \geq q_\infty(u).$$

In our case, the Hilbert space will be $H := \mathrm{H_0}^1(L(1), g_0)$. We consider the problem for the reference family first. Note that $f \in E_0 \cap \mathcal{D}(Q^R_{0,\varepsilon})$ implies that $f(s,w) = u_0(w)\,h(s)$ with $h \in \mathrm{H}^1(\mathbb{R})$. We want to prove that $Q^R_{0,\varepsilon}$ converges to

$$Q^R_0(f) := \begin{cases} \int_{\mathbb{R}} ds|\partial_s h|^2, & f \in \mathcal{N}, \\ \infty, & \text{else.} \end{cases}$$

in the sense of epiconvergence with respect to the weak topology of $\mathrm{H}^1(L(1), g_0)$. The first requirement on epiconvergence follows from pointwise convergence of the forms, i.e. we may use the sequence $f_n \equiv f$ and obtain

$$
\begin{aligned}
\lim_{\varepsilon\to 0} Q^R_{0,\varepsilon}(f) &= \lim_{\varepsilon\to 0} Q^+_{0,\varepsilon}(f) - \frac{\lambda_0}{\varepsilon^2}\langle f, f\rangle_{L^2(L(1),g_0)} + \int_{\mathbb{R}} ds \int_{-1}^{1} dw\, |\partial_s \otimes 1(f)|^2 \\
&= \begin{cases} \int_{\mathbb{R}} ds \int_{-1}^{1} dw\, |\partial_s \otimes 1(f)|^2 & , f \in \mathcal{N} \\ \infty & , \text{else} \end{cases} \\
&= \begin{cases} \int_{-1}^{1} dw\, |u_0|^2 \int_{\mathbb{R}} ds |\partial_s h|^2 & , f \in \mathcal{N} \\ \infty & , \text{else} \end{cases} \\
&= \begin{cases} \int_{\mathbb{R}} ds |\partial_s h|^2, & f \in \mathcal{N}, \\ \infty, & \text{else} \end{cases} \\
&= Q^R_0(f)
\end{aligned}
$$

since the eigenfunction $\int_{-1}^{1} dw\, |u_0|^2 = 1$ is normalised. For the second requirement, we have to prove another result first.

Lemma 1 *Let $\alpha > \lambda_0$. Then we have for all $f \in \mathrm{H_0}^1(L(1), g_0)$ and all $\varepsilon > 0$*

$$Q^R_{0,\varepsilon}(f) + \alpha\langle f, f\rangle_{L^2(L(1),g_0)} \geq \|f\|^2_{\mathrm{H_0}^1(L(1),g_0)}.$$

Proof: We have for $\varepsilon^2 < 1 - \lambda_0/\lambda_1 < 1 - \lambda_0/\lambda_k$ for all k

$$
\begin{aligned}
& Q^R_{0,\varepsilon}(f) + \alpha\langle f, f\rangle_{L^2(L(1),g_0)} \\
= \;& Q^+_{0,\varepsilon}(f) + \left(\alpha - \frac{\lambda_0}{\varepsilon^2}\right)\langle f, f\rangle_{L^2(L(1),g_0)} + \int_{\mathbb{R}} ds \int_{-1}^{1} dw\, |\partial_s \otimes 1(f)|^2 \\
= \;& \sum_{k\geq 1} \frac{\lambda_k - \lambda_0}{\varepsilon^2} \int_{\mathbb{R}} ds\, |\langle u_k, f_s\rangle_{L^2(F_s)}|^2 + \int_{\mathbb{R}} ds \int_{-1}^{1} dw\, |\partial_s \otimes 1(f)|^2 \\
& + \alpha\langle f, f\rangle_{L^2(L(1),g_0)} \\
\geq \;& \sum_{k\geq 1} \lambda_k \int_{\mathbb{R}} ds\, |\langle u_k, f_s\rangle_{L^2(F_s)}|^2 + \int_{\mathbb{R}} ds \int_{-1}^{1} dw\, |\partial_s \otimes 1(f)|^2 + \lambda_0\langle f, f\rangle_{L^2(L(1),g_0)} \\
\geq \;& \sum_{k\geq 0} \lambda_k \int_{\mathbb{R}} ds\, |\langle u_k, f_s\rangle_{L^2(F_s)}|^2 + \int_{\mathbb{R}} ds \int_{-1}^{1} dw\, |\partial_s \otimes 1(f)|^2 \\
= \;& \int_{\mathbb{R}} ds \int_{-1}^{1} dw\, |1 \otimes \partial_w(f)|^2 + \int_{\mathbb{R}} ds \int_{-1}^{1} dw\, |\partial_s \otimes 1(f)|^2 \\
= \;& \|f\|^2_{\mathrm{H_0}^1(L(1),g_0)}.
\end{aligned}
$$

∎

This inequality means that the sequence $Q^R_{0,\varepsilon}$, $\varepsilon > 0$, is *equicoercive* ([1], [3]). To establish the second property of epiconvergence, the lim inf-inequality, we consider a slightly different estimate, namely for $\varepsilon < 1 - \lambda_0/\lambda_1 < 1 - \lambda_0/\lambda_k$,

$$\begin{aligned}
& Q^R_{0,\varepsilon}(f) \\
= \; & Q^+_{0,\varepsilon}(f) - \frac{\lambda_0}{\varepsilon^2}\langle f, f\rangle_{L^2(L(1),g_0)} + \int_{\mathbb{R}} ds \int_{-1}^{1} dw\, |\partial_s \otimes 1(f)|^2 \\
= \; & \sum_{k\geq 1} \frac{\lambda_k - \lambda_0}{\varepsilon^2} \int_{\mathbb{R}} ds\, |\langle u_k, f_s\rangle_{L^2(F_s)}|^2 + \int_{\mathbb{R}} ds \int_{-1}^{1} dw\, |\partial_s \otimes 1(f)|^2 \\
\geq \; & \sum_{k\geq 1} \frac{\lambda_k}{\varepsilon} \int_{\mathbb{R}} ds\, |\langle u_k, f_s\rangle_{L^2(F_s)}|^2 + \int_{\mathbb{R}} ds \int_{-1}^{1} dw\, |\partial_s \otimes 1(f)|^2 \\
= \; & \frac{1}{\varepsilon} \int_{\mathbb{R}} ds \int_{-1}^{1} dw\, |1 \otimes \partial_w(E_0^{\perp} f)|^2 + \int_{\mathbb{R}} ds \int_{-1}^{1} dw\, |\partial_s \otimes 1(f)|^2
\end{aligned}$$

where we denote by $E_0^{\perp}$ the $L^2(L(1), g_0)$-orthogonal projection onto the orthogonal complement of $\mathcal{N}$. Now note that this sum consists of a linear combination of two quadratic forms

$$Q^R_{0,\varepsilon}(f) \geq \frac{1}{\varepsilon} Q_1(f) + Q_2(f)$$

which are all *continuous* and *non-negative* on ${\mathrm{H}_0}^1(L(1), g_0)$. That implies by *polarisation*

$$Q_i(f_n) - 2B_i(f_n, f) + Q_i(f) = Q_i(f_n - f) \geq 0$$

and hence $Q_i(f_n) \geq 2B_i(f_n, f) - Q_i(f)$ which by continuity of the bilinear forms B_i associated to Q_i and by weak convergence of f_n to f finally implies

$$\liminf_{\varepsilon\to 0} Q_i(f_n) \geq Q_i(f)$$

for $i = 1, 2$. Now, to finally prove the second assertion, note that we have established so far that

$$\liminf_{\varepsilon\to 0} Q^R_{0,\varepsilon}(f_n) \geq Q_2(f) + \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon} Q_1(f_\varepsilon).$$

For $f \in \mathcal{N}$ there is nothing else to prove since by $Q_1(f_\varepsilon)/\varepsilon \geq 0$ we then have

$$\liminf_{\varepsilon\to 0} Q^R_{0,\varepsilon}(f_n) \geq Q_2(f) = Q^R_0(f).$$

For $f \notin \mathcal{N}$ we have $\liminf Q_1(f_n) = a > 0$ and hence for some suitable $a > \delta > 0$

$$\liminf_{\varepsilon\to 0} Q^R_{0,\varepsilon}(f_n) \geq Q_2(f) + \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon}(a - \delta) = \infty.$$

That establishes *epiconvergence* of the family $Q^R_{0,\varepsilon}$, $\varepsilon > 0$.

The reason to consider epiconvergence of quadratic forms is that it implies strong resolvent convergence of the self-adjoint operators associated to these forms. This will now be explained.

2.5 Strong convergence of the generators

So far we proved that the quadratic forms epiconverge in the weak topology of ${\mathrm{H}_0}^1(L(1), g_0)$. But what is this good for — we actually want to prove that the associated operators converge in the strong resolvent sense in $L^2(L(1), g_0)$? It turns out that this follows by a rather general chain of arguments that we will present in the sequel. We again use the setup from the beginning of the last subsection, referring to the implications to our special situation inbetween. The reasoning consists essentially of four steps:

1. Epiconvergence in the weak topology implies convergence of the minimisers of the quadratic forms q_n in the following sense: Denote by u_n^* a minimiser of q_n. Then we have the following statement ([3]):

   Lemma 2 *Let the sequence of quadratic forms $q_{n,n\geq 1}$ epiconverge to q_∞ in the weak topology on H. If a sequence $u^*_{n,n\geq 1}$ of minimisers of the q_n converges in the weak sense to some $u^* \in H$, then u^* is a minimiser of q_∞.*

2. The question of whether the sequence $u^*_{n,n\geq 1}$ of minimisers really converges is in general very hard to answer. However, if as in our situation (see Lemma 1) the sequence of quadratic forms is *equicoercive*, meaning that

   $$q_n(u) \geq A\|u\|_H^2 + B,$$

   with some $A > 0$, $B \in \mathbb{R}$, we have that for all $t \in \mathbb{R}$, that the set

   $$K_t := \bigcap_{n\geq 1} \{u \in H \,:\, q_n(u) \leq t\} \subset H$$

   is relatively bounded in H. If we now assume that the family does not degenerate in the sense that we exclude the case $\lim_{n\to\infty} q_n(u) = \infty$ for all $u \in H$ then there is some $t_0 \in \mathbb{R}$ such that K_t is non-empty for all $t > t_0$. But Hilbert spaces are always *reflexive* so that we can apply the *Banach–Alaoglu theorem* also to the weak topology on H. Hence the sets $K_t \subset\subset H$ are relatively compact in H with respect to the weak topology. That means that every sequence of points in K_t — and $u^*_{n,n\geq 1} \subset K_t$ is such a sequence if $t > t_0$ — contains a weakly convergent subsequence. Thus, if the minimiser u^*_∞ of the limiting quadratic form q_∞ is *unique*, Lemma 2 implies that all these weakly convergent subsequences must converge to u^*_∞. That means nothing but:

   Lemma 3 *Assume that the family of quadratic forms $q_{n,n\geq 1}$ epicoverges to the quadratic form q_∞ and that there is one $t \in \mathbb{R}$ with $K_t \neq \emptyset$. If the minimiser u^*_∞ of q_∞ is unique then every sequence $u^*_{n,n\geq 1}$ of minimisers of the $q_{n,n\geq 1}$ converges weakly to u^*_∞.*

   In our case, the minimiser of the limiting quadratic form is unique since Q_0^R is strictly convex on the subset $\mathcal{N}$ where it is not infinite.

3. Weak convergence of the sequence of minimisers also implies weak convergence of the resolvents of the operators that are associated to the quadratic forms. To see this, let $v \in H$ and consider the functions

$$Q_{n,v}(u) := \frac{1}{2} q_n(u) - \langle v, u \rangle_X .$$

By the *Cauchy–Schwarz inequality*, equicoercivity of $q_{n,n\geq 1}$ also implies equicoercivity of $Q_{n,v},\ n \geq 1$ for all $v \in H$. On the other hand, the limiting function $Q_{\infty,v}$ is still strictly convex on $\mathcal{N}$ and therefore has a unique minimiser. Since $u \mapsto \langle v, u \rangle_X$ is just a bounded linear map, weak epiconvergence of the forms $q_{n,n\geq 1}$ to q_∞ implies epiconvergence of the functions $Q_{n,v}$ to $Q_{\infty,v}$. Thus we can draw the same conclusions as before for all sequences of minimisers $u^*_{n,v},\ n \geq 1$ of the functions $Q_{n,v}$. But by *Friedrich's construction* ([6]) we can associate to every quadratic form q_n a closed unbounded operator A_n on H_n which is densely defined on some domain $\mathcal{D}(A_n) \subset H_n$. The equation that determines the minimiser (meaning the equation for the zeroes of the *subdifferential operator* ([1], [2]) as in the finite dimensional case) is thus given by $u^*_{n,v} \in \mathcal{D}(A_n)$ with $A_n u^*_{n,v} = v$ or

$$u^*_{n,v} = A_n^{-1} v.$$

which implies that for all $v \in H$ the inverse operators associated to the quadratic forms converge pointwise in the weak topology of H. By Lemma 1, even the sequence $Q^R_{0,\varepsilon}(f) + \alpha \langle f, f \rangle_{L^2(L(1),g_0)}$ is equicoercive for $\alpha > \lambda_0$. Thus if we denote the operator associated to $Q^R_{0,\varepsilon}$ by $\Delta_0(\varepsilon)$, the minimiser of

$$\frac{1}{2}(Q^R_{0,\varepsilon}(f) + \alpha \langle f, f \rangle_{L^2(L(1),g_0)}) - \langle v, f \rangle_{L^2(L(1),g_0)}$$

is given by the resolvent $f^*_{\varepsilon,v,\alpha} = (\Delta_0(\varepsilon) + \alpha)^{-1} v$ at $\alpha > \lambda_0$. Now we compute the operator associated to the epi-limit $Q^R_0(f) + \alpha \langle f, f \rangle_{L^2(L(1),g_0)}$. Since the limiting quadratic form is infinite outside $\mathcal{N}$, we may restrict ourselves to the calculation of the minimiser on this subspace. Denoting again the $L^2(L(1), g_0)$-orthogonal projection on $\mathcal{N}$ by E_0 we have

$$\begin{aligned}
& \frac{1}{2}\left\{Q^R_{0,\varepsilon}(f) + \alpha \langle f, f \rangle_{L^2(L(1),g_0)}\right\} - \langle v, f \rangle_{L^2(L(1),g_0)} \\
= \ & \frac{1}{2}\left\{Q^R_{0,\varepsilon}(E_0 f) + \alpha \langle E_0 f, E_0 f \rangle_{L^2(L(1),g_0)}\right\} - \langle v, E_0 f \rangle_{L^2(L(1),g_0)} \\
= \ & \frac{1}{2}\int_{\mathbb{R}} ds\ \left\{|\partial_s h|^2 + \alpha |h|^2\right\} - \langle \langle v, u_0 \rangle_{L^2(F_s)}, h \rangle_{L^2(\mathbb{R})}
\end{aligned}$$

since every function $f \in \mathcal{N}$ can be written $f(s,w) = u_0(w)\, h(s)$ and thus

$$\begin{aligned}
\langle v, E_0 f \rangle_{L^2(L(1),g_0)} &= \langle E_0 v, f \rangle_{L^2(L(1),g_0)} = \langle u_0\, \langle u_0, v \rangle_{F_s}, u_0\, h \rangle_{L^2(L(1),g_0)} \\
&= \int_{\mathbb{R}} ds\, \langle u_0, u_0 \rangle_{L^2(F_s)}\, |\langle u_0, v \rangle_{L^2(F_s)}\, h|^2 \\
&= \langle \langle v, u_0 \rangle_{L^2(F_s)}, h \rangle_{L^2(\mathbb{R})}.
\end{aligned}$$

The operator associated to the quadratic form Q_0^R is the laplacian $\Delta_{\mathbb{R}}$ on $\mathbb{R}$. Thus we obtain, differentiating with respect to h and letting the differential be equal to zero:

$$\langle (\Delta_{\mathbb{R}} + \alpha)h, - \rangle_{L^2(\mathbb{R})} = \langle \langle v, u_0 \rangle_{L^2(F_s)}, - \rangle_{L^2(\mathbb{R})}.$$

Multiplying both sides by u_0 and solving for $u_0\, h$ yields finally the more convenient form

$$f^*_{0,v,\alpha} = E_0 \,(\Delta_{\mathbb{R}} + \alpha)^{-1}\, E_0 v = (\Delta_{\mathbb{R}} + \alpha)^{-1}\, E_0 v.$$

Thus, the above general considerations about convergence imply

$$\mathrm{w} - \lim_{\varepsilon \to 0} (\Delta_0(\varepsilon) + \alpha)^{-1} v = E_0 \,(\Delta_{\mathbb{R}} + \alpha)^{-1}\, E_0 v$$

weakly in ${\mathrm{H}_0}^1(L(1), g_0)$.

4. The last observation is now that $\mathrm{H}^1(L(1), g_0) \subset\subset L^2(L(1), g_0)$, i.e. the inclusion is compact by the *Sobolev embedding theorem* ([16]). Thus, weak convergence in the boundary Sobolev space implies strong convergence in $L^2(L(1), g_0)$. That means finally

   Proposition 3 *In $L^2(L(1), g_0)$, we have strong resolvent convergence of the renormalised laplacian associated to the rescaled reference metric on $L(1)$ to an operator*

   $$E_0\, \Delta_{\mathbb{R}}\, E_0 = \Delta_{\mathbb{R}}\, E_0$$

   given by the projection E_0 onto $\mathcal{N}$ followed by the laplacian on $\mathbb{R}$. Here, the laplacian $\Delta_{\mathbb{R}}$ acts on $\mathcal{N}$ by $\Delta_{\mathbb{R}} f = \Delta_{\mathbb{R}} u_0 h = u_0 \Delta_{\mathbb{R}} h$.

   Since the operators $\Delta_0(\varepsilon)$ and $E_0\, \Delta_{\mathbb{R}}\, E_0$ are self-adjoint, strong resolvent convergence of the operators implies strong convergence of the associated semigroups on compact subintervals of $(0, \infty)$ ([6], [2]). Therefore, we finally obtain

   Proposition 4 *For all $v \in L^2(L(1), g_0)$ and all compact subintervals $I \subset\subset (0, \infty)$, we have*

   $$\lim_{\varepsilon \to 0} \sup_{t \in I} \| e^{-\frac{t}{2}\Delta_0(\varepsilon)} v - E_0\, e^{-\frac{t}{2}\Delta_{\mathbb{R}}}\, E_0 v \|_{L^2(L(1), g_0)} = 0.$$

   Thus, the semigroups generated by the rescaled and renormalised Dirichlet laplacians on the reference tube converge to a limit semigroup obtained by *homogenisation* along the fibres. To see this, recall that $E_0 v(s, w) = h(s) u_0(w)$ and note that

   $$\left\{ E_0\, e^{-\frac{t}{2}\Delta_{\mathbb{R}}}\, E_0 v \right\}(s, w) = u_0(w) \left\{ e^{-\frac{t}{2}\Delta_{\mathbb{R}}} h \right\}(s).$$

   The time-dependent part of the dynamic is thus provided by a dynamic on the submanifold alone which modulates the function u_0 on each fibre.

2.6 The result for the induced metric

As said above, the idea is to consider the rescaled and renormalised quadratic forms associated to the induced metric g as perturbations of the forms associated to the reference metric. Recall that the assumption (4) of *geometric finiteness*, i.e. that $|\kappa|$, $|\dot{\kappa}|$ and $|\ddot{\kappa}|$ are uniformly bounded implies that

$$\sup_{s\in\mathbb{R},|w|<1} |W_\varepsilon(s,w) - W_0(s,w)| \leq M_1\varepsilon.$$

On the other hand, by the explicit calculation (3) above, we have that

$$g_\varepsilon(df,df) - g_{0,\varepsilon}(df,df) = (\varepsilon^2 w^2\kappa^2 - 2w\varepsilon\kappa)\,|\partial_s f|^2 \leq M_2\,\varepsilon\,|\partial_s f|^2.$$

Hence

$$\begin{aligned}
& Q^R_\varepsilon(f) + \alpha\langle f,f\rangle_{L^2(L(1),g_0)} - Q^R_{0,\varepsilon}(f) - \int_{L(1)} dV_0\,W_0\,f^2 - \alpha\langle f,f\rangle_{L^2(L(1),g_0)} \\
= \;& Q^R_\varepsilon(f) - Q^R_{0,\varepsilon}(f) - \int_{L(1)} dV_0\,W_0 f^2 \\
= \;& \int_{L(1)} dV_0\,\{g_\varepsilon - g_{0,\varepsilon}\}(df,df) + \int_{L(1)} dV_0\,(W_\varepsilon - W_0)\,f^2 \\
\leq \;& M_2\,\varepsilon \int_{L(1)} dV_0\,|\partial_s f|^2 + M_1\,\varepsilon \int_{L(1)} dV_0\,f^2 \\
\leq \;& \varepsilon\,M\,\left(Q^R_{0,\varepsilon}(f) + \alpha\langle f,f\rangle_{L^2(L(1),g_0)}\right)
\end{aligned}$$

where $M := \max\{M_1, M_2\}$. This inequality provides us with a *Kato-type estimate* ([6]) from perturbation theory for the difference of $Q^R_\varepsilon(f)$ and $Q^R_{0,\varepsilon}(f) + \int_{L(1)} dV_0\,W_0\,f^2$. This estimate enables us to show that both families converge to the same epi-limit. Using essentially the same steps as in Section 2.4, we obtain:

Proposition 5 *The rescaled and renormalised family of quadratic forms*

$$Q^R_\varepsilon(f) + \alpha\langle f,f\rangle_{L^2(L(1),g_0)},\; \varepsilon > 0$$

epiconverges to

$$\begin{aligned}
& Q^R_0(f) + \int_{L(1)} dV_0\,W_0\,f^2 + \alpha\langle f,f\rangle_{L^2(L(1),g_0)} \\
= \;& \begin{cases} \int_{\mathbb{R}} ds\,\left\{|\partial_s h|^2 + (W_0+\alpha)\,h^2\right\}, & f = u_0\,h \in \mathcal{N} \\ \infty, & \text{else} \end{cases}
\end{aligned}$$

as ε tends to zero.

The operator associated to the limit form is the self-adjoint unbounded operator

$$(\Delta_L + W_0 + \alpha)\,E_0 = E_0\,(\Delta_L + W_0 + \alpha)\,E_0$$

on $L^2(\mathbb{R})$ where $\Delta_L + W_0 + \alpha$ again acts on $\mathcal{N}$ by $(\Delta_L + W_0 + \alpha)u_0 h = u_0(\Delta_L + W_0 + \alpha)h$. Recall that $W_0(s, w) = W_0(s)$ only depends on the s-variable. Now denote the operator associated to the quadratic form Q_ε^R by $\Delta(\varepsilon)$. Deducing strong resolvent convergence from epiconvergence by the same mechanism as in Section 2.5 and inserting the expression (5) for W_0, we end up with the final homogenisation result for the dynamics associated to the induced metric

Theorem 1 *For all $v \in L^2(L(1), g_0)$ and all compact subintervals $I \subset\subset (0, \infty)$, we have*

$$\lim_{\varepsilon\to 0} \sup_{t\in I} \|e^{-\frac{t}{2}\Delta(\varepsilon)} v - E_0\, e^{-\frac{t}{2}(\Delta_{\mathbb{R}} - \frac{1}{4}\kappa^2)}\, E_0 v\|_{L^2(L(1),g_0)} = 0.$$

As in the case of the reference measure, we may write

$$\left\{E_0\, e^{-\frac{t}{2}(\Delta_{\mathbb{R}} - \frac{1}{4}\kappa^2)}\, E_0 v\right\}(s, w) = u_0(w)\, \left\{e^{-\frac{t}{2}(\Delta_{\mathbb{R}} - \frac{1}{4}\kappa^2)} h\right\}(s).$$

The limit dynamic along the submanifold is thus generated by a Schrödinger operator with a potential that reflects geometric properties of the embedding.

3 Conditioned Brownian motion

So far, we considered the heat equation on the tube with Dirichlet boundary conditions. This corresponds to Brownian motion on the tube with absorbing boundary conditions, i.e., the Brownian particle only exists until it reaches the boundary for the first time. In this section we will recall some facts that show how intimately the absorbed and the *conditioned* Brownian motions are connected. Let $L \subset M$ be a Riemannian submanifold of the Riemannian manifold M and $\varepsilon > 0$. By the *ε-conditioned Brownian motion with finite time horizon* $T > 0$, we denote the process associated to the measure μ_ε which is the Wiener measure on M conditioned to the event that the paths do not leave the ε-tube $L(\varepsilon)$ up to time T, i.e.

$$\mu_\varepsilon(d\omega) := \mathbb{W}_M(d\omega \,|\, \omega(s) \in L(\varepsilon), \forall_{s\le T}).$$

Using now the Markov property of $\mathbb{W}_M$, we will derive a time dependent version of a well known formula for μ_ε (see [4]). Let $0 < t < T$ and $q \in L(\varepsilon)$ be a *starting point*. In the sequel, we will write $\omega \in \Omega_{u,v}(\varepsilon)$ for the event $\{\omega(s) \in L(\varepsilon), \forall_{u<s\le v}\}$. Then

$$\begin{aligned}
\mu_\varepsilon^q(d\omega) &= \frac{\mathbb{W}_M^q(d\omega, \omega \in \Omega_{0,T}(\varepsilon))}{\mathbb{W}_M^q(\omega \in \Omega_{0,T}(\varepsilon))} \\
&= \frac{\mathbb{W}_M^q(d\omega, \omega \in \Omega_{0,t}(\varepsilon), \omega \in \Omega_{t,T}(\varepsilon))}{\mathbb{W}_M^q(\omega \in \Omega_{0,T}(\varepsilon))} \\
&= \frac{\mathbb{W}_M^q(d\omega, \omega \in \Omega_{t,T}(\varepsilon) \,|\, \omega \in \Omega_{0,t}(\varepsilon))\mathbb{W}_M^q(\omega \in \Omega_{0,t}(\varepsilon))}{\mathbb{W}_M^q(\omega \in \Omega_{0,T}(\varepsilon))} \\
&= \frac{\mathbb{W}_M^{\omega(t)}(d\omega, \omega \in \Omega_{s,T}(\varepsilon))\mathbb{W}_M^q(\omega \in \Omega_{0,t}(\varepsilon))}{\mathbb{W}_M^q(\omega \in \Omega_{0,T}(\varepsilon))}.
\end{aligned}$$

Now let τ_ε be the *first exit time* of a Brownian particle from the tubular neighbourhood $L(\varepsilon)$. Then $\{\omega \in \Omega_{0,t}(\varepsilon)\}$, $\{\omega(t) \in L(\varepsilon)\}$ and $\{\tau_\varepsilon > t\}$ simply denote the same event. Hence if ν_ε denotes the measure associated to the absorbed Brownian motion, it can be described via its restrictions to the sigma algebras $\mathcal{F}_t$ by

$$\nu_\varepsilon(d\omega \cap \mathcal{F}_t) = \mathbb{W}_M(d\omega \cap \mathcal{F}_t, \tau_\varepsilon < t).$$

The key observation is that in fact

$$\mu_\varepsilon^q(d\omega) = \frac{\nu_\varepsilon^q(d\omega \cap \mathcal{F}_t, \omega(t) \in L(\varepsilon))}{\nu_\varepsilon^q(\omega(T) \in L(\varepsilon))} \nu_\varepsilon^{\omega(t)}(d\omega \cap \mathcal{F}_{T-t}, \tau_\varepsilon > T - t) \tag{9}$$

and hence all terms on the right hand side can be expressed in terms of the absorbed Brownian motion. That implies that if, for instance, we are interested in the distribution of the position of the conditioned particle at time $t < T$ given that the particle starts at time $s < t$ in q or, equivalently, in the associated flow $P^\varepsilon_{s,t} f$ for some bounded measurable f, we obtain

$$\begin{aligned}
P^\varepsilon_{s,t} f(q) &= \int_{\Omega(\varepsilon)} \mu_\varepsilon(d\omega \,|\, \omega(s) = q) f(\omega(t)) \\
&= \int_{L(\varepsilon)} \frac{\nu_\varepsilon^q(\omega(t-s) \in dx)}{\nu_\varepsilon^q(\omega(T-s) \in L(\varepsilon))} \nu_\varepsilon^{\omega(t)}(\tau_\varepsilon > T-t) f(\omega(t)) \\
&= \frac{\int_{L(\varepsilon)} \int_{L(\varepsilon)} p^\varepsilon_{t-s}(q, dx) f(x) p^\varepsilon_{T-t}(x, dy)}{\int_{L(\varepsilon)} p^\varepsilon_{T-s}(q, dz)}
\end{aligned}$$

where $p^\varepsilon_t(x, dy)$ denotes the transition kernel of Brownian motion absorbed at the boundary of the tube. Since the generator of this process is the Dirichlet laplacian $-\frac{1}{2}\Delta_\varepsilon$ on the tube, we may also write

$$P^\varepsilon_{s,t} f = \frac{e^{-\frac{t-s}{2}\Delta_\varepsilon}\left(f\, e^{-\frac{T-t}{2}\Delta_\varepsilon} 1\right)}{e^{-\frac{T-s}{2}\Delta_\varepsilon} 1}. \tag{10}$$

This relation shows that in order to understand the limit of the conditioned Brownian motions as ε tends to zero, one has to understand the properties of the absorbed Brownian motion for small tube diameters. This establishes the connection to what was considered before, namely the renormalised limit of Dirichlet operators.

But there is another property of the path measure that is rather immediate from (10). The associated process is a *time-inhomogeneous Markov process.* To see this, we formally compute a time-dependent generator, namely

$$G_\varepsilon(s) f \;=\; \frac{d}{dh}\left[P^\varepsilon_{s,s+h} f\right]_{h=0} = \frac{1}{e^{-\frac{T-s}{2}\Delta_\varepsilon} 1} \frac{d}{dh}\left[e^{-\frac{h}{2}\Delta_\varepsilon}\left(f\, e^{-\frac{T-s-h}{2}\Delta_\varepsilon} 1\right)\right]_{h=0}.$$

In the sequel, we will write

$$\pi^\varepsilon_u(q) = e^{-\frac{u}{2}\Delta_\varepsilon} 1(q)$$

which denotes the *probability that a particle starting from point q is not absorbed after time u*. Now by

$$2\frac{d}{dh}e^{-\frac{h}{2}\Delta_\varepsilon}(f\,\pi^\varepsilon_{T-s-h}) = -e^{-\frac{h}{2}\Delta_\varepsilon}\Delta_\varepsilon(f\,\pi^\varepsilon_{T-s-h}) + e^{-\frac{h}{2}\Delta_\varepsilon}(f\,\Delta_\varepsilon\pi^\varepsilon_{T-s-h}),$$

the question of which functions f will belong to the domain can be answered by considering the expression $\Delta_\varepsilon(f\,\pi^\varepsilon_{T-s-h})$ in detail. For that, we first assume that $f \in C^2(M)$ and note that by elliptic regularity, for $t > 0$ the absorption probability π^ε_{T-s-h}, which solves the heat equation with Dirichlet boundary conditions for initial value $\pi^\varepsilon_0 = 1$, is smooth in x and zero on the boundary $\partial L(\varepsilon)$. Hence

$$f\,\pi^\varepsilon_{T-s-h} \in \{g \in C^2(L(\varepsilon))\,:\, h|_{\partial L(\varepsilon)} = 0\} \subset \mathcal{D}(\Delta_\varepsilon)$$

even though $f \notin \mathcal{D}(\Delta_\varepsilon)$ and we actually have

$$\Delta_\varepsilon(f\,\pi^\varepsilon_{T-s-h}) = \pi^\varepsilon_{T-s-h}\,\Delta f + 2g(df, d\pi^\varepsilon_{T-s-h}) + f\,\Delta_\varepsilon\pi^\varepsilon_{T-s-h}$$

where the expression Δf is understood as simply applying the laplacian to the function and no conditions are imposed on the behaviour of f at the boundary. Thus, after dividing by π^ε_u which is positive on $L(\varepsilon)$, the generator is formally given by

$$G_\varepsilon(s)f = -\frac{1}{2}\Delta f + g(df, d\log\pi^\varepsilon_{T-s})$$

and describes a Brownian motion with time-dependent drift given by the *logarithmic derivative of the probability to stay within the tube for a given time.* Of course, this relation also holds for Brownian motion conditioned to an arbitrary set with sufficiently smooth boundary. The vector field is singular at the boundary pushing the particle back into the tube.

4 The case of Riemannian submanifolds

In this section, we consider the sequence of measures $\mu_{\varepsilon,\varepsilon>0}$ on the path space of M. As shown above, they are supported by the path spaces of the respective tubes $L(\varepsilon)$ and correspond to time-inhomogeneous Markovian processes given by the Brownian motion conditioned to the tubes. As ε tends to zero, it seems natural to ask whether these measures converge in a suitable sense to a measure supported by the path space of L. In the case that $L \subset M$ is a *closed* (i.e., compact without boundary) Riemannian submanifold, we can answer this question to the affirmative and provide an explicit description of the limit measure. Namely, we have the following statements

1. The sequence μ_ε converges *weakly* on the path space of M to a measure μ_0 which is supported by the path space of the submanifold L.

2. μ_0 is equivalent to the Wiener measure $\mathbb{W}_L$ on the submanifold.

3. The Radon–Nikodym density depends on geometric properties of the submanifold and of the embedding and is of Feynman–Kac Gibbs type given by

$$\frac{d\mu_0}{d\mathbb{W}_L}(\omega) = \frac{1}{Z}\exp\left(-\int_0^T ds\, W(\omega(s))\right) \tag{11}$$

where the *effective potential* $W \in C^\infty(L)$ is given by

$$W = \frac{1}{4}\mathrm{Scal}_L - \frac{1}{8}\|\tau\|^2 - \frac{1}{12}\left(\mathrm{Scal}_M + \overline{\mathrm{Ric}}_{M|L} + \overline{\mathrm{R}}_{M|L}\right),$$

Scal denotes the *scalar curvature*, τ the *tension vector field* and $\overline{\mathrm{Ric}}_{M|L}$, $\overline{\mathrm{R}}_{M|L}$ denote the traces of the Riemannian curvature and Ricci tensor of M only with respect to the subbundle $TL \subset TM$. To be precise, let $j : TL \subset TM$ be the embedding and $j_x : T_xL \subset T_xM$ the induced map on the fibres. Then

$$\overline{\mathrm{Ric}}_{M|L}(x) := \mathrm{tr}(\mathrm{Ric}_x \circ j_x \otimes j_x) = \sum_{r=1}^{l} \mathrm{Ric}_x(e_r, e_r),$$

$$\overline{\mathrm{R}}_{M|L} := \mathrm{tr}(\mathrm{R}_x \circ j_x \otimes j_x \otimes j_x \otimes j_x) = \sum_{r,s=1}^{l} \mathrm{R}_x(e_r, e_s, e_r, e_s)$$

where $e_1, \ldots, e_l$ denotes an orthonormal base of T_xL. Z is a normalisation constant.

Thus, the limit of the conditioned Brownian motions which we can in fact consider as a version of the regular conditional probability given the sigma algebra generated by the distance on the path space, is an intrinsic Brownian motion on the submanifold subject to a certain potential that depends on the geometry of the submanifold and of the embedding.

The proof of these facts consists of a combination of the representation (10) with the homogenisation result Theorem 1 for convergence in finite dimensional distributions together with a tightness result based on a moment estimate. For details, we refer to the forthcoming paper [12].

To understand the significance of the different terms in this density, we consider several examples.

Example 1. Submanifolds $\mathbf{L} \subset \mathbb{R}^{\mathbf{n}}$ in euclidean space. In this case, the density simplifies to

$$\frac{d\mu_0}{d\mathbb{W}_L}(\omega) = \frac{1}{Z}\exp\left(\int_0^T ds\left[\frac{1}{8}\|\tau\|^2 - \frac{1}{4}\mathrm{Scal}_L\right](\omega(s))\right)$$

which is an expression depending on the norm of the tension vector field which depends on the embedding and the scalar curvature which is an intrinsic geometrical property of the submanifold. The other terms vanish due to the fact that euclidean space is flat.

Example 2. The unit sphere $\mathbf{S^{n-1}} \subset \mathbb{R}^\mathbf{n}$. For the unit sphere, the norm of the tension vector field which is proportional to the mean curvature of the sphere hypersurface is constant. The scalar curvature of a round sphere is constant, too. Thus, the integral along the path that shows up in the Radon–Nikodym density is given by a constant time T and does not depend on the given path any more. Therefore, by normalisation, the density is in fact equal to one. Hence, Brownian motion on the sphere can be constructed as the weak limit of the conditioned processes on the tubular neighbourhoods.

Example 3. Totally geodesic submanifolds. First of all, due to our compactness assumption on the submanifolds, we have to make sure that there are relevant cases where our result can be applied. For example, *large spheres* $S^l \subset S^m$, $l < m$, in spheres are closed and totally geodesic. Due to the validity of the *Gauss embedding equations*, there are several equivalent ways to simplify the effective potential. We choose the expression

$$W = \frac{1}{4}\left(\sigma + \frac{1}{3}\mathrm{Scal}^{\perp}\right)$$

where $\mathrm{Scal}^{\perp}_P$ is the scalar curvature of the fibre $\pi^{-1}(p)$ at p and

$$\sigma_p := \sum_{k=1,\ldots,l;\alpha=1,\ldots,m-l} K(e_k \wedge n_\alpha)$$

is the sum of the *sectional curvatures* $K(-)$ of all two-planes $e_k \wedge n_\alpha \subset T_pM$ which are spanned by one element of the orthonormal base $e_1, \ldots, e_l$ of T_pL and one element of the orthonormal base $n_1, \ldots, n_{m-l}$ of N_pL. Note that if $L \subset M$ is a totally geodesic submanifold and M is *locally symmetric*, a property that implies that the curvature tensor R of M is parallel, then the potential W is constant implying, as in the case of the sphere in euclidean space, that $\mu \equiv \mathbb{W}_L$. This holds, for instance, for the large spheres mentioned above.

Example 4. Plane curves. Let $\varphi : S^1 \to \mathbb{R}^2$ be an isometric embedding, i.e., the curve is parametrised by arc length. The intrinsic curvature of a one-dimensional object is zero, hence the effective potential is given by

$$W = -\frac{1}{8}\|\tau\|^2$$

and since φ is parametrised by arc length, the tension vector field is actually given by $\tau = \ddot{\varphi}$. In total, that yields a density for the conditioned motion given by

$$\rho(\omega) = \frac{1}{Z}\exp\left(\frac{1}{8}\int_0^T ds\, \|\ddot{\varphi}\|^2(\omega(s))\right).$$

For an ellipse $\Psi(s) = (a\cos s, b\sin s)$, $s \in [0, 2\pi)$ (note that this is not a parametrisation by arc-length), we will have, for instance,

$$\|\tau\|^2 = \frac{a^2 b^2}{(a^2 \sin^2 s + b^2 \cos^2 s)}.$$

where $2a, 2b > 0$ are the lengths of the major axes. The *sojourn probability* of the Brownian particle is thus largest at the intersection of the ellipse and the longer major axis, where we have the largest curvature and lowest at the intersection of the ellipse and the smaller major axis.

5 Two limits

So far we presented the quadratic form approach to the surface measure because it provides us with the fastest approach to the calculation of the effective potential, directly emphasizing the role played by the reference metric and the logarithmic density $\log\rho$. However, the first approach to surface measures was different.

In [14], the following scheme was introduced. Let $\mathcal{P} = \{0 = t_0 < \cdots < t_k = T\}$ be a partition of the time interval and let x_0 be a fixed point in a smooth compact l-dimensional Riemannian manifold L isometrically embedded into the euclidean space $\mathbb{R}^m$. Then the measure $\mathbb{W}_{\mathcal{P}}$ on $C_{x_0}([0,T], \mathbb{R}^m)$ is defined as the law $\mathbb{W}$ of a Brownian motion in $\mathbb{R}^m$ which is conditioned to be in the manifold L at all times t_i. More precisely, given a cylinder set

$$B := B^{s_1,\ldots,s_m}_{A_1,\ldots,A_m} = \{\omega \in C_{x_0}([0,T], \mathbb{R}^m) \colon \omega_{s_i} \in A_i \; 1 \le i \le m\},$$

with all s_j being different from all t_i, its measure $\mathbb{W}_{\mathcal{P}}$ is defined by

$$\mathbb{W}_{\mathcal{P}}(B) = c_{\mathcal{P}} \int_{\underline{B}} p(\Delta u_0, \Delta x_0) \cdots p(\Delta u_{m+k-1}, \Delta x_{m+k-1}) \xi_1 \otimes \cdots \otimes \xi_{m+k}(dx),$$

where $\mathcal{U} = \{0 = u_0 < \cdots < u_{m+n} = T\}$ is the union of the partitions $\mathcal{P}$ and $\mathcal{S} = \{0 < s_1 \le \cdots < s_m\}$, $x = (x_1, \ldots, x_{m+k})$, $\Delta u_i = u_{i+1} - u_i$, $\Delta x_i = x_{i+1} - x_i$, $\underline{B} = B_1 \times \cdots \times B_{m+k}$ and

$$B_i = \begin{cases} A_j, & \text{if } u_i = s_j \\ L, & \text{if } u_i = t_j \text{ for some } j, \end{cases} \qquad \lambda^i = \begin{cases} V_{\mathbb{R}^m}, & \text{if } u_i = s_j \text{ for some } j, \\ V_L, & \text{if } u_i = t_j \text{ for some } j \end{cases}$$

$V_{\mathbb{R}^n}$ and V_L are the Riemannian volumes on $\mathbb{R}^m$ and L, respectively, $p(t, x, y)$ is the density of the n-dimensional normal distribution $\mathrm{N}(\|y - x\|, t)$, and the normalisation constant $c_{\mathcal{P}}$ is chosen so that $\mathbb{W}_{\mathcal{P}}$ becomes a probability measure.

The following theorem has been proved in a more general setting in [14].

Theorem 2 *As $|\mathcal{P}| \to 0$, the measures $\mathbb{W}_{\mathcal{P}}$ converge weakly to a probability measure $\mathbb{S}_{\rm bb}$ on $C_a([0,T], L)$, which is absolutely continuous with respect to the Wiener measure $\mathbb{W}_L$ on that space, and its Radon–Nykodim density is given by*

$$\rho(\omega) = \frac{1}{Z} \exp \int_0^T ds \left[\frac{||\tau||^2}{8} - \frac{\mathrm{Scal}_L}{4} \right] (\omega(s), \tag{12}$$

where Scal_L *is a scalar curvature of L, τ is the tension vector field of the embedding $L \hookrightarrow \mathbb{R}^m$, and Z is the normalising constant.*

Note that this is exactly the same expression as in Example 1 above. Thus, we obtain the same limiting surface measure by a completely different ansatz. In the sequel, we will discuss the difference between the two approaches and why they yield the same result. The full result from Section 4 can also be obtained from this method. This is explained in [15].

The subscript "bb" in $\mathbb{S}_{\rm bb}$ refers to Brownian bridges, which are a cornerstone of the construction. The idea of the proof is to first show that as the *mesh* $|\mathcal{P}|$ of the partition tends to zero, the corresponding marginal measures on cylinder functions tend to the marginals of the limit measure which is supported by the path space of the submanifold. The main tool is a careful analysis of the short-time asymptotic of the semigroup associated to the conditioned kernel using heat kernel estimates based on the *Minakshisundaram-Pleijel expansion* ([7]).

For some fixed partition, the full conditioned measure is given by the marginals constructed as explained above together with *interpolating Brownian bridges* in euclidean space which fix the path measure between those time-points, where the particle is pinned to the submanifold. Along these Brownian bridges, the particle may still leave the submanifold. Finally, a Large-deviation result for Brownian bridges implies tightness for every sequence $\mathbb{W}_{\mathcal{P}_n}$ of conditioned measures for which the meshes $|\mathcal{P}_n|$ tend to zero. Thus, the sequence of measures converges in the weak sense and the limit measure is determined by the limit of the marginals.

Let us now go back to the surface measure $\mathbb{S}_{\rm hc}$ corresponding to a particle moving under hard constraints. Recall that it is defined as the weak limit

$$\mathbb{S}_{\rm hc} = \lim_{\varepsilon \to 0} \mathbb{W}_\varepsilon, \tag{13}$$

where

$$\mathbb{W}_\varepsilon = \mathbb{W}\left(\,\cdot\, | \, \omega_t \in L(\varepsilon) \text{ for all } t \in [0,T] \right)$$

is the law of a flat Brownian motion conditioned to stay in the ε-neighbourhood of the manifold for the whole time. The following theorem has been proven in [11].

Theorem 3 *The limit in* (13) *exists and so the surface measure* $\mathbb{S}_{\mathrm{hc}}$ *is well-defined. It is absolutely continuous with respect to the Wiener measure* $\mathbb{W}_L$ *on that space, and its Radon–Nykodim density is given by* (12)*.*

Theorems 2 and 3 hence imply immediately that $\mathbb{S}_{\mathrm{hc}} = \mathbb{S}_{\mathrm{bb}}$.

Before explaining the main idea of the proof of Theorem 3, let us discuss the reason why the two measures $\mathbb{S}_{\mathrm{hc}}$ and $\mathbb{S}_{\mathrm{bb}}$ turn out to be equal: Note that the probability for a Brownian motion to be in L at a certain fixed time is zero, and the measures $\mathbb{W}_{\mathcal{P}}$ have been defined using iterated integral of the heat kernel in order to avoid conditioning of the flat Wiener measure to a set of measure zero. Alternatively, one can first force a particle to be in the ε-neighbourhood $L(\varepsilon)$ of L at all times $t_i \in \mathcal{P}$ and then let ε go to zero. More precisely, define

$$\mathbb{W}_{\mathcal{P},\varepsilon} = \mathbb{W}\left(\,\cdot\, | \, \omega_{t_i} \in L(\varepsilon) \text{ for all } t_i \in \mathcal{P} \right).$$

Then, in the weak sense, $\mathbb{W}_{\mathcal{P}} = \lim_{\varepsilon \to 0} \mathbb{W}_{\mathcal{P},\varepsilon}$, and Theorem 2 can be reformulated as

$$\lim_{|\mathcal{P}| \to 0} \lim_{\varepsilon \to 0} \mathbb{W}_{\mathcal{P},\varepsilon} = \mathbb{S}_{\mathrm{bb}}. \tag{14}$$

On the other hand, by the continuity of paths, $\mathbb{W}_{\varepsilon} = \lim_{|\mathcal{P}| \to 0} \mathbb{W}_{\mathcal{P},\varepsilon}$, and hence Theorem 3 is equivalent to

$$\lim_{\varepsilon \to 0} \lim_{|\mathcal{P}| \to 0} \mathbb{W}_{\mathcal{P},\varepsilon} = \mathbb{S}_{\mathrm{hc}}. \tag{15}$$

Thus, (14) and (15) illustrate the fact that $\mathbb{S}_{\mathrm{bb}}$ and $\mathbb{S}_{\mathrm{hc}}$ are obtained by interchanging the two limits. This suggests a more general definition of a surface measure

$$\mathbb{S} = \lim_{\substack{|\mathcal{P}| \to 0 \\ \varepsilon \to 0}} \mathbb{W}_{\mathcal{P},\varepsilon}.$$

It has been proven in [10] that this general limit exists and then, of course, is also absolutely continuous with respect to the Wiener measure $\mathbb{W}_L$ with the Radon–Nykodim density given by (12), since it must coincide with both particular limits $\mathbb{S}_{\mathrm{bb}}$ and $\mathbb{S}_{\mathrm{hc}}$. The proof of this statement follows along the same lines of the proof of Theorem 3 in [11] using additionally the continuity of paths of a Brownian motion.

The intuition behind the proof is the decomposition of the generator $\Delta/2$ of a flat Brownian motion into three components: an operator close to the half of the Laplace–Beltrami operator $\Delta_L/2$, the half of the Laplace operator along the fibres of the tubular neighbourhood, and a differential operator of the first order. More precisely, let us assume without loss of generality that the radius of curvature of L is greater than one and so the orthogonal projection π is well-defined on $L(1)$. For each $x \in L(1)$, denote by $L_x \subset L(1)$ an l-dimensional

Riemannian manifold containing x and parallel to L, that is, for any $y \in L_x$ the tangent space $T_y L_x$ is parallel to $T_{\pi(y)}L$. Further, denote by $\tau(x)$ the tension vector of the embedding $L_x \hookrightarrow \mathbb{R}^m$ at the point x. Then, for any $f \in C^2(\mathbb{R}^m)$ and $x \in L(1)$,

$$(\Delta f)(x) = \left(\Delta_{L_x} f + \Delta_{N_x L} f\right)(x) + \langle \tau, \nabla f \rangle (a), \tag{16}$$

where $N_x L$ denotes the orthogonal space to L at $\pi(x)$. Note that for this decomposition the parallel manifolds L_x need only be defined locally in a neighbourhood of x. Now the projections of the process with generator $\left(\Delta_{L_x} + \Delta_{N_x L}\right)/2$ yield almost a Brownian motion on the manifold, and precisely a Brownian motion in the orthogonal direction, and those two components are almost independent. This makes it plausible that conditioning this process to $L(\varepsilon)$ would lead, in the limit as $\varepsilon \to 0$, to a Brownian motion on the manifold. Finally, the first order term $\langle \tau, \nabla f \rangle$ can be dealt with by a Girsanov transformation, and it would lead to a non-trivial density.

However, there are certain difficulties in realizing this program, in particular the fact that the parallel manifolds L_x do not always exist. Namely, they exist if and only if the normal bundle NL is flat, which is always the case for embeddings into $\mathbb{R}^2$ and $\mathbb{R}^3$ but rather exceptional for the higher-dimensional spaces. Hence, the operators Δ_{L_x} and the vector field τ are not well-defined. In fact, this is also the basic observation that yielded to the construction of the unitary rescaling map in Section 2.2. By this transformation, the vector field is automatically removed. We will now present an alternative ansatz to overcome this difficulty together with an alternative and purely probabilistic proof.

It turns out that there is a vector field v on $L(1)$ such that $(\Delta - l\langle v, \nabla\rangle)/2$ is the generator of a stochastic process which converges to a Brownian motion on the manifold, even though it can no longer be written in the form $\left(\Delta_{L_x} + \Delta_{N_x L}\right)/2$ as in the decomposition (16) above. It is defined by

$$v(x) = \nabla \phi(x) \quad \text{ with } \quad \phi(x) = \log \frac{dV_{\mathbb{R}^n}}{dV_0} = \log \rho,$$

where V_0 is the Riemannian volume associated to the reference metric which can also be thought of as the product measure on $L(1)$ defined by

$$V_0(A) = \int_{\pi(A)} V_{\mathbb{R}^{m-l}}(A_x) dV_L(x), \qquad A \subset L(1)\text{Borel},$$

where $A_x = \pi^{-1}(x) \cap A$. In the particular case when the normal bundle NL is flat and the parallel manifolds exist, v coincides with the vector field τ. Moreover, even without that assumption, $v(x)$ coincides with τ on the manifold L where the tension field is defined.

In contrast to the analytical approach above, where one uses mainly perturbation theory for the variational representation of the generators, the approach here is purely probabilistic and one mainly works with stochastic differential equations and convergence of their solutions. However, in both approaches, the Radon–Nikodym density of the Riemannian volumes associated to the induced and reference metrics plays a crucial role. That is, of course, no surprise since the effective potential is computed from this density.

Let us start by writing down the process $(Y_t)_{t \le T}$ with generator $(\Delta - \langle v, \nabla \rangle)/2$ as a solution of the equation

$$\begin{cases} dY_t = dB_t - \dfrac{1}{2} v(Y_t) dt, \\ Y_0 = a. \end{cases} \tag{17}$$

In order to be able to condition (Y_t) to $L(\varepsilon)$ it would be convenient to first decompose it into its component along the manifold and the orthogonal component. The first one can be naturally defined by $X_t = \pi(Y_t)$, where from now on we assume without loss of generality that (Y_t) denotes the solution of (17) stopped at the time when it leaves $L(1)$. The second component has to describe the difference $Y_t - X_t$, which, at every time t, is an element of the $(m-l)$-dimensional orthogonal space $N_{X_t}L$. If there were a smooth globally defined family of orthonormal bases $(e_i(x)_{1 \le i \le m})_{x \in L}$ then the orthogonal component Z_t of Y_t could be defined by the coordinates of $Y_t - X_t$ with respect to $(e_i(X_t)_{l+1 \le i \le m})$. However, such a global family of bases in general does not exist and a way out is to fix an orthonormal basis $(e_i(a)_{1 \le i \le m})$ at the starting point a and move it along the semimartingale (X_t) using the notion of stochastic parallel translation. Then the initial basis will be transformed to a basis at X_t by an orthogonal matrix U_t, which, as a matrix-valued process, is a solution of the Stratonovich equation of stochastic parallel transport

$$\begin{cases} dU_t = \Gamma_{X_t}(\delta X_t), \\ U_0 = I, \end{cases}$$

where Γ is the Levi-Civita connection on L. Now (Z_t) can be defined as a $\mathbb{R}^{m-l}$-valued process defined by the last $m-l$ coordinates of the vector $Y_t - X_t$ with respect to the moving basis $(e_i(X_t)_{l+1 \le i \le m})$, or, equivalently, by the last $n-l$ coordinates of the vector $U_t^{-1}(Y_t - X_t)$ with respect to the fixed basis $(e_i(a)_{l+1 \le i \le n})$. The $M \times \mathbb{R}^{m-l}$-valued process (X_t, Z_t) fully characterises (Y_t) and is called its Fermi decomposition.

The next step is to replace the Brownian motion (B_t) by another Brownian motion $(\hat{B}_t)$, which is better adjusted to the moving frames. We define it by

$$\hat{B}_t = \int_0^t U_s^{-1} dB_s.$$

It turns out that with respect to this new Brownian motion, the triple (X_t, Z_t, U_t)

satisfies the system of stochastic differential equations

$$\begin{cases} dX_t &= \sigma(X_t, Z_t, U_t) d\hat{B}'_t + c(X_t, Z_t, U_t) dt \\ dZ_t &= d\hat{B}''_t \\ dU_t &= \Gamma_{X_t}(\delta X_t), \end{cases}$$

where $\hat{B}'_t$ are the first l and $\hat{B}''_t$ the last $m-l$ coordinates of $\hat{B}_t$, and the coefficients σ and c can be computed explicitly. The main feature of this system is that the processes (X_t) and (Z_t) are driven by independent Brownian motions and hence conditioning (Z_t) does not affect the Brownian motion driving the process (X_t). Hence, in order to prove that the law of the process (Y_t) conditioned to be in $L(\varepsilon)$ at all times $t_i \in \mathcal{P}$ converges to the Wiener measure on paths in L, it suffices to show that the solution $(X_t^{\mathcal{P},\varepsilon})$ of

$$\begin{cases} dX_t^{\mathcal{P},\varepsilon} &= \sigma(X_t^{\mathcal{P},\varepsilon}, Z_t^{\mathcal{P},\varepsilon}, U_t^{\mathcal{P},\varepsilon}) d\hat{B}'_t + c(X_t^{\mathcal{P},\varepsilon}, Z_t^{\mathcal{P},\varepsilon}, U_t^{\mathcal{P},\varepsilon}) dt \\ dU_t^{\mathcal{P},\varepsilon} &= \Gamma_{X_t^{\mathcal{P},\varepsilon}}(\delta X_t^{\mathcal{P},\varepsilon}), \end{cases}$$

converges to a Brownian motion on L, where $(Z_t^{\mathcal{P},\varepsilon})$ is a $(m-l)$-dimensional Brownian motion conditioned to be in the ε-disc around zero at all times $t_i \in \mathcal{P}$. This can be done using moment estimates, the continuity of paths of $(Z_t^{\mathcal{P},\varepsilon})$, and the explicit form of the coefficients of the equation.

Once it is proven that the surface measure corresponding to the process (Y_t) is the Wiener measure $\mathbb{W}_L$, the rest can be done using the Girsanov transformation. Since the law $\mathcal{L}(Y)$ and the Wiener measure are equivalent with

$$\begin{aligned} \frac{d\mathbb{W}}{d\mathcal{L}(Y)}(\omega) &= \exp\left\{ \frac{1}{2} \int_0^T \langle \nabla\phi(\omega_t), d\omega_t \rangle + \frac{1}{8} \int_0^T ||\nabla\phi(\omega_t)||^2 dt \right\} \\ &= \exp\left\{ \frac{\phi(\omega_1) - \phi(\omega_0)}{2} + \int_0^T \left(-\frac{\Delta\phi(\omega_t)}{4} + \frac{||\nabla\phi(\omega_t)||^2}{8} \right) dt \right\}, \end{aligned}$$

it suffices to show that $\phi|_L = 0$, $\Delta\phi|_L = \mathrm{Scal}_L$, and $\nabla\phi_L = \tau$, which is a routine computation. This leads to the non-trivial density (12) in the surface measure.

6 Two open problems

So far, we considered surface measures for Brownian motion which turned out to be regular in the sense that they are equivalent to the Wiener measure of the respective submanifold. Further investigations indicate that this situation is in fact exceptional and that new interesting phenomena appear. Therefore, we want to conclude the paper with two open problems where the limit measures most likely show exceptional, or, better, non-regular behaviour.

6.1 Tubes with fibers of variable shape

In the previous sections, we considered surface measures constructed by conditioning a Brownian motion to neighbourhoods of constant diameter, meaning that all fibres $L_p(\varepsilon) := \pi^{-1}(p) \cap L(\varepsilon)$, $p \in L$ of the tube were balls in $\pi^{-1}(p)$ centered around p with radius ε. It seems that the situation changes drastically if the radius of each $L_p(\varepsilon)$ depends on p, or, if the fibers $L_p(\varepsilon)$ are even of variable shape. For example, one can take a smooth potential $V\colon \mathbb{R}^n \to [0,\infty)$ such that $V|_L = 0$ and define the ε-neighbourhoods by $L^V(\varepsilon) = \{x \in \mathbb{R}^n \colon V(x) \le \varepsilon\}$, which would correspond to a natural conditioning of a Brownian motion B to the event $\{\sup_{t\le T} V(B_t) \le \varepsilon\}$. If, for instance, the potential grows quadratically with respect to the distance from the submanifold, the fibers $L_p^V(\varepsilon)$ are *ellipses* whose principal axes, in general, depend on the base point. This is connected to considering *soft constraints* (not the hard-wall potential) forcing the particle to remain on the submanifold. Another natural example is to take a smooth function $\alpha : L \to (0,\infty)$ and consider tubular neighbourhoods such that their radius over a point $x \in L$ is given by $\varepsilon\alpha(x)$.

First of all, it is not even clear if the corresponding surface measures exist. However, the leading term of the energy of the conditioned Brownian particle in the second example is believed to be

$$E_\varepsilon(\omega) = -\frac{\lambda}{\varepsilon^2}\int_0^T \frac{dt}{\alpha(\omega_t)^2},$$

where $\lambda < 0$ is the largest eigenvalue of $\Delta_{B_p(1)}/2$ in the ball of radius 1 in the orthogonal space. Hence, as ε becomes smaller, the particle should try to minimise the energy E_ε and is expected to spend more and more time in the regions where the neighbourhood is wide. Thus, the limit measure is expected to be singular with respect to the Wiener measure and to be concentrated on the paths staying in (probably local) maxima of α. In particular, one needs to pass to the Skorokhod space in order to study Brownian motion not starting in the maxima of α, since we expect a Brownian particle to jump instantaneously to one of the minima as ε tends to zero.

6.2 Non-smooth manifolds

Another challenging question is to study conditioning to non-smooth manifolds. Since the projection to the manifold is no longer well-defined, the techniques discussed in the previous sections break down. Moreover, similarly to the previously considered case of non-uniform neighbourhoods, in most natural cases the surface measures are believed not to be supported by the space of continuous functions alone. The situation is not even clear for one-dimensional manifolds such as, for example, polygons in $\mathbb{R}^2$. For L-shaped domains, it has been proved (see [9]) that the conditioned Brownian motions converge in finite-dimensional distributions to the Dirac measure on the path staying in the corner. In particular, if the surface measure exists it is in this case the Dirac measure. The main reason for degeneration of the surface measure is the fact that the integrated curvature of the manifold, which is the main ingredient of the Radon–Nykodim density, is infinite for such manifolds. There are also other reasons for non-smoothness than singular curvature. For example, for a cross of two orthogonal lines, the particle will try to escape to infinity, and there will be no limit at all. The intuitive reason for this difference in behaviour is that the amount of space in the ε-neighbourhood around the singularity compared to the amount of manifold is large for an L-shaped domain and small for the cross.

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