Periodic, almost periodic, and not periodic at all problems

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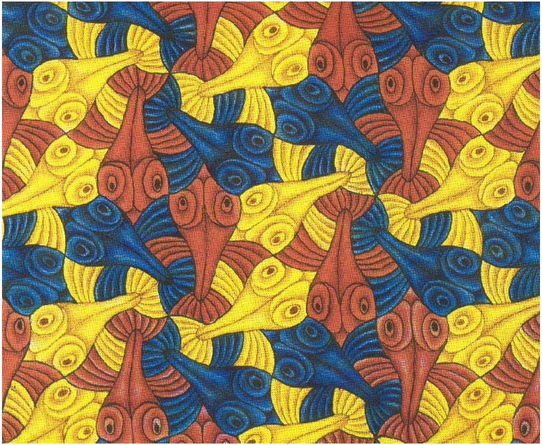

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Periodic differential operators

Periodic Schrödinger operator

$$H = -\Delta + V$$

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If $d = 1$, then the number of gaps is almost always infinite.

Bethe-Sommerfeld conjecture

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If $d \geq 2$, the number of gaps is always finite.

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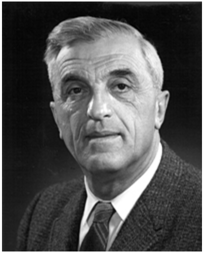
We can also consider the magnetic Schrödinger operator $H = (i\nabla + A)^2 + V$, where $A = A(x)$ is a periodic vector-function with the same lattice of periods Γ. Then the Bethe-Sommerfeld conjecture was proved only for $d = 2$ (A. Mohamed, 1997).

Important tool when working with periodic problems:
Floquet-Bloch decomposition.



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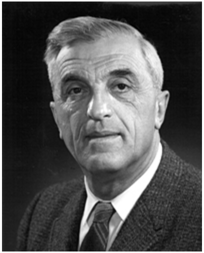

$$H = \int_{\oplus} H(k) dk,$$

where $H(k) := (i\nabla + k)^2 + V$ acts in $L^2(\mathbb{R}^d/\Gamma)$, $k \in \mathbb{R}^d/\Gamma'$, and Γ' is the (analytical) dual to Γ. This means that

$$\sigma(H) = \cup_{k \in \mathbb{R}^d/\Gamma'} \sigma(H(k)).$$

The spectrum of $H(k)$ consists of eigenvalues:

$$\sigma(H(k)) = \{\lambda_1(k) \leq \lambda_2(k) \leq \dots\}.$$

Now we can define

$$\ell_j := \cup_{k \in \mathbb{R}^d/\Gamma'} \lambda_j(k)$$

as the n-th spectral band, so that $\sigma(H) = \cup_j \ell_j$. Then for each λ we can define two functions:

$$m(\lambda) = \#\{j : \lambda \in \ell_j\}$$

(the multiplicity of overlapping) and

$$\zeta(\lambda) = \max_j \max\{t : [\lambda - t, \lambda + t] \subset \ell_j\}.$$

(the overlapping function)

Theorem. (A.Sobolev,LP, 2001)

Let $d = 2, 3, 4$. Then for sufficiently large λ we have:

Dimension	$m(\lambda) \gg$	$\zeta(\lambda) \gg$
2	$\lambda^{\frac{1}{4}}$	$\lambda^{\frac{1}{4}}$
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Unfortunately, the method does not work for $d \geq 5$!

If we want to prove the conjecture, we need to study the eigenvalues of $H(k)$. There are two types of eigenvalues of these operators: stable (corresponding to perturbations of simple eigenvalues, lying not too close to other eigenvalues) and unstable (corresponding to perturbations of a cluster of eigenvalues lying close together). It is relatively straightforward to compute stable eigenvalues with high precision. Unstable eigenvalues cause the main problem.

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Theorem. (G.Barbatis,LP, 2009)

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Theorem. (A.Sobolev,LP, 2010)

Let $d \geq 2$. Then the Bethe-Sommerfeld conjecture holds for operators $H = (-\Delta)^m + q$ with periodic pseudo-differential operators q of order smaller than $2m$. In particular, this conjecture holds for periodic magnetic Schrödinger operators.

Another line of research: the asymptotic behaviour of the (integrated) density of states. The density of states of $H = -\Delta + V$ can be defined by the formula

$$N(\lambda) = N(\lambda; H) := \lim_{L\to\infty} \frac{N(\lambda; H_D^{(L)})}{(2L)^d}.$$

Here, $H_D^{(L)}$ is the restriction of H to the cube $[-L, L]^d$ with the Dirichlet boundary conditions, and $N(\lambda; H_D^{(L)}) = \#\{\lambda_j(H_D^{(L)}) \leq \lambda\}$ is the counting function of the discrete spectrum of $H_D^{(L)}$. More convenient definition:

$$N(\lambda) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d/\Gamma'} N(\lambda, H(k))dk$$

If we put $H_0 = -\Delta$, for positive λ we have

$$N(\lambda; H_0) = C_d \lambda^{d/2},$$

where

$$C_d = \frac{w_d}{(2\pi)^d}$$

and

$$w_d = \frac{\pi^{d/2}}{\Gamma(1 + d/2)}$$

is a volume of the unit ball in $\mathbb{R}^d$.

There is a long-standing conjecture that the density of states of H enjoys the following asymptotic behaviour as $\lambda \to \infty$:

$$N(\lambda) \sim \lambda^{d/2}\Big(C_d + \sum_{j=1}^{\infty} e_j \lambda^{-j}\Big), \tag{1}$$

meaning that for each $K \in \mathbb{N}$ one has

$$N(\lambda) = \lambda^{d/2}\Big(C_d + \sum_{j=1}^{K} e_j \lambda^{-j}\Big) + R_K(\lambda) \tag{2}$$

with $R_K(\lambda) = o(\lambda^{\frac{d}{2}-K})$.

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Yu.Karpeshina (2000) has shown that formula (2) is valid with $K = 1$ (two terms) and $R(\lambda) = O(\lambda^{-\frac{1}{105}})$ when $d = 3$ and $R(\lambda) = O(\lambda^{\frac{d-3}{2}} \ln \lambda)$ when $d > 3$.

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Theorem. (R.Shterenberg,LP, 2008–2010)

Formula (1) *holds in all dimensions.*

Almost-periodic problems

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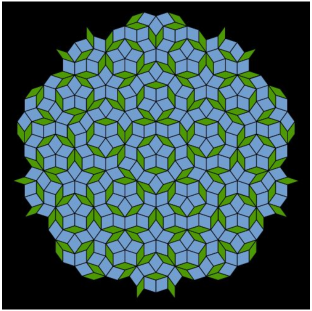

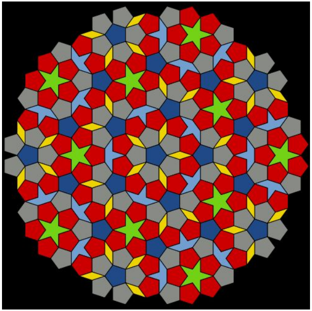

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We want to study the density of states of quasi-periodic operators. First, we need to impose additional condition: let $Z(\Theta)$ be the collection of all linear combination of elements from Θ with integer coefficients. Let $\theta_1, \dots, \theta_d \in Z(\Theta)$. Then either $\{\theta_j\}$ are linearly independent, or $\sum_{j=1}^{d} n_j \theta_j = 0$, where $n_j \in \mathbb{Z}$ and not all n_j are zeros.

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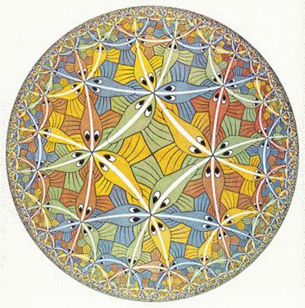

Random problems

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